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Properties of a Jacobian mate

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Abstract

A polynomial $f \in \mathbb{C}[x, y]$ is a Jacobian mate if the Jacobian $\mathrm{J}(f, g) = 1$ for some $g \in \mathbb{C}[x, y]$. It is not known that then $\mathbb{C}[f, g] = \mathbb{C}[x, y]$ and a conjecture that this is the case is the Jacobian conjecture (JC). In this note we will assume that a counterexample to JC exists and obtain additional restrictions on f.

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Key words: Jacobian mate, Jacobian conjecture.

Introduction.

Assume that $f \in \mathbb{C}[x, y]$ (where $\mathbb{C}$ is the field of complex numbers) satisfies $\mathrm{J}(f, g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = 1$ for some $g \in \mathbb{C}[x, y]$. The JC (Jacobian conjecture) implies that then $\mathbb{C}[f, g] = \mathbb{C}[x, y]$ (see [K]).

Recall that if $p \in \mathbb{C}[x, y]$ is a polynomial in 2 variables and each monomial of p is represented by a lattice point on the plane with the coordinate vector equal to the degree vector of this monomial then the convex hull $\mathcal{N}(p)$ of the points so obtained is called the Newton polygon of p.

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†This author implemented the algorithm described in the paper.

It is known for many years that for a potential counterexample to JC there exists an automorphism ξ of $\mathbb{C}[x,y]$ such that the Newton polygon $\mathcal{N}(\xi(f))$ of $\xi(f)$ contains a vertex $v=(m,n)$ where $n>m>0$ and is included in a trapezoid with the vertex v, edges parallel to the y axis and to the bisectrix of the first quadrant adjacent to v, and two edges belonging to the coordinate axes (see [A1], [A2], [AO], [GGV], [H], [J], [L], [M], [MW], [Na1], [Na2], [NN1], [NN2], [Ok]). This was improved with a completely new approach by Pierrette Cassou-Noguès who showed that $\mathcal{N}(f)$ does not have an edge parallel to the bisectrix (see [CN] and [ML1]). From now on we will assume that f is "shaped" like this, i.e. that $\mathcal{N}(f)$ has a vertex (m,n) and is included in a trapezoid described above. We will call v the *leading* vertex and the right edge containing this vertex the *leading* edge.

Our goal is to obtain additional restrictions on f under assumption that g exists. In order to do this we apply to f the Newton resolution process. Here is a brief reminder of this process.

Newton introduced the polygon which we call the Newton polygon in order to find a solution y of $p(x,y)=0$ in terms of x for a polynomial $p(x,y)=\sum_{(i,j)\in\mathcal{N}(p)} p_{ij}x^iy^j$ (see [N]). Here is the process of obtaining such a solution. Consider an edge e of $\mathcal{N}(p)$ which is not parallel to the x axis. Denote by $p(e)=\sum_{(i,j)\in e} p_{ij}x^iy^j$. The form $p(e)$ allows to determine the first summand of a solution as follows. Consider the equation $p(e)=0$. Since $p(e)$ is a homogeneous form relative to a *weight* given by $w(x)=\alpha(\neq 0),\ w(y)=\beta,\ w(x^iy^j)=i\alpha+j\beta$, solutions of this equation are $y=c_ix^{\frac{\beta}{\alpha}}$ and $c_i\in\mathbb{C}$. Choose any solution $c_ix^{\frac{\beta}{\alpha}}$ and replace $p(x,y)$ by $p_1(x,y)=p(x,c_ix^{\frac{\beta}{\alpha}}+y)$. Though p_1 is not necessarily a polynomial in x we can define the Newton polygon of p_1 in the same way as it was done for the polynomials; the only difference is that p_1 may contain monomials $x^\mu y^\nu$ where $\mu\in\mathbb{Q}$ rather than in $\mathbb{Z}$. The polygon $\mathcal{N}(p_1)$ contains the *degree* vertex v of e, i.e. the vertex with ordinate equal to $\deg_y(p(e))$ and an edge e' which is a modification of e (e' may collapse to v). Take the *order* vertex v_1 of e', i.e. the vertex with ordinate equal to the order of $p_1(e')$ as a polynomial in y (if $e'=v$ take $v_1=v$). Use the edge e_1 for which v_1 is the degree vertex to determine the next summand and so on. After possibly a countable number of steps we obtain a vertex v_μ and the edge e_μ for which v_μ is not the degree vertex, i.e. either e_μ is horizontal or the degree vertex of e_μ has a larger ordinate than the ordinate of v_μ. It is possible only if $\mathcal{N}(p_\mu)$ does not have any vertices on the x axis. Therefore $p_\mu(x,0)=0$ and a solution is obtained.

When characteristic is zero the process of constructing a solution is more straightforward then it may seem from this description. The denominators of fractional powers of x (if denominators and numerators of these rational numbers are assumed to be relatively prime) do not exceed $\deg_y(p)$. Indeed, for any initial edge there are at most $\deg_y(p)$ solutions while a summand $cx^{\frac{M}{N}}$ can be replaced by $c\varepsilon^M x^{\frac{M}{N}}$ where $\varepsilon^N = 1$ which gives at least N different solutions.

There are two sets of solutions: by increasing powers of x and by decreasing powers of x. We will be using solutions by decreasing powers. After such a solution is obtained we can talk about the corresponding Newton polygon. It will have a finite chain of non-horizontal edges and possibly infinite horizontal edges going to $-\infty$.

If we apply this process to a Jacobian mate f with the leading vertex $x^m y^n$ we will obtain n solutions. The Newton polygon which we will be using has a chain of vertices $v_i = (\mu_i, \nu_i),\ 0 \le i \le s+1$ starting with the leading vertex $v_0 = (\mu_0, \nu_0) = (m, n)$ and ending with the *principal* vertex $v_{s+1} = (\mu_{s+1}, \nu_{s+1}) = (\mu_{s+1}, 1)$.

All vertices v_i are above the bisectrix of the first quadrant, i.e. $\mu_i < \nu_i$, also $\mu_{i+1} < \mu_i,\ \nu_{i+1} < \nu_i,\ 0 \le i \le s$ and the lines connecting v_i and v_{i+1} intersect the x axis in positive points ρ_i. The edge connecting vertices v_i and v_{i+1} is denoted by e_i.

The edge e_s connecting the vertex v_s and the principal vertex v_{s+1} is the *principal* edge. The slope of the principal edge is larger than 1.

If the leading edge is vertical then $f(e_0) = \phi_0(x)y^n$ and we can assume that the polynomial ϕ_0 has at least two different roots. (If $\phi_0(x) = c_0(x-c_1)^m$ we can make a substitution $x \to x + c_1,\ y \to y$ and get rid of this edge.)

Existence of a solution satisfying these conditions was shown in [ML2]. In the process of obtaining a solution we will get intermediate expressions which are polynomials in y and Laurent polynomials in fractional powers of x.

Denote by f_i expression for which the Newton polygon $\mathcal{N}(f_i)$ contains an edge e'_i from which the edge e_i is obtained after the next step of resolution and by f_{s+1} the first expression for which $\mathcal{N}(f_{s+1})$ contains all vertices v_i.

We will be using only these expressions. To obtain a solution we may need infinitely many additional steps.

Integrality conditions.

The Newton polygon $\mathcal{N}(f)$ contains an edge e'_0 from which the edge e_0 is obtained after the first step of resolution (e_0 cannot collapse to a vertex). Similarly, $\mathcal{N}(f_i)$ contains an edge e'_i from which the edge e_i is obtained after the $r_i + 1$ steps of resolution. Generally specking r_i can be not equal to i. For example, it is possible that $f_i = f_{i+1}$. This happens if $e_i = e'_i$. It is also possible that $r_i > i$ because in the resolution process some edges can collapse to a vertex.

The edges e'_i and e_i define a weight w_i for which all points on these edges have the same weight. If we put $w_i(x) = 1$ then this weight is uniquely defined and $w_i(f_i) = \rho_i > 0$. This is one of the properties of a solution we have chosen.

Similarly, we can define the weight w_e for a non-horizontal edge e. If an expression p can be presented as the sum of w_e homogeneous forms, denote by $p(e)$ the leading form of this expression, i.e. the summand with the maximal weight w_e.

An expression f_i described in the introduction is obtained from f by substituting $y_i = y + \sum_{j=1}^{r_i} c_j x^{\frac{\epsilon_j}{\delta_j}}$, $c_j \in \mathbb{C}$, $\epsilon_j, \delta_j \in \mathbb{Z}$, $\delta_j > 0$ into f : $f_i = f(x, y_i)$, so $f_0 = f \in \mathbb{C}[x, y]$ and $f_i \in C_i = \mathbb{C}[x^{\pm\frac{1}{\Delta_i}}, y]$, $\Delta_i > 0$ and minimal possible.

Consider the form $f_i(e'_i)$ and an algebra A_i with elements $a_i = \sum_j \psi_{i,j}$ where $\psi_{i,j} \in C_i[f_i(e'_i)^{-1}]$ are forms homogeneous relative to w_i and $w_i(\psi_{i,j+1}) < w_i(\psi_{i,j})$.

Lemma on radical. If $r \in \mathbb{Q}$ and $f_i(e'_i)^r \in C_i$ then $f_i^r \in A_i$.

Proof. By the Newton binomial theorem $f_i^r = f_i(e_i')^r \sum_{j=0}^{\infty} \binom{r}{j} (\frac{f_i}{f_i(e_i')} - 1)^j$. $\square$

Since we assumed that f is a Jacobian mate there exists a $g \in \mathbb{C}[x, y]$ for which $\mathrm{J}(f, g) = 1$. (It is shown in [ML2] that g can be recovered if f is known.) Consider $g_i = g(x, y_i)$. Clearly $g_i \in C_i$ and $\mathrm{J}(f_i, g_i) = 1$. Therefore $\mathrm{J}(f_i(e_i'), g_i(e_i'))$ is either 0 or 1.

If $\mathrm{J}(f_i(e_i'), g_i(e_i')) = 0$ then $g_i(e_i') = a_{i,0} f_i(e_i')^{\lambda_{i,0}}$ where $a_{i,0} \in \mathbb{C}$, $\lambda_{i,0} \in \mathbb{Q}$, and $g_{i,1} = g_i - a_{i,0} f_i^{\lambda_{i,0}} \in A_i$. If $\mathrm{J}(f_i(e_i'), g_{i,1}(e_i')) = 0$ we can find $a_{i,1} f_i(e_i')^{\lambda_{i,1}}$ such that $g_{i,1}(e_i') = a_{i,1} f_i(e_i')^{\lambda_{i,1}}$ and define $g_{i,2} = g_{i,1} - a_{i,1} f_i^{\lambda_{i,1}}$, etc.. After several steps like that we will obtain g_{i,t_i} for which $\mathrm{J}(f_i(e_i'), g_{i,t_i}(e_i')) = 1$.

$$g_i = \sum_{j=0}^{t_i - 1} a_{i,j} f_i^{\lambda_{i,j}} + g_{i,t_i};\ a_{i,j} \in \mathbb{C},\ \lambda_{i,j} \in \mathbb{Q}$$

Because of the properties of the chosen solution $w_i(xy) > 0$. It is known that then $h_i = f_i(e_i') g_{i,t_i}(e_i') \in B_i = \mathbb{C}[x^{\frac{1}{\Delta_i}}, y]$ (see [D] or [ML1]).

In the introduction we defined the degree and the order vertices of a non-horizontal edge e. Denote by $\mathrm{dv}(p(e))$ the degree vertex and by $\mathrm{ov}(p(e))$ the order vertex of the edge e_p supporting $p(e)$. (The edges e and e_p are parallel.)

The degree vertex of e_i' is v_i and $\mathrm{dv}((h_i(e_i')))$ is proportional to v_i because $\mathrm{J}(f_i(e_i'), h_i) = f_i(e_i')$ and e_i doesn't collapse to a vertex by the definition of f_i. Thus $\mathrm{dv}((h_i(e_i'))) = c_i(\mu_i, \nu_i)$, $c_i \in \mathbb{Q}$ and ordinate $c_i \nu_i$ of $c_i v_i$ is an integer while abscissa $c_i \mu_i$ of $c_i v_i$ belongs to $\frac{1}{\Delta_i}\mathbb{Z}$. Denote $c_i \nu_i$ by k_i.
Our *first* integrality condition is that $k_i \in \mathbb{Z}$.

If we know $v_i = (\mu_i, \nu_i)$, k_i, and ν_{i+1}' we can find μ_{i+1}' as follows. We can find ρ_i because vectors $\langle \mu_i, \nu_i \rangle - \langle \rho_i, 0 \rangle$ and $c_i \langle \mu_i, \nu_i \rangle - \langle 1, 1 \rangle$ are parallel. Since $c_i w_i(\mu_i, \nu_i) = w_i(1, 1)$ and $w_i(x) = 1$ we see that $w_i(y) = -\frac{c_i \mu_i - 1}{c_i \nu_i - 1}$ and $\rho_i = \mu_i - \frac{c_i \mu_i - 1}{c_i \nu_i - 1} \nu_i = \frac{\nu_i - \mu_i}{c_i \nu_i - 1}$.

$$\rho_i = \frac{\nu_i - \mu_i}{k_i - 1}$$

Now, $f_i(e_i') = x^{\rho_i} p_i(x^{\alpha_i} y)$ where $\alpha_i = -w_i(y) = \frac{c_i \mu_i - 1}{c_i \nu_i - 1} = \frac{k_i \mu_i - \nu_i}{\nu_i (k_i - 1)}$.

$$\alpha_i = \frac{k_i \mu_i - \nu_i}{\nu_i (k_i - 1)}$$

Hence the vertex $v'_{i+1} = (\rho_i, 0) + (\alpha_i, 1)\nu'_{i+1}$ and
$\nu'_{i+1} - \mu'_{i+1} = (1-\alpha_i)\nu'_{i+1} - \rho_i = \frac{(\nu_i-\mu_i)k_i}{\nu_i(k_i-1)}\nu'_{i+1} - \frac{\nu_i-\mu_i}{k_i-1} = \frac{(\nu_i-\mu_i)(k_i\nu'_{i+1}-\nu_i)}{\nu_i(k_i-1)}$.
Thus

$$\mu'_{i+1} = \nu'_{i+1} - \frac{(\nu_i-\mu_i)(k_i\nu'_{i+1}-\nu_i)}{\nu_i(k_i-1)}$$

Similarly for v_{i+1}

$$\mu_{i+1} = \nu_{i+1} - \frac{(\nu_i-\mu_i)(k_i\nu_{i+1}-\nu_i)}{\nu_i(k_i-1)}$$

The *second* integrality condition is that $\mu'_{i+1} \in \frac{1}{\Delta_i}\mathbb{Z}$ and $\mu_{i+1} \in \frac{1}{\Delta_{i+1}}\mathbb{Z}$.
The *third* integrality condition is that $c_i\mu_i \in \frac{1}{\Delta_i}\mathbb{Z}$ because $h_i \in B_i$, i.e. $\frac{k_i}{\nu_i}\mu_i \in \frac{1}{\Delta_i}\mathbb{Z}$ since $c_i = \frac{k_i}{\nu_i}$.

We see that if $\mu_0,\ \nu_0 \in \mathbb{Z}$ are given and we know integers $\nu_j,\ k_j\ \ 1 \le j \le i \in \mathbb{Z}$ and $\nu_{i+1} \in \mathbb{Z}$ then we can recover vertices $v_j,\ 1 \le j \le i+1$.

Additional restriction on data are inequalities:
$k_i > 1$ since $\mathrm{dv}(h_i) \neq (1,1)$ because e'_i doesn't collapse to a vertex (if $\mathrm{dv}(h_i) = (1,1)$ then $h_i = c_0x(y+c_1x^\tau)$ and $f_i(e'_i) = c_2x^\mu(y+c_1x^\tau)^\nu$); $k_i\mu_i - \nu_i > 0$ because $\alpha_i > 0$; $k_i\nu_{i+1} - \nu_i > 0$ because $\nu_{i+1} > \mu_{i+1}$.

Also k_i is not divisible by ν_i:

Lemma on divisibility. If $\mathrm{J}(f_i(e'_i), h_i) = f_i(e'_i)$ then $w_i(f_i)$ doesn't divide $w_i(h_i)$.
Proof. Since $\mathrm{dv}(h_i) \neq (1,1)$ the degree vertices of $\mathrm{dv}(h_i)$ and $f_i(e'_i)$ are proportional. If $\frac{w_i(h_i)}{w_i(f_i)} = k \in \mathbb{Z}$ then we can find $c \in \mathbb{C}$ for which $\mathrm{dv}(h_i - cf_i(e'_i)^{\frac{k_i}{\nu_i}}) \neq \mathrm{dv}(h_i)$. Since $\mathrm{J}(f_i(e'_i), h_i - cf_i(e'_i)^k) = f_i(e'_i)$ these implies that $h_i - cf_i(e'_i)^k = c_0x(y+c_1x^\tau)$. But then e_i is a vertex contrary to our assumption. $\square$

Remark. If e_i is the principal edge then $h_i = f_i(e'_i)g_i(e'_i)$ and $w_i(g_i)$ doesn't divide $w_i(h_i) = w_i(xy)$. $\square$

Polynomiality conditions

In the previous section we checked that $g_i = \sum_{j=0}^{t_i-1} a_{i,j} f_i^{\lambda_{i,j}} + g_{i,t_i}$ where $a_{i,j} \in \mathbb{C}$, $\lambda_{i,j} \in \mathbb{Q}$ and $\mathrm{J}(f_i(e'_i), g_{i,t_i}(e'_i)) = 1$. We can call this the expansion relative to the edge e'_i.

The expansion relative to the edge e'_s is very short since $\mathrm{J}(f_s(e'_s), g_s(e'_s)) = 1$ for the principal edge e'_s. It is shown in [ML2] that $\lambda_0 = \frac{w_s(g_s)}{w_s(f_s)}$ and therefore

$$\lambda_0 = \frac{w_s(xy)}{w_s(f_s)} - 1$$

because $w_s(f_s g_s) = w_s(xy)$.

Lemma on sub-expansion. The expansion relative to the edge e'_i is a sub-expansion of the expansion relative to the edge e'_{i-1}, i.e. $t_{i-1} > t_i$, $\lambda_{i-1,j} = \lambda_{i,j}$, $a_{i-1,j} = a_{i,j}$ for $j < t_i$.

Proof. Consider algebra $D = \mathbb{C}[f_s, f_s g_s]$. It is clear that the expansion of $f_s g_s$ relative to the edge e_i of f_s is

$$f_s g_s = \sum_{j=0}^{t_i-1} a_{i,j} f_s^{\lambda_{i,j}+1} + f_s g_{s,t_i}$$

and that $f_s(e_i) g_{s,t_i}(e_i) = h_i$. Since $w_i(h_i) = w_i(xy) > 0$ because the slope of e_i is larger than the slope of e_s if $i < s$ all $\lambda_{i,j} + 1 > 0$.

As we know $\mathrm{dv}(h_i)$ is proportional to $\mathrm{dv}(f_s(e_i))$; also $\mathrm{ov}(h_i) = (1,1)$ because $\mathrm{ov}(f_s(e_i))$ has a positive ordinate.

We can find ψ which is a polynomial in y such that $f_s(e_i) = \psi^d$ and d is maximal possible. If $p \in D$ then

$$p(e_i) = \sum_{k w_i(\psi) + l w_i(h_i) = w_i(p)} c_{k,l} \psi^k h_i^l$$

Therefore $\mathrm{ov}(p(e_i)) = l_m(1,1) + k_m(\mathrm{ov}(\psi))$ where l_m is the largest value of l since the slope of $< 1, 1 >$ is smaller that the slope of $< \mu_{i+1}, \nu_{i+1} >$. Hence if $\mathrm{ov}(p(e_i))$ is proportional to v_{i+1} then $p(e_i) = c\phi^{d_p}$ and $\mathrm{dv}(p(e_i))$ is proportional to v_i with the same proportionality coefficient $\frac{d_p}{d}$.

Now we can repeat this consideration for the edge e_{i-1}, etc., to confirm that if $\mathrm{ov}(p(e_i))$ is proportional to v_{i+1} then $\mathcal{N}(p)$ has a chain of edges which is homothetic to the chain of edges $e_0, \dots, e_i$.

Since $\mathrm{ov}(f_s(e_{s-1})g_s(e_{s-1}))$ is proportional to v_s with the coefficient $\lambda_0 + 1 = \frac{w_s(xy)}{w_s(f_s)}$ we see that $\lambda_{i,0} = \lambda_0$ and $a_{i,0} = a_0$ for $i < s$.

If $\lambda_0 + 1 = \frac{\theta_0}{\kappa_0}$ and $p_1 = (f_s g_s)^{\kappa_0} - a_0^{\kappa_0} f_s^{\theta_0}$ then

$$p_1 = \kappa_0 (a_0 f_s^{\lambda_0+1})^{\kappa_0 - 1} (\sum_{j=1}^{t_i - 1} a_{i,j} f_s^{\lambda_{i,j}+1} + f_s g_{s,t_i}) + \dots$$

Consider the largest i_1 for which $a_{i_1,1} \neq 0$. The order vertex $\mathrm{ov}(p_1(e_{i_1}))$ is proportional to v_{i_1+1}. Hence $\lambda_{i,1} = \lambda_1$ and $a_{i,1} = a_1$ for $i \leq i_1$.

We can construct p_2, etc. in a similar fashion to check the claim of the Lemma. $\square$

Corollary Since $\mathrm{dv}(h_j)$ is proportional to $\mathrm{dv}(f_s(e_j))$ with the coefficient $\frac{k_j}{\nu_j}$ we have the *polynomiality conditions*:

$$f_i(e_i)^{\frac{k_j}{\nu_j}} \in B_{i+1} \text{ if } j > i.$$

Complexity of a counterexample

Recall that the leading vertex of f is (m, n). Hence the total degree of f is $D = m + n$. It seems that the primary decomposition of D may be a measure of complexity of a counterexample.

If f is a Jacobian mate then $f(e_0') = \phi_0^{d_0}$ where $d_0 > 1$. Hence D cannot be a prime number. Also if $d_0 = \gcd(m, n)$ then f is not a Jacobian mate because in this case $\mathrm{J}(\phi_0, h) = \phi_0$ is impossible by Lemma on divisibility. Indeed, since ϕ_0 is not a monomial $\mathrm{J}(\phi_0, h) = \phi_0$ is possible only if $\mathrm{dv}(h) = \kappa \mathrm{dv}(\phi_0),\ \kappa \in \mathbb{Q}$, but if $\mathrm{dv}(\phi_0) = (m_1, n_1)$ and $\gcd(m_1, n_1) = 1$ then $\kappa \in \mathbb{Z}$. On the other hand if $\mathrm{J}(\phi_0^{d_0}, h) = \phi_0^{d_0}$ then $\mathrm{J}(\phi_0, d_0 h) = \phi_0$.

Therefore $(m_1, n_1) = \delta(a_0, b_0),\ \delta > 1,\ \gcd(a_0, b_0) = 1,\ b_0 > a_0 \geq 1$,

$$D = d_0 \delta (a_0 + b_0)$$

and D is the product of at least three primes.

If $f(e_0') = \phi_0^{d_0}$ and ϕ_0 is a polynomial which is not a power of a polynomial then the expansion relative to the zero edge is

$$g = \sum_{j=0}^{t_0-1} c_j f^{\lambda_j} + g_{t_0};\ c_j \in \mathbb{C},\ \lambda_j \in \frac{1}{d_0}\mathbb{Z},\ \lambda_i > \lambda_{i+1}$$

and expansions relative to the other edges are sub-expansions of this expansion.

It was shown in [ML2] that λ_0 is neither an integer nor the reciprocal of an integer.

If $\lambda_0 = \frac{\epsilon_0}{d_0}$ and $\gcd(\epsilon_0, d_0) = 1$ then e_1' is the principal edge. Indeed, $\nu_1 = d_0\nu_{1,1},\ \nu_{1,1} \in \mathbb{Z}$ because $f(e_0) = \phi_0^{d_0}$ where ϕ_0 is a polynomial in y. Hence $\frac{k_1}{\nu_1} = \frac{l_1}{d_0}$ because $\frac{k_1}{\nu_1} = \lambda_j$ for some j in the expansion relative to e_0. Therefore $k_1 = \nu_{1,1}l_1$. If e_1' is not the principal edge then $f_1(e_1')^{\lambda_0}$ is a polynomial in y, i.e. $\phi_1 = f_1(e_1')^{\frac{1}{d_0}}$ is a polynomial in y and $\mathrm{dv}(h_1) = l_1\mathrm{dv}(\phi_1)$ which, as we know, is impossible for a Jacobian mate. Hence if d_0 is a prime number then e_1' is the principal edge, which is a rather strong restriction.

If $f_1(e_1') = \phi_1^{d_1'}$ where d_1' is maximal possible if ϕ_1 is a polynomial in y and e_1' is not the principal edge then $d_1 = \gcd(d_0, d_1')$ is a proper factor of d_0: if it is d_0 we have a contradiction as above and if it is 1 then $\lambda_0 = \frac{\epsilon_0}{d_0} = \frac{\epsilon_1}{d_1'}$ which is possible only if $\lambda_0 \in \mathbb{Z}$. If we expand g_1 relative to the edge e_1' then all corresponding $\lambda_j \in \frac{1}{d_1}\mathbb{Z}$.

We defined d_0 and d_1. Similarly, we can define d_i: if d_{i-1} is defined and $f_i(e_i') = \phi_i^{d_i'}$ where d_i' is maximal possible if ϕ_i is a polynomial in y and e_i' is not the principal edge then $d_i = \gcd(d_{i-1}, d_i')$ is a proper factor of d_{i-1}. If we expand g_i relative to the edge e_i' then all corresponding $\lambda_j \in \frac{1}{d_i}\mathbb{Z}$. Because of that if e_i' is not the principal edge and d_i is a prime number then e_{i+1}' is the principal edge.

Algorithm

We can use the technique developed in the previous sections in order to construct potential counterexamples which this technique doesn't reject.

The total degree $m+n$ of f was denoted by D. We can assume that $f(e_0') = \phi_0^{d_0}$ where $\phi_0 \in \mathbb{C}[x,y]$ and $d_0 > 1$ is maximal possible, $\mathrm{dv}(\phi_0) = \delta(a_0, b_0)$, $\gcd(a_0, b_0) = 1$, $b_0 > a_0$, and $\delta > 1$. (See Complexity of a counterexample.) Therefore

$$D = d_0\delta(a_0 + b_0);\ \mu_0 = d_0\delta a_0,\ \nu_0 = d_0\delta b_0$$

and D is the product of at least three prime numbers.

Possible vertices v_1'.

Assume that $\deg_y(h_0) = k_0$. Since $\rho_0 = \frac{\nu_0-\mu_0}{k_0-1}$ we have a bound on values of k_0 because $\rho_0 \geq 2$. If $\rho_0 < 2$ then $f(x,0) = c_0x + c_1$. If $c_0 \neq 0$ we can find a polynomial $p(t) \in \mathbb{C}[t]$ such that $g(x,0) - p(f(x,0)) = 0$. Then $\mathrm{J}(f,\tilde{g}) = 1$ where $\tilde{g} = g - p(f)$. If e is the non-horizontal edge of $\mathcal{N}(f)$ with the vertex $(1,0)$ then $\mathrm{J}(f(e),\tilde{g}(e)) \neq 0$ since $\mathrm{ov}(\tilde{g}(e)) \neq (k,0)$. Hence $\mathrm{J}(f(e),\tilde{g}(e)) = 1$ an $\mathrm{ov}(\tilde{g}(e)) = (0,1)$. But then the degree vertices of $f(e)$ and $\tilde{g}(e)$ cannot be proportional and $\mathrm{J}(f(e),\tilde{g}(e)) \neq 1$. If $c_0 = 0$ we can assume that $f(x,0) = 0$. Since $\mathrm{J}(f,g) = 1$ this implies that $g_x(x,0) = c \in \mathbb{C}^*$ because $(f_xg_y - f_yg_x)|_{y=0} = -f_y|_{y=0}g_x(x,0) = 1$ and we will get the same contradiction as above.

Therefore $\frac{\nu_0-\mu_0}{k_0-1} \geq 2$ and $k_0 \leq \frac{\nu_0-\mu_0+2}{2}$. Additionally $k_0 = l_0b_0$ because $\mathrm{dv}(\phi_0)$ and $\mathrm{dv}(h_0)$ are proportional. Thus

$$l_0 \leq \frac{\nu_0 - \mu_0 + 2}{2b_0}$$

and by Lemma on divisibility $\gcd(\delta, l_0) \neq \delta$.

The choice of l_0 determines the slope of the edge e_0' because e_0' is parallel to $\mathcal{N}(h_0)$, thus to the edge with vertices $(1,1)$ and $l_0(a_0,b_0)$. So the vector with components $< l_0a_0 - 1, l_0b_0 - 1 >$ is going in the right direction. The shortest vector with integral components going in this direction is $< \beta_0, \gamma_0 >$ where

$$\beta_0 = \frac{l_0a_0 - 1}{\epsilon_0},\ \gamma_0 = \frac{l_0b_0 - 1}{\epsilon_0};\ \epsilon_0 = \gcd(l_0a_0 - 1, l_0b_0 - 1).$$

Possible vertices v_1' are given by

$$v_1' = (\mu_1', \nu_1') = v_0 - t_1d_0(\beta_0, \gamma_0) = (\mu_0 - t_1d_0\beta_0, \nu_1' = \nu_0 - t_1d_0\gamma_0)$$

where $1 \leq t_1 \leq \frac{\nu_0}{d_0\gamma_0}$ because $\nu_1' \geq 0$ and $t_1 \neq \frac{\nu_0-\mu_0}{d_0(\gamma_0-\beta_0)} = \frac{\epsilon_0(\nu_0-\mu_0)}{d_0 l_0(b_0-a_0)} = \frac{\epsilon_0 d_0 \delta}{d_0 l_0} = \frac{\epsilon_0\delta}{l_0}$ because $\mu_1' \neq \nu_1'$. The coefficient d_0 appears because coordinates of v_1' must be divisible by d_0.

The inequality $t_1 \leq \frac{\nu_0}{d_0\gamma_0}$ allows to get another bound for l_0 as follows: $\frac{\nu_0}{d_0\gamma_0} = \frac{\epsilon_0\nu_0}{d_0(l_0b_0-1)} \leq \frac{(b_0-a_0)\nu_0}{d_0(l_0b_0-1)}$ because $\epsilon_0 = \gcd(l_0a_0-1, l_0b_0-1) = \gcd(l_0(b_0-a_0), l_0b_0-1) = \gcd(b_0-a_0, l_0b_0-1) \leq b_0-a_0$.

Since t_1 can be defined only if $\frac{\nu_0}{d_0\gamma_0} \geq 1$ we have $(b_0-a_0)\nu_0 \geq d_0(l_0b_0-1)$ and $l_0 \leq \frac{(b_0-a_0)\nu_0+d_0}{d_0b_0}$. So

$$1 \leq l_0 \leq \frac{(b_0-a_0)\nu_0+d_0}{d_0b_0}$$

Possible vertices v_1.

We need a Newton modification $y \to y + cx^{\frac{-\beta_0}{\gamma_0}}$ if $\nu_1' < \mu_1'$. Possible vertices v_1 are given by

$$v_1 = v_1' + (d_0t_2 - \nu_1')(\frac{\beta_0}{\gamma_0}, 1) = (\mu_1' + (d_0t_2-\nu_1')\frac{\beta_0}{\gamma_0}, d_0t_2)$$

where $\frac{\mu_1'\gamma_0-\nu_1'\beta_0}{(\gamma_0-\beta_0)d_0} < t_2 \leq \frac{\nu_0-\nu_1'}{d_0\gamma_0}$. The lower limit for t_2 insures that $\mu_1 < \nu_1$, the upper limit is obtained as follows: $f(e_0') = \phi_0^{d_0}$, $\phi_0 = x^{\mu_{1,1}'}y^{\nu_{1,1}'}q_0(x^{\beta_0}y^{\gamma_0})$ where $\mu_{1,1}' = \frac{\mu_1'}{d_0}$, $\nu_{1,1}' = \frac{\nu_1'}{d_0}$ and $\deg(q_0) = \frac{\delta b_0-\nu_{1,1}'}{\gamma_0} = \frac{\nu_0-\nu_1'}{d_0\gamma_0}$. After such a modification we obtain $v_1 = (\mu_1, \nu_1)$.

The Newton polygon of $f(x, y + cx^{\frac{-\beta_0}{\gamma_0}})$ contains points with fractional abscissae and we can take $\Gamma_0 = \gamma_0$ as the common denominator of these fractions.

If $\nu_1' > \mu_1'$ then we can take $v_1 = v_1'$. In this case we will put $\Gamma_0 = 1$.

Is e_1' the principal edge?

If d_0 is a prime number then e_1' must be the principal edge. It also can be the principal edge if d_0 is not prime. To check whether this is possible the formula

$$\nu_2' - \mu_2' = \frac{(\nu_1-\mu_1)(k_1\nu_2'-\nu_1)}{\nu_1(k_1-1)} \qquad (1)$$

can be used. Here $k_1 = \deg_y(h_1)$ and since $f(e_0)^{\frac{k_1}{\nu_1}} \in B_1$ we have $\frac{k_1}{\nu_1} = \frac{l_1}{d_0}$. Thus $k_1 = l_1\nu_{1,1}$ where $\nu_{1,1} = \frac{\nu_1}{d_0}$.

Using this formula we can check whether it is possible that the vertex v_2' is either $(\mu_2', 0)$ or $(\mu_2', 1)$ where $0 < \mu_2' < 1$ and $\Gamma_0\mu_2' \in \mathbb{Z}$ and that e_2' intersects the line $y = 1$ in a point with abscissa between 0 and 1.

Since (1) can be rewritten as

$$\nu_2' - \mu_2' = \frac{(\nu_1 - \mu_1)(l_1\nu_{1,1}\nu_2' - d_0\nu_{1,1})}{\nu_1(l_1\nu_{1,1} - 1)} \qquad (2)$$

the condition that e_2' intersects the line $y = 1$ in a point with abscissa between 0 and 1 means that $l_1 > d_0$ (also $\gcd(l_1, d_0) < d_0$ by Lemma on divisibility).

If $\nu_2' = 0$ then $\Gamma_0\frac{\nu_1-\mu_1}{\nu_{1,1}l_1-1} = z \in \mathbb{Z}$ where $z < \Gamma_0\frac{\nu_1-\mu_1}{\nu_{1,1}d_0-1} = \Gamma_0\frac{\nu_1-\mu_1}{\nu_1-1}$ since $l_1 > d_0$. We can find l_1 for all possible values of z:

$$l_1 = \frac{\Gamma_0\nu_1 - \Gamma_0\mu_1 + z}{z\nu_{1,1}}$$

and we have a potential counterexample if $l_1 \in \mathbb{Z}$ and $\frac{l_1}{d_0} \notin \mathbb{Z}$, $\frac{d_0}{l_1-d_0} \notin \mathbb{Z}$ (otherwise λ_0 is either an integer or the reciprocal of an integer).

If $\nu_2' = 1$ then $\Gamma_0\frac{(\nu_1-\mu_1)(\nu_{1,1}l_1-\nu_1)}{\nu_1(\nu_{1,1}l_1-1)} = z \in \mathbb{Z}$ where $z < \Gamma_0\frac{\nu_1-\mu_1}{\nu_1}$ since $\Gamma_0\frac{(\nu_1-\mu_1)(\nu_{1,1}l_1-\nu_1)}{\nu_1(\nu_{1,1}l_1-1)} = \Gamma_0\frac{\nu_1-\mu_1}{\nu_1}\frac{\nu_{1,1}l_1-\nu_1}{\nu_{1,1}l_1-1} < \Gamma_0\frac{\nu_1-\mu_1}{\nu_1}$ because $\nu_1 > 1$.

If we know z then

$$l_1 = \frac{\nu_1(\Gamma_0\nu_1 - \Gamma_0\mu_1 - z)}{\nu_{1,1}(\Gamma_0\nu_1 - \Gamma_0\mu_1 - z\nu_1)} = d_0\frac{\Gamma_0\nu_1 - \Gamma_0\mu_1 - z}{\Gamma_0\nu_1 - \Gamma_0\mu_1 - z\nu_1}$$

As above, we have a potential counterexample when $l_1 \in \mathbb{Z}$, $\frac{l_1}{d_0} \notin \mathbb{Z}$, $\frac{d_0}{l_1-d_0} \notin \mathbb{Z}$.

If d_0 is prime then e_1' must be principal and we can look at the next possible vertex v_1.

If d_0 is not prime then the case when e_1' is not principal should be considered. In this case $f_1(e_1') = \phi_1^d$ where $d > 1$ is maximal possible. If d is divisible by d_0 the Jacobian condition $\mathrm{J}(f_1(e_1'), h_1) = f_1(e_1')$ cannot be satisfied because $\deg_y(h_1) = l_1\nu_{1,1}$ will be then divisible by $\deg_y(f_1(e_1')^{\frac{1}{d}})$ (see Lemma on divisibility).

Since the expansion relative to e_1' is a sub-expansion of the expansion relative to e_0 we can take $d_1 = \gcd(d_0, d)$ as the common denominator of the powers of f_1 in this expansion and record $f_1(e_1') = \phi_1^{d_1}$ where d_1 is a proper divisor of d_0.

Now we should find vertices v_2. This is similar to finding v_1, the differences are that only the second upper bound for l_1 can be used and that the unit of measurement along the x axis is $\frac{1}{\Gamma_0}$. The search of the vertex v_{i+1} when the previous vertices and Γ_{i-1} are known and e'_i is not the principal edge is the same as this one.

Since $k_1 = l_1\nu_{1,1}$ the choice of l_1 determines the slope of the edge e_1 because $\mathrm{dv}(h_1) = l_1(\mu_{1,1}, \nu_{1,1})$ where $\mu_{1,1} = \frac{\mu_1}{d_0},\ \nu_{1,1} = \frac{\nu_1}{d_0}$ and $\mathcal{N}(h_1)$ contains a point $(1,1)$.

We need to define the shortest vector proportional to the vector $< l_1\mu_{1,1} - 1, l_1\nu_{1,1} - 1 >$ where the measurement unit along x axis is $\frac{1}{\Gamma_0}$. Components $\beta_1,\ \gamma_1$ are computed similarly to components $\beta_0,\ \gamma_0$
$\epsilon_1 = \gcd(\Gamma_0(l_1\mu_{1,1} - 1), l_1\nu_{1,1} - 1),\ \beta_1 = \frac{l_1\mu_{1,1}-1}{\epsilon_1},\ \gamma_1 = \frac{l_1\nu_{1,1}-1}{\epsilon_1}$.
$v'_2 = (\mu'_2, \nu'_2) = v_1 - t_1 d_1(\beta_1, \gamma_1) = (\mu_1 - t_1 d_1\beta_1, \nu_1 - t_1 d_1\gamma_1)$, where $1 \leq t_1 \leq \frac{\nu_1}{d_1\gamma_1}$ because $\nu'_2 \geq 0$ and $t_1 \neq \frac{\nu_1-\mu_1}{d_1(\gamma_1-\beta_1)} = \frac{\epsilon_1(\nu_1-\mu_1)}{d_1 l_1(\nu_{1,1}-\mu_{1,1})} = \frac{\epsilon_1 d_0}{d_1 l_1}$ because $\mu'_2 \neq \nu'_2$.

The inequality $t_1 \leq \frac{\nu_1}{d_1\gamma_1}$ allows to bound l_1 as follows:
$\frac{\nu_1}{d_1\gamma_1} = \frac{\epsilon_1\nu_1}{d_1(l_1\nu_{1,1}-1)} \leq \frac{\Gamma_0(\nu_{1,1}-\mu_{1,1})\nu_1}{d_1(l_1\nu_{1,1}-1)}$ because $\epsilon_1 = \gcd(\Gamma_0 l_1\mu_{1,1} - \Gamma_0, l_1\nu_{1,1} - 1) = \gcd(\Gamma_0 l_1(\mu_{1,1} - \nu_{1,1}), l_1\nu_{1,1} - 1) = \gcd(\Gamma_0(\mu_{1,1} - \nu_{1,1}), l_1\nu_{1,1} - 1) \leq \Gamma_0(\nu_{1,1} - \mu_{1,1})$. Since t_1 can be defined only if $\frac{\nu_1}{d_1\gamma_1} \geq 1$ we have $\Gamma_0(\nu_{1,1} - \mu_{1,1})\nu_1 \geq d_1(l_1\nu_{1,1} - 1)$ and $l_1 \leq \frac{\Gamma_0(\nu_{1,1}-\mu_{1,1})\nu_1+d_1}{d_1\nu_{1,1}}$. So

$$1 \leq l_1 \leq \frac{\Gamma_0(\nu_{1,1} - \mu_{1,1})\nu_1 + d_1}{d_1\nu_{1,1}}$$

Additional restriction on l_1 is $\gcd(l_1 d_1, d_0) < d_0$. Indeed, $f_1(e'_1) = \phi_1^{d_1}$. Hence $\deg_y(\phi_1) = \frac{\nu_1}{d_1}$ while $\deg_y(h_1) = l_1\frac{\nu_1}{d_0}$ and $l_1\frac{\nu_1}{d_0} \div \frac{\nu_1}{d_1} \notin \mathbb{Z}$ by Lemma on divisibility.

Now we can define v_2.
If $\nu'_2 > \mu'_2$ then $v_2 = v'_2,\ \Gamma_1 = \Gamma_0$.
If $\nu'_2 < \mu'_2$ then $v_2 = (\mu'_2, \nu'_2) + (d_1 t_2 - \nu'_2)(\frac{\beta_1}{\gamma_1}, 1) = (\mu_2, \nu_2)$,
$\frac{\gamma_1\mu'_2 - \beta_1\nu'_2}{d_1(\gamma_1-\beta_1)} < t_2 \leq \frac{\nu_1-\nu'_2}{d_1\gamma_1}$.

We had fractions with the denominator Γ_0. The denominator of $\frac{\beta_1}{\gamma_1} = \frac{\Gamma_0\beta_1}{\Gamma_0\gamma_1}$ is $\frac{\Gamma_0\gamma_1}{\gcd(\Gamma_0\beta_1,\Gamma_0\gamma_1)}$. Therefore now we have fractions with the denominator

$$\Gamma_1 = \mathrm{lcm}(\Gamma_0, \frac{\Gamma_0\gamma_1}{\gcd(\Gamma_0\beta_1, \Gamma_0\gamma_1)})$$

Here is a similar description of the algorithm after the vertex v_i is found. At this stage we also know d_{i-1} and Γ_{i-1}.

Next step: check if e_i is the principal edge. This is based on the formula $\nu'_{i+1} - \mu'_{i+1} = \frac{(\nu_i-\mu_i)(k_i\nu'_{i+1}-\nu_i)}{\nu_i(k_i-1)}$ and properties of the principal edge already discussed. Since $f_{i-1}(e_{i-1})^{\frac{k_i}{\nu_i}} \in B_i$ we have $\frac{k_i}{\nu_i} = \frac{l_i}{d_{i-1}}$. Thus $k_i = l_i\nu_{i,1}$ where $\nu_{i,1} = \frac{\nu_i}{d_{i-1}}$.

$$\nu'_{i+1} - \mu'_{i+1} = \frac{(\nu_i - \mu_i)(l_i\nu_{i,1}\nu'_{i+1} - \nu_i)}{\nu_i(l_i\nu_{i,1} - 1)} = \frac{(\nu_i - \mu_i)(l_i\nu'_{i+1} - d_{i-1})}{d_{i-1}(l_i\nu_{i,1} - 1)}$$

If $\nu'_{i+1} = 0$ then $\Gamma_{i-1}\frac{\nu_i-\mu_i}{\nu_{i,1}l_i-1} = z \in \mathbb{Z}$ where $z < \Gamma_{i-1}\frac{\nu_i-\mu_i}{\nu_{i,1}d_{i-1}-1} = \Gamma_{i-1}\frac{\nu_i-\mu_i}{\nu_i-1}$ since $l_i > d_{i-1}$.

We have a potential counterexample if for some admissible z

$$l_i = \frac{\Gamma_{i-1}\nu_i - \Gamma_{i-1}\mu_i + z}{z\nu_{i,1}} \in \mathbb{Z}$$

and $\frac{l_i}{d_{i-1}} \notin \mathbb{Z}$, $\frac{d_{i-1}}{l_i-d_{i-1}} \notin \mathbb{Z}$.

If $\nu'_{i+1} = 1$ then $\Gamma_{i-1}\frac{(\nu_i-\mu_i)(l_i-d_{i-1})}{d_{i-1}(l_i\nu_{i,1}-1)} = z \in \mathbb{Z}$ where $z < \Gamma_{i-1}\frac{\nu_i-\mu_i}{\nu_i}$ since $\Gamma_{i-1}\frac{(\nu_i-\mu_i)(\nu_{i,1}l_i-\nu_i)}{\nu_i(\nu_{i,1}l_i-1)} = \Gamma_{i-1}\frac{\nu_i-\mu_i}{\nu_i}\frac{\nu_{i,1}l_i-\nu_i}{\nu_{i,1}l_i-1} < \Gamma_{i-1}\frac{\nu_i-\mu_i}{\nu_i}$ because $\nu_i > 1$.

Again, we have a potential counterexample if for some admissible z

$$l_i = d_{i-1}\frac{\Gamma_{i-1}\nu_i - \Gamma_{i-1}\mu_i - z}{\Gamma_{i-1}\nu_i - \Gamma_{i-1}\mu_i - z\nu_i} \in \mathbb{Z}$$

and $\frac{l_i}{d_{i-1}} \notin \mathbb{Z}$, $\frac{d_{i-1}}{l_i-d_{i-1}} \notin \mathbb{Z}$.

If e_i is the principal edge then $\lambda_0 + 1 = \frac{w_i(xy)}{w_i(f_i)} = \frac{w_i(h_i)}{w_i(f_i)} = \frac{k_i}{\nu_i} = \frac{l_i}{d_{i-1}}$. Hence

$$\lambda_0 = \frac{l_i}{d_{i-1}} - 1$$

If d_{i-1} is prime then e'_i must be principal and we can look at the next possible vertex v_i.

If d_{i-1} is not prime then the case when e'_i is not principal should be considered. In this case $f_i(e'_i) = \phi_i^{d_i}$ where d_i is a proper divisor of d_{i-1} and d_i can be taken as the common denominator of the powers of f_i in the expansion of g_i relative to the edge e_i.

Now we should find the vertex v_{i+1}.

Since $k_i = l_i\nu_{i,1}$ the choice of l_i determines the slope of the edge e_i because $\mathrm{dv}(h_i) = l_i(\mu_{i,1}, \nu_{i,1})$ where $\mu_{i,1} = \frac{\mu_i}{d_{i-1}},\ \nu_{i,1} = \frac{\nu_i}{d_{i-1}}$ and $\mathcal{N}(h_i)$ contains a point $(1, 1)$.

We need to define the shortest vector proportional to the vector $< l_i\mu_{i,1} - 1, l_i\nu_{i,1} - 1 >$ where the measurement unit along x axis is $\frac{1}{\Gamma_{i-1}}$. Components $\beta_i,\ \gamma_i$ are computed similarly to components $\beta_1,\ \gamma_1$.
$\epsilon_i = \gcd(\Gamma_{i-1}(l_i\mu_{i,1} - 1), l_i\nu_{i,1} - 1),\ \beta_i = \frac{l_i\mu_{i,1}-1}{\epsilon_i},\ \gamma_i = \frac{l_i\nu_{i,1}-1}{\epsilon_i}$.
$v'_{i+1} = v_i - t_1 d_i(\beta_i, \gamma_i);\ \mu'_{i+1} = \mu_i - t_1 d_i\beta_i,\ \nu'_{i+1} = \nu_i - t_1 d_i\gamma_i$, where $1 \le t_1 \le \frac{\nu_i}{d_i\gamma_i}$ because $\nu'_{i+1} \ge 0$ and $t_1 \ne \frac{\nu_i-\mu_i}{d_i(\gamma_i-\beta_i)} = \frac{\epsilon_i(\nu_i-\mu_i)}{d_i l_i(\nu_{i,1}-\mu_{i,1})} = \frac{\epsilon_i d_{i-1}}{d_i l_i}$ because $\mu'_{i+1} \ne \nu'_{i+1}$.

The inequality $t_1 \le \frac{\nu_i}{d_i\gamma_i}$ allows to bound l_i as follows:
$\frac{\nu_i}{d_i\gamma_i} = \frac{\epsilon_i\nu_i}{d_i(l_i\nu_{i,1}-1)} \le \frac{\Gamma_{i-1}(\nu_{i,1}-\mu_{i,1})\nu_i}{d_i(l_i\nu_{i,1}-1)}$ because $\epsilon_i = \gcd(\Gamma_{i-1}l_i\mu_{i,1} - \Gamma_{i-1}, l_i\nu_{i,1} - 1) = \gcd(\Gamma_{i-1}l_i(\mu_{i,1} - \nu_{i,1}), l_i\nu_{i,1} - 1) = \gcd(\Gamma_{i-1}(\mu_{i,1} - \nu_{i,1}), l_i\nu_{i,1} - 1) \le \Gamma_{i-1}(\nu_{i,1} - \mu_{i,1})$.
Since t_1 can be defined only if $\frac{\nu_i}{d_i\gamma_i} \ge 1$ we have $\Gamma_{i-1}(\nu_{i,1}-\mu_{i,1})\nu_i \ge d_i(l_i\nu_{i,1}-1)$ and $l_i \le \frac{\Gamma_{i-1}(\nu_{i,1}-\mu_{i,1})\nu_i+d_i}{d_i\nu_{i,1}}$. So

$$1 \le l_i \le \frac{\Gamma_{i-1}(\nu_{i,1} - \mu_{i,1})\nu_i + d_i}{d_i\nu_{i,1}}$$

Additional restriction on l_i is $\gcd(l_i d_i, d_{i-1}) < d_{i-1}$ (see Lemma on divisibility).

Now we can define v_{i+1}.
If $\nu'_{i+1} > \mu'_{i+1}$ then $v_{i+1} = v'_{i+1},\ \Gamma_i = \Gamma_{i-1}$.
If $\nu'_{i+1} < \mu'_{i+1}$ then $v_{i+1} = (\mu'_{i+1}, \nu'_{i+1}) + (d_i t_2 - \nu'_{i+1})(\frac{\beta_i}{\gamma_i}, 1) = (\mu_{i+1}, \nu_{i+1})$,
$\frac{\gamma_i\mu'_{i+1}-\beta_i\nu'_{i+1}}{d_i(\gamma_i-\beta_i)} < t_2 \le \frac{\nu_i-\nu'_{i+1}}{d_i\gamma_i}$.
The denominator of $\frac{\beta_i}{\gamma_i} = \frac{\Gamma_{i-1}\beta_i}{\Gamma_{i-1}\gamma_i}$ is $\frac{\Gamma_{i-1}\gamma_i}{\gcd(\Gamma_{i-1}\beta_i,\Gamma_{i-1}\gamma_i)}$, hence $\Gamma_i = \mathrm{lcm}(\Gamma_{i-1}, \frac{\Gamma_{i-1}\gamma_i}{\gcd(\Gamma_{i-1}\beta_i,\Gamma_{i-1}\gamma_i)})$.

Results

In the papers [H] and [M] the authors assisted by a computer considered the cases when $\deg(f)$ and $\deg(g)$ do not exceed 100. Here we consider the possibilities for f when $\deg(f) \le 100$.

Computer search gives the following 19 possibilities for D:

$$D \in \{42, 48, 50, 56, 60, 63, 64, 66, 70, 72, 75, 80, 84, 88, 90, 96, 98, 99, 100\}$$

In this section each of the cases is described. The leading vertex will be written as $d_0 \times \delta \times (a_0, b_0)$. Recall that $f(e'_0) = \phi_0^{d_0}$. Next, the coordinates of the further vertices of non-principal edges before and after the Newton resolution steps and of the principal edge before the Newton resolution are presented. The forms supported by the leading edge and further edges, but not by the principal edge will be also described. A description of the form supported by the principal edge requires additional computations (see [D] or [ML1]). Of course, the ratio λ_0 of the degrees of g and f is also given.

$D = 42$. $v_0 = 2 \times 3 \times (2, 5),\ v'_1 = (2, 0),\ v_1 = 2(\frac{7}{3}, 4),\ v'_2 = (\frac{2}{3}, 1),\ \lambda_0 = \frac{7}{2}$, $\phi_0 = cx(xy^3 - r_1)^4(xy^3 - r_2),\ r_1r_2 \neq 0,\ r_1 \neq r_2$[1].

$D = 48$. $v_0 = 3 \times 4 \times (1, 3),\ v'_1 = (3, 0),\ v_1 = 3(\frac{7}{4}, 3),\ v'_2 = (\frac{3}{4}, 1),\ \lambda_0 = \frac{4}{3}$,
$\phi_0 = cx(xy^4 - r_1)^3$.
$v_0 = 6 \times 2 \times (1, 3),\ v_1 = v'_1 = 2 \times 3 \times (2, 5),\ v'_2 = (2, 0),\ v_2 = 2(\frac{7}{3}, 4)$,
$v'_3 = (\frac{2}{3}, 1),\ \lambda_0 = \frac{7}{2},\ \phi_0 = cx^2y^5(y - r_1),\ f_1(e'_1) = \phi_1^2$,
$\phi_1 = cx(xy^3 - r_1)^4(xy^3 - r_2)$

$D = 50$. $v_0 = 2 \times 5 \times (1, 4),\ v'_1 = (2, 0)$.
$v_1 = 2(\frac{8}{5}, 3),\ v'_2 = (\frac{1}{5}, 0),\ \lambda_0 = \frac{3}{2},\ \phi_0 = cx(xy^5 - r_1)^3(xy^5 - r_2)$;
$v_1 = 2(\frac{7}{5}, 2),\ v'_2 = (\frac{4}{5}, 1),\ \lambda_0 = \frac{3}{2},\ \phi_0 = x(xy^5 - r_1)^2q_0(xy^5),\ \deg(q_0) = 2$.
(Hereinafter $q_i(0)q_i(r_1) \neq 0$.)

$D = 56$. $v_0 = 2 \times 7 \times (1, 3)$.
$v'_1 = (4, 2),\ v_1 = 2(3, 5),\ v'_2 = (\frac{3}{4}, 1),\ \lambda_0 = \frac{3}{2},\ \phi_0 = cx^2y(xy^4 - r_1)^5$.
$v'_1 = (4, 2),\ v_1 = 2(\frac{11}{4}, 4),\ v'_2 = (\frac{3}{4}, 1),\ \lambda_0 = \frac{7}{2},\ \phi_0 = cx^2y(xy^4 - r_1)^4(xy^4 - r_2)$.
{Since $v'_1 = (4, 2)$ we have $\deg_x(f(x, 0)) < 2$. This leads to a contradiction.}

[1]Further on r_i are not equal to zero and different if the indexes are different.

$v_1' = (2,0),\ v_1 = 2(\frac{13}{7},3),\ v_2' = (\frac{5}{7},1),\ \lambda_0 = \frac{5}{2},\ \phi_0 = cx(x^2y^7 - r_1)^3.$

$v_1' = (2,0),\ v_1 = 2(\frac{11}{7},2),\ v_2' = (\frac{6}{7},1),\ \lambda_0 = \frac{3}{2},\ \phi_0 = cx(x^2y^7 - r_1)^2(x^2y^7 - r_2).$

$D = 60,\ v_0 = 2\times 3\times(3,7),\ v_1' = (4,0),\ v_1 = 2(\frac{13}{3},7),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{13}{2},$
$\phi_0 = cx^2(xy^3 - r_1)^7.$

$v_0 = 6\times 2\times(1,4),\ v_1' = v_1 = 2\times 3\times(2,5),\ v_2' = (2,0),\ v_2 = 2(\frac{7}{3},4),\ v_3' = (\frac{2}{3},1),$
$\lambda_0 = \frac{7}{2},\ \phi_0 = x^2y^5q_0(y),\ \deg(q_0) = 3,\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx(xy^3 - r_1)^4(xy^3 - r_2).$

$D = 63.$ $v_0 = 3\times 3\times(2,5),\ v_1' = (3,0).$

$v_1 = 3(\frac{8}{3},5),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{3},\ \phi_0 = cx(xy^3 - r_1)^5;$

$v_1 = 3(\frac{7}{3},4),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{11}{3},\ \phi_0 = cx(xy^3 - r_1)^4(xy^3 - r_2).$

$D = 64.$ $v_0 = 4\times 4\times(1,3),\ v_1' = (4,0),\ v_1 = 4(\frac{7}{4},3).$

$v_2' = (\frac{1}{4},0),\ \lambda_0 = \frac{3}{4},\ f(e_0') = \phi_0^4,\ \phi_0 = cx(xy^4 - r_1)^3;$

$v_2' = v_2 = (\frac{11}{2},8),\ v_3' = (\frac{3}{4},1),\ \lambda_0 = \frac{7}{2}\ \phi_0 = cx(xy^4 - r_1)^3,\ f_1(e_1') = \phi_1^2,$
$\phi_1 = x^{\frac{11}{4}}y^4(x^{\frac{3}{4}}y^2 - r_1);$

$v_2' = v_2 = (\frac{5}{2},4),\ v_3' = (\frac{3}{4},1),\ \lambda_0 = \frac{3}{2},\ \phi_0 = cx(xy^4 - r_1)^3,\ f_1(e_1') = \phi_1^2,$
$phi_1 = x^{\frac{5}{4}}y^2(x^{\frac{9}{4}}y^4 - r_1).$

$D = 66.$ $v_0 = 2\times 3\times(3,8),\ v_1' = (2,0).$

$v_1 = 2(\frac{11}{3},8),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{3}{2},\ \phi_0 = cx(xy^3 - r_1)^8;$

$v_1 = 2(\frac{7}{3},4),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{2},\ \phi_0 = x(xy^3 - r_1)^4q_0(xy^3),\ \deg(q_0) = 4.$

$D = 70.$ $v_0 = 2\times 5\times(2,5),\ v_1' = (4,2).$

$v_1 = 2(\frac{13}{3},8),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{5}{2},\ \phi_0 = cx^2y(xy^3 - r_1)^8;$

$v_1 = 2(\frac{11}{3},6),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{11}{2},\ \phi_0 = x^2y(xy^3 - r_1)^6q_0(xy^3),\ \deg(q_0) = 2.$

{Since $v_1' = (4,2)$ we have $\deg_x(f(x,0)) < 2$. This leads to a contradiction.}

$D = 72.$ $v_0 = 2\times 4\times(2,7),\ v_1' = (2,0).$

$v_1 = 2(\frac{11}{4},7),\ v_2' = (\frac{1}{4},0),\ \lambda_0 = \frac{3}{2},\ \phi_0 = cx(xy^4 - r_1)^7;$

$v_1 = 2(\frac{5}{2},6),\ v_2' = (\frac{1}{2},1),\ \lambda_0 = \frac{11}{2}.$ $\phi_0 = cx(xy^4 - r_1)^6(xy^4 - r_1).$

$v_0 = 6\times 2\times(1,5),\ v_1' = v_1 = 2\times 3\times(2,5),\ v_2' = (2,0),\ v_2 = 2(\frac{7}{3},4),\ v_3' = (\frac{2}{3},1),$
$\lambda_0 = \frac{7}{2},\ \phi_0 = x^2y^5q_0(y),\ \deg(q_0) = 5,\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx(xy^3-r_1)^4(xy^3-r_2).$

$v_0 = 6\times 3\times(1,3).$

$v_1' = v_1 = 2\times 3\times(3,7),\ v_2' = (4,0),\ v_2 = 2(\frac{13}{3},7),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{13}{2},$
$\phi_0 = x^3y^7q_0(y),\ \deg(q_0) = 2,\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx^2(xy^3-r_1)^7.$

$v_1' = v_1 = 2\times 3\times(3,8),\ v_2' = (2,0),\ v_2 = 2(\frac{7}{3},4),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{2},$
$\phi_0 = x^3y^8(y-r_1),\ f_1(e_1') = \phi_1^2,\ \phi_1 = x(xy^3-r_1)^4q_1(xy^3),\ \deg(q_1) = 4.$

$v_1' = v_1 = 2\times 3\times(3,8),\ v_2' = (2,0),\ v_2 = 2(\frac{11}{3},8),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{3}{2},$
$\phi_0 = x^3y^8(y-r_1),\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx(xy^3-r_1)^8.$

$v_0 = 9\times 2\times(1,3),\ v_1' = v_1 = 3\times 3\times(2,5),\ v_2' = (3,0).$

$v_2 = 3(\frac{7}{3},4),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{11}{3},\ \phi_0 = x^2y^5(y-r_1),\ f_1(e_1') = \phi_1^3,$
$\phi_1 = cx(xy^3-r_1)^4(xy^3-r_2);$

$v_2 = 3(\frac{8}{3},5),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{3},\ \phi_0 = x^2y^5(y-r_1),\ f_1(e_1') = \phi_1^3,$
$\phi_1 = cx(xy^3-r_1)^5.$

$D = 75.$ $v_0 = 3\times 5\times(1,4),\ v_1' = (3,0).$

$v_1 = 3(\frac{8}{5},3),\ v_2' = (\frac{3}{5},1),\ \lambda_0 = \frac{16}{3},\ \phi_0 = cx(xy^5-r_1)^3(xy^5-r_2);$

$v_1 = 3(\frac{8}{5},3),\ v_2' = (\frac{4}{5},1),\ \lambda_0 = \frac{2}{3},\ \phi_0 = cx(xy^5-r_1)^3(xy^5-r_2);$

$v_1 = 3(\frac{7}{5},2),\ v_2' = (\frac{4}{5},1),\ \lambda_0 = \frac{5}{3},\ \phi_0 = x(xy^5-r_1)^2q_0(xy^5),\ \deg(q_0) = 2;$

$v_1 = 3(\frac{7}{5},2),\ v_2' = (\frac{1}{5},0),\ \lambda_0 = \frac{2}{3},\ \phi_0 = x(xy^5-r_1)^2q_0(xy^5),\ \deg(q_0) = 2.$

$D = 80.$ $v_0 = 2\times 4\times(3,7),\ v_1' = (6,2),\ v_1 = 2(\frac{17}{3},9),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{17}{2},$
$\phi_0 = cx^3y(xy^3-r_1)^9.$

$v_0 = 5\times 4\times(1,3),\ v_1' = (5,0),\ v_1 = 5(\frac{7}{4},3),\ v_2' = (\frac{3}{4},1),\ \lambda_0 = \frac{7}{5},$
$\phi_0 = cx(xy^4-r_1)^3.$

$v_0 = 8\times 2\times(1,4),\ v_1' = v_1 = 2\times 4\times(2,7),\ v_2' = (2,0).$

$v_2 = 2(\frac{11}{4},7),\ v_3' = (\frac{1}{4},0),\ \lambda_0 = \frac{3}{2},\ \phi_0 = x^2y^7(y-r_1),\ f_1(e_1') = \phi_1^2,$
$\phi_1 = cx(xy^4-r_1)^7;$

$v_2 = 2(\frac{5}{2},6),\ v_3' = (\frac{1}{2},1),\ \lambda_0 = \frac{11}{2},\ \phi_0 = x^2y^7(y-r_1),\ f_1(e_1') = \phi_1^2,$
$\phi_1 = cx(xy^4-r_1)^6(xy^4-r_2).$

$v_0 = 10\times 2\times(1,3),\ v_1' = v_1 = 2\times 5\times(2,5),\ v_2' = (4,2).$

$v_2 = 2(\frac{13}{3}, 8),\ v_3' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{5}{2},\ \phi_0 = x^2y^5(y - r_1),\ f_1(e_1') = \phi_1^2,$

$\phi_1 = cx^2y(xy^3 - r_1)^8;$

$v_2 = 2(\frac{11}{3}, 6),\ v_3' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{11}{2},\ \phi_0 = x^2y^5(y - r_1),\ f_1(e_1') = \phi_1^2,$

$\phi_1 = x^2y(xy^3 - r_1)^6 q_0(xy^3),\ \deg(q_0) = 2.$

{Since $v_2' = (4, 2)$ we have $\deg_x(f(x, 0)) < 2$. This leads to a contradiction.}

$D = 84$. $v_0 = 2 \times 6 \times (2, 5),\ v_1' = (4, 0),\ v_1 = 2(\frac{13}{3}, 7),\ v_2' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{13}{2},$

$\phi_0 = x^2(xy^3 - r_1)^7 q_0(xy^3),\ \deg(q_0) = 3.$

$v_0 = 2 \times 7 \times (1, 5),\ v_1' = (4, 0).$

$v_1 = 2(\frac{19}{7}, 5),\ v_2' = (\frac{5}{7}, 1),\ \lambda_0 = \frac{3}{2},\ \phi_0 = cx^2(xy^7 - r_1)^5;$

$v_1 = 2(\frac{19}{7}, 5),\ v_2' = (\frac{4}{7}, 1),\ \lambda_0 = \frac{27}{2},\ \phi_0 = cx^2(xy^7 - r_1)^5;$

$v_1 = 2(\frac{18}{7}, 4),\ v_2' = (\frac{5}{7}, 1),\ \lambda_0 = \frac{7}{2},\ \phi_0 = cx^2(xy^7 - r_1)^4(xy^7 - r_2);$

$v_1 = 2(\frac{17}{7}, 3),\ v_2' = (\frac{6}{7}, 1),\ \lambda_0 = \frac{5}{2},\ \phi_0 = cx^2(xy^7 - r_1)^3 q_1(xy^7),\ \deg(q_1) = 2.$

$v_0 = 3 \times 7 \times (1, 3).$

$v_1' = (6, 3),\ v_1 = 3(3, 5),\ v_2' = (\frac{1}{4}, 0),\ \lambda_0 = \frac{2}{3},\ \phi_0 = cx^2y(xy^4 - r_1)^5.$

$v_1' = (6, 3),\ v_1 = 3(\frac{11}{4}, 4),\ v_2' = (\frac{3}{4}, 1),\ \lambda_0 = \frac{11}{3},\ \phi_0 = cx^2y(xy^4 - r_1)^4(xy^4 - r_2).$

$v_1' = (3, 0),\ v_1 = 3(\frac{13}{7}, 3),\ v_2' = (\frac{5}{7}, 1),\ \lambda_0 = \frac{8}{3},\ \phi_0 = cx(x^2y^7 - r_1)^3.$

$v_1' = (3, 0),\ v_1 = 3(\frac{11}{7}, 2),\ v_2' = (\frac{6}{7}, 1),\ \lambda_0 = \frac{5}{3},\ \phi_0 = cx(x^2y^7 - r_1)^2(x^2y^7 - r_2).$

$v_1' = (3, 0),\ v_1 = 3(\frac{11}{7}, 2),\ v_2' = (\frac{1}{7}, 0),\ \lambda_0 = \frac{2}{3},\ \phi_0 = cx(x^2y^7 - r_1)^2(x^2y^7 - r_2).$

$v_0 = 4 \times 3 \times (2, 5),\ v_1' = (4, 0).$

$v_1 = 4(\frac{7}{3}, 4),\ v_2' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{15}{4},\ \phi_0 = cx(xy^3 - r_1)^4(xy^3 - r_2);$

$v_1 = 4(\frac{8}{3}, 5),\ v_2 = v_2' = (\frac{22}{3}, 12),\ v_3' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{11}{2},$

$\phi_0 = cx(xy^3 - r_1)^5,\ f_1(e_1') = \phi_1^2,\ \phi_1 = x^{\frac{11}{3}}y^6(x^{\frac{5}{3}}y^4 - s_1);$

$v_1 = 4(\frac{8}{3}, 5),\ v_2' = (\frac{2}{3}, 0),\ v_2 = 2(\frac{7}{3}, 4),\ v_3' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{7}{2},$

$\phi_0 = cx(xy^3 - r_1)^5,\ f_1(e_1') = \phi_1^2,\ \phi_1 = x^{\frac{1}{3}}(xy^2 - s_1)^4(xy^2 - s_2);$

$v_1 = 4(\frac{8}{3}, 5),\ v_2' = (\frac{2}{3}, 0),\ v_2 = 2(\frac{11}{6}, 3),\ v_3' = (\frac{1}{6}, 0),\ \lambda_0 = \frac{3}{2},$

$\phi_0 = cx(xy^3 - r_1)^5,\ f_1(e_1') = \phi_1^2,\ \phi_1 = x^{\frac{1}{3}}(xy^2 - s_1)^3 q_1(xy^2),\ \deg(q_1) = 2;$

$v_1 = 4(\frac{8}{3}, 5),\ v_2' = v_2 = 2(\frac{13}{3}, 8),\ v_3' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{5}{2},$

$\phi_0 = cx(xy^3 - r_1)^5,\ f_1(e_1') = \phi_1^2,\ \phi_1 = x^{\frac{13}{3}}y^8(xy^2 - s_1);$

$v_1 = 4(\frac{8}{3}, 5),\ v_2' = v_2 = 2(\frac{7}{3}, 4),\ v_3' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{7}{2},$

$\phi_0 = cx(xy^3 - r_1)^5,\ f_1(e_1') = \phi_1^2,\ \phi_1 = x^{\frac{7}{3}}y^4 q_1(xy^2),\ \deg(q_1) = 3.$

$v_0 = 6\times 2\times (1,6),\ v_1' = v_1 = 2\times 3\times (2,5),\ v_2' = (2,0),\ v_2 = 2(\frac{7}{3},4),\ v_3' = (\frac{2}{3},1),$
$\lambda_0 = \frac{7}{2},\ \phi_0 = x^2y^5q_0(y),\ \deg(q_0) = 7,\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx(xy^3-r_1)^4(xy^3-r_2).$

$D = 88.$ $v_0 = 2\times 11\times (1,3).$ $v_1' = (6,2).$
$v_1 = 2(\frac{19}{4},8),\ v_2' = (\frac{3}{4},1),\ \lambda_0 = \frac{3}{2},\ \ \phi_0 = cx^3y(xy^4-r_1)^8;$
$v_1 = 2(\frac{17}{4},6),\ v_2' = (\frac{3}{4},1),\ \lambda_0 = \frac{11}{2},\ \ \phi_0 = cx^3y(xy^4-r_1)^6q_0(xy^4),\ \deg(q_0) = 2.$

$D = 90,\ v_0 = 2\times 3\times (4,11),\ v_1' = (2,0).$
$v_1 = 2(\frac{11}{3},8),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{3}{2},\ \ \phi_0 = x(xy^3-r_1)^8q_0(xy^3),\ \deg(q_0) = 3;$
$v_1 = 2(\frac{7}{3},4),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{2}.\ \ \phi_0 = x(xy^3-r_1)^4q_0(xy^3),\ \deg(q_0) = 7.$
$v_0 = 2\times 9\times (1,4),\ v_1' = (2,0).$
$v_1 = 2(\frac{17}{9},4),\ v_2' = (\frac{2}{3},1)$ or $(\frac{2}{9},0),\ \lambda_0 = \frac{3}{2},\ \phi_0 = x(x^2y^9-r_1)^4;$
$v_1 = 2(\frac{5}{3},3),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{5}{2},\ \phi_0 = x(x^2y^9-r_1)^3(x^2y^9-r_2).$
$v_0 = 3\times 3\times (3,7),\ v_1' = (6,0),\ v_1 = 3(\frac{13}{3},7),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{20}{3},$
$\phi_0 = x^2(xy^3-r_1)^7.$
$v_0 = 6\times 3\times (1,4).$
$v_1' = v_1 = 2\times 3\times (2,5),\ v_2' = (2,0),\ v_2 = 2(\frac{7}{3},4),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{2},$
$\phi_0 = x^2y^5(xy^7-r_1),\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx(xy^3-r_1)^4(xy^3-r_2).$
$v_1' = v_1 = 2\times 3\times (3,7),\ v_2' = (4,0),\ v_2 = 2(\frac{13}{3},7),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{13}{2},$
$\phi_0 = x^3y^7q_0(y),\ \deg(g_0) = 5,\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx^2(xy^3-r_1)^7.$
$v_1' = v_1 = 2\times 3\times (3,8),\ v_2' = (2,0),\ v_2 = 2(\frac{7}{3},4),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{2},$
$\phi_0 = x^3y^8q_0(y),\ \deg(q_0) = 4,\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx(xy^3-r_1)^4q_1(xy^3),$
$\deg(q_1) = 4.$
$v_1' = v_1 = 2\times 3\times (3,8),\ v_2' = (2,0),\ v_2 = 2(\frac{11}{3},8),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{3}{2},$
$\phi_0 = x^3y^8q_0(y),\ \deg(q_0) = 4,\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx(xy^3-r_1)^8.$
$v_0 = 9\times 2\times (1,4),\ v_1' = v_1 = 3\times 3\times (2,5),\ v_2' = (3,0).$
$v_2 = 3(\frac{7}{3},4),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{11}{3},\ \phi_0 = x^2y^5q_0(y),\ \deg(q_0) = 3,$
$f_1(e_1') = \phi_1^3,\ \phi_1 = cx(xy^3-r_1)^4(xy^3-r_2);$
$v_2 = 3(\frac{8}{3},5),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{3},\ \phi_0 = x^2y^5q_0(y),\ \deg(q_0) = 3,$
$f_1(e_1') = \phi_1^3,\ \phi_1 = cx(xy^3-r_1)^5.$

$D=96$. $v_0=3\times 8\times(1,3),\ v_1'=(6,0),\ v_1=3(\frac{13}{4},5),\ v_2'=(\frac{3}{4},1),\ \lambda_0=\frac{7}{3}$,
$\phi_0=x^2(xy^4-r_1)^5(xy^4-r_2)$.

$v_0=6\times 2\times(1,7),\ v_1'=v_1=2\times 3\times(2,5),\ v_2'=(2,0),\ v_2=2(\frac{7}{3},4),\ v_3'=(\frac{2}{3},1)$,
$\lambda_0=\frac{7}{2},\ \phi_0=x^2y^5q_0(y),\ \deg(q_0)=9,\ f_1(e_1')=\phi_1^2,\ \phi_1=cx(xy^3-r_1)^4(xy^3-r_2)$.

$v_0=6\times 2\times(3,5),\ v_1'=(6,0),\ v_1=6(\frac{7}{2},5),\ v_2'=(1,0),\ v_2=2(\frac{19}{6},4)$,
$v_3'=(\frac{5}{6},1),\ \lambda_0=\frac{7}{2},\ \phi_0=cx(xy^2-r_1)^5,\ f_1(e_1')=\phi_1^2$,
$\phi_1=x^{\frac{1}{2}}(x^2y^3-r_1)^4(x^2y^3-r_2)$.

$v_0=6\times 4\times(1,3),\ v_1'=(6,0),\ v_1=6(\frac{7}{4},3),\ \phi_0=cx(xy^4-r_1)^3$.
$v_2'=(\frac{15}{4},0),\ v_2=3(\frac{19}{8},3),\ v_3'=(\frac{7}{8},1),\ \lambda_0=\frac{4}{3},\ f_1(e_1')=\phi_1^3$,
$\phi_1=cx^{\frac{5}{4}}(x^{\frac{3}{4}}y^2-s_1)^3$;
$v_2'=(\frac{9}{2},2),\ v_2=2(\frac{27}{8},4),\ v_3'=(\frac{7}{8},1),\ \lambda_0=\frac{7}{2},\ f_1(e_1')=\phi_1^2$,
$\phi_1=cx^{\frac{9}{4}}y(x^{\frac{3}{4}}y^2-s_1)^4$;
$v_2=v_2'=3(\frac{11}{4},4),\ v_3'=(\frac{3}{4},1),\ \lambda_0=\frac{11}{3},\ f_1(e_1')=\phi_1^3,\ \phi_1=cx^{\frac{11}{4}}y^4(x^{\frac{3}{4}}y^2-s_1)$;
$v_2=v_2'=2(\frac{19}{4},8),\ v_3'=(\frac{3}{4},1),\ \lambda_0=\frac{3}{2},\ f_1(e_1')=\phi_1^2,\ \phi_1=cx^{\frac{19}{4}}y^8(x^{\frac{1}{2}}y-s_1)$;
$v_2=v_2'=2(\frac{11}{4},4),\ v_3'=(\frac{3}{4},1),\ \lambda_0=\frac{7}{2},\ f_1(e_1')=\phi_1^2,\ \phi_1=cx^{\frac{11}{4}}y^4(x^{\frac{1}{2}}y-s_1)^5$;
$v_2=v_2'=2(3,5),\ v_3'=(\frac{3}{4},1),\ \lambda_0=\frac{3}{2},\ f_1(e_1')=\phi_1^2,\ \phi_1=cx^3y^5(x^{\frac{9}{4}}y^4-s_1)$;
$v_2=v_2'=3(\frac{5}{4},2),\ v_3'=(\frac{3}{4},1),\ \lambda_0=\frac{5}{3}$ or $v_3'=(\frac{1}{4},0),\ \lambda_0=\frac{2}{3},\ f_1(e_1')=\phi_1^3$,
$\phi_1=cx^{\frac{5}{4}}y^2(x^{\frac{9}{4}}y^4-s_1)$.

$v_0=6\times 4\times(1,3)$.
$v_1=v_1'=2\times 3\times(2,5),\ v_2'=(2,0),\ v_2=2(\frac{7}{3},4),\ v_3'=(\frac{2}{3},1),\ \lambda_0=\frac{7}{2}$,
$\phi_0=cx^2y^5(x^2y^7-r_1),\ f_1(e_1')=\phi_1^2,\ \phi_1=cx(xy^3-r_1)^4(xy^3-r_2)$.
$v_1=v_1'=2\times 3\times(3,7),\ v_2'=(4,0),\ v_2=2(\frac{13}{3},7),\ v_3'=(\frac{2}{3},1),\ \lambda_0=\frac{13}{2}$,
$\phi_0=cx^3y^7(xy^5-r_1),\ f_1(e_1')=\phi_1^2,\ \phi_1=cx^2(xy^3-r_1)^7$.
$v_1=v_1'=2\times 3\times(3,8),\ v_2'=(2,0),\ v_2=2(\frac{7}{3},4),\ v_3'=(\frac{2}{3},1),\ \lambda_0=\frac{7}{2}$,
$\phi_0=cx^3y^8(xy^4-r_1),\ f_1(e_1')=\phi_1^2,\ \phi_1=cx(xy^3-r_1)^4q_1(xy^3),\ \deg(q_1)=4$.
$v_1=v_1'=2\times 3\times(3,8),\ v_2'=(2,0),\ v_2=2(\frac{11}{3},8),\ v_3'=(\frac{2}{3},1),\ \lambda_0=\frac{3}{2}$,
$\phi_0=cx^3y^8(xy^4-r_1),\ f_1(e_1')=\phi_1^2,\ \phi_1=cx(xy^3-r_1)^8$.
$v_1=v_1'=2\times 3\times(4,11),\ v_2'=(2,0),\ v_2=2(\frac{7}{3},4),\ v_3'=(\frac{2}{3},1),\ \lambda_0=\frac{7}{2}$,
$\phi_0=cx^4y^{11}(y-r_1),\ f_1(e_1')=\phi_1^2,\ \phi_1=x(xy^3-r_1)^4q_1(xy^3),\ \deg(q_1)=7$.
$v_1=v_1'=2\times 3\times(4,11),\ v_2'=(2,0),\ v_2=2(\frac{11}{3},8),\ v_3'=(\frac{2}{3},1),\ \lambda_0=\frac{3}{2}$,
$\phi_0=cx^4y^{11}(y-r_1),\ f_1(e_1')=\phi_1^2,\ \phi_1=x(xy^3-r_1)^8q_1(xy^3),\ \deg(q_1)=3$.

$v_1 = v_1' = 2 \times 6 \times (2,5),\ v_2' = (4,0),\ v_2 = 2(\frac{13}{3},7),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{13}{2},$
$\phi_0 = cx^4y^{10}q_0(y),\ \deg(q_0) = 2,\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx^2(xy^3 - r_1)^7 q_1(xy^3),$
$\deg(q_1) = 3.$

$v_0 = 8 \times 2 \times (1,5),\ v_1' = v_1 = 2 \times 4 \times (2,7),\ v_2' = (2,0).$

$v_2 = 2(\frac{5}{2},6),\ v_3' = (\frac{1}{2},1),\ \lambda_0 = \frac{11}{2},\ \phi_0 = cx^2y^7q_1(y),\ \deg(q_1) = 3,$
$f_1(e_1') = \phi_1^2,\ \phi_1 = cx(xy^4 - r_1)^6(xy^4 - r_1);$

$v_2 = 2(\frac{11}{4},7),\ v_3' = (\frac{1}{4},0),\ \lambda_0 = \frac{3}{2},\ \phi_0 = cx^2y^7q_1(y),\ \deg(q_1) = 3,$
$f_1(e_1') = \phi_1^2,\ \phi_1 = cx(xy^4 - r_1)^7.$

$v_0 = 8 \times 3 \times (1,3),\ v_1' = v_1 = 2 \times 4 \times (3,7),\ v_2' = (6,2),\ v_2 = 2(\frac{17}{3},9),\ v_3' = (\frac{2}{3},1),$
$\lambda_0 = \frac{17}{2},\ \phi_0 = cx^3y^7q_1(y),\ \deg(q_1) = 2,\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx^3y(xy^3 - r_1)^9.$

$v_0 = 12 \times 2 \times (1,3).$

$v_1' = v_1 = 2 \times 6 \times (2,5),\ v_2' = (4,0),\ v_2 = 2(\frac{13}{3},7),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{13}{2},$
$\phi_0 = cx^2y^5(y - r_1),\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx^2(xy^3 - r_1)^7q_1(xy^3),\ \deg(q_1) = 3.$

$v_1' = v_1 = 4 \times 3 \times (2,5),\ \phi_0 = cx^2y^5(y - r_1),\ v_2' = (4,0).$

$v_2 = 4(\frac{7}{3},4),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{15}{4},\ f_1(e_1') = \phi_1^4,\ \phi_1 = cx(xy^3 - r_1)^4(xy^3 - r_2);$

$v_2 = 4(\frac{8}{3},5),\ v_3' = v_3 = 2(\frac{11}{3},6),\ v_4' = (\frac{2}{3},1),\ \lambda_0 = \frac{11}{2},\ f_1(e_1') = \phi_1^4,$
$\phi_1 = cx(xy^3 - s_1)^5,\ f_2(e_2') = \phi_2^2,\ \phi_2 = x^{\frac{11}{3}}y^6(x^{\frac{5}{3}}y^4 - t_1);$

$v_2 = 4(\frac{8}{3},5),\ v_3' = v_3 = 2(\frac{13}{3},8),\ v_4' = (\frac{2}{3},1),\ \lambda_0 = \frac{5}{2},\ f_1(e_1') = \phi_1^4,$
$\phi_1 = x(xy^3 - s_1)^5,\ f_2(e_2') = \phi_2^2,\ \phi_2 = x^{\frac{13}{3}}y^8(xy^2 - t_1);$

$v_2 = 4(\frac{8}{3},5),\ v_3' = v_3 = 2(\frac{7}{3},4),\ v_4' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{2},\ f_1(e_1') = \phi_1^4,$
$\phi_1 = x(xy^3 - s_1)^5,\ f_2(e_2') = \phi_2^2,\ \phi_2 = x^{\frac{7}{3}}y^4q_2(xy^2),\ \deg(q_2) = 3;$

$v_2 = 4(\frac{8}{3},5),\ v_3' = (\frac{2}{3},0),\ v_3 = 2(\frac{11}{6},3),\ v_4' = (\frac{1}{6},0),\ \lambda_0 = \frac{3}{2},\ f_1(e_1') = \phi_1^4,$
$\phi_1 = x(xy^3 - s_1)^5,\ f_2(e_2') = \phi_2^2,\ \phi_2 = x^{\frac{1}{3}}(xy^2 - t_1)^3q_2(xy^2),\ \deg(q_2) = 2;$

$v_2 = 4(\frac{8}{3},5),\ v_3' = (\frac{2}{3},0),\ v_3 = (\frac{14}{3},8),\ v_4' = (\frac{2}{3},1),\ \lambda_0 = \frac{7}{2},\ f_1(e_1') = \phi_1^4,$
$\phi_1 = x(xy^3 - s_1)^5,\ f_2(e_2') = \phi_2^2,\ \phi_2 = x^{\frac{1}{3}}(xy^2 - t_1)^4(xy^2 - t_2).$

$D = 98.$ $v_0 = 2 \times 7 \times (1,6),\ v_1' = (2,0),\ v_1 = 2(\frac{13}{7},6),\ v_2' = (\frac{2}{7},0)$ or
$v_2' = (\frac{4}{7},1),\ \lambda_0 = \frac{3}{2},\ \phi_0 = cx(xy^7 - r_1)^6.$

$v_0 = 2 \times 7 \times (2,5),\ v_1' = (6,4),\ v_1 = 2(5,8),\ v_2' = (\frac{2}{3},1),\ \lambda_0 = \frac{15}{2},$
$\phi_0 = cx^3y^2(xy^3 - r_1)^8q_0(xy^3), \deg(q_0) = 3.$

{Since $v_1' = (6,4)$ we have $\deg_x(f(x,0)) < 2$. This leads to a contradiction.}

$D = 99.\ v_0 = 3 \times 3 \times (3, 8),\ v_1' = (3, 0).$

$v_1 = 3(\frac{11}{3}, 8),\ v_2' = (\frac{1}{3}, 0),\ \lambda_0 = \frac{2}{3},\ \phi_0 = cx(xy^3 - r_1)^8;$

$v_1 = 3(\frac{10}{3}, 7),\ v_2' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{5}{3},\ \phi_0 = cx(xy^3 - r_1)^7(xy^3 - r_2);$

$v_1 = 3(\frac{8}{3}, 5),\ v_2' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{7}{3},\ \phi_0 = cx(xy^3 - r_1)^5 q_0,\ \deg(q_0) = 3;$

$v_1 = 3(\frac{7}{3}, 4),\ v_2' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{11}{3},\ \phi_0 = cx(xy^3 - r_1)^4 q_0,\ \deg(q_0) = 4.$

$D = 100.\ v_0 = 2 \times 5 \times (3, 7).$

$v_1' = (2, 0),\ v_1 = 2(\frac{19}{5}, 7),\ v_2' = (\frac{3}{5}, 1),\ \lambda_0 = \frac{13}{2},\ \phi_0 = cx(x^2y^5 - r_1)^7.$

$v_1' = (2, 0),\ v_1 = 2(\frac{11}{5}, 3),\ v_2' = (\frac{4}{5}, 1),\ \lambda_0 = \frac{5}{2},\ \phi_0 = x(x^2y^5 - r_1)^3 q_0,$
$\deg(q_0) = 4.$

$v_1' = (8, 4),\ v_1 = 2(7, 11),\ v_2' = (\frac{2}{3}, 1),\ \lambda_0 = \frac{21}{2},\ \phi_0 = cx^4y^2(xy^3 - r_1)^{11}.$

$v_0 = 2 \times 10 \times (1, 4),\ v_1' = (4, 0).$

$v_1 = 2(\frac{18}{5}, 8),\ v_2' = (\frac{3}{5}, 1),\ \lambda_0 = \frac{5}{2},\ \phi_0 = cx^2(xy^5 - r_1)^8;$

$v_1 = 2(\frac{16}{5}, 6),\ v_2' = (\frac{3}{5}, 1),\ \lambda_0 = \frac{11}{2},\ \phi_0 = x^2(xy^5 - r_1)^6 q_0,\ \deg(q_0) = 2.$

$v_0 = 4 \times 5 \times (1, 4),\ v_1' = (4, 0).$

$v_1 = 4(\frac{9}{5}, 4),\ v_2' = (\frac{3}{5}, 1),\ \lambda_0 = \frac{5}{2},\ \phi_0 = cx(xy^5 - r_1)^4;$

$v_1 = 4(\frac{8}{5}, 3),\ v_2' = (\frac{3}{5}, 1),\ \lambda_0 = \frac{11}{2},\ \phi_0 = x(xy^5 - r_1)^3(xy^5 - r_2);$

$v_1 = 4(\frac{7}{5}, 2),\ v_2' = (\frac{4}{5}, 1),\ \lambda_0 = \frac{7}{4},\ \phi_0 = x^2(xy^5 - r_1)^2 q_0,\ \deg(q_0) = 2.$

$v_1 = 4(\frac{8}{5}, 3),\ \phi_0 = x(xy^5 - r_1)^3(xy^5 - r_2),$

$v_2' = (\frac{14}{5}, 0),\ v_2 = 2(\frac{23}{10}, 3),\ v_3' = (\frac{1}{10}, 0),\ \lambda_0 = \frac{3}{2},\ f_1(e_1') = \phi_1^2,$
$\phi_1 = x^{\frac{7}{5}}(x^{\frac{3}{5}}y^2 - s_1)^3;$

$v_2' = (\frac{2}{5}, 0),\ v_2 = 2(\frac{17}{10}, 3),\ v_3' = (\frac{1}{10}, 0),\ \lambda_0 = \frac{7}{2},\ f_1(e_1') = \phi_1^2,$
$\phi_1 = x^{\frac{1}{5}}(xy^2 - s_1)^3;$

$v_2' = (\frac{2}{5}, 0),\ v_2 = 2(\frac{7}{10}, 1),\ v_3' = (\frac{1}{10}, 0),\ \lambda_0 = \frac{5}{2},\ f_1(e_1') = \phi_1^2,$
$\phi_1 = x^{\frac{1}{5}}(xy^2 - s_1)q_1,\ \deg(q_1) = 2.$

$v_1 = 4(\frac{9}{5}, 4),\ \phi_0 = x(xy^5 - r_1)^4.$

$v_2' = v_2 = 2(2, 4),\ v_3' = (\frac{3}{5}, 1),\ \lambda_0 = \frac{7}{2},\ f_1(e_1') = \phi_1^2,\ \phi_1 = x^2y^4q_1(x^{\frac{2}{5}}y),$
$\deg(q_1) = 4;$

$v_2' = v_2 = 2(\frac{8}{5}, 3),\ v_3' = (\frac{1}{5}, 0),\ \lambda_0 = \frac{3}{2},\ f_1(e_1') = \phi_1^2,\ \phi_1 = x^{\frac{8}{5}}y^3q_1(x^{\frac{2}{5}}y),$
$\deg(q_1) = 5.$

$v_0 = 10\times 2\times(1,4),\ v_1 = 2\times 5\times(2,5),\ \phi_0 = x^2y^5q_0(y),\ \deg(q_0)=3, v_2' = (4,2);$
$v_2 = 2(\frac{13}{3},8),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{5}{2},\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx^2y(xy^3-r_1)^8.$
$v_2 = 2(\frac{11}{3},6),\ v_3' = (\frac{2}{3},1),\ \lambda_0 = \frac{11}{2},\ f_1(e_1') = \phi_1^2,\ \phi_1 = cx^2y(xy^3-r_1)^6q_1(xy^3),$
$\deg(q_1)=2.$
{Since $v_2' = (4,2)$ we have $\deg_x(f(x,0)) < 2$. This leads to a contradiction.}

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