On the Bass Note of Compact Minimal Immersions

by

Steen Markvorsen

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26

D-5300 Bonn 3
Federal Republic of Germany

MPI/SFB 85-18

1. Introduction

Let M^m be a compact, connected, m-dimensional Riemannian manifold with boundary ∂M^m. The bass note $\lambda_1(M^m)$ is then defined as the first eigenvalue of the fixed membrane (Dirichlet) problem

$$(1.1) \qquad \Delta\psi + \lambda\psi = 0 \quad , \quad \psi \equiv 0 \text{ on } \partial M ,$$

where Δ is the Laplace operator acting on functions ψ by

$$\Delta\psi = \operatorname{div}(\operatorname{grad} \psi) .$$

Now assume that M^m admits a minimal isometric immersion $\Phi: M^m \longrightarrow N^n$, where N^n is an n-dimensional Riemannian manifold. By abuse of notation we will say that M^m is a compact minimal immersion and write $M^m \subset N^n$.

The purpose of this paper is to point out a simple method by which lower and upper bounds for the bass note of M can be obtained in terms of the exterior size of M in N. In most of our results we control this size by assuming M^m is immersed into a regular ball in N. We consider two types of regular balls:

<u>Definition.</u> Let p be a point in N. Then

$$B_R(p) = \left\{ x \in N \mid \operatorname{dist}_N(p,x) \leq R \right\}$$

is a <u>regular ball of radius R</u> around p iff $R \leq \pi/2\sqrt{b}$ and $R < i_N(p)$, where

$$b = \sup \left\{ K(\omega) \mid \omega \text{ a two-plane in } TB_R(p) \right\}$$

$\pi/2\sqrt{b} = \infty$ if $b \leq 0$, and

$i_N(p)$ = injectivity radius of p in N.

Let V^{n-1} be a totally geodesic hypersurface in N. Then <u>a regular ball (or tube) of radius R</u> around V is defined similarly by

$$B_R(V) = \left\{ x \in N \mid \mathrm{dist}_N(V,x) \leq R \right\},$$

where $R \leq \pi/2\sqrt{b}$ and $R < i_N(V)$.

The following theorem due to Hoffman [6] is of the type mentioned above and is obtained from an isoperimetric inequality together with a well known estimate of λ_1 due to Cheeger (cf. [1]).

<u>Theorem A</u> (Hoffman). Let $M^m \subset B_R(p) \subset N^n$ be a compact minimal immersion into a regular ball in N. If $K_N \leq 0$, then

$$(1.2) \qquad \lambda_1(M^m) \geq \frac{1}{4}\left(\frac{m}{R}\right)^2 .$$

We note, that a slight modification of Hoffman's proof actually gives

<u>Theorem 1.</u> If again $M^m \subset B_R(p) \subset N^n$ is compact and minimal and if now $K_N \leq b \geq 0$, then

$$(1.3) \qquad \lambda_1(M^m) \geq \frac{1}{4} m^2 b \cot^2(\sqrt{b}\, R) .$$

These lower bounds are only close to being optimal when m is large. We now state the main result of the present paper. (Note: from now on, whenever we write $M \subset B_R$ we also tacitly assume $M \not\subset \partial B_R$) .

<u>Theorem 2.</u> Let $M^m \subset B_R(p) \subset N^n$ be a compact minimal immersion into a regular ball in any N . Then

$$\lambda_1(M^m) \geq \frac{m\,\pi^2}{4\,R^2} \quad . \tag{1.4}$$

<u>Remarks.</u>

i) If $\dim(M) = m \leq 9$, then (1.4) is always better than (1.2) and (1.3).

ii) The lower bound in (1.4) is sharp for geodesic segments $\gamma = M^1 \subset B_R(p)$ and is generally best possible in the following sense (which will be made precise in Proposition 10):

If there are sufficiently many compact minimal immersions $M_j \subset B_R(p)$ such that $\lambda_1(M_j) = \dim(M_j)\,\frac{\pi^2}{4R^2}$ for all j , then $B_R(p)$ is a standard hemisphere of constant curvature $K = \left(\frac{\pi}{2R}\right)^2$.

iii) Our method of proof does not apply any isoperimetric inequality but relies heavily on a well known beautiful observation (Theorem C) due to Barta. We also use this observation to get <u>upper</u> bounds on $\lambda_1(M)$ in many cases. It is furthermore possible to give corresponding results for compact immersions with just a uniform bound on the length of the mean curvature vector H . We shall not pursue this further, but only state the following

<u>Proposition 3.</u> Let $M^m \subset B_R(p) \subset N^n$ be a compact immersion into a regular ball. If $K_N \leq b \geq 0$ and $\|H\| \leq \frac{m-1}{m} \sqrt{b} \cot \sqrt{b} R$, then

$$\lambda_1(M^m) \geq \frac{\pi^2}{4 R^2} . \tag{1.5}$$

iv) In most of our results we may have $m = n$, in which case M^n is a compact <u>domain</u> in N^n and the adjective "minimal" should therefore be suppressed (i.e. $\alpha = H = 0$ and $T_q M = T_q N$ in Propos. 8). Cf. Li and Yau [10] and Kasue [9] for more general results on domains.

If the receiving space N is a space form, one can say much more much sharper than Theorem 2. Using heat kernel comparison theory Cheng, Li and Yau have shown the following result.

<u>Theorem B</u> ([4]). Let $M^m \subset B_R(p) \subset \tilde{N}^n$ be a compact minimal immersion into a regular ball in a space form $\tilde{N}^n$. Let D_R^m be the totally geodesic disc of dimension m and radius R in $\tilde{N}^n$.
Then

$$\lambda_1(M^m) \geq \lambda_1(D_R^m) , \tag{1.6}$$

and equality occurs if and only if $M^m = D^m_R$.

Remark. For comparison of this with Theorems 1 and 2 we note, that if $\tilde{N}^n = \mathbb{R}^n$, then $\lambda_1(D^m_R) = (j_k/R)^2$, where $j_k(\sim k \sim \frac{1}{2}m$ as $m \longrightarrow \infty)$ is the smallest positive zero of the Bessel function J_k of order $k = \frac{1}{2}(m-2)$.

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2. Some preliminary results.

We assume that N^n allows a totally geodesic hypersurface V and consider a regular tube $B_R(V) \subset N^n$. In this tube the distance function $\eta(x) = \mathrm{dist}_N(V,x)$ is well defined and smooth outside V. Let $T = \mathrm{grad}(\eta)$ and $q \in B_R(V) - V$. Then $T(q) = \dot{\gamma}(\eta(q))$ where $\gamma : [0,\eta(q)] \longrightarrow N$ is the unique unit speed minimal geodesic from V to q. Now let X be a unit vector in T_qN and $X(s)$ be the Jacobi vector field along γ generated in the usual way by X through minimal geodesics to V such that $X(\eta(q)) = X$ and $X'(0) = \nabla_{\dot{\gamma}(0)} X = 0$. Then (cf. [2] pp. 20-21).

$$\text{(2.1)} \qquad \mathrm{Hess}_N(\eta)\Big|_q(X,X) = I_\gamma(X,X) - \int_0^\eta (T\langle X,T\rangle)^2 ds \quad ,$$

where I_γ is the index form along γ. If we define

$$X^\perp(s) = X(s) - \langle T(s), X(s)\rangle\, T(s) \;,$$

so that

$$\|\nabla_T X\|^2 = \|\nabla_T X^\perp\|^2 + (T\langle X,T\rangle)^2$$

we finally get

$$\text{(2.2)} \qquad \mathrm{Hess}_N(\eta)\Big|_q(X,X) = I_\gamma(X^\perp,X^\perp) \quad \text{for all} \quad X \in T_qN \,.$$

Standard index comparison theory now gives

Proposition 4. If $K_N \leqq b$ (respectively $K_N \geqq b$), $b \in \mathbb{R}$, then for every X in the unit bundle $S^1(B_R(V) - V)$ we have

$$\text{(2.3)} \qquad \mathrm{Hess}_N(\eta)(X,X) \geq (\leq) \; f_b(\eta)\cdot(1 - \langle T,X\rangle^2) ,$$

where

$$\text{(2.4)} \qquad f_b(t) = \begin{cases} -\sqrt{b}\tan(\sqrt{b}\,t) & \text{if} \quad b > 0 \\ 0 & \text{if} \quad b = 0 \\ +\sqrt{-b}\tanh(\sqrt{-b}\,t) & \text{if} \quad b < 0 . \end{cases}$$

In the same way we get for the distance function $\rho(x) = \mathrm{dist}_N(p,x)$ in a regular ball $B_R(p)$:

Proposition 5. If $K_N \leqq b$ (respectively $K_N \geqq b$), $b \in \mathbb{R}$, then for every $X \in S^1(B_R(p)-p)$ we have

$$\text{(2.5)} \qquad \mathrm{Hess}_N(\rho)(X,X) \geq (\leq) \; h_b(\rho)\cdot(1 - \langle T,X\rangle^2) ,$$

where

$$\text{(2.6)} \qquad h_b(t) = \begin{cases} \sqrt{b}\,\cot\sqrt{b}\,t & \text{if} \quad b > 0 \\ \dfrac{1}{t} & \text{if} \quad b = 0 \\ \sqrt{-b}\,\coth\sqrt{-b}\,t & \text{if} \quad b < 0 . \end{cases}$$

Remark. In [8] Kasue proved similar comparison theorems for more general distance functions and showed that in some cases they hold true in a distributional sense across the respective cut loci (cf. Theorem 18, where we shall use this approach).

The idea is now to restrict suitable modifications of the functions ρ and η to the minimal immersions $M^m \subset N^n$. In order to do so we need the following lemmata.

Let $\mu : N \longrightarrow \mathbb{R}$ be any smooth function on N. Then the restriction $\tilde{\mu} = \mu|_M$ is a smooth function on M and the Hessians $\mathrm{Hess}_N(\mu)$ and $\mathrm{Hess}_M(\tilde{\mu})$ are related as follows.

<u>Lemma 6</u> (cf. [7] p. 713).

(2.7) $$\mathrm{Hess}_M(\tilde{\mu})(X,X) = \mathrm{Hess}_N(\mu)(X,X) + \langle \mathrm{grad}_N(\mu), \alpha(X,X) \rangle$$

for all $X \in TM^m \subset TN^n$, where α is the second fundamental form of M in N.

If we modify μ to $F \circ \mu$ by a smooth $F : \mathbb{R} \longrightarrow \mathbb{R}$ we get

<u>Lemma 7</u>

(2.8) $$\mathrm{Hess}_N(F \circ \mu)(X,X) = F''(\mu) \langle \mathrm{grad}_N(\mu), X \rangle^2 + F'(\mu)\, \mathrm{Hess}_N(\mu)(X,X)$$

for all $x \in TN^n$.

In the following we write $\mu = \tilde{\mu}$. Combining (2.7) and (2.8) we obtain

(2.9) $$\mathrm{Hess}_M(F \circ \mu)(X,X) = F''(\mu) \langle \mathrm{grad}_N(\mu), X \rangle^2 + F'(\mu)\, \mathrm{Hess}_N(\mu)(X,X) + \langle \mathrm{grad}_N(\mu), \alpha(X,X) \rangle$$

Now again restrict attention to a regular ball B_R in N of either type. Assume $M^m \subset B_R$ and let μ be the corresponding distance function in B_R. We will always choose F such that $F \circ \mu$ is smooth on M. Now $\mathrm{grad}_N(\mu) = T$ and combining (2.9) with (2.5) and (2.3) we get the following inequalities, where $(\sim)_i$ means either $(\geqq)$ or $(\leqq)$ for each $i = 1,2,3$.

<u>Proposition 8.</u> Let $M^m \subset B_R(V) \subset N^n$ be an isometric immersion (not necessarily minimal) into a ball $B_R(V)$ with $R < i_N(V)$. Assume that $F \circ \eta$ is smooth on M^m and that $F'(\sim)_1 0$ throughout $[0,R]$.

If $K_N(\sim)_2\, b$ for some constant $b \in \mathbb{R}$, then for every $X \in TM \subset TN$

$$(2.10)\quad \mathrm{Hess}_M(F\circ\eta)(X,X) \quad (\sim)_3 \quad (F''(\eta) - F'(\eta)f_b(\eta)) < T,X >^2 + F'(\eta)\Big(f_b(\eta) + < T,\alpha(X,X) > \Big)$$

If $\{X_1,\dots,X_m\}$ is an orthonormal basis of $T_qM \subset T_qN$, we therefore get

$$(2.11)\quad \Delta_M(F\circ\eta)\Big|_q \quad (\sim)_3 \quad (F'' - F'f_b)\cdot\Big(\sum_{j=i}^{m} < T,X_j >^2\Big) + mF'(f_b + < T,H(q) >) \quad ,$$

where $H(q)$ is the mean curvature vector of M at q in N.

If $M^m \subset B_R(p) \subset N^n$ we get similarly

$$(2.12)\qquad \mathrm{Hess}_M(F\circ\rho)(X,X) \quad (\sim)_3 \quad (F''(\rho) - F'(\rho)h_b(\rho)) < T,X >^2 + F'(\rho)\left(h_b(\rho) + <T, \alpha(X,X)>\right) \quad ,$$

and thus

$$(2.13)\qquad \Delta_M(F\circ\rho)\Big|_q \quad (\sim)_3 \quad (F'' - F'h_b)\cdot\left(\sum_{j=1}^{m} < T,X_j >^2\right) \quad + \quad mF'(h_b + < T,H(q)>) \quad .$$

Clearly $(\sim)_3$ depends uniquely on $(\sim)_1$ and $(\sim)_2$ as follows: Let $(\geq)$ correspond to $+1$ and $(\leq)$ correspond to -1 ; Then $(\sim)_3 = -(\sim)_1\cdot(\sim)_2$.

We can now prove Theorem 1 using only a modification of Hoffman's argument.

3. Proof of Theorem 1.

Since $M^m \subset B_R(p)$ is minimal, we have $H \equiv 0$ and now choosing $F(t) = 1 - \cos(\sqrt{b}\,t)$ we get $F'' - F'h_b \equiv 0$. Thus from (2.13)

$$\Delta_M(F\circ\rho) \quad \geq \quad mF'(\rho)h_b(\rho) = m\,b\cos\sqrt{b}\,\rho$$

Now let $X = \mathrm{grad}_M(F\circ\rho) \in \mathcal{X}(M)$ and let Ω be a domain in M^m with outward pointing unit normal vector ξ on $\partial\Omega$.

Then

$$F'(R)\,\mathrm{vol}(\partial\Omega) \geq \int_{\partial\Omega} < X, \xi > *1 = \int_{\Omega} \mathrm{div}_M(X) *1$$

$$= \int_{\Omega} \Delta_M(F\circ\rho) * 1 \geq m\,b\cos(\sqrt{b}R)\,\mathrm{vol}(\Omega) .$$

Therefore $\mathrm{vol}(\partial\Omega)/\mathrm{vol}(\Omega) \geq m\sqrt{b}\cot(\sqrt{b}\,R)$,

so that the Cheeger constant (cf. [1])

$$h = \inf\left\{\mathrm{vol}(\partial\Omega)/\mathrm{vol}(\Omega) \;\middle|\; \Omega \text{ is a compact domain on } M \text{ with } \Omega \cap \partial M = \emptyset \right\}$$

satisfies $h \geq m\sqrt{b}\cot(\sqrt{b}\,R)$. The theorem then follows from Cheeger's inequality.

□

4. Direct estimates of λ_1

A direct two sided bound on λ_1 of a compact manifold M is obtained by the following result of Barta (cf. [3]) and Kasue's generalization thereof.

Theorem C (Barta). Let M be a Riemannian manifold with boundary ∂M . Let ϕ be a smooth function on M which is positive in $\overset{\circ}{M}$ and zero on ∂M .

Then

$$\inf(-\Delta_M \phi/\phi) \leq \lambda_1(M) \leq \sup(-\Delta_M\phi/\phi) \quad , \tag{4.1}$$

and either one of the equalities occurs if and only if ϕ is the first eigenfunction.

<u>Theorem D</u> (Kasue, [9]). Let M be a Riemannian manifold with boundary ∂M . Let ψ be a <u>continuous</u> function on M which is positive in $\mathring{M}$ (but not necessarily zero on ∂M) . If ψ satisfies

$$\Delta_M\psi + \kappa\psi \leq 0 \quad \text{as a distribution on } \mathring{M} \tag{4.2}$$

for some constant κ , then

$$\lambda_1(M) \geq \kappa \quad . \tag{4.3}$$

If ψ is smooth on an open dense subset of M , then equality holds in (4.3) if and only if ψ is the first eigenfunction.

<u>Remark.</u> It follows in particular, that the lower bound in (4.1) is still valid even if ϕ does not vanish on ∂M .

<u>5. Applications to minimal immersions</u>.

We now prove Theorem 2 by establishing the following more general result.

<u>Theorem 9.</u> Let $M^m \subset B_R(p) \subset N^n$ be a compact minimal immersion into a regular ball. Assume that $K_N \leq b \in \mathbb{R}$, and that $M^m \cap B_r(p) = \emptyset$, $0 \leq r < R$. Then

$$\lambda_1(M) \geq \kappa(m,r,R) = (m-1)\frac{\pi}{2R}\tan\left(\frac{\pi}{2}\frac{r}{R}\right)h_b(r) + \frac{\pi^2}{4R^2} \tag{5.1}$$

$$\geq m\frac{\pi^2}{4R^2} \tag{5.2}$$

$$\geq m\,b \tag{5.3}$$

<u>Remark.</u> The inequality (5.2) - which corresponds to Theorem 2 - only depends on b implicitly through the regularity assumption for $B_R(p)$. It follows from (5.1) by letting $r \longrightarrow 0$. Clearly (5.3) is only interesting if $b > 0$, in which case it follows from $R \leq \pi/2\sqrt{b}$.

<u>Proof of Theorem 9.</u> We only prove (5.1). Let $F(t) = \cos(k\,t)$, $k = \pi/2R$. Then $F''(\rho) - F'(\rho)\,h_b(\rho) \geq 0$ for all $\rho \leq R$, and Proposition 8 (2.13) gives

$$\Delta_M(F\circ\rho) \leq (m-1)\,F'(\rho)h_b(\rho) + F''(\rho) \tag{5.4}$$

so that

$$\begin{aligned}\Delta_M(F\circ\rho) + \kappa(m,r,R)\,(F\circ\rho) &\leq (m-1)F'(\rho)h_b(\rho) + \\ &\quad F''(\rho) + \kappa(m,r,R)F(\rho) \leq 0\end{aligned} \tag{5.5}$$

for all $\rho \in [r,R]$. The theorem now follows from Kasue's inequality (4.3).

□

As mentioned in the introduction, we also get a rigidity result:

Proposition 10. Let $B_R(p)$ be a regular ball in a Riemannian manifold N with $K_N \leq b \in \mathbb{R}$. Assume that for every $q \in \partial B_R(p)$ there exist compact minimal immersions

$$M_j^{m_j} \subset B_R(p) - B_{r_j}(p) \ , \quad 0 \leq r_j < R \ ,$$

such that

$$T_q N = \operatorname{span}\left\{ (\partial B_R(p))_q^{\perp} \cup \bigcup_j T_q M_j^{m_j} \right\}$$

and such that

$$\lambda_1(M_j) = \kappa(m_j, r_j, R) \quad \text{for all } j \ .$$

Then $\lambda_1(M_j) = m_j b$ for all j and $B_R(p)$ is isometric to a standard hemisphere of constant curvature $K \equiv b > 0$.

Proof. With the notation of the proof of Theorem 9 we now have for every M_j that $F \circ \rho|_{M_j}$ is the first eigenfunction of Δ_{M_j}. In particular the last inequality in (5.5) is an equality for all $\rho \in [r_j, R]$. This is only possible if $b > 0$ and $k = \pi/2R = \sqrt{b}$. But then $F'' - F'h_b \equiv 0$ so that from equalities in (5.4), (2.13) and (2.12) we finally also have equality in (2.5). This is only possible if the sectional curvature along the Jacobi field from p to $X \in TM_j$ is identically b. By assumption we can span T_qN by such vectors X. It follows that $K_{B_R(p)} \equiv b$, where $R = \pi/2\sqrt{b}$, and the proposition is proved.

□

As a corollary to this proof we also obtain

<u>Proposition 11.</u> Let $M^{n-1} \subset B_R(p) - B_r(p) \subset N^n$ be a compact minimal hyper-immersion such that M^{n-1} is everywhere transversal to the vector field $T = \mathrm{grad}(\mathrm{dist}_N(p,\cdot))$. If $K_N \leq b$, and $\lambda_1(M) = \kappa(n-1,r,R)$, then $\lambda_1(M) = (n-1)b$ and M^{n-1} is isometric to a compact minimal immersion into a standard hemisphere $B_{\pi/2\sqrt{b}}(p)$ of constant curvature $K \equiv b$.

□

If K_N is nonpositive it is possible to give a lower bound for λ_1 which is better than (5.1) when r is positive and close to R.

<u>Proposition 12.</u> Let $M^m \subset B_R(p) - B_r(p) \subset N^n$ be a compact minimal immersion into a regular ball with $K_N \leq -b \leq 0$.

Then

$$\lambda_1(m) \geq \tau(m,r,R) \tag{5.6}$$

$$= mb \frac{\cosh(\sqrt{b}r)}{\cosh(\sqrt{b}R) - \cosh(\sqrt{b}r)}$$

$$\geq \frac{mb}{\cosh(\sqrt{b}R) - 1} .$$

In particular, if $b = 0$, then

$$\lambda_1(M) \geq \frac{2m}{R^2 - r^2} . \tag{5.7}$$

<u>Proof.</u> For F we now choose $F(t) = \cosh(\sqrt{b}R) - \cosh(\sqrt{b}t)$, so that $F'' - F'h_{-b} \equiv 0$ and

$$\Delta_M(F\circ\rho) \leq m F'(\rho) h_{-b}(\rho) = - mb \cosh\sqrt{b}\rho .$$

Thus $\Delta_M(F\circ\rho) + \tau(m,r,R)\ (F\circ\rho) \leq 0$ for all $\rho \in [r,R]$, and the result follows from (4.3).

□

Using Barta's Theorem C we obtain an upper bound on $\lambda_1(M)$ in the following somewhat special situation. This result is also best possible in a sense analoguous to Proposition 10 and 11.

Proposition 13. Let $M^m \subset B_{\pi/2\sqrt{b}}(p) \subset N$ be a compact minimal immersion into N with $K_N \geq b > 0$. The ball $B_{\pi/2\sqrt{b}}$ need not be regular, but we still assume $\pi/2\sqrt{b} < i_N(p)$.

If $\partial M^m \subset \partial B_{\pi/2\sqrt{b}}$, then

(5.8) $$\lambda_1(M) \leq m b .$$

Proof. Let $F(t) = \cos(\sqrt{b}t)$. Then again $F'' - F'h_b \equiv 0$ and (2.13) now gives

$$\Delta_M(F\circ\rho) \geq m F'(\rho) h_b(\rho) = -mb\cos\sqrt{b}\rho$$

so that

$$\Delta_M(F\circ\rho) + mb(F\circ\rho) \geq 0 .$$

Thus we have:

$$\sup\Big(-\Delta_M(F\circ\rho)/(F\circ\rho)\Big) \leq mb ,$$

and the result follows from Theorem C.

□

We now turn attention to compact minimal immersions into balls (tubes) around totally geodesic hypersurfaces $V^{n-1} \subset N^n$. For this we first note the following fact.

Propositon 14. Let $M^m \subset B_R(V) \subset N^n$ be a compact minimal immersion into a regular tube $B_R(V)$ in N with $K_N \leq 0$. If $\partial M \subset V$, then $M \subset V$.

Proof. Recall that $\eta = \mathrm{dist}_N(V,\cdot)$. Now let $F(t) = \frac{1}{2}t^2$. Then by Proposition 8 (2.11) , $\Delta_M(F\circ\eta) \geq m > 0$ which contradicts the existence of an interior maximum of $F\circ\eta|_M$ if we had both $\partial M \subset V$ and $M \not\subset V$.

□

Remark. Similarly, if $K_N > 0$ and $\partial M = \emptyset$, then $M \cap V \neq \emptyset$.

If V is two-sided in N , then V separates every $B_R(V)$ into two connected components. In such a case we denote either one of these components by $B_R^+(V)$.

Proposition 15. Let $M^m \subset B_R^+(V) \subset N^n$ be a compact minimal immersion into a regular (half-) tube such that $\partial M \subset V \not\supset M$. Assume $K_N \leq b$.

Then $b > 0$ and

$$\lambda_1(M) \leq (m-1)b + \frac{\pi^2}{4R^2} . \tag{5.9}$$

Proof. The first claim follows from Propos. 14. Now let $F(t) = \sin(kt)$, $k = \pi/2R \geq \sqrt{b}$. Then $F''(\eta) - F'(\eta) f_b(\eta) \leq 0$ for all $\eta \leq R$ and from (2.11)

$$\Delta_M(F \circ \eta) \geq (m-1) F'(\eta) f_b(\eta) + F''(\eta) .$$

The proof may now be completed as the proof of Propos. 13.

□

In a similar way, using $F(t) = \cos(kt)$, $k = \pi/2R$ one obtains:

Proposition 16. Let $M^m \subset B_R(V) \subset N^n$ be a compact minimal immersion into a (full) tube $B_R(V)$. The tube need not be regular, but we still assume $R \leq \pi/2\sqrt{b}$ and $R < i_N(V)$. If $K_N \geq b \geq 0$ and if $\partial M \subset \partial B_R(V)$, then

$$\lambda_1(M) \leq \frac{\pi^2}{4R^2} . \tag{5.10}$$

□

We close this section by establishing corresponding lower bounds for $\lambda_1(M)$ in case of negatively curved (resp., positively curved) ambient spaces.

Proposition 17. Let $M^m \subset B_R^+(V) - B_r^+(V) \subset N^n$, $0 \leq r < R$, be a compact minimal immersion into a regular tube (-annulus) in N. If $K_N \leq -b < 0$, then

(5.11) $$\lambda_1(M) \geq m\,b \frac{\sinh(\sqrt{b}r)}{\sinh(\sqrt{b}R)-\sinh(\sqrt{b}r)}$$

Proof. The proof is more or less a repetition of the proof of Propos. 12 only now with $F(t) = \sinh(\sqrt{b}R) - \sinh(\sqrt{b}t)$.

□

The next result is stated somewhat differently from the previous ones since it applies to a much more general class of minimal immersions.

Theorem 18. Let $M^m \subset N$ be a compact minimal immersion (with or without boundary) into N . Assume that $K_N \geq b > 0$ and that N admits a totally geodesic hypersurface V which separates N .
Let M_i be any connected component of $M - (M \cap V)$.
Then

(5.12) $$\lambda_1(\bar{M}_i) \geq m\,b \ .$$

Remark. It should be noted, that the following remains an open problem: If $M^{n-1} \subset S_1^n$ is a compact $(\partial M = \emptyset)$ embedded minimal hypersurface of the standard unit sphere, is it then true that $\lambda_1(M^{n-1}) = n-1$?

Proof. Following the work of A. Kasue [9] we only have to establish inequality (4.2) in the distributional sense on $\mathring{M}_i$.

For the continuous function $\psi : \overset{\circ}{M}{}^m \longrightarrow \mathbb{R}$ we choose $\psi = F(\eta) = \sin(\sqrt{b}\eta)$. It then follows from Kasue's arguments on pp. 325-326 of [8] (suitably restricted to work on $M^n \subset N$) , that (2.11) holds for this $F \circ \eta$ in the distributional sense. Thus from $F'' - F'f_b \leq 0$ we get $\Delta_M(F\circ\eta) + m\,b(F\circ\eta) \leq 0$ as a distribution on $\overset{\circ}{M}$. The result then follows from Theorem D.

□

6. Some related estimates.

There is no direct relation between the volume $\mathrm{vol}(M^m)$ and the bass note $\lambda_1(M^m)$ except for dimension $m = 1$ where the relation is always sharp: Let M^1 be compact with boundary; then $\lambda_1(M^1) = \pi^2/\mathrm{length}^2(M^1)$. Thus every estimate in the previous section corresponds immediately to an upper or lower bound on the length of geodesic segments in the respective balls.

For general dimensions, however, we can use the result of Theorem 18 to obtain a relation between $\mathrm{vol}(M^m)$ and the transversality $T(M,V)$, which we now define.

<u>Definition</u>. Let V be a hypersurface in N and $M \subset N$ an isometric immersion. Then

$$(6.1) \qquad T(M,V) = \int_{M\cap V} \cos \measuredangle\,(M,V^{\perp}) * 1 \;,$$

where $\measuredangle(M,V^{\perp})(q)$ is the angle between any normal ξ to V and its projection $\pi_M(\xi)$ to M at $q \in M \cap V \subset N$.

Theorem 19. Let $M^m \subset N^n$ be a compact minimal immersion without boundary. Assume that N is positively K-pinched by $0 < \sigma \leq K_N \leq \delta$ for some constants σ, δ and assume that N admits a totally geodesic hypersurface V which separates N. If M^m is contained in a regular ball around V, then

$$(6.2) \qquad \mathrm{vol}(M^m) > \frac{2\sqrt{m\sigma+\delta}}{m\delta} T(M,V) \quad .$$

Proof. Let M_i be a closed connected component of $M - (M \cap V) \neq M$. Then from Thm. 18 we have

$$(6.3) \qquad \lambda_1(M_i) \geq m\sigma \quad .$$

The minimum principle (cf. [3]) gives

$$(6.4) \qquad \lambda_1(M_i) \leq \int_{M_i} \|\mathrm{grad}_{M_i}(\sin\sqrt{\delta}\,\eta)\|^2 * 1 \Big/ \int_{M_i} \sin^2\sqrt{\delta}\,\eta * 1$$

$$\leq \delta \int_{M_i} \cos^2\sqrt{\delta}\,\eta * 1 \Big/ \int_{M_i} \sin^2\sqrt{\delta}\,\eta * 1$$

$$= \delta\left(\mathrm{vol}(M_i) \Big/ \int_{M_i} \sin^2\sqrt{\delta}\,\eta * 1\right) - \delta$$

Combining this with (6.3) we get

$$(6.5) \qquad (m\frac{\sigma}{\delta} + 1) \int_{M_i} \sin^2\sqrt{\delta}\,\eta \;*1 \;\leq\; \mathrm{vol}(M_i)$$

Now Schwartz' inequality implies

$$(6.6) \qquad \sqrt{m\frac{\sigma}{\delta} + 1} \int_{M_i} \sin\sqrt{\delta}\,|\eta| \;*1 \;<\; \mathrm{vol}(M_i)\ .$$

On the other hand, if we let $F(t) = \sin\sqrt{\delta}\,t$ we get $F''(\eta) - F'(\eta)f_\delta(\eta) \equiv 0$ so that from Propos. 8 (2.11):

$$\Delta_{M_i}(F\circ\eta) \geq mF'(\eta)f_\delta(\eta) = -m\,\delta\sin\sqrt{\delta}\,\eta$$

Therefore

$$(6.7) \qquad m\delta \int_{M_i} \sin\sqrt{\delta}\,\eta \;*1 \;\geq\; -\int_{M_i} \Delta_{M_i}(F\circ\eta) * 1$$

$$= -\int_{\partial M_i} < \mathrm{grad}\,(F\circ\eta),\ \zeta_{out} > *1$$

$$= -\int_{\partial M_i} F'(0) < \xi_{in}, \zeta_{out} > *1$$

$$= +\sqrt{\delta}\ T(M_i,V)\ ,$$

where ξ_{in} is the unit normal of V pointing towards M_i

at $\partial M_i \subset V$ and ζ_{out} is the outward pointing unit normal vector of M_i at ∂M_i .

Summing over i gives

$$(6.8) \qquad \int_M \sin\sqrt{\delta}\,|\eta| * 1 \geq \frac{2}{m\sqrt{\delta}}\, T(M,V) .$$

With (6.6) we therefore have

$$\sqrt{m\,\frac{\sigma}{\delta} + 1}\;\frac{2}{m\sqrt{\delta}}\, T(M,V) < \mathrm{vol}(M) \quad ,$$

which is the desired inequality.

$\square$

<u>Remark.</u> In [5] Choi and Wang show that if M^2 is a compact orientable <u>embedded</u> minimal hypersurface of a compact orientable Riemannian 3-manifold N^3 with $\mathrm{Ric}_N \geq 2\sigma > 0$, then $\mathrm{vol}(M^2) \leq 8\pi(g+1)/\sigma$, where g is the genus of M^2 . Consequently, if the assumptions of Theorem 19 is also satisfied, then there is an upper bound on $T(M^2,V^2)$ in terms of g, η and σ .

As a final application of the results in section 5 we now prove the following theorem on minimal immersions into $\mathbb{R}^n$ which have their boundaries on parallel hyperplanes (such as a parallelly truncated catenoid in $\mathbb{R}^3$) .

<u>Theorem 20.</u> Let $M^m \subset \mathbb{R}^n$ be a compact minimal immersion with $\partial M^m \subset \pi_1 \cup \pi_2$, where π_1 and π_2 are parallel hyperplanes in $\mathbb{R}^n$ with distance L . Assume that $M^m \subset B_R(p) - B_r(p)$ for some $p \in \mathbb{R}^n$, $0 \leq r < R$.
Then

$$(6.9) \qquad L^2 \leq \min \frac{1}{2m} \left\{ \pi^2 (R^2 - r^2) \ , \ 8R^2 \right\} ,$$

and

$$(6.10) \qquad \mathrm{vol}(M^m) \geq L^m \frac{(4\pi)^{m/2}}{e\,\pi^m} .$$

<u>Proof.</u> The situation corresponds to those in Theorem 2, Proposition 12 and Proposition 16. Thus we get

$$(6.11) \qquad \max \left\{ \frac{m\pi^2}{4R^2} , \frac{2m}{R^2 - r^2} \right\} \leq \lambda_1 (M^m) \leq \frac{\pi^2}{L^2} ,$$

and (6.9) follows immediately. The volume bound follows from

$$(6.12) \qquad \frac{(4\pi)^{m/2}}{e\ \mathrm{vol}(M)} \leq \left(\lambda_1 (M^m) \right)^{m/2} \text{and} \quad \lambda_1 (M^m) \leq \frac{\pi^2}{L^2} ,$$

where the first inequality is a consequence of [4] (Corollary 4 p. 1057).

□

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