# Computations of Siegel modular forms of genus two

by

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## B.2 Normierte und euklidische Vektorräume, metrische Räume, Konvergenz

6. Es bezeichne $(X, |\cdot|)$ einen normierten Vektorraum. Zeigen Sie, daß die beiden folgenden Aussagen äquivalent sind:
   (a) Es gibt ein positiv definites Skalarprodukt $\langle\ ,\ \rangle$ auf $X$, sodaß $|x| = \sqrt{\langle x, x\rangle}$ für alle $x \in X$.
   (b) Die Norm $|\cdot|$ erfüllt die Parallelogrammregel, d.h. es ist $|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$ für alle $x, y \in X$.

7. Für eine reelle Zahl $p \geq 1$ und einen Vektor $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ setzen wir

$$|x|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}.$$

   (i) Beweisen Sie die *Höldersche Ungleichung* : es gilt

$$x \cdot y \leq |x|_p \cdot |y|_q$$

   für alle $x, y \in \mathbf{R}^n$ und alle positiven reellen Zahlen $p, q$ mit $\frac{1}{p} + \frac{1}{q} = 1$. (Hinweis: beweisen Sie zunächst die für alle $a, b \geq 0$ geltende Ungleichung $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, und wenden Sie diese auf die einzelnen Summanden der Summe $\sum \frac{|x_i|}{|x|_p} \frac{|y_i|}{|y|_q}$ an.)

   (ii) Zeigen Sie, daß $x \mapsto |\cdot|_p$ eine Norm auf dem $\mathbf{R}^n$ definiert. (Hinweis zur Dreiecksungleichung: schreiben Sie $\sum(x_i + y_i)^p = \sum x_i(x_i + y_i)^{p-1} + \sum y_i(x_i + y_i)^{p-1}$ und wenden sie auf die Summmen auf der rechten Seite jeweils die Höldersche Ungleichung an.)

8. (i) Zeigen Sie in den Bezeichnungen der vorstehenden Aufgabe, daß durch

$$|x|_\infty := \lim_{p \to \infty} |x|_p$$

   eine Norm auf dem $\mathbf{R}^n$ definiert wird. Berechnen Sie den Grenzwert $|\cdot|_\infty$.

   (ii) Skizzieren Sie die Einheitskreise $\left\{x \in \mathbf{R}^2 \mid |x|_p = 1\right\}$.

## B.3 Stetigkeit

9. Sei $A$ Teilmenge eines metrischen Raumes $X$ mit der Eigenschaft, daß jede stetige Funktion $f: A \to \mathbf{R}$ beschränkt ist. Zeigen Sie, daß $A$ dann eine abgeschlossene Teilmenge von $X$ ist.

10. Sei $n \geq 2$; finden Sie eine Funktion $f: \mathbf{R}^n \to \mathbf{R}$, die im Punkt 0 unstetig ist, sodaß aber die Einschränkung $f|\Gamma$ für jede Gerade $\Gamma$ durch 0 im Punkt 0 stetig ist.

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## Introduction

In 1978 Kurokawa computed explicit examples of Siegel modular forms of genus 2 ([Ku]). These examples led to the Saito-Kurokawa conjecture whose proof focussed the attention to Jacobi forms which were then first studied by Eichler and Zagier ([E-Z]). Meanwhile Jacobi forms have been extensively studied and, in the case of genus 1, they are quite well understood. In contrast to that there are still many gaps in the theory of Siegel modular forms of higher genus, and even in the case of genus 2 many questions are still not answered: Is a Hecke eigenform uniquely determined by its eigenvalues? What is the arithmetic nature of its Fourier coefficients? What is the relation between Hecke eigenforms and Galois representations? Are there Hecke eigenforms of even weight and on the full modular group whose first Fourier Jacobi coefficient vanishes? What is the relation between the eigenvalues of a Hecke eigenform and the scalar products of its Fourier Jacobi coefficients (cf. [K-S])?

At the time of Kurokawa's paper it took much effort and tricky manipulations to produce explicit examples of Siegel modular forms at all. This was mainly due to the lack of computationally realistic formulas for Siegel modular forms. In the past decade there has been much progress in the theory of Jacobi forms as well as in computer hard- and software development. Exploiting this it is nowadays rather easy to go beyond Kurokawa's computations and to produce explicit examples of degree-two forms.

The purpose of this paper is first of all to point out how such calculations can be done. Moreover, we performed such calculations. It turned out that there are two striking, to our knowledge so far unobserved phenomena which might deserve further attention. The second purpose of this paper is to describe these.

We computed the Siegel cusp Hecke eigenforms of genus 2 and even weight on the full Siegel modular group which do not belong to the Maaß-Spezialschar. The first of these forms occurs in weight 20, and for weight 20 up to weight 32, which is the range of our computations, the dimensions of the subspaces spanned by such forms is $1, 1, 2, 2, 3, 4, 5$, respectively. Quite expectedly these Hecke eigenforms can be distinguished by their Hecke eigenvalues (even by the eigenvalue of $T(2)$, the second Hecke operator) and their first Fourier Jacobi coefficient does not vanish. Let $\Upsilon 20$, $\Upsilon 22$, $\Upsilon 24a$, $\Upsilon 24b$, etc. denote these Hecke eigenforms (suitably normalized). Then, in complete analogy to the case of elliptic modular forms, it turned out that for weight $k = 28, 30, 32$ the corresponding Hecke eigenforms are conjugate to each other, i.e. $\Upsilon kb$, $\Upsilon kc$, etc. are obtained by applying an automorphism of $\mathbb{C}$ to the Fourier coefficients of $\Upsilon ka$.

The first of the phenomena mentioned above is that this does not hold true for the two eigenforms in weight 24 and weight 26, respectively: these eigenforms have rational Fourier coefficients. This is striking and contradicts the common expectation.

The second, though less striking phenomenon is the existence of congruences modulo various primes (or prime powers) between the Hecke eigenforms $\Upsilon*$ and Hecke eigenforms from the Maaß-Spezialschar. These congruences are trivial in the sense that they can be rather simply verified. On the other hand they extend to congruences between the corresponding Andrianov (or Spinor) zeta functions and might have some less trivial implications in the (so far void) theory of Galois representations associated to the Hecke eigenforms $\Upsilon*$.

In the course of the numerical computations we had to handle quite large integers at a reasonable speed (multiplication, factorization) and we needed a certain amount of linear algebra (multiplication of matrices, inversion, characteristic polynomials). All these computations could easily be performed using the software package PARI (cf. [P]). I am very grateful to H.Cohen for introducing me to this system and helping me to take the first steps in using this great piece of software. More extensive tables of the examples considered in this paper will appear in [C-S-Z].

## Notations

Throughout we shall use the following notations:

- $\mathbf{Z}, \mathbf{Q}, \mathbf{C}$ = integers, rational and complex numbers, $\mathbf{H}$ = Poincaré upper half plane, $\Gamma_1 = \mathrm{SL}_2(\mathbf{Z})$ = elliptic modular group, $\Gamma_2 = \mathrm{Sp}_2(\mathbf{Z})$ = Siegel modular group of genus 2
- $M_k(\Gamma_2)$ = space of Siegel modular forms of genus 2 and weight $k$ on the full Siegel modular group $\Gamma_2$
- $M_k(\Gamma_1)$ = space of elliptic modular forms of weight $k$ on the full modular group $\Gamma_1$
- $J_{k,m}$ = space of Jacobi forms on $SL_2(\mathbf{Z})$ of index $m$ and weight $k$.
- When the $M$ or $J$ above is replaced by $S$, we always mean the corresponding subspace of cusp forms.
- $q = e^{2\pi i\tau}, \quad \zeta = e^{2\pi i z}, \quad q' = e^{2\pi i\tau'} \quad (\tau, \tau' \in \mathbf{H}, z \in \mathbf{C})$
- Special elliptic modular forms:

$$\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \quad \Delta = \eta^{24}, \quad E_{2k} = 1 - \frac{B_{2k}}{2k}\sum_{\ell=1}^{\infty}\sigma_{2k-1}(\ell)q^{\ell}$$

($B_k$ = $k$–th Bernoulli number).

— Special Jacobi forms:

$$\phi_{10} = -\Delta \cdot \eta^{-6} \sum_{\substack{r,s\in\mathbf{Z} \\ r\not\equiv s \bmod 2}} (-1)^r q^{\frac{s^2+r^2}{4}} \zeta^r$$

$$\phi_{12} = \Delta \cdot \eta^{-6} \cdot \left( 6 \sum_{\substack{r,s\in\mathbf{Z} \\ r\not\equiv s \bmod 2}} s^2(-1)^r q^{\frac{s^2+r^2}{4}} \zeta^r - E_2 \sum_{\substack{r,s\in\mathbf{Z} \\ r\not\equiv s \bmod 2}} (-1)^r q^{\frac{s^2+r^2}{4}} \zeta^r \right)$$

— Special Siegel modular forms of genus two

$$\chi_{10} = V(\phi_{10}), \quad \chi_{12} = V(\phi_{12}), \quad \chi_{14} = V(\phi_{10}E_4), \quad \chi_{16a} = V(\phi_{10}E_6).$$

The operator $V$ (mapping Jacobi forms to Siegel modular forms) and the fact that $\phi_{10}$ and $\phi_{12}$ are elements of $S_{10,1}$ and $S_{12,1}$ will be explained below (cf. the second theorem and the proposition in section 1). As basic reference for Siegel modular forms we refer to [F], for Jacobi forms cf. [E-Z].

## §1. The relevant Theorems for computing Siegel modular forms

We are interested in Siegel modular forms of even integral weight on the full modular group. Any such form $F$ has a Fourier development of the form

$$F = \sum_{\substack{r,n,m\in\mathbf{Z} \\ r^2-4mn\leq 0 \\ n,m\geq 0}} a_F(n,r,m) q^n \zeta^r q'^m,$$

i.e. it has a Fourier development and only those Fourier coefficients $a_F(n,r,m)$ are possibly non-zero where the binary quadratic form $[n,r,m]$ (i.e. the form $nX^2 + rXY + mY^2$) is positive semi-definite. Moreover, the Fourier coefficient $a_F(n,r,m)$ depends only on the $\mathrm{GL}_2(\mathbf{Z})$-equivalence class of the binary quadratic form $[n,r,m]$. Thus, one wants to compute the Fourier coefficients $a_F(Q)$ for all positive semi-definite $\mathrm{GL}_2(\mathbf{Z})$-reduced quadratic forms $Q$. The essential ingredient to tabulate these Fourier coefficients is the following theorem of Igusa which describes the structure of the graded ring of all Siegel modular forms of even weight on $\mathrm{Sp}_2(\mathbf{Z})$.

**Theorem ([I]).** *Let $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ be non-zero forms in the one dimensional spaces $M_4(\Gamma_2), M_6(\Gamma_2), S_{10}(\Gamma_2), S_{12}(\Gamma_2)$, respectively. Then*

$$M_{2*}(\Gamma_2) := \bigoplus_{k\in\mathbf{Z}} M_{2k}(\Gamma_2) = \mathbf{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}],$$

*i.e. the modular forms $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ are algebraically independent and any element of $M_{2*}(\Gamma_2)$ can be written as a polynomial in these functions.*

According to Igusa's theorem we have to seek for a good method to compute the $\psi_4, \ldots$ etc.. The most convenient method is provided by the following theorem which is essentially due to Maaß. Recall that any element $\phi$ of $J_{k,1}$ has a Fourier development of the form

$$\phi = \sum_{\substack{D,r\in\mathbb{Z},D\leq 0\\ D\equiv r^2 \bmod 4}} C_\phi(D) q^{\frac{r^2-D}{4}} \zeta^r$$

(cf. [E-Z], Theorem 2.2).

**Theorem ([M]).** *For any integer $k \geq 0$ the map*

$$\phi = \sum_{\substack{D,r\in\mathbb{Z},D\leq 0\\ D\equiv r^2 \bmod 4}} C_\phi(D) q^{\frac{r^2-D}{4}} \zeta^r \mapsto \sum_{\substack{n,r,m\in\mathbb{Z}\\ r^2-4mn\leq 0\\ n,m\geq 0}} a(n,r,m) q^n \zeta^r q'^m,$$

$$a(n,r,m) := \sum_{a|(n,r,m)} a^{k-1} C_\phi\left(\frac{r^2-4mn}{a^2}\right), \qquad a(0,0,0) := -\frac{2k}{B_{2k}} \cdot C_\phi(0)$$

*defines a Hecke equivariant embedding*

$$V\colon J_{k,1} \hookrightarrow M_k(\Gamma_2).$$

*It maps cusp forms to cusp forms and Eisenstein series to Eisenstein series.*

The Siegel modular forms occurring in the image of $V$ are called *Maaß-Spezialformen*. To compute such forms, we want to compute Jacobi forms of index 1. Via the following proposition this is reduced to the computation of elliptic modular forms on the full modular group.

**Proposition ([S]).** *Let*

$$A = \eta^{-6} \sum_{\substack{r,s\in\mathbb{Z}\\ r\not\equiv s \bmod 2}} s^2 (-1)^r q^{\frac{s^2+r^2}{4}} \zeta^r = 2 + q(2\zeta^2 - 8\zeta + 12 - 8\zeta^{-1} + 2\zeta^{-2}) + \cdots,$$

$$B = \eta^{-6} \sum_{\substack{r,s\in\mathbb{Z}\\ r\not\equiv s \bmod 2}} (-1)^r q^{\frac{s^2+r^2}{4}} \zeta^r = 2 - \zeta - \zeta^{-1} + q(2\zeta^2 - 8\zeta + 12 - 8\zeta^{-1} + 2\zeta^{-2}) + \cdots.$$

*Then, for any integer $k$, the map*

$$(f,g) \mapsto \frac{k}{2} f A - \left(q\frac{d}{dq} f\right) B + g B$$

*defines an isomorphism*

$$I\colon M_k(\Gamma_1) \oplus S_{k+2}(\Gamma_1) \xrightarrow{\approx} J_{k,1}.$$

**Proof.** For the convenience of the reader we sketch the short proof of this theorem. Set

$$\vartheta_\rho = \sum_{\substack{r\in\mathbb{Z}\\ r\equiv \rho \bmod 2}} q^{\frac{r^2}{4}} \zeta^r \quad (\rho = 0, 1).$$

Using these fundamental theta functions any $\phi \in J_{k,1}$ can be written as

$$\phi = h_0\vartheta_0 + h_1\vartheta_1, \qquad h_\rho = \sum_{\substack{D \leq 0, \\ D \equiv \rho \bmod 2}} C_\phi(D) q^{-\frac{D}{4}} \quad (\rho = 0, 1).$$

From this equation we obtain

$$(h_0, h_1)W = (\phi(\tau, 0), \phi_{zz}(\tau, 0)), \qquad W = \begin{pmatrix} \vartheta_0(\tau,0) & \vartheta_{0,zz}(\tau,0) \\ \vartheta_1(\tau,0) & \vartheta_{1,zz}(\tau,0) \end{pmatrix},$$

where the subindex $zz$ indicates second partial derivative with respect to $z$. But $W$ is invertible: Namely, using $\det(W) = 2(2\pi i)^2 q^{\frac{1}{4}} + \mathcal{O}(q^{\frac{5}{4}})$, and the well-known transformation laws satisfied by the $\vartheta_\rho$ (cf. [E-Z], §5) it is easily verified that $\det(W)^4$ is an element of $S_{12}(\Gamma_1)$, i.e. equals $\Delta$ up to multiplication by a scalar; whence

$$\det(W) = 2(2\pi i)^2 \eta^6.$$

Using this to write down the inverse matrix of $W$ we find

$$W^{-1} \begin{pmatrix} \vartheta_0 \\ \vartheta_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} A \\ \frac{-1}{2(2\pi i)^2} B \end{pmatrix}.$$

Summing up we finally have

$$\phi(\tau, z) = \frac{1}{2}\phi(\tau, 0)A - \frac{1}{2(2\pi i)^2}\phi_{zz}(\tau, 0)B,$$

which can also be written as

$$\phi = \frac{k}{2} f A - (q\frac{d}{dq} f)B + gB,$$

where

$$f = \frac{\phi}{k}, \quad g = \frac{1}{2\pi i k}\phi_\tau(\tau, 0) - \frac{1}{2(2\pi i)^2}\phi_{zz}(\tau, 0).$$

From the transformation laws of $\phi$ under $\Gamma_1$ it is easily deduced that $f$ and $g$ are elliptic modular forms on the full modular group of weight $k$ and $k+2$, respectively, and $g$ is even a cusp form. Vice versa it can be shown, using the transformation laws for the $\vartheta_\rho$, that the right of the last equation for $\phi$ always defines an element of $J_{k,1}$ if $f \in M_k(\Gamma_1)$ and $g \in S_{k+2}(\Gamma_1)$. This completes the proof of the proposition.

Note that $\phi$ is a cusp form if and only if $f$ is a cusp form. Hence the first Jacobi cusp forms of index 1 occur in weight 10 and 12; these are the two special Jacobi forms listed in Notations. In fact, one has

$$\phi_{10} = I(0, -\Delta), \quad \phi_{12} = I(\Delta, 0).$$

Note that these Jacobi forms have integral Fourier coefficients and that they are normalized in the sense $C_{\phi_{10}}(-3) = C_{\phi_{10}}(-3) = 1$.

Moreover, the proposition and its supplement concerning cusp forms shows that $\dim S_{k,1} = \dim S_k(\Gamma_1) + \dim S_{k+2}(\Gamma_1) = \dim M_{k-12}(\Gamma_1) + \dim M_{k-10}(\Gamma_1)$. Since $\phi_{10}$ and $\phi_{12}$ are obviously linearly independent over the ring

$$M_*(\Gamma_1) := \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_1) = \mathbb{C}[E_4, E_6]$$

($\phi_{12}/\phi_{10}$ does depend on $z$) we can conclude

$$S_{k,1} = M_{k-10}(\Gamma_1)\phi_{10} \oplus M_{k-12}(\Gamma_1)\phi_{12}.$$

To sum up, by the proposition or the last equation we have explicit formulas for Jacobi forms of index 1. Via Maaß's theorem we then have as well explicit formulas for the generators of the ring $M_{2*}(\Gamma_2)$, namely,

$$\psi_4 = V(I(E_4, 0)), \quad \psi_6 = V(I(E_6, 0)), \quad \chi_{10} = V(\phi_{10}), \quad \chi_{12} = V(\phi_{12}).$$

And hence we have such explicit formulas for any Siegel modular form. These formulas are easily implemented on a computer to tabulate the Fourier coefficient of a basis of Siegel modular forms of given weight $k$. The only parts of this procedure which are computationally expensive are the multiplications of Siegel modular forms. To avoid some of these multiplications it is reasonable to generate at least the Maaß-Spezialschar of a given weight directly, i.e. by applying Maaß's theorem and the above proposition directly instead of writing members of the Spezialschar as polynomials in $\psi_4$ to $\chi_{12}$. We followed this procedure for our numerical calculations (cf §4).

## §2. Hecke theory

In this section we recall the theorems concerning the Hecke theory of genus 2 forms which are necessary to handle and to compute Hecke eigenforms.

**Theorem ([A,p.228, Ex.4.2.10.]).** *Let $k, \ell$ integers, $\ell \geq 1$; let*

$$F = \sum_{Q=[n,r,m]\geq 0} a(Q) q^n \zeta^r q'^m, \qquad T(\ell)F = \sum_{Q=[n,r,m]\geq 0} a^*(Q) q^n \zeta^r q'^m,$$

*where $F$ is an element of $M_k(\Gamma_2)$ and $T(\ell)$ denotes the $\ell$–th Hecke Operator on this space. Then*

$$a^*(Q) = \sum_{t_2 | t_1 | \ell} t_1^{k-2} t_2^{k-1} \sum_{\substack{V \in \Gamma^0\left(\frac{t_1}{t_2}\right)\backslash\Gamma_1 \\ Q((X,Y)V) = [n',r',m'] \\ t_1 | n', t_2 | r', m'}} a\left([\frac{\ell n'}{t_1^2}, \frac{\ell r'}{t_1 t_2}, \frac{\ell m'}{t_2^2}]\right)$$

*where the inner sum is over a complete set of representatives* $V$ *for* $\Gamma^0(\frac{t_1}{t_2})\backslash\Gamma_1$ *satisfying the stated conditions, and where* $\Gamma^0(N) := \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \Gamma_1$*).*

We mention some special cases of the above theorem which are important for our numerical computations:

To begin with, assume that we have computed sufficiently many coefficients of a basis of a Hecke invariant subspace of Siegel modular forms on $\Gamma_2$, and that we want to compute the Hecke eigenforms. The obvious method is to compute the matrix of $T(p)$ for some small prime number $p$ and to diagonalize it. Thus, one needs in particular the formula for the action of $T(p)$ on the Fourier coefficients $a(Q)$ of a given form $F$. By the above theorem it is easily verified that such an explicit formula can be given as follows:

$$\begin{aligned} a_{T(p)F}(n,r,m) = {} & a(p[n,r,m]) + p^{2k-3}a(\frac{p}{t^2}[n,r,m]) + p^{k-2}a(\frac{m}{p},r,pn) \\ & + p^{k-2}\sum_{\nu \bmod p} a(\frac{n+r\nu+m\nu^2}{p}, r+2m\nu, pm). \end{aligned}$$

Here $p$ is any prime number and we set $a(Q) = 0$ if $Q$ is not integral.

Secondly, assume that we have computed sufficiently many Fourier coefficients $a(Q)$ of a Hecke eigenform $F$ and that we want to compute the $p$-th Euler factor of the Andrianov zeta function of $F$. As explained in the next theorem we thus need to compute the eigenvalues $\lambda_p$ and $\lambda_{p^2}$ of $T(p)$ and $T(p^2)$, respectively. By the above theorem we find the formulas

$$\begin{aligned} \lambda_p a(1,1,1) &= a(p,p,p) + p^{k-2}(1 + \left(\frac{p}{3}\right))a(1,1,1), \\ \lambda_p a(p,p,p) &= a(p^2,p^2,p^2) + p^{2k-3}a(1,1,1) + p^{k-2}a(1,p,p^2) \\ &\quad + p^{k-2}\sum_{\nu \bmod p} a(1+\nu+\nu^2, p(1+2\nu), p^2), \\ \lambda_{p^2} a(1,1,1) &= a(p^2,p^2,p^2) + p^{k-2} \sum_{\substack{\nu \bmod p \\ 1+\nu+\nu^2 \equiv 0 \bmod p}} a(1+\nu+\nu^2, p(1+2\nu), p^2) \\ &\quad + p^{2k-4}(\left(\frac{p}{3}\right) + \left(\frac{p^2}{3}\right))a(1,1,1). \end{aligned}$$

The eigenvalue $\lambda_p$ can be computed from the first of these equations (if $a(1,1,1) \neq 0$). However, it is computationally expensive to compute $\lambda_{p^2}$ directly from the third equation since one would need to compute $a(p^2,p^2,p^2)$, i.e. one would need to compute a Fourier coefficients $a(Q)$ where the discriminant of $Q$ is of order $p^4$. To avoid this one should eliminate the $a(p^2,p^2,p^2)$ in the third formula using the second one. One can go even one step further and eliminate then the $a(p,p,p)$ using the first formula so to obtain a formula expressing $\lambda_{p^2}$ in terms of $a(1,1,1)$ and $a(Q)$ with $Q$ primitive and of discriminant $-3p^2$.

The precise formula that one obtains in this way is

$$\begin{aligned}\lambda_{p^2}a(1,1,1) = {} & [\lambda_p^2 - \lambda_p p^{k-2}(1+\left(\frac{p}{3}\right)) - p^{2k-3} + p^{2k-4}(\left(\frac{p}{3}\right)+\left(\frac{p^2}{3}\right))]a(1,1,1) \\ & - p^{k-2}a(1,p,p^2) - p^{k-2} \sum_{\substack{\nu \bmod p \\ 1+\nu+\nu^2 \not\equiv 0 \bmod p}} a(1+\nu+\nu^2, p(1+2\nu), p^2).\end{aligned}$$

The arithmetically interesting object associated to a Hecke eigenform is the Andrianov or Spinor zeta function:

**Theorem ([A], p.165, Prop.3.3.35.; [A2], Theorem 1.3.4, Theorem 2.2.1 and Corollary).** *The space $M_k(\Gamma_2)$ has a basis consisting of simultaneous eigenforms for all Hecke operators $T(\ell)$ ($\ell \in \mathbb{N}$). If $F$ is a simultaneous eigenform with eigenvalues $\lambda_\ell$, then the Andrianov zeta function*

$$Z_F(s) := \zeta(2s-2k+4)\sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{\ell^s}$$

*has an Euler product of the form $Z_F(s) = \prod_p Q_p(p^{-s})^{-1}$, where $Q_p(X)$ is a polynomial of degree 4:*

$$Q_p(X) = 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3}X^3 + p^{4k-6}X^4.$$

Note that we can write the Euler factor $Q_p(X)$ in a more symmetric way as

$$Q_p(X) = (1-(\frac{\lambda_p}{2}+\sqrt{d_p})X + p^{2k-3}X^2)(1-(\frac{\lambda_p}{2}-\sqrt{d_p})X + p^{2k-3}X^2),$$

where

$$d_p = -\frac{3}{4}\lambda_p^2 + \lambda_{p^2} + p^{2k-4} + 2p^{2k-3}.$$

This Euler factor is said to fulfill the *Ramanujan-Petersson conjecture* if all its roots have absolute value $p^{3-2k}$, i.e. if

$$(\frac{\lambda_p}{2} \pm \sqrt{d_p})^2 < 4p^{2k-3}.$$

For the sake of completeness we mention the

**Theorem ([A2], Theorem 2.4.1 and Theorem 3.1.1; [A], Theorem 4.3.16).** *The Dirichlet series $Z_F(s)$ is absolutely convergent for $\Re(s) \gg 0$. It can be meromorphically continued to the complex plane and satisfies*

$$Z_F^*(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s) = (-1)^k Z_F^*(2k-2-s).$$

*If $D < 0$ is a fundamental discriminant, and $\chi$ a character of the group $K(D)$ of positive quadratic forms modulo $SL_2(\mathbf{Z})$ with discriminant $D$, then*

$$\sum_Q \chi(Q) \sum_{\ell=1}^{\infty} \frac{a_F(\ell Q)}{l^s} = A_\chi L(s-k+2,\chi)^{-1} Z_F(s),$$

*where*

$$L(s,\chi) = \sum_Q \chi(Q) \sum_{\ell=1}^{\infty} \frac{r_Q(\ell)}{\ell^s}, \quad A_\chi = \sum_Q \chi(Q) a_F(Q)$$

*($r_Q(\ell)$ = number of representations of $\ell$ by $Q$). Here the $Q$-sums are always over a complete set of representatives $Q$ for $K(D)$.*

## §3. Hecke invariant splittings

If we write an element $f$ of $M_k(\Gamma_2)$ in the form

$$F = \sum_{m=0}^{\infty} \phi_m q'^m,$$

then the $\phi_m$ are known to be elements of $J_{k,m}$; the above development is the so-called Fourier Jacobi development of $F$. The space of cusp forms $S_k(\Gamma_2)$ is the space of all such $F$ with $\phi_0 = 0$. This subspace is invariant under all Hecke operators. It contains the Hecke invariant subspace $VS_{k,1}$. This subspace, in turn, is Hecke equivariantly isomorphic to $S_{2k-2}(\Gamma_1)$ ([E-Z], §5).

By a result of Oda and Evdokimov the subspace $VS_{k,1}$ can be characterized as the subspace of $S_k(\Gamma_2)$ which is spanned by all those Hecke eigenforms whose $Z_F(s)$ has a pole (cf. [O]). From this it is clear that there exists one and only one Hecke invariant complement of $VS_{k,1}$ in $S_k(\Gamma_2)$, namely the subspace spanned by all Hecke eigenforms $F$ with holomorphic $Z_F(s)$. We denote this space by $S_k^?(\Gamma_2)$.

Finally, for any elliptic cusp form $f$ in $S_k(\Gamma_1)$ one can form the *Klingen-Eisenstein series*

$$Kf = \sum_{g \in \mathcal{C}\backslash\Gamma_2} \tilde{f}|_k g \qquad (\tilde{f}(\tau,z,\tau') := f(\tau),\ \mathcal{C} = \mathcal{C}_{2,1} \text{ as in [Kl]}).$$

The map $f \mapsto Kf$ defines a Hecke equivariant embedding

$$K\colon S_k(\Gamma_1) \hookrightarrow M_k(\Gamma_2).$$

It has the property that the 0-th Fourier-Jacobi coefficient of $Kf$ is $f$. In particular, we see that the dimension of $\mathbb{C} \cdot VE_k \oplus KS_k(\Gamma_1)$ equals the codimension of the subspace of cusp forms in $M_k(\Gamma_2)$, that this space contains no cusp forms, and hence, that this space is a Hecke invariant complement of the subspace of cusp forms.

Summarizing, one has the Hecke invariant splitting

$$M_k(\Gamma_2) = KS_k(\Gamma_1) \oplus VJ_{k,1} \oplus S_k^?(\Gamma_2).$$

For the Andrianov zeta functions associated to the Hecke eigenforms in the former two spaces one knows the following (cf. [Kl], [E-Z]):

$$\begin{aligned} Z_F(s) &= L_f(s)L_f(s-k+2) \quad \text{for} \quad F = Kf \\ Z_F(s) &= L_f(s)L_{E_2}(s-k+2) \quad \text{for} \quad F = V\phi, \end{aligned}$$

where, in the latter case, $f$ denotes a suitable Hecke eigenform in $S_{2k-2}(\Gamma_1)$ and where, for any elliptic mopdular form $f$ (or $f = E_2$), we use

$$L_f(s) = a_f(1)^{-1} \sum_{\ell=1}^{\infty} a_f(\ell)\ell^{-s}$$

if $f = \sum_{\ell=0}^{\infty} a_f(\ell)q^\ell$ and $a_f(1) \neq 0$. The above identity shows in particular that the Ramanujan-Peterson conjecture fails for the Hecke eigenforms in $VS_{k,1}$. The common expectation is that the Hecke eigenforms which satisfy the Ramanujan-Petersson conjecture are exactly those in the space $S_k^?(\Gamma_2)$.

Thus, it is clear that the attention has to be focussed to the space of *interesting Siegel modular forms* $S_k^?(\Gamma_2)$.

## §4. Computation of the first interesting Hecke eigenforms

From Igusa's theorem we deduce that the dimension of $M_k(\Gamma_2)$ equals the number of quadruples $(a, b, c, d)$ of nonnegative integers satisfying $4a + 6b + 10c + 12d = k$, i.e.

$$\sum_{k=0}^{\infty} \dim M_{2k}(\Gamma_2) X^{2k} = \frac{1}{(1-X^4)(1-X^6)(1-X^{10})(1-X^{12})}.$$

Similarily, using

$$S_*(\Gamma_1) := \bigoplus_{k \in \mathbb{Z}} S_k(\Gamma_1) = \Delta \cdot \mathbb{C}[E_4, E_6]$$

and the fact that $K, V$ are injective and that $S_{2k,1}$ is isomorphic to $S_{2k-2}(\Gamma_1)$ we find

$$\begin{aligned} \sum_{k=0}^{\infty} \dim KS_{2k}(\Gamma_1)\, X^{2k} &= \frac{x^{12}}{(1-X^4)(1-X^6)} \\ \sum_{k=0}^{\infty} \dim VS_{2k,1}\, X^{2k} &= \frac{X^{10}(1+X^2)}{(1-X^4)(1-X^6)}, \end{aligned}$$

and finally, using the results quoted in the preceding section,

$$\sum_{k=0}^{\infty} \dim S^{?}_{2k}(\Gamma_2) X^{2k} = \frac{X^{20}(1+X^2+X^4-X^{12}+X^{14})}{(1-X^4)(1-X^6)}.$$

**Table 1** in §6 lists the dimensions of the first few Hecke invariant subspaces.

The first candidate for an interesting Hecke eigenform, i.e. a non-Maaß-Spezialschar cusp eigenform is found in weight 20. Since $S^{?}_{20}(\Gamma_2)$ is 1-dimensional this first non-Spezialschar cusp eigenform is uniquely determined (up to multiplication by scalars) — we call it $\Upsilon 20$. In [Ku] its first few Hecke eigenvalues have been computed (our $\Upsilon 20$ equals $-\frac{1}{2}$ times Kurokawa's $\chi^{(3)}_{20}$). To write down a formula for it we note first of all that the cusp form $\chi^2_{10}$ is not a Maaß-Spezialscharform: in fact, its Fourier-Jacobi development starts as $\phi^2_{10}q'^2 + \ldots$, i.e. its first Fourier-Jacobi coefficient vanishes, whereas the first Fourier Jacobi coefficient of a Maaß-Spezialform $V\phi$ is $\phi$ itself.

Thus, $\chi^2_{10}$ equals $\Upsilon 20$ plus a Maaß-Spezialscharform, i.e. $\Upsilon 20$ can be obtained by adding a suitable cusp Maaß-Spezialscharform to $\chi^2_{10}$. The subspace $VS_{20,1}$ of Spezialscharformen in $S_{20}(\Gamma_2)$ is 2–dimensional; it is spanned by $V(\phi_{10}E_4E_6)$ and $V(\phi_{12}E_4^2)$. Hence, up to normalisation,

$$\Upsilon 20 \approx \chi^2_{10} + aV(\phi_{10}E_4E_6) + bV(\phi_{12}E_4^2)$$

for suitable constants $a, b$. To find $a, b$ we computed sufficiently many coefficients of $\chi^2_{10}$, $V(\phi_{10}E_4E_6)$, $V(\phi_{12}E_4^2)$. Then we applied $T(2)$ to these forms using the formula for $T(p)$ in section 2. This enabled us to find the matrix $M$, which is uniquely determined by $T(2)B = MB$ where $B = (\chi^2_{10}, V(\phi_{10}E_4E_6), V(\phi_{12}E_4^2))^t$. By well-known algebra we have $\chi(M)B = v \cdot \Upsilon 20$ with a suitable complex column vector $v$, where $\chi(X)$ is the characteristic polynomial of the restriction of $T(2)$ to $VS_{20,1}$. From the formula for the Andrianov zeta function of a Maaß-Spezialform quoted in section 3 one verifies for the latter polynomial the identity $\chi(X) = \widetilde{\chi}(X + 2^{k-2} + 2^{k-1})$, where $\widetilde{\chi}(X)$ denotes the characteristic polynomial of the Hecke operator $T(2)$ on the space $S_{38}(\Gamma_1)$ of elliptic cusp forms of weight 38. The latter can be computed by well-known procedures.

The other first few Hecke eigenforms $\Upsilon 22$, $\Upsilon 24a$, $\Upsilon 24b$, $\Upsilon 26a$, $\Upsilon 26b \cdots$ of weights 22 to 32 can be found similarly The particular results are given in **Table 2**. In **Table 3** we listed the first few Fourier coefficients of these forms.

Note that Table 2 shows in particular that all the forms $\Upsilon 22$ up to $\Upsilon 26b$ have rational Fourier coefficients. For the Hecke eigenforms in $S^{?}_{28}(\Gamma_2)$, $S^{?}_{30}(\Gamma_2)$, $S^{?}_{32}(\Gamma_2)$ this is not true; their Fourier coefficients generate (after suitable normalisation) a cubic, quartic, quintic number field, respectively. This is easily deduced from the fact, that the characteristic polynomials $H\,k(X)$ of $T(2)$ on $S^{?}_{k}(\Gamma_2)$ ($k = 28, 30, 32$) are irreducible over $\mathbf{Q}$. These characteristic polynomials are listed in **Table 5**. This table gives also the prime decomposition of the discriminants $fd\,k$ of the fields $\mathbf{Q}[X]/(H\,k(X))$. Note, that these discriminants contain only a small number of primes as compared to their impressive size. It may be worthwhile to investigate whether this is part of a more general phenomenon.

Finally, using the formulas for $\lambda_p$ and $\lambda_{p^2}$ from section 2 it is possible to compute the first few Euler factors $Q_p(X)$ of the Andrianov zeta function of $\Upsilon 22$ up to $\Upsilon 26b$. The resulting values of $\lambda_p$ and $d_p$ are given in **Table 4**.

We checked within the range of the table 4 that the roots of $x^2 - (\frac{\lambda_p}{2} \pm \sqrt{d_p})X + p^{2k-3}$ are complex conjugate. Thus all roots of $Q_p(X)$ have absolute value $p^{3-2k}$, i.e. within the range of our computations the eigenforms $\Upsilon 20$ to $\Upsilon 26b$ satisfy the generalized Petersson-Ramanujan conjecture.

## §5. Congruences for the interesting Hecke eigenforms

A Siegel modular form is said *to be defined over* $R$ (a subring of $\mathbf{C}$) if all its Fourier coefficients are contained in $R$, i.e. if its Fourier development can be viewed as an element of $R[q,\zeta,q']$. Two Siegel modular forms which are defined over $\mathbf{Z}$ are said to be congruent modulo $N$ ($\in \mathbf{Z}$) if they have the same image under the projection map

$$\mathbf{Z}[q,\zeta,q'] \to \mathbf{Z}[\![q,\zeta,q']\!]/N\mathbf{Z}[q,\zeta,q'].$$

A similar obvious terminology will be applied to Jacobi forms, elliptic modular forms and Dirichlet series.

Using this terminology we note

**Proposition.** *All the Siegel modular forms* $\Upsilon 20$ – $\Upsilon 26b$ *listed in Table 2 are defined over* $\mathbf{Z}$. *One has*

$$a_{\Upsilon *}(1,1,1) = \begin{cases} 1 & \text{if } * = 20, 22, 24a, 26a \\ 3 & \text{if } * = 24b, 26b \end{cases}.$$

*For each of these forms the g.c.d. of its Fourier coefficients is 1.*

**Proof.** These assertions are easily read off from Table 2 and Table 3. For the integrality asertions one uses the following obvious facts: The Jacobi forms $\phi_{10}$ and $\phi_{12}$ and the elliptic modular forms occurring in Table 2 are defined over $\mathbf{Z}$. The $V$-operator maps forms defined over $\mathbf{Z}$ to forms defined over $\mathbf{Z}$. Therefore all Maaß-Spezialscharformen occurring in Table 2 are defined over $\mathbf{Z}$. Thus the $\Upsilon *$ are $\frac{1}{6}\mathbf{Z}$-linear combinations of forms defined over $\mathbf{Z}$, i.e. they have rational Fourier coefficients with denominators at most equal to 6. That the denominator 6 does not really occur has to be checked case by case using the fact that $E_4$ and $E_6$ are congruent to 1 modulo 24 and $\phi_{10}$ and $\phi_{12}$ are congruent modulo 12. The latter is immediately clear from the formulas in Notations.

This proposition together with Table 2 immediately implies that $\Upsilon 20$ is congruent modulo $2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ to the Spezialscharform $V(\frac{1}{2}\phi_{12}E_4^2 + \frac{1}{2}\phi_{10}E_4E_6)$, that $\Upsilon 22$ is congruent to a Spezialscharform modulo $2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 1423$, etc. Even more, it is clear from the closed formulas in Table 2 that the number $2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ is the largest integer $M$ such that $\Upsilon 20$ is congruent modulo $N$ to a Spezialscharform, and similar statements hold for the other specimen in Table 2 too.

It is not hard to prove that congruences as the ones just considered imply congruences for the Andrianov zeta functions. More precisely, one has

**Proposition.** *Let $N$ be a positive integer and let $F \in VJ_{k,1}$ be defined over $\mathbb{Z}$ and such that the g.c.d. of its Fourier coefficients is prime to $N$. Assume that $F$ is a Hecke eigenform modulo $N$, i.e. that $T(\ell)F \equiv \lambda_\ell F \bmod N$ for all $\ell$ and with suitable integers $\lambda_\ell$. Then there exists a $f \in S_{2k-2}(\Gamma_1)$, which is defined over $\mathbb{Z}$ and is a Hecke eigenform modulo $N$, such that*

$$\zeta(2s-2k+4)\sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{\ell^s} \equiv L_f(s)L_{E_2}(s-k+2) \bmod N.$$

**Proof.** If $F$ is a Hecke eigenform one has for any $Q$

$$\zeta(2s-2k+4)\sum_{\ell=1}^{\infty} a_{T(\ell)F}(Q)\ell^{-s} = a_F(Q)Z_F(s),$$

whence

$$\zeta(2s-2k+4)\sum_{\ell=1}^{\infty} a_{T(\ell)F}\ell^{-s} = a_F(Q)L_f(s)L_{E_2}(s-k+2)$$

with a suitable elliptic modular form $f$ from $M_{2k-2}(\Gamma_1)$. Since $VJ_{k,1}$ has a basis of Hecke eigenforms and by linearity the latter identity is true for any element in $VJ_{k,1}$. In particular it holds true for the $F$ as in the proposition, and since $a_{T(\ell)F}(Q) \equiv \lambda_\ell a_F(Q) \bmod N$ we conclude

$$\zeta(2s-2k+4)a_F(Q)\sum_{\ell=1}^{\infty}\lambda_\ell\ell^{-s} \equiv a_F(Q)L_f(s)L_{E_2}(s-k+2) \bmod N.$$

Note that by assumption on $F$ and the foregoing identity $f$ is defined over $\mathbb{Z}$. Since by assumption the g.c.d. of the $a_F(Q)$ and $N$ are relatively prime we deduce from the last identity the asserted one. This identity shows in particular that $L_f(s) \bmod N$ has an Euler product, and by well-known arguments this implies that $f$ is a Hecke eigenform modulo $N$.

As we saw above the $\Upsilon*$ are congruent to Spezialscharformen modulo certain $N$. These Spezialscharformen are then Hecke eigenforms modulo $N$ and its Fourier coefficients are even relatively prime (cf. Table 3), i.e. they fulfill exactly the assumptions of the proposition. Thus, the proposition shows that to each $\Upsilon*$ and its $N$ there corresponds an elliptic modular form $f$, which is a Hecke eigenform modulo $N$, such that

$$Z_{\Upsilon*}(s) \equiv L_f(s)L_{E_2}(s-k+2) \bmod N.$$

Note that this identity implies $a_f(p) \equiv \lambda_f - p^{k-2} - p^{k-1} \bmod N$, where $p$ denotes any prime and $\lambda_p$ the eigenvalue of $\Upsilon k*$ with respect to $T(p)$. Thus, given $N$ and the first few eigenvalues $\lambda_p$ of $\Upsilon k*$, we can immediately identify the modular form $f \bmod N$ with respect to any $\mathbb{Z}$-basis of the lattice of elements of $S_{2k-2}(\Gamma_1)$ which are defined over $\mathbb{Z}$. The particular $f$ (and $N$) corresponding to the $\Upsilon*$ are listed in **Table 6**.

Note that these congruences together with the theory of congruences for elliptic modular forms imply further congruences. E.g. from the first row of Table 6 we can deduce the congruences

$$\begin{aligned} Z_{\Upsilon 20}(s) &\equiv L_{E_2}(s-18)L_{E_2}(s-4) \bmod 5, \\ Z_{\Upsilon 20}(s) &\equiv L_{E_2}(s-18)L_{E_2}(s-3) \bmod 7, \\ Z_{\Upsilon 20}(s) &\equiv L_{E_2}(s-18)L_{E_4}(s-2) \bmod 11. \end{aligned}$$

To prove these congruences recall first of all that for any prime $p$ one has $E_2 \equiv E_{p+1}$ mod $p$ and that $\theta := q\frac{d}{dq}$ maps $\widetilde{M}_k(\Gamma_1)$ to $\widetilde{S}_{k+p+1}(\Gamma_1)$, preserving Hecke eigenforms. Here $\widetilde{M}_k(\Gamma_1)$ and $\widetilde{S}_{k+p+1}(\Gamma_1)$ denote the reduction modulo $p$ of the $\mathbf{Z}$-modules of modular forms in $M_k(\Gamma_1)$ and $S_{k+p+1}(\Gamma_1)$, respectively, which have Fourier coefficients in $\mathbf{Z}$. From this it is immediately clear that $\theta^2 E_2$ mod 5, $\theta^4 E_2$ mod 5 or $\theta^3 E_2$ mod 7, $\theta^2 E_4$ mod 7 or $\theta^2 E_4$ mod 11, $\theta E_6$ mod 11 are Hecke eigenforms in $\widetilde{S}_{38}(\Gamma_1)$ for $p = 5, 7, 11$, respectively. Since the latter spaces are 2-dimensional (over $\mathbf{Z}/p\mathbf{Z}$) these are all Hecke eigenforms in these spaces, and hence the $f$ in Table 6 has to be congruent modulo $5, 7, 11$ to one of these eigenforms (up to multiplication by a scalar), respectively. The particular congruences, which one finds in each of these cases, are just the ones listed above.

We leave it to the reader to verify similar congruences for the other $\Upsilon *$.

Finally, we mention another kind of congruence which can immediately be read off from Table 2. Namely, if we look at the $2 \times 5$ matrix which has as rows the rows of Table 2 corresponding to $\Upsilon 24a$ and $\Upsilon 24b$ then we recognize that the g.c.d. of its $2 \times 2$ minors is $4 \cdot 31$. This indicates that there should be a congruence between $\Upsilon 24a$ and $\Upsilon 24b$ modulo $4 \cdot 31$, and that $4 \cdot 31$ is the largest integer for which such a congruence holds true. In fact, consulting Table 2 the congruence

$$3 \cdot \Upsilon 24a \equiv \Upsilon 24b \bmod 4 \cdot 31$$

is easily verified: the coefficients $3 \cdot (-2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 157)$ and $-2^7 \cdot 3 \cdot 7 \cdot 13^2 \cdot 83$ in the formulas for $3 \cdot \Upsilon 24a$ and $\Upsilon 24b$ in front of $\chi_{10}\chi_{14}$ are congruent modulo 31, and the same is true for the corresponding coefficients in front of $\chi_{12}^2$, $\phi_{12}E_6^2$, etc.. The claimed congruence modulo 4 can be verified similarily where one has to use additionally that $\phi_{12}E_6^2$ is congruent modulo 4 to $\phi_{10}E_4^2E_6$. In the same way it is deduced that

$$3 \cdot \Upsilon 26a \equiv \Upsilon 26b \bmod 4 \cdot 37$$

and that $4 \cdot 37$ is the largest integer for which such a congruence holds true. It is easily checked (e.g. by using the formula expressing the Spinor zeta function in terms of Fourier coefficients as quoted in the last theorem in §2) that these congruences imply corresponding congruences for the Spinor zeta functions.

## §6. Tables

Table 1 : Dimensions of $M_k(\Gamma_2)$ and subspaces for $0 \leq k \leq 50$

| $k$ | $M_k(\Gamma_2)$ | $KS_k(\Gamma_1)$ | $VS_{k,1}$ | $S_k^?(\Gamma_2)$ |
|---|---|---|---|---|
| 0 | 1 | – | – | – |
| 2 | – | – | – | – |
| 4 | 1 | – | – | – |
| 6 | 1 | – | – | – |
| 8 | 1 | – | – | – |
| 10 | 2 | – | 1 | – |
| 12 | 3 | 1 | 1 | – |
| 14 | 2 | 0 | 1 | – |
| 16 | 4 | 1 | 2 | – |
| 18 | 4 | 1 | 2 | – |
| 20 | 5 | 1 | 2 | 1 |
| 22 | 6 | 1 | 3 | 1 |
| 24 | 8 | 2 | 3 | 2 |
| 26 | 7 | 1 | 3 | 2 |
| 28 | 10 | 2 | 4 | 3 |
| 30 | 11 | 2 | 4 | 4 |
| 32 | 12 | 2 | 4 | 5 |
| 34 | 14 | 2 | 5 | 6 |
| 36 | 17 | 3 | 5 | 8 |
| 38 | 16 | 2 | 5 | 8 |
| 40 | 21 | 3 | 6 | 11 |
| 42 | 22 | 3 | 6 | 12 |
| 44 | 24 | 3 | 6 | 14 |
| 46 | 27 | 3 | 7 | 16 |
| 48 | 31 | 4 | 7 | 19 |
| 50 | 31 | 3 | 7 | 20 |

Table 2 : Closed formulas for the interesting Hecke eigenforms $\Upsilon 20 - \Upsilon 26b$

$$\begin{aligned}
\Upsilon 20 &= -2^9\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot \chi_{10}^2 + V\big(\frac{1}{2}\phi_{12}E_4^2 + \frac{1}{2}\phi_{10}E_4E_6\big)\\
\Upsilon 22 &= -2^5\cdot 3\cdot 5\cdot 7\cdot 1423\cdot \chi_{10}\chi_{12}\\
&\quad + V\big(-\frac{5}{2\cdot 3}\phi_{12}E_4E_6 + \frac{11}{2\cdot 3}\phi_{10}E_6^2 + 2^4\cdot 3\cdot 61\cdot \phi_{10}\Delta\big)\\
\Upsilon 24a &= -2^5\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 157\cdot \chi_{10}\chi_{14} + 2^5\cdot 3\cdot 5^2\cdot 11\cdot 157\cdot \chi_{12}^2\\
&\quad + V\big(-\frac{7}{2\cdot 3}\phi_{12}E_6^2 - 2^4\cdot 3\cdot 67\cdot \phi_{12}\Delta + \frac{13}{2\cdot 3}\phi_{10}E_4^2E_6\big)\\
\Upsilon 24b &= -2^7\cdot 3\cdot 7\cdot 13^2\cdot 83\cdot \chi_{10}\chi_{14} - 2^6\cdot 3\cdot 7\cdot 13^2\cdot 83\cdot \chi_{12}^2\\
&\quad + V\big(\frac{41}{2\cdot 3}\phi_{12}E_6^2 + 2^6\cdot 3\cdot 5\cdot 7\cdot \phi_{12}\Delta - \frac{23}{2\cdot 3}\phi_{10}E_4^2E_6\big)\\
\Upsilon 26a &= -2^6\cdot 3^3\cdot 5^2\cdot 11\cdot 29\cdot \chi_{10}\chi_{16a} - 2^6\cdot 3^4\cdot 5^2\cdot 11\cdot 29\cdot \chi_{12}\chi_{14}\\
&\quad + V\big(-\frac{1}{2}\phi_{12}E_4^2E_6 + \frac{3}{2}\phi_{10}E_4^4 - 2^5\cdot 3^2\cdot 31\cdot \phi_{10}\Delta E_4\big)\\
\Upsilon 26b &= -2^6\cdot 3^3\cdot 5^3\cdot 7\cdot 13^2\cdot \chi_{10}\chi_{16a} + 2^5\cdot 3^3\cdot 5^2\cdot 7^2\cdot 13^2\cdot \chi_{12}\chi_{14}\\
&\quad + V\big(\frac{5\cdot 13}{2\cdot 3}\phi_{12}E_4^2E_6 - \frac{47}{2\cdot 3}\phi_{10}E_4^4 + 2^4\cdot 3\cdot 5\cdot 251\cdot \phi_{10}\Delta E_4\big)
\end{aligned}$$

Table 3 : The first few Fourier coefficients of the $\Upsilon 20 - \Upsilon 26b$

| $\Upsilon *$ | 1,1,1 | 1,0,1 | 1,1,2 | 1,0,2 | 1,1,3 |
|---|---|---|---|---|---|
| $\Upsilon 20$ | 1 | $2^2$ | $2^3\cdot 7$ | $2^3\cdot 3\cdot 109$ | $-3\cdot 11\cdot 1669$ |
| $\Upsilon 22$ | 1 | $-2^2\cdot 3$ | $2^6\cdot 3\cdot 7$ | $2^3\cdot 3^3$ | $3^4\cdot 5059$ |
| $\Upsilon 24a$ | 1 | $-2^4$ | $-2^3\cdot 11\cdot 23$ | $-2^5\cdot 3\cdot 11\cdot 19$ | $-3\cdot 11\cdot 23\cdot 563$ |
| $\Upsilon 24b$ | 3 | $2^2\cdot 19$ | $-2^3\cdot 7\cdot 11$ | $-2^3\cdot 3\cdot 11^2$ | $3\cdot 11\cdot 131\cdot 491$ |
| $\Upsilon 26a$ | 1 | $-2^3$ | $-2^5\cdot 233$ | $2^4\cdot 3\cdot 317$ | $-3\cdot 11\cdot 83\cdot 431$ |
| $\Upsilon 26b$ | 3 | $2^2\cdot 31$ | $2^4\cdot 7\cdot 461$ | $-2^3\cdot 3\cdot 17\cdot 269$ | $3\cdot 2433059$ |

**Table 4** : The first few Euler factors of the Andrianov zeta functions of $\Upsilon 20 - \Upsilon 26b$

$$Z_{\Upsilon *}(s) = \prod_p \left(1 - (\frac{\lambda_p}{2} + \sqrt{d_p})p^{-s} + p^{2k-3-2s}\right)^{-1} \cdot \left(1 - (\frac{\lambda_p}{2} - \sqrt{d_p})p^{-s} + p^{2k-3-2s}\right)^{-1}$$

| $\Upsilon *$ | $p$ | $\lambda_p$ | $d_p = -\frac{3}{4}\lambda_p^2 + \lambda_{p^2} + p^{2k-4} + 2p^{2k-3}$ |
|---|---|---|---|
| $\Upsilon 20$ | 2 | $-2^8 \cdot 3^2 \cdot 5 \cdot 73$ | $2^{14} \cdot 3^2 \cdot 7 \cdot 13 \cdot 19 \cdot 241$ |
| | 3 | $2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 5099$ | $2^6 \cdot 3^{10} \cdot 19 \cdot 47 \cdot 150628997$ |
| | 5 | $-2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 166103087$ | $2^8 \cdot 3^2 \cdot 5^6 \cdot 19 \cdot 47 \cdot 1396135808326877$ |
| | 7 | $2^4 \cdot 5^2 \cdot 7^3 \cdot 673 \cdot 28346749$ | $2^8 \cdot 3^6 \cdot 7^6 \cdot 29 \cdot 1097 \cdot 41713094306662453$ |
| $\Upsilon 22$ | 2 | $-2^8 \cdot 3 \cdot 5 \cdot 577$ | $2^{14} \cdot 3^4 \cdot 13^2 \cdot 31 \cdot 439$ |
| | 3 | $-2^3 \cdot 3^5 \cdot 5 \cdot 19 \cdot 97 \cdot 167$ | $2^8 \cdot 3^{10} \cdot 11 \cdot 61 \cdot 8364437759$ |
| | 5 | $2^2 \cdot 3 \cdot 5^3 \cdot 60700091989$ | $2^{10} \cdot 3^4 \cdot 5^6 \cdot 7193 \cdot 9888524030928593$ |
| $\Upsilon 24a$ | 2 | $-2^{11} \cdot 3 \cdot 5 \cdot 181$ | $2^{20} \cdot 3^2 \cdot 7 \cdot 17 \cdot 61559$ |
| | 3 | $-2^3 \cdot 3^6 \cdot 5 \cdot 7 \cdot 23^2 \cdot 491$ | $2^6 \cdot 3^{12} \cdot 7^3 \cdot 413057028823$ |
| | 5 | $-2^2 \cdot 3 \cdot 5^3 \cdot 7 \cdot 29 \cdot 109438961$ | $2^8 \cdot 3^2 \cdot 5^6 \cdot 7^2 \cdot 13 \cdot 19^2 \cdot 157 \cdot 659 \cdot 74293331977811$ |
| $\Upsilon 24b$ | 2 | $-2^9 \cdot 3^2 \cdot 23 \cdot 61$ | $2^{16} \cdot 3^2 \cdot 5 \cdot 11^2 \cdot 97 \cdot 373$ |
| | 3 | $-2^3 \cdot 3^6 \cdot 2328401$ | $2^6 \cdot 3^{12} \cdot 5 \cdot 11^2 \cdot 13^2 \cdot 1163672669$ |
| | 5 | $2^2 \cdot 3^2 \cdot 5^3 \cdot 1562781531383$ | $2^8 \cdot 3^2 \cdot 5^6 \cdot 11^2 \cdot 13^2 \cdot 5036898546360995644l$ |
| $\Upsilon 26a$ | 2 | $-2^{13} \cdot 3^2 \cdot 5 \cdot 7^2$ | $2^{24} \cdot 3^2 \cdot 859 \cdot 5779$ |
| | 3 | $-2^3 \cdot 3^5 \cdot 5 \cdot 307 \cdot 61091$ | $2^{10} \cdot 3^{10} \cdot 107 \cdot 1093 \cdot 16123577711$ |
| | 5 | $-2^2 \cdot 3^2 \cdot 5^5 \cdot 13 \cdot 37 \cdot 293 \cdot 1847 \cdot 3067$ | $2^{12} \cdot 3^2 \cdot 5^{10} \cdot 17 \cdot 373 \cdot 165515489 \cdot 74684067301$ |
| $\Upsilon 26b$ | 2 | $-2^9 \cdot 3^2 \cdot 5 \cdot 229$ | $2^{16} \cdot 3^2 \cdot 7 \cdot 67 \cdot 163 \cdot 33703$ |
| | 3 | $-2^3 \cdot 3^7 \cdot 5 \cdot 7 \cdot 1061 \cdot 1579$ | $2^{10} \cdot 3^{14} \cdot 41 \cdot 1153 \cdot 594719897$ |
| | 5 | $2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 37 \cdot 757 \cdot 2713 \cdot 51713$ | $2^{12} \cdot 3^2 \cdot 5^6 \cdot 11 \cdot 206009 \cdot 1318336479421624233l$ |

**Table 5** : Characteristic polynomials $H\,k$ of $T(2)$ on $S_k^?(\Gamma_2)$ and discriminants $fd\,k$ of $\mathbf{Q}[X]/(H\,k(X))$ for $k = 28, 30, 32$

$$H28 = X^3 + 137681664X^2 + 4794374687293440X + 4100431555335920025600$$

$$fd28 = 5 \cdot 13 \cdot 73693 \cdot 1418741$$

$$\begin{aligned} H30 = {} & X^4 + 374036736X^3 - 38240213642772480X^2 - 1675860454758443227545600X \\ & + 332649478287802168188390604800 0 \end{aligned}$$

$$fd30 = 3 \cdot 7698969562410587331 83$$

$$\begin{aligned} H32 = {} & X^5 + 2026982400X^4 - 1037849863848984576X^3 - 146076577865569625060671488 0X^2 \\ & + 19785068550622402489774561768243200 0X \\ & + 18632364235800427734471441591459840943718400 0 \end{aligned}$$

$$fd32 = 2^2 \cdot 3 \cdot 7 \cdot 17091289294563642107663508479464475 9$$

**Table 6** : Congruences for the Andrianov zeta functions: $Z_{\Upsilon*}(s) \equiv L_{E_2}(s-k+2)L_f(s) \bmod N$

| $\Upsilon*$ | $f$ | $N$ |
|---|---|---|
| $\Upsilon 20$ | $E_4^5 \cdot E_6 \cdot \Delta + 146016 E_4^2 \cdot E_6 \cdot \Delta^2$ | $2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ |
| $\Upsilon 22$ | $E_4^6 \cdot E_6 \cdot \Delta + 4200240 E_4^3 \cdot E_6 \cdot \Delta^2 + 4200240 R \cdot \Delta^3$ | $2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 1423$ |
| $\Upsilon 24a$ | $E_4^7 \cdot E_6 \cdot \Delta + 92736 E_4^4 \cdot E_6 \cdot \Delta^2 + 33120 E_4 \cdot E_6 \cdot \Delta^3$ | $2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 157$ |
| $\Upsilon 24b$ | $E_4^7 \cdot E_6 \cdot \Delta + 18655488 E_4^4 \cdot E_6 \cdot \Delta^2 + 12111936 E_4 \cdot E_6 \cdot \Delta^3$ | $2^6 \cdot 3 \cdot 7 \cdot 13^2 \cdot 83$ |
| $\Upsilon 26a$ | $E_4^8 \cdot E_6 \cdot \Delta + 507600 E_4^5 \cdot E_6 \cdot \Delta^2 + 13694400 E_4^2 \cdot E_6 \cdot \Delta^3$ | $2^6 \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 29$ |
| $\Upsilon 26b$ | $E_4^8 \cdot E_6 \cdot \Delta + 46602000 E_4^5 \cdot E_6 \cdot \Delta^2 + 22420800 E_4^2 \cdot E_6 \cdot \Delta^3$ | $2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13^2$ |

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