Automorphisms of non-compact Riemann surfaces and Riemannian manifolds of negative curvature

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§1. Introduction (1.1) Let S be a Riemann surface of genus g and t ends ("ideal boundary components"), $0 \leq g, t < \infty$. Let $G = \text{Aut } S$. If $t = 0$ and $g \geq 2$, then by a well-known result of Hurwitz $|G| \leq 84(g-1)$ with equality iff ${}_G\backslash^S \approx \mathbb{P}^1$ and the canonical projection $p: S \to {}_G\backslash^S = \mathbb{P}^1$ has precisely three branch points with branching indices $\{2,3,7\}$, cf. e.g. [4] theorem 1.7.2. Now suppose that S is non-compact, so $t \geq 1$. In this case Greenberg proved that if $3 \leq 2g+t$ then G is discrete and if moreover, $2g+t < \infty$ then $|G| < \infty$, cf. [3], theorem 3, cf. also [4] theorem 1.7.1.

It appears however that the following more precise analogue of Hurwitz's theorem is not in the literature. This analogue has an interesting connection with the classical modular group.

(1.2) Theorem. Let S be as above with $3 \leq 2g+t$. Then G is discrete. If $t \geq 1$ and $3 \leq 2g+t < \infty$ then $|G| \leq 12(g-1)+6t$ with equality iff ${}_G\backslash^S$ is homeomorphic to $\mathbb{R}^2$ and the canonical projection $p: S \to {}_G\backslash^S$ has precisely two branch points with branching indices $\{2,3\}$. Moreover in the case of equality, ${}_G\backslash^S$ is biholo-

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morphic to $\mathbb{C}$ iff S is biholomorphic to a compact Riemann surface with finitely many punctures.

This result is better understood in its more geometric formulation which also points out the special role played by the modular group. In case $3 \leq 2g+t$, the universal cover of S may be identified with the upper half plane $\mathbb{H}^2$ and the deck-transformation group $\Delta \cong \pi_1(S)$ acts on $\mathbb{H}^2$ - so $\Delta \leq \mathrm{Aut}\,\mathbb{H}^2 \approx PSL_2(\mathbb{R})$ which is also the full group of orientation-preserving isometries with respect to the hyperbolic metric. Let $\mathcal{N}$ be the normalizer of Δ in $\mathrm{Aut}\,\mathbb{H}^2$. Then $_{\Delta}\backslash^{\mathcal{N}} \cong G$, and the latter part of the above theorem amounts to the following. The equality $|G| = 12(g-1)+6t$ occurs iff $\mathcal{N} \approx \mathbb{Z}_2 * \mathbb{Z}_3$. Suppose $\mathcal{N} = \langle x,y \mid x^2 = y^3 = e\rangle$. Let P,Q be the fixed points of x and y respectively in $\mathbb{H}^2$. Then

(1.2.1) $\begin{cases} \text{the hyperbolic distance } d(P,Q) \geq \ln\sqrt{3} \text{ with equality} \\ \text{iff } \mathcal{N} \text{ is conjugate to the modular group iff the} \\ \text{hyperbolic area of } S \text{ is finite.} \end{cases}$

(1.3) In the proof of (1.2) we reprove the part already obtained by Greenberg by a more geometric method. This proof has the advantage that it partially extends to the isometry groups of arbitrary n-dimensional complete Riemannian manifolds of negative curvature. In this case the notion of the limit set of the deck-transformation group and its convex hull have a meaning. We call such a manifold non-elementary if the corresponding limit set has at least three points.

(1.4) Theorem. Let M^n be a complete, non-elementary Riemannian manifold with negative curvature bounded away from zero. Let $\mathcal{C}$ be the convex hull of the corresponding limit set, and $I(M)$ (resp. $I_0(M)$) denotes the full isometry group of M (resp. the identity component of $I(M)$). Then $I_0(M)$ is compact, and has dimension $\leq \frac{1}{2}\ell(\ell-1)$ where $\ell = n-k$, $k = \dim \mathcal{C}$. In particular if $\dim \mathcal{C} \geq n-1$ then $I(M)$ is discrete.

(1.5) This result partially extends the results of [2], 5.18 and 5.19. In these results, Chen and Eberlein allow the curvature to be non-positive but make a strong hypothesis that the geodesic flow is non-wandering. It appears that already in dimension 3 (1.4) has wider applicability than the results available in the literature. A much more subtle question here is to decide the number of components of $I(M)$. Using deeper results from Kleinian groups and 3-dimensional topology, one can answer this question substantially in dimension 3 cf. (3.7) - (3.9). Of course, making strong geometric hypotheses, e.g. if M is compact or at least has finite volume, this question can be answered in all dimensions. But a weak topological hypothesis "$\pi_1(M)$ is finitely generated", is not sufficient to guarantee an affirmative answer. We discuss some possibilities in §3. All this is of course very far from the precise understanding à la Hurwitz in dimension 2. The only works in higher dimensions in this direction appear to be Huber [6] and Im Hof [7].

§2. Proof of theorem (1.2)

(2.1) Let $\tilde{S}$ be the universal cover of S. Since $t \geq 1$ and $3 \leq 2g+t$ we see that $\pi_1(S) \approx$ a nonabelian free group on $2g+t-1$ generators. It may be realized as a deck-transformation group $\Delta \leq \mathrm{Aut}(\tilde{S})$. If $\mathcal{N}$ is the normalizer of Δ in $\mathrm{Aut}(\tilde{S})$ then $\mathcal{N}/\Delta \approx \mathrm{Aut}\, S = G$. Now $\tilde{S}$, being non-compact is biholomorphic to $\mathbb{C}$ or $\mathbb{H}^2$. But $\mathrm{Aut}(\mathbb{C})$ is solvable and so cannot contain a non-abelian free subgroup. So $\tilde{S}$ may be taken to be $\mathbb{H}^2$, and so G is also the full group of orientation-preserving isometries of S w.r.t. the induced hyperbolic metric. In particular G acts properly i.e. for each compact set $K \subseteq S$, the subset $\{g \in G \mid gK \cap K \neq \emptyset\}$ is compact in G w.r.t. the compact-open topology. We shall also regard $\tilde{S} = \mathbb{H}^2$ as the open unit disk so that $\mathrm{Aut}\, \tilde{S}$ extends to the closed unit disk.

(2.2) We now assert that G is discrete. Indeed, let G_0 resp. $\mathcal{N}_0$ be the identity component of G resp. $\mathcal{N}$. Then $\mathcal{N}_0 / \mathcal{N}_0 \cap \Delta \approx G_0$ and since Δ is discrete, $\mathcal{N}_0$ is connected and normalizes Δ we see that $\mathcal{N}_0$ also centralizes Δ. Let Λ be the limit set of Δ (i.e. the cluster set of an orbit Δp, $p \in \mathbb{H}^2$ in the unit circle). Since $\mathcal{N}_0$ centralizes Δ we see that $\mathcal{N}_0$ fixes Λ pointwise. Since Δ is infinite $\Lambda \neq \emptyset$. If Λ contains ≤ 2 points then Δ would be virtually abelian which is not the case. So Λ contains at least three points. Let p,q,r be any three distinct points of Λ and γ a geodesic of $\mathbb{H}^2$ joining p to q, and s the foot of the perpendicular η from r to γ. Since $\mathcal{N}_0$ fixes p,q,r, it leaves γ and η invariant so it fixes s. Hence in fact it fixes γ pointwise and so also the convex hull $\mathcal{C}$ of Λ in $\mathbb{H}^2$. Since obviously $\mathcal{C}$ has interior it follows that $\mathcal{N}_0 = \{e\}$. So $G_0 = \{e\}$ also.

(2.3) Now assume that $3 \leq 2g+t < \infty$. We claim that G is compact. Since S is homeomorphic to a compact surface of genus g with t points removed, there exists a compact subset $K \subseteq S$ such that its inclusion map is a homotopy equivalence and each component of $S-K$ is an annulus. Then for each $g \in G$ we must have $gK \cap K \neq \emptyset$ for otherwise gK would be contained in an annulus A and the inclusion $gK \hookrightarrow A \hookrightarrow S$ would fail to be a homotopy equivalence. So G is compact.

(2.4) Combining (2.2) and (2.3) we now know that G is finite. Let G_p denote the isotropy subgroup of G at p. The set $\mathcal{S} = \{x \in S | G_x \neq e\}$ is <u>the singular set</u> of the G-action. Let $p: S \to {}_G\backslash^S$ be the canonical projection. The set $\beta = p(\mathcal{S})$ is called the <u>branch set</u> of the G-action. We now assert that $\mathcal{S}$ and hence β are <u>finite</u> sets. Indeed let $\bar{S}$ denote the compact surface obtained from S by adjoining its t ends. The finite group G extends as a group of orientation-preserving homeomorphisms of $\bar{S}$. But then each $g \in G - \{e\}$ can have only finitely many fixed points on $\bar{S}$. So $\mathcal{S}$ and hence β are finite sets.

(2.5) Now we are in a position to apply Hurwitz's analysis. The relation

(2.5.1) $$\chi(S-|\mathcal{S}|) = |G| \; \chi({}_G\backslash^S - |\beta|)$$

where χ denotes the Euler characteristic, leads to the Riemann-

Hurwitz relation.

(2.5.2) $2-2g-t = |G|\{2-2h-u-\sum_{x\in\beta}(1-\frac{1}{n_x})\}$

where h = the genus of ${}_G\backslash^S$, u = the number of ends of ${}_G\backslash^S$ and n_x = the branching index at $x \in \beta$. Since ${}_G\backslash^S$ is non-compact we have $u \geq 1$. If $2-2h-u < 0$ then $|G| \leq (2g+t-2)/(2h+u-2)$. If $2-2h-u = 0$ then since the L.H.S. of (2.5.2) is < 0 we see that $|\beta| \geq 1$ say $\beta = \{x\}$. Since $n_x \geq 2$ we have $|G| \leq (2g+t-2)/(1-\frac{1}{n_x}) \leq 2\cdot(2g+t-2)$. If $2-2h-u = 1$ then $|\beta| \geq 2$ and if $|\beta| = 2$ say $\beta = \{x,y\}$ then we cannot have $(n_x,n_y) = (2,2)$. So if $|\beta| \geq 3$ we have

$$|G| \leq -(2g+2-2)/1-3/2 = 2(2g+t-2)$$

If $|\beta| = 2$ then

$$|G| \leq -(2g+2-2)/1-\frac{1}{2}-\frac{2}{3} = 6(2g+t-2) = 12(g-1)+6t.$$

This proves the first part of theorem 1.

(2.6) The same argument shows that if $|6| = 12(g-1)+6t$ we must have $\chi({}_G\backslash^S) = 1$ i.e. ${}_G\backslash^S$ is homeomorphic to $\mathbb{R}^2$ and there are precisely two branch points with branching indices $\{2,3\}$. This implies that $\mathcal{N}$ is a free product of two elements say x and y of order 2 and 3 respectively.

(2.7) Let P resp. Q be the fixed point of x resp. y in $\mathbb{H}^2$ (which we may consider as the upper half-plane). Let γ_0 be a geodesic passing through P and Q, γ_1 = a geodesic through P perpendicular to γ_0 and γ_2 = a geodesic through Q making an angle $\pi/3$ with γ_0 in the direction towards P. Let σ_i denote the reflections through the geodesics γ_i $i = 0,1,2$. Then clearly $x = \sigma_1\sigma_0$, $y = \sigma_0\sigma_2$. So $xy = \sigma_1\sigma_2$. If γ_1, γ_2 intersect, say at R, then xy would fix R. If xy is of infinite order then $\mathcal{N}$ would not be discrete and if xy has finite order then $\mathcal{N}$ would not be a free product of $\langle x\rangle$ and $\langle y\rangle$. So γ_1 and γ_2 do not intersect. If γ_1, γ_2 are asymptotic then $\mathcal{N}$ is clearly conjugate (in $\mathrm{Aut}(\mathbb{H}^2)$) to the modular group. This happens precisely when $d(P,Q) = \ell n \sqrt{3}$, and then ${}_G\backslash^S$ is biholomorphic to $\mathbb{C}$. Otherwise $d(P,Q) > \ell n \sqrt{3}$ and we must have ${}_G\backslash^S$ homeomorphic but not biholomorphic to $\mathbb{C}$. This finishes the proof of the theorem (1.2), and (1.2.1).

§3. Proof of (1.4) and further comments

(3.1) Proof of (1.4): Let M^n be as in (1.4), $\tilde{M}^n$ its universal cover, $\Delta \approx \pi_1(M)$ the decktransformation group, and $\mathcal{N}$ the normalizer of Δ in $I(\tilde{M})$. So $\mathcal{N}/\Delta \approx I(M)$. If $\mathcal{N}_0$ resp. $I_0(M)$ denote the identity components (w.r.t. compact-open topology) of $\mathcal{N}$ resp. $I(M)$ we have, as in (2.2), $\mathcal{N}_0 / \Delta \cap \mathcal{N}_0 \approx I_0(M)$, and $\mathcal{N}_0$ centralizes Δ. The ideal boundary of $\tilde{M}$ is defined by classes of asymptotic geodesics. We denote this boundary by ∂. It is well-known, cf. [2] and the references there, that w.r.t. an appropriate topology $\partial \approx S^{n-1}$, and $\tilde{M} \cup \partial \approx$ a closed n-disk and $I(M)$ extends continuously to $\tilde{M} \cup \partial$. Given two distinct points x, y in ∂ there exists a unique geodesic in $\tilde{M}$ ending in x and y. The limit set Λ of Δ is the cluster set in ∂ of a Δ-orbit, Δp, $p \in \tilde{M}$. (It is independent of the choice of p.) Let $\mathcal{C}$ be the convex hull of Λ in $\tilde{M}$. Exactly by the argument in (2.2), since by hypothesis Λ contains at least 3 distinct points, we see that $\mathcal{N}_0$ fixes Λ and $\mathcal{C}$ pointwise. Let p be an interior point of any convex simplex σ of maximum dimension (= k say) contained in $\mathcal{C}$. Then $I_0(M)$ fixes the tangent plane to σ at p, and $I_0(M)$ is isomorphic to a closed subgroup of the orthogonal group $O(\ell)$, $\ell = n-k$. This implies all the assertions in (1.4.).

q.e.d.

(3.2) Now we take up the question of determining the number of components of $I(M)$. If $\dim M = 2$ then (1.2) together with the well-known elementary cases $0 \leq 2g+t \leq 2$ give a fair understanding of this question. The question for $\dim M \geq 3$ is quite subtle.

First we note some well-known cases proved in different ways. We derive these from our set-up. In the following, throughout M^n denotes a complete Riemannian manifold with negative curvature bounded away from zero.

(3.3) Proposition. If M^n has finite volume, (in particular if it is compact), then $I(M^n)$ is finite.

Proof. It is easy to see (e.g. from the shape of the Dirichlet fundamental domain) that in this case $\Lambda = \partial$, so $\dim \mathcal{C} = n$. Hence by (3.1) $I(M)$ is discrete. If M^n is compact then $I(M)$ is also compact, so $I(M)$ is finite. Now suppose only that M^n has finite volume. For $\varepsilon > 0$ consider the set $M_\varepsilon = \{x \in M^n \mid$ the injectivity radius at $x \geq \varepsilon\}$, i.e. for each $x \in M_\varepsilon$ the closed metric ball $B_x(\varepsilon)$ of radius ε with center x is homeomorphic to a closed disk. If ε is small then $M_\varepsilon \neq \emptyset$. On the other hand if M_ε were non-compact it would clearly contain infinitely many mutually disjoint balls of radius ε. Each such ball has volume at least equal to that of the Euclidean ball of radius ε. But then M_ε and hence M would have infinite volume. It follows that M_ε is compact and as it is clearly invariant under $I(M)$, we have $I(M) = I(M_\varepsilon)$. The latter is clearly compact.

q.e.d.

(3.4) The argument of (3.3) has wider applicability. E.g. if for some $a,b \in \mathbb{R}_{>0}$, $M_{a,b} = \{x \in M \mid$ the injectivity radius at x lies in $[a,b]\}$ is nonempty and compact then $I(M)$ is compact.

(3.5) Next consider the case when M is elementary, i.e. Λ has

≤ 2 points. If $\Lambda = \emptyset$ then M is simply connected. This case is well-understood, cf. [2] and the references there.

Proposition. If Λ contains two points then either $I(M)$ is compact or else $I(M)$ has a subgroup of index 2 which is isomorphic to $A \times B$ where A is compact and $B \approx \mathbb{R}$ or $\mathbb{Z}$.

Proof. Let $\Lambda = \{x,y\}$ and $\hat{\gamma}$ the geodesic in $\tilde{M}$ joining x and y. Then $I(M)$ leaves Λ and γ invariant and so it has a subgroup G of index ≤ 2 which fixes x and y. Let $p \in \gamma$. Clearly G_p is compact and in fact $G_p = \{g \in G | g$ fixes γ pointwise$\}$. Each $g \in G$ induces an orientation-preserving isometry of γ. The group $I_O(\gamma)$ of orientation-preserving isometries of γ is $\approx \mathbb{R}$. Since G acts properly (but possibly ineffectively) on γ, we see that the image of the canonical homomorphism $G \to I_O(\gamma)$ is either e, $\mathbb{Z}$ or $\mathbb{R}$. So either G and hence $I(M)$ is compact or else it is easily seen that $G \approx G_p \times B$ with G_p compact and $B \approx \mathbb{Z}$ or $\mathbb{R}$.

q.e.d.

(3.6) An interesting case is when $\Lambda = \{\text{a point}\}$. In this case there is a significant contrast between $\dim M = 2$, and $\dim M \geq 3$. An easy curvature calculation shows that if N is a complete Riemannian manifold with non-positive curvature then $M = \mathbb{R} \times N$ with the metric $dt^2 + e^{2t} ds_N^2$ has curvature ≤ -1, and moreover $I(M) \approx I(N)$ but $\Lambda = \{\text{a point}\}$. This shows that $\Lambda = \{\text{a point}\}$ implies no predictable general restrictions on $I(M)$ - especially if $\dim M \geq 4$ and indicates that the action of $I(\tilde{M})$ on ∂ would not be in general smoothable. However in case M is locally homogeneous then Heintze's results cf. [5] imply that I(M) is virtually solvable.

(3.7) Now we restrict to the case $\dim M = 3$.

Proposition. Let M^3 be a non-elementary complete Riemannian manifold with constant curvature $= -1$, and finitely generated π_1. Assume that $\Lambda \neq \partial$. Then $I(M)$ is finite.

Proof. In this case ∂ may be identified with the Riemann sphere S^2 and (modulo passing to an orientable double cover if necessary) the deck-transformation group $\Delta \approx \pi_1(M)$ acts on S^2 by Möbius transformations. The action of Δ on $S^2-\Lambda$ is free and properly discontinuous. Since Δ is finitely generated, by Ahlfors' finiteness theorem cf. [1],[8], $\Delta\backslash(S^2-\Lambda)$ has finitely many components and each component is a Riemann surface whose genus g and number of ends t satisfy $3 \leq 2g+t < \infty$. Now $\Delta\backslash(S^2-\Lambda)$ may be considered as an "ideal boundary" of M. The action of $I(M)$ on M naturally extends to $\Delta\backslash(S^2-\Lambda)$ and the latter action is by conformal transformations. Using the above remarks and (1.2) we see that $I(M)$ has a subgroup H of finite index which acts trivially on $\Delta\backslash(S^2-\Lambda)$. Now $H \approx \Delta\backslash\mathcal{H}$ where $\mathcal{H}$ is a certain subgroup of the normalizer of Δ in $I(\tilde{M})$. Since H acts trivially on $\Delta\backslash(S^2-\Lambda)$ we see that each $h \in \mathcal{H}$ has the form $h = a \cdot h_1$ where $a \in \Delta$ and h_1 acts trivially on $\partial-\Lambda$. But then h_1 acts trivially on the convex hull of $\partial-\Lambda$ in $\tilde{M}$. Since this convex hull has a non-empty (3-dimensional) interior we see that $h_1 = e$ i.e. $\mathcal{H} = \Delta$ or $H = \{e\}$. Thus $I(M)$ is finite.

q.e.d.

(3.8) Here is a variant of (3.7) for M^3 with variable curvature. Note that a 3-dimensional manifold with finitely generated π_1 has a well-defined Euler characteristic, cf. [8], [9].

<u>Proposition</u>. Let M^3 be a complete Riemannian manifold with negative curvature bounded away from zero and with finitely generated π_1. Assume that the Euler characteristic $\chi(M^3) < 0$, and $\Lambda \neq \partial$. Then $I(M)$ is finite.

<u>Proof</u>. We apply the topological analogue of the Ahlfors' finiteness theorem developed in [8] to $\tilde{M}^3 \cup \partial$. For this application note that the action of the deck-transformation group $\Delta \approx \pi_1(M)$ is free and properly discontinuous on $\tilde{M}^3 \cup (\partial - \Lambda)$, and this is the maximal subset of $\tilde{M}^3 \cup \partial$ on which the action is properly discontinuous. Now the results in [8] imply that $_{\Delta}\backslash^{(\partial-\Lambda)}$ have only finitely many components with negative Euler characteristic. The condition $\chi(M^3) < 0$ ensures that there is at least one component of $_{\Delta}\backslash^{(\partial-\Delta)}$ with negative Euler characteristic. Moreover again $_{\Delta}\backslash^{(\partial-\Lambda)}$ may be considered as an "ideal boundary" of M, and the action of $I(M)$ extends to $_{\Delta}\backslash^{(\partial-\Lambda)}$ and is <u>proper</u> on $M \cup {_{\Delta}\backslash^{(\partial-\Lambda)}}$. Now the argument proceeds exactly as in (3.7) once we notice that (1.2) is really valid for any group acting properly.

q.e.d.

(3.9) <u>Remark</u>. It may happen that M^3 <u>is a complete Riemannian manifold with constant negative curvature and with finitely generated</u> π_1 <u>and still</u> $I(M)$ <u>is discrete but infinite</u>. In this case, of course, one must have $\Lambda = \partial$. Indeed let N^3 be a fiber bundle over S^1 with fiber an orientable (possibly non-compact) surface with negative Euler characteristic such that the monodromy is pseudo-Anosov. Then N^3 admits a complete Riemannian metric of negative constant curvature and finite volume, cf. [11] - in fact, this is a crucial case in Thurston's hyperbolization program for 3-manifolds. Let M^3 be the regular covering of N^3 corresponding to π_1 (fiber). Then $\pi_1(M) \approx \pi_1$ (fiber) is finitely generated, $\Lambda = \partial$ so $I(M)$ is discrete by (3.1). But obviously $I(M)$ contains the covering transformation group $\approx \mathbb{Z}$ of the covering $M^3 \to N^3$.

In some sense these examples are typical. For indeed suppose M^3 is a complete Riemannian manifold with negative curvature bounded away from zero, with finitely generated π_1, and suppose that $I(M)$ contains a discrete subgroup $A \approx \mathbb{Z}$. Then A acts freely and properly discontinuously on M. Let $N = {}_A\backslash^{M}$. Since $\pi_1(N)$ is finitely generated by [9] N has a compact submanifold N_1 possibly with boundary such that the inclusion $N_1 \hookrightarrow N$ is a homotopy equivalence. Then clearly one has a surjective homomorphism $\pi_1(N_1) \to \mathbb{Z}$ with <u>finitely generated</u> kernel. In this situation [10] shows that N_1 in fact fibers over S^1.

References

1) Ahlfors, L., Finitely generated Kleinian groups, Amer. J. Math. 86 (1964), 413 - 429

2) Chen S.S. and Eberlein P., Isometry groups of simply connected manifolds of nonpositive curvature, Illinois Jour. of Math. 24 (1980), 73 - 103.

3) Greenberg, L., Conformal Transformations of Riemann Surfaces, Amer. Jour. of Math. 82 (1960), 749 - 760

4) ———————., Finiteness Theorems for Fuchsian and Kleinian Groups, in Discontinuous Groups and Automorphic Functions, ed. by W.J. Harvey, Academic Press (1977)

5) Heintze, E., On homogeneous manifolds of negative curvature, Math. Ann. 211 (1974), 23 - 34

6) Huber, H., Über die Isometriegruppe einer kompakten Mannigfaltigkeit mit negativer Krümmung, Helv. Phys. Acta 45 (1972), 277 - 288.

7) Im Hof H.C., Über die Isometriegruppe bei kompakten Mannigfaltigkeiten negativer Krümmung, Comment. Math. Helv. 48 (1973), 14 - 30

8) Kulkarni, R.S. and Shalen P.B., On Ahlfors' finiteness theorem (to appear)

9) Scott G.P., Finitely generated 3-manifold groups are finitely presented, J. London Math. Soc. (2) 6 (1973), 437 - 440

10) Stallings, J., On fibering certain 3-manifolds, Prentice-Hall, Engelwood Cliffs, N.J. (1962), 95 - 100

11) Sullivan, D., Travaux de Thurston sur les groupes quasifuchsiens et les variétés hyperbolique de dimension 3 fibrées sur S^1. Sem. Bourbaki, No. 554 1980, Lecture Notes in Math. No. 842, Springer Verlag 1981