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BISMUT-ZHANG THEOREM AND ANOMALY FORMULA FOR THE RAY-SINGER METRIC FOR SPACES WITH ISOLATED CONICAL SINGULARITIES

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ABSTRACT. In this article we extend to spaces with isolated conical singularities Bismut and Zhang's generalisation of the Cheeger-Müller Theorem, *i.e.* the comparison formula between analytic torsion and Milnor torsion of a smooth compact manifold equipped with an arbitrary flat Hermitian vector bundle. We also establish anomaly formulas for all three terms appearing in our Bismut-Zhang formula for a space with isolated conical singularities, in particular we generalise Bismut and Zhang's anomaly formula for the Ray-Singer metric to this singular context.

1. INTRODUCTION

The Cheeger-Müller theorem, the comparison of analytic (or Ray-Singer) and topological (or Reidemeister-Franz) torsion for smooth compact manifolds equipped with a unitary flat vector bundle, is one of the most important comparison theorems in global analysis. It has been conjectured by Ray and Singer and has been independently proved by Cheeger [Che79] and Müller [Mül78]. In [Mül93] Müller extended the result to the case of odd dimensional manifolds, where only the metric on the determinant of the flat vector bundle is required to be flat, the so-called unimodular case. In the same time, in [BZ92], Bismut and Zhang combined the Witten deformation ([Wit82, HS85]) and local index techniques to generalise the result of Cheeger and Müller to arbitrary flat vector bundles with arbitrary Hermitian metrics. Bismut and Zhang compare the analytic torsion with the Milnor torsion: If the flat vector bundle is not unitary or unimodular the two torsions are no longer equal and the difference between them can be expressed in terms of the Mathai-Quillen current. In this article we refer to this most general version of the comparison theorem of torsions as the Bismut-Zhang theorem.

The question of extending the Cheeger-Müller theorem to spaces with conical singularities has been raised by Dar [Dar87] nearly 40 years ago, very early in the development of global analysis of these spaces. She proved well-definedness of analytic torsion on singular spaces with *isolated conical singularities* and also defined the intersection Reidemeister torsion for general stratified pseudomanifolds (for any perversity function in the sense of Goresky and MacPherson). For an even-dimensional oriented space with isolated conical singularities, she then proved equality between analytic and intersection Reidemeister torsion (with middle perversity $\overline{m}$); this case is easy due to a duality argument. Following an idea suggested by Lesch in [Les98, Problem 5.3], namely to study the problem of a Cheeger-Müller theorem for singular spaces via gluing formulas, several articles have computed and studied the analytic torsion on a truncated cone [Ver09], [MV14], [HS10], [HS11], [HS16]. The recent preprint [HS20] seems to carry out the

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gluing. However, a problem which the authors face in this approach is the missing interpretation of the analytic correction term, stemming from the singularities of the space, to the comparison formula in the odd dimensional case.

Although a Cheeger-Müller theorem for spaces with isolated conical singularities was still missing, it was expected that the Ray-Singer metric is no longer a topological invariant in general. Only partial results on anomaly formulas for the Ray-Singer metric for spaces with isolated conical singularities exist so far: In [MV14] an asymptotic variation formula for the analytic torsion of a truncated odd dimensional cone is given.

In [MV12] Mazzeo and Vertman prove the well-definedness of analytic torsion for *incomplete edge spaces* (sometimes also called wedge spaces in the literature), *i.e.* spaces with a singular stratum of positive dimension and a cone-like metric near the singular stratum. The authors also prove topological invariance of the analytic torsion for an odd dimensional wedge space with an odd dimensional singular stratum. The Cheeger-Müller theorem on an odd dimensional wedge space with odd dimensional singular stratum equipped with a unimodular bundle satisfying an additional acyclicity condition has been studied in [ARS22]. The strategy in [ARS22] consists in the study of analytic torsion via degeneration of smooth metrics into conical metrics. The assumptions made in [ARS22] exclude the case of odd dimensional spaces with isolated singularities, they also exclude the case of the trivial bundle.

Yet another strategy to approach the Cheeger-Müller theorem for spaces with singularities, proposed by the author, is to attack the question, by generalising the strategy of Bismut and Zhang in [BZ92] to the singular setting. This approach has been successful, providing in [Lud20a] an answer to the long open question. The generalisation of the Cheeger-Müller theorem for singular spaces with isolated conical singularities in [Lud20a] has been proved in the case of unitary vector bundles and under the assumption of the Witt and an additional spectral Witt condition. The comparison theorem in [Lud20a] establishes the equality of the Ray-Singer metric and a metric also defined in [Lud20a] and which henceforth we will call the Bismut-Zhang metric (its definition is recalled in Section 2.9). The Bismut-Zhang metric is a Milnor like metric with an analytic correction term from the singular points of the space. This analytic correction term is precisely the analytic torsion of a model operator on the infinite cone over the link of the singularity. This model operator on the infinite cone has been introduced in [Lud17b] and is the generalisation of Witten's famous harmonic oscillator [Wit82].

The *aim of this article* is to fully profit from the strength of the Bismut-Zhang approach to the study of torsion and complete the program of the study of the comparison of torsions on spaces with isolated conical singularities started in [Lud20a]: Firstly we manage to prove the most general version of the comparison theorem of torsions for spaces with isolated conical singularities, *i.e.* the Bismut-Zhang theorem for these spaces. Secondly we provide anomaly formulas for all three terms in the Cheeger-Müller/Bismut-Zhang theorem, in particular for the Ray-Singer metric on spaces with isolated conical singularities – a task not yet addressed neither in [Lud20a] nor elsewhere.

In the *first part* of this article we prove a Bismut-Zhang formula for spaces with isolated conical singularities, *i.e.* we treat the case, where the flat bundle is not unitary. We are also able to remove the assumption made in [Lud20a], that the space satisfies the Witt and a spectral Witt condition:

Theorem I. *Let (X, g^{TX}) be a space with isolated conical singularities and let $\overline{q} \in \{\overline{m}, \overline{n}\}$ be the lower resp. upper middle perversity. Let (F, ∇^F, g^F) be a flat vector bundle over the smooth stratum X_{sm}. Let $f : X \to \mathbb{R}$ be an anti-radial Morse function, g_0^{TX} a Riemannian metric on X, coinciding with g^{TX} in a neighbourhood of the singular set $\mathrm{Sing}(X)$ and such that the pair (f, g_0^{TX}) is Morse-Smale. Set $Y := \nabla_{g_0} f$. Then:*

$$\log \left(\frac{\| \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}}{\| \ \|^{Y,g^{TX},g^F}_{\det IH^\bullet_{\overline{q}}(X,F)}} \right)^2 = - \int_X \theta(F, g^F) Y^* \Psi(TX, \nabla^{TX}), \tag{1.1}$$

where $\| \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}$ denotes the Ray-Singer metric (see Section 2.6), $\| \ \|^{Y,g^{TX},g^F}_{\det IH^\bullet_{\overline{q}}(X,F)}$ denotes the Bismut-Zhang metric (see Section 2.9), $\theta(F, g^F)$ is a closed 1-form measuring the obstruction to the existence of a flat metric on $\det(F)$ (see Section 2.2, (2.4)*) and $\Psi(TX, \nabla^{TX})$ is the Mathai-Quillen current (see Section 3.2).*

Theorem I generalises the smooth Bismut-Zhang theorem [BZ92, Theorem 0.2] as well as the Cheeger-Müller theorem for spaces with isolated conical singularities in [Lud20a, Section 2.11]. In the case of an even dimensional space with isolated conical singularities the two middle perversities coincide, $\overline{m} = \overline{n}$. For an odd dimensional space which does not satisfy the Witt condition, we get two comparison formulas according to the two middle perversities for intersection cohomology; the two formulas are related by Poincaré duality. Using the methods of this paper the formula (1.1) can also be extended to all mezzo-perversities in the sense of Albin, Banagl, Leichtnam, Mazzeo and Piazza [ABL+15].

Let us comment on the proof of Theorem I. As in [BZ92] and in [Lud20a], local index techniques and the Witten deformation play a major role in this article. Most of the intermediate results, which are the core of the proof of Theorem I, typically consist of two steps: localisation and a local computation near the critical points of the Morse function. The main work which hence remains to be done here, consists in extending the study of the local model near singular points in [Lud20a], where one now has to deal in addition with the different *ibcs à la Cheeger* - this is done in Section 4. Once the local model is understood, the proofs of most of the intermediate results can be generalised by following closely the proof of the corresponding results in [Lud20a]. We will not repeat the details of these proofs here; but instead shortly indicate, why the proofs in [Lud20a] carry through to this more general situation.

In this article we develop a crucial new tool, which consists in a combination of local index techniques à la Bismut-Zhang with the Singular Asymptotic Lemma (SAL) of Brüning and Seeley [BS85]. This tool comes into play, when treating the Cheeger type invariants which appear as the contribution of the singularities of X in the small time asymptotics of the supertrace of operators related to the heat operator. It is this tool, which allows to push the analysis beyond what has been done in [Lud20a] and to treat the more general situation in this article. More concretely this tool is used in the proof of one of the intermediate results, namely Theorem 5.5. The proof of the corresponding result in [Lud20a] in case of an even dimensional space was assuming orientability of the space, relied on a simple duality argument and does not generalise to the present situation.

The above mentioned combination of local index techniques and SAL developed here, indeed allows to replace most of the explicit computations in the local model by abstract arguments. We therefore expect that it will be a useful tool in further studies of secondary invariants in a more general singular setting.

The *second aim of this article* is to study anomaly formulas for all three terms in the Bismut-Zhang formula (1.1), *i.e.* we study their behaviour under change of the Riemannian conical metric g^{TX} and of the Hermitian metric g^F on the flat bundle. This is done in Sections 3 and 7.

For a Euclidean vector bundle (E, ∇^E) with metric connection, we denote by $e(E, \nabla^E)$ the Euler form in Chern-Weil theory. We denote by $\rho : X \times \mathbb{R} \to X$ the canonical projection. The following theorem generalises the smooth anomaly formula for the Ray-Singer metric in [BZ92, Section IV] to spaces with isolated conical singularities:

Theorem II. *Let $\mathbb{R} \ni l \to (g_l^{TX}, g_l^F)$ be a family of metrics on TX, F satisfying the spectral gap condition* (7.1). *Then, the variation of the Ray-Singer metric is given by*

$$
\begin{aligned}
&\partial_l \log \left(\left(\| \quad \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),l} \right)^2 \right) \\
&= \int_X \operatorname{Tr}\left[(g_l^F)^{-1} \frac{\partial g_l^F}{\partial l} \right] e(TX, \nabla_l^{TX}) + \int_X \iota_{\partial_l} e(\rho^* TX, \nabla^{TX,\mathrm{tot}}) \theta(F, g_l^F) \\
&\quad + \sum_{p \in \mathrm{Sing}(X)} (c^{\overline{q}}_{p,l} + \widetilde{c}^{\overline{q}}_{p,l}),
\end{aligned}
\tag{1.2}
$$

where the connection $\nabla^{TX,tot}$ on the pull back bundle ρ^*TX is defined in (3.12) *and the contributions of the singularities $c^{\overline{q}}_{p,l}$, $\widetilde{c}^{\overline{q}}_{p,l}$, $p \in \mathrm{Sing}(X)$, are the Cheeger type invariants defined in* (7.12).

Theorem II generalises the anomaly formula for the Ray-Singer metric in case of a smooth compact manifold in [BZ92, Theorems 4.14 and 4.20]; the two first terms on the right hand side of (1.2) – the interior contribution – appear already in the smooth formulas.

We also prove an *anomaly formula for the Bismut-Zhang metric* (see Theorem 7.8) and a *variation formula for the right hand side in the Bismut-Zhang formula* (1.1) (see Theorem 3.4), the latter generalises the corresponding smooth result in [BZ92, Section VI]. We do not state these theorems in this introduction, since more notation is required for their statement. Let us just emphasise that, unlike the Milnor metric of a smooth compact manifold, the Bismut-Zhang metric of a singular space is in general not a purely topological invariant of the space, even if the flat bundle is unitary.

The study of anomaly formulas in Section 7 is new, and had not been addressed in [Lud20a] (or elsewhere). To deal with the change of domains of the operators in a family, we first adapt a trick used by Cheeger for manifolds with boundary to spaces with singularities. The above mentioned tool of combining the local index techniques of Bismut and Zhang with SAL is again key for relating the contribution of the singularities in the anomaly formulas of the Bismut-Zhang and the Ray-Singer metric, e.g. the contributions $c^{\overline{q}}_{p,l}$, $\widetilde{c}^{\overline{q}}_{p,l}$, $p \in \mathrm{Sing}(X)$ in (1.2) are of this type.

The article is organised as follows: In Section 2 we explain the notation and recall, for convenience of the reader, some basic definitions and facts on singular spaces with isolated conical singularities used in the article. In Section 2.6 (resp. Section 2.9) we recall from [Dar87] resp. [Lud20a] the definition of the Ray-Singer (resp. the Bismut-Zhang) metric for a space with isolated conical singularities, the two metrics compared in Theorem I. In Section 3 we study the Berezin integral formalism as well as the Mathai-Quillen current for singular spaces and anti-radial Morse functions. This will be used for the proof of the variation formulas of the r.h.s. in (1.1) (in Section 3.6) as well as in the proofs of the anomaly formulas in Section 7. In Section 4 we study the Witten deformation: We first shortly recall from [Lud17b] the Witten deformation for singular spaces with isolated conical singularities and anti-radial Morse functions in Section 4.1. In Sections 4.2-4.3, we study the local model Witten Laplacian and adapt results from [Lud20b, Lud20a] to the situation, where the space is no longer Witt and the flat bundle is no longer unitary. In Section 5 we state nine intermediate results, which are the analogues of the nine intermediate results in [BZ92, Section VII] and [Lud20a, Section 5]. Once the nine intermediate results are achieved in our more general situation, the proof of the Bismut-Zhang Theorem (Theorem I) is completely analogous to the proof in [BZ92, Section VII] and in [Lud20a, Section 6], hence we omit it here. Section 6 deals with the proofs of the nine intermediate results: Once the local model is understood, the proofs of most of the intermediate results follow closely those of [Lud20a]. We will not repeat the details of these proofs here but rather explain, why they carry through in this more general situation. As already pointed out, the proof of Theorem 5.5 is completely new both in ideas and technique, and is the only proof of the intermediate results, which we give in detail here. In the last section, Section 7, we study anomaly formulas for the Ray-Singer and the Bismut-Zhang metric of a space with isolated conical singularities; in particular the proof of Theorem II can be found in this section.

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2. Preliminaries

2.1. Singular spaces with isolated conical singularities.

For a smooth manifold L and $\delta > 0$, we denote by

$$c_\delta L := \left([0,\delta) \times L\right)/_{(0,x)\sim(0,y)} \tag{2.1}$$

the (open) truncated cone over L.

Let X be a connected topological space, $\mathrm{Sing}(X) \subset X$ a finite set of points, such that $X_{sm} := X \backslash \mathrm{Sing}(X)$ is a smooth manifold of dimension $n \geqslant 2$. We denote by TX (resp. by T^*X) the tangent bundle (resp. the cotangent bundle) of X_{sm}. Let g^{TX} be a Riemannian metric on X_{sm}. We assume that (X, g^{TX}) is a space with isolated conical singularities of dimension n, *i.e.*

(1) For $p \in \mathrm{Sing}(X)$, there exist an open neighbourhood $B_\delta(p)$ of p, a smooth compact connected manifold L_p of dimension $\dim L_p = n-1$ and a diffeomorphism $\varphi_p : B_\delta(p)\backslash\{p\} \simeq c_\delta L_p \backslash \{0\}$. The diffeomorphism φ_p extends to a homeomorphism, still

denoted by φ_p,

$$\varphi_p : B_\delta(p) \simeq c_\delta L_p \text{ and } g^{TX}_{|B_\delta(p)\backslash\{p\}} = \varphi_p^* \left(dr^2 + r^2 g^{TL_p}\right), \tag{2.2}$$

where r is the radial coordinate and g^{TL_p} is a Riemannian metric on the manifold L_p (not depending on r).

(2) The set

$$X\backslash \left(\bigcup_{p\in \mathrm{Sing}(X)} B_\delta(p) \right)$$

is a smooth compact manifold of dimension n with boundary $\bigcup_{p\in \mathrm{Sing}(X)} L_p$.

The set $\mathrm{Sing}(X)$ is called the singular set of X. For $p \in \mathrm{Sing}(X)$, the manifold L_p is called the link of X at p. Let us emphasise that the radial coordinate r in (2.2) is fixed throughout this article.

2.2. **Flat vector bundles over** X. Let (F, ∇^F, g^F) be a flat vector bundle over X_{sm} with canonical flat connection ∇^F and (not necessarily flat) Hermitian metric g^F.

We make the following assumption: For $p \in \mathrm{Sing}(X)$, we denote by $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$ the restriction of (F, ∇^F, g^F) to the link L_p. We assume that the restriction of (F, ∇^F, g^F) to a punctured neighbourhood of $p \in \mathrm{Sing}(X)$ can be identified with the pull back bundle of the vector bundle $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$.

We denote by F^* the flat bundle dual to F, and by $F^*_{L_p}$ its restriction to L_p, $p \in \mathrm{Sing}(X)$.

Let $\omega(F, g^F)$ be the 1-form on X with values in the self-adjoint endomorphisms of F,

$$\omega(F, g^F) = (g^F)^{-1}\nabla^F g^F. \tag{2.3}$$

Note that $\omega(F, g^F) = 0$, in case g^F is flat. We denote by $\theta(F, g^F)$ the following closed 1-form on X

$$\theta(F, g^F) = \mathrm{Tr}[\omega(F, g^F)]. \tag{2.4}$$

The cohomology class $[\theta(F, g^F)]$ measures the obstruction to the existence of a flat volume form on F. By our assumption, near $p \in \mathrm{Sing}(X)$, the form $\theta(F, g^F)$ does not depend on the radial coordinate r.

2.3. **Local model near** $p \in \mathrm{Sing}(X)$. Let $p \in \mathrm{Sing}(X)$. We denote by $cL_p := \big([0,\infty) \times L_p\big)/_{(0,x)\sim(0,y)}$ the infinite cone over L_p, by 0 the cone tip and by

$$Z_p := cL_p\backslash\{0\} \simeq \mathbb{R}_{>0} \times L_p \tag{2.5}$$

the punctured infinite cone. We write $x \in Z_p$ in its polar coordinates $x = (r, y)$, where r is the radial coordinate and y is the coordinate on the link. We equip Z_p with the conical metric $g^{TZ_p} = dr^2 + r^2 g^{TL_p}$. The flat bundle $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$ can be extended in a trivial way to a flat bundle $(F_{Z_p}, \nabla^{F_{Z_p}}, g^{F_{Z_p}})$ over Z_p. If no confusion can occur, we still write F for F_{Z_p}.

Let us denote by ∇^{TZ_p} the Levi-Civita connection on (TZ_p, g^{TZ_p}) and by $R^{TZ_p} = \left(\nabla^{TZ_p}\right)^2$ its curvature. We denote by ∇^{TL_p} the Levi-Civita connection on (TL_p, g^{TL_p}) and by $R^{TL_p} = \left(\nabla^{TL_p}\right)^2$ its curvature. The curvatures R^{TZ_p} and R^{TL_p} are related by the Gauss equation (see e.g. [BC90, Proposition 1.2] or [O'N83, page 210]): Let X, Y, V be smooth vector fields on the link manifold L_p. We still denote by X, Y, V their $\mathbb{R}_+$-invariant extension to smooth vector fields on Z_p. Then

$$R^{TZ_p}\left(\frac{\partial}{\partial r}, \right) = 0;\ R^{TZ_p}(X,Y)V = R^{TL_p}(X,Y)V - g^{TL_p}(Y,V)X + g^{TL_p}(X,V)Y. \tag{2.6}$$

2.4. **Intersection cohomology.** We denote by $\overline{m}$ (resp. $\overline{n}$) the lower middle (resp. upper middle) perversity in the sense of Goresky and MacPherson [GM80, GM83]. A perversity $\overline{q}$ is a tuple of non negative natural numbers. For a space with isolated singularities however, the only relevant information is the last entry in this tuple. Hence by slight abuse of notation we will identify a perversity $\overline{q}$ with this last entry, more concretely,

$$\overline{q} = \begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{for } \overline{q} = \overline{m}, \\ \lfloor \frac{n-1}{2} \rfloor & \text{for } \overline{q} = \overline{n}. \end{cases} \tag{2.7}$$

Note that for n even, the two middle perversities coincide, $\overline{m} = \overline{n}$.

For $\overline{q} \in \{\overline{m}, \overline{n}\}$ we denote by $IH^\bullet_{\overline{q}}(X, F)$ the intersection cohomology of X with perversity $\overline{q}$ and coefficients in the local system associated to the flat bundle F. For an even dimensional space with isolated singularities lower and upper middle perversity coincide and hence also the two intersection cohomologies are the same. More generally, if X is a Witt space, *i.e.* if $H^{\frac{n-1}{2}}(L_p, F_{L_p}) = 0$ for all $p \in \mathrm{Sing}(X)$, then by [Sie83, Theorem 3.4]

$$IH^\bullet_{\overline{m}}(X, F) \simeq IH^\bullet_{\overline{n}}(X, F). \tag{2.8}$$

In this paper, we will not assume the Witt condition, and hence in the odd dimensional case, we have to distinguish two cases.

Let $H^\bullet(L_p, F_{L_p})$ denote the singular cohomology of the link manifold L_p. For the relative intersection cohomology of the cone cL_p with values in the flat bundle F, denoted by $IH^\bullet_{\overline{q}}(cL_p, L_p, F))$, we have, see [GM83, Section 2.4],

$$IH^k_{\overline{q}}(cL_p, L_p, F) = \begin{cases} H^{k-1}(L_p, F_{L_p}) & \text{for } k \geqslant n - \overline{q}, \\ 0 & \text{else}. \end{cases} \tag{2.9}$$

2.5. L^2**-cohomology.**

2.5.1. *Maximal and minimal extension of the de Rham complex.* Throughout this article, we use the language of Hilbert complexes as introduced by Brüning and Lesch in [BL92].

We denote by $\langle\ ,\ \rangle$ the L^2-inner product on the space of sections of $\Lambda(T^*X) \otimes F$ induced from the metrics g^{TX}, g^F. We denote by $\Omega^\bullet_c(X, F)$ (resp. by $\mathrm{L}^2(\Lambda(T^*X) \otimes F)$) the graded vector space of smooth compactly supported sections (resp. of L^2-sections) of $\Lambda(T^*X) \otimes F$. We denote by d_c the exterior derivative acting on $\Omega^\bullet_c(X, F)$.

The de Rham complex $(\Omega^\bullet_c(X, F), d_c, \langle\ ,\ \rangle)$ admits several closed extensions (in the Hilbert space of L^2-forms) into a Hilbert complex, a choice of which is called an ideal boundary condition (shortly ibc) by Cheeger. Here we focus on the maximal (resp. minimal) extension, denoted by $(\mathcal{C}^\bullet_{\max}, d_{\max}, \langle\ ,\ \rangle)$ (resp. $(\mathcal{C}^\bullet_{\min}, d_{\min}, \langle\ ,\ \rangle)$), where $d_{\max}$ (resp. $d_{\min}$) denotes the maximal (resp. the minimal) closed extension of d_c in $\mathrm{L}^2(\Lambda(T^*X) \otimes F)$. By [BL93, Theorems 3.7 and 3.8],

$$\mathrm{dom}(d_{\max})/\,\mathrm{dom}(d_{\min}) \simeq \bigoplus_{p \in \mathrm{Sing}(X)} H^{\frac{n-1}{2}}(L_p, F_{L_p}). \tag{2.10}$$

Hence for a Witt space and in particular for an even dimensional space, the extension of the de Rham complex $(\Omega_c^\bullet(X,F), d_c, \langle\ ,\ \rangle)$ into a Hilbert complex is unique, $d_{\min} = d_{\max}$.

The cohomology of the maximal Hilbert complex $(\mathcal{C}^\bullet_{\max}, d_{\max}, \langle\ ,\ \rangle)$, is called the L^2-cohomology of X with values in F,

$$H^\bullet_{(2)}(X,F) := H^\bullet_{(2),\overline{m}}(X,F) := H^\bullet\big((\mathcal{C}^\bullet_{\max}, d_{\max}, \langle\ ,\ \rangle)\big). \tag{2.11}$$

We can also define the cohomology of the minimal extension $(\mathcal{C}^\bullet_{\min}, d_{\min}, \langle\ ,\ \rangle)$,

$$H^\bullet_{(2),\overline{n}}(X,F) := H^\bullet\big((\mathcal{C}^\bullet_{\min}, d_{\min}, \langle\ ,\ \rangle)\big). \tag{2.12}$$

The two cohomologies introduced here only depend on the quasi-isometry class of the Riemannian metric. From (2.10), we have for a Witt space $H^\bullet_{(2),\overline{m}}(X,F) = H^\bullet_{(2),\overline{n}}(X,F)$.

2.5.2. L^2-*Hodge-de Rham Theorem.* By a result of Cheeger, Goresky and MacPherson [CGM82, Section 3.4], integration of L^2-forms over intersection chains induces a de Rham isomorphism

$$H^\bullet_{(2),\overline{q}}(X,F) \simeq IH^\bullet_{\overline{q}}(X,F). \tag{2.13}$$

This shows in particular, that the minimal and maximal L^2-cohomology are indeed topological invariants of X.

We denote by δ_c the (formal) adjoint of the operator d_c w.r.t. $\langle\ ,\ \rangle$ acting on compactly supported forms and by $\delta_{\min/\max}$ its minimal resp. maximal extension, which is the adjoint of $d_{\max/\min}$ w.r.t. $\langle\ ,\ \rangle$. We denote by $D^{\overline{m}}$ (resp. $D^{\overline{n}}$) the first order self-adjoint operator associated to the Hilbert complex $(\mathcal{C}^\bullet_{\max}, d_{\max}, \langle\ ,\ \rangle)$ (resp. $(\mathcal{C}^\bullet_{\min}, d_{\min}, \langle\ ,\ \rangle)$). We have

$$D^{\overline{m}} = d_{\max} + \delta_{\min} \text{ (resp. } D^{\overline{n}} = d_{\min} + \delta_{\max}). \tag{2.14}$$

For $\overline{q} \in \{\overline{m}, \overline{n}\}$, we denote by $\Delta^{\overline{q}} := (D^{\overline{q}})^2$ the Laplace operators associated to the two Hilbert complexes and by $\Delta^{\overline{q},(i)}$ their restriction to i-forms. By a standard result on Hilbert complexes, the following L^2-Hodge isomorphism holds (see [Che80, Section 1 and Theorem 5.1], [BL92, Lemma 2.2 and Corollary 2.5])

$$\ker(\Delta^{\overline{q}}) =: \mathscr{H}^\bullet_{(2),\overline{q}}(X,F) \simeq H^\bullet_{(2),\overline{q}}(X,F),\ \overline{q} \in \{\overline{m}, \overline{n}\}. \tag{2.15}$$

2.6. **The Ray-Singer metric on** $\det IH^\bullet_{\overline{q}}(X,F)$**.** We denote by $\Delta^{\overline{q},\perp}$ the restriction of $\Delta^{\overline{q}}$ to $\big(\ker \Delta^{\overline{q}}\big)^\perp$. We denote by N the number operator acting on sections of the bundle $\Lambda(T^*X) \otimes F$ by multiplication with the form degree. For $s \in \mathbb{C}$, $\Re(s) > \frac{n}{2}$, set

$$\zeta_{\overline{q}}(s) := -\mathrm{Tr_s}\big[N(\Delta^{\overline{q},\perp})^{-s}\big]. \tag{2.16}$$

By a result of A. Dar [Dar87, Section 4], the function $\zeta_{\overline{q}}$ extends to a meromorphic function on the whole complex plane, which is holomorphic at $s = 0$. The result in [Dar87, Section 4] has been proved in the case of unitary flat vector bundles, but the same proof also works in the current situation. Incidentally, the holomorphicity of $\zeta_{\overline{q}}$ at 0 also follows by using the Mellin transform and Theorem 5.5 below.

For a complex vector space V of dimension 1 we denote by V^{-1} its dual. For a vector space V we denote by $\det V$ the maximal exterior power of V. We denote by $\det IH^\bullet_{\overline{q}}(X,F)$ the complex line

$$\det IH^\bullet_{\overline{q}}(X,F) := \bigotimes_{k=0}^{n} \big(\det IH^k_{\overline{q}}(X,F)\big)^{(-1)^k}. \tag{2.17}$$

The L^2-metric on sections of $\Lambda(T^*X)\otimes F$ restricts to a metric on the space of L^2-harmonic forms $\mathcal{H}^\bullet_{(2),\overline{q}}(X,F)$. Using the isomorphisms (2.13) and (2.15) we get an induced metric on the line $\det IH^\bullet_{\overline{q}}(X,F)$, which we denote by $|\ \ |^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}$.

Definition 2.1. The Ray-Singer metric $\|\ \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}$ on the line $\det IH^\bullet_{\overline{q}}(X,F)$ is defined as

$$\|\ \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)} := |\ \ |^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)} \exp\left(\frac{1}{2}\zeta'_{\overline{q}}(0)\right). \tag{2.18}$$

We will discuss the anomaly formulas for $\|\ \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}$, *i.e.* the dependence of the Ray-Singer metric $\|\ \ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}$ on the choices of the metrics g^{TX}, g^F, in Section 7.

2.7. **Anti-radial Morse functions.** Anti-radial Morse functions have been introduced in [Lud17b]. One inspiration stems from Marie-Hélène Schwartz's radial vector fields [Sch86].

Definition 2.2. A continuous function $f : X \to \mathbb{R}$ is called an anti-radial Morse function, if the following two conditions hold:

(a) The restriction $f_{sm} := f_{|X_{sm}}$ is a smooth Morse function.
(b) Near a singular point $p \in \mathrm{Sing}(X)$ the function f has the following normal form in the local coordinates (2.2):

$$f(r,y) = f(p) - \frac{1}{2}r^2. \tag{2.19}$$

For an anti-radial Morse function f, we denote by $\mathrm{Crit}(f_{sm})$ (resp. by $\mathrm{Crit}_k(f_{sm})$, $k = 0,\dots,n$) the set of critical points of f_{sm} (resp. the set of critical points of f_{sm} of index k). Set

$$\mathrm{Crit}(f) := \mathrm{Crit}(f_{sm}) \cup \mathrm{Sing}(X). \tag{2.20}$$

Let g^{TX} be a conical metric on X_{sm}. The vector field $-\nabla f := -\nabla_{g^{TX}} f$ induces a well-defined smooth flow on X_{sm}, which extends to a continuous flow $\Phi : X \times \mathbb{R} \to X$. For $p \in \mathrm{Crit}(f)$ we denote by $W^{u/s}(p)$ the unstable resp. stable set of p; their intersection with X_{sm} are submanifolds.

Definition 2.3. We call a pair (f, g^{TX}) consisting of an anti-radial Morse function and a conical metric

(a) an anti-radial Morse-Smale pair, if the Morse-Smale transversality condition holds for $-\nabla f$, *i.e.* all stable and unstable manifolds w.r.t. the negative gradient flow Φ intersect transversally.
(b) an anti-radial standard Morse-Smale pair, if in addition, for $p \in \mathrm{Crit}_k(f_{sm})$ in local Morse coordinates $x_1,\dots,x_n$ of an open neighbourhood $U(p)$,

$$(\nabla f)_{|U(p)} = -x_1\frac{\partial}{\partial x_1} - \dots - x_k\frac{\partial}{\partial x_k} + x_{k+1}\frac{\partial}{\partial x_{k+1}} + \dots + x_n\frac{\partial}{\partial x_n}. \tag{2.21}$$

With other words, we assume that the Riemannian metric g^{TX} is the standard Euclidean metric in the Morse coordinates near $\mathrm{Crit}(f_{sm})$.

The existence of anti-radial (standard) Morse-Smale pairs on a singular space can be proved by an adaptation of the smooth proofs, see [Lud20a, Section 2.8] and the references therein.

2.8. **The singular Morse-Thom-Smale complex computing intersection cohomology.** We recall the singular Morse-Thom-Smale complex defined in [Lud17b, Section 6] and [Lud17a], which is an important ingredient in the definition of the Bismut-Zhang metric. For a singular point $p \in \mathrm{Sing}(X)$, we denote by $o(TL_p)$ the orientation bundle of L_p. Moreover, we have the following notation: $\Omega^\bullet(L_p, F^*_{L_p} \otimes o(TL_p))$ is the space of smooth de Rham forms on L_p with values in the flat bundle $F^*_{L_p} \otimes o(TL_p)$, $H^\bullet(L_p, F^*_{L_p} \otimes o(TL_p))$ is the cohomology of L_p. Let

$$\Xi^k_p \subset \Omega^k(L_p, F^*_{L_p} \otimes o(TL_p)) \tag{2.22}$$

be a set of closed forms on L_p, whose cohomology classes form a basis of $H^k(L_p, F^*_{L_p} \otimes o(TL_p))$, $\mathrm{span}(\Xi^k_p) \simeq H^k(L_p, F^*_{L_p} \otimes o(TL_p))$.

To a given anti-radial Morse-Smale pair (f, g^{TX}) and the sets Ξ^{n-k}_p, $p \in \mathrm{Sing}(X)$, $k \geqslant n - \overline{q}$, one can associate a geometric complex, which computes the intersection homology of X with values in F^* and perversity $\overline{q}$, $IH^{\overline{q}}_\bullet(X, F^*)$.

For $p, q \in \mathrm{Crit}(f_{sm})$ with $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$, by Morse-Smale transversality, the space of trajectories of Φ starting in p and ending in q is a finite set, which we denote by $\Gamma(p, q)$. Choosing orientations on the unstable manifolds of points in $\mathrm{Crit}(f_{sm})$ and using the flow Φ, induces orientations on $\Gamma(p, q)$ (see [Lau92, Section (c)] for more details). With other words, to each trajectory $\gamma \in \Gamma(p, q)$ we can assign $n_\gamma(p, q) \in \{\pm 1\}$. We denote by $\tau_\gamma : F^*_p \to F^*_q$ the map induced from parallel transport w.r.t. the flat connection along the trajectory γ.

Definition/Proposition 2.4. We denote by $(C^{\overline{q}}_\bullet(X, f, g^{TX}, F^*), \partial_\bullet)$ the following complex:

$$\begin{aligned} &C^{\overline{q}}_k(X, f, g^{TX}, F^*) \\ &= \begin{cases} \left(\bigoplus_{p \in \mathrm{Crit}_k(f_{sm})} \langle [W^u(p)] \rangle \otimes F^*_p \right) \oplus \left(\bigoplus_{p \in \mathrm{Sing}(X)} \mathrm{span}(\Xi^{n-k}_p) \right) & \text{if } k \geqslant n - \overline{q}, \\ \bigoplus_{p \in \mathrm{Crit}_k(f_{sm})} \langle [W^u(p)] \rangle \otimes F^*_p & \text{else.} \end{cases} \end{aligned} \tag{2.23}$$

The boundary operator $\partial_\bullet$ is defined as follows: For $p \in \mathrm{Crit}_k(f_{sm})$, $h \in F^*_p$:

$$\partial_k \left([W^u(p)] \otimes h \right) = \sum_{q \in \mathrm{Crit}_{k-1}(f_{sm})} \sum_{\gamma \in \Gamma(p,q)} n_\gamma(p, q) \cdot [W^u(q)] \otimes \tau_\gamma(h) \in C^{\overline{q}}_{k-1}(X, f, g^{TX}, F^*). \tag{2.24}$$

For $p \in \mathrm{Sing}(X)$, $k \geqslant n - \overline{q}$, $\xi^{n-k}_p \in \Xi^{n-k}_p$:

$$\partial_k[\xi^{n-k}_p] = \sum_{q \in \mathrm{Crit}_{k-1}(f_{sm})} \left(\int_{W^s(q) \cap L_p} \xi^{n-k}_p \right) \cdot [W^u(q)] \in C^{\overline{q}}_{k-1}(X, f, g^{TX}, F^*). \tag{2.25}$$

The complex $\left(C^{\overline{q}}_\bullet(X, f, g^{TX}, F^*), \partial_\bullet\right)$ is well-defined, *i.e.* $\partial^2_\bullet = 0$.

Remark 2.5. In [Lud17a] the well-definedness of $(C^{\overline{q}}_\bullet(X, f, g^{TX}, F^*), \partial_\bullet)$ has been proved for the case, where the pair (f, g^{TX}) is an anti-radial standard Morse-Smale pair. Using a perturbation argument in [HS85, Proposition 5.1], we can extend the result to the case of an anti-radial Morse-Smale pair (f, g^{TX}), which is not necessarily standard. For $\epsilon > 0$

one can construct an an anti-radial standard Morse-Smale pair $(f_\epsilon, g_\epsilon^{TX})$ coinciding with (f, g^{TX}) on $X \backslash (\cup_{p \in \mathrm{Crit}(f_{sm})} B_\epsilon(p))$ and such that the complexes $(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet)$ and $(C_\bullet^{\overline{q}}(X, f_\epsilon, g_\epsilon^{TX}, F^*), \partial_\bullet)$ coincide.

The complex defined in Definition 2.4 computes the intersection homology of X:

$$H_\bullet\left(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet\right) \simeq IH_\bullet^{\overline{q}}(X, F^*). \tag{2.26}$$

For an anti-radial standard Morse-Smale pair (f, g^{TX}) this has been proved in [Lud17a, Theorem 6.2]. By Remark 2.5 the isomorphism (2.26) also holds if the pair is not standard.

Let $(C_\bullet^{sm}, \partial_\bullet)$ denote the subcomplex of $(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet)$ generated by $\mathrm{Crit}(f_{sm})$. There is an exact sequence of complexes

$$0 \to (C_\bullet^{sm}, \partial_\bullet) \to (C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet) \to ((C^{\overline{q}}(X, f, g^{TX}, F^*)/C^{sm})_\bullet, \partial_\bullet) \to 0. \tag{2.27}$$

By the local calculation for intersection homology (2.9) and Poincaré duality on the link manifold L_p, we have

$$H_\bullet((C^{\overline{q}}(X, f, g^{TX}, F^*)/C^{sm})_\bullet, \partial_\bullet) \simeq \bigoplus_{p \in \mathrm{Sing}(X)} IH_\bullet^{\overline{q}}(cL_p, L_p, F^*). \tag{2.28}$$

From (2.26), (2.27) and (2.28), we get a natural isomorphism

$$\begin{aligned} \det H_\bullet(C_\bullet^{sm}, \partial_\bullet) \otimes \det(\oplus_{p \in \mathrm{Sing}(X)} IH_\bullet^{\overline{q}}(cL_p, L_p, F^*)) &\simeq \det H_\bullet(C_\bullet^{\overline{q}}(X, f, g^{TX}, F^*), \partial_\bullet) \\ &\simeq \det IH_\bullet^{\overline{q}}(X, F^*) \simeq (\det IH_{\overline{q}}^\bullet(X, F))^{-1}. \end{aligned} \tag{2.29}$$

2.9. **The Bismut-Zhang metric on** $\det IH_{\overline{q}}^\bullet(X, F)$. In [Lud20a, Section 2.10] the Bismut-Zhang metric has been defined for a Witt space equipped with a unitary bundle (*i.e.* g^F is flat). The definition can be generalised easily to the current situation.

We denote by $\mathbf{\Delta}_T^{p,\overline{q}}$ the model Witten Laplacian on the infinite cone (see Section 4.2, (4.12)) and by $\mathbf{\Delta}_T^{p,\overline{q},\perp}$ its restriction to $(\ker \mathbf{\Delta}_T^{p,\overline{q}})^\perp$. Using the spectral properties of $\mathbf{\Delta}_T^{p,\overline{q}}$, computed in Section 4.2.3, one can show that, for $\Re(s) >> 0$, the zeta function

$$s \mapsto \zeta_T^{p,\overline{q}}(s) := -\mathrm{Tr_s}\left[N\left(\mathbf{\Delta}_T^{p,\overline{q},\perp}\right)^{-s}\right] \tag{2.30}$$

is a well-defined holomorphic function. Arguing as in the corresponding statement for a Witt space equipped with a unitary bundle in [Lud20a, Proposition 4.14], we get that the function $\zeta_T^{p,\overline{q}}$, $T > 0$, extends to a meromorphic function on $\mathbb{C}$, which is holomorphic at $s = 0$.

The L^2-metric on sections of $\Lambda(T^*Z_p) \otimes F$ restricts to a metric on $\ker \mathbf{\Delta}_T^{p,\overline{q}} \simeq IH_{\overline{q}}^\bullet(cL_p, L_p, F)$. We denote the induced metric on the line $\det IH_{\overline{q}}^\bullet(cL_p, L_p, F)$ by $|\ \ |^{RS}_{\det IH_{\overline{q}}^\bullet(cL_p, L_p, F), T}$. We denote by

$$\|\ \ \|^{RS}_{\det IH_{\overline{q}}^\bullet(cL_p, L_p, F), T} := |\ \ |^{RS}_{\det IH_{\overline{q}}^\bullet(cL_p, L_p, F), T} \exp\left(\frac{1}{2}(\zeta_T^{p,\overline{q}})'(0)\right). \tag{2.31}$$

Set $\zeta_p^{\overline{q}} := \zeta_1^{\overline{q},p}$.

Definition 2.6. For $p \in \mathrm{Sing}(X)$, the Ray-Singer metric on the line $\det IH^{\bullet}_{\overline{q}}(cL_p, L_p, F)$ is defined as

$$\| \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(cL_p,L_p,F)} := \| \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(cL_p,L_p,F),1} = | \ |^{RS}_{\det IH^{\bullet}_{\overline{q}}(cL_p,L_p,F),1} \exp\left(\frac{1}{2}(\zeta^{\overline{q}}_p)'(0)\right). \tag{2.32}$$

Remark 2.7. Using the scaling property of the model Witten Laplacian (4.15) and proceeding as in the proof of [Lud20a, Proposition 4.15], one can prove that the Definition 2.6 is independent of the choice of $T > 0$ (here $T = 1$).

The metrics g^{F_p} on the fibre F_p, $p \in \mathrm{Crit}(f_{sm})$, induce a metric on $(C^{sm}_{\bullet}, \partial_{\bullet})$. We get an induced metric on $\det H_{\bullet}(\mathrm{Hom}((C^{sm}_{\bullet}, \partial_{\bullet}), \mathbb{C}))$ (see [Mil66], [BZ92, Section I (d)]).

Definition 2.8. We denote by $\| \ \|^{\nabla f, g^{TX}, g^F}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$ the metric on the line $\det IH^{\bullet}_{\overline{q}}(X,F)$ induced via the natural isomorphisms (2.29) from the metric on $\det H^{\bullet}(\mathrm{Hom}((C^{sm}_{\bullet}, \partial_{\bullet}), \mathbb{C}))$ and the metrics $\| \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(cL_p,L_p,F)}$, $p \in \mathrm{Sing}(X)$. We call $\| \ \|^{\nabla f, g^{TX}, g^F}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$ the Bismut-Zhang metric associated to ∇f and the pair of metrics g^{TX}, g^F.

Remark 2.9.
(a) Unlike in the smooth situation, even in case g^F flat, $\| \ \|^{\nabla f, g^{TX}, g^F}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$ is not a purely topological invariant in general. Anomaly formulas for the Bismut-Zhang metric will be discussed in Section 7.3.
(b) As remarked in [Lud20a, Remark 2.11 (c)], in case that L_p is the standard sphere S^{n-1}, *i.e.* p is a smooth point, g^{TX} is Euclidean near p and g^F is flat near p, then the Bismut-Zhang metric is equal to the Milnor metric defined in [BZ92, Definition 1.9].

3. The Berezin integral formalism on a space with isolated conical singularities

In this section we study the Berezin integral formalism and the Mathai-Quillen current for a space with isolated conical singularities equipped with an anti-radial Morse function. We define the third term (the right hand side) in the Bismut-Zhang formula (Theorem I) and study, in Section 3.6, its variation with respect to the metrics g^{TX} and g^F.

In Sections 3.1-3.4, for convenience of the reader, we recall basic results related to the Berezin integral formalism and the Mathai-Quillen current. In Section 3.5 we provide some explicit formulas for the Berezin integral formalism and the Mathai-Quillen current on the infinite cone. They will be used in the study of the anomaly formulas in Section 3.6 and Section 7.

3.1. The Berezin integral.

We shortly recall the definition of the Berezin integral (see [BZ92, Section III]). Let E be an oriented Euclidean vector space of dimension n, and let V be a finite dimensional vector space. Let $e_1, \dots, e_n$ be an oriented orthonormal basis of E, and let $e^1, \dots, e^n$ be the corresponding dual basis of E^*. We denote by $\widehat{\otimes}$ the $\mathbb{Z}_2$-graded tensor product for $\mathbb{Z}_2$-graded algebras. The Berezin integral $\int^B$ is the linear map $\int^B : \Lambda(V^*)\widehat{\otimes}\Lambda(E^*) \longrightarrow \Lambda(V^*)$ characterised by the property that for $\alpha \in \Lambda(V^*)$ and $\beta \in \Lambda(E^*)$

$$\int^B \alpha\beta = \begin{cases} 0 & \text{if } \deg\beta < n, \\ \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^{n/2}}\alpha & \text{if } \beta = e^1 \wedge \dots \wedge e^n. \end{cases} \tag{3.1}$$

In case of a non-oriented Euclidean vector space E with orientation line $o(E)$, the Berezin integral is a map $\int^B : \Lambda(V^*)\widehat{\otimes}\Lambda(E^*) \longrightarrow \Lambda(V^*)\otimes o(E)$.

For an antisymmetric endomorphism C of E, we identify C with an element of $\Lambda^2(E^*)$ given by

$$\dot{C} = \frac{1}{2}\sum_{1\leqslant i,j\leqslant n}\langle e_i, Ce_j\rangle e^i \wedge e^j. \tag{3.2}$$

3.2. **Vector bundles and the Berezin integral formalism: the Mathai-Quillen-Thom forms.** Let $\pi_E : E \to X_{sm}$ be a real vector bundle of rank $\mathrm{rk}(E)$. Let g^E be a Euclidean metric on E and ∇^E a Euclidean connection on (E, g^E). We identify the curvature $R^E = (\nabla^E)^2$ with a smooth section $\dot{R}^E$ of the bundle $\Lambda^2(T^*X)\otimes\Lambda^2(E^*)$. By pullback, we get the Euclidean bundle $\pi_E^*(E, g^E)$ with Euclidean connection $\pi_E^*\nabla^E$ and curvature $\pi_E^*R^E$. The connection ∇^E defines a horizontal subspace T^HE of TE such that $TE = T^HE \oplus E$. We denote by $P^E : TE \to E$ the canonical projection and identify E with E^* by the metric g^E. Then P^E, which is a section of $T^*E\otimes E$, can be identified with a section $\dot{P}^E$ of $T^*E\otimes E^*$. Let Y be the generic element of E.

For $T \geqslant 0$, let A_T be the section of $\Lambda(T^*E)\widehat{\otimes}\pi_E^*\Lambda(E^*)$ on E given by

$$A_T := \frac{1}{2}\pi_E^*\dot{R}^E + \sqrt{T}\dot{P}^E + T|Y|^2. \tag{3.3}$$

The Berezin formalism applied to $V = TE$ yields a map from sections of the bundle $\Lambda(T^*E)\widehat{\otimes}\pi_E^*\Lambda(E^*)$ to sections of $\Lambda(T^*E)\otimes\pi_E^*o(E)$. In the following, we usually decorate with an $\widehat{}$ elements in the second factor of $\Lambda(T^*E)\widehat{\otimes}\pi_E^*\Lambda(E^*)$.

We define the following differential forms on E with values in $\pi_E^*o(E)$:

$$a_T := \int^B \exp(-A_T), T \geqslant 0; \quad b_T := \int^B \frac{\widehat{Y}}{2\sqrt{T}}\exp(-A_T), T > 0. \tag{3.4}$$

For $T \geqslant 0$ the forms a_T are closed forms of degree $\mathrm{rk}(E)$ and their cohomology class does not depend on T. For $T > 0$, the forms b_T have degree $\mathrm{rk}(E) - 1$, and

$$b_T = -\frac{1}{2T}\iota_Y a_T, \quad \frac{\partial a_T}{\partial T} = -db_T. \tag{3.5}$$

The Mathai-Quillen current has been defined in [MQ86, Section 7] (see also [BZ92, Definition 3.6]). It is the following well-defined current of degree $\mathrm{rk}(E) - 1$ on E with values in $\pi_E^*o(E)$

$$\Psi(E, \nabla^E) := \int_0^\infty b_T dT. \tag{3.6}$$

We denote by

$$e(E, \nabla^E) := \mathrm{Pf}\left(\frac{R^E}{2\pi}\right) := \int^B \exp\left(-\frac{\dot{R}^E}{2}\right) \tag{3.7}$$

the closed form of degree $\mathrm{rk}(E)$ on X with values in $o(E)$ representing the rational Euler class of E in Chern-Weil theory. Clearly, in case $\mathrm{rk}(E)$ odd, $e(E, \nabla^E) = 0$. The following identity of currents on E holds (see [BZ92, Theorem 3.7]):

$$d\Psi(E, \nabla^E) = \pi_E^*e(E, \nabla^E) - \delta_X. \tag{3.8}$$

3.3. **The Berezin integral formalism and the Mathai-Quillen current on TX.** We will mostly consider the Berezin integral formalism and the Mathai-Quillen current for the tangent space, *i.e.* we choose, with the notation of the previous section, $E = TX$ equipped with the conical Riemannian metric g^{TX} and its Levi-Civita connection ∇^{TX}. We denote by $o(TX)$ the orientation bundle of X. Let $e_1, \dots, e_n$ be an orthonormal basis of TX, and let $e^1, \dots, e^n$ be the corresponding dual basis of T^*X. We identify the curvature $R^{TX} = (\nabla^{TX})^2$ with a smooth section $\dot{R}^{TX}$ of the bundle $\Lambda^2(T^*X) \otimes \Lambda^2(T^*X)$. Let $f : X \to \mathbb{R}$ be an anti-radial Morse function. We define the following section of $\Lambda(T^*X)\widehat{\otimes}\Lambda(T^*X)$, see [BZ92, Proposition 3.10],

$$B_T := (\nabla f)^* A_T = \frac{\dot{R}^{TX}}{2} + \sqrt{T}\sum_{i=1}^{n} e^i \wedge \widehat{\nabla_{e_i}^{TX}\nabla} f + T|df|^2. \tag{3.9}$$

The Berezin integral formalism defines a map from smooth sections of $\Lambda(T^*X)\widehat{\otimes}\Lambda(T^*X)$ to smooth sections of $\Lambda(T^*X) \otimes o(TX)$. By [BZ92, Remark 3.8],

$$(\nabla f)^* \Psi(TX, \nabla^{TX}) = \int_0^\infty \left(\int^B \frac{\widehat{df}}{2\sqrt{T}} \exp(-B_T) \right) dT \tag{3.10}$$

is a well-defined locally integrable current on X_{sm} with values in $o(TX)$, smooth on $X \backslash \mathrm{Crit}(f)$. Moreover, by [BZ92, (6.1)], we have the following identity of currents on X_{sm}

$$d(\nabla f)^* \Psi(TX, \nabla^{TX}) = e(TX, \nabla^{TX}) - \sum_{p \in \mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)} \delta_p. \tag{3.11}$$

3.4. **Secondary Euler class.** Let g^{TX}, $g^{'TX}$ be two conical metrics on TX. Let $\mathbb{R} \ni l \mapsto g_l^{TX}$ be a family of conical metrics connecting $g^{TX} = g_0^{TX}$ and $g^{'TX} = g_1^{TX}$. We assume that near $p \in \mathrm{Sing}(X)$ the metric g_l^{TX}, $l \in \mathbb{R}$, is of the form $g^{TX}_{l|B_\delta(p)} = dr^2 + r^2 g_l^{TL_p}$, where $\mathbb{R} \ni l \mapsto g_l^{TL_p}$ is a family of Riemannian metrics on L_p and r is the radial coordinate. Let ∇_l^{TX} denote the Levi-Civita connection on (TX, ∇_l^{TX}), and R_l^{TX} the curvature of ∇_l^{TX}. Let $\rho : X \times \mathbb{R} \to X$ be the canonical projection. Let $g^{TX,\mathrm{tot}}$ be the metric on ρ^*TX which coincides with g_l^{TX} over $X \times \{l\}$. Let $\nabla^{TX,\mathrm{tot}}$ be the connection on ρ^*TX,

$$\nabla^{TX,\mathrm{tot}} = \rho^*\nabla_l^{TX} + dl \left(\frac{\partial}{\partial l} + \frac{1}{2}(g_l^{TX})^{-1} \frac{\partial g_l^{TX}}{\partial l} \right). \tag{3.12}$$

Then $\nabla^{TX,\mathrm{tot}}$ preserves the metric $g^{TX,\mathrm{tot}}$. The curvature $R^{TX,\mathrm{tot}} = (\nabla^{TX,\mathrm{tot}})^2$ is given by (see [BZ92, (4.51)])

$$R^{TX,\mathrm{tot}} = \rho^* R_l^{TX} + dl \left(\frac{\partial}{\partial l} \nabla_l^{TX} - \frac{1}{2} \left[\nabla_l^{TX}, (g_l^{TX})^{-1} \frac{\partial g_l^{TX}}{\partial l} \right] \right). \tag{3.13}$$

We denote by

$$\widetilde{e}(TX, \nabla_l^{TX}) := \int_0^1 dl \iota_{\partial_l} e(\rho^*TX, \nabla^{TX,\mathrm{tot}}) \in \Omega^{n-1}(X, o(TX)). \tag{3.14}$$

Since the Euler form $e(\rho^*TX, \nabla^{TX,\mathrm{tot}})$ is a closed form on $X \times \mathbb{R}$, we get

$$d\widetilde{e}(TX, \nabla_l^{TX}) = e(TX, \nabla^{'TX}) - e(TX, \nabla^{TX}), \tag{3.15}$$

hence $\widetilde{e}(TX, \nabla_l^{TX})$ is a secondary Euler class in the sense of Chern-Simons.

By [BZ92, (3.34)] the following identity holds modulo exact currents

$$\Psi(TX, \nabla'^{TX}) - \Psi(TX, \nabla^{TX}) = \pi^* \widetilde{e}(TX, \nabla_l^{TX}), \tag{3.16}$$

where $\pi : TX \to X_{sm}$ is the canonical projection.

3.5. **The Berezin integral formalism and the Mathai-Quillen current on the infinite cone.** In this section we study in more detail the notions introduced in Sections 3.3-3.4 locally near a singular point $p \in \mathrm{Sing}(X)$, *i.e.* on the punctured infinite cone Z_p equipped with the conical metric $g^{TZ_p} = dr^2 + r^2 g^{TL_p}$. We denote by $o(TZ_p)$ the orientation bundle of Z_p.

Let $e_1, \ldots, e_n$ be an orthonormal basis of TZ_p with $e_1 = e_r := \frac{\partial}{\partial r}$. Let $e^1, \ldots, e^n$ be the corresponding dual basis of T^*Z_p. We denote by ∇^{TZ_p} the Levi-Civita connection on (TZ_p, g^{TZ_p}). We again identify the curvature $R^{TZ_p} = (\nabla^{TZ_p})^2$ with a smooth section $\dot{R}^{TZ_p}$ of $\Lambda^2(T^*Z_p)\widehat{\otimes}\Lambda^2(T^*Z_p)$.

We denote by ∇^{sp} the Levi-Civita connection on $(TZ_p, g^{sp} = dr^2 + g^{TL_p})$. We denote by $e(TZ_p, \nabla^{TZ_p})$ (resp. by $e(TZ_p, \nabla^{sp})$) the Euler form associated with (TZ_p, ∇^{TZ_p}) (resp. with (TZ_p, ∇^{sp})). We denote by $\widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp})$ the Chern-Simons class of smooth forms on Z_p with values in $o(TZ_p)$ of degree $n-1$, which is defined modulo exact forms, such that

$$d\widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp}) = e(TZ_p, \nabla^{sp}) - e(TZ_p, \nabla^{TZ_p}). \tag{3.17}$$

In case n odd, clearly,

$$e(TZ_p, \nabla^{TZ_p}) = 0,\ e(TZ_p, \nabla^{sp}) = 0 \ \text{ and } \ \widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp}) = 0. \tag{3.18}$$

In case n even, due to the flat radial direction on the cone (more precisely from (2.6)) resp. since ∇^{sp} is the product connection, the first two identities in (3.18) also hold, hence $\widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp})$ is a closed form.

We denote by $f^p : Z_p \to \mathbb{R}$, $f^p(r, y) = f(p) - \frac{1}{2}r^2$, the model anti-radial Morse function on the infinite cone Z_p. For $T \geqslant 0$, as in (3.9), we have the following smooth section of $\Lambda(T^*Z_p)\widehat{\otimes}\Lambda(T^*Z_p)$ over Z_p,

$$B_T^p := (\nabla f^p)^* A_T^p = \frac{\dot{R}^{TZ_p}}{2} - \sqrt{T}\sum_{i=1}^n e^i \wedge \widehat{e^i} + Tr^2. \tag{3.19}$$

We can define the Mathai-Quillen current on the infinite cone $\Psi(TZ_p, \nabla^{TZ_p})$ as in (3.6). The Berezin integral formalism gives a map $\int^{B,p}$ from smooth sections of $\Lambda(T^*Z_p)\widehat{\otimes}\Lambda(T^*Z_p)$ to smooth sections of $\Lambda(T^*Z_p) \otimes o(TZ_p)$. Hence

$$\begin{aligned}(\nabla f^p)^*\Psi(TZ_p, \nabla^{TZ_p}) &= \int_0^\infty \int^{B,p} \left(\frac{\widehat{df^p}}{2\sqrt{T}} \exp(-B_T^p) \right) dT \\ &= -\int_0^\infty \int^{B,p} \left(\frac{r\widehat{e^r}}{2\sqrt{T}} \exp(-B_T^p) \right) dT.\end{aligned} \tag{3.20}$$

Since $TZ_p \simeq \mathbb{R} \times TL_p$ and fixing the orientation by $e^r := \frac{\partial}{\partial r}$ on the first factor, we get an identification of the orientation lines $o(TZ_p)$ and $o(TL_p)$. We denote by $e(L_p, \nabla^{TL_p}) \in \Omega^{n-1}(L_p, o(TL_p))$ the Euler form of (TL_p, ∇^{TL_p}), which by the above identification can be seen as a form in $\Omega^{n-1}(Z_p, o(TZ_p))$.

Proposition 3.1. *Let* $p \in \mathrm{Sing}(X)$.

(a) *The form* $(\nabla f^p)^*\Psi(TZ_p, \nabla^{TZ_p})$ *is an* $(n-1)$*-form on* Z_p *not depending on the radial coordinate. Moreover the following identity holds modulo exact currents,*

$$\eta_p := (\nabla f^p)^*\Psi(TZ_p, \nabla^{TZ_p}) = \begin{cases} -\widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp}) & \text{if } n \text{ is even,} \\ \frac{1}{2} e(TL_p, \nabla^{TL_p}) & \text{if } n \text{ is odd.} \end{cases} \tag{3.21}$$

(b) *Let* $h : Z_p \to \mathbb{R}, (r, y) \mapsto h(y)$*, be a function not depending on the radial coordinate* r*. Then the following integral is well-defined and does not depend on* $T > 0$*:*

$$\int_{Z_p} h(y) \int^{B,p} \exp(-B_T^p). \tag{3.22}$$

Moreover

$$\int_{Z_p} h(y) \int^{B,p} \exp(-B_T^p) = -\int_{L_p} h(y)\eta_p. \tag{3.23}$$

(c) *Setting* $h \equiv 1$ *in (b), we have the following well-defined integral, not depending on* $T > 0$*:*

$$\alpha_p := \int_{Z_p} \int^{B,p} \exp(-B_T^p). \tag{3.24}$$

Moreover

$$\alpha_p = -\int_{L_p} \eta_p = \begin{cases} \int_{L_p} \widetilde{e}(TZ_p, \nabla^{TZ_p}, \nabla^{sp}) & \text{for } n \text{ even}, \\ -\frac{1}{2}\chi(L_p, \mathbb{C}) & \text{for } n \text{ odd}. \end{cases} \tag{3.25}$$

Proof. For $a > 0$, let h_a be the radial scaling $r \mapsto ar$. We have the following scaling properties (see [Lud20b, Lemma 6.5]):

$$\begin{aligned} h_a^* \int^{B,p} \left(\frac{\widehat{df^p}}{2\sqrt{T}} \exp(-B_T^p) \right) &= a^2 \int^{B,p} \left(\frac{\widehat{df^p}}{2a\sqrt{T}} \exp(-B_{a^2T}^p) \right), \\ h_a^* \int^{B,p} \exp(-B_T^p) &= \int^{B,p} \exp(-B_{a^2T}^p). \end{aligned} \tag{3.26}$$

(a) Using (3.26) and the change of variables $T \rightsquigarrow Tr^2$ in the integral (3.20) we get the first claim. The second claim has been proved in [Lud20b, Proposition 6.6].

(c) The statement is a consequence of part (a) and (b), and has been proved in [Lud20b, Theorem 6.7]. (b) The fact that the integral (3.22) does not depend on $T > 0$ follows from the scaling property (3.26). Well-definedness of the integral at $r = \infty$ follows due to the Gaussian factor $\exp(-Tr^2)$ in $\int^{B,p} \exp(-B_T^p)$; well-definedness at $r = 0$ follows using (2.6), (3.19). For the proof of (3.23), we use a transgression argument very similar to [Lud20b, Theorem 6.7]: here it is crucial that the function h does not depend on the radial coordinate.

□

Starting with two conical metrics g^{TZ_p}, g'^{TZ_p} and a family $\mathbb{R} \ni l \mapsto g_l^{TZ_p}$ connecting them, we can define all the notions introduced in Section 3.4: $\rho_p : Z_p \times \mathbb{R} \to Z_p$, $g^{TZ_p,\mathrm{tot}}$, $\nabla^{TZ_p,\mathrm{tot}}$, $\widetilde{e}(TZ_p, \nabla_l^{TZ_p})$, etc. We hereby apply the formalism introduced in Section 3.2 to the Euclidean vector bundle $(\rho_p^* T^*Z_p, g^{TZ_p,\mathrm{tot}})$ over $Z_p \times \mathbb{R}$, with Euclidean connection $\nabla^{TZ_p,\mathrm{tot}}$.

We denote by $\widetilde{\nabla f^p} = -r\frac{\partial}{\partial r}$ the section on $\rho_p^* TZ_p$ induced from ∇f^p. Set

$$\widetilde{e}_\Psi(Z_p, \nabla_l^{TZ_p}) := \int_0^1 dl \iota_{\partial_l} (\widetilde{\nabla f^p})^* \Psi(\rho^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) \in \Omega^{n-2}(Z_p, o(TZ_p)). \tag{3.27}$$

Similarly to Proposition 3.1 one can see that $\widetilde{e}_\Psi(Z_p, \nabla_l^{TZ_p})$ does not depend on the radial coordinate.

We define, for $T \geqslant 0$, the following smooth section of $\Lambda(T^*(Z_p \times \mathbb{R})) \widehat{\otimes} \Lambda(\rho_p^* T^* Z_p)$,

$$\widetilde{B}_T^p := (\widetilde{\nabla f^p})^* A_T^p = \frac{\dot{R}^{TZ_p,\mathrm{tot}}}{2} - \sqrt{T} \sum_{i=1}^n e^i \wedge \widehat{e^i} + Tr^2, \tag{3.28}$$

and

$$e_T(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) := \int^{B,p} \exp\left(-\widetilde{B}_T^p\right) \in \Omega^n(Z_p \times \mathbb{R}, o(\rho_p^* TZ_p)); \tag{3.29}$$

in particular $e_0(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) = e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}})$.

Proposition 3.2. *Let* $p \in \mathrm{Sing}(X)$.

(a) *We have the following refinement of* (3.16)*, locally near* p*,*

$$(\nabla f^p)^* \Psi(TZ_p, \nabla^{TZ_p}) - (\nabla f^p)^* \Psi(TZ_p, \nabla'^{TZ_p}) = -\widetilde{e}(TZ_p, \nabla_l^{TZ_p}) - d\widetilde{e}_\Psi(Z_p, \nabla_l^{TZ_p}). \tag{3.30}$$

(b) *The form*

$$(\widetilde{\nabla f^p}^*) \Psi(\rho^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) \tag{3.31}$$

is an $(n-1)$*-form on* $Z_p \times \mathbb{R}$ *not depending on the radial coordinate* r*. Moreover, we have the following identity*

$$\int_{Z_p} \theta(F, g^F) \wedge \iota_{\partial_l} e_1(\rho^* TZ_p, \nabla^{TZ,\mathrm{tot}}) = -\int_{L_p} \theta(F, g^F) \wedge \iota_{\partial_l} (\widetilde{\nabla f^p})^* \Psi(\rho^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}). \tag{3.32}$$

Proof. (a) Since $\widetilde{\nabla f^p}$ is nowhere vanishing on $Z_p \times \mathbb{R}$, we get from [BZ92, Remark 3.8]

$$d^{Z_p \times \mathbb{R}} (\widetilde{\nabla f^p})^* \Psi(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) = e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}). \tag{3.33}$$

By comparing the coefficients of dl in (3.33), and integrating over $l \in [0,1]$, we get (3.30).

(b) Using [BC90, Proposition 1.2], (2.6) and (3.13) we have

$$R^{TZ_p,\mathrm{tot}}(e_r, \ ) = 0. \tag{3.34}$$

For $a > 0$, let again h_a be the radial scaling $r \to ar$; then using (2.6), (3.13), (3.34), we have $h_a^* \dot{R}^{TZ_p,\mathrm{tot}} = \dot{R}^{TZ_p,\mathrm{tot}}$. Hence, together with (3.28), we get the following scaling property

$$h_a^* \int^{B,p} \left(\frac{\widehat{\widetilde{df^p}}}{2\sqrt{T}} \exp(-\widetilde{B}_T^p) \right) = a^2 \int^{B,p} \left(\frac{\widehat{\widetilde{df^p}}}{2a\sqrt{T}} \exp(-\widetilde{B}_{a^2T}^p) \right). \tag{3.35}$$

Using (3.35) we can now argue as in the proof of Proposition 3.1 (a) to prove the first claim.

In view of (3.34) and the definition of the Berezin integral, the n-form on $Z_p \times \mathbb{R}$

$$e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) = \int^{B,p} \exp\left(-\frac{\dot{R}^{TZ_p,\mathrm{tot}}}{2}\right) \tag{3.36}$$

does not contain e^r. Since, as seen in Section 2.2, the form $\theta(F, g^F)$ on Z_p does not depend on the radial coordinate, the form $\theta(F, g^F) \wedge \iota_{\partial l} e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}})$ is an n-form on Z_p (depending on the parameter l) not containing e^r. Hence

$$\theta(F, g^F) \wedge \iota_{\partial l} e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) = 0. \tag{3.37}$$

From (3.5) we get

$$\partial_T e_T(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) = d^{Z_p\times\mathbb{R}} \int^{B,p} \frac{r\widehat{e^r}}{2\sqrt{T}} \exp\left(-\widetilde{B}_T^p\right). \tag{3.38}$$

Using (3.35) and (3.38) we get

$$\begin{aligned} &e_1(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) - e(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) \\ &\qquad = d^{Z_p\times\mathbb{R}} \left(\int_0^1 dT \int^{B,p} \frac{r\widehat{e^r}}{2\sqrt{T}} \exp\left(-\widetilde{B}_T^p\right) \right) \\ &\qquad = d^{Z_p\times\mathbb{R}} \left(\int_0^{r^2} dT \left(\int^{B,p} \frac{r\widehat{e^r}}{2\sqrt{T}} \exp\left(-\widetilde{B}_T^p\right) \right)_{|r=1} \right) \\ &\qquad =: d^{Z_p\times\mathbb{R}} (dl \wedge \gamma_1(l,r,y) + \gamma_2(l,r,y)). \end{aligned} \tag{3.39}$$

Using (3.34) and the definition of the Berezin integral, one sees that γ_1 (resp. γ_2) is a form of degree $n-2$ (resp. of degree $n-1$) on $Z_p \times \mathbb{R}$ not containing e^r. Therefore

$$\iota_{\partial_l} d^{Z_p\times\mathbb{R}} \left(\int_0^1 dT \left(\int^{B,p} \frac{r\widehat{e^r}}{2\sqrt{T}} \exp\left(-\widetilde{B}_T^p\right) \right) (r,y) \right) = -d^{L_p}\gamma_1 - e^r \wedge \frac{\partial \gamma_1}{\partial r} + \frac{\partial \gamma_2}{\partial l}. \tag{3.40}$$

Since $\theta(F, g^F)$ does not depend on the radial coordinate and is a closed form, we get using (3.35), (3.37), (3.39), (3.40) and Stokes' Theorem

$$\begin{aligned} \int_{Z_p} \theta(F, g^F) \wedge \iota_{\partial_l} e_1(\rho_p^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}) &= -\int_{Z_p} \theta(F, g^F) \wedge e^r \wedge \frac{\partial \gamma_1}{\partial r} \\ &= -\int_{L_p} \theta(F, g^F) \wedge \iota_{\partial_l} (\widetilde{\nabla f^p})^* \Psi(\rho^* TZ_p, \nabla^{TZ_p,\mathrm{tot}}). \end{aligned} \tag{3.41}$$

□

3.6. **Variation formula for the integral** $-\int_X \theta(F, g^F)(\nabla f)^* \Psi(TX, \nabla^{TX})$**.** The current $(\nabla f)^* \Psi(TX, \nabla^{TX})$ has been defined in Section 3.3 and is a locally integrable current with values in $o(TX)$, smooth on $X \backslash \mathrm{Crit}(f)$. We have

$$\theta(F, g^F)(\nabla f)^* \Psi(TX, \nabla^{TX}) = 0 \text{ near } \mathrm{Sing}(X), \tag{3.42}$$

since it is a form of top degree not containing e^r. The integral

$$-\int_X \theta(F, g^F)(\nabla f)^* \Psi(TX, \nabla^{TX}) \tag{3.43}$$

appearing in as the right hand side in the Bismut-Zhang formula in Theorem I is hence well-defined.

The aim of this section is to study the dependence of the integral (3.43) with respect to the metrics g^F and g^{TX}.

We denote by $g^{'TX}$ a second conical metric on X, and by $g^{'F}$ a second metric on F satisfying the assumption explained in Section 2.2. The Levi-Civita connection associated to $g^{'TX}$ is denoted by $\nabla^{'TX}$ and we denote by $\nabla' f$ the gradient vector field of f with respect to the conical metric $g^{'TX}$. Let $\mathbb{R} \ni l \mapsto g_l^{TX}$ be a family of conical metrics on X connecting g^{TX}, $g^{'TX}$ as explained in Section 3.4.

For $\epsilon > 0$ small enough, we denote by $X_\epsilon := X \backslash (\cup_{p \in \mathrm{Sing}(X)} B_\epsilon(p))$. We identify the orientation bundle of X_ϵ and the orientation bundle of ∂X_ϵ using the Stokes' convention. Note that the Stokes' convention and the convention on orientation at the beginning of Proposition 3.1 differ by a sign.

The next proposition generalises [BZ92, Theorem 6.1] to our singular setting:

Proposition 3.3. *The following identity holds:*

$$\int_X \theta(F, g^F)(\nabla f)^* \Psi(TX, \nabla^{TX}) = \int_X \theta(F, g^F)(\nabla' f)^* \Psi(TX, \nabla^{TX}). \tag{3.44}$$

Proof. We denote by $\nabla_l f$ the gradient of f w.r.t. the metric g_l^{TX}; we have for $l \in [0,1]$,

$$\nabla_l f = -r\partial_r \text{ loc. near } \mathrm{Sing}(X). \tag{3.45}$$

We define the homotopy $H : X \times [0,1] \to TX, \quad H_l = \nabla_l f$. Then

$$\begin{aligned}(\nabla f)^* \Psi(TX, \nabla^{TX}) &- (\nabla' f)^* \Psi(TX, \nabla^{TX}) \\ &= d\left(\int_0^1 dl \iota_{\partial_l} H^* \Psi(TX, \nabla^{TX})\right) + \int_0^1 dl \iota_{\partial_l} H^* d\Psi(TX, \nabla^{TX}).\end{aligned} \tag{3.46}$$

By (3.8) and (3.45), the last term on the right hand side of (3.46) vanishes. By (3.45), we also have $\iota_{\partial_l} H^* \Psi(TX, \nabla^{TX}) = 0$ near $\mathrm{Sing}(X)$. Therefore we get from (3.46), that $(\nabla f)^* \Psi(X, \nabla^{TX}) - (\nabla' f)^* \Psi(X, \nabla^{TX})$ is an exact current and can be written as

$$(\nabla f)^* \Psi(X, \nabla^{TX}) - (\nabla' f)^* \Psi(X, \nabla^{TX}) = d\sigma, \tag{3.47}$$

for a form σ which vanishes near $\mathrm{Sing}(X)$. Since $\theta(F, g^F)$ is closed, we get using (3.47) and Stokes' Theorem, for $\epsilon > 0$ small enough,

$$\begin{aligned}\int_{X_\epsilon} \theta(F, g^F)(\nabla f)^* \Psi(TX, \nabla^{TX}) &- \int_{X_\epsilon} \theta(F, g^F)(\nabla' f)^* \Psi(TX, \nabla^{TX}) = \\ &= \int_{X_\epsilon} \theta(F, g^F) \wedge d\sigma = -\int_{\partial X_\epsilon} \theta(F, g^F) \wedge \sigma = 0.\end{aligned} \tag{3.48}$$

The claim of the proposition follows by taking the limit $\epsilon \searrow 0$ in (3.48). □

Recall that η_p, $\widetilde{e}_\Psi(Z_p, \nabla_l^{TZ_p})$, $p \in \mathrm{Sing}(X)$, have been defined in (3.21), (3.27). The following theorem generalises [BZ92, Theorem 6.3] to our situation:

Theorem 3.4. *The following identity holds*

$$
\begin{aligned}
&\int_X \theta(F,g^F)(\nabla f)^*\Psi(TX,\nabla^{TX}) - \int_X \theta(F,g^{'F})(\nabla' f)^*\Psi(TX,\nabla^{'TX}) \\
&= \int_X \log\left(\frac{\|\ \|^{'2}_{\det F}}{\|\ \|^2_{\det F}}\right) e(TX,\nabla^{TX}) - \sum_{p\in \mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)} \log\left(\frac{\|\ \|^{'2}_{\det F_p}}{\|\ \|^2_{\det F_p}}\right) \\
&\quad - \int_X \theta(F,g^{'F})\widetilde{e}(TX,\nabla^{TX}_l) \\
&\quad + \sum_{p\in \mathrm{Sing}(X)} \left(\int_{L_p} \log\left(\frac{\|\ \|^{'2}_{\det F}}{\|\ \|^2_{\det F}}\right)\wedge \eta_p - \int_{L_p} \theta(F,g^{'F})\wedge \widetilde{e}_\Psi(Z_p,\nabla^{TZ_p}_l)\right).
\end{aligned}
\tag{3.49}
$$

Remark 3.5. (a) The only difference between (3.49) and the corresponding smooth formula in [BZ92, Theorem 6.3] are the last two terms on the right hand side, which are the contribution of the singularities of X.

(b) Note that the two integrals over X on the right hand side of (3.49) are well-defined: The first integrand vanishes near $\mathrm{Sing}(X)$ by (3.18), the second integrand vanishes near $\mathrm{Sing}(X)$ by (3.14), (3.37).

Proof. We have, by definition of $\theta(F,g^F)$,

$$
\theta(F,g^F) - \theta(F,g^{'F}) = -d\log\left(\frac{\|\ \|^{'2}_{\det F}}{\|\ \|^2_{\det F}}\right).
\tag{3.50}
$$

Using the identity of currents (3.11), as well as (3.18), (3.21), (3.42), (3.50) and Stokes' Theorem, we get

$$
\begin{aligned}
&\int_X (\theta(F,g^F) - \theta(F,g^{'F}))(\nabla f)^*\Psi(TX,\nabla^{TX}) \\
&= \int_{X_\epsilon} (\theta(F,g^F) - \theta(F,g^{'F}))(\nabla f)^*\Psi(TX,\nabla^{TX}) \\
&= \int_{X_\epsilon} \log\left(\frac{\|\ \|^{'2}_{\det F}}{\|\ \|^2_{\det F}}\right) e(TX,\nabla^{TX}) - \sum_{p\in \mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)} \log\left(\frac{\|\ \|^{'2}_{\det F_p}}{\|\ \|^2_{\det F_p}}\right) \\
&\quad - \int_{\partial X_\epsilon} \log\left(\frac{\|\ \|^{'2}_{\det F}}{\|\ \|^2_{\det F}}\right) \wedge (\nabla f)^*\Psi(TX,\nabla^{TX}) \\
&= \int_X \log\left(\frac{\|\ \|^{'2}_{\det F}}{\|\ \|^2_{\det F}}\right) e(TX,\nabla^{TX}) - \sum_{p\in \mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)} \log\left(\frac{\|\ \|^{'2}_{\det F_p}}{\|\ \|^2_{\det F_p}}\right) \\
&\quad + \sum_{p\in \mathrm{Sing}(X)} \int_{L_p} \log\left(\frac{\|\ \|^{'2}_{\det F}}{\|\ \|^2_{\det F}}\right)\wedge \eta_p.
\end{aligned}
\tag{3.51}
$$

Using Proposition 3.2, (3.16), (3.42), $d\theta(F, g^F) = 0$ and Stokes' Theorem, we get, for $\epsilon > 0$ small enough,

$$\begin{aligned}
&\int_X \theta(F, g'^F)\Big((\nabla f)^*\Psi(TX, \nabla^{TX}) - (\nabla f)^*\Psi(TX, \nabla'^{TX})\Big) \\
&\quad = \int_{X_\epsilon} \theta(F, g'^F)\Big((\nabla f)^*\Psi(TX, \nabla^{TX}) - (\nabla f)^*\Psi(TX, \nabla'^{TX})\Big) \\
&\quad = -\int_{X_\epsilon} \theta(F, g'^F)\widetilde{e}(TX, \nabla_l^{TX}) - \sum_{p\in \mathrm{Sing}(X)} \int_{L_p} \theta(F, g'^F) \wedge \widetilde{e}_\Psi(Z_p, \nabla_l^{TZ_p}) \\
&\quad = -\int_{X} \theta(F, g'^F)\widetilde{e}(TX, \nabla_l^{TX}) - \sum_{p\in \mathrm{Sing}(X)} \int_{L_p} \theta(F, g'^F) \wedge \widetilde{e}_\Psi(Z_p, \nabla_l^{TZ_p}).
\end{aligned} \tag{3.52}$$

The claim of Theorem 3.4 follows from Proposition 3.3, (3.51) and (3.52). □

4. Witten deformation for singular spaces with isolated conical singularities using anti-radial Morse functions

An important role in the extension of the Cheeger-Müller theorem by Bismut and Zhang [BZ92] is played by the Witten deformation. The Witten deformation is an analytic proof of the Morse inequalities proposed by [Wit82]. Rigorous proofs have been given by Helffer and Sjöstrand in [HS85] using semi-classical analysis. In [BZ94, Section 6] Bismut and Zhang gave a different proof of the hard part of Witten's program using a result of Laudenbach [Lau92]. For singular spaces with iterated conical singularities and anti-radial Morse functions the easy part of the Witten deformation has been generalised in [ÁC17] and [Lud17b], the hard part – which is a crucial ingredient in the proof of the Cheeger-Müller theorem – has been adressed in [Lud17b].

Section 4 is organised as follows: In Section 4.1 we shortly recall from [Lud17b] the Witten deformation for singular spaces with isolated conical singularities and anti-radial Morse functions. Most of the proofs of the intermediate results of Section 5.2 consist of two steps: localisation and a local computation near $p \in \mathrm{Crit}(f)$. In Sections 4.2–4.3 we deal with the second step for $p \in \mathrm{Sing}(X)$. To this purpose, we generalise the study of the spectral properties and the local index techniques for the local model Witten Laplacian in [Lud20b] and [Lud20a, Section 4] to the present situation.

4.1. **Witten deformation.** Let $f : X \to \mathbb{R}$ be an anti-radial Morse function and $T \geqslant 0$. The de Rham complex $(\Omega_c^\bullet(X, F), d_c, \langle\ ,\ \rangle)$ can be deformed by deforming the differential d_c via

$$d_{T,c} := e^{-Tf} d_c e^{Tf}. \tag{4.1}$$

The complex $(\Omega_c^\bullet(X, F), d_{T,c}, \langle\ ,\ \rangle)$ admits a maximal and a minimal extension denoted by $(\widetilde{\mathcal{C}}^\bullet_{T,\max/\min}, d_{T,\max/\min}, \langle\ ,\ \rangle)$. The Hilbert complex $(\widetilde{\mathcal{C}}^\bullet_{T,\max/\min}, d_{T,\max/\min}, \langle\ ,\ \rangle)$ still computes $H^\bullet_{(2),\overline{m}}(X, F)$ (resp. $H^\bullet_{(2),\overline{n}}(X, F)$). We denote by $\delta_{T,c}$ the (formal) adjoint of $d_{T,c}$ w.r.t. the L^2-inner product $\langle\ ,\ \rangle$. We again have the first order operator $\widetilde{D}_T^{\overline{m}} := d_{T,\max} + \delta_{T,\min}$ (resp. $\widetilde{D}_T^{\overline{n}} := d_{T,\min} + \delta_{T,\max}$) associated to the Hilbert complex $(\widetilde{\mathcal{C}}^\bullet_{T,\max}, d_{T,\max}, \langle\ ,\ \rangle)$ (resp. $(\widetilde{\mathcal{C}}^\bullet_{T,\min}, d_{T,\min}, \langle\ ,\ \rangle)$). We denote by $\widetilde{\Delta}_T^{\overline{q}} := (\widetilde{D}_T^{\overline{q}})^2$, $\overline{q} \in \{\overline{m}, \overline{n}\}$, the second order operators associated to the two Hilbert complexes.

We denote by $\widehat{c}(\nabla f)$ the Clifford multiplication acting on sections α of $\Lambda(T^*X)\otimes F$ by

$$\widehat{c}(\nabla f)\alpha = df \wedge \alpha + \nabla f \,\lrcorner\, \alpha. \tag{4.2}$$

We denote by $L_{\nabla f}$ the Lie derivative in direction of ∇f and by $L^*_{\nabla f}$ its adjoint with respect to the L^2-inner product $\langle\ ,\ \rangle$. The operator $L_{\nabla f} + L^*_{\nabla f}$ acting on $\mathrm{L}^2(\Lambda(T^*X)\otimes F)$ is a bounded operator of order 0. This follows from the smooth theory combined with the local model near $\mathrm{Sing}(X)$, see Section 4.2.2 for more details.

We have, see [BZ92, Proposition 5.5], [Lud17b, Propositions 3.8 and 5.1],

$$\begin{aligned} \widetilde{D}^{\overline{q}}_T &= D^{\overline{q}} + T\widehat{c}(\nabla f), \quad \mathrm{dom}(\widetilde{D}^{\overline{q}}_T) = \mathrm{dom}(D^{\overline{q}}), \\ \widetilde{\Delta}^{\overline{q}}_T &= \Delta^{\overline{q}} + T(L_{\nabla f} + L^*_{\nabla f}) + T^2|\nabla f|^2, \quad \mathrm{dom}(\widetilde{\Delta}^{\overline{q}}_T) = \mathrm{dom}(\Delta^{\overline{q}}). \end{aligned} \tag{4.3}$$

Using the local model form of the Witten Laplacian $\widetilde{\Delta}^{\overline{q}}_T$ near $p \in \mathrm{Sing}(X)$ (see (4.12)), one can show inductively, that for $\overline{q} \in \{\overline{m}, \overline{n}\}$, $l \in \mathbb{N}$, $T \geqslant 0$:

$$\mathrm{dom}((\widetilde{\Delta}^{\overline{q}}_T)^l) = \mathrm{dom}((\Delta^{\overline{q}})^l). \tag{4.4}$$

There is a second, equivalent way of describing the Witten deformation: The de Rham complex $(\Omega^\bullet_c(X,F), d_c, \langle\ ,\ \rangle)$ can also be deformed by deforming the L^2-inner product $\langle\ ,\ \rangle$ via

$$\langle \alpha, \beta\rangle_T := \int_X \langle \alpha, \beta\rangle_{\Lambda(T^*X)\otimes F}(x) e^{-2Tf(x)} d\mathrm{vol}_X(x); \tag{4.5}$$

here $d\mathrm{vol}_X$ denotes the Riemannian volume form on (X, g^{TX}).

The deformed complex $(\Omega^\bullet_c(X,F), d_c, \langle\ ,\ \rangle_T)$ also admits a maximal and minimal extension into a Hilbert complex $(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$. We denote by $\delta'_{T,c}$ the (formal) adjoint of d_c w.r.t. the twisted L^2-inner product $\langle\ ,\ \rangle_T$ and by $\delta'_{T,\max/\min}$ its maximal resp. its minimal extension. The first order operator associated to the Hilbert complex $(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$ (resp. $(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$) is given by

$$D^{\overline{m}}_T = d_{\max} + \delta'_{T,\min} \ (\text{resp. } D^{\overline{n}}_T = d_{\min} + \delta'_{T,\max}). \tag{4.6}$$

We denote by $\Delta^{\overline{q}}_T := (D^{\overline{q}}_T)^2$, $\overline{q} \in \{\overline{m}, \overline{n}\}$ the second order self-adjoint operator associated to the two Hilbert complexes.

4.2. **The model Witten Laplacian $\boldsymbol{\Delta}^{p,\overline{q}}_T$, $p \in \mathrm{Sing}(X)$, $\overline{q} \in \{\overline{m}, \overline{n}\}$, $T \geqslant 0$.** We now study the situation locally near $p \in \mathrm{Sing}(X)$, *i.e.* on the punctured infinite cone Z_p. As in Section 3.5, we denote by $f^p : cL_p \longrightarrow \mathbb{R}, (r,y) \mapsto f(p) - \frac{1}{2}r^2$ the model anti-radial Morse function on the infinite cone.

4.2.1. *A useful unitary transformation.* Let $\pi : Z_p \simeq \mathbb{R}_{>0} \times L_p \to L_p$ be the projection into the second factor. We denote by $\mathrm{L}^2(\Lambda^k(T^*L_p)\otimes F_{L_p})$ the space of L^2-sections of $\Lambda^k(T^*L_p)\otimes F_{L_p}$ with respect to the L^2-metric induced from g^{TL_p} and $g^{F_{L_p}}$. We denote by $\mathrm{L}^2(\Lambda^k(T^*Z_p)\otimes F)$ the space of L^2-sections of $\Lambda^k(T^*Z_p)\otimes F$ with respect to the L^2-metric $\langle\ ,\ \rangle$ induced from g^{TZ_p} and g^F.

For $k = 0, \ldots n$, the bijective maps

$$\begin{aligned} U_k :\ \ & C^\infty_c(\mathbb{R}_{>0}, \Omega^{k-1}(L_p, F_{L_p}) \oplus \Omega^k(L_p, F_{L_p})) \longrightarrow \Omega^k_c(Z_p, F) \\ & (\phi_{k-1}, \phi_k) \mapsto r^{k-1-(n-1)/2}\pi^*\phi_{k-1} \wedge dr + r^{k-(n-1)/2}\pi^*\phi_k, \end{aligned} \tag{4.7}$$

extend to unitary maps from $\mathrm{L}^2\big(\mathbb{R}_{>0}, \mathrm{L}^2((\Lambda^{k-1}(T^*L_p)\oplus\Lambda^k(T^*L_p))\otimes F_{L_p})\big)$ to $\mathrm{L}^2\big(\Lambda^k(T^*Z_p)\otimes F\big)$.

We denote by d_{L_p} the exterior derivative on $\Omega^\bullet(L_p, F_{L_p})$, and by δ_{L_p} its adjoint with respect to the L^2-metric on $\Omega^\bullet(L_p, F_{L_p})$ induced from the metrics g^{TL_p} and $g^{F_{L_p}}$. We denote by S_p the following self-adjoint elliptic operator on the link L_p:

$$S_p := \begin{pmatrix} c_0 & \delta_{L_p} & 0 & \dots & 0 \\ d_{L_p} & c_1 & & \ddots & \vdots \\ 0 & & & & 0 \\ \vdots & \ddots & & c_{n-2} & \delta_L \\ 0 & \dots & 0 & d_{L_p} & c_{n-1} \end{pmatrix} \text{ where } c_k := (-1)^k\left(k - \frac{n-1}{2}\right). \tag{4.8}$$

For the Laplace operator $\mathbf{\Delta}^{p,\mathrm{ev/odd}}$ on the infinite cone acting on compactly supported even (resp. odd) forms we have (see e.g. [BS87, Section 5]):

$$U^{-1}\mathbf{\Delta}^{p,\mathrm{ev/odd}}U = -\frac{\partial^2}{\partial r^2} + r^{-2}\left[\left(S \pm \frac{1}{2}\right)^2 - \frac{1}{4}\right]. \tag{4.9}$$

4.2.2. *Definition of the model Witten Laplacian $\mathbf{\Delta}_T^{p,\overline{q}}$, $p \in \mathrm{Sing}(X)$, $\overline{q} \in \{\overline{m}, \overline{n}\}$.* We denote by $e_1, \dots, e_n$ an orthonormal basis of TZ_p, by $e^1, \dots, e^n$ the dual basis of T^*Z_p. We denote by

$$c(e_k) := e^k - \iota_{e_k}, \widehat{c}(e_k) := e^k + \iota_{e_k}, k = 1, \dots, n, \tag{4.10}$$

the Clifford operators. Let us denote again by N the number operator acting by multiplication with the form degree. We have (see [BZ92, (11.1)]):

$$N = \frac{1}{2}\sum_{i=1}^{n} c(e_i)\widehat{c}(e_i) + \frac{n}{2}. \tag{4.11}$$

The action of the Witten Laplacian $\widetilde{\Delta}_T^{\overline{q}}$, $T > 0$, on forms with support in a neighbourhood of $p \in \mathrm{Sing}(X)$ can be identified with the action of the model Witten Laplacian $\mathbf{\Delta}_T^{p,\overline{q}}$ on the infinite cone Z_p, which we now define: Let $\mathbf{\Delta}_T^p$ denote the following operator acting on compactly supported forms on Z_p with values in the bundle F:

$$\mathbf{\Delta}_T^p := \mathbf{\Delta}^p + T(n - 2N) + T^2 r^2 = \mathbf{\Delta}^p - T\sum_{i=1}^{n} c(e_i)\widehat{c}(e_i) + T^2 r^2, \tag{4.12}$$

where for the last identity we have used (4.11).

We denote by $(\Omega_c^\bullet(Z_p, F), d_{T,c})$, where $d_{T,c}\omega := d_c\omega + Tdf^p \wedge \omega = d_c\omega - Trdr \wedge \omega$, the deformed de Rham complex of smooth compactly supported forms on the infinite cone Z_p. We consider the maximal extension $(\mathcal{C}_{T,\max}^\bullet(Z_p, F), d_{T,\max}, \langle\ ,\ \rangle)$ resp. the minimal extension $(\mathcal{C}_{T,\min}^\bullet(Z_p, F), d_{T,\min}, \langle\ ,\ \rangle)$ of the deformed de Rham complex on the infinite cone. The model Witten Laplacian $\mathbf{\Delta}_T^{p,\overline{m}}$ (resp. $\mathbf{\Delta}_T^{p,\overline{n}}$) is the closed self-adjoint extension of the operator $\mathbf{\Delta}_T^p$ associated to the Hilbert complex $(\mathcal{C}_{T,\max}^\bullet(Z_p, F), d_{T,\max}, \langle\ ,\ \rangle)$ (resp. $(\mathcal{C}_{T,\min}^\bullet(Z_p, F), d_{T,\min}, \langle\ ,\ \rangle)$).

For $T \geqslant 0$, we denote by $\mathbf{\Delta}_{T,\max}^p$ the maximal extension of $\mathbf{\Delta}_T^p$. The domain of the model Witten Laplacian can be described as follows:

$$\mathrm{dom}(\mathbf{\Delta}_T^{p,\overline{m}}) = \{\omega \in \mathrm{dom}(\mathbf{\Delta}_{T,\max}^p) \mid \omega \in \mathrm{dom}\,(\delta_{T,\min}d_{T,\max}) \cap \mathrm{dom}\,(d_{T,\max}\delta_{T,\min}) \text{ loc. at } r = 0\} \tag{4.13}$$

resp.
(4.14)
$$\operatorname{dom}(\boldsymbol{\Delta}_T^{p,\overline{n}}) = \{\omega \in \operatorname{dom}(\boldsymbol{\Delta}_{T,\max}^{p}) \mid \omega \in \operatorname{dom}(\delta_{T,\max} d_{T,\min}) \cap \operatorname{dom}(d_{T,\min}\delta_{T,\max}) \text{ loc. at } r=0\}.$$

The boundary conditions in (4.13), (4.14) at $r = 0$ are inherited from the boundary condition near $p \in \operatorname{Sing}(X)$ for the Witten Laplacian $\widetilde{\Delta}_T^{\overline{q}}$. By completeness of the infinite cone, at $r = \infty$ we do not need to specify boundary conditions.

For $a > 0$, let h_a be the operator acting on sections w of $\Lambda(T^*Z_p) \otimes F$ by radial scaling, $h_a w(r, y) = w(ar, y)$. For $\overline{q} \in \{\overline{m}, \overline{n}\}$, $T \geqslant 0$ from (4.12), (4.13) and (4.14) we get the following scaling property for the model Witten Laplacian

$$h_a^{-1} \boldsymbol{\Delta}_T^{p,\overline{q}} h_a = a^2 \boldsymbol{\Delta}_{T/a^2}^{p,\overline{q}}. \tag{4.15}$$

4.2.3. *Spectral data for the model Witten Laplacian* $\boldsymbol{\Delta}_T^{p,\overline{q}}$. The separation of variables for the Laplacian $\boldsymbol{\Delta}^{p,\overline{q}}$ on the infinite cone has first been used by Cheeger (see e.g. [Che83, Section 3]) to split the computations of the spectral data of $\boldsymbol{\Delta}^{p,\overline{q}}$ according to the spectrum of the transversal Laplacian Δ_{L_p}. In [Ver09] the analytic torsion of a truncated cone has been computed also using the separation of variables trick. In view of (4.9), (4.12) we can apply here the same principle to study the spectral properties of $\boldsymbol{\Delta}_T^{p,\overline{q}}$.

We denote by $\operatorname{Spec}\big(\Delta_{L_p,ccl}^{(k)}\big)$, $k = 0, \dots, n-2$, the co-closed spectrum of $\Delta_{L_p}^{(k)}$. For $\mu \in \operatorname{Spec}\big(\Delta_{L_p,ccl}^{(k)}\big)$ we denote by

$$\mathscr{H}_{\mu,ccl}^k(L_p, F_{L_p}) := \{\psi \in \Omega^k(L_p, F_{L_p}) \mid \Delta_{L_p}^{(k)}\psi = \mu\psi, \delta_{L_p}\psi = 0\} \tag{4.16}$$

the space of co-closed eigenforms of $\Delta_{L_p}^{(k)}$ to the eigenvalue μ. In particular $\mathscr{H}_{0,ccl}^k(L_p, F_{L_p}) = \mathscr{H}^k(L_p, F_{L_p})$ is the space of harmonic k-forms on the link L_p. We have the following orthogonal decomposition

$$\begin{aligned} &\mathrm{L}^2\left(\Lambda(T^*L_p) \otimes F_{L_p}\right) \\ &= \Big(\bigoplus_{k=0}^{n-1} \mathscr{H}^k(L_p, F_{L_p})\Big) \bigoplus \Big(\bigoplus_{\substack{0 \leqslant k \leqslant n-2 \\ \mu \in \operatorname{Spec}\left(\Delta_{L_p,ccl}^{(k)}\right) \setminus \{0\}}} \Big(\mathscr{H}_{\mu,ccl}^k(L_p, F_{L_p}) \oplus d_{L_p}\mathscr{H}_{\mu,ccl}^k(L_p, F_{L_p})\Big)\Big). \end{aligned} \tag{4.17}$$

For $k = -1, \dots, n-2$, set

$$\alpha_k := (k + 1 - n/2). \tag{4.18}$$

For $k = 0, \dots, n-2$, and $\mu \in \operatorname{Spec}\left(\Delta_{L_p,ccl}^{(k)}\right) \setminus \{0\}$, set

$$\beta(\mu) := \beta_k(\mu) := \sqrt{\alpha_k^2 + \mu}. \tag{4.19}$$

We can split all spectral computations for $\boldsymbol{\Delta}_T^{p,\overline{q}}$ into computations on subcomplexes of the Hilbert complexes $(C^\bullet_{T,\max/\min}(Z_p, F), d_{T,\max/\min}, \langle\ ,\ \rangle)$.

Subcomplex of type 1: Let $\mu \in \operatorname{Spec}\left(\Delta^{(k)}_{L_p,ccl}\right)\backslash\{0\}$. For $0 \neq \psi \in \mathscr{H}^k_{\mu,ccl}(L_p, F_{L_p})$, we denote by

$$\begin{aligned}
\xi_1 &= \xi_1(\psi) := (0, \psi) \in \Omega^{k-1}(L_p, F_{L_p}) \oplus \Omega^k(L_p, F_{L_p}),\\
\xi_2 &= \xi_2(\psi) := \big(\psi, 0\big) \in \Omega^k(L_p, F_{L_p}) \oplus \Omega^{k+1}(L_p, F_{L_p}),\\
\xi_3 &= \xi_3(\psi) := \big(0, \mu^{-1/2} d_{L_p}\psi\big) \in \Omega^k(L_p, F_{L_p}) \oplus \Omega^{k+1}(L_p, F_{L_p}),\\
\xi_4 &= \xi_4(\psi) := (\mu^{-1/2} d_{L_p}\psi, 0) \in \Omega^{k+1}(L_p, F_{L_p}) \oplus \Omega^{k+2}(L_p, F_{L_p}).
\end{aligned} \tag{4.20}$$

We still denote by $\xi_1 \in C^\infty(\mathbb{R}_{>0}, \Omega^{k-1}(L_p, F_{L_p}) \oplus \Omega^k(L_p, F_{L_p}))$ the constant function with value ξ_1. Similarly for $\xi_2, \xi_3 \in C^\infty(\mathbb{R}_{>0}, \Omega^k(L_p, F_{L_p}) \oplus \Omega^{k+1}(L_p, F_{L_p}))$ and $\xi_4 \in C^\infty(\mathbb{R}_{>0}, \Omega^{k+1}(L_p, F_{L_p}) \oplus \Omega^{k+2}(L_p, F_{L_p}))$. The subcomplex of type 1 associated to $0 \neq \psi \in \mathscr{H}^k_{\mu,ccl}(L_p, F_{L_p})$ is the subcomplex:

$$0 \to \langle U_k(\xi_1)\rangle \xrightarrow{d_T} \langle U_{k+1}(\xi_2), U_{k+1}(\xi_3)\rangle \xrightarrow{d_T} \langle U_{k+2}(\xi_4)\rangle \to 0. \tag{4.21}$$

By the proof of [Lud17b, Theorem 4.2] it is known already that the subcomplex (4.21) does not yield any contribution to $\ker(\mathbf{\Delta}^{p,\bar{q}}_T)$. Therefore, by the Hodge theorem, to study the eigenequation

$$\mathbf{\Delta}^{p,\bar{q}}_T \omega = \lambda\omega,\ \lambda \neq 0, \tag{4.22}$$

on the subcomplex (4.21) it is sufficient to study the eigenequation (4.22) on $\langle U_k(\xi_1)\rangle$ and on $\langle U_{k+2}(\xi_4)\rangle$. On $\langle U_k(\xi_1), U_{k+2}(\xi_4)\rangle$, using the unitary transformation (4.7), the action of the model Witten Laplacian can be identified with the action of the following regular singular operator on $\mathrm{L}^2(\mathbb{R}_{>0})$:

$$L_\mu := -\partial_r^2 + r^{-2}\left(\beta_k(\mu)^2 - \frac{1}{4}\right) + T(n - 2N) + T^2 r^2, \tag{4.23}$$

where N is the degree operator on $\langle U(\xi_i(\psi))\rangle$, $i = 1, 4$. The operator L_μ is in the limit point case at ∞. Moreover, at $r = 0$, the operator L_μ is in the limit point case iff $\beta_k(\mu)^2 \geqslant 1$. If $0 < \beta_k(\mu)^2 < 1$ however, one has to choose boundary conditions. Hence, we have to study the eigenequation on the half-line $\mathbb{R}_{>0}$:

$$L_\mu g = \left(-\partial_r^2 + r^{-2}\left(\beta_k(\mu)^2 - \frac{1}{4}\right) + T(n - 2N) + T^2 r^2\right) g = \lambda g, \tag{4.24}$$

imposing appropriate boundary conditions at $r = 0$, induced from the boundary conditions (4.13), (4.14) for $\mathbf{\Delta}^{p,\bar{q}}_T$. Arguing as in [Ver09, Section 4.1], the boundary conditions (4.13), (4.14) can both be translated into the following boundary condition for g:

$$g(r) = \mathcal{O}(r^{1/2}) \text{ as } r \to 0, \text{ and } g(r) \in \mathrm{L}^2(\mathbb{R}_{>0}). \tag{4.25}$$

It is at this stage of the computation, where we profit from the fact that we have to study the eigenequation (4.22) only on $\langle U_k(\xi_1), U_{k+2}(\xi_4)\rangle$, where one can translate the boundary conditions (4.13), (4.14) easily, namely into (4.25).

We denote by L^β_j the Laguerre polynomial. The subcomplex of type 1 corresponding to $0 \neq \psi \in \mathscr{H}^k_{\mu,ccl}(L_p, F_{L_p})$, $\mu \neq 0$, yields the following eigenvalues and eigenforms of the model Witten Laplacian $\mathbf{\Delta}^{p,\bar{q}}_T$ (each of multiplicity 1), $j \in \mathbb{N}_0$:

eigenvalue of $\mathbf{\Delta}_T^{p,\overline{q}}$	eigenform of $\mathbf{\Delta}_T^{p,\overline{q}}$
$\big(4j+n-2k+2\beta+2\big)T$	$\phi_1 := r^{\beta+\frac{1}{2}} \exp\left(\frac{-Tr^2}{2}\right) L_j^{\beta}(Tr^2)\cdot U_k(\xi_1) \in \Omega^k(Z_p,F)$
$\big(4j+n-2k+2\beta-2\big)T$	$\phi_4 := r^{\beta+\frac{1}{2}} \exp\left(\frac{-Tr^2}{2}\right) L_j^{\beta}(Tr^2)\cdot U_{k+2}(\xi_4) \in \Omega^{k+2}(Z_p,F)$
$\big(4j+n-2k+2\beta+2\big)T$ $\big(4j+n-2k+2\beta-2\big)T$	$d_T\phi_1 \in \langle U_{k+1}(\xi_2), U_{k+1}(\xi_3)\rangle \subset \Omega^{k+1}(Z_p,F)$ $\delta_T\phi_4 \in \langle U_{k+1}(\xi_2), U_{k+1}(\xi_3)\rangle \subset \Omega^{k+1}(Z_p,F)$

Subcomplex of type 2: Let $0 \neq \eta \in \mathscr{H}^k(L_p, F_{L_p})$, $k = 0,\dots,n-1$, be a harmonic form on L_p. As before, $(0,\eta) \in C^\infty(\mathbb{R}_{>0}, \Omega^{k-1}(L_p,F_{L_p})\oplus\Omega^k(L_p,F_{L_p}))$ denotes the constant function with value $(0,\eta)$, and similarly for $(\eta,0) \in C^\infty(\mathbb{R}_{>0}, \Omega^{k}(L_p,F_{L_p})\oplus\Omega^{k+1}(L_p,F_{L_p}))$. We have the following subcomplex of type 2:

$$0 \to \langle U_k(0,\eta)\rangle \xrightarrow{d_T} \langle U_{k+1}(\eta,0)\rangle \to 0. \tag{4.26}$$

On $\langle(0,\eta)\rangle$ (resp. on $\langle(\eta,0)\rangle$), using the unitary transformation (4.7), the action of the model Witten Laplacian can be identified with the action of the following regular singular operator on $\mathrm{L}^2(\mathbb{R}_{>0})$

$$(-\partial_r^2 + r^{-2}(c_k^2 \pm (-1)^k c_k) + T(n-2N) + T^2r^2)g = \lambda g, \tag{4.27}$$

with the boundary conditions induced from (4.13), (4.14).

Case 1: Let us first consider the case, where either n is even, or n is odd and $k \neq \lfloor n/2\rfloor$.

By [Ver09, Proposition 7.1], we have to study the equation (4.27) with the boundary condition

$$g = O(r^{1/2}) \text{ as } r\to 0, \text{ and } g(r)\in \mathrm{L}^2(\mathbb{R}_{>0}). \tag{4.28}$$

Hence, the same computation as in [Lud20b, Section 4] shows, that the subcomplex of type 2 corresponding to $0 \neq \eta \in \mathscr{H}^k(L_p,F_{L_p})$, yields the following eigenvalues (each of multiplicity 1) and eigenforms of the model Witten Laplacian, $j\in\mathbb{N}_0$:

eigenvalue of $\mathbf{\Delta}_T^{p,\overline{q}}$	eigenform of $\mathbf{\Delta}_T^{p,\overline{q}}$
$\big(4j-2\alpha_k+4+2\lvert\alpha_k\rvert\big)T$	$r^{\lvert\alpha_k\rvert+1/2}\exp\left(\frac{-Tr^2}{2}\right) L_j^{\lvert\alpha_k\rvert}(Tr^2)U_k((0,\eta)) \in \Omega^k(Z_p,F)$
$\big(4j-2\alpha_k+2(\lvert\alpha_{k-1}\rvert+1)\big)T$	$r^{\lvert\alpha_{k-1}\rvert+1/2}\exp\left(\frac{-Tr^2}{2}\right) L_j^{\lvert\alpha_{k-1}\rvert}(Tr^2)U_{k+1}((\eta,0)) \in \Omega^{k+1}(Z_p,F)$

Case 2: For $n=2\nu+1$ odd and $k=\nu$, the two boundary conditions in (4.13), (4.14) do differ on a subcomplex of type 2.

Case 2 (a): Let $\overline{q}=\overline{n}$. For $k=\nu$, on $\langle(0,\eta)\rangle$, by (4.8), (4.27) and [Ver09, Propositions 7.1 and 7.3], one is reduced to study the eigenequation on $\mathrm{L}^2(\mathbb{R}_{>0})$

$$(-\partial_r^2 + T + T^2r^2)g = \lambda g, \tag{4.29}$$

with boundary condition

$$g = O(r^{1/2}) \text{ and } (\partial_r - Tr)g = O(1) \text{ as } r\to 0;\ g(r)\in\mathrm{L}^2(\mathbb{R}_{>0}). \tag{4.30}$$

The only solutions of (4.29) in $\mathrm{L}^2(\mathbb{R}_{>0})$ occur for $\lambda_j = 2jT, j\in\mathbb{N}$, with eigenfunction $g_{\lambda_j} = \exp(-Tr^2/2)H_{j-1}(\sqrt{T}r)$, where H_j denotes the Hermite polynomial. The following recurrence relation of Hermite polynomials holds $H_j' = 2rH_j - H_{j+1}$. Moreover, as $r\to 0$, we have $H_j(r) = O(1)$ if j is even, and $H_j(r) = O(r)$ if j is odd. Hence, taking into account the boundary conditions at $r\to 0$ in (4.30), only the eigenvalues $\lambda_j = 2jT, j\in\mathbb{N}$, j even, appear.

On $\langle(\eta, 0)\rangle$ one is reduced to study the eigenequation on $\mathrm{L}^2(\mathbb{R}_{>0})$

$$(-\partial_r^2 - T + T^2 r^2) g = \lambda g, \tag{4.31}$$

with boundary condition

$$g = O(1) \text{ and } (\partial_r - Tr) g = O(r^{1/2}) \text{ as } r \to 0;\ g(r) \in \mathrm{L}^2(\mathbb{R}_{>0}), \tag{4.32}$$

and argue similarly. The above arguments show that the subcomplex of type 2 corresponding to $0 \neq \eta \in \mathcal{H}^\nu(L_p, F_{L_p})$, yields the following eigenvalues (each of multiplicity 1) and eigenforms of the model Witten Laplacian:

eigenvalue of $\mathbf{\Delta}_T^{p,\overline{n}}$	eigenform of $\mathbf{\Delta}_T^{p,\overline{n}}$
$4jT, j \in \mathbb{N}$	$\exp\left(\frac{-Tr^2}{2}\right) H_{2j-1}(\sqrt{T}r) U_\nu((0,\eta)) \in \Omega^\nu(Z_p, F)$ $= \exp\left(\frac{-Tr^2}{2}\right) r L_{j-1}^{1/2}(Tr^2) U_\nu((0,\eta))$
$4jT, j \in \mathbb{N}_0$	$\exp\left(\frac{-Tr^2}{2}\right) H_{2j}(\sqrt{T}r) U_{\nu+1}((\eta,0)) \in \Omega^{\nu+1}(Z_p, F)$ $= \exp\left(\frac{-Tr^2}{2}\right) L_j^{-1/2}(Tr^2) U_{\nu+1}((\eta,0))$

Case 2 (b): Let $\overline{q} = \overline{m}$. This time we have the boundary condition (4.32) on $\langle(0,\eta)\rangle$ (resp. (4.30) on $\langle(\eta,0)\rangle$).

Arguing similarly as in Case 2 (a), this time we get

eigenvalue of $\mathbf{\Delta}_T^{p,\overline{m}}$	eigenform of $\mathbf{\Delta}_T^{p,\overline{m}}$
$(4j+2)T, j \in \mathbb{N}_0$	$\exp\left(\frac{-Tr^2}{2}\right) H_{2j}(\sqrt{T}r) U_\nu((0,\eta)) \in \Omega^\nu(Z_p, F)$ $= \exp\left(\frac{-Tr^2}{2}\right) L_j^{-1/2}(Tr^2) U_\nu((0,\eta))$
$(4j+2)T, j \in \mathbb{N}_0$	$\exp\left(\frac{-Tr^2}{2}\right) H_{2j+1}(\sqrt{T}r) U_{\nu+1}((\eta,0)) \in \Omega^{\nu+1}(Z_p, F)$ $= \exp\left(\frac{-Tr^2}{2}\right) r L_j^{1/2}(Tr^2) U_{\nu+1}((\eta,0))$

Remark 4.1. (a) From the above computation together with the local calculation for the relative intersection cohomology of a cone (recalled in (2.9)), we get

$$\ker \mathbf{\Delta}_T^{p,\overline{q},(k)} \simeq IH_{\overline{q}}^k(cL_p, L_p, F),\ k = 0, \ldots, n. \tag{4.33}$$

Note that only harmonic forms on the link L_p do contribute to $\ker \mathbf{\Delta}_T^{p,\overline{q}}$.

(b) Note that for n odd, the only difference in the spectral data of $\mathbf{\Delta}_T^{p,\overline{m}}$ and $\mathbf{\Delta}_T^{p,\overline{n}}$ stems from the harmonic forms on L_p of degree $\lfloor n/2 \rfloor$.

(c) Under the Witt and spectral Witt assumptions made in [Lud20b], the model Witten Laplacian is the Friedrichs extension of (4.12) and the boundary condition at $r \to 0$ always translates into (4.28). Also, for an odd dimensional Witt space, case 2 in the above discussion does not appear.

4.2.4. *Heat kernel of the model Witten Laplacian.*

Definition 4.2. Let $\overline{q} \in \{\overline{m}, \overline{n}\}$ the lower middle (resp. upper middle) perversity.

(a) For $(t,T) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geqslant 0}$, we denote by $Q_{t,T}^{p,\overline{q}}(x,x')$, $x, x' \in Z_p$, the kernel of the operator $\exp\left(-t\mathbf{\Delta}_T^{p,\overline{q}}\right)$ with respect to $d\mathrm{vol}_{Z_p}$. Set $Q_t^{p,\overline{q}}(x,x') := Q_{t,0}^{p,\overline{q}}(x,x')$, $x, x' \in Z_p$. We denote by $Q_{t,T}^{p,\overline{q},(k)}(x,x')$, $x, x' \in Z_p$, the kernel of the operator $\exp\left(-t\mathbf{\Delta}_T^{p,\overline{q}}\right)$ restricted to k-forms, $k \in \{0, \ldots, n\}$.

(b) For $(t,T) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geqslant 0}$, we denote by $U^{p,\overline{q}}_{t,T}(x,x')$, $x,x' \in Z_p$, the kernel of the operator $\exp\big(-t^2\mathbf{\Delta}^{p,\overline{q}}_{T/t}\big)$ with respect to $d\mathrm{vol}_{Z_p}$.

Note that, for $(t,T) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geqslant 0}$, we have directly from Definition 4.2,

$$U^{p,\overline{q}}_{t,T}(x,x') = Q^{p,\overline{q}}_{t^2,T/t}(x,x'),\ x,x' \in Z_p. \tag{4.34}$$

Using the computation of the spectral data for $\mathbf{\Delta}^{p,\overline{q}}_T$ we can generalise [Lud20a, Proposition 4.5]. The heat kernel of the model operator for $T > 0$ and for $T = 0$ are related as follows:

Proposition 4.3. *Let* $(t,T) \in \mathbb{R}^2_{>0}$, $(r,y),(r',y') \in Z_p$. *Then*

$$\begin{aligned} &Q^{p,\overline{q},(k)}_{t,T}\big((r,y),(r',y')\big) \\ &= e^{-(n-2k)tT}T^{n/2}\exp\left(-T\tanh(tT)\frac{r^2+r'^2}{2}\right)Q^{p,\overline{q},(k)}_{\sinh(2tT)/2}\Big(\big(\sqrt{T}r,y\big),\big(\sqrt{T}r',y'\big)\Big). \end{aligned} \tag{4.35}$$

Remark 4.4. The heat kernel on the infinite cone $Q^{p,\overline{q}}_t(x,x')$, $x,x' \in Z_p$, can be expressed in terms of the spectral data of the transversal Laplacian Δ_{L_p} and involves the modified Bessel functions, see [Che83], [Les97, Proposition 2.3.11].

4.3. **The Cheeger invariant and a local index theorem for the infinite cone.**

4.3.1. *The Cheeger invariant.* The Cheeger invariant of X at $p \in \mathrm{Sing}(X)$ is the contribution of the singularity p to Cheeger's Chern-Gauss-Bonnet theorem for spaces with isolated conical singularities [Che83, Theorem 5.1].

Definition/Lemma 4.5. Let $\overline{q} \in \{\overline{m},\overline{n}\}$, $t > 0$. The following integral expression

$$\begin{aligned} \gamma^{\overline{q}}_p(F) &:= \int_{Z_p} \mathrm{Tr_s}\left[Q^{p,\overline{q}}_t((r,y),(r,y))\right] d\mathrm{vol}_{Z_p} \\ &= \frac{1}{2}\int_0^\infty \frac{du}{u}\int_{L_p} \mathrm{Tr_s}\left[Q^{p,\overline{q}}_u((1,y),(1,y))\right] d\mathrm{vol}_{L_p} \end{aligned} \tag{4.36}$$

is well-defined and will be called the Cheeger invariant of X at p.

Proof. In this proof we follow arguments in [Che83, Section 2], [BC90, Section 1(f)], [Les97, Lemma 2.2.4], [Lud20b, Section 7]. We recall these arguments in some detail here, since similar arguments will play an important role in the study of all Cheeger-type invariants appearing in this paper (see in particular Sections 6.2 and 7).

We first prove the equality of the two integral expressions in (4.36), for $t > 0$: The scaling properties (4.15) of $\mathbf{\Delta}^{p,\overline{q}}$ on the infinite cone imply (see e.g. [Che80, Section 2] and [BC90, Proposition 1.7])

$$Q^{p,\overline{q}}_t((r,y),(r,y)) = \frac{1}{r^n}Q^{p,\overline{q}}_{t/r^2}((1,y),(1,y)). \tag{4.37}$$

The equality of the two integrals in (4.36) follows from (4.37) using the change of variables $u = t/r^2$.

It is now enough to prove the well-definedness of the second integral in (4.36). Using local index techniques we get the following asymptotic expansion, as $u \searrow 0$,

$$\mathrm{Tr_s}\left[Q^{p,\overline{q}}_u((r,y),(r,y))\right] d\mathrm{vol}_{Z_p} = \mathrm{rk}(F)\cdot e(TZ_p,\nabla^{TZ_p}) + \mathcal{O}(\sqrt{u}); \tag{4.38}$$

the expansion is uniform on compact sets of Z_p. By (3.18) the constant term in the asymptotic expansion (4.38) vanishes and therefore, as $u \searrow 0$, $\mathrm{Tr_s}\left[Q_u^{p,\overline{q}}\big((1,y),(1,y)\big)\right] = \mathcal{O}(\sqrt{u})$. This gives well-definedness of the second integral in (4.36) at 0.

Using the characterisation of $\mathrm{dom}(\mathbf{\Delta}^{p,\overline{q}})$ near the cone point (see e.g. [Les97, Lemma 2.2.4]), there exists $a > 0$ such that, as $u \to \infty$

$$\int_{L_p} \mathrm{Tr_s}\left[Q_u^{p,\overline{q}}\big((1,y),(1,y)\big)\right] d\mathrm{vol}_{L_p} = \mathcal{O}(u^{-a}). \tag{4.39}$$

This gives well-definedness of the second integral in (4.36) at ∞. □

4.3.2. *An index formula for the infinite cone.* The invariant α_p has been defined in (3.24) and studied further in Section 3.5. The following theorem generalises the index theorem for the infinite cone [Lud20b, Theorem II] to the current situation.

Theorem 4.6. *For* $(t,T) \in \mathbb{R}^2_{>0}$,

$$\begin{aligned} I\chi^{\overline{q}}(cL_p, L_p, F) &= \mathrm{Tr_s}\left[\exp(-t\mathbf{\Delta}_T^{p,\overline{q}})\right] = \int_{Z_p} \mathrm{Tr_s}\left[Q_{t,T}^{p,\overline{q}}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} \\ &= \mathrm{rk}(F)\int_{Z_p}\int^{B,p} \exp(-B^p_{T^2}) + \gamma_p^{\overline{q}}(F) = \mathrm{rk}(F)\alpha_p + \gamma_p^{\overline{q}}(F). \end{aligned} \tag{4.40}$$

Remark 4.7. On the right hand side of (4.40) two terms appear: the interior contribution $\mathrm{rk}(F)\alpha_p$ does not depend on the perversity $\overline{q} \in \{\overline{m}, \overline{n}\}$, while the contribution of the singular point $\gamma_p^{\overline{q}}(F)$ does. On spaces with conical singularities, this is a general phenomenon in the study of the asymptotic behaviour of the supertrace of the heat and related operators. We also encounter it e.g. in Cheeger's Chern-Gauss-Bonnet Theorem [Che83, Theorem 5.1] (recalled in (6.12)) as well as in the anomaly formulas in Section 7.

Proof. The theorem has been proved under the Witt and a spectral Witt condition for unitary flat vector bundles in [Lud20b, Theorem II]; this proof relies on local index techniques. In the present situation, we can give a prove by using a combination of local index techniques and the Singular Asymptotic Lemma (SAL) of Brüning and Seeley [BS85], similarly to the proofs of Theorem 7.7 and Proposition 6.3. □

5. Bismut-Zhang Theorem (Theorem I): The nine intermediate results

The core of the proof of the Bismut-Zhang Theorem (Theorem I) are the nine intermediate results, which we state in this section. Once the nine intermediate results are achieved in our more general situation, the proof of the Bismut-Zhang Theorem is completely analogous to the proof in [BZ92, Section VII] and in [Lud20a, Section 6], and we omit to repeat it here.

5.1. **Simplifying assumption.** Using the anomaly formulas of Theorems II, 3.4, 7.8 as well as Proposition 3.3 and Remark 2.5, it is clear that to establish Theorem I, it is enough to establish it for the case, where $g^{TX} = g_0^{TX}$, g^F is flat near $\mathrm{Crit}(f_{sm})$ and moreover the pair (f, g^{TX}) is an anti-radial standard Morse-Smale pair.

5.2. **Nine intermediate results.** We denote the intersection Euler characteristic of X with perversity $\overline{q}$ and coefficients in F by

$$I\chi_{\overline{q}} := I\chi_{\overline{q}}(X,F) := \sum_{k=0}^{n}(-1)^k \dim IH^k_{\overline{q}}(X,F). \tag{5.1}$$

For $p \in \mathrm{Sing}(X)$, we denote by $b^k(L_p, F_{L_p}) := \dim H^k(L_p, F_{L_p})$, $k = 0, \ldots, n-1$, the Betti numbers for the link L_p. For $k = 0, \ldots, n$, we denote by $c_k(f_{sm}) := \#\mathrm{Crit}_k(f_{sm})$ and we define the "number of critical points of the anti-radial Morse function f" by

$$\begin{aligned} c_k^{\overline{q}}(f) := c_k^{\overline{q}}(f,F) &:= \mathrm{rk}(F)\cdot c_k(f_{sm}) + \sum_{p\in \mathrm{Sing}(X)} IH^k_{\overline{q}}(cL_p, L_p, F) \\ &= \begin{cases} \mathrm{rk}(F)\cdot c_k(f_{sm}) + \sum_{p\in\mathrm{Sing}(X)} b^{k-1}(L_p, F_{L_p}) & \text{for } k \geqslant n-\overline{q}, \\ \mathrm{rk}(F)\cdot c_k(f_{sm}) & \text{else.} \end{cases} \end{aligned} \tag{5.2}$$

For $p \in \mathrm{Sing}(X)$, we denote by by $I\chi_{\overline{q}}(cL_p, L_p, F)$ the relative intersection Euler characteristic of the cone. From the Spectral Gap Theorem [Lud17b, Theorem I], we get the following Poincaré-Hopf formula

$$I\chi_{\overline{q}} = \mathrm{rk}(F)\cdot \sum_{k=0}^{n}(-1)^k c_k(f_{sm}) + \sum_{p\in\mathrm{Sing}(X)} I\chi_{\overline{q}}(cL_p, L_p, F) = \sum_{k=0}^{n}(-1)^k c_k^{\overline{q}}(f). \tag{5.3}$$

We denote by

$$\begin{aligned} I\chi'_{\overline{q}} &:= I\chi'_{\overline{q}}(X,F) := \sum_{k=0}^{n}(-1)^k k \dim IH^k_{\overline{q}}(X,F), \\ I\widetilde{\chi}'_{\overline{q}} &:= I\widetilde{\chi}'_{\overline{q}}(X,F) := \sum_{k=0}^{n}(-1)^k k c_k^{\overline{q}}(f), \\ \mathrm{Tr_s}[f,F,\overline{q}] &:= \mathrm{rk}(F) \sum_{p\in\mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)} f(p) + \sum_{p\in\mathrm{Sing}(X)} f(p)\cdot I\chi_{\overline{q}}(cL_p, L_p, F). \end{aligned} \tag{5.4}$$

Moreover

$$\chi_{sm} := \mathrm{rk}(F)\sum_{k=0}^{n}(-1)^k c_k(f_{sm}), \text{ and } \widetilde{\chi}'_{sm} := \mathrm{rk}(F)\sum_{k=0}^{n}(-1)^k k c_k(f_{sm}). \tag{5.5}$$

Let $T \geqslant 0$. We denote by $P_T^{\overline{q},]1,\infty[}$ the orthogonal projection w.r.t. $\langle\ ,\ \rangle_T$ to the space of eigenforms of the Laplacian $\Delta_T^{\overline{q}} = (D_T^{\overline{q}})^2$ to eigenvalues in $]1,\infty[$. We denote by $D_T^{\overline{q},2,]0,1]}$ (resp. $D_T^{\overline{q},2,[0,1]}$) the restriction of $\Delta_T^{\overline{q}}$ to the eigenspace of $\Delta_T^{\overline{q}}$ associated to eigenvalues in the interval $]0,1]$ (resp. in the interval $[0,1]$). By the Hodge theorem for the complex $(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$ and the Cheeger-Goresky-MacPherson Theorem (2.13), we have canonical isomorphisms

$$\ker \Delta_T^{\overline{q}} \simeq H^\bullet(\mathcal{C}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T) \simeq IH^\bullet_{\overline{q}}(X,F). \tag{5.6}$$

The twisted L^2-metric $\langle\ ,\ \rangle_T$ restricted to $\ker \Delta_T^{\overline{q}} \simeq IH^\bullet_{\overline{q}}(X,F)$ induces a metric on the line $\det IH^\bullet_{\overline{q}}(X,F)$, which we denote by $|\ \ |^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),T}$.

The following intermediate results are the analogues of [BZ92, Theorems 7.6–7.14] resp. [Lud20a, Theorems 5.4–5.12]. Compared to [Lud20a, Theorems 5.4–5.12], in case n odd, we take into account the two perversities $\overline{q} \in \{\overline{m}, \overline{n}\}$. Moreover, since here g^F is not assumed to be flat, an additional term appears in Theorem 5.7.

Theorem 5.1. *The following identity holds, for $T \to \infty$,*

(5.7)

$$\mathrm{Tr_s}\left[N \log\left(D_T^{\overline{q},2,]0,1]}\right)\right] - \log\left(\frac{\|\ \|^{\nabla f, g^{TX}, g^F}_{\det IH^\bullet_{\overline{q}}(X,F)}}{|\ |^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),T}}\right)^2 + 2T\mathrm{Tr_s}[f, F, \overline{q}]$$
$$+ \left(\frac{n}{2} I\chi_{\overline{q}} - I\widetilde{\chi}'_{\overline{q}}\right)\log(T) - \left(\frac{n}{2}\chi_{sm} - \widetilde{\chi}'_{sm}\right)\log(\pi) + \sum_{p \in \mathrm{Sing}(X)} \left(\zeta_p^{\overline{q}}\right)'(0) = \mathcal{O}(\exp(-cT)).$$

Theorem 5.2. *Given ϵ, A with $0 < \epsilon < A < \infty$, there exists $C > 0$ such that, for $t \in [\epsilon, A]$, $T \geqslant 1$,*

$$\left|\mathrm{Tr_s}[N \exp(-t(D_T^{\overline{q}})^2)] - I\widetilde{\chi}'_{\overline{q}}\right| \leqslant \frac{C}{\sqrt{T}}. \tag{5.8}$$

Theorem 5.3. *For any $t > 0$,*

$$\lim_{T\to\infty} \mathrm{Tr_s}\left[N \exp(-t(D_T^{\overline{q}})^2) P_T^{\overline{q},]1,\infty[}\right] = 0. \tag{5.9}$$

Moreover there exist $c > 0$, $C > 0$ such that, for $t \geqslant 1$, $T \geqslant 0$,

$$\left|\mathrm{Tr_s}\left[N \exp(-t(D_T^{\overline{q}})^2) P_{\overline{q},T}^{]1,\infty[}\right]\right| \leqslant C\exp(-ct). \tag{5.10}$$

We denote by $(\mathcal{S}^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$ the Witten complex, *i.e.* the subcomplex of $(C^\bullet_{T,\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle_T)$ generated by the eigenforms of $\Delta_T^{\overline{m}/\overline{n}}$ to eigenvalues in $[0, 1]$.

Theorem 5.4. *For $T > 0$ large enough and $k = 0, \dots, n$,*

$$\dim \mathcal{S}^k_{T,\max} = c_k^{\overline{m}}(f), \quad \dim \mathcal{S}^k_{T,\min} = c_k^{\overline{n}}(f). \tag{5.11}$$

Moreover $\lim_{T\to\infty} \mathrm{Tr}\left[D_T^{\overline{q},2,[0,1]}\right] = 0.$

Let $e_1, \dots, e_n$ be an orthonormal basis of TX, $e^1, \dots, e^n$ the dual basis of T^*X. Let W be the smooth section of $\Lambda(T^*X)\widehat{\otimes}\Lambda(T^*X)$ defined by

$$W := \frac{1}{2}\sum_{i=1}^n e^i \wedge \widehat{e^i}. \tag{5.12}$$

Note that W does not depend on the choice of orthonormal basis $e_1, \dots, e_n$.

Theorem 5.5. *The following asymptotic expansion holds, as $t \searrow 0$,*

(5.13)

$$\mathrm{Tr_s}\big[N \exp(-t(D^{\overline{q}})^2)\big] = \mathrm{rk}(F)\int_X \int^B W \exp\left(-\frac{\dot{R}^{TX}}{2}\right)\sqrt{t}^{-1} + \frac{n}{2}\cdot I\chi_{\overline{q}}(X, F) + \mathcal{O}(\sqrt{t}).$$

Note, that the leading coefficient in the expansion (5.13) vanishes in case n even. This is due to the fact, that the integrand in the Berezin integral $\int_X \int^B W \exp\left(-\frac{\dot{R}^{TX}}{2}\right)$ is a sum of forms of degree (k, k), with k odd.

Theorem 5.6. *Let $0 < t < 1$ be small enough. Then there exists $c > 0$ such that, as $T \to \infty$,*

$$\mathrm{Tr_s}\big[f\exp(-t(D_T^{\overline{q}})^2)\big] = \mathrm{Tr_s}[f,F,\overline{q}] + \left(\frac{n}{4}I\chi_{\overline{q}} - \frac{1}{2}I\widetilde{\chi}'_{\overline{q}}\right)\frac{1}{T} + \mathcal{O}(\exp(-cT)). \tag{5.14}$$

Theorem 5.7. *For any $d > 0$, there exists $C > 0$ such that, for $0 < t \leqslant 1$, $0 \leqslant T \leqslant \frac{d}{t}$,*

$$\Big|\mathrm{Tr_s}\Big[f\exp\Big(-\big(tD^{\overline{q}} + T\widehat{c}(\nabla f)\big)^2\Big)\Big] - \mathrm{rk}(F)\int_X f\int^B \exp(-B_{T^2})$$
$$+\frac{t}{2}\int_X \theta(F,g^F)\int^B \widehat{df}\exp(-B_{T^2}) - \sum_{p\in\mathrm{Sing}(X)} f(p)\gamma_p^{\overline{q}}(F)\Big| \leqslant Ct^2,$$

where $\gamma_p^{\overline{q}}(F)$, $p\in\mathrm{Sing}(X)$, is the Cheeger invariant defined in (4.36).

Theorem 5.8. *For any $T > 0$, the following identity holds,*

$$\lim_{t\to 0}\frac{1}{t^2}\left(\mathrm{Tr_s}\left[f\exp\left(-\left(tD^{\overline{q}} + \frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] - \mathrm{Tr_s}[f,F,\overline{q}]\right) =$$
$$= \left(\frac{n}{4}\chi_{sm} - \frac{1}{2}\widetilde{\chi}'_{sm}\right)\frac{1}{T\tanh(T)}$$
$$-\frac{1}{2T}\sum_{p\in\mathrm{Sing}(X)}\left(\mathrm{Tr_s}[N\exp(-\boldsymbol{\Delta}_T^{p,\overline{q},\perp})] - \sum_{k\geqslant n-\overline{q}}(-1)^k\left(\frac{n}{2}-k\right)b^{k-1}(L_p,F_{L_p})\right).$$

Theorem 5.9. *There exist $t_0 > 0$, $c > 0$, $C > 0$, such that, for $t\in]0,t_0]$ and $T \geqslant 1$,*

$$\left|\frac{1}{t^2}\left(\mathrm{Tr_s}\left[f\exp\left(-\left(tD^{\overline{q}} + \frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] - \mathrm{Tr_s}[f,F,\overline{q}] - \frac{t^2}{T}\left(\frac{n}{4}I\chi_{\overline{q}} - \frac{1}{2}I\widetilde{\chi}'_{\overline{q}}\right)\right)\right|$$
$$\leqslant C\exp(-cT).$$

6. Proof of the nine intermediate results

In this section we deal with the proofs of the nine intermediate results of Section 5.2. For most of the proofs of the intermediate results we will just recall the main arguments of the proofs in [Lud20a, Section 7] and explain, why they carry through to the more general situation treated in this article. The bigger part of this section is devoted to the proof of Theorem 5.5 for even dimensional spaces, here the corresponding proof in [Lud20a] cannot be adapted to the present situation.

In Section 6.1 we give a sketch of the proofs of Theorems 5.1–5.4. Section 6.2 is devoted to the proof of Theorem 5.5. In Section 6.3 we comment on the proofs of Theorems 5.6-5.9.

6.1. Proof of Theorems 5.1-5.4.

The proofs of Theorems 5.1–5.4 rely on the Witten deformation.

Proof of Theorem 5.1. The main ingredient of the proof of Theorem 5.1 is the hard part of the Witten deformation, *i.e.* the comparison result of the Witten complex $(\mathcal{S}_{T,\max/\min}, d_{\max,\min}, \langle\ ,\ \rangle_T)$ and the singular Morse-Thom-Smale complex $(C_\bullet^{\overline{q}}(X,f,g^{TX},F^*),\partial_\bullet)$, defined in Definition 2.4. This comparison result has been established in [Lud17b, Theorem II], without assuming the Witt condition, for the lower middle perversity $\overline{m}$, the case of the upper middle perversity $\overline{n}$ being similar. The proof of Theorem 5.1 follows precisely the proof of the corresponding statement in [Lud20a, Theorem 5.6]. For n odd and the upper middle perversity $\overline{n}$ we have only to additionally take into account in all computations the contribution of $H^{(n-1)/2}(L_p, F_{L_p})$, $p \in \mathrm{Sing}(X)$. □

Proof of Theorems 5.2-5.4. Using the Witten deformation for the anti-radial Morse function $f : X \to \mathbb{R}$, more precisely [Lud17b, Section 5], one can proceed as in the proof of [BZ92, Theorems 7.7 and 7.8] to get the claims of Theorems 5.2 and 5.3. The proofs in [Lud17b, Section 5] are only given for the lower middle perversity $\overline{m}$, but for deformation parameter $T \in \mathbb{R}$. The Hodge star operator $*^F : \Lambda^k(T^*X) \otimes F \to \Lambda^{n-k}(T^*X) \otimes F^* \otimes o(TX)$ induces isomorphisms of Hilbert complexes $(\widetilde{\mathcal{C}}(X,F)^\bullet_{T,\max/\min}, d_{T,\max/\min}, \langle\ ,\ \rangle) \to (\widetilde{\mathcal{C}}^\bullet_{-T,\min/\max}(X, F^* \otimes o(TX)), d_{-T,\min/\max}, \langle\ ,\ \rangle)$. Using this duality, the easy part of the Witten deformation (in particular the Spectral Gap Theorem, [Lud17b, Theorem I]) hold for the perversity $\overline{n}$ as well.

The claim of Theorem 5.4 follows from the Spectral Gap Theorem [Lud17b, Theorem I]. □

6.2. **Proof of Theorem 5.5: The asymptotics of $\mathrm{Tr}_s[N\exp(-t\Delta^{\overline{q}})]$ as $t \searrow 0$.** Theorem 5.5 generalises the corresponding statement [Lud20a, Theorem 5.8] to the more general situation of this article. In case n odd, the proof in [Lud20a] generalises directly. In case n even, in [Lud20a] the space was assumed to be oriented and (F, ∇^F, g^F) is unitary. The proof of [Lud20a, Theorem 5.8] relies on a Poincaré duality argument, which fails here.

In this section we give a proof of Theorem 5.5 in case n even. For the rest of this section we always assume that n is even. Recall that for n even, the two middle perversities coincide $\overline{m} = \overline{n}$. We will therefore omit the sub- and superscipt $\overline{q}$.

Definition 6.1. Let $t > 0$.

(a) We denote by $S_t(x,x')$, $x, x' \in X_{sm}$, the kernel of the operator $N\exp(-t\Delta)$ with respect to $d\mathrm{vol}_X$.
(b) Let $p \in \mathrm{Sing}(X)$. For $T \geqslant 0$, we denote by $S^p_{t,T}(x,x')$, $x, x' \in Z_p$, the kernel of the operator $N\exp(-t\mathbf{\Delta}^p_T)$ with respect to $d\mathrm{vol}_{Z_p}$. For $T = 0$, we have $S^p_t(x,x') = S^p_{t,0}(x,x')$.

Proof of Theorem 5.5 in case n even: We proceed very similarly to the proof of Cheeger's Chern-Gauss-Bonnet Theorem for spaces with isolated conical singularities [Che83, Theorem 5.1]. Using local index techniques as in the proofs of [BZ92, Theorem 4.20 and Theorem 7.10] and (4.11), one has the following pointwise asymptotic expansion, as $t \searrow 0$,

$$\begin{aligned}&\mathrm{Tr}_s[S_t(x,x)]d\mathrm{vol}_X\\ &= \left(\frac{n}{2}\mathrm{rk}(F)e(TX,\nabla^{TX}) + \frac{1}{2}\int^B W\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\nabla^{TX}\widehat{\theta}(F,g^F)\right)(x) + \mathcal{O}(t)\\ &=: a_0(x)d\mathrm{vol}_X + \mathcal{O}(t).\end{aligned} \tag{6.1}$$

The asymptotic expansion (6.1) is uniform on compact sets, the coefficients depend only on local geometrical data of X_{sm}.

Since ∇^{TX} is torsionfree, we have $\nabla^{TX}W = 0$. Hence, using [BGV04, Proposition 1.50] and the Bianchi identity,

$$\begin{aligned}(6.2)\qquad &\int^B W \exp\left(-\frac{\dot{R}^{TX}}{2}\right) \nabla^{TX}\widehat{\theta}(F,\nabla^F)\\ &= d\left(\int^B W\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\widehat{\theta}(F,\nabla^F)\right) - \int^B \left(\nabla^{TX}W\right)\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\widehat{\theta}(F,\nabla^F)\\ &= d\left(\int^B W\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\widehat{\theta}(F,g^F)\right).\end{aligned}$$

Recall that, near $\mathrm{Sing}(X)$, the form $\theta(F,g^F)$ does not depend on the radial coordinate. Using (2.6), (3.18), (5.12) and (6.2) we get that locally near $p \in \mathrm{Sing}(X)$:

$$(6.3)\qquad a_0(r,y)d\mathrm{vol}_X = \frac{1}{2}d\left(\int^B W\exp\left(-\frac{\dot{R}^{TX}}{2}\right)\widehat{\theta}(F,g^F)\right) = d_{L_p}\beta_p \wedge r^{-1}dr,$$

with $\beta_p \in \Omega^{n-2}(L_p, F_{L_p})$ a smooth form not depending on the radial coordinate r.

Unlike in the case of a smooth manifold, it is not enough to just integrate the pointwise asymptotic expansion (6.1) to get $\mathrm{Tr_s}[N\exp(-t\Delta)]$; indeed the integrals over the local coefficients are not always defined and we have to take the finite part of these integrals instead. For $0 < \epsilon < \delta$, denote by $X_\epsilon := X\backslash(\cup_{p\in\mathrm{Sing}(X)}B_\epsilon(p))$. The finite part of the integral over the constant coefficient in the expansion (6.1) is just given by

$$(6.4)\qquad \int_{X_\epsilon} a_0(x)d\mathrm{vol}_X.$$

Using (3.18), (6.1)–(6.4) and Stokes' Theorem we get

$$(6.5)\qquad \int_{X_\epsilon} a_0(x)d\mathrm{vol}_X = \frac{n}{2}\mathrm{rk}(F)\int_X e(TX,\nabla^{TX}).$$

In addition, there will also be contributions to $\mathrm{Tr_s}[N\exp(-t\Delta)]$ coming from the singularities $p \in \mathrm{Sing}(X)$, which we now explain: From the pointwise asymptotic expansion (6.1) and from (6.3), we get for the kernel of the operator $N\exp(-t\boldsymbol{\Delta}^p)$ on Z_p, as $u \searrow 0$,

$$(6.6)\qquad \int_{L_p}\mathrm{Tr_s}[S^p_u((1,y),(1,y))]d\mathrm{vol}_{L_p} = \int_{L_p} a_0(1,y)d\mathrm{vol}_{L_p} + \mathcal{O}(u) = \mathcal{O}(u).$$

Using the scaling property (4.15),

$$(6.7)\qquad S^p_t((r,y),(r,y)) = \frac{1}{r^n}S^p_{t/r^2}((1,y),(1,y)).$$

Using (6.6), (6.7) and arguing as in Section 4.3.1 the following integral is well-defined:

$$(6.8)\qquad \gamma_p^{tors}(F) := \frac{1}{2}\int_0^\infty\frac{du}{u}\int_{L_p}\mathrm{Tr_s}[S^p_u((1,y),(1,y))]d\mathrm{vol}_{L_p}.$$

From (6.6), (6.7), (6.8) and using the change of variable $u = t/r^2$, as $t \searrow 0$,

$$\begin{aligned}(6.9)\qquad \int_{c_\epsilon L_p}\mathrm{Tr_s}[S^p_t((r,y),(r,y))]d\mathrm{vol}_{Z_p} &= \frac{1}{2}\int_{\epsilon^{-2}t}^\infty\frac{du}{u}\int_{L_p}\mathrm{Tr_s}[S^p_u((1,y),(1,y))]d\mathrm{vol}_{L_p}\\ &= \gamma_p^{tors}(F) + \mathcal{O}(t).\end{aligned}$$

Using Duhamel's principle and the singular elliptic estimates of Lesch [Les97, Section 1.4, more precisely Theorem 1.4.11], one has that, for $l > 0$, there exists an L^2-integrable function $\rho : X \to \mathbb{R}$, such that for $x \in Z_p \cap \{r < \epsilon\} \simeq B_\epsilon(p)\backslash\{p\}$,

$$|S_t(x,x) - S_t^p(x,x)| \leqslant \rho^2(x)t^l. \tag{6.10}$$

Using (6.1), (6.5), (6.9) and (6.10) we get, as $t \searrow 0$,

$$\begin{aligned}
\mathrm{Tr_s}[N\exp(-t\Delta)] &= \int_{X_\epsilon} \mathrm{Tr_s}[S_t(x,x)]d\mathrm{vol}_X + \sum_{p\in\mathrm{Sing}(X)} \int_{B_\epsilon(p)} \mathrm{Tr_s}[S_t(x,x)]d\mathrm{vol}_X \\
&= \int_{X_\epsilon} \mathrm{Tr_s}[S_t(x,x)]d\mathrm{vol}_X + \sum_{p\in\mathrm{Sing}(X)} \int_{c_\epsilon L_p} \mathrm{Tr_s}[S_t^p(x,x)]d\mathrm{vol}_{Z_p} + \mathcal{O}(t) \\
&= \int_{X_\epsilon} a_0(x)d\mathrm{vol}_X + \gamma_p^{tors}(F) + \mathcal{O}(t) \\
&= \frac{n}{2}\mathrm{rk}(F)\int_X e(TX,\nabla^{TX}) + \gamma_p^{tors}(F) + \mathcal{O}(t).
\end{aligned} \tag{6.11}$$

Cheeger's Chern-Gauss-Bonnet Theorem for spaces with isolated conical singularities [Che83, Theorem 5.1] states that

$$I\chi(X,F) = \mathrm{rk}(F)\int_X e(TX,\nabla^{TX}) + \gamma_p(F), \tag{6.12}$$

where the Cheeger invariant $\gamma_p(F)$ has been defined in (4.36). Comparing (6.11) and (6.12) and using the below Proposition 6.3 (b) we get the claim.

□

Remark 6.2. (a) In the asymptotic expansions, as $t \searrow 0$, of $\mathrm{Tr}[\exp(-t\Delta^{(k)})]$, $k = 0,\dots,n$, there are logarithmic terms $\log(t)$ appearing, which by taking the alternating weighted sum do cancel out. More precisely, the coefficient of $\log(t)$ in the expansion (6.11) of $\mathrm{Tr_s}[N\exp(-t\Delta)]$ is

$$\sum_{p\in\mathrm{Sing}(X)} \int_{L_p} a_0(1,y)d\mathrm{vol}_{L_p}, \tag{6.13}$$

which vanishes by (6.3).

(b) Note that using the asymptotic expansion in Theorem 5.5 and the Mellin transform one can show that the torsion zeta function (2.16) extends to a meromorphic function on $\mathbb{C}$, which is holomorphic at 0. For this latter result, it is crucial that no logarithmic term appears in the asymptotic expansion of $\mathrm{Tr_s}[N\exp(-t\Delta)]$.

Proposition 6.3. *Let n be even and $p \in \mathrm{Sing}(X)$.*

(a) *We have, for $T > 0$ fixed, as $t \searrow 0$,*

$$\mathrm{Tr_s}[N\exp(-t\boldsymbol{\Delta}_T^p)] = t^{-1}\mathrm{rk}(F)\int_{Z_p}\int^{B,p} W\exp(-B_{T^2}^p) + \gamma_p^{tors}(F) + \frac{n}{2}\mathrm{rk}(F)\alpha_p + \mathcal{O}(t^{1/2}). \tag{6.14}$$

(b) *We have the following relation between the Cheeger invariant and the torsion Cheeger invariant:*

$$\gamma_p^{tors}(F) = \frac{n}{2}\gamma_p(F). \tag{6.15}$$

Proof. The proof relies on a combination of local index techniques as developed in [BZ92] with the Singular Asymptotic Lemma (SAL) of Brüning and Seeley [BS85]; here we use the version given in [Les97, Theorem 2.1.11].

Preliminaries on local index techniques: In the following we will apply local index techniques as in [BZ92, Theorem 13.4] and in [Lud20b, Theorem II], which we shortly recall here for further use: We fix a point $z = (r, y) \in Z_p$. For $\epsilon > 0$ small enough, we identify $B_\epsilon(z)$ with $B_\epsilon^{T_zZ_p}(0)$ using geodesic coordinates centred at z. For $x \in B_\epsilon(z) \simeq B_\epsilon^{T_zZ_p}(0)$, we identify T_xZ_p, F_x with T_zZ_p, F_z by parallel transport along the geodesic $t \in [0,1] \to tx$ with respect to the connections ∇^{TZ_p}, ∇^F. The operator $t^2\mathbf{\Delta}_{T/t}^{p,\overline{q}}$ is now seen as an operator acting on sections of $(\Lambda(T^*Z_p) \otimes F)_z$ over $B_\epsilon^{T_zZ_p}(0)$. We consider the operator we get from $t^2\mathbf{\Delta}_{T/t}^{p,\overline{q}}$ by rescaling via $x \to x/t$ and then replacing the Clifford operators $c(e_k)$, $\widehat{c}(e_k)$, defined in (4.10), with

$$c_t(e_k) := \frac{e^k}{\sqrt{t}} - \sqrt{t} i_{e_k}, \quad \widehat{c}_t(e_k) := \frac{\widehat{e^k}}{\sqrt{t}} + \sqrt{t} i_{\widehat{e_k}}, \tag{6.16}$$

for $k = 1, \dots, n$.

As in [BZ92, Theorem 13.4 and Theorem 13.5], we get for $T \in [0, 1/t]$ as $t \searrow 0$, uniformly on compact subsets of Z_p the following pointwise asymptotic expansion

(6.17)
$$\begin{aligned}
&\operatorname{Tr_s}\left[U_{t,T}^{p,\overline{q}}((r,y),(r,y))\right] d\mathrm{vol}_{Z_p}\\
&= \operatorname{rk}(F) \int^{B,p} \exp\left(-B_{T^2}^p\right) + t\left(\int^{B,p} \left(\frac{1}{2}\nabla^{TZ_p} + \iota_{T\widehat{\nabla f}}\right) \widehat{\theta}(F,\nabla^F) \exp(-B_{T^2}^p)\right) + \mathcal{O}(t^2)\\
&= \operatorname{rk}(F) \int^{B,p} \exp\left(-B_{T^2}^p\right) + \frac{1}{2} t d\left(\int^{B,p} \widehat{\theta}(F,\nabla^F) \exp(-B_{T^2}^p)\right) + \mathcal{O}(t^2)\\
&=: a_0(T,r)dr + t a_1(T,r)dr + \mathcal{O}(t^2).
\end{aligned}$$

Moreover, asymptotic expansions for the derivatives both with respect to t and (r, y) are obtained by differentiating (6.17).

The coefficients in the asymptotic expansion (6.17) have exponential decay. Moreover, from the scaling properties of the Berezin integral (3.26) and the fact, that $\theta(F, \nabla^F)$ does not depend on the radial coordinate, we have

$$a_0(T,r) = r^{-1} a_0(Tr, 1), \quad a_1(T,r) = r^{-2} a_1(Tr, 1) = d_{L_p}\beta(T,r), \tag{6.18}$$

where β is an $(n-2)$-form on the link L_p, depending smoothly on the radial coordinate $r > 0$ and on $T > 0$.

(a) Let $T > 0$ be fixed.

Step 1: Splitting the integral: Let us denote by $\widetilde{S}_{t,T}^p(x,y) = S_{t^2,T/t}^p(x,y)$, $x, y \in Z_p$, the kernel of $N\exp(-t^2\mathbf{\Delta}_{T/t}^p)$ w.r.t. $d\mathrm{vol}_{Z_p}$. The scaling property (4.15) of the model Witten Laplacian implies

$$S_{t,T}^p((r,y),(r,y)) = t^{n/2} \widetilde{S}_{t,T}^p\left((\sqrt{t}r, y), (\sqrt{t}r, y)\right). \tag{6.19}$$

Let $\varphi : \mathbb{R}_{\geqslant 0} \to [0,1]$ be a cut-off function, with $\operatorname{supp}\varphi \subset [0,1]$ and $\varphi \equiv 1$ in $[0,1/2]$. From (6.19) and using the change of variables $r \to \sqrt{t}r$, we get

(6.20)
$$\begin{aligned}
&\mathrm{Tr_s}\left[N\exp(-t\boldsymbol{\Delta}_T^p)\right] = \int_{Z_p} \mathrm{Tr_s}\left[S^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}\\
&= \int_{Z_p} \mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\left((\sqrt{t}r,y),(\sqrt{t}r,y)\right)\right] t^{n/2} d\mathrm{vol}_{Z_p} = \int_{Z_p} \mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}\\
&= \int_{Z_p} \varphi(r)\mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} + \int_{Z_p} (1-\varphi(r))\mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}.
\end{aligned}$$

Step 2: We establish a pointwise asymptotic expansion, which will be used in Step 3 and Step 4: We use local index techniques as in the preliminaries; *i.e.* we apply to the operator $t^2\boldsymbol{\Delta}^p_{T/t}$the (local) scaling $x \to tx$ and replace the Clifford variables $c(e_k), \widehat{c}(e_k)$ by $c_t(e_k), \widehat{c}_t(e_k)$ (see (6.16)). We denote by C_t the operator we get from $\frac{1}{2}\sum c(e_i)\widehat{c}(e_i)$ by the above scaling. We have

$$tC_t \xrightarrow{t\searrow 0} W = \frac{1}{2}\sum_{i=1}^n e^i \wedge \widehat{e}^i. \tag{6.21}$$

Using (4.11), (6.21) and proceeding as in [BZ92, Theorem 13.4 and Theorem 13.5], we get the following pointwise asymptotic expansion as $t \searrow 0$ and $T \in [0,1/t]$, uniformly on compact sets of Z_p,

$$\begin{aligned}
&\mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p}\\
&= \frac{1}{t}\mathrm{rk}(F)\int^{B,p} W\exp(-B^p_{T^2}) + \int^{B,p} W\left(\frac{1}{2}\nabla^{TZ_p} + \iota_{T\widehat{\nabla f}}\right)\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\\
&\quad + \frac{n}{2}\mathrm{rk}(F)\int^{B,p}\exp(-B^p_{T^2}) + \mathcal{O}(t)\\
&= \frac{1}{t}\mathrm{rk}(F)\int^{B,p} W\exp(-B^p_{T^2}) + \frac{1}{2}d\left(\int^{B,p} W\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right)\\
&\quad + \frac{n}{2}\mathrm{rk}(F)\int^{B,p}\exp(-B^p_{T^2}) + \mathcal{O}(t)\\
&=: \frac{1}{t}\widetilde{a}_{-1}(T,r)dr + \widetilde{a}_0(T,r)dr + \mathcal{O}(t),
\end{aligned}\tag{6.22}$$

where for the last equality we have used [BZ92, (3.23)], [BGV04, Proposition 1.50], $\nabla^{TZ_p}W = 0$ and the fact that $\hat{\theta}$ does not contain $\hat{e^r}$.

The $(n-1)$-form $\int^{B,p} W\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})$ contains e^r, hence we have

$$d\left(\int^{B,p} W\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right) = e^r \wedge d_{L_p}\widetilde{\beta}(r,T), \tag{6.23}$$

for an $(n-2)$-form $\widetilde{\beta}$ on the link L_p, which depends on the radial coordinate r.

For $a > 0$, we denote by h_a the radial scaling $r \to ar$. We have the following scaling properties

(6.24)
$$h_a^* \int^{B,p} W \exp(-B^p_{T^2}) = a \int^{B,p} W \exp(-B^p_{a^2T^2}),$$
$$h_a^* \int^{B,p} W \widehat{\theta}(F, g^F) \exp(-B^p_{T^2}) = \int^{B,p} W \widehat{\theta}(F, g^F) \exp(-B^p_{a^2T^2}),$$

which, together with (3.26) imply

(6.25)
$$\widetilde{a}_{-1}(T, r) = \widetilde{a}_{-1}(Tr, 1), \ \widetilde{a}_0(T, r) = r^{-1}\widetilde{a}_0(Tr, 1).$$

The coefficients appearing in the asymptotic expansion (6.22) have exponential decay as $r \to \infty$. Together with Proposition 3.1 (c) and (6.24), this shows that $\widetilde{a}_{-1}(T, -)dr$, $\widetilde{a}_0(T, -)dr$ are integrable over the infinite cone.

Step 3: We study the first integral on the right hand side of (6.20) *as* $t \searrow 0$ *using SAL*: The scaling property (4.15) of the model Witten Laplacian implies,

(6.26)
$$\widetilde{S}^p_{t,T}((r, y), (r, y)) = r^{-n} \widetilde{S}^p_{t/r,Tr}((1, y), (1, y)).$$

Using (6.26), we get for the first integral on the right hand side of (6.20):

(6.27)
$$\int_{Z_p} \varphi(r) \mathrm{Tr}_s \left[\widetilde{S}^p_{t,T}\left((r, y), (r, y)\right)\right] d\mathrm{vol}_{Z_p} = \int_0^\infty \frac{dr}{r} \varphi(r) \int_{L_p} \mathrm{Tr}_s \left[\widetilde{S}^p_{t/r,Tr}\left((1, y), (1, y)\right)\right] d\mathrm{vol}_{L_p}$$
$$= z \int_0^\infty \varphi(r) \widetilde{\sigma}(r, rz) dr,$$

where $z := t^{-1}$ and

(6.28)
$$\widetilde{\sigma}(r, \xi) := \frac{1}{\xi} \int_{L_p} \mathrm{Tr}_s \left[\widetilde{S}^p_{\xi^{-1},Tr}\left((1, y), (1, y)\right)\right] d\mathrm{vol}_{L_p}.$$

We have, using (6.8), $\widetilde{\sigma}(0, \xi) \in L^1(\mathbb{R}_{\geqslant 0})$,

(6.29)
$$\int_0^\infty \widetilde{\sigma}(0, \xi) d\xi = \int_0^\infty \frac{d\xi}{\xi} \int_{L_p} \mathrm{Tr}_s \left[\widetilde{S}^p_{\xi^{-1},0}\left((1, y), (1, y)\right)\right] d\mathrm{vol}_{L_p}$$
$$= \int_0^\infty \frac{d\xi}{\xi} \int_{L_p} \mathrm{Tr}_s \left[S^p_{\xi^{-2},0}\left((1, y), (1, y)\right)\right] d\mathrm{vol}_{L_p}$$
$$= \frac{1}{2} \int_0^\infty \frac{du}{u} \int_{L_p} \mathrm{Tr}_s \left[S^p_u\left((1, y), (1, y)\right)\right] d\mathrm{vol}_{L_p}$$
$$= \gamma^{tors}_p(F).$$

We define

(6.30)
$$\widetilde{\sigma}_0(r) := \int_{L_p} \widetilde{a}_{-1}(Tr, 1), \quad \widetilde{\sigma}_{-1}(r) := \int_{L_p} \widetilde{a}_0(Tr, 1).$$

From the discussion in Step 2, we have $\widetilde{\sigma}_0(r), r^{-1}\widetilde{\sigma}_{-1}(r) \in L^1(\mathbb{R}_{\geqslant 0})$. Moreover

(6.31)
$$\int_0^\infty \varphi(r) \widetilde{\sigma}_0(r) dr = \mathrm{rk}(F) \int_{Z_p} \varphi(r) \int^{B,p} W \exp(-B^p_{T^2}),$$

and, using also (6.23),

$$\int_0^\infty \varphi(r) r^{-1} \widetilde{\sigma}_{-1}(r) dr = \frac{n}{2}\mathrm{rk}(F) \int_{Z_p} \varphi(r) \int^{B,p} \exp(-B^p_{T^2}) \text{ and } \widetilde{\sigma}_{-1}(0) = 0. \tag{6.32}$$

From (6.22), we get, as $\xi \to \infty$ and $0 \leqslant r \leqslant 1$, $j = 0, 1$,

$$|\partial_r^j [\widetilde{\sigma}(r,\xi) - \widetilde{\sigma}_0(r) - \xi^{-1}\widetilde{\sigma}_{-1}(r)]| = \mathcal{O}(\xi^{-2}). \tag{6.33}$$

Using the explicit expression for the heat kernel $\exp(-t\boldsymbol{\Delta}_T^{p,\overline{q}})$ in Proposition 4.3, Remark 4.4 and the asymptotic behaviour of the modified Bessel functions, one can prove the following integrability condition for $\widetilde{\sigma}$:

$$\int_0^1 \int_0^1 s |\partial_r \widetilde{\sigma}(\theta s t, s)| ds dt = \mathcal{O}(\theta^{-1/2}), \qquad \text{for } 0 < \theta \leqslant 1. \tag{6.34}$$

The assumptions in SAL [Les97, Theorem 2.1.11] are hence fulfilled. Applying SAL, as $z \to \infty$,

$$\begin{aligned} &z \int_0^\infty \widetilde{\sigma}(r, rz) dr \\ &= \int_0^\infty \widetilde{\sigma}(0,\xi) d\xi + \int_0^\infty \varphi(r) \widetilde{\sigma}_0(r) dr \, z + \int_0^\infty \varphi(r) \widetilde{\sigma}_{-1}(r) r^{-1} dr + \widetilde{\sigma}_{-1}(0) \log z + \mathcal{O}(z^{-1/2}), \end{aligned} \tag{6.35}$$

where we have also used that, as discussed in (6.29), (6.31) and (6.32), the infinite integrals appearing in (6.35) are well-defined.

From (6.27), (6.28), (6.31), (6.32) and (6.35), as $t \searrow 0$,

$$\begin{aligned} &\int_{Z_p} \varphi(r) \mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} \\ &= t^{-1}\mathrm{rk}(F) \int_{Z_p} \varphi(r) \int^{B,p} W \exp(-B^p_{T^2}) \\ &\quad + \gamma_p^{tors}(F) + \frac{n}{2}\mathrm{rk}(F) \int_{Z_p} \varphi(r) \int^{B,p} \exp(-B_{T^2}) dr + \mathcal{O}(t^{1/2}) \end{aligned} \tag{6.36}$$

<u>*Step 4:*</u> *We study the second integral on the right hand side of* (6.20) *as* $t \searrow 0$: Using (6.26) and the scaling properties of the Berezin integral (3.26), the pointwise asymptotic expansion (6.22) - which is uniform on compact sets of Z_p and for $0 \leqslant tT < 1$, and the

change of variables $u = t/r$, we get

(6.37)

$$\begin{aligned}
&\int_{Z_p} (1-\varphi(r))\mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} \\
&= \int_{\{r\geqslant 1/2\}\times L_p} (1-\varphi(r))\mathrm{Tr_s}\left[\widetilde{S}^p_{t,T}\big((r,y),(r,y)\big)\right] d\mathrm{vol}_{Z_p} \\
&= \int_0^{2t} \frac{du}{u}\int_{L_p} (1-\varphi(t/u))\mathrm{Tr_s}\left[\widetilde{S}^p_{u,\frac{Tt}{u}}\big((1,y),(1,y)\big)\right] d\mathrm{vol}_{L_p} \\
&= \int_0^{2t} \frac{du}{u}(1-\varphi(t/u))\int_{L_p}\left\{u^{-1}\widetilde{a}_{-1}\left(\frac{Tt}{u},1\right)+\widetilde{a}_0\left(\frac{Tt}{u},1\right)+\mathcal{O}(u)\right\} \\
&= t^{-1}\mathrm{rk}(F)\int_{Z_p}(1-\varphi(r))\int^{B,p} W\exp(-B^p_{T^2}) + \mathrm{rk}(F)\frac{n}{2}\int_{Z_p}(1-\varphi(r))\int^{B,p}\exp(-B^p_{T^2}) \\
&\quad + \mathcal{O}(t).
\end{aligned}$$

<u>Step 5:</u> *We finish the proof:* By putting together (6.20), (6.36) and (6.37), we get the claim in part (a) of the proposition.

(b) <u>Step 1:</u> *We study* $\mathrm{Tr_s}[r^2\exp(-t^2\mathbf{\Delta}^p_{T/t})]$ *by applying SAL:* Let $T > 0$ be fixed; in this step we proceed similarly to part (a). Using a cut-off function $\varphi : \mathbb{R}_{\geqslant 0} \to [0,1]$ as in part (a), we write

$$\begin{aligned}
\mathrm{Tr_s}[r^2\exp(-t^2\mathbf{\Delta}^p_{T/t})] &= \int_{Z_p}\varphi(r)r^2\mathrm{Tr_s}\left[U^p_{t,T}\left((r,y),(r,y)\right)\right]d\mathrm{vol}_{Z_p} \\
&\quad + \int_{Z_p}(1-\varphi(r))r^2\mathrm{Tr_s}\left[U^p_{t,T}\left((r,y),(r,y)\right)\right]d\mathrm{vol}_{Z_p}.
\end{aligned} \tag{6.38}$$

The scaling property (4.15) of the model Witten Laplacian implies

$$U^{p,\overline{q}}_{t,T}((r,y),(r,y)) = r^{-n}U^{p,\overline{q}}_{t/r,Tr}\big((1,y),(1,y)\big). \tag{6.39}$$

For the first integral in (6.38), using (6.39), we get

(6.40)

$$\begin{aligned}
\int_{Z_p}\varphi(r)r^2\mathrm{Tr_s}\left[U^p_{t,T}\left((r,y),(r,y)\right)\right]d\mathrm{vol}_{Z_p} &= \int_0^\infty\varphi(r)rdr\int_{L_p}\mathrm{Tr_s}\left[U^p_{t/r,Tr}\left((1,y),(1,y)\right)\right]d\mathrm{vol}_{L_p} \\
&= z\int_0^\infty\varphi(r)\sigma(r,rz)dr,
\end{aligned}$$

where $z := t^{-1}$ and

$$\sigma(r,\xi) := \frac{r^2}{\xi}\int_{L_p}\mathrm{Tr_s}\left[U^p_{\xi^{-1},Tr}\left((1,y),(1,y)\right)\right]d\mathrm{vol}_{L_p}. \tag{6.41}$$

With the notations introduced in (6.17), we define

$$\sigma_{-1}(r) := r^2\int_{L_p}a_0(Tr,1), \quad \sigma_{-2}(r) := r^2\int_{L_p}a_1(Tr,1) = 0, \tag{6.42}$$

where the vanishing of σ_{-2} follows from the second identity in (6.18). From (6.17), we get, as $\xi\to\infty$ and $0\leqslant r\leqslant 1$, $j=0,1,2$,

$$|\partial^j_r[\sigma(r,\xi)-\xi^{-1}\sigma_{-1}(r)-\xi^{-2}\sigma_{-2}(r)]| = \mathcal{O}(\xi^{-3}). \tag{6.43}$$

Using the explicit expression for the heat kernel $\exp(-t\mathbf{\Delta}_T^{p,\bar{q}})$ in Proposition 4.3 and the asymptotic behaviour of the modified Bessel functions, one can prove the following integrability condition for $\sigma(r,\xi)$:

$$\int_0^1\int_0^1 s^2|\partial_r^2\sigma(\theta st,s)|dsdt = \mathcal{O}(\theta^{1/2}),\qquad \text{for } 0<\theta\leqslant 1. \tag{6.44}$$

Application of SAL, in a combined version of [Les97, Theorem 2.1.11] and [BS85], gives, as $z\to\infty$,

$$\begin{aligned}
&\int_{Z_p}\varphi(r)r^2\mathrm{Tr_s}\left[U_{t,T}\left((r,y),(r,y)\right)\right]d\mathrm{vol}_{Z_p} = z\int_0^\infty\varphi(r)\sigma(r,rz)dr\\
&=\int_0^\infty\sigma(0,\xi)d\xi+\int_0^\infty\xi\partial_r\sigma(0,\xi)d\xi z^{-1}\\
&\quad+\int_0^\infty\varphi(r)\sigma_{-1}(r)r^{-1}dr+\int_0^\infty\varphi(r)\sigma_{-2}(r)r^{-2}drz^{-1}\\
&\quad+\sigma_{-1}(0)\log z+\partial_r\sigma_{-2}(0)z^{-1}\log z+\mathcal{O}(z^{-2})\\
&=\mathrm{rk}(F)\int_{Z_p}\varphi(r)r^2\int^{B,p}\exp(-B^p_{T^2})+\mathcal{O}(t^2).
\end{aligned}\tag{6.45}$$

Proceeding as in Step 4 of part (a), we get for the second integral on the right hand side of (6.38),

$$\begin{aligned}
&\int_{Z_p}(1-\varphi(r))r^2\mathrm{Tr_s}\left[U^p_{t,T}\left((r,y),(r,y)\right)\right]d\mathrm{vol}_{Z_p}\\
&\qquad=\mathrm{rk}(F)\int_{Z_p}(1-\varphi(r))r^2\int^{B,p}\exp(-B^p_{T^2})+\mathcal{O}(t^2).
\end{aligned}\tag{6.46}$$

Putting together (6.38), (6.45), (6.46) and using the scaling properties of the Berezin integral (3.26), we get

$$\begin{aligned}
\mathrm{Tr_s}[r^2\exp(-t^2\mathbf{\Delta}^p_{T/t})] &= \int_{Z_p}r^2\mathrm{Tr_s}\left[U^p_{t,T}\left((r,y),(r,y)\right)\right]d\mathrm{vol}_{Z_p}\\
&=\mathrm{rk}(F)\int_{Z_p}r^2\int^{B,p}\exp(-B^p_{T^2})+\mathcal{O}(t^2)\\
&=\frac{\mathrm{rk}(F)}{T^2}\int_{Z_p}r^2\int^{B,p}\exp(-B^p_1)+\mathcal{O}(t^2).
\end{aligned}\tag{6.47}$$

Step 2: Deriving a second asymptotic expansion for $\mathrm{Tr_s}\left[N\exp(-t\mathbf{\Delta}^p_1)\right]$: As in the proof of [BZ92, Theorem 5.6], one can show that the form

$$\frac{dt}{2t}\mathrm{Tr_s}\left[N\exp(-t\mathbf{\Delta}^p_T)\right]-dT\mathrm{Tr_s}\left[-\frac{r^2}{2}\exp(-t\mathbf{\Delta}^p_T)\right] \tag{6.48}$$

is a closed form on $\mathbb{R}^2_{>0}$, from which we deduce

$$\mathrm{Tr_s}\left[-\frac{r^2}{2}\exp(-t\mathbf{\Delta}^p_T)\right]=-\frac{1}{2T}\Big(\mathrm{Tr_s}\left[N\exp(-t\mathbf{\Delta}^p_T)\right]-\frac{n}{2}I\chi(cL_p,L_p,F)\Big). \tag{6.49}$$

Using (6.47) (with $t = 1$), (6.49) and the fact, that $\mathrm{Tr_s}\left[N\exp(-t\mathbf{\Delta}^p_T)\right]$ is symmetric w.r.t. interchanging (t, T), we get the following asymptotic expansion as $t \searrow 0$:

$$\mathrm{Tr_s}\left[N\exp(-t\mathbf{\Delta}^p_1)\right] = \frac{1}{t}\mathrm{rk}(F)\int_{Z_p} r^2 \int^{B,p} \exp(-B^p_1) + \frac{n}{2} I\chi(cL_p, L_p, F) + \mathcal{O}(t). \tag{6.50}$$

Using Theorem 4.6 and comparing the constant coefficient in the asymptotic expansions (6.14) and (6.50) we get the claim. (Equality of the leading coefficients in the two expansions is a consequence of (3.5).) □

6.3. **Proof of Theorems 5.6-5.9.** The proofs of Theorems 5.6-5.9 consist of two steps: a localisation argument and an explicit computation using the local model operator near $\mathrm{Crit}(f)$; the latter has been taken care of in Section 4.2. The localisation argument in the proofs of Theorems 5.6-5.9 do mostly rely on the singular elliptic estimates of Lesch [Les97, Section 1.4] for $\Delta^{\overline{q}}$, the Spectral Gap Theorem, Theorem [Lud17b, Theorem I], for the operator $\widetilde{\Delta}^{\overline{q}}_T$, and the fact that $\mathrm{dom}(\Delta^{\overline{q}})^l = \mathrm{dom}(\widetilde{\Delta}^{\overline{q}}_T)^l$, $l \in \mathbb{N}$ (see (4.4)). All these ingredients hold without assuming the Witt condition and without assuming that g^F is flat; the singular elliptic estimates of Lesch are available for every closed extension of a symmetric elliptic differential operator of Fuchs type.

Hence the proofs of Theorems 5.6-5.9 follow by a direct generalisation of the proofs of the corresponding statements in [Lud20a].

7. Anomaly formulas

In Section 7.2 we prove, for $\overline{q} \in \{\overline{m}, \overline{n}\}$, the anomaly formulas for the Ray-Singer metric $\|\ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}$ stated in Theorem II of the introduction. In Section 7.3 we prove anomaly formulas for the Bismut-Zhang metric $\|\ \|^{Y,g^{TX},g^F}_{\det IH^\bullet_{\overline{q}}(X,F)}$. The anomaly formula for the Ray-Singer metric $\|\ \|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F)}$ generalises the anomaly formula of Bismut and Zhang in the smooth setting [BZ92, Section 4]. In Remark 7.9 we show that the Bismut-Zhang formula in Theorem I and the three anomaly formulas for the three terms in it (Theorems II, 3.4, and 7.8 are compatible, as it should be!

In this section we will always consider a family of metrics $l \in \mathbb{R} \to (g^{TX}_l, g^F_l)$ on TX, F depending smoothly on the parameter l. We assume moreover that the metrics g^{TX}_l are conical, *i.e.* for $p \in \mathrm{Sing}(X)$, there is a family of Riemannian metrics $l \in \mathbb{R} \to g^{TL_p}_l$, such that $g^{TX}_l = dr^2 + r^2 g^{TL_p}_l$ near p. Similarly, we assume that near $p \in \mathrm{Sing}(X)$, the metric g^F_l is of the form explained in Section 2.2.

In the following we use a sub- resp. a superscript l to characterise operators associated to the pair (g^{TX}_l, g^F_l).

7.1. **Spectral gap condition.** In this section we explain an additional assumption on the metrics (g^{TX}, g^F), which will be in place for most of the results in Section 7. It is used in particular in the proof of Proposition 7.3 to deal with the boundary terms appearing near $p \in \mathrm{Sing}(X)$ when applying Stokes' Theorem.

For $p \in \mathrm{Sing}(X)$, the following spectral gap condition for the first order differential operator S_p on the link L_p, defined in (4.8), will be assumed:

$$\mathrm{Spec}(S_p) \cap (-1/2, 1/2) = \{0\}. \tag{7.1}$$

By [BL93, Corollary 2.3] condition (7.1) is equivalent to the following spectral gap condition for the transversal Laplacian Δ_{L_p} on the link L_p:
(7.2)
$$\begin{cases} \mathrm{Spec}(\Delta^{(\nu-1)}_{L_p,ccl}) \cap (0,1) = \varnothing & \text{if } n = 2\nu \text{ is even,} \\ \Big(\mathrm{Spec}(\Delta^{(\nu-1)}_{L_p,ccl}) \cap (0,3/4)\Big) \cup \Big(\mathrm{Spec}(\Delta^{(\nu)}_{L_p,ccl}) \cap (0,3/4)\Big) = \varnothing & \text{if } n = 2\nu+1 \text{ is odd.} \end{cases}$$

Condition (7.1) (and hence (7.2)) can be achieved by a rescaling of the metric g^{TL_p} into $c^2 g^{TL_p}$ with $c > 0$ sufficiently small.

We denote by $D_{\min}$ (resp. $D_{\max}$) the minimal (resp. the maximal) closed extension of the first order operator $D_c = d_c + \delta_c$. By [BS88, Theorem 3.2 and Lemma 3.2] we have

$$\mathrm{dom}(D_{\min}) = \left\{ \omega \in \mathrm{dom}(D_{\max}) \,\middle|\, \begin{array}{c} \|U^{-1}(\omega)\|_{\mathrm{L}^2(\Lambda(TL_p)\otimes F)} = o(r^{1/2}|\log r|^{1/2}) \\ \text{locally near } p \in \mathrm{Sing}(X) \end{array} \right\}, \tag{7.3}$$

where U is the unitary transformation defined in Section 4.2.1. Also by [BS88, Theorem 3.2 and Lemma 3.2], in case n even, assuming the spectral gap condition (7.1) the operator D_c is essentially self-adjoint, hence

$$D^{\overline{m}} = D^{\overline{n}} = D_{\min} = D_{\max}. \tag{7.4}$$

In case $n = 2\nu + 1$ odd, assuming (7.1), we have

$$\mathrm{dom}(D^{\mathrm{ev}}_{\max}) / \mathrm{dom}(D^{\mathrm{ev}}_{\min}) \simeq \bigoplus_{p \in \mathrm{Sing}(X)} \mathscr{H}^{\nu}(L_p, F_{L_p}). \tag{7.5}$$

By [BS88, Lemma 3.2], [ALMP18, Section 5], there are continuous linear functionals $a, b : \mathrm{dom}(D_{\max}) \to \bigoplus_{p \in \mathrm{Sing}(X)} \mathscr{H}^{\nu}(L_p, F_{L_p})$ such that, for $\omega \in \mathrm{dom}(D_{\max})$, locally near $p \in \mathrm{Sing}(X)$,

$$\omega - a(\omega) - b(\omega) \wedge dr \in \mathrm{dom}(D_{\min}). \tag{7.6}$$

Moreover using [ALMP18, Lemma 5.2] we can characterise the extensions $D^{\overline{q}}$, $\overline{q} \in \{\overline{m}, \overline{n}\}$, by

$$\begin{aligned} \mathrm{dom}(D^{\overline{m}}) &= \{\omega \in \mathrm{dom}(D_{\max}) \mid b(\omega) = 0\}, \\ \mathrm{dom}(D^{\overline{n}}) &= \{\omega \in \mathrm{dom}(D_{\max}) \mid a(\omega) = 0\}. \end{aligned} \tag{7.7}$$

7.2. **Anomaly formula for the Ray-Singer metric $\| \quad \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$. Proof of Theorem II.** Let $e_1, \ldots, e_n$ be an ONB of (TX, g^{TX}_l). We denote by $*_l$, $*^F_l$ the Hodge star operators associated to the metrics g^{TX}_l, (g^{TX}_l, g^F_l). By [BZ92, Proposition 4.15], we have

$$*_l^{-1} \frac{\partial *_l}{\partial l} = -\frac{1}{2} \sum_{1 \leqslant i,j \leqslant n} \left\langle (g^{TX}_l)^{-1} \frac{\partial g^{TX}_l}{\partial l} e_i, e_j \right\rangle_{g^{TX}_l} c(e_i) \widehat{c}(e_j). \tag{7.8}$$

Set

$$\dot{\omega}^X_l := -\frac{1}{2} \sum_{1 \leqslant i,j \leqslant n} \left\langle (g^{TX}_l)^{-1} \frac{\partial g^{TX}_l}{\partial l} e_i, e_j \right\rangle_{g^{TX}_l} e^i \wedge \widehat{e}^j. \tag{7.9}$$

Similarly to (7.9) we define, for $p \in \mathrm{Sing}(X)$,

$$\begin{aligned}\dot{\omega}_l^{Z_p} &:= -\frac{1}{2}\sum_{1\leqslant i,j\leqslant n}\left\langle (g_l^{TZ_p})^{-1}\frac{\partial g_l^{TZ_p}}{\partial l}e_i, e_j\right\rangle_{g_l^{TZ_p}} e^i \wedge \widehat{e}^j \\ &= -\frac{1}{2}\sum_{2\leqslant i,j\leqslant n}\left\langle (g_l^{TZ_p})^{-1}\frac{\partial g_l^{TZ_p}}{\partial l}e_i, e_j\right\rangle_{g_l^{TZ_p}} e^i \wedge \widehat{e}^j.\end{aligned} \tag{7.10}$$

Note that the coefficients in the above sum do not depend on the radial coordinate.

The characteristic class $e(\rho^*TX, \nabla^{TX,\mathrm{tot}})$ associated to the family of conical Riemannian metrics $(g_l^{TX})_l$ has been defined in Section 3.4 and is vanishing in case n odd.

Theorem 7.1. *Let $\mathbb{R} \ni l \to (g_l^{TX}, g_l^F)$ be a family of metrics on TX, F as explained at the beginning of Section 7. Then we have the following asymptotic expansion as $t \searrow 0$:*

$$\begin{aligned}&\mathrm{Tr_s}\left[\left(*_l^{-1}\frac{\partial *_l}{\partial l} + (g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right)\exp(-t(D_l^{\overline{q}})^2)\right] \\ &= \int_X \mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right] e(TX, \nabla_l^{TX}) \\ &+ \frac{1}{\sqrt{t}}\mathrm{rk}(F)\int_X \int^B \dot{\omega}_l^X \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right) + \int_X \iota_{\partial_l} e(\rho^*TX, \nabla^{TX,\mathrm{tot}})\theta(F, g_l^F) \\ &+ \sum_{p\in\mathrm{Sing}(X)} (c_{p,l}^{\overline{q}} + \widetilde{c}_{p,l}^{\overline{q}}) + \mathcal{O}(t^{1/2}),\end{aligned} \tag{7.11}$$

where the contributions of the singularities $c_{p,l}^{\overline{q}}$, $\widetilde{c}_{p,l}^{\overline{q}}$, $p \in \mathrm{Sing}(X)$, are given by the following well-defined integrals:

$$\begin{aligned}c_{p,l}^{\overline{q}} &:= \frac{1}{2}\int_0^\infty \frac{du}{u}\int_{L_p} \mathrm{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right] d\mathrm{vol}_{L_p}, \\ \widetilde{c}_{p,l}^{\overline{q}} &:= \frac{1}{2}\int_0^\infty \frac{du}{u}\int_{L_p} \mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right] d\mathrm{vol}_{L_p}.\end{aligned} \tag{7.12}$$

Remark 7.2. (a) The first three terms on the right hand side of the formula (7.11) are the interiour contribution, familiar from the anomaly formula for the Ray-Singer metric on a smooth compact manifold [BZ92, Theorem 4.14, Theorem 4.20]. They do not depend on the chosen extension of $D_{l,c} = d_c + \delta_{l,c}$. The contributions of the singularities of X to the formula (7.11), $c_{p,l}^{\overline{q}}$, $\widetilde{c}_{p,l}^{\overline{q}}$, $p \in \mathrm{Sing}(X)$, do depend on the chosen extension $D_l^{\overline{q}}$.

(b) One can establish a corresponding formula for every other closed self-adjoint extension of the Laplacian, which is invariant under radial scaling near $\mathrm{Sing}(X)$.

(c) Note the following vanishing properties for the coefficients in (7.11): If n is even, $\int^B \dot{\omega}_l^X \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right) = 0$ since the integrand is a sum of forms of type (k,k), k odd. If n is odd, clearly from their definition, $e(TX, \nabla_l^{TX}) = 0$, $e(\rho^*TX, \nabla^{TX,\mathrm{tot}}) = 0$.

Proof. In the following we fix $0 < \epsilon < \delta$ and set $X_\epsilon := X\backslash(\cup_{p\in\mathrm{Sing}(X)}B_\epsilon(p))$. We identify a neighbourhood of a singular point $p \in \mathrm{Sing}(X)$ with a neighbourhood of the tip point in the infinite cone over L_p. We denote by $\exp(-t(D_l^{\overline{q}})^2)(x,y)$, $x,y \in X_{sm}$, $t>0$, the kernel of the heat operator $\exp(-t(D_l^{\overline{q}})^2)$.

Step 1: Variation of the metric on F. Using local index techniques as in [BZ92, Theorem 4.20] (more precisely [BZ92, (4.61)]), we get the following pointwise asymptotic expansion as $t \searrow 0$:

$$\begin{aligned}
&\operatorname{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\exp(-t(D_l^{\overline{q}})^2)(x,x)\right]d\mathrm{vol}_X(x)\\
&= \left(\operatorname{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]\int^B \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\right)(x) + \mathcal{O}(t^{1/2})\\
&= \left(\operatorname{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]e(TX,\nabla_l^{TX})\right)(x) + \mathcal{O}(t^{1/2}).
\end{aligned} \tag{7.13}$$

The expansion (7.13) is uniform on compact sets; the coefficients do only depend on local geometric data and do not depend on the chosen extension of $D_{l,c} := d_c + \delta_{l,c}$. Since by (3.18), $e(TX,\nabla_l^{TX})$ vanishes near $p \in \mathrm{Sing}(X)$, the coefficient of t^0 in the above pointwise asymptotic expansion (7.13) vanishes near the singularities of X.

For $p \in \mathrm{Sing}(X)$, we define

$$\begin{aligned}
c_{p,l}^{\overline{q}} &:= \lim_{t\to 0}\int_{\{0\leqslant r\leqslant \epsilon\}\times L_p}\operatorname{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_t^{p,l,\overline{q}}((r,y),(r,y))\right]d\mathrm{vol}_{Z_p}\\
&= \lim_{t\to 0}\frac{1}{2}\int_{\epsilon^{-2}t}^{\infty}\frac{du}{u}\int_{L_p}\operatorname{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right]d\mathrm{vol}_{L_p}\\
&= \frac{1}{2}\int_0^{\infty}\frac{du}{u}\int_{L_p}\operatorname{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right]d\mathrm{vol}_{L_p}.
\end{aligned} \tag{7.14}$$

To prove the well-definedness of the integrals in (7.14) we use the same arguments as in the proof of the well-definedness of the Cheeger invariant $\gamma_p^{\overline{q}}(F)$ in Section 4.3.1: For the first identity in (7.14) we have used the change of variables $u = t/r^2$, the scaling property (4.37) for the heat kernel on the infinite cone and the fact that, on Z_p, the operator $\left((g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right)$ does not depend on the radial coordinate. From (3.18) and (7.13) we have, as $u \searrow 0$,

$$\operatorname{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right] \sim \mathcal{O}(u^{1/2}), \tag{7.15}$$

which shows the well-definedness of the last integral in (7.14) at $u=0$. The well-definedness at $u=\infty$ follows using the characterisation of $\mathrm{dom}(\boldsymbol{\Delta}^{p,l,\overline{q}})$, see (4.39).

Proceeding with Cheeger's strategy [Che83, Section 2] (which has already been used in the proof of Theorem 5.5 in Section 6.2), we get from (7.13), (7.14) and Duhamel's principle, as $t \searrow 0$:

$$
\begin{aligned}
&\operatorname{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\exp(-t(D_l^{\overline{q}})^2)\right]\\
&=\int_{X_\epsilon}\operatorname{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\exp(-t(D_l^{\overline{q}})^2)(x,x)\right]d\mathrm{vol}_X\\
&+\sum_{p\in\mathrm{Sing}(X)}\int_{Z_{p,\epsilon}}\operatorname{Tr_s}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}Q_t^{p,l,\overline{q}}(x,x)\right]d\mathrm{vol}_{Z_p}+\mathcal{O}(t)\\
&=\int_X \operatorname{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]e(TX,\nabla_l^{TX})+\sum_{p\in\mathrm{Sing}(X)}c_{p,l}^{\overline{q}}+\mathcal{O}(t^{1/2}).
\end{aligned}
\tag{7.16}
$$

Note that it is due to the vanishing of the Euler form $e(TX, \nabla_l^{TX})$ near $\mathrm{Sing}(X)$, that the first integral on the right hand side of (7.16) is well-defined and moreover no logarithmic term $\log(t)$ appears in the asymptotic expansion (7.16).

Step 2: Variation of the metric on TX. Using local index techniques as in [BZ92, Theorem 4.20], we get the following pointwise asymptotic expansion uniformly on compact sets, as $t \searrow 0$,

$$
\begin{aligned}
&\operatorname{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t(D_l^{\overline{q}})^2)(x,x)\right]d\mathrm{vol}_X=\\
&=\begin{cases}\frac{1}{2}\left(\int^B\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\nabla_l^{TX}\widehat{\theta}(F,g_l^F)\right)(x)+\mathcal{O}(t) & \text{if } n \text{ is even},\\[2ex] \frac{1}{\sqrt{t}}\mathrm{rk}(F)\left(\int^B\dot{\omega}_l^X\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\right)(x)+\mathcal{O}(t^{1/2}) & \text{if } n \text{ is odd}.\end{cases}
\end{aligned}
\tag{7.17}
$$

Indeed the above asymptotics has been worked out in [BZ92, Theorem 4.20] for n even. As remarked in [BM06, (4.23b)], proceeding as in [BZ92, (4.55)-(4.63)] one gets (7.17) for the case n odd as well. Note that the leading coefficients in the above expansions (7.17) vanish near $\mathrm{Sing}(X)$: Using [BC90, Proposition 1.2], (2.6), (7.10) and the fact that $\theta(F, g^F)$ does not depend on the radial coordinate, one has that the integrands in the Berezin integrals appearing in (7.17), near $\mathrm{Sing}(X)$, are a sum of summands not containing either e^r or $\widehat{e}^r$. Similarly, also the $(n-1)$-form $\int^B \exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\widehat{\theta}(F, g_l^F)$ vanishes near $\mathrm{Sing}(X)$.

From the above discussion, using Stokes' Theorem, the Bianchi identity and [BZ92, (4.74)-(4.86)] we have, for $\epsilon > 0$ small enough,

(7.18)

$$
\begin{aligned}
&\frac{1}{2}\int_X\int^B\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\nabla_l^{TX}\widehat{\theta}(F,g_l^F)=\frac{1}{2}\int_{X_\epsilon}\int^B\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\nabla_l^{TX}\widehat{\theta}(F,g_l^F)\\
&=-\frac{1}{2}\int_{X_\epsilon}\int^B\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)(\nabla_l^{TX}\dot{\omega}_l^X)\widehat{\theta}(F,g_l^F)+\frac{1}{2}\int_{\partial X_\epsilon}\int^B\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)\dot{\omega}_l^X\widehat{\theta}(F,g_l^F)\\
&=-\frac{1}{2}\int_X\int^B\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right)(\nabla_l^{TX}\dot{\omega}_l^X)\widehat{\theta}(F,g_l^F)+\int_X\iota_{\partial_l}e(\rho^*TX,\nabla^{TX,\mathrm{tot}})\theta(F,g_l^F).
\end{aligned}
$$

Recall that, in (3.37), we have seen that the integrand on the right hand side of (7.18) vanishes near $\mathrm{Sing}(X)$.

We now define

$$
\begin{aligned}
\widetilde{c}^{\overline{q}}_{p,l} &:= \lim_{t\to 0}\int_{\{0\leqslant r\leqslant \epsilon\}\times L_p} \mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_t^{p,l,\overline{q}}((r,y),(r,y))\right] d\mathrm{vol}_{Z_p} \\
&= \lim_{t\to 0}\frac{1}{2}\int_{\epsilon^{-2}t}^{\infty}\frac{du}{u}\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right] d\mathrm{vol}_{L_p} \\
&= \frac{1}{2}\int_0^{\infty}\frac{du}{u}\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_u^{p,l,\overline{q}}((1,y),(1,y))\right] d\mathrm{vol}_{L_p}.
\end{aligned}
\tag{7.19}
$$

The well-definedness of the term (7.19) follows with analogous arguments as for the well-definedness of the integral in (7.14). Again, to get well-definedness of the integral on the right hand side of (7.19) at $u = 0$ the vanishing of the leading term in the asymptotic expansion (7.17) near $\mathrm{Sing}(X)$ is crucial.

Using (7.17), (7.18), (7.19) and Duhamel's principle and proceeding with Cheeger's strategy as in Section 6.2, we get

$$
\begin{aligned}
&\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t(D_l^{\overline{q}})^2)\right] \\
&= \int_{X_\epsilon}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t(D_l^{\overline{q}})^2)(x,x)\right] d\mathrm{vol}_X \\
&\quad + \sum_{p\in\mathrm{Sing}(X)}\int_{Z_{p,\epsilon}}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_t^{p,l,\overline{q}}((r,y),(r,y))\right] d\mathrm{vol}_{Z_p} + \mathcal{O}(t) \\
&= \frac{1}{\sqrt{t}}\mathrm{rk}(F)\int_X\int^B \dot{\omega}_l^X\exp\left(-\frac{1}{2}\dot{R}_l^{TX}\right) + \int_X \iota_{\partial_l}e(\rho^*TX,\nabla^{TX,\mathrm{tot}})\theta(F,g_l^F) \\
&\quad + \sum_{p\in\mathrm{Sing}(X)}\widetilde{c}^{\overline{q}}_{p,l} + \mathcal{O}(t^{1/2}).
\end{aligned}
\tag{7.20}
$$

Note again that the fact that there is no term in $\log(t)$ appearing in (7.20), is due to the vanishing of the coefficient of t^0 in the pointwise asymptotic expansion (7.17) near $\mathrm{Sing}(X)$. All integrals in (7.20) are well-defined by the proceeding discussion.

□

In the next proposition we adapt a trick explained by Cheeger in [Che79, Theorem 3.10] for manifolds with boundary (and absolute or relative boundary conditions at the boundary) to our situation. We decompose the action of the Laplacian according to the Hodge decomposition for the complex $(\mathcal{C}_{\max/\min}, d_{\max/\min}, \langle\ ,\ \rangle)$ into its action on exact, coexact and harmonic forms

$$
\Delta^{\overline{q}} = \Delta^{\overline{q}}_{ex} + \Delta^{\overline{q}}_{cex} + \Delta^{\overline{q}}_{harm}.
\tag{7.21}
$$

We denote by

$$
\sigma_l := (*_l^F)^{-1}\frac{\partial *_l^F}{\partial l} = *_l^{-1}\frac{\partial *_l}{\partial l} + (g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}.
\tag{7.22}
$$

Proposition 7.3. *Let* $\mathbb{R}\ni l\to(g_l^{TX}, g_l^F)$ *be a family of metrics on* TX, F *as explained at the beginning of Section 7 and such that the spectral gap condition* (7.1) *is satisfied.*

(a) *The following holds:*

$$\begin{aligned}
&\frac{d}{dl}\mathrm{Tr}[\exp(-t\Delta_l^{\overline{q},(k)})]\\
&=-\,t\Big\{\mathrm{Tr}[\Delta_l^{\overline{q},(k+1)}\exp(-t\Delta_{l,ex}^{\overline{q},(k+1)})\sigma_l]-\mathrm{Tr}[\Delta_l^{\overline{q},(k)}\exp(-t\Delta_{l,cex}^{\overline{q},(k)})\sigma_l]\\
&\qquad+\mathrm{Tr}[\Delta_l^{\overline{q},(k)}\exp(-t\Delta_{l,ex}^{\overline{q},(k)})\sigma_l]-\mathrm{Tr}[\Delta_l^{\overline{q},(k-1)}\exp(-t\Delta_{l,cex}^{\overline{q},(k-1)})\sigma_l]\Big\}\\
&=t\frac{d}{dt}\Big\{\mathrm{Tr}[\exp(-t\Delta_{l,ex}^{\overline{q},(k+1)})\sigma_l]-\mathrm{Tr}[\exp(-t\Delta_{l,cex}^{\overline{q},(k)})\sigma_l]\\
&\qquad+\mathrm{Tr}[\exp(-t\Delta_{l,ex}^{\overline{q},(k)})\sigma_l]-\mathrm{Tr}[\exp(-t\Delta_{l,cex}^{\overline{q},(k-1)})\sigma_l]\Big\}.
\end{aligned}\tag{7.23}$$

(b) *The following holds:*

$$\frac{d}{dl}\mathrm{Tr_s}[N\exp(-t\Delta_l^{\overline{q}})]=-t\frac{d}{dt}\mathrm{Tr_s}[\exp(-t\Delta_l^{\overline{q}})\sigma_l].\tag{7.24}$$

Remark 7.4. We have

$$\dot{\Delta}=d\dot{\delta}+\dot{\delta}d,\quad \dot{\delta}=-\sigma\delta+\delta\sigma.\tag{7.25}$$

For a smooth compact manifold, since the operators d and δ_l commute with $\exp(-t\Delta_l^{(k)})$, the first identity in (7.23) is equivalent to

$$\frac{d}{dl}\mathrm{Tr}[\exp(-t\Delta_l^{(k)})]=-t\mathrm{Tr}[\dot{\Delta}_l^{(k)}\exp(-t\Delta_l^{(k)})].\tag{7.26}$$

In the presence of singularities the commutation property only holds on the domain of the Laplacian $\Delta_l^{\overline{q}}$, which however is not invariant under σ_l.

Proof. (a) Denote by $\pi_{1,2}:X\times X\to X$ the two projections. We denote by $\Box_l^{\overline{q}}=\partial_t+\Delta_l^{\overline{q}}$ the heat operator on X. We denote by $P_t^{l,\overline{q}}(x,y)$, $t>0$, the fundamental solution for the heat equation associated to $\Delta_l^{\overline{q},(k)}$. The fundamental solution $P_t^{l,\overline{q}}(x,y)$ is a smooth double form in $\pi_1^*(\Lambda^k(T^*X)\otimes F)\otimes\pi_2^*(\Lambda^k(T^*X)\otimes F)$ satisfying the heat equation in each variable. We denote simply by $P_t^{\overline{q}}(x,y)$, etc. the operators associated to $l=0$.

For $\alpha>0$, we denote by $X_\alpha:=X\setminus\cup_{p\in\mathrm{Sing}(X)}B_\alpha(p)$. In the following we use the following abbreviating notation, for two double forms ω,ω':

$$\langle\omega(x,z),\omega'(z,x)\rangle_\alpha:=\int_{X_\alpha}\omega(x,z)\wedge *_z^F\omega'(z,x).\tag{7.27}$$

In the following all operations are applied to the variable z and correspond to $l=0$. We have

$$\begin{aligned}
&\langle P_\epsilon^{l,\overline{q}}(x,z),P_{t-\epsilon}^{\overline{q}}(z,x)\rangle_\alpha-\langle P_{t-\epsilon}^{l,\overline{q}}(x,z),P_\epsilon^{\overline{q}}(z,x)\rangle_\alpha\\
&=\int_\epsilon^{t-\epsilon}\partial_s(\langle P_{t-s}^{l,\overline{q}}(x,z),P_s^{\overline{q}}(z,x)\rangle_\alpha)ds\\
&=\int_\epsilon^{t-\epsilon}\langle\partial_sP_{t-s}^{l,\overline{q}}(x,z),P_s^{\overline{q}}(z,x)\rangle_\alpha ds+\int_\epsilon^{t-\epsilon}\langle P_{t-s}^{l,\overline{q}}(x,z),\partial_sP_s^{\overline{q}}(z,x)\rangle_\alpha ds\\
&=-\int_\epsilon^{t-\epsilon}\langle\Box^{\overline{q}}P_{t-s}^{l,\overline{q}}(x,z),P_s^{\overline{q}}(z,x)\rangle_\alpha ds+\int_\epsilon^{t-\epsilon}\langle\Delta^{\overline{q}}P_{t-s}^{l,\overline{q}}(x,z),P_s^{\overline{q}}(z,x)\rangle_\alpha ds\\
&\quad+\int_\epsilon^{t-\epsilon}\langle P_{t-s}^{l,\overline{q}}(x,z),\Box^{\overline{q}}P_s^{\overline{q}}(z,x)\rangle_\alpha ds-\int_\epsilon^{t-\epsilon}\langle P_{t-s}^{l,\overline{q}}(x,z),\Delta^{\overline{q}}P_s^{\overline{q}}(z,x)\rangle_\alpha ds.
\end{aligned}\tag{7.28}$$

Applying Stokes' Theorem and using $\square^{\overline{q}} P^{\overline{q}}_t = 0$ in (7.28), we have

$$\begin{aligned}
&\langle P^{l,\overline{q}}_{t-\epsilon}(x,z), P^{\overline{q}}_{\epsilon}(z,x)\rangle_\alpha - \langle P^{l,\overline{q}}_{\epsilon}(x,z), P^{\overline{q}}_{t-\epsilon}(z,x)\rangle_\alpha \\
&\quad = \int_\epsilon^{t-\epsilon} \langle \square^{\overline{q}} P^{l,\overline{q}}_{t-s}(x,z), P^{\overline{q}}_s(z,x)\rangle_\alpha ds \\
&\quad - \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} \delta P^{l,\overline{q}}_{t-s}(x,z) \wedge *^F P^{\overline{q}}_s(z,x) \right\} ds \\
&\quad \pm \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} *^F dP^{l,\overline{q}}_{t-s}(x,z) \wedge P^{\overline{q}}_s(z,x) \right\} ds \\
&\quad \pm \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} *^F P^{l,\overline{q}}_{t-s}(x,z) \wedge \delta P^{\overline{q}}_s(z,x) \right\} ds \\
&\quad - \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} P^{l,\overline{q}}_{t-s}(x,z) \wedge *^F dP^{\overline{q}}_s(z,x) \right\} ds.
\end{aligned} \tag{7.29}$$

We now consider the second boundary integral in (7.29): Let n be odd. We have $dP^{l,\overline{q}}_{t-s}(x,-) \in \mathrm{dom}(D^{l,\overline{q}})$ and $P_s(-,x) \in \mathrm{dom}(D^{\overline{q}})$. Therefore, by (7.6) and (7.7), locally near $p \in \mathrm{Sing}(X)$, we have expansions

$$dP^{l,\overline{q}}_{t-s} = a_l + b_l \wedge dr + \omega_l, \quad P^{\overline{q}}_s = a + b \wedge dr + \omega, \tag{7.30}$$

with a, b, a_l, b_l as in (7.7) and $\omega \in \mathrm{dom}(D_{\min})$, $\omega_l \in \mathrm{dom}(D_{l,\min})$; a_l, b_l, ω_l depending smoothly on the parameter l. For the leading term of $[*^F dP^{l,\overline{q}}_{t-s}(x,z) \wedge P^{\overline{q}}_s(z,x)]_{|\partial X_\alpha}$ we hence have, using (7.7) and (7.30),

$$[*^F(a_l + b_l \wedge dr) \wedge (a + b \wedge dr)]_{|\partial X_\alpha} = \widetilde{*}^F b_l \wedge a = 0, \tag{7.31}$$

where $\widetilde{*}^F$ denotes the Hodge star operator on the link L_p associated to the metrics $g^{TL_p}, g^{F_{L_p}}$.

From (7.3), (7.4), (7.30) and (7.31) we get, for both n even or odd,

$$\int_{\partial X_\alpha} *^F dP^{l,\overline{q}}_{t-s}(x,z) \wedge P^{\overline{q}}_s(z,x) = o(\alpha^{1/2}|\log\alpha|^{1/2}). \tag{7.32}$$

By similar arguments, the third and fourth boundary integral in (7.29) are also $o(\alpha^{1/2}|\log\alpha|^{1/2})$. Note that, since $\mathrm{dom}\,\delta_{\min/\max} \neq \mathrm{dom}\,\delta_{l,\min/\max}$, we can not argue in the same fashion for the first boundary integral in (7.29), but we will treat this term later in (7.36).

Differentiating $\square_l P^{l,\overline{q}}_t = 0$ in l, we get

$$\dot{\square}^{\overline{q}}_l P^{l,\overline{q}}_t + \square^{\overline{q}}_l \dot{P}^{l,\overline{q}}_t = 0. \tag{7.33}$$

Thus differentiating (7.29) in l, setting $l = 0$ and using (7.32) and (7.33), we get

$$\begin{aligned}
&\langle \dot{P}^{\overline{q}}_{t-\epsilon}(x,z), P^{\overline{q}}_{\epsilon}(z,x)\rangle_\alpha - \langle \dot{P}^{\overline{q}}_{\epsilon}(x,z), P^{\overline{q}}_{t-\epsilon}(z,x)\rangle_\alpha + o(\alpha^{1/2}|\log\alpha|^{1/2}) \\
&\quad = -\int_\epsilon^{t-\epsilon} \langle \dot{\square}^{\overline{q}} P^{\overline{q}}_{t-s}(x,z), P^{\overline{q}}_s(z,x)\rangle_\alpha ds - \int_\epsilon^{t-\epsilon} \left\{ \int_{\partial X_\alpha} \delta \dot{P}^{\overline{q}}_{t-s}(x,z) \wedge *^F P^{\overline{q}}_s(z,x) \right\} ds.
\end{aligned} \tag{7.34}$$

Using (7.25) and applying Stokes' Theorem we get from (7.34)

$$
\begin{aligned}
&\langle \dot{P}^{\overline{q}}_{t-\epsilon}(x,z), P^{\overline{q}}_{\epsilon}(z,x)\rangle_\alpha - \langle \dot{P}^{\overline{q}}_{\epsilon}(x,z), P^{\overline{q}}_{t-\epsilon}(z,x)\rangle_\alpha + o(\alpha^{1/2}|\log\alpha|^{1/2})\\
&= -\int_\epsilon^{t-\epsilon}\Big\{\langle \dot{\delta} dP^{\overline{q}}_{t-s}(x,z), P^{\overline{q}}_s(z,x)\rangle_\alpha + \langle \dot{\delta} P^{\overline{q}}_{t-s}(x,z), \delta P^{\overline{q}}_s(z,x)\rangle_\alpha\Big\}\, ds\\
&\quad - \int_\epsilon^{t-\epsilon}\left\{\int_{\partial X_\alpha} (\delta \dot{P}^{\overline{q}}_{t-s}(x,z) + \dot{\delta} P^{\overline{q}}_{t-s}(x,z)) \wedge *^F P^{\overline{q}}_s(z,x)\right\} ds\\
&= \int_\epsilon^{t-\epsilon}\{\langle \sigma\delta dP^{\overline{q}}_{t-s}(x,z), P^{\overline{q}}_s(z,x)\rangle_\alpha - \langle \sigma dP^{\overline{q}}_{t-s}(x,z), dP^{\overline{q}}_s(z,x)\rangle_\alpha\}\, ds\\
&\quad + \int_\epsilon^{t-\epsilon}\{\langle \sigma\delta P^{\overline{q}}_{t-s}(x,z), \delta P^{\overline{q}}_s(z,x)\rangle_\alpha - \langle \sigma P^{\overline{q}}_{t-s}(x,z), d\delta P^{\overline{q}}_s(z,x)\rangle_\alpha\}\, ds\\
&\quad - \int_\epsilon^{t-\epsilon}\left\{\int_{\partial X_\alpha} (\delta \dot{P}^{\overline{q}}_{t-s}(x,z) + \dot{\delta} P^{\overline{q}}_{t-s}(x,z)) \wedge *^F P^{\overline{q}}_s(z,x)\right\} ds\\
&\quad \pm \int_\epsilon^{t-\epsilon}\left\{\int_{\partial X_\alpha} *^F \sigma dP^{\overline{q}}_{t-s}(x,z) \wedge P^{\overline{q}}_s(z,x)\right\} ds\\
&\quad \pm \int_\epsilon^{t-\epsilon}\left\{\int_{\partial X_\alpha} *^F \sigma P^{\overline{q}}_{t-s}(x,z) \wedge \delta P^{\overline{q}}_s(z,x)\right\} ds
\end{aligned}
\tag{7.35}
$$

We now treat the first boundary integral on the right hand side of (7.35): Since $\delta_l P^{l,\overline{q}}_{t-s} \in \mathrm{dom}(D^{\overline{q}}_l)$ and arguing as in (7.30)-(7.32), we have

$$\int_{\partial X_\alpha} \delta_l P^{l,\overline{q}}_{t-s}(x,z) \wedge *^F P^{\overline{q}}_s(z,x) = o(\alpha^{1/2}|\log\alpha|^{1/2}). \tag{7.36}$$

Differentiating (7.36) and setting $l = 0$, we get

$$\int_{\partial X_\alpha} (\delta \dot{P}^{\overline{q}}_{t-s}(x,z) + \dot{\delta} P^{\overline{q}}_{t-s}(x,z)) \wedge *^F P^{\overline{q}}_s(z,x) = o(\alpha^{1/2}|\log\alpha|^{1/2}). \tag{7.37}$$

We now treat the second boundary integral in (7.35): Let $n = 2\nu + 1$ be odd. Since $dP^{\overline{q}}_{t-s}, P^{\overline{q}}_s \in \mathrm{dom}(D^{\overline{q}})$ we have, locally near $p \in \mathrm{Sing}(X)$,

$$P^{\overline{q}}_s - (a + b \wedge dr),\ dP^{\overline{q}}_{t-s} - (a' + b' \wedge dr) \in \mathrm{dom}(D_{\min}). \tag{7.38}$$

with a, a', b, b' as in (7.7). By the assumption on the metrics explained at the beginning of Section 7, the operator σ is an operator on the link (not depending on r). Hence from (7.3), (7.38) we get that

$$\|U^{-1}(\sigma dP^{\overline{q}}_{t-s} - (f(a') + f(b') \wedge dr))\|_{L^2(\Lambda(TL_p)\otimes F_{L_p})} = o(r^{1/2}|\log r|^{1/2}), \tag{7.39}$$

where $f : \mathcal{H}^\nu(L_p, F_{L_p}) \to \Omega^\nu(L_p, F_{L_p})$ is a $\mathbb{C}$-linear map. Using (7.7), (7.38), (7.39) the leading term in the expansion of $[*^F \sigma dP^{\overline{q}}_{t-s}(x,z) \wedge P^{\overline{q}}_s(z,x)]_{|\partial X_\alpha}$ is

$$[*^F(f(a') + f(b') \wedge dr) \wedge (a + b \wedge dr)]_{|\partial X_\alpha} = \tilde{*}^F f(b') \wedge a = 0. \tag{7.40}$$

Hence from (7.3), (7.4), (7.38), (7.39) and (7.40) we get, for both n even or odd,

$$\int_{\partial X_\alpha} *^F \sigma dP^{\overline{q}}_{t-s}(x,z) \wedge P^{\overline{q}}_s(z,x) = o(\alpha^{1/2}|\log\alpha|^{1/2}). \tag{7.41}$$

We can argue similarly for the third boundary term in (7.35).

Using (7.37) and (7.41), by taking the limit $\alpha \to 0$ in (7.35) we get:

$$\begin{aligned}
&\langle \dot{P}^{\overline{q}}_{t-\epsilon}(x,z), P^{\overline{q}}_{\epsilon}(z,x)\rangle - \langle \dot{P}^{\overline{q}}_{\epsilon}(x,z), P^{\overline{q}}_{t-\epsilon}(z,x)\rangle \\
&\quad = \int_{\epsilon}^{t-\epsilon} \left\{ \langle \sigma\delta d P^{\overline{q}}_{t-s}(x,z), P^{\overline{q}}_{s}(z,x)\rangle - \langle \sigma d P^{\overline{q}}_{t-s}(x,z), dP^{\overline{q}}_{s}(z,x)\rangle \right\} ds \\
&\qquad + \int_{\epsilon}^{t-\epsilon} \left\{ \langle \sigma\delta P^{\overline{q}}_{t-s}(x,z), \delta P^{\overline{q}}_{s}(z,x)\rangle - \langle \sigma P^{\overline{q}}_{t-s}(x,z), d\delta P^{\overline{q}}_{s}(z,x)\rangle \right\} ds.
\end{aligned} \tag{7.42}$$

Taking the trace with respect to x and the limit $\epsilon \to 0$ on the left hand side of (7.42), and using the semi-group property for $\exp(-t\Delta^{\overline{q}})$, we get the left hand side of (7.23). On the right hand side of (7.42) we do reverse the order of integration (w.r.t. x and z) and take the limit $\epsilon \to 0$; we get the right hand side of (7.23).

(b) The statement follows from (a) by taking the alternating weighted sum. □

Proof of Theorem II: Let $\mathbb{R} \ni l \to (g^{TX}_l, g^F_l)$ be a family of metrics on TX, F as explained at the beginning of Section 7 and such that the spectral gap condition (7.1) is satisfied. Using Proposition 7.3 (b) we can proceed as in the smooth situation (see [BGS88, (1.114)–(1.122)]) to get that, the variation

$$\partial_l \log \left(\left(\| \quad \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F),l} \right)^2 \right) \tag{7.43}$$

is given by the coefficient of t^0 in the asymptotic expansion for $t \searrow 0$ of

$$\mathrm{Tr_s}\left[\left(*_l^{-1} \frac{\partial *_l}{\partial l} + (g^F_l)^{-1} \frac{\partial g^F_l}{\partial l} \right) \exp(-(D^{\overline{q}}_l)^2) \right]. \tag{7.44}$$

The claim of Theorem II then follows from Theorem 7.1. □

Remark 7.5. Let X be an even dimensional oriented space with isolated conical singularities and (F, ∇^F, g^F) a unitary flat vector bundle on X. It has been proved in [Dar87], by the usual Poincaré duality argument, that in this case the Ray-Singer torsion is trivial. Let $\mathbb{R} \ni l \to g^{TX}_l$ be a family of conical metrics on TX. Then, clearly by Dar's result

$$\partial_l \log \left(\left(\| \quad \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F),l} \right)^2 \right) = 0. \tag{7.45}$$

The result (7.45) can be recovered using Theorem II, since in this case, again by a duality argument,

$$\mathrm{Tr_s}\left[*_l^{-1} \frac{\partial *_l}{\partial l} \exp(-(D^{\overline{q}}_l)^2) \right] = 0. \tag{7.46}$$

7.3. **Anomaly formula for the Bismut-Zhang metric** $\| \quad \|^{Y,g^{TX},g^F}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$. The aim of this section is the study of anomaly formulas for the Bismut-Zhang metric (see Theorem 7.8). The next two theorems give anomaly formulas for the metric $\| \quad \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(cL_p,L_p,F)}$, $p \in \mathrm{Sing}(X)$, which has been introduced in Definition 2.6 and is the contribution of the singular points of X to the Bismut-Zhang metric.

Theorem 7.6. *Let $p \in \mathrm{Sing}(X)$. Let $l \in \mathbb{R} \to (g^{TZ_p}, g^F_l)$ be a family of metrics on TZ_p, F_{Z_p} as explained at the beginning of Section 7, $g^{TZ_p} = dr^2 + r^2 g^{TL_p}$ being a fixed conical metric.*

We assume that the spectral gap condition (7.1) *is satisfied. Then*

(7.47)
$$\begin{aligned}\partial_l \log\left(\left(\|\ \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(cL_p,L_p,F),l}\right)^2\right) &= \lim_{t\to 0}\mathrm{Tr_s}\left[(g^F_l)^{-1}\frac{\partial g^F_l}{\partial l}\exp(-t\mathbf{\Delta}^{\overline{q}}_1)\right]\\ &= c^{\overline{q}}_{p,l} - \int_{L_p}\mathrm{Tr}\left[(g^F_l)^{-1}\frac{\partial g^F_l}{\partial l}\right]\wedge \eta_p,\end{aligned}$$

where η_p *(resp.* $c^{\overline{q}}_{p,l}$*) is as defined in* (3.21) *(resp.* (7.12)*).*

Proof. The operator $(g^F_l)^{-1}\frac{\partial g^F_l}{\partial l}$ on the infinite cone Z_p does not depend on the radial coordinate. Proceeding as in the proof of Proposition 6.3 and using Proposition 3.1 (b), we get as $t \searrow 0$,

(7.48)
$$\begin{aligned}&\mathrm{Tr_s}\left[(g^F_l)^{-1}\frac{\partial g^F_l}{\partial l}\exp(-t\mathbf{\Delta}^{p,\overline{q}}_1)\right]\\ &\xrightarrow[t\searrow 0]{} c^{\overline{q}}_{p,l} + \int_{Z_p}\mathrm{Tr}\left[(g^F_l)^{-1}\frac{\partial g^F_l}{\partial l}\right]\int^{B,p}\exp(-B^p_1) = c^{\overline{q}}_{p,l} - \int_{L_p}\mathrm{Tr}\left[(g^F_l)^{-1}\frac{\partial g^F_l}{\partial l}\right]\wedge\eta_p.\end{aligned}$$

□

Theorem 7.7. *Let* $p\in\mathrm{Sing}(X)$*. Let* $l\in\mathbb{R}\to(g^{TZ_p}_l = dr^2+r^2g^{TL_p}_l, g^F)$ *be a family of conical metrics on the infinite cone* Z_p*; the metric* g^F *on the flat bundle* F_{Z_p} *is fixed. We assume that the spectral gap condition* (7.1) *holds. Then*

(7.49)
$$\partial_l \log\left(\left(\|\ \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(cL_p,L_p,F),l}\right)^2\right) = \widetilde{c}^{\overline{q}}_{p,l} + \int_{L_p}\theta(F,\nabla^F)\wedge\iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}),$$

with $\widetilde{c}^{\overline{q}}_{p,l}$ *as defined in* (7.12).

Proof. Step 1: Proceeding as in the proof of Theorem II we can prove that

(7.50)
$$\partial_l \log\left(\left(\|\ \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(cL_p,L_p,F),l}\right)^2\right)$$

is given by the coefficient of t^0 in the asymptotic expansion of $\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t\mathbf{\Delta}^{p,l,\overline{q}}_1)\right]$ as $t\searrow 0$. In Step 2-Step 6 we compute this asymptotic expansion proceeding as in the proof of Proposition 6.3.

Step 2: Splitting the integral: Let $T>0$ be fixed. We denote by $U^{p,l,\overline{q}}_{t,T}(x,x')$, $x,x'\in Z_p$, the heat kernel of the operator $t^2\mathbf{\Delta}^{p,l,\overline{q}}_{T/t}$ w.r.t. $d\mathrm{vol}_{Z_p}$. Let $\varphi:\mathbb{R}_{\geqslant 0}\to[0,1]$ be a cut-off function, with $\mathrm{supp}\,\varphi\subset[0,1]$ and $\varphi\equiv 1$ in $[0,1/2]$. Using the scaling properties of the model Witten Laplacian (4.15) and the fact that the operator $*_l^{-1}\frac{\partial *_l}{\partial l}$ on the infinite cone

Z_p does not depend on the radial coordinate, we write

$$\begin{aligned}
&\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t\mathbf{\Delta}_T^{p,l,\overline{q}})\right] = \int_{Z_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q_{t,T}^{p,l,\overline{q}}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\
&= \int_{Z_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U_{t,T}^{p,l,\overline{q}}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\
&= \int_{Z_p}\varphi(r)\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U_{t,T}^{p,l,\overline{q}}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\
&\quad + \int_{Z_p}(1-\varphi(r))\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U_{t,T}^{p,l,\overline{q}}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}.
\end{aligned} \tag{7.51}$$

<u>Step 3:</u> *We establish a pointwise asymptotic expansion, which will be used in Step 4 and* <u>Step 5:</u> We use local index techniques as in the proof of Proposition 6.3; *i.e.* we apply to $t^2\mathbf{\Delta}_{T/t}^{p,l,\overline{q}}$ the (local) scaling $x \to tx$ and replace the Clifford variables $c(e_k), \widehat{c}(e_k)$ by $c_t(e_k), \widehat{c}_t(e_k)$. We denote by C_t the operator we get from $*_l^{-1}\frac{\partial *_l}{\partial l}$ by the above scaling. We have

$$tC_t \xrightarrow{t\searrow 0} \dot{\omega}_l^{Z_p}, \tag{7.52}$$

where $\dot{\omega}_l^{Z_p}$ has been defined in (7.10). Using (7.52) and proceeding as in [BZ92, Theorem 13.4], we get the following pointwise asymptotics as $t \searrow 0$, uniformly on compact sets and for $T \in [0, 1/t]$,

(7.53)

$$\begin{aligned}
&\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U_{t,T}^{p,l,\overline{q}}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\
&= \frac{\mathrm{rk}(F)}{t}\int^{B,p}\dot{\omega}_l^{Z_p}\exp(-B_{T^2}^p) + \int^{B,p}\dot{\omega}_l^{Z_p}\left(\left(\frac{1}{2}\nabla_l^{TZ_p} + \iota_{\widehat{T\nabla f^p}}\right)\widehat{\theta}(F,g^F)\right)\exp(-B_{T^2}^p) + \mathcal{O}(t)\\
&=: \frac{1}{t}a_{-1}(T,r)dr + a_0(T,r)dr + \mathcal{O}(t).
\end{aligned}$$

Recall that the form $e_T(\rho^*TZ_p, \nabla^{TZ_p,\mathrm{tot}})$ has been defined in (3.29). From (7.10) and $\nabla_{g_l^{TZ_p}}f^p = -r\partial_r$, we have $\iota_{\widehat{T\nabla f^p}}\dot{\omega}_l^{Z_p} = 0$. Hence, using [BGV04, Proposition 1.50], [BZ92, Theorem 3.2], [BZ92, Theorem 3.13] and proceeding as in [BZ92, (4.74)-(4.86)] we get

(7.54)

$$\begin{aligned}
&\int^{B,p}\dot{\omega}_l^{Z_p}\left(\left(\frac{1}{2}\nabla_l^{TZ_p} + \iota_{\widehat{T\nabla f^p}}\right)\widehat{\theta}(F,g^F)\right)\exp(-B_{T^2}^p)\\
&= \frac{1}{2}d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B_{T^2}^p)\right) - \int^{B,p}\left(\left(\frac{1}{2}\nabla_l^{TZ_p} + \iota_{\widehat{T\nabla f^p}}\right)\dot{\omega}_l^{Z_p}\right)\widehat{\theta}(F,g^F)\exp(-B_{T^2}^p)\\
&= \frac{1}{2}d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B_{T^2}^p)\right) - \int^{B,p}\left(\frac{1}{2}\nabla_l^{TZ_p}\dot{\omega}_l^{Z_p}\right)\widehat{\theta}(F,g^F)\exp(-B_{T^2}^p)\\
&= \frac{1}{2}d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B_{T^2}^p)\right) - \int^{B,p}\theta(F,g^F)\left(-\frac{1}{2}\widehat{\nabla_l^{TZ_p}\omega_l^{Z_p}}\right)\exp(-B_{T^2}^p)\\
&= \frac{1}{2}d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B_{T^2}^p)\right) - \theta(F,g^F)\iota_{\partial l}e_{T^2}(\rho^*TZ_p, \nabla^{TZ_p,\mathrm{tot}}).
\end{aligned}$$

The form $\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})$ is an $(n-1)$-form containing e^r. Hence

$$d\left(\int^{B,p}\dot{\omega}_l^{Z_p}\widehat{\theta}(F,g^F)\exp(-B^p_{T^2})\right)=e^r\wedge d_{L_p}\beta(r,T),\tag{7.55}$$

for an $(n-2)$-form β on L_p, which depends on the radial coordinate.

The coefficients in the asymptotic expansion in (7.53) have exponential decay as $r\to\infty$ and enjoy scaling properties analogous to those described in (6.24), (6.25).

Step 4: We study the first integral on the right hand side of (7.51) *as* $t\searrow 0$: Let $T>0$ be fixed. Using (6.39) and the fact, that the operator $*_l^{-1}\frac{\partial *_l}{\partial l}$ does not depend on the radial coordinate, we get for the first integral on the right hand side of (7.51):

$$\begin{aligned}&\int_{Z_p}\varphi(r)\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U^{p,l,\overline{q}}_{t,T}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\&=\int_0^\infty\frac{dr}{r}\varphi(r)\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U^{p,l,\overline{q}}_{t/r,Tr}\left((1,y),(1,y)\right)\right]d\mathrm{vol}_{L_p}\\&=z\int_0^\infty\varphi(r)\sigma(r,rz)dz,\end{aligned}\tag{7.56}$$

where $z:=t^{-1}$ and

$$\sigma(r,\xi):=\frac{1}{\xi}\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U^{p,l,\overline{q}}_{\xi^{-1},Tr}\left((1,y),(1,y)\right)\right]d\mathrm{vol}_{L_p}.\tag{7.57}$$

We have, using (7.19), $\sigma(0,\xi)\in L^1(\mathbb{R}_{\geqslant 0})$,

$$\begin{aligned}\int_0^\infty\sigma(0,\xi)d\xi&=\int_0^\infty\frac{d\xi}{\xi}\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U^{p,l,\overline{q}}_{\xi^{-1},0}\left((1,y),(1,y)\right)\right]d\mathrm{vol}_{L_p}\\&=\int_0^\infty\frac{d\xi}{\xi}\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q^{p,l,\overline{q}}_{\xi^{-2},0}\left((1,y),(1,y)\right)\right]d\mathrm{vol}_{L_p}\\&=\frac{1}{2}\int_0^\infty\frac{du}{u}\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}Q^{p,l,\overline{q}}_{u}\left((1,y),(1,y)\right)\right]d\mathrm{vol}_{L_p}\\&=\widetilde{c}^{\overline{q}}_{p,l}.\end{aligned}\tag{7.58}$$

We define

$$\sigma_0(r):=\int_{L_p}a_{-1}(Tr,1),\quad\sigma_{-1}(r):=\int_{L_p}a_0(Tr,1),\tag{7.59}$$

with a_0,a_{-1} as defined in (7.53).

From the discussion in Step 3, we have $\sigma_0(r),r^{-1}\sigma_{-1}(r)\in L^1(\mathbb{R}_{\geqslant 0})$. Moreover

$$\int_0^\infty\varphi(r)\sigma_0(r)dr=\mathrm{rk}(F)\int_{Z_p}\varphi(r)\int^{B,p}\dot{\omega}_l^{Z_p}\exp(-B^p_{T^2}),\tag{7.60}$$

and, using also (7.54),

$$\int_0^\infty\varphi(r)r^{-1}\sigma_{-1}(r)dr=-\int_{Z_p}\varphi(r)\theta(F,g^F)\iota_{\partial l}e_{T^2}(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}})\text{ and }\sigma_{-1}(0)=0.\tag{7.61}$$

From (7.53), we get, as $\xi\to\infty$ and $0\leqslant r\leqslant 1$, $j=0,1$,

$$\left|\partial_r^j\left[\sigma(r,\xi)-\sigma_0(r)-\xi^{-1}\sigma_{-1}(r)\right]\right|=\mathcal{O}(\xi^{-2}).\tag{7.62}$$

The operator $*_l^{-1}\frac{\partial *_l}{\partial l}$ does not depend on the radial coordinate, hence as in (6.34) we can prove the integrability condition for $\sigma(r,\xi)$. The asymptions in SAL [Les97, Theorem 2.1.11] are hence satisfied. Applying SAL, as $z\to\infty$,

(7.63)
$$z\int_0^\infty \varphi(r)\sigma(r,rz)dz$$
$$=\int_0^\infty \sigma(0,\xi)d\xi+\int_0^\infty \varphi(r)\sigma_0(r)dr\,z+\int_0^\infty \varphi(r)\sigma_{-1}(r)r^{-1}dr+\sigma_{-1}(0)\log z+\mathcal{O}(z^{-1/2}),$$

where we have also used that, as discussed before, the infinite integrals appearing in (7.63) exist.

From now on, set $T=1$. Using (7.58), (7.60), (7.61) and (7.63), we get for the first integral on the right hand side of (7.51) as $t\searrow 0$,

(7.64)
$$\int_{Z_p}\varphi(r)\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U_{t,1}^{p,l,\overline{q}}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}=\frac{\mathrm{rk}(F)}{t}\int_{Z_p}\varphi(r)\int^{B,p}\dot{\omega}_l^{Z_p}\exp(-B_1^p)$$
$$+\widetilde{c}_{p,l}^{\overline{q}}-\int_{Z_p}\varphi(r)\theta(F,g^F)\iota_{\partial l}e_1(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}})+\mathcal{O}(t^{1/2}).$$

<u>Step 5:</u> *We study the second integral on the right hand side of* (7.51) *as* $t\searrow 0$: Using the scaling properties of the model Witten Laplacian and the Berezin integrals, the asymptotic expansion (7.53), as well as the change of variables $u=t/r$, we get

(7.65)
$$\begin{aligned}&\int_{Z_p}(1-\varphi(r))\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U_{t,1}^{p,l,\overline{q}}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\&\quad=\int_{\{r\geqslant 1/2\}\times L_p}(1-\varphi(r))\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U_{t,1}^{p,l,\overline{q}}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\&\quad=\int_0^{2t}\frac{du}{u}(1-\varphi(t/u))\int_{L_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U_{u,\frac{t}{u}}^{p,l,\overline{q}}\big((1,y),(1,y)\big)\right]d\mathrm{vol}_{L_p}\\&\quad=\int_0^{2t}\frac{du}{u}(1-\varphi(t/u))\left\{u^{-1}\int_{L_p}a_{-1}\left(\frac{t}{u},1\right)+\int_{L_p}a_0\left(\frac{t}{u},1\right)+\mathcal{O}(u)\right\}\\&\quad=\frac{\mathrm{rk}(F)}{t}\int_{Z_p}(1-\varphi(r))\int^{B,p}\dot{\omega}_l^{Z_p}\exp(-B_1^p)\\&\qquad-\int_{Z_p}(1-\varphi(r))\theta(F,g^F)\iota_{\partial l}e_1(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}})+\mathcal{O}(t).\end{aligned}$$

Step 6: We finish the proof: Using Proposition 3.2 (b), (7.51), (7.64) and (7.65) we get as $t \searrow 0$,

(7.66)
$$\begin{aligned}
&\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t\mathbf{\Delta}_1^{p,l,\overline{q}})\right] = \int_{Z_p}\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}U_{t,1}^{p,l,\overline{q}}\big((r,y),(r,y)\big)\right]d\mathrm{vol}_{Z_p}\\
&=\frac{\mathrm{rk}(F)}{t}\int_{Z_p}\int^{B,p}\dot{\omega}_l^{Z_p}\exp(-B_1^p)+\widetilde{c}_{p,l}^{\overline{q}}-\int_{Z_p}\theta(F,g^F)\iota_{\partial l}e_1(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}})+\mathcal{O}(t^{1/2})\\
&=\frac{\mathrm{rk}(F)}{t}\int_{Z_p}\int^{B,p}\dot{\omega}_l^{Z_p}\exp(-B_1^p)+\widetilde{c}_{p,l}^{\overline{q}}+\int_{L_p}\theta(F,g^F)\wedge\iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}})\\
&\quad+\mathcal{O}(t^{1/2}).
\end{aligned}$$

From (7.66) we get that the coefficient of t^0 in the asymptotic expansion as $t \searrow 0$ of $\mathrm{Tr_s}\left[*_l^{-1}\frac{\partial *_l}{\partial l}\exp(-t\mathbf{\Delta}_1^{p,l,\overline{q}})\right]$, is given by

$$\widetilde{c}_{p,l}^{\overline{q}}+\int_{L_p}\theta(F,g^F)\wedge\iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}). \tag{7.67}$$

The claim of the theorem follows putting together Step 1 and (7.67).

□

Let $l\in\mathbb{R}\to(g_l^{TX},g_l^F)$ be a family of metrics on TX, F as at the beginning of Section 7. We denote by $\|\ \|_{\det IH_{\overline{q}}^\bullet(X,F)}^{Y,g_l^{TX},g_l^F}$ the associated Bismut-Zhang metric.

Theorem 7.8. *Let $\mathbb{R}\ni l\to(g_l^{TX},g_l^F)$ be a family of metrics on TX, F as explained at the beginning of Section 7 and satisfying the spectral gap condition* (7.1). *Then*

(7.68)
$$\begin{aligned}
\partial_l\log\left(\big(\|\ \|_{\det IH_{\overline{q}}^\bullet(X,F)}^{Y,g_l^{TX},g_l^F}\big)^2\right) &= \sum_{p\in\mathrm{Crit}(f_{sm})}(-1)^{\mathrm{ind}(p)}\partial_l\log(\|\ \|^2_{\det F_p,l})\\
&+\sum_{p\in\mathrm{Sing}(X)}\left(-\int_{L_p}\mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]\wedge\eta_{p,l}+c_{p,l}^{\overline{q}}+\widetilde{c}_{p,l}^{\overline{q}}\right)\\
&+\sum_{p\in\mathrm{Sing}(X)}\int_{L_p}\theta(F,g_l^F)\wedge\iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^*TZ_p,\nabla^{TZ_p,\mathrm{tot}}),
\end{aligned}$$

where $\eta_{p,l}$ (resp. $c_{p,l}^{\overline{q}}$ and $\widetilde{c}_{p,l}^{\overline{q}}$) is as defined in (3.21) *(resp.* (7.12)*).*

Proof. The proof follows from the definition of the Bismut-Zhang metric (Definition 2.8), and Theorems 7.6 and 7.7. □

Remark 7.9. Putting together Theorems II and 7.8, we have

$$\begin{aligned}\partial_l \log\left(\frac{\|\ \|^{RS}_{\det IH^\bullet_{\bar q}(X,F),l}}{\|\ \|^{Y,g_l^{TX},g_l^F}_{\det IH^\bullet_{\bar q}(X,F)}}\right)^2 &= \int_X \mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right] e(TX,\nabla_l^{TX}) \\ &+ \int_X \iota_{\partial_l} e(\rho^* TX, \nabla^{TX,\mathrm{tot}})\theta(F,g_l^F) \\ &- \sum_{p\in \mathrm{Crit}(f_{sm})} (-1)^{\mathrm{ind}(p)}\partial_l \log(\|\ \|^2_{\det F_p,l}) + \sum_{p\in\mathrm{Sing}(X)}\int_{L_p}\mathrm{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right]\wedge \eta_{p,l} \\ &- \sum_{p\in\mathrm{Sing}(X)}\int_{L_p}\theta(F,g_l^F)\wedge \iota_{\partial_l}(\widetilde{\nabla f^p})^*\Psi(\rho^* TZ_p,\nabla^{TZ_p,\mathrm{tot}}).\end{aligned} \tag{7.69}$$

Integrating (7.69) over $l\in[0,1]$ and comparing with (3.49) we have

$$\begin{aligned}&\log\left(\frac{\|\ \|'^{RS}_{\det IH^\bullet_{\bar q}(X,F)}}{\|\ \|^{Y,g'^{TX},g'^F}_{\det IH^\bullet_{\bar q}(X,F)}}\right)^2 - \log\left(\frac{\|\ \|^{RS}_{\det IH^\bullet_{\bar q}(X,F)}}{\|\ \|^{Y,g^{TX},g^F}_{\det IH^\bullet_{\bar q}(X,F)}}\right)^2 = \\ &= -\int_X \theta(F,g'^F)(\nabla' f)^*\Psi(TX,\nabla'^{TX}) + \int_X \theta(F,g^F)(\nabla f)^*\Psi(TX,\nabla^{TX}),\end{aligned} \tag{7.70}$$

which shows that the variations of the three terms in the Bismut-Zhang formula w.r.t. the two metrics (g^{TX},g^F) are consistent with the Bismut-Zhang formula.

The result in [BZ92, Theorem 16.1] can also be generalised to this setting: Let us fix a flat Hermitian vector bundle (F,∇^F,g^F). Let (f,g_0^{TX}), $(f',g_0'^{TX})$ be anti-radial Morse-Smale pairs, we assume that the conical metrics g_0^{TX}, $g_0'^{TX}$ coincide in an open neighbourhood of $\mathrm{Sing}(X)$. We denote by $Y=\nabla_{g_0^{TX}} f$, $Y'=\nabla_{g_0'^{TX}} f'$ the gradient vector fields. Let $\|\ \|^{Y',g_0'^{TX},g^F}_{\det IH^\bullet_{\bar q}(X,F)}$, $\|\ \|^{Y,g_0^{TX},g^F}_{\det IH^\bullet_{\bar q}(X,F)}$ denote the associated Bismut-Zhang metrics on $\det IH^\bullet_{\bar q}(X,F)$.

Let g^{TX} be a further arbitray conical Riemannian metric on X, which does also coincide with the conical metrics g_0^{TX}, $g_0'^{TX}$ in an open neighbourhood of $\mathrm{Sing}(X)$; we denote by ∇^{TX} the Levi-Civita connection of (TX,g^{TX}).

Theorem 7.10. *In the situation described above we have*

$$\log\left(\frac{\|\ \|^{Y',g_0'^{TX},g^F}_{\det IH^\bullet_{\bar q}(X,F)}}{\|\ \|^{Y,g_0^{TX},g^F}_{\det IH^\bullet_{\bar q}(X,F)}}\right)^2 = \int_X \theta(F,g^F)(Y')^*\Psi(TX,\nabla^{TX}) - \int_X\theta(F,g^F)Y^*\Psi(TX,\nabla^{TX}). \tag{7.71}$$

Proof. The theorem is a consequence of the Bismut-Zhang theorem, Theorem I. It can be proved independently by an easy generalisation of [BZ92, Section XVI]. ☐

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