Max-Planck-Institut für Mathematik
Bonn

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by

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Max-Planck-Institut für Mathematik
Preprint Series 2010 (80)

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MPIM 10-80

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October 28, 2010

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Abstract

One hopes that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}$ is generated by values of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$. These numbers are also called multiple zeta values. In this note, assuming motivic formalism, we give a proof, that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$, which are unramified everywhere. The main subject of the paper is however the l-adic Galois analogue of the above result. We shall also discuss some other examples in the l-adic Galois setting.

0 Introduction

One hopes that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}$ is generated by values of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$. These numbers are also called multiple zeta values. In modern times these numbers first appeared in the Deligne paper [4]. In more explicit form they appeared in the article of Zagier (see [22]), though they were already studied by Euler (see [9]).

In this note we give a brief proof, assuming motivic formalism, that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$, which are unramified everywhere. We explain what it means for a linear combination of such iterated integrals to be unramified everywhere. We give also a criterion when a linear combination with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ is unramified everywhere. Such a result may be useful even if finally one shows that iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ generate the $\mathbb{Q}$-algebra of mixed Tate motives over $Spec\,\mathbb{Z}$.

These results have their analogues in l-adic Galois realizations. In fact we shall study l-adic situation first and in more details. The l-adic situation is easier conceptually, because the Galois group G_K of a number field K and its various weighted Tate $\mathbb{Q}_l$-completions replace the motivic fundamental group of the category of mixed Tate motives over $Spec\,\mathcal{O}_{K,S}$, which is perhaps still a conjectural object.

Let S be a finite set of finite places of K. We shall consider weighted Tate representations of $\pi_1^{\text{et}}(Spec\,\mathcal{O}_{K,S}; Spec\bar{K})$ in finite dimensional $\mathbb{Q}_l$-vector spaces. The universal proalgebraic group over $\mathbb{Q}_l$ by which such representations factorize we shall denote by $\mathcal{G}(\mathcal{O}_{K,S}; l)$.

The kernel of the projection $\mathcal{G}(\mathcal{O}_{K,S}; l) \to \mathbb{G}_m$ we denote by $\mathcal{U}(\mathcal{O}_{K,S}; l)$. The associated graded Lie algebra of $\mathcal{U}(\mathcal{O}_{K,S}; l)$ with respect of the weight filtration we denote by $L(\mathcal{O}_{K,S}; l)$.

We assume that S contains all finite places of K lying over (l). Then the group $\mathcal{G}(\mathcal{O}_{K,S}; l)$ is isomorphic to the conjectural motivic fundamental group of the Tannakian category of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,S}$ tensored with $\mathbb{Q}_l$ (see [10] and [11]).

Hain and Matsumoto also considered the case when S does not contain all finite places of K lying over (l). However the construction of the corresponding universal group is decidedly more complicated in this case and we do not understand it well. We shall present in this paper a simpler, more explicit version though only for weighted Tate representations and only on the level of graded Lie algebras. The construction is described briefly below.

Let S be a finite set of finite places of K. Every non trivial l-adic weighted Tate representation of G_K is ramified at all finite places of K which lie over (l). Therefore we must consider the weighted Tate $\mathbb{Q}_l$-completion of $\pi_1^{\mathrm{et}}(Spec\,\mathcal{O}_{K,S_l};Spec\bar{K})$, where S_l is the union of S and all finite places of K lying over (l). This has an effect that the Lie algebra $L(\mathcal{O}_{K,S_l};l)$ has more generators in degree 1 than the corresponding Lie algebra of the Tannakian category of mixed Tate motives over $Spec\,\mathcal{O}_{K,S}$. To get rid of these additional generators in degree 1 we shall define a homogeneous Lie ideal $\langle \mathfrak{l} \mid l\rangle_{K,S}$ of $L(\mathcal{O}_{K,S_l};l)$ and then the quotient Lie algebra

$$L_l(\mathcal{O}_{K,S}) := L(\mathcal{O}_{K,S};l)/\langle \mathfrak{l} \mid l\rangle_{K,S}\ .$$

We shall show that the Lie algebra $L_l(\mathcal{O}_{K,S})$ is also graded, i.e.

$$L_l(\mathcal{O}_{K,S}) = \bigoplus_{i=1}^{\infty} L_l(\mathcal{O}_{K,S})_i.$$

and that it has a correct number of generators.

Let us define

$$(L_l(\mathcal{O}_{K,S}))^{\diamond} := \oplus_{i=1}^{\infty}\mathrm{Hom}(L_l(\mathcal{O}_{K,S})_i,\mathbb{Q}_l).$$

We shall call $(L_l(\mathcal{O}_{K,S}))^{\diamond}$ the dual of $L_l(\mathcal{O}_{K,S})$.

The vector space $(L_l(\mathcal{O}_{K,S}))^{\diamond}$ is an l-adic analogue of the generators of the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathcal{O}_{K,S}$.

Ihara in [12] and Deligne in [4] studied the action of the Galois group $G_{\mathbb{Q}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\};\overrightarrow{01})$. The pair $(\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\},\overrightarrow{01})$ has good reduction everywhere. Hence after passing to associated graded Lie algebras we get a Lie algebra representation

$$L(\mathbb{Z}[\frac{1}{l}];l) \longrightarrow Der^{*}Lie(X,Y)$$

which factors through

$$L_l(\mathbb{Z}) \longrightarrow Der^{*}Lie(X,Y). \tag{0.1}$$

It is not known, at least to the author of this article, if the last morphism is injective. (This question was studied very much by Ihara and his students.) Hence we do not know if the vector space $L_l(\mathbb{Z})^{\diamond}$ is generated by the coefficients of the representation (0.1). This is the l-adic analogue of the problem about the multiple zeta values stated at the beginning of the section.

In [16] we have studied the action of $G_{\mathbb{Q}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,-1,\infty\}; \overrightarrow{01})$. After the standard embedding of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,-1,\infty\}; \overrightarrow{01})$ into the $\mathbb{Q}_l$-algebra of non-commutative formal power series $\mathbb{Q}_l\{\{X, Y_0, Y_1\}\}$ and passing to the associated graded Lie algebra we get a Lie algebra representation

$$\Phi_{\overrightarrow{01}} : L(\mathbb{Z}[\frac{1}{2l}], l) \longrightarrow Der^* Lie(X, Y_0, Y_1),$$

where $Der^* Lie(X, Y_0, Y_1)$ is the Lie algebra of special derivations of the free Lie algebra $Lie(X, Y_0, Y_1)$. The Lie ideal $\langle \mathfrak{l} \mid l \rangle_{\mathbb{Q},(2)}$ is contained in the kernel of $\Phi_{\overrightarrow{01}}$. Hence we get a morphism

$$\Phi_{\overrightarrow{01}} : L_l(\mathbb{Z}[\frac{1}{2}]) \to Der^* Lie(X, Y_0, Y_1).$$

Theorem 15.5.3 from [16] can be interpreted in the following way.

Theorem A. The vector space $(L_l(\mathbb{Z}[\frac{1}{2}]))^{\diamond}$ is generated by the coefficients of the representation $\Phi_{\overrightarrow{01}}$.

We shall show that the natural map

$$L_l(\mathbb{Z}[\frac{1}{2}]) \longrightarrow L_l(\mathbb{Z}),$$

induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}]$, is a surjective morphism of Lie algebras. Let $I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})$ be its kernel.

We say that $f \in (L_l(\mathbb{Z}[\frac{1}{2}]))^{\diamond}$ is unramified everywhere if $f(I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})) = 0$. Our next result is then the immediate consequence of Theorem A.

Corollary B. The vector space $(L_l(\mathbb{Z}))^{\diamond}$ is generated by these linear combinations of coefficients of the representation $\Phi_{\overrightarrow{01}}$, which are unramified everywhere.

The result mentioned at the beginning of the section is the Hodge–de Rham analogue of Corollary B.

We shall also consider the following situation. Let L be a finite Galois extension of K. We assume that a pair (V_L, v) or a triple (V_L, z, v) is defined over L. Then we get a representation of G_L on $\pi_1(V_{\bar{L}}; v)$ or $\pi(V_{\bar{L}}; z, v)$. We shall define what it means that a coefficient of a such representation is defined over K.

Then, working in Hodge–de Rham realization and assuming motivic formalism, one can show that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}[\frac{1}{3}]$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus (\{0,\infty\} \cup \mu_3)$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$, $\frac{dz}{z-\xi_3}$, $\frac{dz}{z-\xi_3^2}$ ($\xi_3 = e^{\frac{2\pi i}{3}}$) from $\overrightarrow{01}$ to $\overrightarrow{10}$, which are defined over $\mathbb{Q}$. However in this paper we shall show only an l-adic analogue of that result.

Remark. A pair $(\mathbb{P}^1 \setminus \{0,1,\infty\}, \overrightarrow{03})$ ramifies only at (3), hence periods of a mixed Tate motive associated with $\pi_1(\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}; \overrightarrow{03})$ are periods of mixed Tate motives over $Spec\,\mathbb{Z}[\frac{1}{3}]$. However one can easily show that in this way we shall not get all such periods.

The final aim is to show that the vector space $(L_l(\mathcal{O}_{K,S}))^\diamond$ is generated by linear combinations of coefficients, which are unramified outside S and defined over K of representations of G_L - for various L finite Galois extensions of K - on fundamental groups or on torsors of paths of a projective line minus a finite number of points or perhaps some other algebraic varieties. This will imply (by the very definition) that all mixed Tate representations of $L_l(\mathcal{O}_{K,S})$ are of geometric origin. We are however very far from this aim.

Then we must pass from Lie algebra representations of $L_l(\mathcal{O}_{K,S})$ to the representation of the corresponding group in order to show that any mixed Tate representation of G_K is of geometric origin. This part of the problem is not studied here.

The results of this paper where presented in a seminar talk in Lille in May 2009 and then at the end of my lectures at the summer school at Galatasaray University in Istanbul in June 2009.

While finishing this paper the author has a delegation in CNRS in Lille at the *Laboratoire*, Paul Painlevé and he would like to thank very much the director, Professor Jean D'Almeida for accepting him in the Painleve Laboratory. Thanks are also due to Professor J.-C. Douai who helped me to get this delegation.

Parts of this paper were written during our visits in Max-Planck-Institut für Mathematik in Bonn and during the visit in Isaac Newton Institute for Mathematical Sciences in Cambridge during the program "Non-Abelian Fundamental Groups in Arithmetic Geometry". We would like to thank very much both these institutes for support.

1 Weighted Tate completions of Galois groups

Let K be a number field and let S a finite set of finite places of K. Let $\mathcal{O}_{K,S}$ be the ring of S-integers in K, i.e.

$$\mathcal{O}_{K,S} := \{\frac{a}{b} \mid a, b \in \mathcal{O}_K,\, b \notin \mathfrak{p} \text{ for all } \mathfrak{p} \notin S\}.$$

Let us fix a rational prime l. We denote by $\{\mathfrak{l} \mid l\}_K$ the set of finite places of K lying over the prime ideal (l) of $\mathbb{Z}$.

We introduce here some standard notation concerning Lie algebras that we shall use frequently.

Let L be a Lie algebra. The Lie subalgebras $\Gamma^n L$ of the lower central series of L are defined recursively by $\Gamma^1 L := L$, $\Gamma^{n+1}L := [\Gamma^n L, L]$, $n = 1, 2, 3, \dots$. If L is graded then $L^{ab} = L/[L, L]$, $\Gamma^n L$ and $L/\Gamma^n L$ are also graded.

Let $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ be the weighted Tate $\mathbb{Q}_l$-completion of the étale fundamental group $\pi_1^{\mathrm{et}}(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K})$. The group $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is an affine, proalgebraic group over $\mathbb{Q}_l$ equipped with the homomorphism

$$\pi_1^{\mathrm{et}}(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K}) \longrightarrow \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)(\mathbb{Q}_l)$$

with a Zariski dense image, such that any weighted Tate finite dimensional $\mathbb{Q}_l$-representation of $\pi_1^{\mathrm{et}}(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K})$ factors through $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$. We point out that weighted Tate finite dimensional $\mathbb{Q}_l$-representations of $\pi_1^{\mathrm{et}}(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K})$ provide weighted Tate finite dimensional $\mathbb{Q}_l$- representations of G_K unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$ and vice versa.

There is an exact sequence

$$1 \to \mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathbb{G}_m \to 1.$$

The kernel $\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is a prounipotent proalgebraic affine group over $\mathbb{Q}_l$ equipped with the weight filtration $\{W_{-2i}\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)\}_{i\in\mathbb{N}}$ (see [10] and [11].)

Let us define

$$L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_i := W_{-2i}\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)/W_{-2(i+1)}\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$$

and

$$L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) := \bigoplus_{i=1}^{\infty} L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_i.$$

The Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is a free Lie algebra. In degree 1 there are functorial isomorphisms

(1.1.a)

$$\mathrm{Hom}(L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1; \mathbb{Q}_l) \approx H^1(Spec\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathbb{Q}_l(1)) \approx \mathcal{O}^{\times}_{K,S\cup\{\mathfrak{l}|l\}_K} \otimes \mathbb{Q}_l$$

and

$$L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1 \approx \mathrm{Hom}(\mathcal{O}^{\times}_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathbb{Q}_l). \tag{1.1.b}$$

In degree $i > 1$ there are functorial isomorphisms

$$\mathrm{Hom}\big((L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)^{ab})_i; \mathbb{Q}_l\big) \approx H^1(G_K; \mathbb{Q}_l(i)). \tag{1.1.c}$$

(see [10] Theorem 7.2.).

Let us assume that a pair (V, v) is defined over K and has good reduction outside S. The representation of G_K on the pro-l quotient of $\pi_1^{\mathrm{et}}(V_{\bar{K}}; v)$ is unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$ and if it is non-trivial, it is ramified at all finite places of K, which lie over (l). This has an effect that the Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ has more generators in degree 1 than the corresponding Lie algebra of the Tannakian category of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,S}$.

We shall show below how to kill these additional generators corresponding to finite places of K lying over (l), which are not in S.

Let $u\in\mathcal{O}^{\times}_{K,S\cup\{\mathfrak{l}|l\}_K}$ and let $\kappa(u):G_K\to\mathbb{Z}_l$ be the Kummer character of u. The representation

$$G_K\ni\sigma\longrightarrow\begin{pmatrix}1 & 0\\ \kappa(u)(\sigma) & \chi(\sigma)\end{pmatrix}\in GL_2(\mathbb{Q}_l)$$

is an l-adic weighted Tate representation of G_K unramified outside $S\cup\{\mathfrak{l}\mid l\}_K$, i.e. it is an l-adic weighted Tate representation of $\pi_1^{\mathrm{et}}(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};Spec\bar{K})$. By (1.1.a) the Kummer character $\kappa(u)$ we can view also as a homomorphism

$$\kappa(u):L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1\to\mathbb{Q}_l.$$

Let us set

$$(\mathfrak{l}\mid l)_{K,S}:=\bigcap_{u\in\mathcal{O}^{\times}_{K,S}}\big(Ker\big(\kappa(u):L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1\to\mathbb{Q}_l\big)\big).$$

Let $\langle\mathfrak{l}\mid l\rangle_{K,S}$ be the Lie ideal of $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ generated by elements of $(\mathfrak{l}\mid l)_{K,S}$.

Definition 1.2. We set

$$L_l(\mathcal{O}_{K,S})=L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)/\langle\mathfrak{l}\mid l\rangle_{K,S}.$$

Observe that $L_l(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K})=L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$.

Proposition 1.3.

i) The Lie algebra $L_l(\mathcal{O}_{K,S})$ is graded.

ii) For i greater than 1 there are functorial isomorphisms

$$\mathrm{Hom}\big((L_l(\mathcal{O}_{K,S})^{ab})_i;\mathbb{Q}_l\big)\approx H^1(G_K;\mathbb{Q}_l(i)).$$

iii) In degree 1 there is a functorial isomorphism

$$\mathrm{Hom}(L_l(\mathcal{O}_{K,S})_1;\mathbb{Q}_l)\approx\mathcal{O}^{\times}_{K,S}\otimes\mathbb{Q}_l.$$

iv) The Lie algebra $L_l(\mathcal{O}_{K,S})$ is free, freely generated by $n_1=\dim_{\mathbb{Q}}(\mathcal{O}^{\times}_{K,S}\otimes\mathbb{Q})$ elements in degree 1 and by $n_i=\dim_{\mathbb{Q}_l}(H^1(G_K;\mathbb{Q}_l(i))$ elements in degree $i>1$.

Proof. The Lie ideal $\langle \mathfrak{l} \mid l\rangle_{K,S}$ of the Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ is generated by elements of degree 1, hence it is homogeneous. Therefore the quotient Lie algebra $L_l(\mathcal{O}_{K,S})$ has a natural grading induced from that of $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$.

Let us choose $u_1,\dots,u_p \in \mathcal{O}^\times_{K,S}$ ($p=\dim\mathcal{O}^\times_{K,S}\otimes\mathbb{Q}$) such that $u_1\otimes 1,\dots,u_p\otimes 1$ is a base of $\mathcal{O}^\times_{K,S}\otimes\mathbb{Q}$. Let $z_1,\dots,z_q \in \mathcal{O}^\times_{K,S\cup\{\mathfrak{l}|l\}_K}$ be such that $u_1\otimes 1,\dots,u_p\otimes 1, z_1\otimes 1,\dots,z_q\otimes 1$ is a base of $(\mathcal{O}^\times_{K,S\cup\{\mathfrak{l}|l\}_K})\otimes\mathbb{Q}$. Let $\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q$ be the base of $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1$ dual to the Kummer characters $\kappa(u_1),\dots,\kappa(u_p)$, $\kappa(z_1),\dots,\kappa(z_q)$. Then $\beta_1,\dots,\beta_q$ generate the Lie ideal $\langle \mathfrak{l} \mid l\rangle_{K,S}$. The points ii), iii) and iv) follow now immediately from the fact that the Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ is free, freely generated by the elements $\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q$ in degree 1 and by n_i generators in degrees $i>1$ (see [10] Theorem 7.2.) and from the functorial isomorphisms (1.1.a) and (1.1.c). $\Box$

Definition 1.4. Let $L=\bigoplus_{i=1}^{\infty} L_i$ be a graded Lie algebra over a field k such that $\dim L_i<\infty$ for every i. We define

$$L^\diamond := \bigoplus_{i=1}^{\infty} \mathrm{Hom}(L_i,k).$$

We call $L^\diamond$ the dual of L. The vector space $L^\diamond$ is graded and

$$(L^\diamond)_i = (L_i)^\diamond := \mathrm{Hom}(L_i,k).$$

The Lie bracket $[\, ,\,]$ of the Lie algebra L induces a morphism

$$d: L^\diamond \to L^\diamond \otimes L^\diamond,$$

whose image is contained in the subspace of $L^\diamond\otimes L^\diamond$ generated by all anti-symmetric tensors of the form $a\otimes b - b\otimes a$.

Definition 1.5. The $\mathbb{Q}_l$-vector space $(L_l(\mathcal{O}_{K,S}))^\diamond$ we shall call the vector space of coefficients on $L_l(\mathcal{O}_{K,S})$.

Remark 1.5.1. We consider the $\mathbb{Q}_l$-vector space $(L_l(\mathcal{O}_{K,S}))^\diamond$ as an analogue of generators of the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\mathcal{O}_{K,S}$.

The morphism

$$d: (L_l(\mathcal{O}_{K,S}))^\diamond \to (L_l(\mathcal{O}_{K,S}))^\diamond \otimes (L_l(\mathcal{O}_{K,S}))^\diamond$$

induced by the Lie bracket of $L_l(\mathcal{O}_{K,S})$ we denote by $d_{\mathcal{O}_{K,S}}$. We set

$$\mathcal{L}(\mathcal{O}_{K,S};l) := ker(d_{\mathcal{O}_{K,S}}).$$

Observe that

$$\mathcal{L}(\mathcal{O}_{K,S};l) = \{f\in (L_l(\mathcal{O}_{K,S}))^\diamond \mid f(\Gamma^2 L_l(\mathcal{O}_{K,S}))=0\} \approx (L_l(\mathcal{O}_{K,S})^{ab})^\diamond.$$

The vector space $\mathcal{L}(\mathcal{O}_{K,S};l)$ inherits grading from $(L_l(\mathcal{O}_{K,S}))^\diamond$ and we have

$$\mathcal{L}(\mathcal{O}_{K,S};l) = \bigoplus_{i=1}^{\infty} \mathcal{L}(\mathcal{O}_{K,S};l)_i.$$

It follows from Proposition 1.3 that there are natural isomorphisms

$$(1.5.a) \qquad \mathcal{L}(\mathcal{O}_{K,S};l)_i = ker(d_{\mathcal{O}_{K,S}})_i \approx H^1(G_K;\mathbb{Q}_l(i)) \quad \text{for} \quad i > 1$$

and

$$(1.5.b) \qquad \mathcal{L}(\mathcal{O}_{K,S};l)_1 = ker(d_{\mathcal{O}_{K,S}})_1 = (L_l(\mathcal{O}_{K,S})_1)^\diamond \approx \mathcal{O}_{K,S}^\times \otimes \mathbb{Q}_l.$$

We finish this section with the study of the dual of the Lie bracket of the Lie algebra $L_l(\mathcal{O}_{K,S})$. To simplify the notation we denote $d_{\mathcal{O}_{K,S}}$ by d. The operators

$$d^{(n)} : (L_l(\mathcal{O}_{K,S}))^\diamond \longrightarrow \bigotimes_{i=1}^{n+1} (L_l(\mathcal{O}_{K,S}))^\diamond$$

are defined recursively by $d^{(1)} := d$, $\quad d^{(n+1)} := (d \otimes (\otimes_{i=1}^n Id_{(L_l(\mathcal{O}_{K,S}))^\diamond})) \circ d^{(n)}$, $\quad n = 1, 2, 3, \dots$. The linear maps

$$pr_{n+1} : \otimes_{i=1}^{n+1} L_l(\mathcal{O}_{K,S}) \longrightarrow L_l(\mathcal{O}_{K,S})$$

are defined recursively by $pr_1(u_1) := u_1$, $\ pr_{n+1}(u_1 \otimes u_2 \otimes \ldots \otimes u_n \otimes u_{n+1}) := [pr_n(u_1 \otimes u_2 \otimes \ldots \otimes u_n), u_{n+1}]$, $\quad n = 1, 2, 3, \dots$.

Lemma 1.6. We have:

i) $(pr_{n+1})^\diamond = d^{(n)}$.

ii) $f \in (L_l(\mathcal{O}_{K,S}))^\diamond$ vanishes on $\Gamma^{n+1}(L_l(\mathcal{O}_{K,S}))$ if and only if $d^{(n)}(f) = 0$.

iii) Let $f \in (L_l(\mathcal{O}_{K,S}))^\diamond$ be such that $d^{(k+1)}(f) = 0$. Then $d^{(k)}(f) \in \bigotimes_{i=1}^{k+1} \mathcal{L}(\mathcal{O}_{K,S};l)$.

Proof. The point i) is clear and ii) follows from i). It rests to show the point iii). It follows from ii) that f vanishes on $\Gamma^{k+2}L_l(\mathcal{O}_{K,S})$ hence it factors by $L_l(\mathcal{O}_{K,S})/\Gamma^{k+2}L_l(\mathcal{O}_{K,S})$. The map $d^{(k)}f = f \circ pr_{k+1}$ is then equal to the composition of the following two maps

$$\otimes_{i=1}^{k+1} L_l(\mathcal{O}_{K,S}) \to \otimes_{i=1}^{k+1} L_l(\mathcal{O}_{K,S})^{ab} \to \Gamma^{k+1}L_l(\mathcal{O}_{K,S})/\Gamma^{k+2}L_l(\mathcal{O}_{K,S})$$

and

$$\Gamma^{k+1}L_l(\mathcal{O}_{K,S})/\Gamma^{k+2}L_l(\mathcal{O}_{K,S}) \hookrightarrow L_l(\mathcal{O}_{K,S})/\Gamma^{k+2}L_l(\mathcal{O}_{K,S}) \xrightarrow{f} \mathbb{Q}_l.$$

The isomorphism $\mathcal{L}(\mathcal{O}_{K,S};l) \approx (L_l(\mathcal{O}_{K,S})^{ab})^\diamond$ implies that $d^{(k)}(f) \in \bigotimes_{i=1}^{k+1} \mathcal{L}(\mathcal{O}_{K,S};l)$.
$\square$

2 Functorial properties of weighted Tate completions

Let K be a number field and let L be a finite extension of K. Let S be a set of finite places of K and let T be a set of finite places of L containing all places lying over elements of S. The inclusion of fields $K \subset L$ induces the inclusion of rings

$$\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K} \hookrightarrow \mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}. \tag{2.1}$$

The morphism of rings (2.1) induces a morphism of groups

$$\pi_1^{\mathrm{et}}(Spec\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}; Spec\bar{L}) \to \pi_1^{\mathrm{et}}(Spec\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K}).$$

Therefore we get morphisms of affine proalgebraic groups over $\mathbb{Q}_l$

$$\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K} : \mathcal{G}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}; l) \longrightarrow \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$$

and

$$\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K} : \mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}; l) \longrightarrow \mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l).$$

Passing to associated graded Lie algebras we get a morphism of graded Lie algebras

$$L(\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) : L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}; l) \longrightarrow L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l).$$

Lemma 2.1. For each $i > 1$ we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{L}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_i & \longrightarrow & \mathcal{L}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}; l)_i \\
\approx\downarrow & & \approx\downarrow \\
H^1(K; \mathbb{Q}_l(i)) & \longrightarrow & H^1(L; \mathbb{Q}_l(i)).
\end{array}$$

In degree 1 there is the following commutative diagram

$$\begin{array}{ccc}
(L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1)^{\diamond} & \longrightarrow & (L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}; l)_1)^{\diamond} \\
\approx\downarrow & & \approx\downarrow \\
\mathcal{O}^{\times}_{K,S\cup\{\mathfrak{l}|l\}_K} \otimes \mathbb{Q}_l & \longrightarrow & \mathcal{O}^{\times}_{L,T\cup\{\mathfrak{l}|l\}_L} \otimes \mathbb{Q}_l.
\end{array}$$

Proof. The lemma follows from the existence of the functorial isomorphisms (1.1.a) and (1.1.c) and from the functoriality of weighted Tate completions. □

Lemma 2.2. The morphism of graded Lie algebras

$$L(\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) : L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}; l) \longrightarrow L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l).$$

maps the Lie ideal $\langle \mathfrak{l} \mid l \rangle_{L,T}$ of $L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)$ into the Lie ideal $\langle \mathfrak{l} \mid l \rangle_{K,S}$ of $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$.

Proof. The Lie ideal $\langle \mathfrak{l} \mid l \rangle_{L,T}$ is generated by all elements $z \in L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)_1$ satisfying $\kappa(u)(z) = 0$ for all $u \in \mathcal{O}^{\times}_{L,T}$. We have $\mathcal{O}^{\times}_{K,S} \subset \mathcal{O}^{\times}_{L,T}$. Hence it follows from the second part of Lemma 2.1 that $\kappa(u)(L(\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K};l)(z)) = 0$ for all $u \in \mathcal{O}^{\times}_{K,S}$. Hence $L(\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K};l)(z)$ belongs to the set $(\mathfrak{l} \mid l)_{K,S}$ of generators of the Lie ideal $\langle \mathfrak{l} \mid l \rangle_{K,S}$. $\square$

It follows from Lemma 2.2 that $L(\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ induces

$$L_l(\pi^{L,T}_{K,S}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S}).$$

Proposition 2.3. We have:

i) The morphism
$$L_l(\pi^{L,T}_{K,S}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$$
is a surjective morphism of graded Lie algebras.

ii) For each $i > 1$ there is the following commutative diagram
$$\begin{array}{ccc} \mathcal{L}(\mathcal{O}_{K,S};l)_i & \longrightarrow & \mathcal{L}(\mathcal{O}_{L,T};l)_i \\ \approx\big\downarrow & & \approx\big\downarrow \\ H^1(K;\mathbb{Q}_l(i)) & \longrightarrow & H^1(L;\mathbb{Q}_l(i)). \end{array}$$

iii) In degree 1 there is the following commutative diagram
$$\begin{array}{ccc} (L_l(\mathcal{O}_{K,S})_1)^{\diamond} & \longrightarrow & (L_l(\mathcal{O}_{L,T})_1)^{\diamond} \\ \approx\big\downarrow & & \approx\big\downarrow \\ \mathcal{O}^{\times}_{K,S} \otimes \mathbb{Q}_l & \longrightarrow & \mathcal{O}^{\times}_{L,T} \otimes \mathbb{Q}_l. \end{array}$$

Proof. By the very definition the ideals $\langle \mathfrak{l} \mid l \rangle_{K,S}$ and $\langle \mathfrak{l} \mid l \rangle_{L,T}$ are generated by elements of degree 1. Hence it follows from Lemma 2.2 that $L_l(\pi^{L,T}_{K,S}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$ is a morphism of graded Lie algebras.

The points ii) and iii) follow from Lemma 2.1.

It rests to show that the morphism of graded Lie algebras $L_l(\pi^{L,T}_{K,S}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$ is surjective. The inclusion of number fields $K \subset L$ induces injective morphisms in Galois cohomology

$$H^1(G_K;\mathbb{Q}_l(i)) \to H^1(G_L;\mathbb{Q}_l(i))$$

for $i > 1$. It follows from this fact and from the parts ii) and iii) of the proposition already proved that the map

$$\mathcal{L}(\mathcal{O}_{K,S}; l) \to \mathcal{L}(\mathcal{O}_{L,T}; l)$$

is injective. Hence the homomorphism

$$L_l(\mathcal{O}_{L,T})^{ab} \to L_l(\mathcal{O}_{K,S})^{ab}$$

is surjective. Therefore the morphism of graded Lie algebras

$$L_l(\pi_{K,S}^{L,T}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$$

is surjective. □

Definition 2.4. We define

$$I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S}) := ker\big(L_l(\pi_{K,S}^{L,T}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})\big).$$

Proposition 2.5. We have:

i) The Lie ideal $I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$ is generated by homogeneous elements.

ii) The quotient Lie algebra $L_l(\mathcal{O}_{L,T})/I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$ is a graded Lie algebra.

iii) The induced morphism

$$L_l(\mathcal{O}_{L,T})/I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S}) \to L_l(\mathcal{O}_{K,S})$$

is an isomorphism of graded Lie algebras.

Proof. The morphism $L_l(\pi_{K,S}^{L,T}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$ is a surjective morphism of graded Lie algebras. Therefore $ker(L_l(\pi_{K,S}^{L,T})) = I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$ is a graded Lie ideal. Hence one can choose homogeneous set of generators of $I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$. Therefore the points ii) and iii) are clear. □

The surjective morphism of graded Lie algebras

$$L_l(\pi_{K,S}^{L,T})_l : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$$

induces an injective map of graded vector spaces

$$\Pi_{L,T}^{K,S} : L_l(\mathcal{O}_{K,S})^\diamond \to L_l(\mathcal{O}_{L,T})^\diamond.$$

Hence we get the following description of coefficients on $L_l(\mathcal{O}_{K,S})$.

Corollary 2.6. The map $\Pi_{L,T}^{K,S}$ induces an isomorphism

$$(L_l(\mathcal{O}_{K,S}))^\diamond \approx \{f \in (L_l(\mathcal{O}_{L,T}))^\diamond \mid f(I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})) = 0\}.$$

We indicate two important special cases.

Let S and S_1 be finite disjoint sets of finite places of K. The inclusion of rings

$$\mathcal{O}_{K,S} \hookrightarrow \mathcal{O}_{K,S\cup S_1}$$

induces the surjective morphism of graded Lie algebras

$$\pi_{K,S}^{K,S\cup S_1} : L_l(\mathcal{O}_{K,S\cup S_1}) \longrightarrow L_l(\mathcal{O}_{K,S}).$$

Definition 2.7. Let S and S_1 be finite disjoint sets of finite places of K. We say that $f \in (L_l(\mathcal{O}_{K,S\cup S_1}))^\diamond$ is unramified outside S_1 if $f(I(\mathcal{O}_{K,S\cup S_1} : \mathcal{O}_{K,S})) = 0$.

Corollary 2.6 in this special case can be formulated in the following suggestive form.

Corollary 2.8. The vector space of coefficients on $L_l(\mathcal{O}_{K,S})$ is the subspace of the vector space of coefficients on $L_l(\mathcal{O}_{K,S\cup S_1})$ consisting of elements which are unramified outside S_1.

The following observation will be useful.

Lemma 2.9. The Lie ideal $I(\mathcal{O}_{K,S\cup S_1} : \mathcal{O}_{K,S})$ is generated by elements of degree 1.

The second important case is the following one. Let K be a number field and let S be a set of finite places of K. Let L be a finite Galois extension of K and let T be a set of finite places of L lying over elements of S. The inclusion of rings of algebraic integers

$$\mathcal{O}_{K,S} \hookrightarrow \mathcal{O}_{L,T}$$

induces the surjective morphism of graded Lie algebras

$$L_l(\pi_{K,S}^{L,T}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S}).$$

Definition 2.10. We say that $f \in (L_l(\mathcal{O}_{L,T}))^\diamond$ is defined over K if $f(I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})) = 0$.

In this special case we reformulate Corollary 2.6 in the following way.

Corollary 2.11. The vector space of coefficients on $L_l(\mathcal{O}_{K,S})$ is the subspace of the vector space of coefficients on $L_l(\mathcal{O}_{L,T})$ consisting of elements which are defined over K.

3 Geometric coefficients

Let $a_1, \ldots, a_n \in K$ and let $V := \mathbb{P}^1_K \setminus \{a_1, \ldots, a_n, \infty\}$. Let v and z be K-points of V or tangential points defined over K. Let S be a finite set of finite places of K. Let l be a fixed rational prime.

We denote by $\pi_1(V_{\bar{K}}; v)$ the pro-l completion of the étale fundamental group of $V_{\bar{K}}$ based at v and by $\pi(V_{\bar{K}}; z, v)$ the $\pi_1(V_{\bar{K}}; v)$-torsor of pro-l paths from v to z.

The Galois group G_K acts on $\pi_1(V_{\bar{K}}; v)$ and on $\pi(V_{\bar{K}}; z, v)$. After the standard embedding of $\pi_1(V_{\bar{K}}; v)$ into the $\mathbb{Q}_l$-algebra $\mathbb{Q}_l\{\{X_1, \ldots, X_n\}\}$ of formal power series in non-commuting variables we get two Galois representations

$$\varphi_v = \varphi_{V,v} : G_K \longrightarrow \mathrm{Aut}(\mathbb{Q}_l\{\{X_1, \ldots, X_n\}\})$$

and

$$\psi_{z,v} = \psi_{V,z,v} : G_K \longrightarrow GL(\mathbb{Q}_l\{\{X_1, \ldots, X_n\}\})$$

deduced from actions of G_K on π_1 and on the π_1-torsor (see [14], section 4).

Let us assume that a pair (V, v) and a triple (V, z, v) have good reduction outside S. Then the representations $\varphi_{V,v}$ and $\psi_{V,z,v}$ factor through the weighted Tate $\mathbb{Q}_l$-completion $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ of $\pi_1^{\mathrm{et}}(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K})$ because the representations $\varphi_{V,v}$ and $\psi_{V,z,v}$ are weighted Tate $\mathbb{Q}_l$-representations unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$ (see [18] Proposition 1.0.3). Passing to associated graded Lie algebras with respect to the weight filtrations we get morphisms of graded Lie algebras

$$gr_W Lie\varphi_v : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to Der^*Lie(X_1, \ldots, X_n)$$

and

$$gr_W Lie\psi_{z,v} : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to Lie(X_1, \ldots, X_n)\tilde{\times}Der^*Lie(X_1, \ldots, X_n),$$

where $Der^*Lie(X_1, \ldots, X_n)$ is the Lie algebra of special derivations of $Lie(X_1, \ldots, X_n)$ (see the definition of the Lie algebra $Der^*Lie(X_1, \ldots, X_n)$ and the semi-direct product $Lie(X_1, \ldots, X_n)\tilde{\times}Der^*Lie(X_1, \ldots, X_n)$ in [14], p.134).

Theorem 3.1. Let $a_1, \ldots, a_{n+1}$ be K-points of $\mathbb{P}^1_K$ and let $V := \mathbb{P}^1_K \setminus \{a_1, \ldots, a_{n+1}\}$. Let z and v be K-points of V or tangential points defined over K. Let us assume that the pair (V, v) (resp. the triple (V, z, v)) has good reduction outside S. Then the morphism of graded Lie algebras

$$gr_W Lie\varphi_{V,v} : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to Der^*Lie(X_1, \ldots, X_n)$$

(resp. $gr_W Lie\psi_{V,z,v} : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to Lie(X_1, \ldots, X_n)\tilde{\times}Der^*Lie(X_1, \ldots, X_n)$)

deduced from the action of G_K on $\pi_1(V_{\bar{K}}; v)$ (resp. on $\pi(V_{\bar{K}}; z, v)$) factors through the Lie algebra $L_l(\mathcal{O}_{K,S})$.

Proof. Let us assume that a pair (V, v) (resp. a triple (V, z, v)) has good reduction outside S. We shall show in the next lemma that then the morphism $gr_W Lie\varphi_{V,v}$ (resp. $gr_W Lie\psi_{V,z,v}$) in degree 1 is given by Kummer characters of elements belonging to $\mathcal{O}^{\times}_{K,S}$. This implies that the morphism vanishes on $(\mathfrak{l} \mid l)_{K,S}$, hence it vanishes on $\langle \mathfrak{l} \mid l \rangle_{K,S}$. Hence the theorem follows immediately. $\square$

Lemma 3.1.1. Let us assume that a pair (V,v) (resp. a triple (V,z,v)) has good reduction outside S. Then the morphism $gr_W Lie\varphi_{V,v}$ (resp. $gr_W Lie\psi_{V,z,v}$) in degree 1 is given by the Kummer characters of elements belonging to $\mathcal{O}_{K,S}^\times$.

Proof. For simplicity we shall consider only a pair (V,v), where v is a K-point. The definition of good reduction at a finite place $\mathfrak{p}$ depends only on the isomorphism class of (V,v) over K (see [17], definition 17.5), hence we can assume that $a_1=0$, $a_2=1$ and $a_{n+1}=\infty$.

The morphism $gr_W Lie\varphi_{V,v}$ is given in degree 1 by the Kummer characters $\kappa(\frac{a_i-a_k}{v-a_k})$ for $i\neq k$ and $i,k\in\{1,2,\dots,n\}$ (see [17], 17.10.a). Let $\mathcal{S}(V,v)$ be a set of finite places $\mathfrak{p}$ of K such that there exists a pair (i,k) satisfying $i\neq k$ and such that $\mathfrak{p}$ valuation of $\frac{a_i-a_k}{v-a_k}$ is different from 0. Then clearly $\frac{a_i-a_k}{v-a_k}\in\mathcal{O}_{K,\mathcal{S}(V,v)}^\times$ for all pair (i,k) with $i\neq k$.

For the pair (V,v) the notion of good reduction at $\mathfrak{p}$ and strong good reduction at $\mathfrak{p}$ coincide (see [17], Definitions 17.4, 17.5 and Corollary 17.18). It follows from Lemma 17.15 in [17] that $\mathfrak{p}\notin S$ implies $\mathfrak{p}\notin\mathcal{S}(V,v)$. Hence $\mathcal{S}(V,v)\subset S$. Therefore $\frac{a_i-a_k}{v-a_k}\in\mathcal{O}_{K,S}^\times$ for all pairs (i,k) with $i\neq k$. □

We shall denote by

$$L_l(\varphi_v): L_l(\mathcal{O}_{K,S})\to Der^*Lie(X_1,\dots,X_n)$$

$$\text{(resp. by } L_l(\psi_{z,v}): L_l(\mathcal{O}_{K,S})\to Lie(X_1,\dots,X_n)\tilde{\times}Der^*Lie(X_1,\dots,X_n)\text{)}$$

the morphism induced by

$$gr_W Lie\varphi_{V,v}: L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)\to Der^*Lie(X_1,\dots,X_n)$$

$$\text{(resp. by } gr_W Lie\psi_{V,z,v}: L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)\to Lie(X_1,\dots,X_n)\tilde{\times}Der^*Lie(X_1,\dots,X_n))\text{).}$$

Let $\langle X_i\rangle$ be a one dimensional vector subspace of $Lie(X_1,\dots,X_n)$ generated by X_i. The Lie algebra $Der^*Lie(X_1,\dots,X_n)$ is isomorphic as a vector space to the direct sum $\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle$ (see [14], p.138). The Lie bracket of $Der^*Lie(X_1,\dots,X_n)$ induces the new Lie bracket, denoted by $\{,\}$, on the direct sum. The vector space $\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle$ equipped with the Lie bracket $\{,\}$ we shall denote by $(\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle;\{\,\})$. Passing to dual vector spaces and substituting $Der^*Lie(X_1,\dots,X_n)$ by $(\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle;\{\,\})$ we get morphisms

$$\Phi^v:=(L_l(\varphi_v))^\diamond:\big(\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle;\{\,\}\big)^\diamond\to(L_l(\mathcal{O}_{K,S}))^\diamond$$

and

$$\Psi^{z,v}:=(L_l(\psi_{z,v}))^\diamond:\big(Lie(X_1,\dots,X_n)\tilde{\times}(\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle;\{\,\})\big)^\diamond\to(L_l(\mathcal{O}_{K,S}))^\diamond.$$

Definition 3.2. We set

$$Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,v) := Image\,(\Phi^v)$$

and

$$Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,z,v) := Image\,(\Psi^{z,v}).$$

The vector subspace $Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,v)$ (resp. $Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,z,v)$) of $(L_l(\mathcal{O}_{K,S}))^\diamond$ we shall call the vector space of geometric coefficients on $L_l(\mathcal{O}_{K,S})$ coming from (V,v) (resp. (V,z,v)).

Let us fix a Hall base $\mathcal{B}$ of the free Lie algebra $Lie(X_1,\dots,X_n)$. If $e\in\mathcal{B}$ then e^* denotes the dual linear form in $Lie(X_1,\dots,X_n)^\diamond$ with respect to the base $\mathcal{B}$. Let

$$pr_{i_0} : \bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle \longrightarrow Lie(X_1,\dots,X_n)/\langle X_{i_0}\rangle$$

be the projection on the i_0-th component. Let

$$p : Lie(X_1,\dots,X_n)\tilde{\times}(\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle) \longrightarrow Lie(X_1,\dots,X_n)$$

be the projection on the first factor.

We set

$$\{z,v\}_{e^*} := e^*\circ p\circ L_l(\psi_{z,v}) = \Psi^{z,v}(e^*\circ p). \tag{3.3}$$

Let $e\in\mathcal{B}$ be different from X_i. Let $\overrightarrow{a_i}$ be any tangential point defined over K at a_i. Then we have

$$\{\overrightarrow{a_i},v\}_{e^*} = e^*\circ pr_i\circ L_l(\varphi_v) = \Phi^v(e^*\circ pr_i). \tag{3.4}$$

The geometric coefficients $\{z,v\}_{e^*}$ considered here are the l-adic iterated integrals from [14]. We use here the notation $\{z,v\}_{e^*}$ because it is more convenient for our study.

If $\psi \in \big(Lie(X_1,\dots,X_n)\tilde{\times}(\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle)\big)^\diamond$ then $\Psi^{z,v}(\psi) = \psi\circ L_l(\psi_{z,v})$ is a linear combination of symbols (3.3) and (3.4).

Elements of $(L_l(\mathcal{O}_{K,S}))^\diamond$ which belong to $Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,v)$ are of geometric origin, hence they are motivic. For few rings of algebraic integers one can show that

$$(L_l(\mathcal{O}_{K,S}))^\diamond = Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,v) \tag{3.5}$$

for a convenable choice of a pair (V,v). In the next sections we shall indicate these examples. They follow easily from our paper [16]. The Hodge–de Rham side was presented by P. Deligne on the conference in Schloss Ringberg for $Z[\frac{1}{2}]$ (see [5]), which motivated our study in [16], and in his recent preprint (see [6]).

One cannot expect to show the equality (3.5) for all rings $\mathcal{O}_{K,S}$. Examples in Zagier paper [21] suggests a way to follow.

Let K be a number field and let L be a finite extension of K. Let S be a finite set of finite places of K and let T be a finite set of finite places of L containing all places lying over elements of S. The inclusion of rings

$$\mathcal{O}_{K,S} \hookrightarrow \mathcal{O}_{L,T}$$

induces the surjective morphism

$$L_l(\pi^{L,T}_{K,S}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S}),$$

whose kernel we have denoted by $I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$.

Definition 3.6. Let $g \in (L_l(\mathcal{O}_{K,S}))^\diamond$. We say that g is geometric if there exists $f \in (L_l(\mathcal{O}_{L,T}))^\diamond$ such that

i) f is a geometric coefficient coming from some pair (V, v) or triple (V, z, v);

ii) $f\big(I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})\big) = 0$;

iii) $g \circ L_l(\pi^{L,T}_{K,S}) = f$.

We shall usually denote f and g by the same letter f.

Let $\mathcal{O}_{F,R}$ be a subring of $\mathcal{O}_{K,S}$. Corollary 2.6, which we recall here in the form

$$(L_l(\mathcal{O}_{F,R}))^\diamond = \{f \in (L_l(\mathcal{O}_{K,S}))^\diamond \ \mid \ f(I(\mathcal{O}_{K,S} : \mathcal{O}_{F,R})) = 0\},$$

implies that for subrings $\mathcal{O}_{F,R}$ of the ring $\mathcal{O}_{K,S}$ satisfying (3.5) we have

$$(L_l(\mathcal{O}_{F,R}))^\diamond = \{f \in Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V, v) \ \mid \ f(I(\mathcal{O}_{K,S} : \mathcal{O}_{F,R})) = 0\}.$$

Examples of such rings we shall also discuss in the next sections. In particular we shall show that

$$(L_l(Z))^\diamond = \{f \in Geom\,Coeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01}) \ \mid \ f(I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})) = 0\}.$$

Hence we shall show that all elements of $\big(L_l(\mathbb{Z})\big)^\diamond$ are geometric in the sense of Definition 3.6.

We hope that for any ring $\mathcal{O}_{K,S}$, all coefficients on $L_l(\mathcal{O}_{K,S})$ are geometric in the sense of Definition 3.6.

Remark 3.7. In [18] we were studying related questions. Starting from the torsor of paths $\pi(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus 0, 1, \infty\}; \xi_p, \overrightarrow{01})$ we have constructed all coefficient on $L_l(\mathbb{Z}[\frac{1}{p}])$. However we have not proved that they are geometric in the sense of

Defition 3.6. In the moment of publishing [18] we were thinking that it was obvious. But this is not the case.

Remark 3.8. The geometric coefficients $\{z,v\}_{e^*}$ coming from (V,z,v) are l-adic Galois analogues of iterated integrals from v to z on $\mathbb{P}^1(\mathbb{C})\setminus\{a_1,\dots,a_n,\infty\}$ of sequences of one-forms $\frac{dz}{z-a_1},\dots,\frac{dz}{z-a_n}$. Geometric coefficients in the sense of Definition 3.6 correspond to linear combinations of such iterated integrals. For example $Li_n(\xi_p^k)$ for $1\le k\le p-1$ are periods of a mixed Tate motive over Spec$\mathbb{Q}(\mu_p)$, but $\sum_{k=1}^{p-1}Li_n(\xi_p^k)$ is a period of a mixed Tate motive over Spec$\mathbb{Q}$.

4 From $\mathbb{P}^1\setminus\{0,1,-1,\infty\}$ to periods of mixed Tate motives over Spec$\mathbb{Z}$

Let $V:=\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,-1,\infty\}$. In [16], 15.5 we have studied the Galois representation

$$\varphi_{\overrightarrow{01}}:G_{\mathbb{Q}}\to \mathrm{Aut}(\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})). \tag{4.0}$$

Observe that the pair $(V,\overrightarrow{01})$ has good reduction outside the prime ideal (2) of $\mathbb{Z}$ (see [18], Definition 2.0). Hence the representation (4.0) is unramified outside prime ideals (2) and (l) (see [17], Corollary 17.17). After the standard embedding of $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})$ into the $\mathbb{Q}_l$-algebra of formal power series in non-commuting variables $\mathbb{Q}_l\{\{X,Y_0,Y_1\}\}$ (see [16], 15.2) we get a representation

$$\varphi_{\overrightarrow{01}}:G_{\mathbb{Q}}\to \mathrm{Aut}(\mathbb{Q}_l\{\{X,Y_0,Y_1\}\}). \tag{4.1}$$

It follows from the universal properties of the weighted Tate $\mathbb{Q}_l$-completion that the morphism (4.1) factors through

$$\varphi_{\overrightarrow{01}}:\mathcal{G}(\mathbb{Z}[\frac{1}{2l}];l)\to \mathrm{Aut}(\mathbb{Q}_l\{\{X,Y_0,Y_1\}\}).$$

Passing to associated graded Lie algebras we get a morphism of graded Lie algebras studied in [16], 15.5,

$$gr_W Lie\varphi_{\overrightarrow{01}}:L(\mathbb{Z}[\frac{1}{2l}];l)\to (Lie(X,Y_0,Y_1),\{\,\}). \tag{4.2}$$

It follows from Theorem 3.1 that the morphism (4.2) induces a morphism of graded Lie algebras

$$L_l(\varphi_{\overrightarrow{01}}):L_l(\mathbb{Z}[\frac{1}{2}])\to (Lie(X,Y_0,Y_1),\{\,\}). \tag{4.3}$$

Proposition 4.4. The morphism of graded Lie algebras

$$L_l(\varphi_{\overrightarrow{01}}):L_l(\mathbb{Z}[\frac{1}{2}])\to (Lie(X,Y_0,Y_1),\{\,\})$$

deduced from the action of $G_{\bar{\mathbb{Q}}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, -1, \infty\}; \overrightarrow{01})$ is injective.

Proof. The proposition follows from [16], Theorem 15.5.3. Below we give a more detailed proof.

We recall that $\{G_i(V, \overrightarrow{01})\}_{i\in\mathbb{N}}$ is a filtration of $G_{\mathbb{Q}}$ associated with the representation (4.0) (see [14], section 3). The pair $(V, \overrightarrow{01})$ has good reduction outside the prime ideal (2) of $\mathbb{Z}$. Hence the natural morphism of graded Lie algebras

$$L(\mathbb{Z}[\frac{1}{2l}]; l) \to \bigoplus_{i=1}^{\infty}(G_i(V, \overrightarrow{01})/G_{i+1}(V, \overrightarrow{01})) \otimes \mathbb{Q} \tag{4.4.1}$$

is surjective (see [17], Proposition 19.1). Moreover the natural morphism

$$\bigoplus_{i=1}^{\infty}(G_i(V, \overrightarrow{01})/G_{i+1}(V, \overrightarrow{01})) \otimes \mathbb{Q} \to (Lie(X, Y_0, Y_1), \{\ \}) \tag{4.4.2}$$

is injective (see [17], Proposition 19.2). The morphism (4.2) is the composition of morphisms (4.4.1) and (4.4.2). It follows from Theorem 3.1 that the morphism (4.2) induces a morphism (4.3) Hence the morphism (4.3) induces a surjective morphism of graded Lie algebras

$$L_l(\mathbb{Z}[\frac{1}{2}]) \to \bigoplus_{i=1}^{\infty}(G_i(V, \overrightarrow{01})/G_{i+1}(V, \overrightarrow{01})) \otimes \mathbb{Q} \tag{4.4.3}$$

The graded Lie algebra $L_l(\mathbb{Z}[\frac{1}{2}])$ is free, freely generated by elements dual to $\kappa(2)$ and $l_{2n+1}(-1)$ for $n > 0$. It follows from [16], Theorem 15.5.3 that the elements dual to $\kappa(2)$ and $l_{2n+1}(-1)$ for $n > 0$ are generators of a free Lie subalgebra of $\bigoplus_{i=1}^{\infty}(G_i(V, \overrightarrow{01})/G_{i+1}(V, \overrightarrow{01})) \otimes \mathbb{Q}$. Therefore the morphism (4.4.3) is an isomorphism. This implies that the morphism

$$L_l(\varphi_{\overrightarrow{01}}) : L_l(\mathbb{Z}[\frac{1}{2}]) \to (Lie(X, Y_0, Y_1), \{\ \})$$

is injective. □

The immediate consequence of Proposition 4.4 is the following corollary.

Corollary 4.5. All coefficients on $L_l(\mathbb{Z}[\frac{1}{2}])$ are geometrical, more precisely

$$(L_l(\mathbb{Z}[\frac{1}{2}]))^{\diamond} = GeomCoeff^{l}_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, -1, \infty\}, \overrightarrow{01}).$$

We recall that the morphism of graded Lie algebras

$$L_l(\pi^{\mathbb{Q},(2)}_{\mathbb{Q},\emptyset}) : L_l(\mathbb{Z}[\frac{1}{2}]) \to L_l(\mathbb{Z})$$

induced by the inclusion of rings $\mathbb{Z} \hookrightarrow \mathbb{Z}[\frac{1}{2}]$ is surjective by Proposition 2.3 and its kernel is by the very definition the Lie ideal $I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})$.

Corollary 4.6. We have

$$(L_l(\mathbb{Z}))^{\diamond} = \{f \in GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}} \setminus 0,1,-1,\infty\},\overrightarrow{01}) \ \mid \ f(I(\mathbb{Z}[\frac{1}{2}]:\mathbb{Z}))=0\},$$

i.e. the vector space of coefficients on $L_l(\mathbb{Z})$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}} \setminus 0,1,-1,\infty\},\overrightarrow{01})$ consisting of all coefficients unramified everywhere.

Proof. The corollary follows from Corollary 4.5 and Corollary 2.6. □

Remark 4.6.1. The corresponding statement in Hodge–de Rham realization says that all periods of mixed Tate motives over Spec$\mathbb{Z}$ are unramified everywhere $\mathbb{Q}$-linear combinations of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ in one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$. It will be proved in section 7.

Now we shall look more carefully at geometric coefficients to see which are unramified everywhere.

The Lie algebra $L_l(\mathbb{Z}[\frac{1}{2}])$ is free, freely generated by one generator z_i in each odd degree. The Lie ideal $I(\mathbb{Z}[\frac{1}{2}]:\mathbb{Z})$ is generated by the generator in degree 1. This generator z_1 can be chosen to be dual to the Kummer character $\kappa(2)$, i.e. $\kappa(2)(z_1)=1$.

Let us choose a Hall base $\mathcal{B}$ of the free Lie algebra $Lie(X,Y_0,Y_1)$. Then the geometric coefficients, elements of the $\mathbb{Q}_l$-vector space $GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,-1,\infty\},\overrightarrow{01})$ are of the form $\{\overrightarrow{10},\overrightarrow{01}\}_{e^*}$ and $\{\overrightarrow{10},\overrightarrow{01}\}_{\psi}$, where $\psi=\sum_{i=1}^k n_i e_i^*$ and $e,e_i \in \mathcal{B}$.

Proposition 4.7. Let $e \in \mathcal{B}$ be a Lie bracket in X and Y_0 only. Then the coefficient $\{\overrightarrow{10},\overrightarrow{01}\}_{e^*}$ is unramified everywhere.

Proof. Let $j : \mathbb{P}^1 \setminus \{0,1,-1,\infty\} \hookrightarrow \mathbb{P}^1 \setminus \{0,1,\infty\}$ be the inclusion. Then j induces

$$j_* : \pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,-1,\infty\};\overrightarrow{01}) \to \pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\};\overrightarrow{01}).$$

After the standard embeddings of the fundamental groups into the $\mathbb{Q}_l$-algebras of non-commutative formal power series $\mathbb{Q}_l\{\{X,Y_0,Y_1\}\}$ and $\mathbb{Q}_l\{\{X,Y\}\}$ we get a morphism of $\mathbb{Q}_l$-algebras

$$j_* : \mathbb{Q}_l\{\{X,Y_0,Y_1\}\} \to \mathbb{Q}_l\{\{X,Y\}\}$$

induced by the morphism of fundamental groups such that $j_*(X)=(X),\ j_*(Y_0)=Y,\ j_*(Y_1)=0$.

Then we have $\{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y_0)^*} = \{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y)^*\circ j_*} = \{j(\overrightarrow{10}),j(\overrightarrow{01})\}_{e(X,Y)^*} = \{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y)^*}$ (see [15] (10.0.6)). The pair $(\mathbb{P}^1 \setminus \{0,1,\infty\},\overrightarrow{01})$ is unramified everywhere, hence the coefficient $\{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y_0)^*}$ belonging to $GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0,1,-1,\infty\},\overrightarrow{01})$ is unramified everywhere. □

There are however coefficients in the $\mathbb{Q}_l$-vector space $GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0,1,-1,\infty\}, \overrightarrow{01})$ which contain Y_1 and which also are unramified everywhere. These coefficients are of course the most interesting in view of Corollary 4.6 as we perhaps still do not know if the inclusion

$$GeomCoeff^l_{\mathbb{Z}}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \overrightarrow{01}) \subset (L_l(\mathbb{Z}))^\diamond$$

is the equality. For example we have the following result.

Proposition 4.8. We have

$$\{\overrightarrow{10}, \overrightarrow{01}\}_{[Y_1, X^{(n-1)}]^*} = \frac{1-2^{n-1}}{2^{n-1}} \cdot \{\overrightarrow{10}, \overrightarrow{01}\}_{[Y_0, X^{(n-1)}]^*}.$$

Proof. It follows immediately from the definition of coefficients $\{\overrightarrow{10}, \overrightarrow{01}\}_{e^*}$ and the definition of l-adic polylogarithms (see [15], Definition 11.0.1) that $\{\overrightarrow{10}, \overrightarrow{01}\}_{[Y_0, X^{(n-1)}]^*} = l_n(1)$.

It follows from [16], Lemma 15.3.1 that $\{\overrightarrow{10}, \overrightarrow{01}\}_{[Y_1, X^{(n-1)}]^*} = l_n(-1)$. The proposition now follows from the distribution relation $2^{n-1}(l_n(-1) + l_n(1)) = l_n(1)$ (see [15] Corollary 11.2.3). $\square$

Below we shall give an inductive procedure to decide which coefficients are unramified everywhere. Let us denote for simplicity

$$\mathcal{L} := \mathcal{L}(\mathbb{Z}[\frac{1}{2}]; l), \quad \mathcal{L}_i := \mathcal{L}(\mathbb{Z}[\frac{1}{2}]; l)_i \quad \text{and} \quad \mathcal{L}_{>1} := \bigoplus_{i=2}^{\infty} \mathcal{L}_i.$$

Lemma 4.9. We have

i) $\mathcal{L}_i = \mathbb{Q}_l$ for i odd and $\mathcal{L}_i = 0$ for i even;

ii) $\mathcal{L}_1$ is generated by the Kummer character $\kappa(2)$;

iii) $\mathcal{L}_{2k+1}$ is generated by $l_{2k+1}(-1)$ for $k > 0$.

Proof. It follows from (1.5.b) that $\mathcal{L}_1 = (L_l(\mathbb{Z}[\frac{1}{2}]))_1^\diamond \approx \mathbb{Z}[\frac{1}{2}]^\times \otimes \mathbb{Q}_l \approx \mathbb{Q}_l$. Hence $\mathcal{L}_1$ is generated by the Kummer character $\kappa(2)$.

For $i > 1$ it follows from (1.5.a) that $\mathcal{L}_i \approx H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(i))$. The group $H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(i)) = 0$ for i even and $H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(i)) \approx \mathbb{Q}_l$ for i odd by the result of Soulé (see [13]) combined with the theorem of A. Borel (see [2]). The cohomology group $H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(2k+1))$ is generated by a Soulé class, which is a rational multiple of $l_{2k+1}(-1)$. $\square$

If $e \in \mathcal{B}$ then $\deg_{Y_i} e$ denotes degree of e with respect to Y_i. We define

$$\deg_Y e \;:=\; \deg_{Y_0} e \;+\; \deg_{Y_1} e.$$

Lemma 4.10. Let $\varphi \in GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0,1,-1,\infty\}, \overrightarrow{01})$ be homogeneous of degree k.

i) If $k = 1$ then $d\varphi = 0$ and φ is a $\mathbb{Q}_l$-multiple of $\kappa(2)$. Hence if $\varphi \neq 0$ then φ ramifies at (2).

ii) If $k > 1$ and $d\varphi = 0$ the φ is unramified everywhere.

iii) If $k > 1$ and $\varphi = \sum_{i=1}^{m} a_i e_i^*$, where $e_i \in \mathcal{B}$ and $\deg_Y e_i^* = 1$ for each i then $d\varphi = 0$ and φ is unramified everywhere.

Proof. In degree 1 there are the following geometric coefficients $\{\overrightarrow{10}, \overrightarrow{01}\}_X = 0$, $\{\overrightarrow{10}, \overrightarrow{01}\}_{Y_0} = 0$ and $\{\overrightarrow{10}, \overrightarrow{01}\}_{Y_1} = \kappa(2)$ – the Kummer character of 2, which ramifies at (2).

If $\deg\varphi = k > 1$ and $d\varphi = 0$ then φ is a $\mathbb{Q}_l$-multiple of $l_k(-1)$ by Lemma 4.9 iii). Hence φ is unramified everywhere by Propositions 4.8 and 4.7.

If $\deg_Y e = 1$ then $e = [Y_0, X^{(k-1)}]$ or $e = [Y_1, X^{(k-1)}]$. In both cases it is clear that $d(e^*) = 0$. □

Proposition 4.11. Let $\varphi \in GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0, 1, -1, \infty\}, \overrightarrow{01})$ be homogeneous of degree greater than 1.

i) If $d^{(k+1)}\varphi = 0$ then $d^{(k)}\varphi \in \otimes_{i=1}^{k}\mathcal{L}$.

ii) Le us assume that $d^{(k+1)}\varphi = 0$. Then φ is unramified everywhere if and only if $d^{(k)}\varphi \in \otimes_{i=1}^{k}\mathcal{L}_{>1}$ and $d^{(j)}\varphi$ is unramified everywhere for $0 < j < k$, i.e. $d^{(j)}\varphi \in \otimes_{i=1}^{j}(L_l(\mathbb{Z}))^{\diamond}$ for $0 < j < k$.

iii) Let $\varphi = \sum_{i=1}^{m} n_i \{\overrightarrow{10}, \overrightarrow{01}\}_{e_i^*}$, where $e_i \in \mathcal{B}$ and $\deg_Y e_i \leq k + 1$ for each $i = 1, 2, \ldots, m$. Then $d^{(k+1)}(\varphi) = 0$.

Proof. Let us write $d^{(k)}\varphi$ in the form $\sum_{i\in I} \beta_i^1 \otimes \alpha_i \otimes \beta_i^2$, where $\beta_i^1 \in \otimes_{t=1}^{s}(L(\mathbb{Z}[\frac{1}{2}]))^{\diamond}$, $\alpha_i \in (L(\mathbb{Z}[\frac{1}{2}]))^{\diamond}$ and $\beta_i^2 \in \otimes_{t=1}^{k-s}(L_l(\mathbb{Z}[\frac{1}{2}]))^{\diamond}$. We can assume that elements $\beta_i^1 \otimes \beta_i^2$, $i \in I$ are linearly independent. Observe that the condition $d^{(k+1)}\varphi = 0$ implies that $\big((\otimes_{t=1}^{s} id) \otimes d \otimes (\otimes_{t=1}^{k-s} id)\big) \circ d^{(k)}\varphi = 0$. Hence we get $d\alpha_i = 0$ for $i \in I$. Therefore $\alpha_i \in \mathcal{L}$ for $i \in I$. We have chosen s arbitrary, hence $d^{(k)}\varphi \in \otimes_{i=1}^{k}\mathcal{L}$.

Now we shall prove the part ii) of the proposition. If $d^{(j)}\varphi \in \otimes_{i=1}^{j}(L_l(\mathbb{Z}))^{\diamond}$ for $0 < j < k$ and $d^{(k)}\varphi \in \otimes_{i=1}^{k}\mathcal{L}_{>1}$ then φ vanishes on all Lie brackets containing z_1 of length d for $2 \leq d \leq k + 1$. The linear form φ has degree greater than 1, hence it vanishes on z_1. The assumption $d^{(k+1)}\varphi = 0$ implies that φ vanishes on $\Gamma^{k+2}L_l(\mathbb{Z}[\frac{1}{2}])$. Hence φ vanishes on the Lie ideal $I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})$. Therefore φ is unramified everywhere. The implication in the opposite direction is clear. The part iii) of the proposition is also clear. □

5 $\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3)$ and periods of mixed Tate motives over Spec$\mathbb{Z}[\frac{1}{3}]$ and Spec$\mathbb{Z}[\mu_3]$

In this section and the next one we present more examples when $(L_l(\mathcal{O}_{K,S}))^{\diamond}$ is given by geometric coefficients though without detailed proofs.

Let $U := \mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3)$. In [16] we have also studied the Galois representation

$$\varphi_{U,\overrightarrow{01}} : G_{\mathbb{Q}(\mu_3)} \longrightarrow \mathrm{Aut}\big(\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_3); \overrightarrow{01})\big).$$

The pair $(U,\overrightarrow{01})$ has good reduction outside the prime ideal $(1-\xi_3)$ of $\mathcal{O}_{\mathbb{Q}(\mu_3)}$, where ξ_3 is a primitive 3rd root of 1. Observe that we have the equality of ideals $(1-\xi_3)^2 = (3)$. Hence we get a morphism of graded Lie algebras

$$(5.0) \qquad gr_W Lie\varphi_{U,\overrightarrow{01}} : L(\mathbb{Z}[\mu_3][\frac{1}{3l}];l) \longrightarrow (Lie(X,Y_0,Y_1,Y_2),\{\ \}).$$

It follows from Theorem 3.1 that the morphism (5.0) induces

$$(5.1) \qquad L_l(\varphi_{U,\overrightarrow{01}}) : L_l(\mathbb{Z}[\mu_3][\frac{1}{3}]) \longrightarrow (Lie(X,Y_0,Y_1,Y_2),\{\ \}).$$

Proposition 5.2. The morphism of graded Lie algebras

$$L_l(\varphi_{U,\overrightarrow{01}}) : L_l(\mathbb{Z}[\mu_3][\frac{1}{3}]) \longrightarrow (Lie(X,Y_0,Y_1,Y_2),\{\ \}).$$

deduced from the action of $G_{\mathbb{Q}(\mu_3)}$ on $\pi_1(U_{\bar{\mathbb{Q}}};\overrightarrow{01})$ is injective.

Proof. The proposition follows from [16], Theorem 15.4.7. □

Corollary 5.3. All coefficients on $L_l(\mathbb{Z}[\mu_3][\frac{1}{3}])$ are geometrical. More precisely we have

$$(L_l(\mathbb{Z}[\mu_3][\frac{1}{3}]))^{\diamond} = GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01}).$$

Proof. The result follows immediately from Proposition 5.2. □

The rings of algebraic S-integers $\mathbb{Z}[\mu_3]$, $\mathbb{Z}[\frac{1}{3}]$ and $\mathbb{Z}$ are subrings of the ring $\mathbb{Z}[\mu_3][\frac{1}{3}]$. The following result follows immediately from Corollaries 2.6 and 5.3.

Corollary 5.4. We have:

i) The vector space $(L_l(\mathbb{Z}[\mu_3]))^{\diamond}$ is equal to the vector subspace of these elements of $GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01})$, which are unramified everywhere, i.e.

$$(L_l(\mathbb{Z}[\mu_3]))^{\diamond} = \{f \in GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01}) \ \mid f\big(I(\mathbb{Z}[\mu_3][\frac{1}{3}] : \mathbb{Z}[\mu_3])\big) = 0\}.$$

ii) The vector space $(L_l(\mathbb{Z}[\frac{1}{3}]))^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01})$ consisting of coefficients which are defined over $\mathbb{Q}$, i.e.

$$(L_l(\mathbb{Z}[\frac{1}{3}]))^\diamond = \{f \in GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01}) \mid f\big(I(\mathbb{Z}[\mu_3][\frac{1}{3}] : \mathbb{Z}[\frac{1}{3}])\big) = 0\}.$$

iii) The vector space $(L_l(\mathbb{Z}))^\diamond$ is equal to the vector subspace of these elements of $GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01})$, which are defined over $\mathbb{Q}$ and unramified everywhere, i.e.

$$(L_l(\mathbb{Z}))^\diamond = \{f \in GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01}) \mid f\big(I(\mathbb{Z}[\mu_3][\frac{1}{3}] : \mathbb{Z})\big) = 0\}.$$

6 $\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4)$ and $\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8)$ and periods of mixed Tate motives over Spec$\mathbb{Z}[i]$, Spec$\mathbb{Z}[\mu_8]$, Spec$\mathbb{Z}[\sqrt{2}][\frac{1}{2}]$, Spec$\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]$, Spec$\mathbb{Z}[\sqrt{2}]$ and Spec$\mathbb{Z}[\sqrt{-2}]$

Let us set $W = \mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4)$ and $Z = \mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8)$. The pair $(W, \overrightarrow{01})$ (resp. $(Z, \overrightarrow{01})$) has good reduction outside the prime ideal $(1-i)$ of $\mathbb{Z}[\mu_4]$ (resp. $(1 - e^{\frac{2\pi i}{8}})$ of $\mathbb{Z}[\mu_8]$) lying over (2). Hence it follows from Theorem 3.1 and from [16], Corollary 15.6.4 and Proposition 15.6.5 that morphisms of graded Lie algebras

$$L_l(\varphi_{W,\overrightarrow{01}}) : L_l(\mathbb{Z}[\mu_4][\frac{1}{2}]) \to (Lie(X, Y_0, Y_1, Y_2, Y_3), \{\ \})$$

and

$$L_l(\varphi_{Z,\overrightarrow{01}}) : L_l(\mathbb{Z}[\mu_8][\frac{1}{2}]) \to (Lie(X, Y_0, Y_1, \dots, Y_8), \{\ \})$$

deduced from the action of $G_{\mathbb{Q}(\mu_4)}$ (resp. $G_{\mathbb{Q}(\mu_8)}$) on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_4); \overrightarrow{01})$ (resp. $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_8); \overrightarrow{01})$) are injective. Hence we get the following theorem.

Theorem 6.1. All coefficients on $L_l(\mathbb{Z}[\mu_4][\frac{1}{2}])$ and on $L_l(\mathbb{Z}[\mu_8][\frac{1}{2}])$ are geometrical, more precisely

$$(L_l(\mathbb{Z}[\mu_4][\frac{1}{2}]))^\diamond = GeomCoeff^l_{\mathbb{Z}[\mu_4][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4), \overrightarrow{01})$$

and

$$(L_l(\mathbb{Z}[\mu_8][\frac{1}{2}]))^\diamond = GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \overrightarrow{01}).$$

The rings of algebraic S-integers $\mathbb{Z}[\mu_4]$, $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}$ are subrings of $\mathbb{Z}[\mu_4][\frac{1}{2}]$, while $\mathbb{Z}[\mu_8]$, $\mathbb{Z}[\sqrt{2}][\frac{1}{2}]$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]$, $\mathbb{Z}[\sqrt{-2}]$ and also $\mathbb{Z}[\mu_4][\frac{1}{2}]$, $\mathbb{Z}[\mu_4]$, $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}$ are subrings of $\mathbb{Z}[\mu_8][\frac{1}{2}]$. Hence we get the following result.

Corollary 6.2.

i) The vector space $(L_l(\mathbb{Z}[\mu_4]))^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_4][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\}\cup\mu_4), \overrightarrow{01})$ consisting of the coefficients which are unramified everywhere, i.e.

$$(L_l(\mathbb{Z}[\mu_4]))^\diamond = \{f \in GeomCoeff^l_{\mathbb{Z}[\mu_4][\frac{1}{2}]}(\mathbb{P}^1\setminus(\{0,\infty\}\cup\mu_4), \overrightarrow{01}) \ \mid f\big(I(\mathbb{Z}[\mu_4][\frac{1}{2}] : \mathbb{Z}[\mu_4])\big) = 0\}.$$

ii) The vector space $(L_l(\mathbb{Z}[\mu_8]))^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\}\cup\mu_8), \overrightarrow{01})$ consisting of the coefficients which are unramified everywhere.

iii) The vector space $(L_l(\mathbb{Z}[\sqrt{2}][\frac{1}{2}]))^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\}\cup\mu_8), \overrightarrow{01})$ consisting of coefficients which are defined over $\mathbb{Q}(\sqrt{2})$.

iv) The vector space $(L_l(\mathbb{Z}[\sqrt{2}]))^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\}\cup\mu_8), \overrightarrow{01})$ consisting of coefficients which are unramified everywhere and defined over $\mathbb{Q}(\sqrt{2})$.

v) The vector space $(L_l(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]))^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\}\cup\mu_8), \overrightarrow{01})$ consisting of coefficients which are defined over $\mathbb{Q}(\sqrt{-2})$.

vi) The vector space $(L_l(\mathbb{Z}[\sqrt{-2}]))^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\}\cup\mu_8), \overrightarrow{01})$ consisting of coefficients which are unramified everywhere and defined over $\mathbb{Q}(\sqrt{-2})$.

7 Periods of mixed Tate motives, Hodge–De Rham side

Assuming the motivic formalism as in [1], we shall show here the result announced at the beginning of th paper.

Theorem 7.1. The $\mathbb{Q}$-algebra of periods of mixed Tate motives over Spec$\mathbb{Z}$ is generated by these linear combinations with $\mathbb{Q}$- coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$ of sequences of one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\vec{01}$ to $\vec{10}$, which are unramified everywhere.

Before giving the proof of the theorem we recall some facts about mixed Tate motives.

As in [1] we assume that the category $\mathcal{MT}_{\mathcal{O}_{K,S}}$ of mixed Tate motives over Spec$\mathcal{O}_{K,S}$ exists and has all good properties. In particular the category $\mathcal{MT}_{\mathcal{O}_{K,S}}$ is a tannakian category over $\mathbb{Q}$. Let $\mathcal{G}(\mathcal{O}_{K,S})$ be the motivic fundamental group of the category $\mathcal{MT}_{\mathcal{O}_{K,S}}$ and let $\mathcal{U}(\mathcal{O}_{K,S}) := ker(\mathcal{G}(\mathcal{O}_{K,S}) \to \mathbb{G}_m)$. We have various realization functors from the category $\mathcal{MT}_{\mathcal{O}_{K,S}}$. In particular we have the Hodge–de Rham realization functor to the category of mixed Hodge structures over Spec$\mathcal{O}_{K,S}$;

$$real_{H-DR} : \mathcal{MT}_{\mathcal{O}_{K,S}} \to MHS_{\mathcal{O}_{K,S}}, \ \ M \to \big((M_{DR}, W, F), \ (M_{B,\sigma}, W)_{\sigma:K\to\mathbb{C}},$$

$$(comp_{M,\sigma} : (M_{B,\sigma}\otimes\mathbb{C}, W)\overset{\approx}{\to}(M_{DR}\otimes_\sigma\mathbb{C}, W))_{\sigma:K\to\mathbb{C}}\big).$$

Let V be a smooth quasi-projective algebraic variety over Spec$K$. Let us assume that V has good reduction outside S. Let M be a mixed motive determined by V. Then $M_{DR} = H^*_{DR}(V)$ equipped with weight and Hodge filtrations. For any $\sigma : K \subset \mathbb{C}$, let $V_\sigma := V \times_\sigma$ Spec$\mathbb{C}$. Let $V_\sigma(C)$ be the set of $\mathbb{C}$-points of V_σ. Then $M_{B,\sigma} = H^*(V_\sigma(\mathbb{C}); \mathbb{Q})$ equipped with weight filtration. The isomorphism $comp_{M,\sigma}$ is the comparison isomorphism $H^*(V_\sigma(\mathbb{C}); \mathbb{Q})\otimes\mathbb{C} \to H^*_{DR}(V)\otimes_\sigma\mathbb{C}$.

From now on we assume that $K = \mathbb{Q}$ and S is a finite set of finite places of $\mathbb{Q}$. Then the ring $\mathcal{O}_{\mathbb{Q},S} = \mathbb{Z}[\frac{1}{m}]$ for some $m \in \mathbb{Z}$. Hence we shall write $\mathbb{Z}[\frac{1}{m}]$ instead of $\mathcal{O}_{\mathbb{Q},S}$.

We have two fiber functors on $\mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}$ with values in vector spaces over $\mathbb{Q}$: the Betty realization functor

$$F_B : \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]} \to Vect_{\mathbb{Q}}; \ M \to M_B$$

and the de Rham realization functor

$$F_{DR} : \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]} \to Vect_{\mathbb{Q}}, \ M \to M_{DR}.$$

These two fiber functors are isomorphic. Let $(s_M)_{M\in Ob\mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}} \in \mathrm{Iso}^{\otimes}(F_{DR}, F_B)$ be an isomorphism between the fiber functors F_{DR} and F_B. For each $M \in \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}$ let α_M be the composition

$$M_{DR}\otimes\mathbb{C} \overset{s_M\otimes id_{\mathbb{C}}}{\longrightarrow} M_B\otimes\mathbb{C} \overset{comp_M}{\longrightarrow} M_{DR}\otimes\mathbb{C}.$$

Then $\alpha := (\alpha_M)_{M\in Ob\mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}}$ is an automorphism of the fiber functor

$$F_{DR}\otimes\mathbb{C} : \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]} \to Vect_{\mathbb{C}}; \ \text{given by} \ (F_{DR}\otimes\mathbb{C})(M) = M_{DR}\otimes\mathbb{C}.$$

Hence $\alpha \in \mathrm{Aut}^{\otimes}(F_{DR} \otimes \mathbb{C})$, the group of automorphisms of the fiber functor $F_{DR} \otimes \mathbb{C}$. The group $\mathrm{Aut}^{\otimes}(F_{DR} \otimes \mathbb{C})$ is the group of $\mathbb{C}$-points of $\mathcal{G}_{DR}(\mathbb{Z}[\frac{1}{m}]) = \mathrm{Aut}^{\otimes}(F_{DR})$. Observe that the group $\mathcal{G}_{DR}(\mathbb{Z}[\frac{1}{m}])(\mathbb{C})$ acts on $M_{DR} \otimes \mathbb{C}$ for any $M \in \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}$ and

$$\alpha(M_{DR}) = comp_M(M_B) \subset M_{DR} \otimes \mathbb{C}. \tag{7.2}$$

We denote the element α by $\alpha_{\mathbb{Z}[\frac{1}{m}]}$. Observe that $comp_M(M_B)$ is the Betty lattice in $M_{DR} \otimes \mathbb{C}$ and its coordinates with respect to any base of the $\mathbb{Q}$-vector space M_{DR} are periods of the mixed Tate motive M.

Definition 7.3. We denote by $Periods(M)$ the $\mathbb{Q}$-algebra generated by periods of a mixed Tate motive M.

It is clear that the $\mathbb{Q}$-algebra $Periods(M)$ does not depend on a choice of a base of M_{DR}.

The element $\alpha_{\mathbb{Z}[\frac{1}{m}]} \in \mathcal{G}_{DR}(\mathbb{Z}[\frac{1}{m}])(\mathbb{C})$. The group scheme $\mathcal{G}_{DR}(\mathbb{Z}[\frac{1}{m}])$ is an affine group scheme over $\mathbb{Q}$, hence

$$\mathcal{G}_{DR}(\mathbb{Z}[\frac{1}{m}]) = \mathrm{Spec}(\mathcal{A}_{\mathbb{Z}[\frac{1}{m}]}),$$

where $\mathcal{A}_{\mathbb{Z}[\frac{1}{m}]}$ is the $\mathbb{Q}$-algebra of polynomial functions on $\mathcal{G}_{DR}(\mathbb{Z}[\frac{1}{m}])$.

Definition 7.4. We set

$$UnivPeriods(\mathbb{Z}[\frac{1}{m}]) := \{f(\alpha_{\mathbb{Z}[\frac{1}{m}]}) \in \mathbb{C} \ \mid \ f \in \mathcal{A}_{\mathbb{Z}[\frac{1}{m}]}\}.$$

The set $UnivPeriods(\mathbb{Z}[\frac{1}{m}])$ is a $\mathbb{Q}$-algebra. Observe that we have a surjective morphism of $\mathbb{Q}$-algebras

$$\mathcal{A}_{\mathbb{Z}[\frac{1}{m}]} \longrightarrow UnivPeriods(\mathbb{Z}[\frac{1}{m}]);\ f \to f(\alpha_{\mathbb{Z}[\frac{1}{m}]}).$$

The usual conjecture about periods is that this morphism of $\mathbb{Q}$-algebras is an isomorphism.

Proposition 7.5. For any mixed Tate motive M over $\mathrm{Spec}\mathbb{Z}[\frac{1}{m}]$, the $\mathbb{Q}$-algebra $Periods(M)$ is a $\mathbb{Q}$-subalgebra of the $\mathbb{Q}$-algebra $UnivPeriods(\mathbb{Z}[\frac{1}{m}])$.

Proof. It follows immediately from the formula (7.2). $\square$

Another easy observation is the following one.

Proposition 7.6. We have

$$UnivPeriods(\mathbb{Z}[\frac{1}{m}]) = \bigcup_{M \in \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}} Periods(M).$$

Now we shall study relations between periods of mixed Tate motives over different subrings of $\mathbb{Q}$.

Proposition 7.7. For any relatively prime positive integers m and n, the $\mathbb{Q}$-algebra $UnivPeriods(\mathbb{Z}[\frac{1}{m}])$ is a $\mathbb{Q}$-subalgebra of the $\mathbb{Q}$-algebra $UnivPeriods(\mathbb{Z}[\frac{1}{m\cdot n}])$.

Proof. Let M be a mixed Tate motive over $\mathrm{Spec}\mathbb{Z}[\frac{1}{m}]$. Then M is also a mixed Tate motive over $\mathrm{Spec}\mathbb{Z}[\frac{1}{m\cdot n}]$. But in both cases the Betty and the De Rham lattices in $M_{DR}\otimes\mathbb{C}$ are the same. Hence the proposition follows from Proposition 7.6. □

Definition 7.8. Let m and n be relatively prime, positive integers. We say that $\lambda \in UnivPeriods(\mathbb{Z}[\frac{1}{m\cdot n}])$ is unramified outside m if $\lambda \in UnivPeriods(\mathbb{Z}[\frac{1}{m}])$.

Examples 7.9. Let $z \in \mathbb{Q}^\times$ be such that $1 - z \in \mathbb{Q}^\times$. The triple $(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z, \overrightarrow{01})$ has good reduction outside the set S of primes which appear in the decomposition of the product $z(1 - z)$. The mixed Hodge structure of the torsor of paths $\pi(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}; z, \overrightarrow{01})$ is described by iterated integrals of sequences of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to z and from $\overrightarrow{01}$ to 10 on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. Hence the numbers $\log z$, $\log(1 - z)$, $Li_2(z), \dots, Li_n(z), \dots$ belong to $UnivPeriods(\mathcal{O}_{\mathbb{Q},S})$.

Let p be a prime number. The pair $(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{0p})$ has good reduction outside p. Using the definition of iterated integrals starting from tangential points (see [20]) one gets that $\int_{\overrightarrow{0p}}^{\overrightarrow{10}} \frac{dz}{z} = \log p$. Hence $\log p \in UnivPeriods(\mathbb{Z}[\frac{1}{p}])$.

Now we restrict our attention to $\mathbb{Z}$ and $\mathbb{Z}[\frac{1}{2}]$. First we present the result of Deligne from the conference in Schlossringberg (see [5]). The result of Deligne is also in his recent preprint (see [6]).

The mixed Hodge structure on $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}; \overrightarrow{01})$ is entirely described by the formal power series $\Lambda_{\overrightarrow{01}}(\overrightarrow{10})$ belonging to $\mathbb{C}\{\{X, Y_0, Y_1\}\}$, whose coefficients are iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$. Observe that the pair $(\mathbb{P}^1\setminus\{0, 1, -1, \infty\}, \overrightarrow{01})$ has good reduction outside (2). Hence the mixed Tate motive associated with $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}; \overrightarrow{01})$ is over $\mathrm{Spec}\mathbb{Z}[\frac{1}{2}]$. The result of Deligne can be formulated in the following way.

Theorem 7.10. The morphism

$$\mathcal{G}_{DR}(\mathbb{Z}[\frac{1}{2}])(\mathbb{C}) \longrightarrow \mathrm{Aut}\big(\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, -1, \infty\}; \overrightarrow{01})_{DR}\otimes\mathbb{C}\big)$$

is injective.

The following corollary is an immediate consequence of the theorem.

Corollary 7.11. The $\mathbb{Q}$-algebra $UnivPeriods(\mathbb{Z}[\frac{1}{2}])$ is generated by all iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$.

Proof. Let us denote by $Motive(\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\}; \overrightarrow{01}))$ the mixed Tate motive over Spec$\mathbb{Z}[\frac{1}{2}]$ associated with the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\}; \overrightarrow{01})$. It follows from Theorem 7.10 that

$$Periods\big(Motive(\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\}; \overrightarrow{01}))\big) = UnivPeriods(\mathbb{Z}[\frac{1}{2}]).$$

By (7.2) the Betty lattice of $\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,-1,\infty\}; \overrightarrow{01})_{DR} \otimes \mathbb{C}$ is given by $\alpha_{\mathbb{Z}[\frac{1}{2}]}\big(\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,-1,\infty\}; \overrightarrow{01})_{DR}\big)$ But on the other side it is explicitely given by the formal power series $\Lambda_{\overrightarrow{01}}(\overrightarrow{10}) \in \mathbb{C}\{\{X, Y_0, Y_1\}\}$. Hence it follows that the algebra $UnivPeriods(\mathbb{Z}[\frac{1}{2}])$ is generated by the coefficients of the formal power series $\Lambda_{\overrightarrow{01}}(\overrightarrow{10})$. □

Proof of Theorem 7.1. It follows from Proposition 7.7 that $UnivPeriods(\mathbb{Z})$ is a $\mathbb{Q}$-subalgebra of $UnivPeriods(\mathbb{Z}[\frac{1}{2}])$. Hence it follows from Corollary 7.11 that the $\mathbb{Q}$-algebra $UnivPeriods(\mathbb{Z})$ is generated by certain products of sums of some iterated integrals of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\}$. A product of iterated integrals is a sum of iterated integrals by the formula of Chen (see [3]), which is also valid for iterated integrals from tangential points to tangential points (see [20]). Hence the $\mathbb{Q}$-algebra $UnivPeriods(\mathbb{Z})$ is generated by certain linear combinations with $\mathbb{Q}$-coefficients of iterated integrals of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\}$. By the very definition (see Definition 7.8) such linear combinations are unramified everywhere. □

8 Relations in the image of the Galois representations on fundamental groups

Let p be an odd prime. The pair $(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0,\infty\} \cup \mu_p), \overrightarrow{01})$ has good reduction outside (p). The Galois group $G_{\mathbb{Q}(\mu_p)}$ acts on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_p); \overrightarrow{01})$. After the standard embedding of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_p); \overrightarrow{01})$ into $\mathbb{Q}_l\{\{X, Y_0, \ldots, Y_{p-1}\}\}$ we get the Galois representation

$$\varphi_{\overrightarrow{01}} : G_{\mathbb{Q}(\mu_p)} \to \mathrm{Aut}\big(\mathbb{Q}_l\{\{X, Y_0, \ldots, Y_{p-1}\}\}\big)$$

(see [16]). It follows from Theorem 3.1 that $\varphi_{\overrightarrow{01}}$ induces the morphism of graded Lie algebras

$$L_l(\varphi_{\overrightarrow{01}}) : L_l(\mathbb{Z}[\mu_p][\frac{1}{p}]) \longrightarrow Der^*_{\mathbb{Z}/p}(Lie(X, Y_0, \ldots, Y_{p-1}))$$

(see [16], where the Lie algebra of derivations $Der^*_{\mathbb{Z}/p}(Lie(X, Y_0, \dots, Y_{p-1}))$ is defined. The following result generalizes our partial results for $p = 5$ (see [17], Proposition 20.5) and for $p = 7$ (see [7], Theorem 4.1).

Proposition 8.1. Let p be an odd prime.

i) In the image of the morphism of graded Lie algebras

$$L_l(\varphi_{\vec{01}}) : L_l(\mathbb{Z}[\mu_p][\frac{1}{p}]) \longrightarrow Der^*_{\mathbb{Z}/p}(Lie(X, Y_0, \dots, Y_{p-1}))$$

there are linearly independent over $\mathbb{Q}_l$ derivations τ_i for $1 \leq i \leq \frac{p-1}{2}$ such that

$$\tau_i(Y_0) = [Y_0, Y_i + Y_{p-i}].$$

ii) There are the following relations between commutators

$$\mathcal{R}_k : \quad [\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i] = 0 \quad \text{for} \quad 1 \leq k \leq \frac{p-1}{2}$$

and between relations

$$\sum_{i=k}^{\frac{p-1}{2}} \mathcal{R}_k = 0.$$

Proof. The equality $\xi_p^i(1-\xi_p^{p-i}) = -(1-\xi_p^i)$ implies that $l(1-\xi_p^{p-i}) = l(1-\xi_p^i)$ on $L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])$. Elements $1 - \xi_p^i$ for $1 \leq i \leq \frac{p-1}{2}$ are linearly independent in the $\mathbb{Z}$-module $\mathbb{Z}[\mu_p]^\times$. Hence the point i) of the proposition follows from [16], Lemma 15.3.2.

To show the point ii) we need to calculate the Lie bracket

$$[\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i]$$

in the Lie algebra of special derivations $Der^*_{\mathbb{Z}/p}(Lie(X, Y_0, \dots, Y_{p-1}))$. We recall that the Lie algebra $Der^*_{\mathbb{Z}/p}(Lie(X, Y_0, \dots, Y_{p-1}))$ is isomorphic to $(Lie(X, Y_0, \dots, Y_{p-1}), \{\})$ (see [16]), hence we can do the calculations in $(Lie(X, Y_0, \dots, Y_{p-1}), \{\})$. We have

$$[\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i] = \{Y_k + Y_{p-k}, \sum_{i=1}^{\frac{p-1}{2}} (Y_i + Y_{p-i})\} = \{Y_k + Y_{p-k}, \sum_{i=0}^{p-1} Y_i\} = [Y_k, \sum_{i=0}^{p-1} Y_i] +$$

$$\sum_{i=0}^{p-1} [Y_i, Y_{i+k}] - \sum_{i=0}^{p-1} [Y_k, Y_{k+i}] + [Y_{p-k}, \sum_{i=0}^{p-1} Y_i] + \sum_{i=0}^{p-1} [Y_i, Y_{i+p-k}] - \sum_{i=0}^{p-1} [Y_{p-k}, Y_{i+p-k}] = 0.$$

The relation $[\sum_{k=1}^{\frac{p-1}{2}} \tau_k, \sum_{i=1}^{\frac{p-1}{2}} \tau_i] = 0$ holds in any Lie algebra, hence we have a relation $\sum_{k=1}^{\frac{p-1}{2}} \mathcal{R}_k = 0$ between the relations. □

9 An example of a missing coefficient

We finish our paper with an example showing that one can deal with a single coefficient. We shall use notations and results from our papers [16] and [17].

Let p be an odd prime. It follows from Proposition 1.3 that

$$(L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])_1)^{\diamond} \approx (\mathbb{Z}[\mu_p][\frac{1}{p}])^{\times} \otimes \mathbb{Q}_l.$$

Observe that the elements $1-\xi_p^i$, $1 \leq i \leq \frac{p-1}{2}$ generate freely a free $\mathbb{Z}$-module of maximal rank in $\mathbb{Z}[\mu_p][\frac{1}{p}]^{\times}$. Hence $\dim(L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])_1) = \frac{p-1}{2}$ and elements $T_1, \ldots, T_{\frac{p-1}{2}}$ dual to the Kummer characters $\kappa(1-\xi_p^1), \ldots, \kappa(1-\xi_p^{\frac{p-1}{2}})$ form a base of $L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])_1$. The elements $T_1, \ldots, T_{\frac{p-1}{2}}$ generate freely a free Lie subalgebra of $L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])$.

The elements $\tau_1, \ldots, \tau_{\frac{p-1}{2}}$ from Proposition 8.1 are also dual to the Kummer characters $\kappa(1-\xi_p^1), \ldots, \kappa(1-\xi_p^{\frac{p-1}{2}})$ by the very construction. Hence we have

$$L_l(\varphi_{\overrightarrow{01}})(T_i) = \tau_i \quad \text{for} \quad 1 \leq i \leq \frac{p-1}{2},$$

where $L_l(\varphi_{\overrightarrow{01}}) : L_l(\mathbb{Z}[\mu_p][\frac{1}{p}]) \to Der^*_{\mathbb{Z}/p}(Lie(X, Y_0, \ldots, Y_{p-1}))$ is the morphism from Proposition 8.1. However we have the relations

$$[\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i] = 0 \quad \text{for} \quad 1 \leq k \leq \frac{p-1}{2}.$$

Therefore in degree 2 we have

$$GeomCoeff^l_{\mathbb{Z}[\mu_p][\frac{1}{p}]}\big(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0,\infty\} \cup \mu_p); \overrightarrow{01}\big)_2 \subset \big(L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])\big)^{\diamond}_2$$

but

$$GeomCoeff^l_{\mathbb{Z}[\mu_p][\frac{1}{p}]}\big(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0,\infty\} \cup \mu_p); \overrightarrow{01}\big)_2 \neq \big(L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])\big)^{\diamond}_2$$

for $p > 3$.

The obvious question is how to construct geometric coefficients in degree 2 (or periods of mixed Tate motives over $\mathrm{Spec}\mathbb{Z}[\mu_p][\frac{1}{p}]$) in degree 2) which are dual to Lie brackets $[T_i, T_j]$ for $(i < j)$. It is clear from Proposition 8.1 that there is not enough coefficients in $GeomCoeff^l_{\mathbb{Z}[\mu_p][\frac{1}{p}]}\big(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0,\infty\} \cup \mu_p), \overrightarrow{01}\big)$ if $p > 3$.

We consider only the simplest case $p = 5$. It follows from Proposition 8.1 (see also [17], Proposition 20.5) that there is a coefficient of degree 2 in

$(L_l(\mathbb{Z}[\mu_5][\frac{1}{5}]))^\diamond$, which does not belong to $GeomCoeff^l_{\mathbb{Z}[\mu_5][\frac{1}{5}]}\big(\mathbb{P}^1_{\mathbb{Q}(\mu_5)}\setminus(\{0,\infty\}\cup\mu_5);\overrightarrow{01}\big)$. We shall construct this missing coefficient using the action of $G_{\mathbb{Q}(\mu_{10})}=G_{\mathbb{Q}(\mu_5)}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus(\{0,\infty\}\cup\mu_{10}),\overrightarrow{01})$. The pair $(\mathbb{P}^1_{\mathbb{Q}(\mu_{10})}\setminus(\{0,\infty\}\cup\mu_{10}),\overrightarrow{01})$ has good reduction outside prime divisors of (10).

Observe that $\dim(\mathbb{Z}[\mu_{10}][\frac{1}{10}]^\times\otimes\mathbb{Q})=3$. Hence $\dim L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1=3$. There are the following relations in $\mathbb{Z}[\mu_{10}][\frac{1}{10}]^\times$ modulo torsion

$$(9.1.a)\quad (1-\xi_{10}^i)=(1-\xi_{10}^{-i}),\quad (1-\xi_5^i)(1-\xi_5^{5+2i})=(1-\xi_5^{2i})\quad\text{and}\quad(1-\xi_{10}^5)=2\,.$$

Hence we get

$$(9.1.b)\qquad (1-\xi_{10}^1)=(1-\xi_{10}^3)^{-1}=(1-\xi_5^1)(1-\xi_5^2)^{-1}\,.$$

Therefore the Kummer characters $l(1-\xi_5^1)$, $l(1-\xi_5^2)$ and $l(2)$ form a base of $(L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}]))_1^\diamond$ and $l(1-\xi_5^1)$, $l(1-\xi_5^2)$ form a base of $(L_l(\mathbb{Z}[\mu_5][\frac{1}{5}]))_1^\diamond$. Let S_1,S_2,N (resp. s_1,s_2) be the base of $L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1$ (resp. $L_l(\mathbb{Z}[\mu_5][\frac{1}{5}])_1$) dual to the base $l(1-\xi_5^1)$, $l(1-\xi_5^2)$ and $l(2)$ (resp. $l(1-\xi_5^1)$, $l(1-\xi_5^2)$). Then the morphism

$$\Pi:=\pi^{\mathbb{Q}[\mu_{10}],\{(5),(2)\}}_{\mathbb{Q}[\mu_5],(5)}:L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])\longrightarrow L_l(\mathbb{Z}[\mu_5][\frac{1}{5}])$$

is given in degree 1 by the formulas

$$\Pi(S_1)=s_1,\quad \Pi(S_2)=s_2,\quad \Pi(N)=0.$$

Hence it follows the following result.

Lemma 9.2 The Lie ideal $I(\mathbb{Z}[\mu_{10}][\frac{1}{10}]:\mathbb{Z}[\mu_5][\frac{1}{5}])$ is generated by the element N.

Let us fix a Hall base $\mathcal{B}$ of the free Lie algebra $Lie(X,Y_0,\dots,Y_9)$. If $e\in\mathcal{B}$ we denote by $e^\diamond$ the dual linear form on $Lie(X,Y_0,\dots,Y_9)$ with respect to $\mathcal{B}$.

We have the following result.

Proposition 9.3. We have:

i) In degree 1 the image of the morphism

$$L_l(\varphi_{\overrightarrow{01}}):L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])\longrightarrow(Lie(X,Y_0,\dots,Y_9),\{\ \})$$

induced by the action of $G_{\mathbb{Q}(\mu_{10})}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus(\{0,\infty\}\cup\mu_{10}),\overrightarrow{01})$ is generated by $\sigma_1:=Y_1+Y_9+Y_2+Y_8-Y_3-Y_7$, $\sigma_2:=-Y_1-Y_9+Y_4+Y_6+Y_3+Y_7$ and $\eta:=Y_5$.

ii) The Lie bracket $\{\sigma_1,\sigma_2\}=$

$$[Y_1,2Y_4+Y_6+2Y_8]+[Y_9,2Y_2+Y_4+2Y_6]-[Y_3,2Y_2+2Y_4+Y_8]-[Y_7,Y_2+2Y_6+2Y_8]+$$
$$[-Y_2-Y_8+Y_4+Y_6-Y_1-Y_9+Y_3+Y_7,Y_5]+2[Y_3+Y_7,Y_1+Y_9].$$

iii) Let $\mathcal{F} := [Y_1, Y_8]^* \circ L_l(\varphi_{\overrightarrow{01}})$. Then $\mathcal{F} \neq 0$ and $\mathcal{F}$ vanishes on the Lie ideal $I(\mathbb{Z}[\mu_{10}][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$. Hence $\mathcal{F}$ defines a non trivial linear form of degree 2 on $L_l(\mathbb{Z}[\mu_5][\frac{1}{5}])$ non vanishing on $\Gamma^2 L_l(\mathbb{Z}[\frac{1}{5}])$, i.e. $\mathcal{F}([s_1, s_2]) \neq 0$.

Proof. Let $S \in L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1$. Then it follows from [16] that

$$L_l(\varphi_{\overrightarrow{01}})(S) = \sum_{i=1}^{9} l(1 - \xi_{10}^{-i})(S) Y_i \, .$$

It follows from the relations (9.1.a) and (9.1.b) and the very definition of the elements S_1, S_2 and N of $L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1$ that $\sigma_1 := L_l(\varphi_{\overrightarrow{01}})(S_1) = Y_1 + Y_9 + Y_2 + Y_8 - Y_3 - Y_7$, $\sigma_2 := L_l(\varphi_{\overrightarrow{01}})(S_2) = -Y_1 - Y_9 + Y_4 + Y_6 + Y_3 + Y_7$ and $\eta := L_l(\varphi_{\overrightarrow{01}})(N1) = Y_5$. The elements S_1, S_2 and N form a base of $L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1$. Hence σ_1, σ_2, η generate the image of $L_l(\varphi_{\overrightarrow{01}})$ in degree 1.

To show the point ii) one calculates the Lie bracket $\{\sigma_1, \sigma_2\}$.

Let $\mathcal{F} := [Y_1, Y_8]^* \circ L_l(\varphi_{\overrightarrow{01}})$. Then $\mathcal{F}([S_1, S_2]) = [Y_1, Y_8]^\diamond(\{\sigma_1, \sigma_2\}) = 2$. Therefore we have $\mathcal{F} \neq 0$.

The Lie ideal $I(\mathbb{Z}[\mu_{10}][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$ has a base $[S_1, N]$, $[S_2, N]$ in degree 2. Observe that $\mathcal{F}([S_i, N]) = [Y_1, Y_8]^\diamond(\{\sigma_i, \eta\}) = 0$ because the Lie brackets $[Y_a, Y_b]$ appearing in $\{\sigma_i, \eta\}$ contain Y_5 or the diffirence $a - b$ is 5 or -5. Therefore $\mathcal{F}$ vanishes on the Lie ideal $I(\mathbb{Z}[\mu_{10}][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$. Hence it follows that $\mathcal{F}$ defines a linear form $\bar{\mathcal{F}}$ on $L_l(\mathbb{Z}[\mu_5][\frac{1}{5}])$ such that $\bar{\mathcal{F}}([s_1, s_2]) = 2$ $\square$

Corollary 9.4. Any element of $\left(L_l(\mathbb{Z}[\mu_5][\frac{1}{5}])\right)_i^\diamond$ for $i \leq 2$ is geometric.

Remark 9.5. There are three linearly independent over $\mathbb{Q}$ periods of mixed Tate motives over $\mathrm{Spec}\mathbb{Z}[\mu_5][\frac{1}{5}]$ in degree 2, $Li_2(\xi_5^1)$, $Li_2(\xi_5^2)$ and the third one, which we denote by λ_2. One cannot get this third period λ_2 as an iterated integral on $\mathbb{P}^1(\mathbb{C}) \setminus (\{0, \infty\} \cup \mu_5)$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ of a sequence of length two of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$, $\frac{dz}{z-\xi_5^k}$ for $k = 1, 2, 3, 4$. One gets λ_2 as a linear combination with $\mathbb{Q}$-coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus (\{0, \infty\} \cup \mu_{10})$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ of sequences of length two of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-\xi_{10}^k}$ for $k = 0, 1, 2, \ldots, 9$.

Note added 9.6. The formula ii) of Proposition 8.1 was communicated by P. Deligne to H. Nakamura in his letter of August 31, 2009.

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