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MAPPING PARTITION FUNCTIONS

DI YANG AND DON ZAGIER

Abstract. We introduce an infinite group action on partition functions of WK type, meaning of the type of the partition function Z^{WK} in the famous result of Witten and Kontsevich expressing the partition function of ψ-class integrals on the compactified moduli space $\overline{\mathcal{M}}_{g,n}$ as a τ-function for the Korteweg–de Vries hierarchy. Specifically, the group which acts is the group $\mathcal{G}$ of formal power series of one variable $\varphi(V) = V + O(V^2)$, with group law given by composition, acting in a suitable way on the infinite tuple of variables of the partition functions. In particular, any $\varphi \in \mathcal{G}$ sends the Witten–Kontsevich (WK) partition function Z^{WK} to a new partition function Z^{φ}, which we call the *WK mapping partition function associated to* φ. We show that the genus zero part of $\log Z^{\varphi}$ is independent of φ and give an explicit recursive description for its higher genus parts (loop equation), and as applications of this obtain relationships of the ψ-class integrals to Gaussian Unitary Ensemble and generalized Brézin–Gross–Witten correlators. In a different direction, we use Z^{φ} to construct a new integrable hierarchy, which we call the *WK mapping hierarchy associated to* φ. We show that this hierarchy is a bihamiltonian perturbation of the Riemann–Hopf hierarchy possessing a τ-structure, and prove that it is a universal object for all such perturbations. Similarly, for any $\varphi \in \mathcal{G}$, we define the *Hodge mapping partition function associated to* φ, prove that it is integrable, and study its role in hamiltonian perturbations of the Riemann–Hopf hierarchy possessing a τ-structure. Finally, we establish a *generalized Hodge–WK correspondence* relating different Hodge mapping partition functions.

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1. Introduction

The Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + \frac{\epsilon^2}{12} \frac{\partial^3 u}{\partial x^3} \tag{1}$$

was discovered in the study of shallow water waves in the 19th century [12, 65]. It was shown [66, 84, 85] in the 1960s that this equation can be extended to a family of pairwise commuting evolutionary PDEs, called the *KdV hierarchy*:

$$\frac{\partial u}{\partial t_i} = \frac{u^i}{i!} \frac{\partial u}{\partial x} + \epsilon^2 K_i\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots, \frac{\partial^{2i+1} u}{\partial x^{2i+1}}, \epsilon\right), \quad i \geq 0 . \tag{2}$$

Here $t_0 = x$, $t_1 = t$, and K_i, $i \geq 0$, are certain polynomials. For more about the KdV hierarchy see e.g. [44, 45, 94, 98]. The *Riemann–Hopf (RH) hierarchy*, aka the dispersionless KdV hierarchy, is defined again as (2) but with ϵ taken to be 0.

In 1990, Witten [98] made a famous conjecture: the partition function $Z^{\rm WK}(\mathbf{t}; \epsilon)$ of ψ-class integrals on the Deligne–Mumford moduli space of algebraic curves [24]

$$Z^{\rm WK}(\mathbf{t}; \epsilon) = \exp\left(\sum_{g,\, n \geq 0} \epsilon^{2g-2} \sum_{i_1, \dots, i_n \geq 0} \frac{t_{i_1} \cdots t_{i_n}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \right), \tag{3}$$

is a τ-function for the KdV hierarchy, and in particular,

$$u^{\rm WK}(\mathbf{t}; \epsilon) := \epsilon^2 \partial_{t_0}^2 (\log Z^{\rm WK}(\mathbf{t}; \epsilon)) \tag{4}$$

satisfies the KdV hierarchy (2). Here $\mathbf{t} = (t_0, t_1, t_2, \dots)$ is an infinite tuple of indeterminates, $\overline{\mathcal{M}}_{g,n}$ denotes the moduli space of stable algebraic curves of genus g with n distinct marked points, and ψ_a $(a = 1, \dots, n)$ denotes the first Chern class of the ath tautological line bundle on $\overline{\mathcal{M}}_{g,n}$. Note that the integral appearing in the right-hand side of (3) vanishes unless the degree-dimension matching condition

$$i_1 + \cdots + i_n = 3g - 3 + n \tag{5}$$

is satisfied. Witten's conjecture, that opens the studies of the deep relations between topology of $\overline{\mathcal{M}}_{g,n}$ and integrable systems, was first proved by Kontsevich [63] and is now known as the *Witten–Kontsevich theorem*. See [6, 22, 61, 62, 83, 91] for several other proofs of this theorem. The function $Z^{\mathrm{WK}}(\mathbf{t};\epsilon)$ is referred to indifferently as the *WK (Witten–Kontsevich) partition function* or as the *WK tau-function*.

The general notion behind the story, which has appeared in combinatorics, statistical physics, matrix models, and other places, is that many interesting partition functions are τ-functions of integrable systems. On one hand, there are axiomatic or constructive ways approaching topologically interesting numbers [45, 49, 64], the partition functions of which would correspond to some integrable systems. In particular, Dubrovin and Zhang [45] gave a constructive way of defining a hierarchy of evolutionary PDEs in $(1+1)$ dimensions[1] associated to essentially any partition function. For instance, their construction applied to the WK partition function gives the KdV hierarchy. The Dubrovin–Zhang hierarchy corresponding to the partition function of Hodge integrals on $\overline{\mathcal{M}}_{g,n}$ depending on an infinite family of parameters [34] will also play an important role in this paper and will be called simply the *Hodge hierarchy*. On the other hand, one is interested in finding certain integrable systems that admit τ-functions, sometimes called possessing a τ-structure[2], which axiomatically leads to certain classification invariants [34, 45, 72]. The deep relations between these two notions is revealed most beautifully when there is a one-to-one correspondence between them, an example being the *Hodge universality conjecture* in the study of Hodge integrals (rank 1 cohomological field theories) and τ-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs [34], which says that the Hodge hierarchy is a universal object for one-component τ-symmetric integrable Hamiltonian perturbations of the RH hierarchy,[3] i.e., conjecturally any such integrable hierarchy is equivalent to the Hodge hierarchy.

In this paper we will study the deep relations from a novel perspective that sheds new light on both sides:

(a) We introduce an infinite group action, different from those of Givental or Sato–Segal–Wilson, on the arguments (infinite tuples) of partition functions. The group which acts is the group $\mathcal{G}$ of power series of one variable $\varphi(V)=V+O(V^2)$, acting on the right on the infinite tuple (denoted $\mathbf{t}\mapsto\mathbf{t}.\varphi$ and defined in equation (16) below). In particular, if we start with the WK partition function then each element φ of the group defines a new partition function $Z^{\varphi}(\mathbf{t};\epsilon) := Z^{\mathrm{WK}}(\mathbf{t}.\varphi^{-1};\epsilon)$, which we will call the *WK mapping partition function associated to φ*. The coefficients of its logarithm provide new and potentially interesting numbers, although we do not know their topological meaning. We show that the genus-zero part of this logarithm

[1]For readers not familiar with some of the terminology, we refer to Section 7 for a brief review.

[2]In literature (see e.g. [34, 45]), "(bi-)hamiltonian τ-structure" is often specialized to τ-symmetry, but "τ-structure" in this paper has the broader meaning (for details see Section 8; cf. [44], [96]).

[3]Here "perturbation of the RH hierarchy" means a hierarchy of evolutionary PDEs whose right-hand sides differ from those of the RH hierarchy by terms with more than one spatial derivative.

is independent of φ, give the dilaton equation and Virasoro constraints, and derive loop equations determining also the higher genus parts. Now applying the Dubrovin–Zhang construction to the WK mapping partition function we obtain a new hierarchy which we will call the *WK mapping hierarchy associated to* φ. We show that this hierarchy can be obtained by a space-time exchange combined with a Miura-type transformation on the KdV hierarchy, and then by using a recent result given by S.-Q. Liu, Z. Wang and Y. Zhang [70], prove the following theorem in Section 9:

Theorem 1. *The WK mapping hierarchy is a bihamiltonian perturbation of the RH hierarchy possessing a* τ*-structure.*

(b) We study the classification of bihamiltonian perturbations of the RH hierarchy possessing a τ-structure under the Miura-type group action. In [34] a related but different classification work was studied and it was conjectured that the universal object for the τ-symmetric integrable hierarchies of bihamiltonian evolutionary PDEs is the Volterra lattice hierarchy. Here, however, we consider a larger class by allowing a weaker form of the τ-symmetry condition used in [34, 45]. It turns out that there is a rich family of such bihamiltonian perturbations, part of which can be seen from Theorem 1, and we propose and prove the *WK mapping universality theorem*: the WK mapping hierarchy is a universal object in one-component bihamiltonian perturbations of the RH hierarchy possessing a τ-structure. This theorem has a precise numerical meaning and we also give verifications of it to high orders in Section 9.

Similarly, we consider the $\mathcal{G}$-action on the Hodge partition function. The resulting power series will be called the *Hodge mapping partition function*, and the Dubrovin–Zhang hierarchy for the Hodge mapping partition function will be called the *Hodge mapping hierarchy.*

Theorem 2. *The Hodge mapping hierarchy is an integrable perturbation of the RH hierarchy possessing a* τ*-structure.*

The proof of a refined version of this theorem is given in Section 11. We expect that this integrable hierarchy is hamiltonian. Note that our proof for the integrability also works for the WK mapping hierarchy, and also that Theorem 2 generalizes part of the result in Theorem 1. We will also propose in Section 11 (see Conjecture 2 there) the *Hodge mapping universality conjecture*: the Hodge mapping hierarchy is a universal object in one-component hamiltonian perturbations of the RH hierarchy possessing a τ-structure (weakening again the τ-symmetry condition from [34, 45]). This conjecture generalizes the Hodge universality conjecture [34].

For the special case when the group element φ is taken to be

$$\varphi_{\text{special}}(V) := \frac{e^{2qV}-1}{2q}\,, \tag{6}$$

by using the loop equation we will prove in Section 10 the *Hodge–WK correspondence* described in the following theorem, which is a relationship between a certain special-Hodge partition function $Z_{\Omega^{\rm special}(q)}(\mathbf{t};q)$ (see (225) and (231) in Section 10 for the definition) and the WK partition function $Z^{\rm WK}(\mathbf{t};\epsilon)$.

Theorem 3. *The following identity holds in* $\mathbb{C}((\epsilon^2))[[q]][[\mathbf{t}]]$:

$$Z_{\Omega^{\rm special}(q)}(\mathbf{t}.\varphi_{\rm special};\epsilon) \,=\, Z^{\rm WK}(\mathbf{t};\epsilon)\,. \tag{7}$$

We note that, although not completely obvious, this theorem is equivalent to a result of Alexandrov [5]; see Section 10 for more details.

As an application of the Hodge–WK correspondence, we will establish in the following two theorems explicit relationships of the WK partition function $Z^{\rm WK}(\mathbf{t};\epsilon)$ to the modified GUE partition function $Z^{\rm meGUE}(x,\mathbf{s};\epsilon)$ and to the generalized BGW partition function $Z^{\rm cBGW}(x,\mathbf{r};\epsilon)$ (see [35, 41] or Section 10 for the definition of $Z^{\rm meGUE}(x,\mathbf{s};\epsilon)$ and see [100] (cf. [3, 13, 53, 82, 100]) for the definition of $Z^{\rm cBGW}(x,\mathbf{r};\epsilon)$). Here and below, "GUE" refers to Gaussian Unitary Ensemble, and "BGW" refers to Brézin–Gross–Witten.

Theorem 4. *The following identity holds true in* $\mathbb{C}((\epsilon^2))[[x-1]][[\mathbf{s}]]$:

$$Z^{\rm WK}\Big(\mathbf{t}^{\rm WK-GUE}(x,\mathbf{s});\epsilon\Big)\, e^{\frac{A(x,\mathbf{s})}{\epsilon^2}} \,=\, Z^{\rm meGUE}\Big(x,\mathbf{s};\frac{\epsilon}{\sqrt{2}}\Big)\,, \tag{8}$$

where $A(x,\mathbf{s})$ *is a quadratic series defined by*

$$\begin{aligned} A(x,\mathbf{s}) \,&= \frac12 \sum_{j_1,j_2\geq 1} \frac{j_1 j_2}{j_1+j_2}\binom{2j_1}{j_1}\binom{2j_2}{j_2}\Big(s_{j_1}-\frac{\delta_{j_1,1}}{2}\Big)\Big(s_{j_2}-\frac{\delta_{j_2,1}}{2}\Big)\\ &\quad +\, x\sum_{j\geq1}\binom{2j}{j}\Big(s_j-\frac{\delta_{j,1}}{2}\Big)\,, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\frac{2^m}{(2m+1)!!}\, t_m^{\rm WK-GUE}(x,\mathbf{s})\\ &\quad = \frac23\,\delta_{m,1} \,+\, \frac{1}{2m+1}\,x \,+\, \sum_{j\geq1}\binom{m+j-1/2}{j-1}\,2^{2j-1}\Big(s_j-\frac{\delta_{j,1}}{2}\Big)\,,\quad m\geq 0\,. \end{aligned} \tag{10}$$

We call (8) the *WK–GUE correspondence.*

Theorem 5. *The following identity holds true in* $\mathbb{Q}((\epsilon^2))[[x+2]][[\mathbf{r}]]$:

$$Z^{\rm WK}\Big(\mathbf{t}^{\rm WK-BGW}(x,\mathbf{r});\sqrt{-4}\epsilon\Big)\, e^{\frac{A_{\rm cBGW}(x,\mathbf{r})}{\epsilon^2}} \,=\, Z^{\rm cBGW}(x,\mathbf{r};\epsilon)\,, \tag{11}$$

where $A_{\rm cBGW}(x,\mathbf{r})$ *is a quadratic function given by*

$$A_{\rm cBGW}(x,\mathbf{r}) \,=\, \frac{1}{2}\sum_{a,b\geq 0}\frac{(r_a-\delta_{a,0})(r_b-\delta_{b,0})}{a!\,b!\,(a+b+1)} - x\sum_{b\geq 0}\frac{r_b-\delta_{b,0}}{b!\,(2b+1)}\,, \tag{12}$$

and

$$t_m^{\rm WK-BGW}(x,\mathbf{r}) \,=\, \delta_{m,1} \,+\, 2\,\delta_{m,0} \,+\, \frac{(2m-1)!!}{2^m}\,x - 2\sum_{j\geq m}\frac{(-1)^m}{(j-m)!}\,r_j\,. \tag{13}$$

We call (11) the *WK–BGW correspondence.*

Using the Hodge–WK correspondence and the $\mathcal{G}$-action we establish in Theorem 15 an explicit relationship between the Hodge mapping partition function with a special choice of its parameters associated to an arbitrarily given group element $\psi\in\mathcal{G}$ (which will be called the *special-Hodge mapping partition function associated to* ψ) and the WK mapping partition function associated to φ, where φ and ψ are related by $\varphi=\varphi_{\rm special}\circ\psi$ with $\varphi_{\rm special}$ as in (6), i.e.,

$$\varphi(V) \,=\, \frac{e^{2\,q\,\psi(V)}-1}{2\,q}\,, \quad \psi(V) \,=\, \frac{\log(1+2\,q\,\varphi(V))}{2\,q}\,. \tag{14}$$

Such a relationship will be called the *generalized Hodge–WK correspondence.*

Organization of the paper. In Section 2 we introduce the infinite group action on infinite tuples, define the WK mapping partition function, and prove Theorem 6: the genus zero part is a fixed point of the group action. In Section 3 we give the dilaton equation and Virasoro constraints for the WK mapping partition function. In Section 4 we give a geometric proof of Theorem 6. In Section 5 we prove the existence of the jet-variable representation for the higher genus WK mapping free energies, and in Section 6 we derive the loop equation. In Sections 7 and 8 we study the classification of hamiltonian and bihamiltonian perturbations of the RH hierarchy possessing a τ-structure. In Section 9 we prove Theorem 1 and prove the WK mapping universality theorem. A particular example is discussed in Section 10, where we prove Theorems 3, 4, 5. In Section 11 we prove Theorem 2 and propose the Hodge mapping universality conjecture. Section 12 is devoted to generalizations.

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2. $\mathcal{G}$-action and the definition of the WK mapping partition function

In this section we define an infinite group action on infinite tuples and the WK mapping partition function, and prove Theorem 6 below.

Fix a ground ring R (which for us will always be a $\mathbb{Q}$-algebra, usually $\mathbb{Q}$ or $\mathbb{C}$ or $\mathbb{Q}[q]$) and let $\mathcal{G} = V + V^2R[[V]]$ be the group of invertible power series of one variable with leading coefficient 1, with the group law given by composition and denoted by $\circ$. We define an affine-linear right action of the group $\mathcal{G}$ on tuples $\mathbf{t}$ by

$$\mathbf{t} = (t_0, t_1, t_2, \dots) \quad \mapsto \quad \mathbf{t}.\varphi = \mathbf{T} = (T_0, T_1, T_2, \dots)\,, \tag{15}$$

where $\mathbf{t}$ and $\mathbf{T}$ are related by

$$B_{\mathbf{T}}(V) = \sqrt{\varphi'(V)}\, B_{\mathbf{t}}(\varphi(V)) \tag{16}$$

with $B_{\mathbf{t}}$ defined for any infinite tuple $\mathbf{t}$ by

$$B_{\mathbf{t}}(v) := v - \sum_{i\geq 0} \frac{t_i}{i!}\, v^i\,. \tag{17}$$

Explicitly, if we write $\varphi(V) = \sum_{k=0}^{\infty} a_k V^k$ with $a_0 = 0$, $a_1 = 1$, then

$$T_0 = t_0\,, \quad T_1 = t_1 + a_2\, t_0\,, \quad T_2 = t_2 + 4\, a_2\, (t_1 - 1) + \big(3a_3 - a_2^2\big)\, t_0\,, \quad \dots\,, \tag{18}$$

$$t_0 = T_0\,, \quad t_1 = T_1 - a_2\, T_0\,, \quad t_2 = T_2 - 4\, a_2\, (T_1 - 1) - \big(3a_3 - 5a_2^2\big)\, T_0\,, \quad \dots\,. \tag{19}$$

Note that if we introduce for any tuple $\mathbf{t}$ the 1-form

$$\omega_{\mathbf{t}}(v) := B_{\mathbf{t}}(v)^2\, dv\,, \tag{20}$$

then the defining equation (16) for the $\mathcal{G}$-action can be stated equivalently as

$$\omega_{\mathbf{T}}(V) = \omega_{\mathbf{t}}(\varphi(V))\,. \tag{21}$$

Let $E(\mathbf{t})$ denote the following power series

$$E(\mathbf{t}) = \sum_{n\geq 1} \frac{1}{n} \sum_{\substack{i_1,\dots,i_n\geq 0\\ i_1+\dots+i_n=n-1}} \frac{t_{i_1}}{i_1!} \cdots \frac{t_{i_n}}{i_n!} = t_0 + t_0\, t_1 + \frac{2\, t_0\, t_1^2 + t_0^2\, t_2}{2} + \cdots\,, \tag{22}$$

which is the unique power-series solution (see [27, 98]) to the RH hierarchy

$$\frac{\partial E(\mathbf{t})}{\partial t_i} = \frac{E(\mathbf{t})^i}{i!}\, \frac{\partial E(\mathbf{t})}{\partial x}\,, \quad i \geq 0 \tag{23}$$

specified by the initial condition $E(x, 0, \dots) = x$, where $x := t_0$. Alternatively, it can be uniquely determined by the following equation:

$$B_{\mathbf{t}}(E(\mathbf{t})) = 0 \tag{24}$$

(see e.g. [27]), sometimes called the *genus zero Euler–Lagrange equation* [29, 45]. It is easily seen (and well known) that the power series $E(\mathbf{t})$ has the property:

$$\frac{\partial^k E(\mathbf{t})}{\partial x^k} \;=\; \delta_{k,1} \,+\, t_k \,+\, \text{higher degree terms}\,. \tag{25}$$

The following two lemmas are important.

Lemma 1. *For any $\varphi \in \mathcal{G}$, we have the identity:*

$$\varphi(E(\mathbf{T})) \;=\; E(\mathbf{t})\,, \tag{26}$$

where $\mathbf{t}$ *and* $\mathbf{T}$ *are related by* (16)*.*

Proof. By definition we have $B_{\mathbf{T}}(E(\mathbf{T})) = 0$. Then by (16) we obtain

$$B_{\mathbf{t}}(\varphi(E(\mathbf{T}))) \;=\; 0\,. \tag{27}$$

Note that $\varphi(E(\mathbf{T}))$ can be viewed as a power series of $\mathbf{t}$. The identity (26) then holds due to the uniqueness of power-series solution to equation (24). □

Remark 1. We can extend the group $\mathcal{G}$ to a semi-direct product consisting of all pairs (m, φ) with power series $m(V) \in 1 + VR[[V]]$ and $\varphi(V) \in V + V^2R[[V]]$, and with the group law $*$ given by

$$(m_1, \varphi_1) * (m_2, \varphi_2) \;=\; (m_1 \cdot (m_2 \circ \varphi_1), \varphi_1 \circ \varphi_2)\,, \tag{28}$$

where "·" denotes multiplication of power series. It acts on tuples $\mathbf{t}$ by sending $\mathbf{t}$ to $\mathbf{T} = \mathbf{t}.(m, \varphi)$, where $B_{\mathbf{T}}(V) = m(V) B_{\mathbf{t}}(\varphi(V))$. One can verify that the identity (26) still holds for φ in this larger group. One could therefore also consider partition functions under the extended group action, but we do not know whether this would have any interesting applications.

Lemma 2. *We have*

$$\frac{\partial E(\mathbf{T})}{\partial t_0} \;=\; \sqrt{\varphi'(E(\mathbf{T}))}\,\frac{\partial E(\mathbf{T})}{\partial T_0}\,, \tag{29}$$

where $\mathbf{t}$ *and* $\mathbf{T}$ *are related by* (16)*.*

Proof. By (16) and (23). □

For convenience, we denote $X \equiv T_0$ and $x \equiv t_0$ as in (23), and write (29) as

$$\frac{\partial E(\mathbf{T})}{\partial x} \;=\; \sqrt{\varphi'(E(\mathbf{T}))}\,\frac{\partial E(\mathbf{T})}{\partial X}\,. \tag{30}$$

By using the identities (26), (30) iteratively, one can obtain the map between the higher x-derivatives of $E(\mathbf{t})$ and X-derivatives of $E(\mathbf{T})$. For instance,

$$\frac{\partial E(\mathbf{t})}{\partial x} = \varphi'(E(\mathbf{T}))^{3/2} \frac{\partial E(\mathbf{T})}{\partial X}, \tag{31}$$

$$\frac{\partial^2 E(\mathbf{t})}{\partial x^2} = 2\,\varphi'(E(\mathbf{T}))\,\varphi''(E(\mathbf{T})) \left(\frac{\partial E(\mathbf{T})}{\partial X}\right)^2 + \varphi'(E(\mathbf{T}))^2 \frac{\partial^2 E(\mathbf{T})}{\partial X^2}. \tag{32}$$

By induction we arrive at the following lemma describing this map.

Lemma 3. *For each $k \geq 0$, there exists a function $M_k(V_0, \dots, V_k)$, which is a polynomial of $V_1, \dots, V_k$, such that*

$$\frac{\partial E(\mathbf{t})}{\partial x^k} = M_k\left(E(\mathbf{T}), \frac{\partial E(\mathbf{T})}{\partial X}, \dots, \frac{\partial^k E(\mathbf{T})}{\partial X^k}\right). \tag{33}$$

Moreover, for $k \geq 1$, the function $M_k(V_0, \dots, V_k)$ satisfies the homogeneity condition:

$$\sum_{j=1}^{k} j\, V_j \frac{\partial M_k(V_0, \dots, V_k)}{\partial V_j} = k\, M_k(V_0, \dots, V_k). \tag{34}$$

The first few M_k are $M_0(V) = V$, $M_1(V, V_1) = \varphi'(V)^{3/2} V_1$, $M_2(V, V_1, V_2) = 2\varphi'(V)\varphi''(V)\, V_1^2 + \varphi'(V)^2 V_2$.

Recall that the free energy $\mathcal{F}^{\mathrm{WK}}(\mathbf{t};\epsilon)$ of ψ-class intersection numbers is defined by

$$\mathcal{F}^{\mathrm{WK}}(\mathbf{t};\epsilon) := \log Z^{\mathrm{WK}}(\mathbf{t};\epsilon). \tag{35}$$

By definition the free energy $\mathcal{F}^{\mathrm{WK}}(\mathbf{t};\epsilon)$ admits the following genus expansion:

$$\mathcal{F}^{\mathrm{WK}}(\mathbf{t};\epsilon) =: \sum_{g\geq 0} \epsilon^{2g-2}\, \mathcal{F}_g^{\mathrm{WK}}(\mathbf{t}). \tag{36}$$

We call $\mathcal{F}_g^{\mathrm{WK}}(\mathbf{t})$ ($g \geq 0$) the *genus g free energy of ψ-class intersection numbers.* Explicitly,

$$\mathcal{F}_g^{\mathrm{WK}}(\mathbf{t}) = \sum_{n\geq 0} \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} t(\psi_1)\cdots t(\psi_n), \qquad t(z) := \sum_{i\geq 0} t_i\, z^i. \tag{37}$$

Definition 1. Let $\varphi \in \mathcal{G}$. The *WK mapping free energy associated to φ* is defined by

$$\mathcal{F}^{\varphi}(\mathbf{T};\epsilon) := \mathcal{F}^{\mathrm{WK}}\big(\mathbf{T}.\varphi^{-1};\epsilon\big), \tag{38}$$

and define the *genus g WK mapping free energy associated to φ*, denoted as $\mathcal{F}_g^{\varphi}(\mathbf{T})$, by

$$\mathcal{F}_g^{\varphi}(\mathbf{T}) := \mathcal{F}_g^{\mathrm{WK}}\big(\mathbf{T}.\varphi^{-1}\big), \quad g \geq 0. \tag{39}$$

Remark 2. By the degree-dimension matching (5), one can deduce that $\mathcal{F}^{\varphi}(\mathbf{T};\epsilon)$ and $\mathcal{F}^{\varphi}_g(\mathbf{T})$ ($g\geq 0$) are well-defined elements in $\epsilon^{-2}R[[\mathbf{T}]][[\epsilon^2]]$ and $R[[\mathbf{T}]]$, respectively. Another consequence of (5) is that we can upgrade our $\mathcal{G}$-action to an action of the full group of units $\mathbb{C}[[V]]^{\times}$ by setting

$$\mathcal{F}^{\varphi}(\mathbf{T};\epsilon) \;=\; \mathcal{F}^{\rm WK}\big(\mathbf{T}.\varphi^{-1},\epsilon/\varphi'(0)^{3/2}\big) \tag{40}$$

and

$$\mathcal{F}^{\varphi}_g(\mathbf{T}) \;=\; \varphi'(0)^{3g-3}\,\mathcal{F}^{\rm WK}_g\big(\mathbf{T}.\varphi^{-1}\big)\,, \tag{41}$$

and similarly for $Z^{\varphi}(\mathbf{T};\epsilon)$ below. Note that formula (40) makes sense even without choosing a square-root of $\varphi'(0)$, because $\mathcal{F}^{\rm WK}$ is an even power series of ϵ.

Definition 2. For $\varphi\in\mathcal{G}$, the *WK mapping partition function associated to* φ is defined by

$$Z^{\varphi}(\mathbf{T};\epsilon) \;:=\; Z^{\rm WK}\big(\mathbf{T}.\varphi^{-1};\epsilon\big)\,. \tag{42}$$

It is clear from the definitions that $\mathcal{F}^{\varphi}(\mathbf{T};\epsilon)$ has a genus expansion

$$\mathcal{F}^{\varphi}(\mathbf{T};\epsilon) \;=\; \sum_{g\geq 0}\epsilon^{2g-2}\,\mathcal{F}^{\varphi}_g(\mathbf{T}) \tag{43}$$

and that

$$Z^{\varphi}(\mathbf{T};\epsilon) \;=\; e^{\mathcal{F}^{\varphi}(\mathbf{T};\epsilon)}\,. \tag{44}$$

Our first main result says that the power series $\mathcal{F}^{\rm WK}_0(\mathbf{t})$ is $\mathcal{G}$-invariant, i.e.,

$$\mathcal{F}^{\rm WK}_0(\mathbf{t}) \;=\; \mathcal{F}^{\rm WK}_0(\mathbf{t}.\varphi)\,,\quad \forall\,\varphi\in\mathcal{G}\,. \tag{45}$$

In view of the definition of the group action (39), we can state this even more compactly in the following way.

Theorem 6. *For any $\varphi\in\mathcal{G}$, we have $\mathcal{F}^{\varphi}_0=\mathcal{F}^{\rm WK}_0$.*

Proof. The ψ-class intersection numbers in genus zero have the well-known formula:

$$\int_{\overline{\mathcal{M}}_{0,n}}\psi_1^{i_1}\cdots\psi_n^{i_n} \;=\; \binom{n-3}{i_1,\dots,i_n}\,,\quad i_1,\dots,i_n\geq 0\,. \tag{46}$$

Writing $(n-3)!$ as $\int_0^\infty s^{n-3}e^{-s}ds$, we find

$$\mathcal{F}_0^{\rm WK}(\mathbf{t}) = \sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{i_1,\dots,i_n\geq 0\\ i_1+\cdots+i_n=n-3}}\prod_{j=1}^n\frac{t_{i_j}}{i_j!}\int_0^\infty s^{i_1+\cdots+i_n}\,e^{-s}\,ds \tag{47}$$
$$= \mathrm{res}_{z=0}\biggl(\int_0^\infty e^{-s}\sum_{n=0}^\infty\sum_{i_1,\dots,i_n=0}^\infty\frac{s^{i_1+\cdots+i_n}}{n!}\,z^{2-n+i_1+\cdots+i_n}\prod_{j=1}^n\frac{t_{i_j}}{i_j!}\,ds\,dz\biggr),$$
$$= \mathrm{res}_{z=0}\biggl(\int_0^\infty e^{-B_{\mathbf{t}}(v)/z}\,dv\,z\,dz\biggr),$$

where $B_{\mathbf{t}}(v)$ is defined by (17). Here, in the last equality we employed the change of variables $v=zs$.

Therefore,

$$\mathcal{F}_0^{\varphi}(\mathbf{T}) := \mathcal{F}_0^{\rm WK}(\mathbf{t}) = \mathrm{res}_{z=0}\biggl(\int_0^\infty e^{-B_{\mathbf{t}}(v)/z}dv\,z\,dz\biggr)$$
$$= \mathrm{res}_{\tilde z=0}\biggl(\int_0^\infty e^{-\sqrt{\varphi'(V)}\,B_{\mathbf{t}}(\varphi(V))/\tilde z}dV\,\tilde z\,d\tilde z\biggr)$$
$$= \mathrm{res}_{z=0}\biggl(\int_0^\infty e^{-B_{\mathbf{T}}(V)/z}dV\,z\,dz\biggr).$$

Here, in the first line we used (39) and (47), in the second line we employed the change of variables $v=\varphi(V)$ and $\tilde z=\sqrt{\varphi'(V)}\,z$, and in the last equality we used the definition (16). The theorem is proved. ☐

We also note that the power series $E(\mathbf{t})$ defined in (22) is equal to the second t_0-derivative of $\mathcal{F}_0^{\rm WK}(\mathbf{t})$, i.e.,

$$E(\mathbf{t}) = \frac{\partial^2\mathcal{F}_0^{\rm WK}(\mathbf{t})}{\partial t_0^2}. \tag{48}$$

Then from (47) we immediately get an integral representation for $E(\mathbf{t})$ as follows:

$$E(\mathbf{t}) = \mathrm{res}_{z=0}\biggl(\int_0^\infty e^{-B_{\mathbf{t}}(v)/z}\,dv\,\frac{1}{z}\,dz\biggr) \tag{49}$$

Before ending this section, we make the following remark.

Remark 3. There is another way to state Theorem 6. For any $\varphi\in\mathcal{G}$, define a modified right action, denoted $\mathbf{t}\mapsto\hat{\mathbf{T}}=\mathbf{t}|\varphi$ by the following formula which is similar to (16), but with the map now being *linear* rather than *affine linear*:

$$\sum_{i\geq 0}\hat T_i\,\frac{V^i}{i!} = \sqrt{\varphi'(V)}\sum_{i\geq 0}t_i\,\frac{\varphi(V)^i}{i!}. \tag{50}$$

It is clear from that (16) and (50) that

$$T_i \;=\; \hat{T}_i + \delta_{i,1} - C_i, \qquad \sum_{i\geq 0} C_i \, \frac{V^i}{i!} \;:=\; \sqrt{\varphi'(V)}\, \varphi(V)\,. \tag{51}$$

It is also easy to deduce from (50) that

$$B_{\mathbf{T}}(V) \;=\; \sqrt{\varphi'(V)}\, B_{\mathbf{t}+\mathbf{d}-\mathbf{c}}(\varphi(V))\,, \quad \sum_{i\geq 0} c_i \, \frac{v^i}{i!} \;:=\; \sqrt{f'(v)}\, f(v)\,, \quad f \;:=\; \varphi^{-1}\,, \tag{52}$$

where $\mathbf{d} = (0, 1, 0, 0, 0, \dots)$ and $\mathbf{c} = (c_0, c_1, c_2, \dots)$. Theorem 6 can then be alternatively stated as follows:

$$\mathcal{F}_0\big(\hat{T}_0, \hat{T}_1, \hat{T}_2, \dots\big) \;=\; \mathcal{F}_0(t_0, t_1, t_2 - c_2, t_3 - c_3, \dots)\,. \tag{53}$$

One can use the modified group action to define a modified WK mapping partition function for any $\varphi \in \mathcal{G}$. It leads to the same WK mapping hierarchy as before. The shifts are nevertheless interesting due to their connection to higher Weil–Petersson volumes [58, 80, 68, 86, 10]) and will be useful for several of the applications later, e.g. in connection with the Alexandrov formula where the c_j (up to a scaling factor q^{j-1}) are specific numbers $(-4, 23, -176, \cdots)$ (Theorem B in Section 10).

3. Virasoro constraints for the WK mapping partition function

In this section, we give the Virasoro constraints for the WK mapping partition function.

Recall from [98] that the free energy $\mathcal{F}^{\rm WK}(\mathbf{t};\epsilon)$ satisfies the following *dilaton* and *string* equations, respectively:

$$\sum_{i\geq 0} t_i \, \frac{\partial \mathcal{F}^{\rm WK}(\mathbf{t};\epsilon)}{\partial t_i} \;+\; \epsilon\, \frac{\partial \mathcal{F}^{\rm WK}(\mathbf{t};\epsilon)}{\partial \epsilon} \;+\; \frac{1}{24} \;=\; \frac{\partial \mathcal{F}^{\rm WK}(\mathbf{t};\epsilon)}{\partial t_1}\,, \tag{54}$$

$$\sum_{i\geq 0} t_{i+1} \, \frac{\partial \mathcal{F}^{\rm WK}(\mathbf{t};\epsilon)}{\partial t_i} \;+\; \frac{t_0^2}{2\,\epsilon^2} \;=\; \frac{\partial \mathcal{F}^{\rm WK}(\mathbf{t};\epsilon)}{\partial t_0}\,. \tag{55}$$

Recall also that the Witten–Kontsevich theorem can be equivalently formulated as an infinite family of linear constraints for $Z^{\rm WK}(\mathbf{t};\epsilon)$, which come from a realization of half of the Virasoro algebra of central charge 1, called the *Virasoro constraints* [26, 45]. More precisely, define the linear operators L_k, $k\geq -1$, by

$$L_k^{\rm WK} = \sum_{i\geq 0} \frac{(2i+2k+1)!!}{2^{k+1}\,(2i-1)!!}\, t_i \frac{\partial}{\partial t_{i+k}} - \frac{(2k+3)!!}{2^{k+1}} \frac{\partial}{\partial t_{1+k}} + \frac{\delta_{k,0}}{16} + \frac{\epsilon^2}{2} \sum_{\substack{i,j\geq 0\\ i+j=k-1}} \frac{(2i+1)!!\,(2j+1)!!}{2^{k+1}} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{t_0^2}{2\,\epsilon^2}\,\delta_{k,-1}\,. \tag{56}$$

These operators satisfy the Virasoro commutation relations:

$$\left[L_{k_1}^{\rm WK}, L_{k_2}^{\rm WK}\right] = (k_1-k_2)\, L_{k_1+k_2}^{\rm WK}\,, \quad \forall\, k_1,k_2\geq -1\,. \tag{57}$$

The Virasoro constraints for $Z^{\rm WK}(\mathbf{t};\epsilon)$ then read

$$L_k^{\rm WK}\big(Z^{\rm WK}(\mathbf{t};\epsilon)\big) = 0\,, \qquad k\geq -1\,. \tag{58}$$

Obviously, the $k=-1$ constraint in (58) is the same as (55).

Proposition 1. *We have*

$$\sum_{i\geq 0} \tilde T_i \frac{\partial Z^{\varphi}(\mathbf{T};\epsilon)}{\partial T_i} + \epsilon \frac{\partial Z^{\varphi}(\mathbf{T};\epsilon)}{\partial \epsilon} + \frac{1}{24} Z^{\varphi}(\mathbf{T};\epsilon) = 0\,, \tag{59}$$

$$\overline{L}_k^{\varphi}\big(Z^{\varphi}(\mathbf{T};\epsilon)\big) = 0\,, \quad k\geq -1\,, \tag{60}$$

where $\tilde T_i := T_i - \delta_{i,1}$, and $\overline{L}_k^{\varphi}$, $k\geq -1$, are linear operators of the form

$$\overline{L}_k^{\varphi} = \epsilon^2 \sum_{i,j\geq 0} a_{ij}^{\varphi}(k) \frac{\partial^2}{\partial T_i \partial T_j} + \sum_{i,j\geq 0} b_{ij}^{\varphi}(k)\, \tilde T_i \frac{\partial}{\partial T_j} + \frac{T_0^2}{2\epsilon^2}\,\delta_{k,-1} + \frac{\delta_{k,0}}{16}\,. \tag{61}$$

Here the coefficients $a_{ij}^{\varphi}(k), b_{ij}^{\varphi}(k)$ depend on φ and k.

Proof. For $\varphi\in\mathcal{G}$, write

$$T_i = \delta_{i,1} + \sum_{m=0}^{i} M_i^m\,(t_m-\delta_{m,1})\,, \quad t_m = \delta_{m,1} + \sum_{i=0}^{m} N_m^i\,(T_i-\delta_{i,1})\,. \tag{62}$$

Here M_i^m and N_m^i have dependence on φ. Then we have

$$\sum_{m\geq 0} (t_m-\delta_{m,1}) \frac{\partial}{\partial t_m} = \sum_{m\geq 0}\sum_{i\geq m} (t_m-\delta_{m,1})\, M_i^m \frac{\partial}{\partial T_i} = \sum_{i\geq 0} (T_i-\delta_{i,1}) \frac{\partial}{\partial T_i}\,. \tag{63}$$

Similarly,

$$L_k^{\mathrm{WK}} = \frac{T_0^2}{2\epsilon^2}\,\delta_{k,-1} + \frac{\delta_{k,0}}{16} + \sum_{i\geq 0}\frac{(2i+2k+1)!!}{2^{k+1}\,(2i-1)!!}\sum_{j=0}^{i} N_i^j\,\tilde{T}_j \sum_{r=0}^{i+k} M_r^{i+k}\frac{\partial}{\partial T_r}$$
$$+ \frac{\epsilon^2}{2}\sum_{\substack{i,j\geq 0\\ i+j=k-1}}\frac{(2i+1)!!\,(2j+1)!!}{2^{k+1}}\sum_{r_1=0}^{i}\sum_{r_2=0}^{j} M_{r_1}^i M_{r_2}^j\frac{\partial}{\partial T_{r_1}}\frac{\partial}{\partial T_{r_2}}.$$

The proposition is proved. □

We call (59) the *dilaton equation* for $Z^\varphi(\mathbf{T};\epsilon)$, and (60) the *Virasoro constraints* for $Z^\varphi(\mathbf{T};\epsilon)$ because the operators $\overline{L}_k^\varphi$ satisfy the Virasoro commutation relations:

$$\left[\overline{L}_k^\varphi\,,\,\overline{L}_\ell^\varphi\right] = (k-\ell)\,\overline{L}_{k+\ell}^\varphi\,,\quad k,\ell\geq -1\,. \tag{64}$$

4. The WK mapping free energy in genus zero

In this section, we give a different and more geometric proof of Theorem 6.

Lemma 4. *The following identity holds:*

$$\frac{\partial^2\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_i\partial t_\ell} = \frac{\partial^2(\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t}.\varphi))}{\partial t_i\partial t_\ell}\,,\quad i,\ell\geq 0\,. \tag{65}$$

Proof. Recall the following well-known identity

$$\frac{\partial^2\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_i\partial t_\ell} = \frac{E(\mathbf{t})^{i+\ell+1}}{i!\,\ell!\,(i+\ell+1)}\,,\qquad \forall\, i,\ell\geq 0\,, \tag{66}$$

(see e.g. [29, 45]) with the $i=\ell=0$ case being the same as (48). From (66) one can easily deduce:

$$\frac{\partial^3\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_i\partial t_\ell\partial t_m} = \frac{E(\mathbf{t})^{i+\ell+m}}{i!\,\ell!\,m!}\frac{\partial v(\mathbf{t})}{\partial t_0}\,, \tag{67}$$

$$\frac{\partial^3(\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t}.\varphi))}{\partial t_i\partial t_\ell\partial t_m} = \sum_{m_1,m_2,m_3\geq 0}\frac{\partial T_{m_1}}{\partial t_i}\frac{\partial T_{m_2}}{\partial t_\ell}\frac{\partial T_{m_3}}{\partial t_m}\frac{E(\mathbf{T})^{m_1+m_2+m_3}}{m_1!\,m_2!\,m_3!}\frac{\partial E(\mathbf{T})}{\partial T_0}\,, \tag{68}$$

where $\mathbf{T}$ and $\mathbf{t}$ are related by $\mathbf{T}=\mathbf{t}.\varphi$, and $i,\ell,m\geq 0$. By using (16) we have

$$\sum_{m=0}^{\infty}\frac{\partial T_m}{\partial t_i}\frac{E(\mathbf{T})^m}{m!} = \sqrt{\varphi'(E(\mathbf{T}))}\,\frac{\varphi(E(\mathbf{T}))^i}{i!}\,. \tag{69}$$

Using (68), (69) and Lemma 1, we obtain that

$$\frac{\partial^3(\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t}.\varphi))}{\partial t_i\partial t_\ell\partial t_m} = \varphi'(E(\mathbf{T}))^{3/2}\,\frac{E(\mathbf{t})^{i+\ell+m}}{i!\,\ell!\,m!}\,\frac{\partial E(\mathbf{T})}{\partial T_0}\,. \tag{70}$$

We conclude from (67), (70), (31) that

$$\frac{\partial^3 \mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_i \partial t_\ell \partial t_m} = \frac{\partial^3 (\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t}.\varphi))}{\partial t_i \partial t_\ell \partial t_m}, \quad i, \ell, m \geq 0. \tag{71}$$

So the two sides of (65) can only differ by a constant. The lemma is then proved by observing that they both vanish when $\mathbf{t} = \mathbf{0}$. □

A second proof of Theorem 6. It follows from the dilaton equation that

$$2\,\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t}) = \sum_{i\geq 0} t_i \frac{\partial \mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_i} - \frac{\partial \mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_1}. \tag{72}$$

Differentiating this identity with respect to t_ℓ we find

$$\frac{\partial \mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_\ell} = \sum_{i\geq 0} t_i \frac{\partial^2 \mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_i \partial t_\ell} - \frac{\partial^2 \mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_1 \partial t_\ell}, \quad \ell \geq 0. \tag{73}$$

From (72), (73) we see that the power series $\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})$ is uniquely determined by $\frac{\partial^2 \mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_i \partial t_\ell}$, $i, \ell \geq 0$. Combined with Lemma 4, this proves the theorem. □

5. The higher genus WK mapping free energies

In this section, we show that the higher genus WK mapping free energies admit jet representations.

It is known that [27, 43, 45, 48] the power series $\mathcal{F}_g^{\mathrm{WK}}(\mathbf{t})$, $g \geq 1$, has the $(3g-2)$-jet representation, i.e., there exists $F_g^{\mathrm{WK}}(v_1, \dots, v_{3g-2})$, such that

$$\mathcal{F}_g^{\mathrm{WK}}(\mathbf{t}) = F_g^{\mathrm{WK}}\left(\frac{\partial E(\mathbf{t})}{\partial t_0}, \dots, \frac{\partial^{3g-2} E(\mathbf{t})}{\partial t_0^{3g-2}}\right), \quad g \geq 1, \tag{74}$$

with

$$F_1^{\mathrm{WK}}(v_1) = \frac{1}{24} \log v_1. \tag{75}$$

Moreover, for $g \geq 2$, $F_g^{\mathrm{WK}}(v_1, \dots, v_{3g-2})$ is a polynomial of $v_2, \dots, v_{3g-2}$ and v_1^{-1} (see e.g. [43, 45]), that satisfies the following two homogeneity conditions:

$$\sum_{k\geq 1} k\, v_k \frac{\partial F_g^{\mathrm{WK}}(v_1, \dots, v_{3g-2})}{\partial v_k} = (2g-2)\, F_g^{\mathrm{WK}}(v_1, \dots, v_{3g-2}), \quad g \geq 2, \tag{76}$$

$$\sum_{k\geq 2} (k-1)\, v_k \frac{\partial F_g^{\mathrm{WK}}(v_1, \dots, v_{3g-2})}{\partial v_k} = (3g-3)\, F_g^{\mathrm{WK}}(v_1, \dots, v_{3g-2}), \quad g \geq 2. \tag{77}$$

By using (39), Lemma 3 and (74)–(77) we arrive at the following proposition.

Proposition 2. *For $g=1$ we have the identity:*

$$\mathcal{F}_1^{\varphi}(\mathbf{T}) \;=\; F_1^{\varphi}\biggl(E(\mathbf{T}), \frac{\partial E(\mathbf{T})}{\partial X}\biggr), \text{ with } F_1^{\varphi}(V,V_1) \;:=\; \frac{1}{24}\log V_1 \;+\; \frac{1}{16}\log\varphi'(V)\,. \tag{78}$$

For each $g\geq 2$, $\mathcal{F}_g^{\varphi}(\mathbf{T})$ is given by

$$\mathcal{F}_g^{\varphi}(\mathbf{T}) \;=\; F_g^{\varphi}\biggl(E(\mathbf{T}),\dots,\frac{\partial^{3g-2}E(\mathbf{T})}{\partial X^{3g-2}}\biggr) \tag{79}$$

for some function $F_g^{\varphi}(V_0,\dots,V_{3g-2})$ which is a polynomial in V_1^{-1}, V_2, $\dots$, V_{3g-2}. Moreover, this polynomial is weighted homogeneous of degree $2g-2$ (where V_i has weight i), i.e.,

$$\sum_{k=1}^{3g-2} k\,V_k\,\frac{\partial F_g^{\varphi}(V_0,\dots,V_{3g-2})}{V_k} \;=\; (2g-2)\,F_g^{\varphi}(V_0,\dots,V_{3g-2})\,. \tag{80}$$

What is more, for each $g\geq 1$ the differences $\mathcal{F}_g^{\varphi}(\mathbf{T})-\mathcal{F}_g^{\rm WK}(\mathbf{T})$ and $F_g^{\varphi}(V_0,\dots,V_{3g-2})-F_g^{\rm WK}(V_1,\dots,V_{3g-2})$, as power series of $a_2,a_3,a_4,\dots$, have vanishing constant terms.

For instance, for $g=2$ we have the following explicit expression for F_2^{φ}:

$$\begin{aligned} F_2^{\varphi}(V,V_1,V_2,V_3,V_4) \;=\;& \frac{V_4}{1152\,V_1^2} - \frac{7\,V_3V_2}{1920\,V_1^3} + \frac{V_2^3}{360\,V_1^4} \\ &+ \frac{\varphi''(V)}{320\,\varphi'(V)}\,\frac{V_3}{V_1} - \frac{11\,\varphi''(V)}{3840\,\varphi'(V)}\,\frac{V_2^2}{V_1^2} + \biggl(\frac{5\,\varphi^{(3)}(V)}{768\,\varphi'(V)} - \frac{29\,\varphi''(V)^2}{7680\,\varphi'(V)^2}\biggr)V_2 \\ &+ \biggl(\frac{\varphi^{(4)}(V)}{384\,\varphi'(V)} + \frac{\varphi''(V)^3}{11520\,\varphi'(V)^3} - \frac{\varphi^{(3)}(V)\,\varphi''(V)}{384\,\varphi'(V)^2}\biggr)V_1^2\,. \end{aligned} \tag{81}$$

In the next section we will show that this function, and also the higher F_g^{φ}, are equal to a power of V_1 times a weighted homogeneous polynomial in the variables V_{i+1}/V_1^{i+1} and $d^i(\log\varphi'(V))/dV^i$ (eqs (116) and (119)).

From the last statement in Proposition 2 we know that for each $g\geq 1$, $\mathcal{F}_g^{\varphi}(\mathbf{T})$ is a deformation of $\mathcal{F}_g^{\rm WK}(\mathbf{T})$, as well as that $F_g^{\varphi}(V_0,\dots,V_{3g-2})$ is a deformation of $F_g^{\rm WK}(V_1,\dots,V_{3g-2})$. For $g=1,2$, this is obvious from (78), (81). An alternative way to see this e.g. in genus $g=1$ is from the identity

$$\mathcal{F}_1^{\varphi}(\mathbf{T})-\mathcal{F}_1^{\rm WK}(\mathbf{T}) \;=\; \frac{1}{16}\,\log\varphi'\biggl(\frac{\partial^2\mathcal{F}_0^{\rm WK}(\mathbf{T})}{\partial T_0^2}\biggr)\,. \tag{82}$$

6. The loop equation for the WK mapping free energy

This section devotes to the derivation of the loop equations for the WK mapping free energy.

Following [45], introduce the following creation and annihilation operators:

$$a_p \;=\; \begin{cases} \epsilon \, \frac{\partial}{\partial t_{p-1/2}} \,, & p>0\,, \\ \epsilon^{-1}\,(-1)^{p+1/2}\,(t_{-p-1/2}-\delta_{p,-3/2})\,, & p<0\,, \end{cases} \tag{83}$$

where p is a half integer. Let

$$f \;=\; \sum_{p\in\mathbb{Z}+\frac12} a_p \int_0^\infty e^{-\lambda z}\, z^{p-1}dz \;=\; \sum_{p\in\mathbb{Z}+\frac12} a_p\,\Gamma(p)\,\lambda^{-p}\,. \tag{84}$$

Then

$$\partial_\lambda(f) \;=\; -\sum_{p\in\mathbb{Z}+\frac12} a_p\,\Gamma(p+1)\,\lambda^{-p-1} \;=:\; -\sqrt{\pi}\,\epsilon\,A \,-\, \frac{\sqrt{\pi}}{\epsilon}\,B\,, \tag{85}$$

with

$$A \;=\; \sum_{m\geq 0} \frac{(2m+1)!!}{2^{m+1}}\,\lambda^{-m-3/2}\,\frac{\partial}{\partial t_m}\,, \tag{86}$$

$$B \;=\; \sum_{m\geq 0} \frac{2^m}{(2m-1)!!}\,\lambda^{m-1/2}\,(t_m-\delta_{m,1})\,. \tag{87}$$

It can then be verified that

$$\sum_{k\geq -1} \frac{L_k}{\lambda^{k+2}} \;:=\; (T(\lambda))_{\leq -1}\,, \tag{88}$$

where L_k, $k\geq -1$, are the operators defined in (56), and

$$T(\lambda) \;:=\; \frac{1}{2\pi}\;:(\partial_\lambda f)^2:\;+\frac{1}{16\,\lambda^2} \;=\; \frac{\epsilon^2}{2}\,A^2 \,+\, B\circ A \,+\, \frac{1}{2\epsilon^2}\,B^2 \,+\, \frac{1}{16\,\lambda^2}\,. \tag{89}$$

The Virasoro constraints (58) can now be written as

$$(T(\lambda))_-(Z(\mathbf{t};\epsilon)) \;=\; 0\,. \tag{90}$$

Here "$-$" means taking the negative power of λ. By definition we have

$$(T(\lambda))_-\big(Z^\varphi(\mathbf{T};\epsilon)\big) \;=\; 0\,. \tag{91}$$

Dividing both sides of (91) by Z^φ and taking the coefficients of ϵ^{-2}, we have

$$(B\circ A)_-\big(\mathcal{F}_0^\varphi(\mathbf{T})\big) \,+\, \frac12\,\big(A\big(\mathcal{F}_0^\varphi(\mathbf{T})\big)\big)^2 \,+\, \frac12\,(B^2)_- \;=\; 0\,. \tag{92}$$

For $k\ge 0$, applying ∂_X^{k+2} on both sides of the equality (92), one obtains

$$
\begin{aligned}
(93)\qquad & \Big(A\big(\mathcal{F}_0^{\varphi}\big)\circ A+(B\circ A)_-\Big)(V_k)\\
&= -\sum_{m=1}^{k}\binom{k}{m}\,\partial_X^{m-1}\big(A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)\,\partial_X^{k-m}(A(V))\\
&\quad -\partial_X^{k}\Big((k+2)\big(B_X\circ A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)_- \,+\, \big(A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)^2 \,+\, \big((B_X)^2\big)_-\Big)\\
&= -\sum_{m=1}^{k}\binom{k}{m}\,\Big(\partial_X^{m-1}\big(B_X+A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)\,\partial_X^{k-m+1}\big(B_X+A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)\Big)_-\\
&\quad -\partial_X^{k}\Big(\big(B_X+A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)^2\Big)_-\,.
\end{aligned}
$$

Here $\mathcal{F}_0^{\varphi}:=\mathcal{F}_0^{\varphi}(\mathbf{T})=\mathcal{F}_0(\mathbf{T})$.

Introduce

$$
(94)\qquad \mathcal{F}_{\text{h.g.}}^{\varphi} \;=\; \mathcal{F}_{\text{h.g.}}^{\varphi}(\mathbf{T};\epsilon) \;:=\; \mathcal{F}^{\varphi}(\mathbf{T};\epsilon)-\epsilon^{-2}\mathcal{F}_0(\mathbf{T}) \;=\; \sum_{g\ge 1}\epsilon^{2g-2}\mathcal{F}_g^{\varphi}(\mathbf{T})\,.
$$

Here and below "h.g." stands for higher genera. Dividing both sides of (91) by Z^{φ} and taking the coefficients of nonnegative power of ϵ, we find

$$
\begin{aligned}
(95)\qquad & (B\circ A)_-\big(\mathcal{F}_{\text{h.g.}}^{\varphi}\big) \,+\, \frac{\epsilon^2}{2}\Big(\big(A\big(\mathcal{F}_{\text{h.g.}}^{\varphi}\big)\big)^2 \,+\, A^2\big(\mathcal{F}_{\text{h.g.}}^{\varphi}\big)\Big)\\
&+\, A\big(\mathcal{F}_0^{\varphi}\big)\,A\big(\mathcal{F}_{\text{h.g.}}^{\varphi}\big) \,+\, \frac{1}{2}\,A^2\big(\mathcal{F}_0^{\varphi}\big) \,+\, \frac{1}{16\,\lambda^2} \;=\; 0\,.
\end{aligned}
$$

Substituting the jet representation (78), (79) for $\Delta\mathcal{F}^{\varphi}$ into (95), we obtain

$$
\begin{aligned}
(96)\qquad & -\sum_{k\ge 0}\sum_{m=1}^{k}\binom{k}{m}\Big(\partial_X^{m-1}\big(B_X+A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)\,\partial_X^{k-m+1}\big(B_X+A(\mathcal{F}_{0X}^{\varphi})\big)\Big)_-\,\frac{\partial\mathcal{F}_{\text{h.g.}}^{\varphi}}{\partial V_k}\\
&-\sum_{k\ge 0}\partial_X^{k}\Big(\big(B_X+A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)^2\Big)_-\,\frac{\partial\mathcal{F}_{\text{h.g.}}^{\varphi}}{\partial V_k}\\
&+\,\frac{\epsilon^2}{2}\sum_{q_1,q_2\ge 0}\partial_X^{q_1+1}\big(A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)\,\partial_X^{q_2+1}\big(A\big(\mathcal{F}_{0X}^{\varphi}\big)\big)\,\Big(\frac{\partial\mathcal{F}_{\text{h.g.}}^{\varphi}}{\partial V_{q_1}}\frac{\partial\mathcal{F}_{\text{h.g.}}^{\varphi}}{\partial V_{q_2}}+\frac{\partial^2\mathcal{F}_{\text{h.g.}}^{\varphi}}{\partial V_{q_1}\partial V_{q_2}}\Big)\\
&+\,\frac{\epsilon^2}{2}\sum_{m\ge 0}A^2(V_m)\,\frac{\partial\mathcal{F}_{\text{h.g.}}^{\varphi}}{\partial V_m} \,+\, \frac{1}{2}\,A^2\big(\mathcal{F}_0^{\varphi}\big) \,+\, \frac{1}{16\,\lambda^2} \;=\; 0\,.
\end{aligned}
$$

Here we also used (93). We note that

$$A\big(\mathcal{F}^{\varphi}_{0X}\big) \;=\; \sum_{a,\,i\ge 0} \frac{(2a+1)!!}{2^{a+1}} \, \frac{E(\mathbf{t})^{a+i+1}}{a!\,i!\,(a+i+1)} \, \frac{\partial t_i}{\partial X} \, \lambda^{-a-3/2}\,, \tag{97}$$

$$A^2\big(\mathcal{F}^{\varphi}_{0}\big) \;=\; \frac{1}{8\,(\lambda-\varphi(E(\mathbf{T})))^2} \;-\; \frac{1}{8\,\lambda^2}\,, \tag{98}$$

$$B_X \;=\; \sum_{i\ge 0} \frac{\partial B}{\partial t_i}\frac{\partial t_i}{\partial X} \;=\; \sum_{b\ge 0} \frac{2^b\,b!}{(2b-1)!!}\,\lambda^{b-1/2}\,\mathrm{Coef}\bigg(x^b, \frac{1}{\sqrt{\varphi'(\varphi^{-1}(x))}}\bigg)\,. \tag{99}$$

It follows from the equality (97) and Lemma 1 that

$$A\big(\mathcal{F}^{\varphi}_{0X}\big) \;=\; \frac{1}{2}\int_0^{E(\mathbf{t})} \frac{1}{(\lambda-x)^{3/2}}\,\frac{dx}{\sqrt{\varphi'(\varphi^{-1}(x))}} \;=\; \frac{1}{2}\int_0^{E(\mathbf{T})} \frac{\sqrt{\varphi'(x)}}{(\lambda-\varphi(x))^{3/2}}\,dx\,. \tag{100}$$

Together with (99) we find via integration by parts that $B_X + A(\mathcal{F}^{\varphi}_{0X})$ admits the following explicit Puiseux expansion as $\lambda\to\varphi(V)$:

$$\begin{aligned} & B_X \;+\; A\big(\mathcal{F}^{\varphi}_{0X}\big) \\ &\quad = \Bigg(\sum_{k\ge -1} \frac{2^{k+1}}{(2k+1)!!}\,(\lambda-\varphi(V))^{k+1/2}\,\bigg(\frac{1}{\varphi'(V)}\,\partial_V\bigg)^{k+1}\bigg(\frac{1}{\sqrt{\varphi'(V)}}\bigg)\Bigg)\Bigg|_{V=E(\mathbf{T})}\,. \end{aligned} \tag{101}$$

Then by noticing that the genus g part of equation (96) admits a Laurent expansion as $\lambda\to\varphi(V)$ and that the vanishing of the coefficients of negative powers in λ is equivalent to the vanishing of the coefficients of negative powers in $\lambda-\varphi(V)$ in the Laurent expansion, we arrive at

$$\begin{aligned} & -\sum_{k\ge 0}\Bigg(\partial^k\big(\mathcal{W}(\lambda)^2\big) \;+\; \sum_{j=1}^{k}\binom{k}{j}\Big(\partial^{j-1}\big(\mathcal{W}(\lambda)\big)\,\partial^{k+1-j}\big(\mathcal{W}(\lambda)\big)\Big)\Bigg)^{-}\,\frac{\partial\mathcal{F}^{\varphi}_{\mathrm{h.g.}}}{\partial V_k} \\ &\quad +\; \frac{\epsilon^2}{2}\sum_{k,\ell\ge 0}\partial^{k+1}\big(\mathcal{W}(\lambda)\big)\,\partial^{\ell+1}\big(\mathcal{W}(\lambda)\big)\bigg(\frac{\partial^2\mathcal{F}^{\varphi}_{\mathrm{h.g.}}}{\partial V_k\partial V_\ell} \;+\; \frac{\partial\mathcal{F}^{\varphi}_{\mathrm{h.g.}}}{\partial V_k}\frac{\partial\mathcal{F}^{\varphi}_{\mathrm{h.g.}}}{\partial V_\ell}\bigg) \\ &\quad +\; \frac{\epsilon^2}{16}\sum_{k\ge 0}\partial^{k+2}\bigg(\frac{1}{(\lambda-\varphi(V))^2}\bigg)\frac{\partial\mathcal{F}^{\varphi}_{\mathrm{h.g.}}}{\partial V_k} \;+\; \frac{1}{16}\,\frac{1}{(\lambda-\varphi(V))^2} \;=\; 0\,, \end{aligned} \tag{102}$$

where $\partial = \sum_k V_{k+1}\partial/\partial V_k$,

$$\mathcal{W}(\lambda) \;:=\; \sum_{s\ge 0}\frac{2^s}{(2s-1)!!}\,(\lambda-\varphi(V))^{s-1/2}\bigg(\frac{1}{\varphi'(V)}\partial_V\bigg)^{s}\bigg(\frac{1}{\sqrt{\varphi'(V)}}\bigg)\,, \tag{103}$$

and $(\bullet)^-$ means taking terms having negative powers of $\lambda-\varphi(V)$.

We have the following theorem.

Theorem 7. *The generating function*

$$F^{\varphi}_{\text{h.g.}} \;=\; F^{\varphi}_{\text{h.g.}}(\epsilon) \;=\sum_{g\geq 1} \epsilon^{2g-2}\, F^{\varphi}_{g}(V_0,\dots,V_{3g-2}) \tag{104}$$

satisfies

$$\begin{aligned} & -\sum_{k\geq 0}\Bigl(\partial^{k}\bigl(W^2\bigr) \,+\, \sum_{j=1}^{k}\binom{k}{j}\,\partial^{j-1}\bigl(W\bigr)\,\partial^{k+1-j}\bigl(W\bigr)\Bigr)^{-}\;\frac{\partial F^{\varphi}_{\text{h.g.}}}{\partial V_k} \\ & +\,\frac{\epsilon^2}{2}\sum_{k,\ell\geq 0}\Bigl(\partial^{k+1}\bigl(W\bigr)\,\partial^{\ell+1}\bigl(W\bigr)\Bigr)^{-}\Bigl(\frac{\partial^2 F^{\varphi}_{\text{h.g.}}}{\partial V_k\partial V_\ell} \,+\, \frac{\partial F^{\varphi}_{\text{h.g.}}}{\partial V_k}\frac{\partial F^{\varphi}_{\text{h.g.}}}{\partial V_\ell}\Bigr) \\ & +\,\frac{\epsilon^2}{16}\sum_{k\geq 0}\partial^{k+2}\Bigl(\frac{1}{\Delta^2}\Bigr)\,\frac{\partial F^{\varphi}_{\text{h.g.}}}{\partial V_k} \,+\, \frac{1}{16}\,\frac{1}{\Delta^2} \;=\; 0\,, \end{aligned} \tag{105}$$

where W is the element in

$$\varphi'(V)^{-1/2}\Delta^{-1/2}\mathbb{Q}\bigl[\varphi'(V)^{\pm 1},\varphi''(V),\varphi'''(V),\dots\bigr][[\Delta]] \tag{106}$$

defined by

$$W \;:=\; \sum_{s\geq 0}\frac{2^s}{(2s-1)!!}\Bigl(\Bigl(\frac{1}{\varphi'(V)}\frac{\partial}{\partial V}\Bigr)^{s}\Bigl(\frac{1}{\sqrt{\varphi'(V)}}\Bigr)\Bigr)\,\Delta^{s-\frac12}\,, \tag{107}$$

the operator ∂ is defined on functions of $\Delta, V_0, V_1, V_2, \dots$ by

$$\partial \;=\; -\varphi'(V)\,V_1\,\frac{\partial}{\partial \Delta} \,+\, \sum_{k\geq 0} V_{k+1}\,\frac{\partial}{\partial V_k}\,, \tag{108}$$

and $(\bullet)^{-}$ means taking terms having negative powers of Δ. Moreover, the solution to (105) is unique up to a sequence of additive constants which can be uniquely fixed by (78) and the following equation:

$$\sum_{k\geq 1} k\,V_k\,\frac{\partial F^{\varphi}_{g}}{\partial V_k} \;=\; (2g-2)\,F^{\varphi}_{g} \,+\, \frac{\delta_{g,1}}{24}\,, \quad g\geq 1\,. \tag{109}$$

Proof. Observing that $\mathcal{W}(\lambda) = W|_{\Delta\mapsto\lambda-\varphi(V)}$, equation (105) is just a rewriting of (102).

To see uniqueness, by taking coefficients of powers of ϵ^{2g-2}, $g \ge 1$, in the loop equation (105) we see that the loop equation (105) is equivalent to

$$\sum_{k\ge 0}\Big(\partial^k(W^2) + \sum_{j=1}^{k}\binom{k}{j}\partial^{j-1}(W)\,\partial^{k+1-j}(W)\Big)^{-}\frac{\partial F_g^\varphi}{\partial V_k} \tag{110}$$
$$= \frac{1}{2}\sum_{k,\ell\ge 0}\big(\partial^{k+1}(W)\,\partial^{\ell+1}(W)\big)^{-}\Big(\frac{\partial^2 F_{g-1}^\varphi}{\partial V_k\partial V_\ell} + \sum_{m=1}^{g-1}\frac{\partial F_m^\varphi}{\partial V_k}\frac{\partial F_{g-m}^\varphi}{\partial V_\ell}\Big)$$
$$+ \frac{1}{16}\sum_{k\ge 0}\partial^{k+2}\Big(\frac{1}{\Delta^2}\Big)\frac{\partial F_{g-1}^\varphi}{\partial V_k} + \frac{1}{16}\frac{1}{\Delta^2}\,\delta_{g,1}\,, \qquad g\ge 1$$

(setting $F_0^\varphi = 0$). For each $g \ge 1$, since F_g^φ is a function of $V_0, \ldots, V_{3g-2}$ the sum $\sum_k$ on the left-hand side of (110) is actually is finite sum, and by comparing coefficients of negatives powers of Δ we find that (110) is equivalent to the following triangular inhomogeneous linear system for the gradients of F_g^φ:

$$C_g\left(\frac{\partial F_g^\varphi}{\partial V_0}, \ldots, \frac{\partial F_g^\varphi}{\partial V_{3g-2}}\right)^T = M_g\,, \tag{111}$$

where M_g is a column vector which is determined by $F_1^\varphi, \ldots, F_{g-1}^\varphi$ and W, and C_g is an upper triangular matrix determined by W. Moreover, by a straightforward calculation we find

$$\det C_g = \prod_{j=0}^{3g-2}\frac{(2j+1)!!}{2^j}\,\varphi'(V)^{j-1} \ne 0\,, \tag{112}$$

which implies that (110) gives a recursive formula for the gradients of F_g^φ, $g \ge 1$. For $g = 1$, equation (110) reads

$$\left(\frac{3}{2}\frac{V_1}{\Delta^2} - \frac{3}{2}\frac{\varphi''(V)}{\varphi'(V)^2}\frac{V_1}{\Delta}\right)\frac{\partial F_1^\varphi}{\partial V_1} + \frac{1}{\varphi'(V)}\frac{1}{\Delta}\frac{\partial F_1^\varphi}{\partial V_0} = \frac{1}{16\,\Delta^2}\,.$$

By equating the coefficients of Δ^{-1} and Δ^{-2} to 0, we get

$$\frac{\partial F_1^\varphi}{\partial V_1} = \frac{1}{24\,V_1}\,, \quad \frac{\partial F_1^\varphi}{\partial V} = \frac{3}{2}V_1\frac{\varphi''(V)}{\varphi'(V)}\frac{\partial F_1^\varphi}{\partial V_1} = \frac{1}{16}\frac{\varphi''(V)}{\varphi'(V)}\,, \tag{113}$$

which agrees with (78). For $g \ge 2$, the homogeneity (109) fixes F_g^φ by its gradients. □

When $g = 2$, the expression of F_2^φ obtained using a computer algorithm designed from the above theorem coincides with the one given by (81). We have made this double check also for $g = 3, 4, 5$ with a simple home computer.

We call (96) or (105) or (110) the *Dubrovin–Zhang type loop equation for the WK mapping partition function*, for short, the *loop equation*.

Remark 4. We note that the existence of solution to the loop equation (96) is a non-trivial fact. Our construction proves this existence.

Remark 5. For the case that $\varphi(V)=V$, the loop equation (105) reduces to

$$\begin{aligned} -\sum_{k\geq 0}\biggl(\partial^k\Bigl(\frac{1}{\Delta}\Bigr)+\sum_{m=1}^{k}\binom{k}{m}\partial^{m-1}\Bigl(\frac{1}{\sqrt{\Delta}}\Bigr)\partial^{k-m+1}\Bigl(\frac{1}{\sqrt{\Delta}}\Bigr)\biggr)\frac{\partial F^{\varphi}_{\rm h.g.}}{\partial V_k} \\ +\frac{\epsilon^2}{2}\sum_{k_1,\,k_2\geq 0}\partial^{k_1+1}\Bigl(\frac{1}{\sqrt{\Delta}}\Bigr)\partial^{k_2+1}\Bigl(\frac{1}{\sqrt{\Delta}}\Bigr)\biggl(\frac{\partial^2 F^{\varphi}_{\rm h.g.}}{\partial V_k\partial V_\ell}+\frac{\partial F^{\varphi}_{\rm h.g.}}{\partial V_k}\frac{\partial F^{\varphi}_{\rm h.g.}}{\partial V_\ell}\biggr) \\ +\frac{\epsilon^2}{16}\sum_{k\geq 0}\partial^{k+2}\Bigl(\frac{1}{\Delta^2}\Bigr)\frac{\partial F^{\varphi}_{\rm h.g.}}{\partial V_k}+\frac{1}{16\,\Delta^2}=0\,. \end{aligned}$$

This loop equation coincides with the loop equation for the Witten–Kontsevich partition function, derived by Dubrovin and Zhang in [45].

Introduce the polynomial ring

$$\mathcal{R}=\mathbb{Q}[w_1,w_2,\dots;\ell_1,\ell_2,\dots]\,. \tag{114}$$

As a vector space $\mathcal{R}$ decomposes into a direct sum of homogeneous subspaces

$$\mathcal{R}=\oplus_{m\geq 0}\mathcal{R}^{[m]}\,, \tag{115}$$

where elements in $\mathcal{R}^{[m]}$ are weighted homogeneous polynomials of degree m in variables w_i and ℓ_i of weight i ($i\geq 1$). In the remainder of this section we will give a more elementary description of the functions F^{φ}_g by showing that

$$F^{\varphi}_g(V,V_1,\dots,V_{3g-2})=V_1^{2g-2}\,P_g\Bigl(\frac{V_2}{V_1^2},\frac{V_3}{V_1^3},\dots;l_1(V),l_2(V),\dots\Bigr)\quad(g\geq 2)\,, \tag{116}$$

where $l_k(V)$ is defined by

$$l_k(V)=\Bigl(\frac{d}{dV}\Bigr)^k(\log\varphi'(V))\,,\quad k\geq 0\,, \tag{117}$$

and $P_g=P_g(w_1,w_2,\dots;\ell_1,\ell_2,\dots)$ is a polynomial in $\mathcal{P}^{[3g-3]}$. For instance, for $g=2$,

$$\begin{aligned} P_2=\frac{w_3}{1152}-\frac{7\,w_1\,w_2}{1920}+\frac{w_1^3}{360}+\frac{\ell_1\,w_2}{320}-\frac{11\,\ell_1\,w_1^2}{3840} \\ +\Bigl(\frac{5\,\ell_2}{768}+\frac{7\,\ell_1^2}{2560}\Bigr)w_1+\Bigl(\frac{\ell_1\,\ell_2}{192}+\frac{\ell_1^3}{11520}+\frac{\ell_3}{384}\Bigr)\,, \end{aligned} \tag{118}$$

which is much simpler to read than the equivalent expression (81) for F_2^φ and belongs to $\mathcal{R}^{[3]}$. More generally, we will show that

$$\frac{\partial F_g^\varphi}{\partial V_k} \;=\; V_1^{2g-2-k}\, P_{g,k}\Big(\frac{V_2}{V_1^2},\frac{V_3}{V_1^3},\dots;l_1(V),l_2(V),\dots\Big) \quad (0\le k\le 3g-2) \tag{119}$$

for some polynomials $P_{g,k}$ in $\mathcal{R}^{[3g-2-k]}$. Notice that this formula, unlike (116), is true also in genus 1, with

$$P_{1,0} \;=\; \frac{\ell_1}{16}\,, \quad P_{1,1} \;=\; \frac{1}{24}\,, \tag{120}$$

as we see immediately from equation (78).

Let us introduce further some notations. First, we can show that $\partial^k(W)$ for every $k\ge 1$ is $\frac{V_1^k}{\sqrt{\varphi'(V)\Delta}}$ times a polynomial in $\frac{1}{y}=\frac{\varphi'(V)}{\Delta}$, namely,

$$\partial^k(W) \;=\; \frac{V_1^k}{\sqrt{\varphi'(V)\Delta}}\sum_{n=1}^{k} M_{k,n}\Big(\frac{V_2}{V_1^2},\dots,\frac{V_{k+1-n}}{V_1^{k+1-n}};l_1(V),\dots,l_{k-n}(V)\Big)\,y^{-n}\,, \tag{121}$$

where $M_{k,n}(w_1,\dots,w_{k-n};\ell_1,\dots,\ell_{k-n})\in\mathcal{R}^{[k-n]}$ with $M_{1,n}=\delta_{n,1}/2$. Second, for $k\ge 0$, we define $Y_k=\sum_{j=0}^k\binom{k+1}{j+1}\partial^j(W)\partial^{k-j}(W)$. Then we have that there exist

$$Y_{k,n}(w_1,\dots,w_{k-1};\ell_1,\dots,\ell_n)\in\mathcal{R}^{[n]}\,,$$

such that

$$Y_k \;=\; \frac{V_1^k}{\varphi'(V)\,\Delta}\sum_{n\ge 0} Y_{k,n}\Big(\frac{V_2}{V_1^2},\dots,\frac{V_k}{V_1^k};l_1(V),\dots,l_n(V)\Big)\,y^{n-k} \tag{122}$$

with $Y_{k,0}=(2k+1)!!/2^k$, $k\ge 0$. Third, for $k\ge 1$ we have

$$\partial^k\Big(\frac{1}{\Delta^2}\Big) \;=:\; \frac{V_1^k}{\varphi'(V)\Delta}\sum_{m=3}^{k+2} Q_{k,m}\Big(\frac{V_2}{V_1^2},\dots,\frac{V_{k+3-m}}{V_1^{k+3-m}};l_1(V),\dots,l_{k+2-m}(V)\Big)\,y^{1-m}\,, \tag{123}$$

for some $Q_{k,m}=Q_{k,m}(w_1,\dots,w_{k+2-m};\ell_1,\dots,\ell_{k+2-m})\in\mathcal{P}^{[k+2-m]}$ with $Q_{1,3}=2$.

We also introduce the first-order differential operators $D_k^{[m]}:\mathcal{R}\to\mathcal{R}$ by

$$D_k^{[m]} \;:=\; D_k \,+\, \delta_{k,1}\,m\,,$$

where D_k is the derivation defined by

$$D_0 \;=\; \sum_{i\ge1}\ell_{i+1}\frac{\partial}{\partial\ell_i}\,,\quad D_1 \;=\; -\sum_{i\ge1}(i+1)w_i\frac{\partial}{\partial w_i}\,,\quad D_k \;=\; \frac{\partial}{\partial w_{k-1}}\ (k\ge2)\,.$$

The operators D_k, $k\ge0$, simply correspond to $V_1^k\frac{\partial}{\partial V_k}$ when applied to the function $P(V_2/V_1^2,V_3/V_1^3,\dots;l_1(V),l_2(V),\dots)$.

Theorem 8. *The functions* F_g^φ *and* $\partial F_g^\varphi/\partial V_k$ *are given by equations* (116) *and* (119), *where the polynomials* $P_{g,k} \in \mathcal{R}^{[3g-2-k]}$ $(0 \le k \le 3g-2)$ *are defined by the initial values* (120) *and the recursion*

(124)
$$\frac{(2k+1)!!}{2^k} P_{g,k} \;=\; \frac{1}{16} \sum_{\ell=0}^{3g-5} Q_{\ell+2,k+1}\, P_{g-1,\ell} - \sum_{j=k+1}^{3g-2} Y_{j,j-k}\, P_{g,j}$$
$$+\; \frac{1}{2} \sum_{k_1,k_2=0}^{3g-5} \sum_{\substack{1\le n_1\le k_1+1\\ 1\le n_2\le k_2+1\\ n_1+n_2=k}} M_{k_1+1,n_1}\, M_{k_2+1,n_2} \left(D_{k_2}^{[2g-4-k_1]}(P_{g-1,k_1}) \,+\, \sum_{m=1}^{g-1} P_{m,k_1} P_{g-m,k_2} \right)$$

for $g \ge 2$, *and the polynomial* $P_g \in \mathcal{R}^{[3g-3]}$ *is defined by*

$$P_g \;=\; \frac{1}{2g-2} \left(P_{g,1} + \sum_{k\ge 2} k\, w_{k-1}\, P_{g,k} \right) \qquad (g \ge 2)\,. \tag{125}$$

Moreover, the polynomials P_g *and* $P_{g,k}$ *are related by*

$$P_{g,k} \;=\; \big(D_k + (2g-2)\delta_{k,1}\big)\, P_g \qquad (g \ge 2)\,. \tag{126}$$

Proof. Substituting (121)–(123) in the loop equation (110), and comparing the coefficients of powers of y we obtain (124). Dividing (109) by V_1^{2g-2}, we find that for $g \ge 2$ the polynomials P_g can be constructed by $P_{g,k}$ by (125). □

Remark 6. Formulas (120) and (124) define $P_{g,k}$ for all g and k explicitly by induction. But the fact that these polynomials and the polynomial P_g defined by (125) are related by (126) is not at all obvious from this definition. This corresponds to our previous Remark 4 about the existence of a solution to the loop equation.

Remark 7. Note that the functions $M_k(V_0, \ldots, V_k)$ given in Lemma 3 for $k \ge 1$ have the following more accurate form

$$\frac{M_k(V_0, \ldots, V_k)}{\varphi'(V)^{\frac{k}{2}+1} V_1^k} \;=\; N_k\left(\frac{V_2}{V_1^2}, \ldots, \frac{V_k}{V_1^k}; l_1(V), \ldots, l_{k-1}(V) \right), \tag{127}$$

where $N_k = N_k(w_1, \ldots, w_{k-1}; \ell_1, \ldots, \ell_{k-1}) \in \mathcal{R}^{[k-1]}$ with $N_1 = 1$. Then by using (39) and (74) we get the expressions of the polynomials P_g, $g \ge 2$, from F_g^{WK}. For the reader's convenience, let us provide here the expressions for the first few N_k, $k \ge 1$:

$$N_1 \;=\; 1\,, \quad N_2 \;=\; 2\,\ell_1 \,+\, w_1\,, \quad N_3 \;=\; \frac{25}{4}\,\ell_1^2 \,+\, \frac{5}{2}\,\ell_2 \,+\, \frac{15}{2}\,\ell_1\, w_1 \,+\, w_2\,. \tag{128}$$

7. Review of Hamiltonian and bihamiltonian evolutionary PDEs

In this section we review basic terminologies about evolutionary PDEs and Poisson structures, referring to [30, 36, 45] for more details. The evolutionary PDEs considered in this paper are always in (1+1) dimensions, meaning that the unknown functions have one space variable and one time variable, and also, the number of unknown functions will be one. (We note that most of the terminology reviewed in this section and the classification projects in this section and the next can be generalized without much difficulty to the case when there are several unknown functions, often called the multicomponent case; we also note that in the last section of this paper, a more general situation is actually briefly discussed.). For readers from other areas, we also recall that "evolutionary" simply means that the PDE expresses the partial derivative of the unknown function with respect to the time variable as a function of the partial derivatives with respect to the space variable.

Let $\mathcal{A}_U = \mathcal{S}(U)[U_1, U_2, \cdots]$ be the differential polynomial ring of U, where $\mathcal{S}(U)$ is some suitable ring of functions on U. For instance, $\mathcal{S}(U)$ could be $\mathcal{O}_c(U)$, the ring of power series in $U - c$ for some constant c (we often take $c = 0$). Let $\partial := \sum_{m\geq 0} U_{m+1}\partial/\partial U_m$ be a derivation. When U is taken as a function of X, we identify U_m with $\partial^m U/\partial X^m$, $m \geq 0$, and ∂ with $\partial/\partial X$. Define a gradation deg on $\mathcal{A}_U$ by the degree assignments $\deg U_m = m$ $(m \geq 1)$, and we use $\mathcal{A}_U^{[k]}$ to denote the set of elements in $\mathcal{A}_U$ that are graded homogeneous of degree k with respect to deg. For $\ell \in \mathbb{Z}$, we also denote

$$\mathcal{A}_U[[\epsilon]]_\ell = \{a \in \mathcal{A}_U[[\epsilon]] \,|\, \mathrm{gr}\, a = \ell\, a\}\,, \tag{129}$$

where

$$\mathrm{gr} \;=\; -\epsilon\,\frac{\partial}{\partial\epsilon} + \sum_{m\geq 1} m\,U_m\,\frac{\partial}{\partial U_m}\,.$$

An element M in $\mathcal{A}_U[[\epsilon]]_0$ can be written in the form $M = \sum_{k\geq 0} M^{[k]}(U, U_1, \ldots, U_k)\epsilon^k$, where $M^{[k]}(U, U_1, \ldots, U_k) \in \mathcal{A}_U^{[k]}$, $k \geq 0$.

A derivation $D : \mathcal{A}_U[[\epsilon]] \to \mathcal{A}_U[[\epsilon]]$ is called *admissible* if it commutes with ∂ and ϵ. Following Dubrovin and Novikov [28, 38, 39, 88], we call an admissible derivation $D : \mathcal{A}_U[[\epsilon]] \to \mathcal{A}_U[[\epsilon]]$ a *derivation of hydrodynamic type* if $D(U)$ has the form

$$D(U) \;=\; S(U)\,U_1\,, \quad S(U) \in \mathcal{S}(U)\,. \tag{130}$$

If we replace D by $\partial/\partial T$ and think of U as a function of X and T, then we get a *one-component evolutionary PDE of hydrodynamic type* [28, 38, 39, 88]:

$$\frac{\partial U}{\partial T} \;=\; S(U)\,\frac{\partial U}{\partial X}\,. \tag{131}$$

We sat that an admissible derivation $D : \mathcal{A}_U[[\epsilon]] \to \mathcal{A}_U[[\epsilon]]$ is of the *Dubrovin–Zhang normal form* if $D(U)$ has the form

$$D(U) = \sum_{k\geq 0} \epsilon^k S_k(U, U_1, \ldots, U_{k+1}), \quad S_k \in \mathcal{A}_U^{[k+1]}, \quad S_0(U, U_1) = S(U)\, U_1 . \tag{132}$$

We also call (132) a *perturbation* of (130), or say that (130) is the *dispersionless limit* (the $\epsilon \to 0$ limit) of (132). A derivation of the Dubrovin–Zhang normal form is called an *infinitesimal symmetry* of (132) if it commutes with (132).

For any $R(U) \in \mathcal{S}(U)$, the derivation of hydrodynamic type D', specified by

$$D'(U) = R(U)\, U_1 , \tag{133}$$

is an infinitesimal symmetry of (130). We call the following family of derivations of hydrodynamic type D_S, $S \in \mathcal{O}_c(U)$, defined by

$$D_S(U) = S(U)\, U_1 , \tag{134}$$

the *abstract local RH hierarchy* (see e.g. [30]), sometimes simply the *abstract RH hierarchy*. It is obvious that the derivations in (134) pairwise commute. When we take the countable subfamily of derivations $(S_i(U) = U^i/i!)_{i\geq 0}$ and consider U as a function of X and T_i, $i \geq 0$, then the equations $\partial U/\partial T_i = S_i(U)\partial U/\partial X$, $i \geq 0$, are nothing but the RH hierarchy (23), where we identify X with T_0 as we do before. In practice, some other interesting countable subfamily of derivations like $(\sin(ku))_{k\geq 1}$, $(e^{ku})_{k\geq 1}$, ..., can also be taken and the resulting family of equations are an integrable hierarchy which we call the *chord RH hierarchy* (sometimes simply still the *RH hierarchy*).

By a *perturbation of the abstract local RH hierarchy* (134), we mean a family of derivations $D_S(U)$, $S \in \mathcal{O}_c(U)$, each being given by (132). We say that $D_S(U)$ is *integrable* if $D_{S_2} D_{S_1}(U) = D_{S_1} D_{S_2}(U)$, $\forall\, S_1, S_2 \in \mathcal{O}_c(U)$.

Denote by $\int : \mathcal{A}_U \to \mathcal{A}_U / \partial \mathcal{A}_U$ the projection, which extends termwise to a projection on $\mathcal{A}_U[[\epsilon]]_0$. Elements in $\mathcal{A}_U[[\epsilon]]_0 / \partial \mathcal{A}_U[[\epsilon]]_{-1} =: \mathcal{F}$ are called *local functionals.* The *variational derivative* of a local function $\int h$ with respective to U is defined by

$$\frac{\delta \int h}{\delta U} = \sum_{k\geq 0} (-\partial)^k \left(\frac{\partial h}{\partial U_k} \right) . \tag{135}$$

Clearly, if $h \in \mathcal{A}_U[[\epsilon]]_0$, then $\frac{\delta \int h}{\delta U} \in \mathcal{A}_U[[\epsilon]]_0$. Also, for $a \in \mathcal{A}_U[[\epsilon]]_0$, it is known that $a \in \partial \mathcal{A}_U[[\epsilon]]_{-1}$ if and only if the right-hand side of (135) with h replaced by a vanishes, so (135) is well defined.

Let P be an operator of the form

$$P = \sum_{k\geq 0} \epsilon^k P^{[k]}, \quad P^{[k]} = \sum_{j=0}^{k+1} A_{k,j}\, \partial^j , \quad A_{k,j} \in \mathcal{A}_U^{[k+1-j]} . \tag{136}$$

Such an operator P defines a bracket $\{\,,\,\}_P : \mathcal{F}\times\mathcal{F}\to\mathcal{F}$ via

$$\left\{\int F, \int G\right\}_P \;=\; \int \frac{\delta F}{\delta u}\, P\left(\frac{\delta G}{\delta u}\right), \quad \forall\, F, G \in \mathcal{A}_U[[\epsilon]]_0\,. \tag{137}$$

This bracket is obviously bilinear. We say that $\{\,,\,\}_P$ is *Poisson* if it is anti-symmetric and satisfies the Jacobi identity. We call that P a *Poisson or hamiltonian operator* if $\{\,,\,\}_P$ is a Poisson bracket. An equivalent criterion of the operator P to be Poisson is that

$$[P,P] \;=\; 0\,,$$

where $[\,,\,]$ denotes the Schouten–Nijenhuis bracket (see e.g. [45]). The part $P^{[0]}$ in (136) is called the *dispersionless limit* of P. Obviously, if P is Poisson then $P^{[0]}$ is Poisson. A Poisson operator like $P^{[0]}$ is called a *Poisson operator of hydrodynamic type*. According to Dubrovin and Novikov [38], a Poisson operator of hydrodynamic type corresponds to a contravariant flat pseudo-Riemannian metric (true also for the multi-component case), i.e., $P^{[0]}$ must have the form

$$P^{[0]} \;=\; g(U)\,\partial_X \,+\, \frac{1}{2}\,g'(U)\,U_X\,, \tag{138}$$

where $g(U)$ is a contravariant metric (automatically flat for our one-component case).

We call (132) a *hamiltonian derivation of the Dubrovin–Zhang normal form* if there exists a Poisson operator P and an element $h\in\mathcal{A}_U[[\epsilon]]_0$, such that

$$D(U) \;=\; P\left(\frac{\delta \int h}{\delta U}\right). \tag{139}$$

We call $\int h$ the *hamiltonian* of (139) and h the *hamiltonian density*. A local functional $\int r$, $r\in\mathcal{A}_U[[\epsilon]]_0$, is called a *Casimir* for the Poisson operator P if

$$P\left(\frac{\delta \int r}{\delta U}\right) \;=\; 0\,, \tag{140}$$

with r being called the *Casimir density*.

We will call any transformation of the form

$$U \;\mapsto\; M \;=\; \sum_{k\geq 0} \epsilon^k M^{[k]}(U,U_1,\dots,U_k) \in \mathcal{A}_U[[\epsilon]]_0\,, \quad M^{[0]}(U)\in\mathcal{S}(U)^\times\,, \tag{141}$$

a *Miura-type transformation*. These transformations form a group, called the *Miura group*, which contains the local diffeomorphism group as a subgroup. For more details about Miura-type transformations see e.g. [45, 93].

A further extension is given by the *quasi-Miura transformations*

$$U \;\mapsto\; Q \;=\; \sum_{k\geq 0} \epsilon^k Q^{[k]}(U,U_1,\dots,U_{N_k})\,, \tag{142}$$

where $Q^{[k]}(U, U_1, \ldots, U_{N_k})$, $k \geq 0$, are usually still required to have polynomial dependence in $U_2, \ldots, U_{N_k}$ for some integers N_k but are now allowed to have rational dependence in U_1.

The class of derivations of the Dubrovin–Zhang normal form, the class of hamiltonian derivations of the Dubrovin–Zhang normal form and the class of Poisson operators are invariant under the Miura-type transformations [45]. In particular, the hamiltonian derivation of the Dubrovin–Zhang normal form (139) under the Miura-type transformation (141) transforms to the hamiltonian derivation of the Dubrovin–Zhang normal form given by

$$D(M) \,=\, \widetilde{P}\biggl(\frac{\delta \int h}{\delta M}\biggr)\,, \tag{143}$$

where h is understood as an element in $\mathcal{A}_M[[\epsilon^2]]_0$, and

$$\widetilde{P} \,=\, \sum_{k,\ell \geq 0} (-1)^\ell \, \frac{\partial M}{\partial U_k} \circ \partial^k \circ P \circ \partial^\ell \circ \frac{\partial M}{\partial U_\ell}\,. \tag{144}$$

To make notations compact we will often write P as $P(U)$ and $\widetilde{P}$ as $P(M)$, when U and M are related by a Miura-type transformation.

If we apply the quasi-Miura transformation (142) to (139), the Poisson operator still transforms under the rules (144) but the resulting operator could have rational dependence in M_1 and the variational derivative of the hamiltonian could have rational dependence in M_1 (here the definition of the variational derivative is extended again with the same rule (135)).

Two Poisson operators P_1, P_2 are called *compatible* if an arbitrary linear combination of P_1, P_2 is a Poisson operator. When P_1, P_2 are compatible, we call $P_2 + \lambda\, P_1$ the *Poisson pencil* associated to P_1, P_2. Following Dubrovin [29], let us start with considering the dispersionless limit $P_2^{[0]} + \lambda\, P_1^{[0]}$. Fix $P_2^{[0]}(U) + \lambda\, P_1^{[0]}(U)$ an arbitrary Poisson pencil of hydrodynamic type. According to Dubrovin [29] (see also [38, 39]), it corresponds to a flat pencil, that is, a pencil of flat contravariant pseudo-Riemannian metrics $g_2(U) + \lambda\, g_1(U)$. The associated *canonical coordinate* of the pencil $u = u(U)$ is defined by $u = g_2(U)/g_1(U)$. The flat pencil in the u-coordinate reads $u\, g(u) + \lambda\, g(u)$ with $g(u) = g_1(U) u'(U)^2$. A Poisson pencil $P_2(U) + \lambda\, P_1(U)$ with the hydrodynamic limit being $P_2^{[0]}(U) + \lambda\, P_1^{[0]}(U)$ is then characterized by the so-called *central invariant*, denoted $c(u)$, defined in [36, 73, 77]. On one hand, two Poisson pencils having the same hydrodynamic limit $P_2^{[0]}(U) + \lambda\, P_1^{[0]}(U)$ (or say the same $g(u)$) are equivalent under Miura-type transformations if and only if they have the same central invariant [36]. On the other hand, it is shown in [20, 21] (see also [76] for the case $g(u) = u$) that, for any given function $c(u)$ there exits a Poisson pencil $P_2(U) + \lambda\, P_1(U)$, with the hydrodynamic limit $P_2^{[0]}(U) + \lambda\, P_1^{[0]}(U)$ and with the central invariant $c(u)$; the proof is based on a subtle computation of the bihamiltonian cohomology introduced in [45] and developed in [36, 73]. For example, the Poisson pencil corresponding to the pair

$$(g(u), c(u)) = \Big(u, \frac{1}{24}\Big) \tag{145}$$

can be obtained from the bihamiltonian structure discovered by Magri [78] of the KdV equation (1) (see e.g. [36]).

For an element $g(U) \in \mathcal{O}_c(U)$ that is not identically zero, the abstract local RH hierarchy (134) can be written in the form:

$$D_S(U) = \Big(g(U)\,\partial + \frac{1}{2}\,g'(U)\,U_1\Big)\bigg(\frac{\delta \int h_S^{[0]}}{\delta U}\bigg), \tag{146}$$

where $h_S^{[0]}$ is a solution to the following ODE

$$g(U)\,\big(h_S^{[0]}\big)'' + \frac{1}{2}\,g'(U)\,\big(h_S^{[0]}\big)' = S(U)\,. \tag{147}$$

Obviously, up to a trivial additive constant, the $h_S^{[0]}$ is unique up to the addition of a Casimir density for the Poisson operator $g(U)\,\partial_X + \frac{1}{2}\,g'(U)\,U_X$.

Let us consider the hamiltonian perturbation of the abstract local RH hierarchy (146):

$$D_S(U) = P(U)\bigg(\frac{\delta \int h_S}{\delta U}\bigg), \quad S \in \mathcal{O}_c(U)\,, \tag{148}$$

where $P(U)$ is a Poisson operator of the form (136) with $P^{[0]}(U) = g(U)\,\partial_X + \frac{1}{2}\,g'(U)\,U_X$, and h_S are hamiltonian densities of the form

$$h_S = \sum_{k\geq 0} \epsilon^k h_S^{[k]}\,, \quad h_S^{[k]} \in \mathcal{A}_U^{[k]}\,, \tag{149}$$

with $h_S^{[0]}$ given by (147). Here, we point out that our labellings of the derivations and of the hamiltonian densities for the abstract local RH hierarchy and for its perturbation are different from the one used in [30].

According to [23, 45, 47], the Darboux theorem holds for the hamiltonian operator $P(U)$, namely, there exists a Miura-type transformation $U \mapsto M$ of the form

$$M = \sum_{k\geq 1} \epsilon^k\, M^{[k]}(U, U_1, \ldots, U_k)\,, \quad M^{[0]}(U) = \int_{U^*}^{U} \frac{1}{\sqrt{g(U')}}\,dU'\,, \tag{150}$$

reducing $P(U)$ to $P(M) = \partial$.

For simplicity, we shall consider in this paper that h_S written in the M-coordinate are power series of ϵ^2.

Before proceeding we introduce some notations. A *partition* is a non-increasing infinite sequence of non-negative integers $\mu = (\mu_1, \mu_2, \dots)$. The number of non-zero components of μ is called the *length* of μ, denoted by $\ell(\mu)$. The sum $\sum_{i\geq 1} \mu_i$ is called the *weight* of μ, denote by $|\mu|$. The set of all partitions is denoted by $\mathcal{P}$, and the set of partitions of weight d, $d \geq 0$, is denoted by $\mathcal{P}_d$. If $\ell(\mu) > 0$, we often write μ as $(\mu_1, \dots, \mu_{\ell(\mu)})$; otherwise, we write μ either as (0) or as $(\,)$. Denote $\mu+1 = (\mu_1+1, \dots, \mu_{\ell(\mu)}+1)$ if $\ell(\mu) > 0$, and $(\,)+1 = (\,)$ otherwise. We use $\mathrm{mult}_i(\mu)$ to denote the multiplicity of i in μ, $i \geq 1$, and denote $\mathrm{mult}(\mu)! = \prod_{i=1}^{\infty} \mathrm{mult}_i(\mu)!$. For any sequence of indeterminates $(y_1, y_2, \dots)$, $y_\mu := \prod_{i=1}^{\ell(\mu)} y_{\mu_i}$ (clearly, $y_{(\,)} = 1$).

S.-Q. Liu and Y. Zhang found (see also [17, 30, 34]) that performing a *canonical* [30] Miura-type transformation $M \mapsto w = M + \cdots$, that is, a Miura-type transformation keeping the Poisson operator ∂ invariant, yields the following unique standard form:

$$D_S(w) \;=\; \partial\!\left(\frac{\delta \int h_S}{\delta w}\right), \tag{151}$$

with the derivation $D_{M^{[0]}(U)}$ satisfying

$$D_{M^{[0]}(U)}(w) \;=\; \partial\!\left(\frac{\delta \int h_{M^{[0]}(U)}}{\delta w}\right), \tag{152}$$

where

$$h_{M^{[0]}(U)} \;=\; \frac{w^3}{6} - \frac{\epsilon^2}{24} a_0(w) w_1^2 \;+\; \sum_{g\geq 2} \epsilon^{2g} \sum_{\substack{\lambda\in\mathcal{P}_{2g} \\ \ell(\lambda)>1,\, \lambda_1=\lambda_2}} \alpha_\lambda(w) w_\lambda\,. \tag{153}$$

Here $a_0(w)$ and $\alpha_\lambda(w)$ with $\lambda \in \mathcal{P}_{2g}$ $(g \geq 2)$, $\ell(\lambda) > 1$, $\lambda_1 = \lambda_2$, are functions of w.

Remark 8. It was conjectured by S.-Q. Liu and Y. Zhang that if one imposes the integrability to the above standard form (153), then the functions $\alpha_\lambda(w)$ appearing in (153) are uniquely determined by $\alpha_{2^m}(w)$, $m \geq 2$, and the functions $\alpha_{2^m}(w)$, $m \geq 2$, are free functional parameters. In [34] (see also [17]) it is indicated that if one further imposes a symmetry condition [34, 45] for the hamiltonian densities (so-called *τ-symmetry*) then the functional parameters $a_0(w)$ and $\alpha_\lambda(w)$ appearing in (153) all become constants. In this paper, we will impose a new condition, as already mentioned in Introduction, i.e., to require the hamiltonian system to possess a τ-structure (see the next section for the details).

In [30], B. Dubrovin considers the bihamiltonian test for the integrable hamiltonian perturbation and obtained the following theorem.

Theorem A (Dubrovin [30]) *For $a_0(w), q(w), q'(w)$ all not identically 0, let P_1, P_2 be the Poisson operators of the form: $P_1 = \partial$, and*

$$P_2(w) \;=\; q(w)\,\partial \,+\, \frac{1}{2}\,q'(w)\,w_1 \,+\, \cdots\,, \tag{154}$$

where "$\cdots$" contains higher order terms in ϵ. The two commutativity properties

$$\left\{\int h_{S_1}, \int h_{S_2}\right\}_{P_1} \;=\; O(\epsilon^6)\,,\;\; \left\{\int h_{S_1}, \int h_{S_2}\right\}_{P_2} \;=\; O(\epsilon^6)\,,\quad \forall\, S_1,\, S_2\,, \tag{155}$$

hold if and only if

$$\alpha_{2^2}(w) \;=\; \frac{a_0(w)^2}{960}\left(5\,\frac{a_0'(w)}{a_0(w)} - \frac{q''(w)}{q'(w)}\right). \tag{156}$$

The explicit expression for the ϵ^2-term in $P_2(w)$ is given in the Appendix of [30]. We have the following proposition.

Proposition 3. *The central invariant for the pencil $P_2 + \lambda P_1$ in Theorem A is*

$$c(u) \;=\; \frac{1}{24}\,\frac{a_0(q^{-1}(u))}{q'(q^{-1}(u))}\,, \tag{157}$$

where $u = q(w)$ is the canonical coordinate for the pencil $P_2(w) + \lambda P_1(w)$.

Proof. Let us perform the following Miura-type transformation:

$$u \;=\; q(w)\,. \tag{158}$$

The Poisson operators P_1, P_2 in the u-coordinate read:

$$P_1(u) \;=\; \frac{1}{2}\,q'(q^{-1}(u))^2 \circ \partial \,+\, \frac{1}{2}\,\partial \circ q'(q^{-1}(u))^2\,, \tag{159}$$

$$P_2(u) \;=\; \frac{1}{2}\,u\,q'(q^{-1}(u))^2 \circ \partial \,+\, \frac{1}{2}\,\partial \circ u \circ q'(q^{-1}(u))^2 \,+\, \cdots\,. \tag{160}$$

Using [36, formula (1.49)] and using [30, Appendix] we obtain the expression (157) of the central invariant $c(u)$. The proposition is proved. □

8. Hamiltonian and bihamiltonian perturbations possessing a τ-structure

Driven by topological field theories and the Witten–Kontsevich theorem (see [26, 27, 29, 45, 63, 98]), the *τ-structure* for the KdV hierarchy (see [10, 25, 44, 45]) becomes an important notion in the theory of integrable systems. It still makes sense to speak of a τ-structure for more general evolutionary systems (we will give a precise definition in a moment). One of our main objects for the rest of the paper is to give conjectural classifications of hamiltonian and of bihamiltonian systems possessing a τ-structure with the help of the group $\mathcal{G}$.

It can be shown (see e.g. [10, 25, 27, 44, 45, 63, 98]) that there exist unique elements $\Omega^{\rm KdV}_{i,j}\in\mathcal{A}_u[\epsilon^2]$, $i,j\geq 0$, such that

$$\epsilon^2\,\frac{\partial^2\log\mathcal{F}^{\rm WK}(\mathbf{t};\epsilon)}{\partial t_i\partial t_j}\;=\;\Omega^{\rm KdV}_{i,j}\big|_{u_k\mapsto\partial_x^k(u^{\rm WK}(\mathbf{t};\epsilon)),\,k\geq 0}\,,\quad i,j\geq 0\,. \tag{161}$$

For instance,

$$\Omega^{\rm KdV}_{0,0}\;=\;u\,,\quad\Omega^{\rm KdV}_{0,1}\;=\;\frac{u^2}{2}+\epsilon^2\,\frac{u_2}{12}\,,\quad\Omega^{\rm KdV}_{1,1}\;=\;\frac{u^3}{3}+\epsilon^2\Big(\frac{uu_2}{6}+\frac{u_1^2}{24}\Big)+\epsilon^4\,\frac{u_4}{144}\,. \tag{162}$$

The polynomials $(\Omega^{\rm KdV}_{i,j})_{i,j\geq 0}$ form a τ-structure for the KdV hierarchy (2) and hence defines τ-functions (see [10, 25, 44, 45, 94]).

Constructively, $\Omega^{\rm KdV}_{i,j}$ can be obtained in the following way. Introduce

$$u^{\rm WK}(\mathbf{t};\epsilon)\;:=\;\epsilon^2\,\frac{\partial^2\mathcal{F}^{\rm WK}(\mathbf{t};\epsilon)}{\partial x^2}\;=\;E(\mathbf{t})\;+\;\sum_{g\geq 1}\epsilon^{2g}\,\frac{\partial\mathcal{F}^{\rm WK}_g(\mathbf{t})}{\partial x^2}\,. \tag{163}$$

Here $x=t_0$. From (74) we know that it leads to the quasi-Miura transformation

$$v\;\mapsto\;u\;=\;v\;+\;\sum_{g\geq 1}\epsilon^{2g}\,\partial^2\big(F^{\rm WK}_g\big)\,, \tag{164}$$

which transforms the abstract local RH hierarchy $D_S(v)=S(v)\,v_1$ to

$$\begin{aligned}D_S(u)\;=&\;S(u)\,u_1\;+\;\Big(\frac{S'(u)}{12}\,u_3+\frac{S''(u)}{6}\,u_1u_2+\frac{S'''(u)}{24}\,u_1^3\Big)\epsilon^2\\&+\Big(\frac{S''(u)}{240}\,u_5+\frac{S^{(3)}(u)}{80}\,u_4u_1+\frac{S^{(3)}(u)}{48}\,u_3u_2+\frac{23\,S^{(4)}(u)}{1440}\,u_3u_1^2\\&\quad+\frac{31\,S^{(4)}(u)}{1440}\,u_2^2u_1+\frac{S^{(5)}(u)}{90}\,u_2u_1^3+\frac{S^{(6)}(u)}{1152}\,u_1^5\Big)\,\epsilon^4\;+\;\cdots\,.\end{aligned} \tag{165}$$

By the Witten–Kontsevich theorem, when $S(u)=u^i/i!$ $(i\geq 0)$, equations (165) are the abstract KdV hierarchy (see (2) and [10, 25, 44, 45]). In general, D_S commutes with $D_{u^i/i!}$, $i\geq 0$. We call (165) the *abstract local KdV hierarchy.* Note that

$$\epsilon^2\,\frac{\partial^2\log\mathcal{F}^{\rm WK}(\mathbf{t};\epsilon)}{\partial t_i\partial t_j}\;=\;\frac{E(\mathbf{t})^{i+j+1}}{i!\,j!\,(i+j+1)}\;+\;\sum_{g\geq 1}\epsilon^{2g}\,\frac{\partial^2\log\mathcal{F}^{\rm WK}_g(\mathbf{t})}{\partial t_i\partial t_j}\,, \tag{166}$$

where we used (36) and (66). By (74) we know that the right-hand side of (166) can be represented by the jets $v,v_1,v_2,\cdots$. Substituting the inverse of the quasi-Miura transformation into the jet representation of the right-hand side of (166) we get $\Omega^{\rm KdV}_{i,j}$.

As in [44] (cf. [10, 45]), we say that a perturbation of the abstract local RH hierarchy (see (132)) possesses a *τ-structure* if there exist $\Omega_{S_1,S_2} \in \mathcal{A}_U[[\epsilon]]_0$, $S_1, S_2 \in \mathcal{O}_c(U)$, such that $\Omega^{[0]}_{1,1} \in \mathcal{O}_c(U)^\times$ and

$$\Omega_{S_1,S_2} \,=\, \Omega_{S_2,S_1}\,, \quad D_{S_1}(\Omega_{S_2,S_3}) \,=\, D_{S_2}(\Omega_{S_1,S_3})\,, \quad \forall\, S_1, S_2, S_3 \in \mathcal{O}_c(U)\,. \tag{167}$$

It can be easily verified that the existence of a τ-structure implies integrability [44] (cf. also [34]). More general setups for this principle are given in [96]. We refer also to a related forthcoming joint work announced in [72].

We are ready to give an axiomatic way to approach the class of (bi)-hamiltonian perturbations of the RH hierarchy possessing a τ-structure.

Lemma 5. *The class of hamiltonian perturbations of the RH hierarchy possessing a τ-structure is invariant under Miura-type transformations.*

Proof. We already know that the class of hamiltonian perturbations of the RH hierarchy is invariant under Miura-type transformations. It is also obvious that under a Miura-type transformation a τ-structure remains a τ-structure. □

Let us now consider hamiltonian perturbations of the RH hierarchy possessing a τ-structure with a fixed choice of $P^{[0]}$. First we use Miura-type transformations reducing the consideration to the standard form (151)–(153). Then with a help of a computer program we find that the requirement of existence of a τ-structure implies that the functions $\alpha_{3^2}(w)$, $\alpha_{2^13^2}(w)$, $\alpha_{4^2}(w)$, $\alpha_{2^23^2}(w)$, $\alpha_{2^14^2}(w)$, $\alpha_{5^2}(w)$ are uniquely determined by $a_0(w)$, $\alpha_{2^2}(w)$, $\alpha_{2^3}(w)$, $\alpha_{2^4}(w)$, $\alpha_{2^5}(w)$ (agreeing with Remark 8), and that the functions $\alpha_{2^2}(w)$, $\alpha_{2^3}(w)$, $\alpha_{2^4}(w)$ must have the expressions

$$\alpha_{2^2} \,=\, \frac{a_0 a_0'}{240} \,+\, q_1 a_0^3\,, \tag{168}$$

$$\alpha_{2^3} \,=\, \frac{31 a_0''' a_0^2}{96768} + \frac{527 q_1 a_0^3 a_0''}{1008} + \frac{1800 q_1^2 a_0^4 a_0'}{7} + \frac{499 q_1 a_0^2 a_0'^2}{336} + \frac{23 a_0'^3}{45360} + \frac{1613 a_0 a_0' a_0''}{967680} + q_2 a_0^6\,, \tag{169}$$

$$
\begin{aligned}
(170)\quad \alpha_{2^4} \;=\;& \frac{913 a_0''''' a_0^3}{46448640} + \frac{1795 q_1 a_0'''' a_0^4}{32256} + \frac{10357 q_1^2 a_0''' a_0^5}{168} + 25920 q_1^3 a_0^6 a_0'' \\
&+ \frac{167 q_2 a_0^6 a_0''}{105} + \frac{23087 q_1 a_0^3 a_0''^2}{40320} + 155520 q_1^3 a_0^5 a_0'^2 + \frac{12528 q_1 q_2 a_0^7 a_0'}{7} \\
&+ \frac{15635 q_1^2 a_0^3 a_0'^3}{14} + \frac{593 q_2 a_0^5 a_0'^2}{70} + \frac{20893 q_1 a_0 a_0'^4}{20160} + \frac{7733 a_0'''' a_0^2 a_0'}{30965760} \\
&+ \frac{212591 a_0''' a_0^2 a_0''}{464486400} + \frac{47953 q_1 a_0''' a_0^3 a_0'}{60480} + \frac{56519 a_0''' a_0 a_0'^2}{66355200} + \frac{48785 q_1^2 a_0^4 a_0' a_0''}{56} \\
&+ \frac{733 q_1 a_0^2 a_0'^2 a_0''}{224} + \frac{70229 a_0 a_0' a_0''^2}{58060800} + \frac{1049357 a_0'^3 a_0''}{1393459200} + q_3 a_0^9 \,,
\end{aligned}
$$

where q_1, q_2, q_3 are arbitrary parameters (independent of w) and where the arguments of the functions $a_0(w)$, $\alpha_{2^2}(w)$, $\alpha_{2^3}(w)$, $\alpha_{2^4}(w)$ have been omitted. More generally, we expect that there are unique expressions giving all α_λ as polynomials in $a_0, a_0', a_0'', \dots$ and constants q_1, q_2, ..., where q_i first appears linearly in $\alpha_{2^{i+1}}(w)$. This implies in particular that if $a_0(w)$ is a constant function, then all of the $\alpha_\lambda(w)$ are constants[4]. Moreover, we expect that except for the term $q_i a_0^{3i}$ all terms in $\alpha_{2^{i+1}}$ contain higher derivatives of a_0, so that when a_0 is a constant, then $\alpha_{2^{i+1}}$ is simply $q_i a_0^{3i}$.

We continue to consider bihamiltonian perturbations of the (local) RH hierarchy possessing a τ-structure. Of course, this class of perturbations is again invariant under Miura-type transformations (see Lemma 5). We reduce the considerations to the standard form as above, and the bihamiltonian axiom will further impose restrictions on q_i's. Note that in Theorem A, B. Dubrovin already did the bihamiltonian test for integrable hamiltonian perturbations (hamiltonian perturbations with a τ-structure belong to this class) up to order 4 in ϵ. So by using (168) and by using formula (156) of Theorem A, we find

$$
(171)\qquad q(w) \;=\; \begin{cases} C_1 \int_{w^*}^{w} a_0(w')\, dw' \,+\, C_2\,, & q_1 \,=\, 0\,, \\[2ex] C_1 \frac{1-\exp\left(-960\, q_1 \int_{w^*}^{w} a_0(w')\, dw'\right)}{960\, q_1} \,+\, C_2\,, & q_1 \,\neq\, 0\,, \end{cases}
$$

where C_1, C_2 are arbitrary constants (that can depend on q_i's) and $C_1 \neq 0$. Continuing Dubrovin's bihamiltonian test, up to the order 8 in ϵ, we find that

$$
(172)\qquad q_2 \;=\; \frac{6400}{3}\, q_1^3\,, \quad q_3 \;=\; 0\,.
$$

We expect that $q_4, q_5, \cdots$ are also determined by q_1 and $a_0(w)$. Note that since q_2, q_3 do not depend on $a_0(w)$, we can further expect this to be true for $q_4, q_5, \cdots$; with this

[4]This occurs, for example, in the presence of τ-symmetry, as shown in [16, 17, 34].

consideration, we can restrict to the simple case $a_0(w) \equiv 1$ and the corresponding bihamiltonian test allows us to compute two more values:

$$q_4 \,=\, -\frac{36805017600000}{77}\, q_1^7\,, \quad q_5 \,=\, -\frac{4561255268352000000}{7007}\, q_1^9\,. \tag{173}$$

We have the following proposition.

Proposition 4. *The central invariant for the pencil $P_2 + \lambda P_1$ is given by*

$$c(u) \,=\, \begin{cases} \frac{1}{24\, C_1}\,, & q_1 \,=\, 0\,, \\ \frac{1}{24\,(C_1 - 960\, q_1\,(u - C_2))}\,, & q_1 \,\neq\, 0\,. \end{cases} \tag{174}$$

Proof. By using Proposition 3 and the expression (171). □

Using equations (168), (169), (172), the above proposition, [36, Theorem 1.8] and [73, Theorems 1 and 2], we arrive at

Theorem 9. *A bihamiltonian perturbation of the RH hierarchy with the central invariant $c(u) \not\equiv 0$ admits a τ-structure up to the ϵ^8 approximation if and only if $1/c(u)$ is an affine-linear function of u.*

The following theorem will be proved in the next section.

Theorem 10. *The statement in Theorem 9 holds to all orders in ϵ.*

Notice that when there is a(n) (approximated) bihamiltonian structure, there is a choice of the associated Poisson pencil. Namely, consider the following change of the choice of Poisson pencil:

$$\widetilde{P}_1 \,:=\, c\, P_2 \,+\, d\, P_1\,, \quad \widetilde{P}_2 \,:=\, a\, P_2 \,+\, b\, P_1\,, \quad ad - bc \neq 0\,. \tag{175}$$

Here $a, b, c, d \in \mathbb{C}$ are constants. The canonical coordinate of $\widetilde{P}_1, \widetilde{P}_2$, denoted $\tilde{u}$, is related to u by

$$\tilde{u} \,=\, \frac{a\, u + b}{c\, u + d}\,.$$

The pair of functions $(\tilde{g}, \tilde{c})$ that characterizes the pencil $\widetilde{P}_2 + \lambda \widetilde{P}_1$ are given by

$$\tilde{g}(\tilde{u}) \,=\, \frac{(ad - bc)^2}{(c\, u + d)^3}\, g(u)\,, \tag{176}$$

$$\tilde{c}(\tilde{u}) \,=\, \frac{c\, u + d}{ad - bc}\, c(u)\,. \tag{177}$$

In particular, formula (177) was obtained in [36]. So, if the central invariant $c(u)$ of a Poisson pencil satisfies that $1/c(u)$ is an affine-linear function of u, then it is always possible to choose properly the pencil so that the central invariant is $1/24$.

Hence the above Theorem 10 can be more compactly reformulated as follows.

Theorem 10′. *A bihamiltonian perturbation of the abstract local RH hierarchy possesses a τ-structure if and only if under a proper choice of the associated Poisson pencil the central invariant is* $1/24$.

Here we recall again that according to [20, 21, 76], the existence of a bihamiltonian perturbation with central invariant $1/24$ is known (actually for arbitrary function $c(u)$ the existence is also known).

Remark 9. Theorem 10′ was known for the case of the flat-exact Poisson pencils [37], where the τ-structure is associated to *τ-symmetry* [34, 45]. The flat-exact condition implies $g(u) = u$ which is a special case in our general consideration.

9. The WK mapping hierarchy and the WK mapping universality

In this section, we introduce the hierarchy of equations associated to the WK mapping partition function, call it the *WK mapping hierarchy*, and prove it to be integrable and bihamiltonian with the central invariant $1/24$. Then we propose and prove the WK mapping universality.

9.1. **The WK mapping hierarchy.** For an arbitrary element $\varphi \in \mathcal{G}$, let $\mathcal{F}^\varphi(\mathbf{T};\epsilon)$ be the WK mapping free energy. Introduce

$$U^\varphi(\mathbf{T};\epsilon) := \epsilon^2 \frac{\partial^2 \mathcal{F}^\varphi(\mathbf{T};\epsilon)}{\partial X^2} = E(\mathbf{T}) + \sum_{g\geq 1} \epsilon^{2g} \frac{\partial \mathcal{F}_g^\varphi(\mathbf{T})}{\partial X^2}. \tag{178}$$

Here $X = T_0$, and we used (43) and Theorem 6. From Proposition 2 we know that (178) leads to a quasi-Miura transformation

$$V \mapsto U^\varphi = V + \sum_{g\geq 1} \epsilon^{2g} \partial^2\big(F_g^\varphi\big). \tag{179}$$

It transforms the abstract local RH hierarchy $D_S(V) = S(V)\, V_1$ to

$$\begin{aligned} D_S(U^\varphi) = {} & S\, U_1^\varphi + \Bigg(\frac{S'}{12} U_3^\varphi + \left(\frac{S''}{6} + \frac{S'}{8}\frac{\varphi''}{\varphi'}\right) U_1^\varphi U_2^\varphi \\ & + \left(\frac{S'''}{24} + \frac{S''}{16}\frac{\varphi''}{\varphi'} + \frac{S'}{16}\left(\frac{\varphi'''}{\varphi'} - \frac{\varphi''^2}{\varphi'^2}\right)\right)\big(U_1^\varphi\big)^3\Bigg)\epsilon^2 + \cdots, \end{aligned} \tag{180}$$

which (by Proposition 2 and the Witten–Kontsevich theorem) is a perturbation of the abstract local KdV hierarchy (165). In (180), $\varphi = \varphi(U^\varphi)$, $S = S(U^\varphi)$, and $\varphi^{(k)} = \varphi^{(k)}(U^\varphi)$, $S^{(k)} = S^{(k)}(U^\varphi)$ for $k \geq 1$. We call (180) the *abstract local WK* *mapping hierarchy associated to* φ, for short the *abstract local WK mapping hierarchy.* In particular, $D_{U^\varphi}(U^\varphi)$ reads as follows:

$$
\begin{aligned}
(181)\quad & D_{U^\varphi}(U^\varphi) \\
&= U^\varphi U_1^\varphi + \epsilon^2\left(\frac{1}{12}U_3^\varphi + \frac{\varphi''}{8\varphi'}U_1^\varphi U_2^\varphi + \left(\frac{\varphi'''}{16\varphi'} - \frac{\varphi''^2}{16\varphi'^2}\right)(U_1^\varphi)^3\right) \\
&+ \epsilon^4\left(\frac{\varphi''}{480\varphi'}U_5^\varphi + \left(\frac{7\varphi'''}{480\varphi'} - \frac{11\varphi''^2}{960\varphi'^2}\right)U_4^\varphi U_1^\varphi + \left(\frac{\varphi'''}{48\varphi'} - \frac{\varphi''^2}{192\varphi'^2}\right)U_3^\varphi U_2^\varphi\right. \\
&\quad + \left(\frac{53\varphi''^3}{2880\varphi'^3} - \frac{11\varphi'''\varphi''}{240\varphi'^2} + \frac{9\varphi^{(4)}}{320\varphi'}\right)U_3^\varphi(U_1^\varphi)^2 \\
&\quad + \left(\frac{17\varphi^{(4)}}{480\varphi'} - \frac{7\varphi''^3}{1440\varphi'^3} - \frac{7\varphi'''\varphi''}{240\varphi'^2}\right)(U_2^\varphi)^2U_1^\varphi \\
&\quad + \left(\frac{5\varphi^{(5)}}{192\varphi'} + \frac{71\varphi''^4}{1920\varphi'^4} - \frac{13\varphi'''^2}{1920\varphi'^2} - \frac{13\varphi^{(4)}\varphi''}{320\varphi'^2} - \frac{\varphi'''\varphi''^2}{64\varphi'^3}\right)U_2^\varphi(U_1^\varphi)^3 \\
&\quad + \left(\frac{\varphi^{(6)}}{384\varphi'} - \frac{23\varphi''^5}{640\varphi'^5} - \frac{\varphi^{(5)}\varphi''}{192\varphi'^2} + \frac{\varphi'''\varphi^{(4)}}{768\varphi'^2}\right. \\
&\quad\left.\left. - \frac{11\varphi^{(4)}\varphi''^2}{1920\varphi'^3} + \frac{11\varphi'''\varphi''^3}{160\varphi'^4} - \frac{33\varphi'''^2\varphi''}{1280\varphi'^3}\right)(U_1^\varphi)^5\right) + \cdots .
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
(182)\quad D_{U^\varphi}(U^\varphi) &= U^\varphi U_1^\varphi + \left(\frac{1}{12}U_3^\varphi + \frac{l_1}{8}U_1^\varphi U_2^\varphi + \frac{l_2}{16}(U_1^\varphi)^3\right)\epsilon^2 \\
&+ \left(\frac{l_1}{480}U_5^\varphi + \left(\frac{7l_2}{480} + \frac{l_1^2}{320}\right)U_4^\varphi U_1^\varphi + \left(\frac{l_2}{48} + \frac{l_1^2}{64}\right)U_3^\varphi U_2^\varphi\right. \\
&\quad + \left(\frac{9l_3}{320} + \frac{37l_1l_2}{960} + \frac{l_1^3}{1440}\right)U_3^\varphi(U_1^\varphi)^2 \\
&\quad + \left(\frac{17l_3}{480} + \frac{37l_1l_2}{480} + \frac{l_1^3}{720}\right)(U_2^\varphi)^2U_1^\varphi \\
&\quad + \left(\frac{5l_4}{192} + \frac{61l_1l_3}{960} + \frac{137l_2^2}{1920} + \frac{l_1^2l_2}{192}\right)U_2^\varphi(U_1^\varphi)^3 \\
&\quad \left. + \left(\frac{l_5}{384} + \frac{l_1l_4}{128} + \frac{7l_2l_3}{256} + \frac{l_1^2l_3}{1280} + \frac{l_1l_2^2}{640}\right)(U_1^\varphi)^5\right)\epsilon^4 + \cdots ,
\end{aligned}
$$

where $l_k = l_k(U^\varphi)$ are defined in (117). The abstract local WK mapping hierarchy (180) reduces to (165) when $\varphi(V) = V$. Recalling that $S(V) \in \mathcal{O}_c(V)$, we note that for $c \neq 0$ one should modify the infinite group $\mathcal{G}$ to $\mathcal{G} = V - c + (V-c)^2R[[V-c]]$, which does not affect the previous formulations. By construction, the power series $U^\varphi(\mathbf{T};\epsilon)$ satisfies the following hierarchy of evolutionary PDEs:

$$\frac{\partial U^\varphi(\mathbf{T};\epsilon)}{\partial T_i} \;=\; D_{(U^\varphi)^i/i!}(U^\varphi(\mathbf{T};\epsilon))\,, \quad i\ge 0\,, \tag{183}$$

which we call the *mapping WK hierarchy associated to* φ. Here on the right-hand side it is understood that one replaces U_k^φ by $\partial U^\varphi(\mathbf{T};\epsilon)/X^k$, $k\ge 1$, with $X=T_0$.

Remark 10. Surprisingly, the rational numbers appearing on the right-hand side of (182) are all *positive*. This may hold to all orders in ϵ.

A priori the coefficient of each power of ϵ^2 on the right-hand side of (180) could be a polynomial in $(U_1^\varphi)^{\pm 1}$, U_2^φ, U_3^φ, ..., but with the help of a general Mathematica package[5] designed by Joel Ekstrand, we find that up to and including the term ϵ^{10} there are never any negative powers of U_1^φ. We will prove the following theorem.

Theorem 11. *The abstract local WK mapping hierarchy* (180) *has polynomiality: for any* S *the right-hand side of* (180) *belongs to* $\mathcal{A}_{U^\varphi}[[\epsilon^2]]_1$.

We first prove a special case of Theorem 11.

Proposition 5. *Theorem 11 holds when* $\varphi=\mathrm{id}$*, i.e., for the local KdV hierarchy* (165)*. More precisely, the abstract local KdV hierarchy* (165) *has the form:*

$$D_S(u) \;=\; \partial\biggl(\int^u S \,+\, \sum_{g\ge 1}\epsilon^{2g}\sum_{\lambda\in\mathcal{P}_{2g}} K_\lambda\, S^{(\ell(\lambda)+g-1)}(u)\, u_\lambda\biggr) \tag{184}$$

$$\;=\; S(u)\,u_1 \,+\, \sum_{g\ge 1}\epsilon^{2g}\sum_{\mu\in\mathcal{P}_{2g+1}} G_\mu\, S^{(\ell(\mu)+g-1)}(u)\, u_\mu\,, \tag{185}$$

where K_λ, G_μ *are rational numbers.*

Proof. For $S(u)=u^i/i!$ $(i\ge 0)$, it is known that (165), i.e., the abstract KdV hierarchy, can be written as

$$D_{u^i/i!}(u) \;=\; \partial(h_{i-1}(u,u_1,u_2,\dots,u_i))\,, \quad i\ge 0\,, \tag{186}$$

where $h_k=h_k(u,u_1,u_2,\dots,u_{k+1})\in\mathbb{Q}[u,u_1,\dots,u_{k+1}][\epsilon^2]_0$, $k\ge -1$, are τ-symmetric hamiltonian densities for the KdV hierarchy [45] (see also [10, 44]), which satisfy

$$h_k-\frac{u^{k+2}}{(k+2)!} \;\in\; \epsilon^2\cdot\mathbb{Q}[u,u_1,\dots,u_{k+1}][\epsilon^2]_{-2}\ (\forall\, k\ge -1), \tag{187}$$

$$h_{-1} \;=\; u\,, \quad D_{u^j/j!}(h_{i-1}) \;=\; D_{u^i/i!}(h_{j-1})\ (\forall\, i,j\ge 0)\,, \tag{188}$$

[5]The package is based on the method given in [31, 45, 76].

and

$$\frac{\partial h_i}{\partial u} \,=\, h_{i-1}\,, \quad i \ge 0\,. \tag{189}$$

For $S(u) = \sum_{m\ge 0} a_m u^m/m!$ with a_m being arbitrarily given constants, we have

$$D_S(u) \,=\, \sum_{m\ge 0} a_m D_{u^m/m!}(u)\,. \tag{190}$$

The expression (184) follows from (189), (190) and (186). For $S \in \mathcal{O}_c(u)$ with $c \neq 0$, the proof is then similar. Equation (185) follows from (184). □

We note that assuming polynomiality the precise form (184) can also be obtained as a result of the quasi-trivial transformation combined with (74)–(77).

Remark 11. Let us briefly describe another way of defining the abstract local KdV hierarchy D_S, $S \in \mathcal{O}_c(u)$. Define D_u as an admissible derivation such that $D_u(u) = uu_1 + \epsilon^2 \frac{u_3}{12}$. Require D_S to be the admissible derivation on $\mathcal{A}_u[\epsilon^2]$ satisfying

$$D_S(u) - S(u)\, u_1 \,\in\, \epsilon \cdot \mathcal{A}_u[\epsilon]_{-2}\,, \quad [D_S, D_u] \,=\, 0\,. \tag{191}$$

For the uniqueness of D_S for any $S \in \mathcal{O}_c(u)$ see e.g. [14, 74]. The existence of $D_{u^i/i!}$, $i \ge 0$, is well known. For $S(u) = \sum_{m\ge 0} a_m u^m/m!$, let $D_S(u)$ be assigned as the right-hand side of (190), then it can be checked that D_S satisfies (191). For a general $S \in \mathcal{O}_c(u)$ the proof of existence is similar.

It is known (see e.g. [10, 14, 25, 27, 44, 45, 63, 98]) that $\Omega^{\rm KdV}_{i,j}$, $i, j \ge 0$, actually all belong to $\mathcal{A}_u[[\epsilon^2]]_0$. Then by an argument similar to the proof of Proposition 5 we have $\Omega^{\rm KdV}_{S_1,S_2} \,=\, \int^u S_1 S_2 + O(\epsilon^2) \,\in\, \mathcal{A}_u[[\epsilon^2]]_0$, $\forall\, S_1, S_2 \,\in\, \mathcal{O}_c(u)$. Here $\Omega^{\rm KdV}_{S_1,S_2}$ is defined as the substitution of the inverse of the quasi-Miura type transformation (164) in $\int^u S_1S_2 + \sum_{g\ge 1} \epsilon^{2g} D_{S_1} D_{S_2}(F^{\rm WK}_g)$ (similar to the definition of $\Omega^{\rm KdV}_{i,j}$; see (166)).

In order to prove Theorem 11 for general φ, we will prove a stronger statement. (In Section 11 we will give a more direct proof of a generalization of Theorem 11.) Define two operators P^φ_1 and P^φ_2 by

$$P^\varphi_1 \,:=\, \sum_{k,\ell\ge 0} \frac{\partial U^\varphi}{\partial V_k} \circ \partial^k \circ \left(\frac{1}{2\,\varphi'(V)} \circ \partial \,+\, \partial \circ \frac{1}{2\,\varphi'(V)} \right) \circ (-\partial)^\ell \circ \frac{\partial U^\varphi}{\partial V_\ell}\,, \tag{192}$$

$$P^\varphi_2 \,:=\, \sum_{k,\ell\ge 0} \frac{\partial U^\varphi}{\partial V_k} \circ \partial^k \circ \left(\frac{\varphi(V)}{2\,\varphi'(V)} \circ \partial \,+\, \partial \circ \frac{\varphi(V)}{2\,\varphi'(V)} \right) \circ (-\partial)^\ell \circ \frac{\partial U^\varphi}{\partial V_\ell}\,, \tag{193}$$

where U^φ is given by the quasi-Miura transformation (179). It readily follows from the definition that the operators P_a^φ, $a=1,2$, have the form:

$$P_a^\varphi(U^\varphi) \;=\; \sum_{g\geq 0} \epsilon^{2g}\, P_a^{\varphi,[g]}\,, \tag{194}$$

$$P_a^{\varphi,[g]} \;=\; \sum_{j=0}^{3g+1} A^\varphi_{2g,j;a}\,\partial^j\,, \quad A^\varphi_{2g,j;a} \in \mathcal{O}_c(U^\varphi)\big[U_1^\varphi,\ldots,U^\varphi_{3g+1},(U_1^\varphi)^{-1}\big]\,, \tag{195}$$

$$P_1^{\varphi,[0]} \;=\; \frac{1}{2\,\varphi'(U^\varphi)}\circ\partial \,+\, \partial\circ\frac{1}{2\,\varphi'(U^\varphi)}\,, \tag{196}$$

$$P_2^{\varphi,[0]} \;=\; \frac{\varphi(U^\varphi)}{2\,\varphi'(U^\varphi)}\circ\partial \,+\, \partial\circ\frac{\varphi(U^\varphi)}{2\,\varphi'(U^\varphi)}\,, \tag{197}$$

$$\sum_{m\geq 1} m\,U_m^\varphi\,\frac{\partial A^\varphi_{2g,j;a}}{\partial U_m^\varphi} \;=\; (2g+1-j)A^\varphi_{2g,j;a}\,. \tag{198}$$

We know that $[P_a^\varphi(U^\varphi),P_b^\varphi(U^\varphi)]=0$, for arbitrary $a,b\in\{1,2\}$. The abstract local WK mapping hierarchy (180) can be written in the following form:

$$D_S(U^\varphi) \;=\; P_1^\varphi(U^\varphi)\left(\frac{\delta\int h^\varphi_{1;S}}{\delta U^\varphi}\right) \;=\; P_2^\varphi(U^\varphi)\left(\frac{\delta\int h^\varphi_{2;S}}{\delta U^\varphi}\right), \quad i\geq 0\,. \tag{199}$$

Here, the hamiltonian densities $h^\varphi_{1;S}$, $h^\varphi_{2;S}$ are understood as the substitutions of the inverse of the quasi-Miura transformation (179) into

$$h^\varphi_{1;S} \;=\; \int_0^V \sqrt{\varphi'(x_2)}\int_0^{x_2} S(x_1)\,\sqrt{\varphi'(x_1)}\,dx_1\,dx_2\,, \tag{200}$$

$$h^\varphi_{2;S} \;=\; \int_0^V \sqrt{\frac{\varphi'(x_2)}{\varphi(x_2)}}\int_0^{x_2} S(x_1)\,\sqrt{\frac{\varphi'(x_1)}{\varphi(x_1)}}\,dx_1\,dx_2\,. \tag{201}$$

A priori, the operators $P_a^\varphi(U^\varphi)$ and the variational derivatives of the hamiltonian densities $h^\varphi_{a;S}$ with respect to U^φ, $a=1,2$, could contain negative powers of U_1^φ, but just as in the remark preceding Theorem 11, we can use Ekstrand's Mathematica pakage to check that up to and including the ϵ^8 term this does not happen. Explicit expressions for P_1, P_2 up to and including ϵ^2 are given as follows:

$$P_1^\varphi(U^\varphi) = \frac{1}{2}\frac{1}{\varphi'}\circ\partial + \frac{1}{2}\partial\circ\frac{1}{\varphi'} + \frac{1}{2}\Biggl(\biggl(\frac{3\varphi'''^2}{16\varphi'^3} - \frac{\varphi^{(5)}}{24\varphi'^2} + \frac{13\varphi^{(4)}\varphi''}{48\varphi'^3} - \frac{15\varphi'''\varphi''^2}{16\varphi'^4} + \frac{\varphi''^4}{2\varphi'^5}\biggr)(U_1^\varphi)^3 + \biggl(\frac{7\varphi'''\varphi''}{8\varphi'^3} - \frac{\varphi^{(4)}}{6\varphi'^2} - \frac{3\varphi''^3}{4\varphi'^4}\biggr)U_1^\varphi U_2^\varphi + \biggl(\frac{\varphi''^2}{6\varphi'^3} - \frac{\varphi'''}{12\varphi'^2}\biggr)U_3^\varphi + \biggl(\Bigl(\frac{3\varphi'''\varphi''}{8\varphi'^3} - \frac{\varphi^{(4)}}{12\varphi'^2} - \frac{\varphi''^3}{4\varphi'^4}\Bigr)(U_1^\varphi)^2 + \Bigl(\frac{\varphi''^2}{3\varphi'^3} - \frac{\varphi'''}{6\varphi'^2}\Bigr)U_2^\varphi\biggr)\circ\partial\Biggr)\epsilon^2 + \cdots \tag{202}$$

and

$$P_2^\varphi(U^\varphi) = \frac{1}{2}\frac{\varphi}{\varphi'}\circ\partial + \frac{1}{2}\partial\circ\frac{\varphi}{\varphi'} + \frac{\epsilon^2}{2}\Biggl(\biggl(-\frac{\varphi\varphi^{(5)}}{24\varphi'^2} - \frac{\varphi^{(4)}}{8\varphi'} + \frac{3\varphi\varphi'''^2}{16\varphi'^3} + \frac{\varphi\varphi''^4}{2\varphi'^5} - \frac{\varphi''^3}{4\varphi'^3} + \frac{13\varphi\varphi^{(4)}\varphi''}{48\varphi'^3} - \frac{15\varphi\varphi'''\varphi''^2}{16\varphi'^4} + \frac{19\varphi'''\varphi''}{48\varphi'^2}\biggr)(U_1^\varphi)^3 + \biggl(-\frac{\varphi\varphi^{(4)}}{6\varphi'^2} - \frac{\varphi'''}{3\varphi'} - \frac{3\varphi\varphi''^3}{4\varphi'^4} + \frac{3\varphi''^2}{8\varphi'^2} + \frac{7\varphi\varphi'''\varphi''}{8\varphi'^3}\biggr)U_1^\varphi U_2^\varphi + \biggl(\frac{\varphi\varphi''^2}{6\varphi'^3} - \frac{\varphi\varphi'''}{12\varphi'^2} - \frac{\varphi''}{12\varphi'}\biggr)U_3^\varphi + \biggl(\Bigl(-\frac{\varphi\varphi^{(4)}}{12\varphi'^2} - \frac{\varphi'''}{6\varphi'} - \frac{\varphi\varphi''^3}{4\varphi'^4} + \frac{\varphi''^2}{8\varphi'^2} + \frac{3\varphi\varphi'''\varphi''}{8\varphi'^3}\Bigr)(U_1^\varphi)^2 + \Bigl(\frac{\varphi\varphi''^2}{3\varphi'^3} - \frac{\varphi\varphi'''}{6\varphi'^2} - \frac{\varphi''}{6\varphi'}\Bigr)U_2^\varphi\biggr)\circ\partial + \frac{\partial^3}{4}\Biggr) + \cdots . \tag{203}$$

The following theorem, which is stronger than Theorem 11, gives a refined version of Theorem 1.

Theorem 12. *For $a = 1, 2$, $g \geq 0$ and $0 \leq j \leq 3g + 1$, the elements $A^\varphi_{2g,j;a}$ all belong to $\mathcal{A}^{[2g+1-j]}_{U^\varphi}$. Moreover, the variational derivatives of the hamiltonians $\int h^\varphi_{1;S}$ and $\int h^\varphi_{2;S}$ with respect to U^φ belong to $\mathcal{A}_{U^\varphi}[[\epsilon]]$.*

Proof. First of all, we have

$$\partial \;=\; \sum_{m\geq 0} \frac{\partial t_m}{\partial X} D_{u^m/m!}\,. \tag{204}$$

Here when we use a function of u, say $f(u)$, to label a derivation $D_{f(u)}$ we mean the corresponding derivation in the abstract local KdV hierarchy. By the definition (16) we can simplify the above equality to

$$\partial \;=\; D_{\sqrt{\varphi'(\varphi^{-1}(u))}}\,. \tag{205}$$

By Proposition 5 we know that $D_{\sqrt{\varphi'(\varphi^{-1}(u))}}(u)$ has polynomiality and of course it commutes with the KdV derivation $D_u(u)$. Then, using the results in [70], we know that by taking $\partial = \partial_X$ as the spatial derivative the abstract local KdV hierarchy is transformed to a bihamiltonian evolutionary system for u and particularly the ∂_x-flow of the abstract local KdV hierarchy after the transformation is bihamiltonian in Dubrovin–Zhang's normal form. Secondly, we note that

$$U^\varphi \;=\; \epsilon^2 \frac{\partial^2 \mathcal{F}^\varphi(\mathbf{T};\epsilon)}{\partial X^2} \;=\; \epsilon^2 \sum_{i,j\geq 0} \frac{\partial t_i}{\partial X}\frac{\partial t_j}{\partial X}\, \Omega^{\mathrm{KdV}}_{i,j} \;=\; \varphi^{-1}(u) \,+\, O(\epsilon^2)\,. \tag{206}$$

Since the ∂_x-flow for u with $\partial = \partial_X$ as the spatial derivative is an evolutionary PDE in Dubrovin–Zhang's normal form and since $\Omega^{\mathrm{KdV}}_{i,j} \in \mathcal{A}_u[[\epsilon^2]]_0$, we find by doing the substitution that $\Omega^{\mathrm{KdV}}_{i,j}$ are power series of ϵ^2 with coefficients being polynomials of $\partial_X(u), \partial_X^2(u), \dots$. This means that equation (206) gives a Miura-type transformation (with the spacial derivative being ∂). The theorem is proved. ☐

We note that an equivalent description of the second statement of Theorem 12 is that the hamiltonian densities $h^\varphi_{a;S}$, $a = 1, 2$, modulo certain total ∂-derivatives, both belong to $\mathcal{A}_{U^\varphi}[[\epsilon]]$ for any S. We also note that the first statement of Theorem 12 implies in particular that $A^\varphi_{2g,j;a} = 0$ for all $j \geq 2g + 2$.

Proof of Theorem 11. The theorem follows from Theorem 12. ☐

Since we have proved Theorem 11, by using the quasi-trivial transformation (179) with Theorem 8, we can further prove that the abstract local WK mapping hierarchy (180) has the more precise form:

$$D_S(U^\varphi) \;=\; \partial\left(\int^{U^\varphi} S + \sum_{g\geq 1} \epsilon^{2g} \sum_{\lambda\in\mathcal{P}_{2g}} \sum_{j=1}^{\ell(\lambda)+g-1} Y^\varphi_{\lambda,j}(l_1(U^\varphi),\dots)\, S^{(j)}(U^\varphi)\, U^\varphi_\lambda\right), \tag{207}$$

where $Y^\varphi_{\lambda,j}(\ell_1,\dots) \in \mathcal{R}^{[\ell(\lambda)+g-1-j]}$, and $l_k(U^\varphi)$ are defined in (117).

For $\varphi \in \mathcal{G}$, we have

$$\epsilon^2 \frac{\partial^2 \log \mathcal{F}^\varphi(\mathbf{T};\epsilon)}{\partial T_i \partial T_j} = \frac{E(\mathbf{T})^{i+j+1}}{i!\,j!\,(i+j+1)} + \sum_{g\geq 1} \epsilon^{2g} \frac{\partial^2 \log \mathcal{F}_g^\varphi(\mathbf{T})}{\partial T_i \partial T_j}\,. \tag{208}$$

According to Proposition 2, the right-hand side of (208) can be represented by the jets $V, V_1, V_2, \ldots$. Substituting the inverse of the quasi-Miura transformation (179) into the jet representation of the right-hand side of (208) we get a power series of ϵ^2, which we denote by $\Omega^\varphi_{i,j}$. We have

$$\Omega^\varphi_{i,j} = \sum_{i_1,j_1\geq 0} \frac{\partial t_{i_1}}{\partial T_i} \frac{\partial t_{j_1}}{\partial T_j}\, \Omega^{\mathrm{KdV}}_{i_1,j_1}\,, \quad i,j\geq 0\,. \tag{209}$$

This implies that $\Omega^\varphi_{i,j}$ belong to $\mathcal{A}_{U^\varphi}[[\epsilon^2]]$. In Section 11 we will give a more general description about this.

Remark 12. Recall that the Hodge hierarchy is a τ-symmetric integrable Hamiltonian perturbation of the RH hierarchy, depending on an infinite sequence of parameters [34] (see also [18, 19]). The Hodge universality conjecture proposed in [34] says that the Hodge hierarchy is a universal object in τ-symmetric Hamiltonian integrable hierarchies, meaning that any τ-symmetric Hamiltonian integrable hierarchy in the sense of [34] is related to the Hodge hierarchy via a Miura-type transformation. The WK mapping hierarchy is integrable, Hamiltonian (actually bihamiltonian) and possesses a τ-structure. However, in general its hamiltonian densities cannot be chosen to satisfy the τ-symmetry condition of [34, 45]. So our result is consistent with [34].

Using the explicit expressions (202)–(203) one can easily compute the central invariant of the pencil $P_2^\varphi + \lambda P_1^\varphi$. Although this invariant can be deduced from the result of [70] or Proposition 3, we give an explicit computation. The canonical coordinate for the pencil $P_2^\varphi(U^\varphi) + \lambda P_1^\varphi(U^\varphi)$ is $\varphi(U^\varphi)$. Perform the following Miura-type transformation to (199):

$$\hat{u} = \varphi(U^\varphi)\,. \tag{210}$$

The Poisson operators P_1, P_2 in the $\hat{u}$-coordinate read:

$$P_1^\varphi(\hat{u}) = \frac{1}{2}\,\varphi'(\varphi^{-1}(\hat{u}))\circ\partial + \frac{1}{2}\,\partial\circ\varphi'(\varphi^{-1}(\hat{u})) + \cdots\,, \tag{211}$$

$$P_2^\varphi(\hat{u}) = \frac{1}{2}\,\hat{u}\,\varphi'((\varphi^{-1}(\hat{u}))\circ\partial + \frac{1}{2}\,\partial\circ\hat{u}\circ\varphi'(\varphi^{-1}(\hat{u})) + \cdots\,, \tag{212}$$

where "$\cdots$" denotes terms containing $\epsilon^2, \epsilon^4, \cdots$. In particular, the terms containing ϵ^2 can be obtained from (202)–(203). Now using [36, formula (1.49)] we find that the central invariant is the constant-valued function $1/24$ for any φ. So the WK mapping hierarchy leads to a construction of the representatives of Poisson pencils for

$$(g,c) \;=\; \Big(\varphi'(\varphi^{-1}(\hat{u})),\, \frac{1}{24}\Big)\,. \tag{213}$$

Let us now give a proof of Theorem 10 (or equivalently Theorem 10′).

Proof of Theorem 10. The necessity is already implied by Theorem 9. For the sufficiency, using the argument given after the statement of Theorem 9, we know that, under a proper choice of the Poisson pencil for the bihamiltonian perturbation under consideration, the central invariant equals 1/24. Since the WK mapping hierarchy, which is bihamiltonian, also has central invariant 1/24, the results [73, Theorems 1 and 2] and [36, Theorem 1.8] then imply that the given bihamiltonian perturbation is Miura equivalent to the WK mapping hierarchy. As the WK mapping hierarchy has a tau-structure, the sufficiency part is proved by recalling Lemma 5. □

The above proof immediately leads also to a proof of the following theorem, which we call the *WK mapping universality theorem*.

Theorem 13. *The abstract local WK mapping hierarchy is a universal object for bihamiltonian perturbations of the abstract local RH hierarchy possessing a τ-structure.*

To make the content of Theorem 13 clearer, and at the same time as an additional check that the statement is correct, we provide some direct verifications up to order ϵ^8 (i.e., up to and including a_3), with q_2 and q_3 given by (172). Namely, the following Miura-type transformation

$$U^\varphi \;\mapsto\; w \;=\; M(U^\varphi) \,+\, \sum_{k=1}^{4} \epsilon^{2k} \sum_{\lambda\in\mathcal{P}_{2k}} C_\lambda(U^\varphi)\, U^\varphi_\lambda \,+\, \mathcal{O}(\epsilon^{10}) \tag{214}$$

transforms the WK mapping hierarchy (180) to the standard form (152), (153) up to and including the ϵ^8 term. Here,

$$M(U^\varphi) \;=\; \int_0^{U^\varphi} \sqrt{\varphi'(y)}\, dy\,, \tag{215}$$

$$a_0(w) \;=\; M'(M^{-1}(w))\,, \tag{216}$$

and C_λ for $|\lambda| \le 8$ are explicit expressions, e.g.,

$$C_{(2)}(U^\varphi) \;=\; -\frac{q}{8}\,(\varphi'(U^\varphi))^{3/2}\,, \tag{217}$$

$$C_{(1^2)}(U^\varphi) \;=\; -\frac{q}{8}\,(\varphi'(U^\varphi))^{1/2}\,\varphi''(U^\varphi) \;+\; \frac{\varphi''(U^\varphi)^2}{24\,\varphi'(U^\varphi)^{3/2}} - \frac{\varphi^{(3)}(U^\varphi)}{48\sqrt{\varphi'(U^\varphi)}}\,, \tag{218}$$

$$C_{(1^8)}(U^\varphi) \;=\; -\frac{107}{185794560}\,\frac{\varphi^{(12)}(U^\varphi)}{\sqrt{\varphi'(U^\varphi)}} \;+\; \text{more than two hundred terms}\,. \tag{219}$$

We end this section with one more remark.

Remark 13. Using the Miura-type transformation we can assume that $\Omega_{1,1} = U$ in the setting for hamiltonian perturbations (148) of the RH hierarchy possessing a τ-structure. When $\Omega_{1,1} = U$, we can define the *normal Miura-type transformation*, which has the form

$$U \;\mapsto\; \tilde{U} \;=\; \sum_{k\ge 0} \tilde{U}^{[k]}\,\epsilon^k \;=\; U \;+\; \epsilon^2\,\partial^2(A(U, U_1, U_2, \dots; \epsilon)) \tag{220}$$

for some $A \in \mathcal{A}_U[[\epsilon]]_0$. So normal Miura-type transformations are a special class, but whenever we use the word "normal" we mean that the transformation does not just act on the hierarchy as usual, but also acts on the τ-structure by

$$\tilde{\Omega}_{S_1,S_2} \;:=\; \Omega_{S_1,S_2} \;+\; \epsilon^2\,D_{S_1}D_{S_2}(A)\,. \tag{221}$$

Since the normal Miura-type transformation changes the resulting τ-function when $A \neq$ constant, it plays the role of choosing different τ-structures. However, in this paper, we do not use this terminology.

10. A special group element and the Hodge–WK correspondence

In this section, we will consider the particular example for the WK mapping hierarchy given by

$$\varphi_{\rm special}(V) \;=\; \frac{e^{2\,q\,V} - 1}{2\,q} \;=\; V \;+\; q\,V^2 \;+\; \frac{2}{3}\,q^2\,V^3 \;+\; \cdots \;\in\; \mathcal{G} \tag{222}$$

(as in (6)) over the ground ring $R = \mathbb{Q}[q]$, where q is a free parameter. The inverse group element is $f_{\rm special}(v) = \log(1 + 2qv)/2q$. We will establish a relationship between the WK mapping partition function associated to (222) and a Hodge partition function.

Before entering into the details, we recall some general terminology for the Hodge side. Let $\mathbb{E}_{g,n}$ be the rank g Hodge bundle on $\overline{\mathcal{M}}_{g,n}$. By *Hodge integrals* we mean integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_{j_1} \cdots \lambda_{j_m} \,, \tag{223}$$

where $\lambda_j := c_j(\mathbb{E}_{g,n})$, $j = 0, \ldots, g$, are Chern classes of $\mathbb{E}_{g,n}$, and $0 \le j_1, \ldots, j_m \le g$. The degree-dimension matching now reads

$$\sum_{a=1}^{n} i_a + \sum_{b=1}^{m} j_b = 3g - 3 + n \,. \tag{224}$$

We also denote by $\mathrm{ch}_j(\mathbb{E}_{g,n})$, $j \ge 0$, the components of the Chern character of $\mathbb{E}_{g,n}$. Mumford's relation tells that even components of the Chern character must vanish. The Hodge partition function [34, 46, 49] is then defined by

$$Z_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon) = \exp\biggl(\sum_{g,\, n\ge 0} \frac{\epsilon^{2g-2}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(\boldsymbol{\sigma}) \cdot t(\psi_1) \cdots t(\psi_n) \biggr) \tag{225}$$

where $\mathbf{t} = (t_0, t_1, t_2, \ldots)$ as before, $\boldsymbol{\sigma} = (\sigma_1, \sigma_3, \sigma_5, \ldots)$ is an infinite tuple of parameters, and

$$\Omega_{g,n}(\boldsymbol{\sigma}) = \exp\biggl(\sum_{j\ge 1} \sigma_{2j-1} \, \mathrm{ch}_{2j-1}(\mathbb{E}_{g,n}) \biggr) \,. \tag{226}$$

Obviously, $Z_{\Omega(\mathbf{0})}(\mathbf{t};\epsilon) = Z^{\mathrm{WK}}(\mathbf{t};\epsilon)$. The logarithm $\log Z_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon) =: \mathcal{F}_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon)$ is called the *free energy of Hodge integrals*, for short the *Hodge free energy*. By definition the free energy $\mathcal{F}_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon)$ admits the genus expansion:

$$\mathcal{F}_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon) =: \sum_{g\ge 0} \epsilon^{2g-2} \, \mathcal{F}_{\Omega_g(\boldsymbol{\sigma})}(\mathbf{t}) \,. \tag{227}$$

We call $\mathcal{F}_{\Omega_g(\boldsymbol{\sigma})}(\mathbf{t})$ $(g \ge 0)$ the *genus g Hodge free energy*.

Faber and Pandharipande in [46] obtain the following explicit formula for the Hodge partition function:

$$Z_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon) = e^{\sum_{k\ge 1} \frac{B_{2k}}{(2k)!} \sigma_{2k-1} D_k} \bigl(Z^{\mathrm{WK}}(\mathbf{t};\epsilon) \bigr) \,, \tag{228}$$

where B_m denote the mth Bernoulli number, and D_k, $k \ge 1$, are operators given by

$$D_k = \frac{\partial}{\partial t_k} - \sum_{i\ge 0} t_i \frac{\partial}{\partial t_{i+2k-1}} + \frac{\epsilon^2}{2} \sum_{m=0}^{2k-2} (-1)^m \frac{\partial^2}{\partial t_m \partial t_{2k-2-m}} \,. \tag{229}$$

This formula is interpreted by Givental [49] as a Givental group action, and is generalized from the viewpoint of Virasoro-like algebra in [72].

It was shown that the Hodge partition function gives rise to a τ-symmetric integrable hierarchy of hamiltonian evolutionary PDEs, called the *Hodge hierarchy*, so that the Hodge partition function is a τ-function for the Hodge hierarchy [18, 19, 34, 45]. Roughly speaking, the Hodge hierarchy is an integrable perturbation of the KdV hierarchy, with σ's being the deformation parameters.

The interest of this section will be focused on the following specialization of the parameters in the Hodge partition function:

$$\sigma_{2j-1}^{\text{special}} = (4^j - 1)\,(2j-2)!\,q^{2j-1}\,, \quad j \geq 1\,. \tag{230}$$

Using relations between the Chern character and the Chern polynomial [34], we have

$$\Omega_{g,n}(\boldsymbol{\sigma}^{\text{special}}) = \Lambda_{g,n}(2q)^2\,\Lambda_{g,n}(-q) =: \Omega_{g,n}^{\text{special}}(q)\,, \tag{231}$$

where $\Lambda_{g,n}(z) = \sum_{j=0}^{g} \lambda_j\, z^j$ denotes the Chern polynomial of $\mathbb{E}_{g,n}$. We call Hodge integrals with $\Omega_{g,n}^{\text{special}}(q)$ the *special-Hodge integrals*, whose significance is manifested by the Gopakumar–Mariño–Vafa conjecture regarding the Chern–Simons/string duality [50, 81, 67], and is discussed in [34, 35, 40, 100] from the viewpoints of bihamiltonian structures and random matrices. We call $Z_{\Omega^{\text{special}}(q)}(\mathbf{t};\epsilon)$ and $\mathcal{F}_{\Omega^{\text{special}}(q)}(\mathbf{t};\epsilon)$ the *special-Hodge partition function* and the *special-Hodge free energy*, respectively.

The Hodge hierarchy [34] under the specialization (230), called the *special-Hodge hierarchy*, has the form

$$\begin{aligned} \frac{\partial w}{\partial t_1} &= w\,w' + \epsilon^2\Big(\frac{1}{12}w''' + \frac{q}{4}w'w''\Big) \\ &\quad + \epsilon^4\Big(\frac{q}{240}w''''' + \frac{q^2}{80}w'\,w'''' + \frac{q^2}{16}w''w''' + \frac{q^3}{180}w'^2w''' + \frac{q^3}{90}w'w''^2\Big) + \mathcal{O}(\epsilon^6)\,, \end{aligned} \tag{232}$$

$$\frac{\partial w}{\partial t_i} = \frac{1}{i!}\,w^i w' + \mathcal{O}(\epsilon^2)\,, \quad i \geq 2\,. \tag{233}$$

Here, $w := \epsilon^2 \partial_{t_0}^2\big(\mathcal{F}_{\Omega^{\text{special}}(q)}(\mathbf{t};\epsilon)\big)$ is the normal coordinate, and, prime, $'$, denotes the derivative with respect to t_0.

10.1. **The Hodge–WK correspondence.** The goal is to connect $Z_{\Omega^{\text{special}}(q)}(\mathbf{t};\epsilon)$ with $Z^{\varphi_{\text{special}}}(\mathbf{t};\epsilon)$. Let $\mathbf{t}$ and $\mathbf{T}$ be related by $\mathbf{T} = \mathbf{t}.\varphi_{\text{special}}$ as in (16) with φ_{special} given by (6) or (222). We have the following lemma.

Lemma 6. *We have*

$$T_i - \delta_{i,1} = \sum_{m=0}^{i} A(i,m)\,q^{i-m}\,(t_m - \delta_{m,1})\,, \qquad i \geq 0\,, \tag{234}$$

where

$$A(i,m) := \frac{1}{2^m\,m!}\sum_{j=0}^{m}(-1)^{m-j}\,(2j+1)^i\binom{m}{j}\,, \tag{235}$$

or equivalently via the following generating series

$$T(z) \,=\, \sum_{m\geq 0} (t_m - \delta_{m,1})\, \frac{z^m}{(1-qz)(1-3qz)\cdots(1-(2m+1)qz)}\,. \tag{236}$$

The inverse map is given by

$$t_m - \delta_{m,1} \,=\, \sum_{i=0}^{m} P(m,i)\, q^{m-i}\, (T_i - \delta_{i,1})\,, \tag{237}$$

where

$$P(m,i) \,:=\, (-1)^{m-i} \sum_{j=i}^{m} (-2)^{m-j} \binom{j}{i}\, s(m,j)\,. \tag{238}$$

Here, $s(m,j)$ denotes the Stirling number of the first kind.

Proof. Formula (234) can be obtained via a direct verification. The rest of the proof is an elementary exercise. □

It follows from the definition of Stirling numbers that the $P(m,i)$ admit the generating function:

$$\sum_{i=0}^{m} P(m,i)\, z^i \,=\, (z-1)(z-3)(z-5)\cdots(z-(2m-1)) \,=\, 2^m \Big(\frac{z}{2} - \frac{2m-1}{2}\Big)_m\,, \tag{239}$$

and also that $P(m,0)$ and $P(m,1)$ have the more explicit expressions

$$P(m,0) \,=\, (-1)^m (2m-1)!!\,,\ \ P(m,1) \,=\, (-1)^{m-1}(2m-1)!! \sum_{j=0}^{m-1} \frac{1}{2j+1}\,, \tag{240}$$

the first values of $P(m,1)$ being 0, 1, -4, 23, -176, 1689.

The loop equation (102) now reads explicitly as follows:

(241)

$$\sum_{k\geq 0}\sum_{m=1}^{k}\binom{k}{m}\left(\partial^{m-1}\left(\frac{\sqrt{1+2qv}}{(1+2\lambda q)\sqrt{\lambda-v}}\right)\partial^{k-m+1}\left(\frac{\sqrt{1+2qv}}{(1+2\lambda q)\sqrt{\lambda-v}}\right)\right)_{-}\frac{\partial F_{\rm h.g.}^{\varphi_{\rm special}}}{\partial V_k}$$
$$+\sum_{k\geq 0}\partial^k\left(\frac{1+2qv}{(1+2\lambda q)^2(\lambda-v)}\right)_{-}\frac{\partial F_{\rm h.g.}^{\varphi_{\rm special}}}{\partial V_k}$$
$$-\frac{\epsilon^2}{2}\sum_{k_1,k_2\geq 0}\left(\partial^{k_1+1}\left(\frac{\sqrt{1+2qv}}{(1+2\lambda q)\sqrt{\lambda-v}}\right)\partial^{k_2+1}\left(\frac{\sqrt{1+2qv}}{(1+2\lambda q)\sqrt{\lambda-v}}\right)\right)_{-} s\big(F_{\rm h.g.}^{\varphi_{\rm special}}, V_{k_1}, V_{k_2}\big)$$
$$-\frac{\epsilon^2}{16}\sum_{k\geq 0}\partial^{k+2}\left(\frac{1}{(\lambda-v)^2}\right)\frac{\partial F_{\rm h.g.}^{\varphi_{\rm special}}}{\partial V_k} \;=\; \frac{1}{16\,(\lambda-v)^2}\,.$$

(Recall that "h.g." stands for "higher genera" and refers to the sum over all contributions from $g>0$, while $\varphi_{\rm special}$ is the function defined in (222).)

We are ready to give a proof of the Hodge–WK correspondence.

Proof of Theorem 3. According to Definition 2,

$$Z^{\varphi_{\rm special}}(\mathbf{T};\epsilon) \;=\; Z^{\rm WK}(\mathbf{t},\epsilon)\,.$$

To show (7), it is equivalent to show

$$Z_{\Omega^{\rm special}(q)}(\mathbf{T};\epsilon) \;=\; Z^{\varphi_{\rm special}}(\mathbf{T};\epsilon)\,, \tag{242}$$

or equivalently to show for all $g\geq 0$, $\mathcal{F}_{\Omega_g^{\rm special}(q)}(\mathbf{T}) = \mathcal{F}_g^{\varphi_{\rm special}}(\mathbf{T})$. Let us first prove their genus zero parts are equal, and by Theorem 6 this is equivalent to the following known fact for Hodge integrals:

$$\mathcal{F}_{\Omega_0^{\rm special}(q)}(\mathbf{T}) \;=\; \mathcal{F}_0(\mathbf{T})\,. \tag{243}$$

Let us continue to show the higher genus parts of $\mathcal{F}_{\Omega^{\rm special}(q)}(\mathbf{T};\epsilon)$ and $\mathcal{F}^{\varphi_{\rm special}}(\mathbf{T};\epsilon)$ are equal. To this end, we first notice that both $\mathcal{F}_{\Omega_g^{\rm special}(q)}(\mathbf{T})$ and $\mathcal{F}_g^{\varphi_{\rm special}}(\mathbf{T})$ for $g\geq 1$ admit the jet representations. Indeed, for $\mathcal{F}_g^{\varphi_{\rm special}}(\mathbf{T})$, the jet representation is given by Proposition 2; for $\mathcal{F}_{\Omega_g^{\rm special}(q)}(\mathbf{T})$, it is known from for example [34, 35] that

$$\mathcal{F}_{\Omega_1^{\rm special}(q)}(\mathbf{T}) \;=\; F_{\Omega_1^{\rm special}(q)}\left(E(\mathbf{T}), \frac{\partial E(\mathbf{T})}{\partial T_0}\right), \tag{244}$$

with

$$F_{\Omega_1^{\rm special}(q)}(V_0,V_1) \;:=\; \frac{1}{24}\,\log V_1 \;+\; \frac{q}{8}\,V_0\,, \tag{245}$$

and that for each $g \geq 2$ there exists

$$F_{\Omega_g^{\rm special}(q)} = F_{\Omega_g^{\rm special}(q)}(V_1, \dots, V_{3g-2}) \ \in \ \mathbb{Q}[q]\big[V_1^{-1}, V_1, V_2, \dots, V_{3g-2}\big]\,,$$

satisfying

$$\sum_{k=1}^{3g-2} k\, V_k \, \frac{\partial F_{\Omega_g^{\rm special}(q)}}{\partial V_k} \ = \ (2g-2)\, F_{\Omega_g^{\rm special}(q)}\,, \tag{246}$$

$$q\, \frac{\partial F_{\Omega_g^{\rm special}(q)}}{\partial q} \ + \ \sum_{k=1}^{3g-2} (k-1)\, V_k \, \frac{\partial F_{\Omega_g^{\rm special}(q)}}{\partial V_k} \ = \ (3g-3)\, F_{\Omega_g^{\rm special}(q)}\,, \tag{247}$$

such that

$$\mathcal{F}_{\Omega_g^{\rm special}(q)}(\mathbf{T}) \ = \ F_{\Omega_g^{\rm special}(q)}\biggl(\frac{\partial V(\mathbf{T})}{\partial T_0}, \dots, \frac{\partial^{3g-2} V(\mathbf{T})}{\partial T_0^{3g-2}}\biggr)\,. \tag{248}$$

So, in order to show $F_{\Omega_g^{\rm special}(q)}(\mathbf{T}) = \mathcal{F}_g^{\varphi_{\rm special}}(\mathbf{T})$, $g \geq 1$, it suffices to show that

$$F_{\Omega_g^{\rm special}(q)} \ = \ F_g^{\varphi_{\rm special}}\,. \tag{249}$$

For $g=1$, (249) is true due to (245), (78), (222).

For $g \geq 2$, let us compare the loop equations. We use $F_{\rm h.g.}^{\Omega^{\rm special}(q)}$ to denote $\sum_{g\geq 1} \epsilon^{2g-2} F_{\Omega_g^{\rm special}(q)}$. The following loop equation for $F_{\rm h.g.}^{\Omega^{\rm special}(q)}$ can be obtained from [35]:

$$\begin{aligned} &\sum_{k\geq 0} \biggl(\partial^k \Bigl(\frac{1}{P^2}\Bigr) + \sum_{r=1}^{k} \binom{k}{r} \partial^{r-1}\Bigl(\frac{1}{P}\Bigr) \partial^{k-r+1}\Bigl(\frac{1}{P}\Bigr)\biggr) \frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_k} \\ &- \frac{\epsilon^2}{2} \sum_{k_1,k_2\geq 0} \partial^{k_1+1}\Bigl(\frac{1}{P}\Bigr) \partial^{k_2+1}\Bigl(\frac{1}{P}\Bigr) \biggl(\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_{k_1}} \frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_{k_2}} + \frac{\partial^2 F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_{k_1} \partial V_{k_2}} \biggr) \\ &- \frac{\epsilon^2}{16} \sum_{k\geq 0} \partial^{k+2} \Bigl(\frac{1}{P^4} + \frac{4q}{P^2} \Bigr) \frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_k} \ - \ \frac{q}{4P^2} \ - \ \frac{1}{16 P^4} \ = \ 0\,, \end{aligned} \tag{250}$$

where $P = \sqrt{-\frac{1}{2q} - \frac{4\, e^{-2qV}}{\lambda}}$ and $\partial = \sum_k V_{k+1} \partial/\partial V_k$. Recall that the loop equation (250) holds identically in λ and so holds identically in P. Note that

$$\partial(P) \ = \ -\Bigl(q\, P + \frac{1}{2P}\Bigr)\, V_1\,. \tag{251}$$

Introduce

$$\widetilde{P} \;=\; e^{-qV}\sqrt{\lambda - \frac{e^{2qV}-1}{2q}}\,, \tag{252}$$

and observe that

$$\partial\big(\widetilde{P}\big) \;=\; -\Big(q\,\widetilde{P} + \frac{1}{2\widetilde{P}}\Big)\,V_1\,, \tag{253}$$

which has the same form as (251). Therefore,

$$\begin{aligned}
&\sum_{k\geq 0}\bigg(\partial^k\Big(\frac{1}{\widetilde{P}^2}\Big) + \sum_{r=1}^{k}\binom{k}{r}\,\partial^{r-1}\Big(\frac{1}{\widetilde{P}}\Big)\,\partial^{k-r+1}\Big(\frac{1}{\widetilde{P}}\Big)\bigg)\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_k}\\
&-\frac{\epsilon^2}{2}\sum_{k_1,k_2\geq 0}\partial^{k_1+1}\Big(\frac{1}{\widetilde{P}}\Big)\partial^{k_2+1}\Big(\frac{1}{\widetilde{P}}\Big)\bigg(\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_{k_1}}\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_{k_2}} + \frac{\partial^2 F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_{k_1}\partial V_{k_2}}\bigg)\\
&-\frac{\epsilon^2}{16}\sum_{k\geq 0}\partial^{k+2}\Big(\frac{1}{\widetilde{P}^4} + \frac{4\,q}{\widetilde{P}^2}\Big)\,\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_k} \;-\; \frac{q}{4\widetilde{P}^2} \;-\; \frac{1}{16\widetilde{P}^4} \;=\; 0
\end{aligned} \tag{254}$$

holds identically in $\widetilde{P}$. Dividing both sides of (254) by $(1+2\lambda q)^2$, we obtain

$$\begin{aligned}
&\sum_k\sum_{m=1}^{k}\binom{k}{m}\,\partial^{m-1}\Bigg(\frac{e^{qV}/(1+2\lambda q)}{\big(\lambda-\frac{e^{2qV}-1}{2q}\big)^{1/2}}\Bigg)\,\partial^{k-m+1}\Bigg(\frac{e^{qV}/(1+2\lambda q)}{\big(\lambda-\frac{e^{2qV}-1}{2q}\big)^{1/2}}\Bigg)\,\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_k}\\
&+\sum_{k\geq 0}\partial^k\Bigg(\frac{e^{2qV}}{(1+2\lambda q)^2\big(\lambda-\frac{e^{2qV}-1}{2q}\big)}\Bigg)\,\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_k}\\
&-\frac{\epsilon^2}{2}\sum_{k_1,k_2\geq 0}\partial^{k_1+1}\Bigg(\frac{e^{qV}/(1+2\lambda q)}{\big(\lambda-\frac{e^{2qV}-1}{2q}\big)^{1/2}}\Bigg)\,\partial^{k_2+1}\Bigg(\frac{e^{qV}/(1+2\lambda q)}{\big(\lambda-\frac{e^{2qV}-1}{2q}\big)^{1/2}}\Bigg)\\
&\qquad\times\Bigg(\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_{k_1}}\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_{k_2}} + \frac{\partial^2 F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_{k_1}\partial V_{k_2}}\Bigg)\\
&-\frac{\epsilon^2}{16}\sum_{k\geq 0}\partial^{k+2}\Bigg(\frac{(1+2\lambda q)^2}{\big(\lambda-\frac{e^{2qV}-1}{2q}\big)^2}\Bigg)\,\frac{\partial F_{\rm h.g.}^{\Omega^{\rm special}(q)}}{\partial V_k} \;=\; \frac{1}{16\big(\lambda-\frac{e^{2qV}-1}{2q}\big)^2} \;-\; \frac{q^2}{4(1+2\lambda q)^2}\,.
\end{aligned}$$

This implies that $F_{\rm h.g.}^{\Omega^{\rm special}(q)}$ satisfies equation (241). Since the solution to (241) is unique with (246), we conclude that $F_{\rm h.g.}^{\varphi_{\rm special}} = F_{\rm h.g.}^{\Omega^{\rm special}(q)}$. The theorem is proved. □

We mention that for $\varphi_{\rm special}$ we have the following explicit $v \leftrightarrow V$ map:

$$M_0(Y;q) \;=\; \frac{e^{2qY_0}-1}{2q}\,, \tag{255}$$

$$\frac{1}{2q} + \sum_{j\geq 1} e^{-(j+2)qY_0}\, M_j(Y;q)\, \frac{\lambda^j}{j!} \;=\; \frac{1}{2q}\, \frac{w(Y;\lambda;q)^2}{\lambda^2} \tag{256}$$

with $w = w(Y;\lambda;q) = \lambda + \cdots$ being the unique element in $\lambda + \mathbb{C}[Y,q][[\lambda]]\lambda$ satisfying

$$w\, e^{-\,q\,R(Y;w)} \;=\; \lambda\,, \quad R(Y;w) \;:=\; \sum_{k\geq 1} Y_k\, \frac{w^k}{k!}\,. \tag{257}$$

Although it is not completely obvious, Theorem 3 is equivalent to the following theorem obtained by Alexandrov.

Theorem B (Alexandrov [5]). *Define an invertible* $\mathbf{t} \to \hat{\mathbf{T}}$ *map by*

$$\hat{T}_i \;=\; S^i(t_0)\,, \quad i \geq 0\,, \tag{258}$$

where S *denotes the following differential operator*

$$S \;=\; \sum_{m\geq 0} t_{m+1}\, \frac{\partial}{\partial t_m} \;+\; q \sum_{m\geq 0} (2m+1)\, t_m\, \frac{\partial}{\partial t_m}\,, \tag{259}$$

and define a sequence of elements $c_m \in \mathbb{Q}[q]$ *by*

$$c_0 \;=\; c_1 \;=\; 0\,, \quad c_m \;=\; P(m,1)\, q^{m-1} \tag{260}$$

with $P(m,1)$ *as in* (240). *Then we have*

$$Z_{\Omega^{\rm special}}\big(\hat{\mathbf{T}};\epsilon\big) \;=\; Z^{\rm WK}(\mathbf{t};\epsilon)|_{t_m\mapsto t_m - c_m,\, m\geq 0}\,. \tag{261}$$

To see the equivalence between Theorem 3 and Theorem B, first of all, it is an elementary exercise to verify that the linear $\mathbf{t} \leftrightarrow \hat{\mathbf{T}}$ map defined by (258) coincides with the one given in equation (50) specialized to $\varphi_{\rm special}(V) = (e^{2qV}-1)/2q$. The equivalence is then proved by using the relation (52).

Since the WK partition function $Z^{\rm WK}(\mathbf{t};\epsilon)$ is a KdV τ-function, Theorem 3 immediately implies the following corollary.

Corollary. The partition function $Z_{\Omega^{\rm special}(q)}(\mathbf{t}.\varphi_{\rm special};\epsilon)$ *is a KdV* τ*-function.*

From (51) we know that the Corollary can be equivalently stated as the following

Proposition A (Alexandrov [4]). *The partition function* $Z_{\Omega^{\rm special}(q)}\big(\mathbf{t}|\varphi_{\rm special};\epsilon\big)$ *is a KdV* τ*-function.*

Remark 14. Proposition A was obtained by Alexandrov in [4]. Alexandrov's proof in [5] of Theorem B was based on Proposition A. In an early version of the current paper, we also deduced Theorem B from Proposition A before [5] appeared on arXiv with a complete proof (different from Alexandrov's), and also sketched a second proof, not based on Proposition A but using the Virasoro constraints (58) instead, which reduces Theorem B to a few elementary identities including e.g.

(262)
$$\sum_{i=m}^{n} \frac{2i+1}{2} A(i,m)\, P(n,i) \;-\; \sum_{j\geq 1} \frac{4^j-1}{2j} B_{2j} \sum_{i=m}^{n-(2j-1)} A(i,m)\, P(n,i+2j-1)$$
$$= \delta_{n,m} \frac{2m+1}{2} + \delta_{n\geq m+1} \frac{(-1)^{n-m-1}}{(n-m)(n-m+1)} \frac{(2n+1)!!}{2\cdot(2m-1)!!}\,, \quad \forall\, n \geq m \geq 0.$$

The proof of Theorem 3 given here provides a yet different proof of Theorem B, again not using Proposition A (and thus self-contained), but using the loop equations instead. In this proof, we used a technique found during the preparation of the paper [100] by Q. Zhang and the first-named author of the present paper of identifying solutions to the loop equations (although in the end that technique was not given there), whereas our earlier and less self-contained proof preceded [100].

10.2. **Two applications.** In this subsection we will give two applications of the Hodge–WK correspondence.

Application I. *The WK–GUE correspondence.* Recall that the Hodge–GUE correspondence was recently obtained in [35, 40] (see also [34]), which establishes a relationship between the special-Hodge partition function with $q=-1/2$ and the even Gaussian Unitary Ensemble (GUE) partition function (see [2, 11, 41, 54, 98, 99]). So

$$\varphi_{\rm special}(V)|_{q=-1/2} \;=\; 1-e^{-V} \tag{263}$$

is now under consideration.

Let

$$Z_n^{\rm eGUE1}(\mathbf{s};\epsilon) \;:=\; 2^{-\frac{n}{2}} (\pi\epsilon)^{-\frac{n^2}{2}} \int_{\mathcal{H}(n)} e^{-\frac{1}{2\epsilon}{\rm tr}\, M^2 + \frac{1}{\epsilon}\sum_{j\geq 1} s_j {\rm tr}\, M^{2j}} dM \tag{264}$$

be the normalized even GUE partition function of size n [11, 54, 98]. Here $\mathbf{s} = (s_1, s_2, \cdots)$ is an infinite tuple of indeterminates, ϵ is an indeterminate, and "normalized" means that $Z_n^{\rm eGUE1}(\mathbf{0};\epsilon)=1$ for all n and ϵ. According to [11, 54, 55, 56], the logarithm $\log Z_n^{\rm eGUE1}(\mathbf{s};\epsilon) =: \mathcal{F}_n^{\rm eGUE1}(\mathbf{s};\epsilon)$ has the expansion:

$$\mathcal{F}_n^{\rm eGUE1}(\mathbf{s};\epsilon) \;=\; \sum_{g\geq 0}\sum_{k\geq 1}\sum_{j_1,\dots,j_k\geq 1} a_g(2j_1,\dots,2j_k)\, s_{j_1}\dots s_{j_k}\, n^{2-2g-k+|j|}\, \epsilon^{|j|-k}\,, \tag{265}$$

where $|j| := j_1+\cdots+j_k$, and

$$a_g(2j_1,\dots,2j_k) \;=\; \sum_{G\in {\rm Ribb}_g(2j_1,\dots,2j_k)} \frac{2j_1\cdots 2j_k}{|{\rm Aut}(G)|}\,. \tag{266}$$

Here $\mathrm{Ribb}_g(2j_1,\dots,2j_k)$ denotes the set of connected ribbon graphs of genus g with k vertices of valencies $2j_1, \dots, 2j_k$. Note that $a_g(2j_1,\dots,2j_k)$ vanishes unless $2-2g-k+|j|>0$. Therefore, $\log Z_n^{\mathrm{eGUE1}}(\mathbf{s};\epsilon)\in n\,\mathbb{Q}[n,\epsilon][[\mathbf{s}]]$.

Let $x=n\epsilon$ denote the t'Hooft coupling constant [55, 56], and introduce

$$\gamma(z)=\frac{z^2}{2}\Big(\log z-\frac{3}{2}\Big)\,-\,\frac{\log z}{12}\,+\,\sum_{g\geq 2}\frac{B_{2g}}{2g(2g-2)\,z^{2g-2}}\,, \tag{267}$$

which satisfies the second-order difference equation

$$\gamma(z+1)\,-\,2\,\gamma(z)\,+\,\gamma(z-1)\,=\,\log z \tag{268}$$

and for z large and integral gives the asymptotic expansion of $\log(1!2!\cdots(z-1)!)$ up to an additive affine-linear function $\zeta'(-1)+\log(2\pi)z/2$ [97]. Following [2, 40, 41], define the *corrected even GUE free energy* (for short the *even GUE free energy*), denoted $\mathcal{F}^{\mathrm{eGUE}}(x,\mathbf{s};\epsilon)$, as follows:

$$\mathcal{F}^{\mathrm{eGUE}}(x,\mathbf{s};\epsilon)\;=C(x,\epsilon)\,+\,\mathcal{F}^{\mathrm{eGUE1}}_{x/\epsilon}(\mathbf{s};\epsilon)\,, \tag{269}$$

where

$$C(x,\epsilon)\;=\gamma(\frac{x}{\epsilon})\,+\,\frac{x^2}{2\,\epsilon^2}\log\epsilon\,-\,\frac{1}{12}\log\epsilon \tag{270}$$

$$=\frac{x^2}{2\,\epsilon^2}\Big(\log x-\frac{3}{2}\Big)\,-\,\frac{\log x}{12}\,+\,\sum_{g\geq 2}\frac{\epsilon^{2g-2}B_{2g}}{2g\,(2g-2)\,x^{2g-2}}\,. \tag{271}$$

We also define the even GUE partition function $Z^{\mathrm{eGUE}}(x,\mathbf{s};\epsilon)$ as $e^{\mathcal{F}^{\mathrm{eGUE}}(x,\mathbf{s};\epsilon)}$. From the definition we know that $\mathcal{F}^{\mathrm{eGUE}}(x,\mathbf{s};\epsilon)\in\epsilon^{-2}\mathbb{Q}[[x-1,\epsilon^2]][[\mathbf{s}]]$ and $Z^{\mathrm{eGUE}}(x,\mathbf{s};\epsilon)\in\mathbb{Q}((\epsilon^2))[[x-1]][[\mathbf{s}]]$. For more details about the even GUE partition function see [11, 32, 35, 40, 41, 54, 98, 99]. For $k\geq 1$, and $j_1,\dots,j_k\geq 1$, introduce the notation:

$$\langle m_{j_1}\dots m_{j_k}\rangle(x,\epsilon)\;=\;\frac{\partial^k\mathcal{F}^{\mathrm{eGUE1}}_{x/\epsilon}(\mathbf{s};\epsilon)}{\partial s_{j_1}\dots\partial s_{j_k}}\bigg|_{\mathbf{s}=\mathbf{0}}\,. \tag{272}$$

Explicitly,

$$\langle m_{j_1}\dots m_{j_k}\rangle(x,\epsilon)\;=\;k!\sum_{0\leq g\leq\frac{|j|}{2}-\frac{k}{2}+\frac{1}{2}}a_g(2j_1,\dots,2j_k)\,x^{2-2g-k+|j|}\,\epsilon^{2g-2}\,. \tag{273}$$

The *modified even GUE partition function* $Z^{\mathrm{meGUE}}(x,\mathbf{s};\epsilon)$ is introduced in [34] (see also [40]) as the unique element in $\mathbb{Q}((\epsilon^2))[[x-1]][[\mathbf{s}]]$ such that

$$Z^{\mathrm{eGUE}}(x,\mathbf{s};\epsilon)\;=\;Z^{\mathrm{meGUE}}\Big(x-\frac{\epsilon}{2},\mathbf{s};\epsilon\Big)\,Z^{\mathrm{meGUE}}\Big(x+\frac{\epsilon}{2},\mathbf{s};\epsilon\Big)\,. \tag{274}$$

This partition function also relates to the Laguerre Unitary Ensemble (LUE) or say to Grothendieck's dessins d'enfant [52, 102, 103]. The logarithm $\log Z^{\mathrm{meGUE}}(x,\mathbf{s};\epsilon) =: \mathcal{F}^{\mathrm{meGUE}}(x,\mathbf{s};\epsilon)$ is called the *modified even GUE free energy*, which has the form

$$\mathcal{F}^{\mathrm{meGUE}}(x,\mathbf{s};\epsilon) \,=\, B(x,\epsilon) \,+\, \sum_{k\geq 1} \frac{1}{k!} \sum_{j_1,\dots,j_k\geq 1} \langle \phi_{j_1}\dots\phi_{j_k}\rangle(x,\epsilon)\, s_{j_1}\cdots s_{j_k}\,, \tag{275}$$

where

$$B(x,\epsilon) \,=\, \gamma\Big(\frac{x}{2\epsilon}+\frac{1}{4}\Big) \,+\, \gamma\Big(\frac{x}{2\epsilon}-\frac{1}{4}\Big) \,+\, \frac{x^2}{4\,\epsilon^2}\log(2\epsilon) \,-\, \frac{5}{48}\log(2\epsilon) \tag{276}$$

$$= \Big(\frac{1}{4}\log x - \frac{3}{8}\Big)\frac{x^2}{\epsilon^2} \,-\, \frac{5}{48}\log x \,-\, \frac{53\,\epsilon^2}{3840\,x^2} \,+\, \frac{599\,\epsilon^4}{64512\,x^4} \,+\, \dots, \tag{277}$$

$$\langle \phi_{j_1}\dots\phi_{j_k}\rangle(x,\epsilon) \,=\, \frac{1}{e^{\epsilon\partial_x/2}+e^{-\epsilon\partial x/2}}\, \langle m_{j_1}\dots m_{j_k}\rangle(x,\epsilon)\,, \quad k, j_1,\dots,j_k\geq 1\,. \tag{278}$$

Recall that the Hodge–GUE correspondence says that the following identity holds true in $\mathbb{C}((\epsilon^2))[[x-1]][[\mathbf{s}]]$:

$$Z_{\Omega^{\mathrm{special}}(-1/2)}\big(\mathbf{T}^{\mathrm{Hodge-GUE}}(x,\mathbf{s});\sqrt{2}\epsilon\big)\, e^{\frac{A(x,\mathbf{s})}{2\epsilon^2}} \,=\, Z^{\mathrm{meGUE}}(x,\mathbf{s};\epsilon)\,, \tag{279}$$

where $A(x,\mathbf{s})$ is the explicit quadratic series given by (9) and $\mathbf{T}^{\mathrm{Hodge-GUE}}(x,\mathbf{s})$ is defined by

$$T_i^{\mathrm{Hodge-GUE}}(x,\mathbf{s}) \,=\, -1 \,+\, \delta_{i,1} \,+\, x\,\delta_{i,0} \,+\, \sum_{j\geq 1} j^{i+1}\binom{2j}{j} s_j\,, \quad i\geq 0\,. \tag{280}$$

We are ready to prove Theorem 4.

Proof of Theorem 4. By using (279) and (7). □

Let us denote

$$f_m(x) \,:=\, \frac{(2m-1)!!}{2^m}\,x - \frac{(2m+1)!!}{2^m} \quad (m\geq 2). \tag{281}$$

We will also use Witten's notation $\langle \tau_{m_1}\dots\tau_{m_n}\rangle_g$:

$$\langle \tau_{m_1}\dots\tau_{m_n}\rangle_g \,=\, \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{m_1}\cdots\psi_n^{m_n}\,. \tag{282}$$

Performing the Taylor expansion with respect to $s_1, s_2, \cdots$ on the logarithms of both sides of (8) and using the dilaton equation (54), we arrive at the following proposition.

Proposition 6. *We have*

$$\frac{\frac14 - x}{\epsilon^2} - \frac{1}{24}\log\frac{3-x}{2} + \sum_{g,\,p\geq 0} \epsilon^{2g-2}\frac{(x-1)^p}{p!}\sum_{\lambda\in\mathcal{P}_{3g-3+p}}\frac{\langle\tau_0^p\tau_{\lambda+1}\rangle_g}{\mathrm{mult}(\lambda)!}\Big(\frac{2}{3-x}\Big)^{2g-2+\ell(\lambda)+p}\prod_{s=1}^{\ell(\lambda)} f_{1+\lambda_s}(x) = B\Big(x,\frac{\epsilon}{\sqrt2}\Big)\,, \tag{283}$$

where $B(x,\epsilon)$ *is defined in* (276)*, and for* $k, j_1,\dots,j_k\geq 1$*,*

$$\frac{\delta_{k,1}}{\epsilon^2}\binom{2j_1}{j_1}\Big(x-\frac{j_1}{j_1+1}\Big) + \frac{\delta_{k,2}}{\epsilon^2}\frac{j_1j_2}{j_1+j_2}\binom{2j_1}{j_1}\binom{2j_2}{j_2} + \sum_{m_1,\dots,m_k\geq 0} U_{m_1,\dots,m_k}(x,\epsilon)\prod_{i=1}^{k} e_{m_i,\,j_i} = \langle\phi_{j_1}\dots\phi_{j_k}\rangle\Big(x,\frac{\epsilon}{\sqrt2}\Big)\,, \tag{284}$$

where

$$e_{m,j} := \frac{(2m+2j-1)!!}{2^{m-j}(j-1)!}\,, \tag{285}$$

and for $m_1,\dots,m_k\geq 0$*,*

$$U_{m_1,\dots,m_k}(x,\epsilon) := \sum_{g\geq 0}\epsilon^{2g-2}\sum_{p\geq 0}\frac{(x-1)^p}{p!} \times \sum_{\lambda\in\mathcal{P}_{3g-3+p+k-|m|}}\frac{\langle\tau_0^p\tau_{\lambda+1}\tau_{m_1}\dots\tau_{m_k}\rangle_g}{\mathrm{mult}(\lambda)!}\Big(\frac{2}{3-x}\Big)^{2g-2+\ell(\lambda)+k+p}\prod_{s=1}^{\ell(\lambda)} f_{1+\lambda_s}(x)\,. \tag{286}$$

For instance, using (46) the coefficient of ϵ^{-2} of the left-hand side of (283) begins

$$\frac14 - x + \frac{(x-1)^3}{3!}\Big[\frac{2}{3-x}\Big] + \frac{(x-1)^4}{4!}\Big[\Big(\frac{2}{3-x}\Big)^3\Big(\frac34 x-\frac{5!!}{4}\Big)\Big] + \frac{(x-1)^5}{5!}\Big[\Big(\frac{2}{3-x}\Big)^4\Big(\frac{5!!}{8}x-\frac{7!!}{8}\Big)+\frac{6}{2!}\Big(\frac{2}{3-x}\Big)^5\Big(\frac34 x-\frac{5!!}{4}\Big)^2\Big] + \cdots\,, \tag{287}$$

which is consistent with the expansion of $\frac{x^2}{4}\log x - \frac38 x^2$ as $x\to 1$. (With the above given terms one can check the agreement up to and including $\mathrm{O}((x-1)^5)$.) Similarly, using $\langle\tau_0\tau_2\rangle_1 = \langle\tau_0^3\tau_3\rangle_1 = 1/24$ and $\langle\tau_0^2\tau_2^2\rangle_1 = 1/6$, the coefficient of ϵ^0 of the left-hand side of (283) begins

$$(288)\quad -\frac{1}{24}\log\frac{3-x}{2} + (x-1)\left[\frac{1}{24}\Big(\frac{2}{3-x}\Big)^2\Big(\frac{3}{4}x-\frac{5!!}{4}\Big)\right] + \frac{(x-1)^2}{2!}\left[\frac{1}{24}\Big(\frac{2}{3-x}\Big)^3\Big(\frac{5!!}{8}x-\frac{7!!}{8}\Big)+\frac{1}{6}\frac{1}{2!}\Big(\frac{2}{3-x}\Big)^4\Big(\frac{3}{4}x-\frac{5!!}{4}\Big)^2\right] + \cdots,$$

which is consistent with the expansion of $\frac{5}{48}\log x$ as $x\to 1$. The coefficient of ϵ^2 of the left-hand side of (283) begins

$$(289)\quad \left[\langle\tau_4\rangle_2\Big(\frac{2}{3-x}\Big)^4\Big(\frac{7!!}{16}x-\frac{9!!}{16}\Big) + \langle\tau_3\tau_2\rangle_2\Big(\frac{2}{3-x}\Big)^5\Big(\frac{5!!}{8}x-\frac{7!!}{8}\Big)\Big(\frac{3}{4}x-\frac{5!!}{4}\Big) + \frac{\langle\tau_2^3\rangle_2}{3!}\Big(\frac{2}{3-x}\Big)^6\Big(\frac{3}{4}x-\frac{5!!}{4}\Big)^3\right] + (x-1)\Big[\cdots\Big] + \cdots,$$

which is consistent with the expansion of $-\frac{1}{2}\frac{53\,\epsilon^2}{3840\,x^2}$ as $x\to 1$ by substituting $\langle\tau_4\rangle_2 = 1/1152$, $\langle\tau_3\tau_2\rangle_2 = 29/5760$ and $\langle\tau_2^3\rangle_2 = 7/240$ [98]. Let us present one more verification. Recall that $\langle m_1\rangle(x,\epsilon) = x^2/\epsilon^2$, giving $\langle\phi_1\rangle(x,\epsilon) = \frac{x^2}{2\epsilon^2}-\frac{1}{8}$. Using (46) the coefficient of ϵ^{-2} of the left-hand side of (284) for $k=1$ and $j_1=1$ begins

$$(290)\quad e_{0,1}\left\{\frac{(x-1)^2}{2!}\left[\frac{2}{3-x}\right] + \frac{(x-1)^3}{3!}\left[\Big(\frac{2}{3-x}\Big)^3\Big(\frac{3}{4}x-\frac{5!!}{4}\Big)\right] + \cdots\right\} + e_{1,1}\left\{\frac{(x-1)^3}{3!}\left[\Big(\frac{2}{3-x}\Big)^2\right] + \cdots\right\} + \cdots + 2\Big(x-\frac{1}{2}\Big),$$

which is consistent with the expansion of x^2 as $x\to 1$.

Application II. *The WK–BGW correspondence.* The Hodge–BGW correspondence was recently found in [100], which gives a relationship between the special-Hodge partition function again with $q=-1/2$ and the generalized Brézin–Gross–Witten (BGW) model (see [3, 13, 53, 82]). Let $Z^{\mathrm{cBGW}}(x,\mathbf{r};\epsilon)$ denote the generalized BGW partition function in the sense of [100] (it is denoted by $Z(x,\mathbf{T};\hbar)$ in [100]), which is an element in $\mathbb{C}((\epsilon^2))[[x+2]][[\mathbf{r}]]$. Here x is the *Alexandrov coupling constant*, ϵ is an indeterminate, and $\mathbf{r}=(r_0,r_1,r_2,\dots)$ is an infinite tuple of indeterminates. In particular, we recall that

$$(291)\quad \log Z^{\mathrm{cBGW}}(x,\mathbf{0};\epsilon) = \frac{x^2}{4\,\epsilon^2}\Big(\log\big(-\frac{x}{2}\big)-\frac{3}{2}\Big) + \frac{\log\big(-\frac{x}{2}\big)}{12} - \sum_{g\geq 2}\frac{\epsilon^{2g-2}(-2)^{g-1}B_{2g}}{2g\,(2g-2)\,x^{2g-2}}$$

For more details about the generalized BGW partition function see e.g. [44, 100, 101] and the references therein.

The Hodge–BGW correspondence says that

$$Z_{\Omega(-1/2)}\big(\mathbf{T}^{\text{Hodge-BGW}}(x,\mathbf{r});\sqrt{-4}\epsilon\big)\, e^{\frac{A_{\text{cBGW}}(x,\mathbf{r})}{\epsilon^2}} \;=\; Z^{\text{cBGW}}(x,\mathbf{r};\epsilon)\,, \tag{292}$$

where $A_{\text{cBGW}}(x,\mathbf{r})$ is the quadratic series given by (12), and

$$\begin{aligned} &T_i^{\text{Hodge-BGW}}(x,\mathbf{r}) \\ &\quad = -\Big(-\frac{1}{2}\Big)^{i-1} + \delta_{i,1} + x\,\delta_{i,0} - 2\sum_{j\geq 0}\frac{1}{j!}\Big(-\frac{2j+1}{2}\Big)^{i} r_j\,, \quad i\geq 0\,. \end{aligned} \tag{293}$$

Both the WK partition function $Z^{\text{WK}}(\mathbf{t};\epsilon)$ and the generalized BGW partition function $Z^{\text{cBGW}}(x,\mathbf{r};\epsilon)$ are particular τ-functions for the KdV hierarchy [44]. However, we would like to remark that the Hodge–WK correspondence (7) is different from the Hodge–BGW correspondence (292). This can be seen from the simple fact that the change of the independent variables (293) in the Hodge–BGW correspondence is NOT invertible, while that in the Hodge–WK correspondence is part of a group action and hence certainly invertible. As far as we know the WK partition function $Z^{\text{WK}}(\mathbf{t};\epsilon)$ cannot be obtained from the generalized BGW partition function $Z^{\text{cBGW}}(x,\mathbf{r};\epsilon)$ by just shifting the independent variables (vice versa). However, these two different correspondences enable us to establish a relationship between $Z^{\text{WK}}(\mathbf{t};\epsilon)$ and $Z^{\text{cBGW}}(x,\mathbf{r};\epsilon)$ (see Theorem 5 below). Of course, some important connections between $Z^{\text{WK}}(\mathbf{t};\epsilon)$ and $Z^{\text{cBGW}}(x,\mathbf{r};\epsilon)$ were already known: the genus zero parts of the WK partition function $Z^{\text{WK}}(\mathbf{t};\epsilon)$ and the generalized BGW partition function $Z^{\text{cBGW}}(x,\mathbf{r};\epsilon)$ are different, but they are equal for genus bigger than or equal to 1 in a non-obvious way. Indeed, for $g\geq 1$, the jet representations of the genus g WK free energy and the genus g generalized BGW free energy are the same, which is actually the content of the Okuyama–Sakai conjecture [92] proved in [100, 101].

We are ready to prove Theorem 5.

Proof of Theorem 5. By using (292) and (7) with $q=-1/2$. □

Denote

$$g_m(x) := \frac{(2m-1)!!}{2^m}\,x \quad (m\geq 2), \tag{294}$$

and denote by $\langle\omega_{j_1}\dots\omega_{j_k}\rangle(x,\epsilon)$ the generalized BGW correlators, i.e.,

$$\langle\omega_{j_1}\dots\omega_{j_k}\rangle(x,\epsilon) \;=\; \frac{\partial^k \log Z^{\text{cBGW}}(x,\mathbf{r};\epsilon)}{\partial r_{j_1}\dots\partial r_{j_k}}\bigg|_{\mathbf{s}=\mathbf{0}}\,. \tag{295}$$

Similarly to Proposition 6 we have the following proposition.

Proposition 7. *There holds that*

$$
\begin{aligned}
(296)\quad & \frac{\frac{x}{3}+\frac{1}{2}}{\epsilon^2} - \frac{1}{24}\log\Big(-\frac{x}{2}\Big) \\
& + \sum_{g,p\geq 0} (\sqrt{-4}\epsilon)^{2g-2} \frac{(x+2)^p}{p!} \sum_{\lambda\in\mathcal{P}_{3g-3+p}} \frac{\langle \tau_0^p \tau_{\lambda+1}\rangle_g}{\mathrm{mult}(\lambda)!} \Big(-\frac{2}{x}\Big)^{2g-2+\ell(\lambda)+p} \prod_{s=1}^{\ell(\lambda)} g_{1+\lambda_s}(x) \\
& = \log Z^{\mathrm{cBGW}}(x,\mathbf{0};\epsilon)\,,
\end{aligned}
$$

where the expression of $\log Z^{\mathrm{cBGW}}(x,\mathbf{0};\epsilon)$ *is given in* (291)*, and for* $k \geq 1$ *and* $j_1,\dots,j_k \geq 0$,

$$
\begin{aligned}
(297)\quad & -\frac{\delta_{k,1}}{\epsilon^2}\frac{1}{j_1!}\Big(\frac{x}{2j_1+1}+\frac{1}{j_1+1}\Big) + \frac{\delta_{k,2}}{\epsilon^2}\frac{1}{j_1!j_2!(j_1+j_2+1)} \\
& + \sum_{0\leq m_1\leq j_1,\dots,0\leq m_k\leq j_k} V_{m_1,\dots,m_k}(x,\epsilon) \prod_{i=1}^{k} E_{m_i,j_i} \\
& = \langle \omega_{j_1}\dots\omega_{j_k}\rangle(x,\epsilon)\,,
\end{aligned}
$$

where

$$
(298)\quad E_{m,j} := -2\,\frac{(-1)^m}{(j-m)!}\,,
$$

and for $0\leq m_1\leq j_1,\ \dots,\ 0\leq m_k\leq j_k$,

$$
\begin{aligned}
(299)\quad V_{m_1,\dots,m_k}(x,\epsilon) &:= \sum_{g\geq 0}(-4)^{g-1}\epsilon^{2g-2}\sum_{p\geq 0}\frac{(x+2)^p}{p!} \\
& \times \sum_{\lambda\in\mathcal{P}_{3g-3+p+k-|m|}} \frac{\langle \tau_0^p\tau_{\lambda+1}\tau_{m_1}\dots\tau_{m_k}\rangle_g}{\mathrm{mult}(\lambda)!}\Big(-\frac{2}{x}\Big)^{2g-2+\ell(\lambda)+k+p}\prod_{s=1}^{\ell(\lambda)} g_{1+\lambda_s}(x)\,.
\end{aligned}
$$

For instance, using (46) the coefficient of ϵ^{-2} of the left-hand side of (296) begins

$$
\begin{aligned}
(300)\quad & x+\frac{1}{2}-\frac{1}{4}\bigg\{\frac{(x+2)^3}{3!}\Big[-\frac{2}{x}\Big] + \frac{(x+2)^4}{4!}\Big[\Big(-\frac{2}{x}\Big)^3\Big(\frac{3}{4}x\Big)\Big] \\
& + \frac{(x+2)^5}{5!}\Big[\Big(-\frac{2}{x}\Big)^4\Big(\frac{5!!}{8}x\Big)+\frac{6}{2!}\Big(-\frac{2}{x}\Big)^5\Big(\frac{3}{4}x\Big)^2\Big] + \cdots\bigg\}\,,
\end{aligned}
$$

which agrees with the expansion of $\frac{1}{4}x^2\log\bigl(-\frac{x}{2}\bigr)-\frac{3}{8}x^2$ as $x\to-2$. Similarly, the coefficient of ϵ^0 of the left-hand side of (296) begins

$$-\frac{1}{24}\log\Bigl(-\frac{x}{2}\Bigr)\,+\,(x+2)\biggl[\frac{1}{24}\Bigl(-\frac{2}{x}\Bigr)^2\Bigl(\frac{3}{4}x\Bigr)\biggr] + \frac{(x+2)^2}{2!}\biggl[\frac{1}{24}\Bigl(-\frac{2}{x}\Bigr)^3\Bigl(\frac{5!!}{8}x\Bigr)+\frac{1}{6}\frac{1}{2!}\Bigl(-\frac{2}{x}\Bigr)^4\Bigl(\frac{3}{4}x\Bigr)^2\biggr]\,+\,\cdots\,, \tag{301}$$

which agrees with the expansion of $\frac{1}{12}\log\bigl(-\frac{x}{2}\bigr)$ as $x\to-2$. The coefficient of ϵ^2 of the left-hand side of (296) begins

$$-4\biggl[\langle\tau_4\rangle_2\Bigl(-\frac{2}{x}\Bigr)^4\Bigl(\frac{7!!}{16}x\Bigr)\,+\,\langle\tau_3\tau_2\rangle_2\Bigl(-\frac{2}{x}\Bigr)^5\Bigl(\frac{5!!}{8}x\Bigr)\Bigl(\frac{3}{4}x\Bigr) + \frac{\langle\tau_2^3\rangle_2}{3!}\Bigl(-\frac{2}{x}\Bigr)^6\Bigl(\frac{3}{4}x\Bigr)^3\biggr]-4\,(x+2)\Bigl[\cdots\Bigr]\,+\,\cdots\,, \tag{302}$$

which agrees with the expansion of $-\frac{1}{480}\frac{1}{x^2}$ as $x\to-2$. Using (46) the coefficient of ϵ^{-2} of the left-hand side of (297) for $k=1$ and $j_1=1$ begins

$$-\frac{1}{4}\,E_{0,1}\biggl\{\frac{(x-1)^2}{2!}\biggl[-\frac{2}{x}\biggr]\,+\,\frac{(x-1)^3}{3!}\biggl[\Bigl(-\frac{2}{x}\Bigr)^3\Bigl(\frac{3}{4}x\Bigr)\biggr]\,+\,\cdots\biggr\} -\frac{1}{4}\,E_{1,1}\biggl\{\frac{(x-1)^3}{3!}\biggl[\Bigl(-\frac{2}{x}\Bigr)^2\biggr]\,+\,\cdots\biggr\}\,+\,\cdots-\Bigl(\frac{x}{3}+\frac{1}{2}\Bigr)\,, \tag{303}$$

which is consistent with the expansion of $x^4/96$ as $x\to-2$.

11. The Hodge mapping partition function

In the previous sections, we have introduced a $\mathcal{G}$-action on infinite tuples, have defined for any $\varphi\in\mathcal{G}$ the WK mapping partition function associated to φ, and have associated to it the WK mapping hierarchy. In this section, just like the previous constructions, for any $\varphi\in\mathcal{G}$ we define the *Hodge mapping partition function* $Z^{\varphi}_{\Omega(\boldsymbol{\sigma})}$ associated to φ by

$$Z^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon)\;=\;Z_{\Omega(\boldsymbol{\sigma})}\bigl(\mathbf{T}.\varphi^{-1};\epsilon\bigr)\,, \tag{304}$$

which as we will see also has several nice properties. Here $Z_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon)$ is the Hodge partition function defined in (225). Obviously, $Z^{\varphi}_{\Omega(\mathbf{0})}(\mathbf{T};\epsilon)=Z^{\varphi}(\mathbf{T};\epsilon)$.

Recall that the Hodge partition function satisfies the dilaton equation:

$$\sum_{i\geq0}t_i\,\frac{\partial Z_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon)}{\partial t_i}\,+\,\epsilon\,\frac{\partial Z_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon)}{\partial\epsilon}\,+\,\frac{1}{24}\,Z_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon)\;=\;\frac{\partial Z_{\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon)}{\partial t_1}\,. \tag{305}$$

It follows the dilaton equation for the Hodge mapping partition function:

$$\sum_{i\geq 0} T_i \frac{\partial Z^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon)}{\partial T_i} + \epsilon \frac{\partial Z^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon)}{\partial \epsilon} + \frac{1}{24} Z^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon) = \frac{\partial Z^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon)}{\partial T_1}. \tag{306}$$

The Virasoro constraints for the Hodge partition function can be found in [72]. So it is possible to translate them to the Hodge mapping partition function, which we will do elsewhere.

As before, the logarithm $\log Z^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon) =: \mathcal{F}^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon)$, called the *Hodge mapping free energy*, has a genus expansion:

$$\mathcal{F}^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon) =: \sum_{g\geq 0} \epsilon^{2g-2} \mathcal{F}^{\varphi}_{\Omega_g(\boldsymbol{\sigma})}(\mathbf{T}). \tag{307}$$

We call $\mathcal{F}^{\varphi}_{\Omega_g(\boldsymbol{\sigma})}(\mathbf{T})$, $g \geq 0$, *the genus g Hodge mapping free energy.* By Theorem 6 and the well-known fact $\mathcal{F}_{\Omega_0(\boldsymbol{\sigma})}(\mathbf{t}) \equiv \mathcal{F}^{\rm WK}_0(\mathbf{t})$ we immediately obtain the following

Proposition 8. *For any $\varphi \in \mathcal{G}$, we have $\mathcal{F}^{\varphi}_{\Omega_0(\boldsymbol{\sigma})} = \mathcal{F}^{\rm WK}_0$.*

For genus bigger than or equal to 1, it is known from e.g. [34, 43] that the genus g ($g \geq 1$) Hodge free energy $\mathcal{F}_{\Omega_g(\boldsymbol{\sigma})}(\mathbf{t})$ has the $(3g-2)$-jet representation, i.e., there exists $F_{\Omega_g(\boldsymbol{\sigma})}(v_0, v_1, \dots, v_{3g-2}; \boldsymbol{\sigma})$, such that

$$\mathcal{F}_{\Omega_g(\boldsymbol{\sigma})}(\mathbf{t}) = F_{\Omega_g(\boldsymbol{\sigma})}\left(E(\mathbf{t}), \frac{\partial E(\mathbf{t})}{\partial t_0}, \dots, \frac{\partial^{3g-2} E(\mathbf{t})}{\partial t_0^{3g-2}}; \boldsymbol{\sigma}\right), \quad g \geq 1, \tag{308}$$

where $E(\mathbf{t})$ is defined by (22). We then have the following

Proposition 9. *For $g = 1$ we have the identity:*

$$\mathcal{F}^{\varphi}_{\Omega_1(\boldsymbol{\sigma})}(\mathbf{T}) = F^{\varphi}_{\Omega_1(\boldsymbol{\sigma})}\left(E(\mathbf{T}), \frac{\partial E(\mathbf{T})}{\partial X}; \boldsymbol{\sigma}\right), \tag{309}$$

with

$$F^{\varphi}_{\Omega_1(\boldsymbol{\sigma})}(V, V_1; \boldsymbol{\sigma}) := \frac{1}{24} \log V_1 + \frac{1}{16} \log \varphi'(V) + \frac{\sigma_1}{24} \varphi(V). \tag{310}$$

For each $g \geq 2$, there exists $F^{\varphi}_{\Omega_g(\boldsymbol{\sigma})}(V_0, \dots, V_{3g-2}; \boldsymbol{\sigma})$ that is a polynomial of $\sigma_1, \dots, \sigma_{2g-1}$, V_2, ..., V_{3g-2} and a rational function of V_1, such that

$$\mathcal{F}^{\varphi}_{\Omega_g(\boldsymbol{\sigma})}(\mathbf{T}) = F^{\varphi}_{\Omega_g(\boldsymbol{\sigma})}\left(E(\mathbf{T}), \dots, \frac{\partial^{3g-2} E(\mathbf{T})}{\partial X^{3g-2}}; \boldsymbol{\sigma}\right). \tag{311}$$

Let

$$U^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon) := \epsilon^2 \frac{\partial^2 \mathcal{F}^{\varphi}_{\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon)}{\partial X^2}, \tag{312}$$

where $X = T_0$. This gives a quasi-Miura transformation

$$V \;\mapsto\; U^{\varphi}_{\Omega(\boldsymbol{\sigma})} \;=\; V \,+\, \sum_{g\geq 1} \epsilon^{2g}\, \partial^2\big(F^{\varphi}_{\Omega_g(\boldsymbol{\sigma})}\big)\,, \tag{313}$$

which transforms the abstract local RH hierarchy $D_S(v) = S(v)\, v_1$ to

$$\begin{aligned} D_S(U) \;=\; S\, U_1 \,+\, \epsilon^2\bigg(\frac{S'}{12} U_3 \,+\, \bigg(\frac{\varphi'' S'}{8\varphi'} + \frac{S''}{6} + \frac{\sigma_1}{12}\varphi' S'\bigg) U_1\, U_2 \\ +\, \bigg(-\frac{\varphi''^2 S'}{16\varphi'^2} + \frac{\varphi''' S' + \varphi'' S''}{16\varphi'} + \frac{S'''}{24} + \frac{\sigma_1}{24}\big(\varphi' S'\big)'\bigg) U_1^3\bigg) \,+\, \cdots\,, \end{aligned} \tag{314}$$

where $U = U^{\varphi}_{\Omega(\boldsymbol{\sigma})}$, and we omitted the arguments $U^{\varphi}_{\Omega(\boldsymbol{\sigma})}$ from $\varphi', \varphi'', \cdots$ and from $S, S', S'', \cdots$. We call (314) the *abstract local Hodge mapping hierarchy associated to* φ. When $\varphi = id$, we call (314) the *abstract local Hodge hierarchy.*

Define $\Omega^{\rm Hodge}_{S_1(w),S_2(w)}$, $S_1(w), S_2(w) \in \mathcal{O}_c(w)$, as the substitution of the inverse of the quasi-Miura type transformation $v \mapsto w = v + \sum_{g\geq 1} \epsilon^{2g}\, \partial^2\big(F_{\Omega_g(\boldsymbol{\sigma})}\big)$ in $\int^w S_1 S_2 + \sum_{g\geq 1} \epsilon^{2g} D_{S_1} D_{S_2}(F_{\Omega_g(\boldsymbol{\sigma})})$. Similarly, define $\Omega^{\varphi,\,\rm Hodge}_{S_1(U),S_2(U)}$, $S_1(U), S_2(U) \in \mathcal{O}_c(U)$, as the substitution of the inverse of (313) in $\int^U S_1 S_2 + \sum_{g\geq 1} \epsilon^{2g} D_{S_1(U)} D_{S_2(U)}(F^{\varphi}_{\Omega_g(\boldsymbol{\sigma})})$.

The following theorem, which is a refinement of Theorem 2, gives a generalization of Theorem 11 and some results in [18, 19, 34].

Theorem 14. *The abstract local Hodge mapping hierarchy* (180) *have polynomiality: for any S the right-hand side of* (180) *belongs to* $\mathcal{A}_{U^{\varphi}_{\Omega(\boldsymbol{\sigma})}}[[\epsilon^2]]_1$*. Moreover, the elements* $\Omega^{\varphi,\,\rm Hodge}_{S_1(U),\,S_2(U)}$, $S_1(U), S_2(U) \in \mathcal{O}_c(U)$*, belong to* $\mathcal{A}_{U^{\varphi}_{\Omega(\boldsymbol{\sigma})}}[[\epsilon^2]]_0$.

Let us first prove Theorem 14 for the case when $\varphi = \mathrm{id}$. Indeed, similarly to the proof of Proposition 5, by using the properties of the τ-symmetric hamiltonian densities of the Hodge hierarchy [34] and the results in [14, 18, 19], we arrive at the following proposition.

Proposition 10. *Theorem 14 holds when $\varphi = \mathrm{id}$. Moreover, the elements* $\Omega^{\rm Hodge}_{S_1(w),\,S_2(w)}$, $S_1(w), S_2(w) \in \mathcal{O}_c(w)$*, belong to* $\mathcal{A}_w[[\epsilon^2]]_0$.

Proof of Theorem 14. First,

$$\partial \;=\; \sum_{m\geq 0} \frac{\partial t_m}{\partial X} D_{w^m/m!} \;=\; D_{\sqrt{\varphi'(\varphi^{-1}(w))}}\,. \tag{315}$$

Here $D_{w^m/m!}$, $m \geq 0$, are derivations of the abstract Hodge hierarchy. By Proposition 10 the element $D_{\sqrt{\varphi'(\varphi^{-1}(w))}}(w)$ has polynomiality. Note that

$$\Omega^{\varphi,\,\rm Hodge}_{U^i/i!,\,U^j/j!} \;=\; \sum_{i_1,\,j_1\geq 0} \frac{\partial t_{i_1}}{\partial T_i} \frac{\partial t_{j_1}}{\partial T_j}\, \Omega^{\rm Hodge}_{w^{i_1}/i_1!,\,w^{j_1}/j_1!}\,, \quad i, j \geq 0\,. \tag{316}$$

By an iteration, the ∂_x-flow for w with $\partial = \partial_X$ as the spatial derivative is an evolutionary PDE in Dubrovin–Zhang's normal form. Since $\Omega^{\rm Hodge}_{w^{i_1}/i_1!,\, w^{j_1}/j_1!} \in \mathcal{A}_w[[\epsilon^2]]_0$ and by substituting the ∂_x-flow, we find that $\Omega^{\rm Hodge}_{w^{i_1}/i_1!,\, w^{j_1}/j_1!}$ are power series of ϵ^2 with coefficients being polynomials of $\partial_X(w)$, $\partial_X^2(w)$, $\dots$, so are $\Omega^{\varphi,\,\rm Hodge}_{U^i/i!,\, U^j/j!}$. This implies in particular that $U = U^{\varphi}_{\Omega(\boldsymbol{\sigma})} = \Omega^{\varphi,\,\rm Hodge}_{1,1} = \varphi^{-1}(w) + \dots$ gives a Miura-type transformation. So $\Omega^{\varphi,\,\rm Hodge}_{U^i/i!,\, U^j/j!} \in \mathcal{A}_U[[\epsilon^2]]_0$, and thus $\Omega^{\varphi,\,\rm Hodge}_{S_1(U),\, S_2(U)} \in \mathcal{A}_U[[\epsilon^2]]_0$. Finally, $D_{S(U)}(U) = D_{S(U)}\big(\Omega^{\varphi,\,\rm Hodge}_{1,\,1}\big) = \partial\big(\Omega^{\varphi,\,\rm Hodge}_{1,\,S(U)}\big) \in \mathcal{A}_U[[\epsilon^2]]_1$. □

We also verified Theorem 14 directly up to and including terms of ϵ^8.

By using again the definition (i.e., using the quasi-Miura map), we find that the abstract local Hodge mapping hierarchy (180) has the more precise form:

(317)
$$D_S(U) \,=\, \partial\left(\int^U S \,+\, \sum_{g\geq 1} \epsilon^{2g} \sum_{\lambda\in\mathcal{P}_{2g}} \sum_{j=1}^{\ell(\lambda)+g-1} Y^{\varphi}_{\lambda,j}(l_1(U),\dots;m_1(U),\dots)\, S^{(j)}(U)\, U_\lambda\right),$$

where $U = U^{\varphi}_{\Omega(\boldsymbol{\sigma})}$, $Y^{\varphi}_{\lambda,j}(\ell_1,\dots;\rho_1,\dots)$ are weighted homogeneous polynomials of degree $\ell(\lambda)+g-1-j$ in variables ℓ_i and ρ_i of weight i ($i\geq 1$), $l_i(U)$ are defined in (117), and $m_i(U) = \sigma_{2i-1}\varphi'(U)^i$.

The abstract local Hodge mapping hierarchy (314) can also be written in the form

$$D_S(U) \,=\, P^{\varphi}_1(U)\left(\frac{\delta\int h^{\varphi}_{1;S}}{\delta U}\right), \quad S\in\mathcal{O}_c\,, \tag{318}$$

where $U = U^{\varphi}_{\Omega(\boldsymbol{\sigma})}$, $P^{\varphi}_1(U)$ is the operator given by

$$P^{\varphi}_1(U) \,:=\, \sum_{k,\ell\geq 0} (-1)^{\ell}\, \frac{\partial U}{\partial V_k}\circ\partial^k\circ\left(\frac{1}{2}\,\frac{1}{\varphi'(V)}\circ\partial \,+\, \frac{1}{2}\,\partial\circ\frac{1}{\varphi'(V)}\right)\circ\partial^{\ell}\circ\frac{\partial U}{\partial V_\ell}\,, \tag{319}$$

and the hamiltonian density $h^{\varphi}_{1;S}$ is understood as the substitution of the inverse of the quasi-Miura transformation (313) into (200). As before, $P^{\varphi}(U)$ has the form:

$$P^{\varphi}_1(U) \,=\, \sum_{g\geq 0} \epsilon^{2g}\, P^{\varphi,[g]}_{1;\Omega(\boldsymbol{\sigma})}\,, \quad P^{\varphi,[0]}_{1;\Omega(\boldsymbol{\sigma})} \,=\, \frac{1}{2\,\varphi'(U)}\circ\partial \,+\, \partial\circ\frac{1}{2\,\varphi'(U)}\,, \tag{320}$$

$$P^{\varphi,[g]}_{1;\Omega(\boldsymbol{\sigma})} \,=\, \sum_{j=0}^{3g+1} A^{\varphi}_{2g,j;\Omega(\boldsymbol{\sigma})}\,\partial^j\,, \quad A^{\varphi}_{2g,j;\Omega(\boldsymbol{\sigma})} \in \mathcal{O}_c(U)\big[U_1,\dots,U_{3g+1},U_1^{-1}\big][\boldsymbol{\sigma}]\,, \tag{321}$$

$$\sum_{m\geq 1} m\, U_m\, \frac{\partial A^{\varphi}_{2g,j;\Omega(\boldsymbol{\sigma})}}{\partial U_m} \,=\, (2g+1-j) A^{\varphi}_{2g,j;\Omega(\boldsymbol{\sigma})}\,. \tag{322}$$

We have the following conjecture.

Conjecture 1. *For $g \geq 0$ and $0 \leq j \leq 3g+1$, the elements $A^{\varphi}_{2g,j;\Omega(\boldsymbol{\sigma})}$ all belong to $\mathcal{A}_U^{[2g+1-j]}$. Moreover, for $i \geq 0$, the variational derivatives of the hamiltonians $\int h^{\varphi}_{1;S}$ with respect to U belong to $\mathcal{A}_U[[\epsilon]]$.*

Motivated by the Hodge universality conjecture proposed in [34] (see Remark 12) and the classification work mentioned in Section 9, we propose the following *Hodge mapping universality conjecture.*

Conjecture 2. *The abstract local Hodge mapping hierarchy is a universal object for hamiltonian perturbations of the abstract local RH hierarchy possessing a τ-structure.*

Conjecture 2 generalizes Theorem 13 as well as the Hodge universality conjecture from [34]. Let us verify Conjecture 2 *directly* up to and including terms of order 8 in ϵ. Indeed, the following Miura-type transformation

$$w \;=\; M(U) \,+\, \sum_{k=1}^{4} \epsilon^{2k} \sum_{\lambda\in\mathcal{P}_{2k}} C_\lambda(U)\, U_\lambda \,+\, \mathcal{O}(\epsilon^{10}) \tag{323}$$

transforms the abstract local Hodge mapping hierarchy (314) to the standard form (152) up to ϵ^8, with $U = U^{\varphi}_{\Omega(\boldsymbol{\sigma})}$,

$$M(U) \;=\; \int_0^U \sqrt{\varphi'(y)}\, dy\,, \tag{324}$$

$$a_0(w) \;=\; M'(M^{-1}(w))\,, \tag{325}$$

and

$$C_{(2)}(U) \;=\; -\frac{\sigma_1}{24}\, \varphi'(U)^{3/2}\,,$$

$$C_{(1^2)}(U) \;=\; -\frac{\sigma_1}{24} \sqrt{\varphi'(U)}\, \varphi''(U) \,+\, \frac{\varphi''(U)^2}{24\, \varphi'(U)^{3/2}} - \frac{\varphi^{(3)}(U)}{48\sqrt{\varphi'(U)}}\,,$$

$$C_{(4)}(U) \;=\; -\frac{\sigma_1}{240} \sqrt{\varphi'(U)}\, \varphi''(U) \,+\, \frac{\sigma_1^2}{1920} \varphi'(U)^{5/2} \,+\, \frac{\varphi''(U)^2}{384\, \varphi'(U)^{3/2}} - \frac{\varphi^{(3)}(U)}{480\sqrt{\varphi'(U)}}\,,$$

$$\cdots\,,$$

$$C_{(1^8)}(U) \;=\; -\frac{107}{185794560} \frac{\varphi^{(12)}(U)}{\sqrt{\varphi'(U)}} \,+\, \text{ more than two hundred terms}\,.$$

Here the beginning relationships between the classification invariants $q_1, q_2, \cdots$ and the Chern-Hodge-Mumford parameters $\sigma_1, \sigma_3, \cdots$ are given by

$$q_1 \;=\; \frac{\sigma_1}{2^5\, 3^2\, 5^1}\,, \qquad q_2 \;=\; \frac{2\,\sigma_1^3 - \sigma_3}{2^{10}\, 3^5\, 5^1}\,, \qquad q_3 \;=\; \frac{16\,\sigma_1^5 - 20\,\sigma_1^2\sigma_3 + \sigma_5}{2^{13}\, 3^6\, 5^2\, 7^1}\,. \tag{326}$$

We note that the relations in (326) coincide with the ones given in [34] (see also [17]). Note that in [34, 17] only the case with $a_0(w) \equiv 1$ (i.e., the case with $\varphi(V) = V$) was considered. But the results in this paper show that the above beginning relations (326) do not depend on φ. In general, this independence of φ is expected. Note that equations (326) specialize to (172) when the σ's are specialized by (230).

Remark 15. For each CohFT, A. Buryak [15] defined the *double ramification (DR) hierarchy*, which is a τ-symmetric hamiltonian system [16, 17]. For the trivial case (the case when the CohFT is given by $\Omega(\mathbf{0}) = 1$), the DR hierarchy coincides with the KdV hierarchy. For the Hodge CohFT $\Omega(\boldsymbol{\sigma})$ (see (226)), it is conjectured in [15] and refined in [16, 34] that the DR hierarchy associated to $\Omega(\boldsymbol{\sigma})$ is normal Miura-type equivalent [34, 45] to the Hodge hierarchy. Later it is shown by Buryak, Dubrovin, Guéré and Rossi [17] that the DR hierarchy is the standard deformation with $a_0(w) \equiv 1$, and moreover, by an explicit computation in the DR side they obtained the following conjectural relations between q's and σ's when $a_0(w) \equiv 1$:

$$q_{g-1} \,=\, (3g-2) \int_{\overline{\mathcal{M}}_{g,0}} \lambda_g \exp\Biggl(\sum_{j\geq 1} \sigma_{2j-1}\,\mathrm{ch}_{2j-1}(\mathbb{E}_{g,0})\Biggr), \quad g \geq 2\,. \tag{327}$$

By the discussion given right above this remark, we conjecture this holds for all φ which makes the discussion more explicit. Using formula (327) and the algorithm in [34] for computing Hodge integrals (or the Hodge–GUE correspondence [34, 35, 40]), we can compute more explicit values for q_i in the following table:

i	1	2	3	4	5	6	7
q_i	$\frac{q}{2^5 3^1 5^1}$	$\frac{q^3}{2^7 3^4 5^1}$	0	$\frac{-13\,q^7}{2^{10} 3^4 5^2 7^1 11^1}$	$\frac{-59\,q^9}{2^5 3^7 5^2 7^2 11^1 13^1}$	$\frac{19\,q^{11}}{2^{11} 3^4 5^1 7^2 11^1 13^1}$	$\frac{1493\,q^{13}}{2^9 3^7 5^3 7^2 13^1 17^1}$

12. The generalized Hodge–WK correspondence

In this section, by using the $\mathcal{G}$-action and the Hodge–WK correspondence we obtain explicit relationships between the WK mapping partition functions and the special-Hodge mapping partition functions, and we investigate bihamiltonian structures for the Hodge mapping hierarchy.

Theorem 15. *The special-Hodge mapping partitions and the WK mapping partition functions are related by*

$$Z^{\psi}_{\Omega^{\mathrm{special}}(q)} \,=\, Z^{\varphi}\,, \tag{328}$$

where the two power series ψ and φ are related by (14)*.*

Proof. Recall from Section 10 that the Hodge–WK correspondence says

$$Z_{\Omega^{\rm special}(q)}\big(\mathbf{T};\epsilon\big) \;=\; Z^{\rm WK}\big(\mathbf{T}.\varphi_{\rm special}^{-1};\epsilon\big)\,,$$

where we recall that $\varphi_{\rm special}$ is defined as in (222). Therefore,

$$Z_{\Omega^{\rm special}(q)}\big(\mathbf{T}.\psi^{-1};\epsilon\big) \;=\; Z^{\rm WK}\big(\mathbf{T}.\psi^{-1}\circ\varphi_{\rm special}^{-1};\epsilon\big) \;=\; Z^{\rm WK}\big(\mathbf{T}.(\varphi_{\rm special}\circ\psi)^{-1};\epsilon\big)\,.$$

The theorem is proved. ☐

We call (328) the *generalized Hodge–WK correspondence*. From the definition, an alternative form of (328) is

$$Z^{\psi}_{\Omega^{\rm special}(q)}(\mathbf{t}.\varphi;\epsilon) \;=\; Z^{\rm WK}(\mathbf{t};\epsilon)\,, \tag{329}$$

where φ and ψ are related by (14).

Let us consider the Poisson geometry behind this theorem. Indeed, via a bihamiltonian test, we find that up to order ϵ^8, the Hodge mapping hierarchy associated to an arbitrarily given group element $\psi\in\mathcal{G}$ is bihamiltonian if and only if its parameters have the specific values

$$\sigma_1 \;=\; 3\,q\,,\quad \sigma_3 \;=\; 30\,q^3\,,\quad \sigma_5 \;=\; 1512\,q^5\,,\quad \sigma_7 \;=\; 183600\,q^7\,. \tag{330}$$

This specialization is remarkable because it does not depend on ψ. For the case when $\psi(V)=V$, we already know that the answer is the special-Hodge specialization (230), conjectured[6] in [34]. So we expect that the Hodge mapping hierarchy associated to $\psi\in\mathcal{G}$ is bihamiltonian if and only if $\sigma_{2j-1} = \sigma_{2j-1}^{\rm special} = (4^j-1)\,(2j-2)!\,q^{2j-1}$, $j\ge1$, of which the first four values are the ones given in equation (330). We call the Hodge mapping hierarchy associated to ψ with this specialization the *special-Hodge mapping hierarchy associated to* ψ. The following corollary gives the sufficiency part.

Corollary 1. *The special-Hodge mapping hierarchy associated to ψ has a bihamiltonian structure with the Poisson pencil $Q_2^{\psi}\big(U^{\psi}_{\Omega^{\rm special}(q)}\big)+\lambda\,Q_1^{\psi}\big(U^{\psi}_{\Omega^{\rm special}(q)}\big)$ given by*

$$Q_1^{\psi}\big(U^{\psi}_{\Omega^{\rm special}(q)}\big) \;:=\; \sum_{k,\ell\ge0}(-1)^{\ell}\,\frac{\partial U^{\psi}_{\Omega^{\rm special}(q)}}{\partial V_k}\circ\partial^k\circ Q^{\psi,[0]}(V)\circ\partial^{\ell}\circ\frac{\partial U^{\psi}_{\Omega^{\rm special}(q)}}{\partial V_\ell}\,, \tag{331}$$

$$Q_1^{\psi,[0]}(V) \;:=\; \frac12\,\frac{e^{-2\,q\,\psi(V)}}{\psi'(V)}\circ\partial \,+\, \frac12\,\partial\circ\frac{e^{-2\,q\,\psi(V)}}{\psi'(V)}\,, \tag{332}$$

$$Q_2^{\varphi}\big(U^{\psi}_{\Omega^{\rm special}(q)}\big) \;:=\; \sum_{k,\ell\ge0}(-1)^{\ell}\,\frac{\partial U^{\varphi}_{\Omega^{\rm special}(q)}}{\partial V_k}\circ\partial^k\circ Q_2^{\psi,[0]}(V)\circ\partial^{\ell}\circ\frac{\partial U^{\psi}_{\Omega^{\rm special}(q)}}{\partial V_\ell}\,, \tag{333}$$

[6] About this known conjecture, the sufficiency part is proved by the Hodge–GUE correspondence [35] but the necessity part is still open.

$$Q_2^{\psi,[0]}(V) \;:=\; \frac{1}{2}\,\frac{1-e^{-2\,q\,\psi(V)}}{2\,q\,\psi'(V)}\circ\partial \;+\; \frac{1}{2}\,\partial\circ\frac{1-e^{-2\,q\,\psi(V)}}{2\,q\,\psi'(V)}\,. \tag{334}$$

Proof. By Theorem 12 and Theorem 15. □

Note that, by definition, the Schouten bracket of $Q_2^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)+\lambda\,Q_1^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$ and itself vanishes identically in λ, so the non-trivial part of the above corollary is about the polynomial dependence of the coefficients of both $Q_1^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$ and $Q_2^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$. We also verified the polynomiality directly up to and including the terms of order 8 in ϵ. It also follows from Theorem 12, Theorem 15 and the computation for (213) that for any $\psi\in\mathcal{G}$, the central invariant of the Poisson pencil $Q_2^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)+\lambda\,Q_1^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$ is $1/24$ identically in q.

There can be choices for $Q_a^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$, $a=1,2$, for a pencil. Our choice satisfies

$$Q_2^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big) \;+\; \frac{1}{2\,q}\,Q_1^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big) \;=\; P^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)\,, \tag{335}$$

where $P^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$ is defined in (319). Note that we did not choose either the Poisson operator $Q_1^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$ or $Q_2^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$ to simply be $P^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$, but we choose them to match with the Poisson pencil for the bihamiltonian structure for the WK mapping hierarchy, along the generalized Hodge–WK correspondence. For the particular case when $\psi(V)=V$, a similar but different choice was made in [34], where $Q_2^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$ was chosen to be $-P^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$ and $Q_1^\psi\big(U^\psi_{\Omega^{\rm special}(q)}\big)$ was chosen the same as above, giving rise also to the central invariant $1/24$.

Before ending the paper, we would like to mention a generalization of part of our constructions to semisimple Frobenius manifolds. This will be studied in a subsequent publication.

Let M be an n-dimensional calibrated semisimple Frobenius manifold. Denote by $Z_M(\mathbf{t})$ and $Z_{M,\Omega(\boldsymbol{\sigma})}(\mathbf{t})$ the topological partition function of M and the Hodge partition function of M, respectively. Here $\mathbf{t}=(t^{\alpha,k})_{\alpha=1,\dots,n,k\ge0}$ is an infinite tuple of indeterminates. Recall that the integrable hierarchies corresponding to the partition functions Z^M and $Z_{M,\Omega(\boldsymbol{\sigma})}$ are the *Dubrovin–Zhang hierarchy of* M (*aka the integrable hierarchy of topological type of* M) and the *Hodge hierarchy of* M, respectively. The logarithm $\log Z_M =: \mathcal{F}_M$ is called the *topological free energy of* M, and $\log Z_{M,\Omega(\boldsymbol{\sigma})} =: \mathcal{F}_{M,\Omega(\boldsymbol{\sigma})}$ the *Hodge free energy of* M. Both $\mathcal{F}_M$ and $\mathcal{F}_{M,\Omega(\boldsymbol{\sigma})}$ have genus expansions:

$$\mathcal{F}_M(\mathbf{t};\epsilon) \;=\; \sum_{g\ge0}\epsilon^{2g-2}\mathcal{F}_{M,g}(\mathbf{t})\,,\quad \mathcal{F}_{M,\Omega(\boldsymbol{\sigma})}(\mathbf{t};\epsilon) \;=\; \sum_{g\ge0}\epsilon^{2g-2}\mathcal{F}_{M,\Omega(\boldsymbol{\sigma}),g}(\mathbf{t})\,. \tag{336}$$

In this more general context, the group $\mathcal{G}$ is replaced by a more general group of affine-linear transformations such that $\mathcal{F}_{M,0}(\mathbf{t})=\mathcal{F}_{M,\Omega(\boldsymbol{\sigma}),0}(\mathbf{t})$ is invariant under the transformation. For any such transformation φ we define the *mapping partition function of M associated to φ* as before by $Z_M^\varphi(\mathbf{T};\epsilon) := Z_M(\mathbf{T}.\varphi^{-1};\epsilon)$ and the *Hodge mapping partition function of M associated to φ* by $Z^\varphi_{M,\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon) := Z_{M,\Omega(\boldsymbol{\sigma})}(\mathbf{T}.\varphi^{-1};\epsilon)$. Since $Z^\varphi_{M,\Omega(\mathbf{0})}(\mathbf{T};\epsilon) = Z^\varphi_M(\mathbf{T};\epsilon)$, it is enough to study the Hodge mapping partition function of M. Let $X = T^{1,0}$ and let

$$U^\varphi_{\alpha,M,\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon) := \epsilon^2 \frac{\partial^2 F^\varphi_{M,\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon)}{\partial X \partial T^{\alpha,0}}, \quad \alpha = 1,\dots,n\,. \tag{337}$$

By the arguments similar to the proof of Theorem 14, we know that $U^\varphi_{\alpha,M,\Omega(\boldsymbol{\sigma})}(\mathbf{T};\epsilon)$, $\alpha = 1,\dots,n$, satisfy an integrable hierarchy of evolutionary PDEs, which we call the *Hodge mapping hierarchy of M associated to φ*. We expect that this hierarchy is hamiltonian. In particular, when $\boldsymbol{\sigma} = \mathbf{0}$ we call it the *Dubrovin–Zhang mapping hierarchy of M associated to φ*, which is bihamiltonian for reasons similar to the proof of Theorem 12 (cf. [45, 69, 70]). We also call $Z^\varphi_{M,\Omega^{\rm special}(q)}(\mathbf{T};\epsilon)$ the *special-Hodge mapping partition function of M associated to φ*, and the integrable hierarchy satisfied by $U^\varphi_{\alpha,M,\Omega^{\rm special}(q)}(\mathbf{T};\epsilon)$ is called the *special-Hodge mapping hierarchy of M associated to φ.*

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