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AUTOMORPHISMS OF THE MAPPING CLASS GROUP OF A REAL PROJECTIVE PLANE WITH n−MARKED POINTS

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ABSTRACT. Let $\mathrm{Mod}(N)$ be the mapping class group of a real projective plane N with $n \geq 4$ marked points. We show that the outer automorphism group of $\mathrm{Mod}(N)$ is trivial.

1. Introduction

Let N be a connected, compact nonorientable surface of genus $g \geq 1$ with n marked points. The mapping class group $\mathrm{Mod}(N)$ of N is the group of isotopy classes of all diffeomorphisms of $N \to N$ which take the set of marked points to itself. Atalan and Szepietowski [5] proved that the outer automorphism group $\mathrm{Out}(\mathrm{Mod}(N))$ is trivial, for $g \geq 5$ with $n \geq 0$ marked points. In this note, we will show that this is also true for the genus $g = 1$ and $n \geq 4$ marked points.

Theorem 1.1. *The outer automorphism group* $\mathrm{Out}(\mathrm{Mod}(N))$ *is trivial for the genus* $g = 1$ *and* $n \geq 4$.

Ivanov [10] proved the analogous theorem for the mapping class group of an orientable surface S of genus $g \geq 3$, each automorphism of $\mathrm{Mod}(S)$ is induced by a diffeomorphism of S, not necessarily orientation preserving. Ivanov and McCarthy [12] proved that any injective endomorphism of $\mathrm{Mod}(S)$ must be an isomorphism. Bell and Margalit [6] prove the analog of the theorem of Ivanov and McCarthy for genus zero surfaces. Castel [8] and Aramayona-Souto [1], any nontrivial endomorphism of $\mathrm{Mod}(S)$ must be an isomorphism. Finally, Irmak and Paris [9] proved that if G is a finite index subgroup of $\mathrm{Mod}(N)$ and $\phi : G \to \mathrm{Mod}(N)$ is an injective homomorphism, then there is $f \in \mathrm{Mod}(N)$ such that $\phi(g) = fgf^{-1}$ for all $g \in G$, where N is a nonorientable surface of the genus $g \geq 5$ with $n \geq 0$.

In this note, our strategy to prove Theorem 1.1 is that an automorphism Φ of $\mathrm{Mod}(N)$ is half-twist preserving and is induced by a diffeomorphism of N. Similarly as in [6], the main key of our proof of Theorem 1.1 is a half-twist preserving from which we obtain that any automorphism of $\mathrm{Mod}(N)$

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maps half-twists on half-twists. However, unlike for the sphere, $\mathrm{Mod}(N)$ is not generated by half-twists (see Subsection 2.1). Finally, we finish the proof of Theorem 1.1 by using the technique of the work in [2].

2. Preliminaries

Hereafter throughout the paper, by N we denote a real projective plane with $n \geq 4$ marked points. A curve a in N is an unoriented simple closed curve. If a regular neighborhood of a is an annulus (respectively, a Möbius strip), a is called two-sided (respectively, one-sided). We say that a is trivial if it bounds either a disc with at most one marked point or a Möbius strip. Otherwise, we call it nontrivial. Let N_a be the surface obtained by cutting N along the curve a. A curve a is nonseparating if N_a is connected and separating otherwise. We note that all two-sided curves are separating in this work. If a two-sided curve bounds two marked points, then we call it a 2-separating curve. By an arc on N is an embedded arc connecting two different marked points. We note that there is a one-to-one correspondence between isotopy classes of 2-separating curves and the isotopy classes of arcs joining two different marked points.

We denote by σ_a the half-twist defined to be the isotopy class of the diffeomorphism of N interchanging two marked points along an arc a as shown on Figure 1, and equal to the identity outside a disc containing these marked points. Here, σ_a^2 is equal to the Dehn twist t_a about 2-separating curve a. We notice that t_a is not possible to recognize between right- and left-handed twists on N, so, we should specify the direction of t_a for every two-sided curve a.

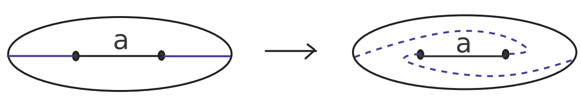

Figure 1. Half-twist about an arc a

If a and b are two 2-separating curves such that the corresponding arcs a' and b' can be chosen disjoint with exactly one common endpoint, then we say that a and b form a simple pair of 2-separating curves and denote it by $\langle a, b\rangle$ (see Figure 2(a)). Similarly, we say that $\langle a', b'\rangle$ is a simple pair of arcs. We also call their isotopy classes simple pairs.

Let $a_1', a_2', \ldots, a_k'$ be embedded pairwise disjoint arcs, z_i and z_{i+1} the end points of a_i', with $z_i \neq z_j$ for $i \neq j$, $0 \leq i, j \leq k$. We say that $\langle a_1', a_2', \ldots, a_k'\rangle$ is a chain of arcs. Similarly, the corresponding 2-separating curves $a_1, a_2, \ldots, a_k$ is a chain of curves (see Figure 2 (b)).

Throughout the paper, we denote curves, arcs, and their isotopy classes by the same letter.

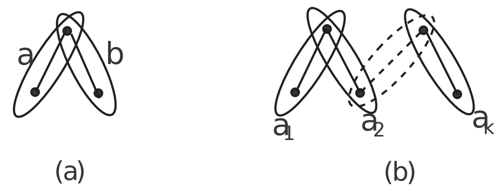

FIGURE 2

2.1. **Generators of** Mod(N).

Theorem 2.1. Mod(N) *is generated by* $\{v_1, \sigma_{a_1}, \sigma_{a_2}, \cdots, \sigma_{a_{n-1}}\}$.

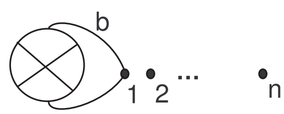

FIGURE 3. A real projective plane with n marked points

In Theorem 2.1 proved by Korkmaz in [13], v_1 is the puncture slide obtained by sliding the first marked point once along the one-sided loop b at the marked point (see Figure 3).

2.2. **Canonical reduction systems and Pure Subgroups.** Let $\mathcal{C}$ be a collection of isotopy classes of disjoint curves on N. Let $N_{\mathcal{C}}$ denote the surface obtained by cutting N along the representatives for $\mathcal{C}$. Let f be a mapping class fixing a collection $\mathcal{C}$ of isotopy classes of disjoint curves on N. Then there is a representative diffeomorphism for f fixing a set of representatives for $\mathcal{C}$. This gives a well-defined element $f_{\mathcal{C}}$ of Mod($N_{\mathcal{C}}$). It is called the reduction of f along $\mathcal{C}$.

We say that a mapping class f is pure if it has a reduction $f_{\mathcal{C}}$ inducing the trivial permutation on the components of $N_{\mathcal{C}}$, acting as the identity on the boundary, (fixing marked points), and restricting to either the identity or a pseudo-Anosov map on every such component. (see [11], [2],[5].) By Corollary 1.8 in [11], there is a subgroup Γ of finite index in the mapping class group of the surface consisting entirely of pure elements. For the case of nonorientable surfaces we recall from [2] (see also [5]) the construction of finite index pure subgroups $\Gamma(m)$ of Mod(N) (see Section 2 of [2] for more details). If $m \geq 3$, then $\Gamma(m)$ is a pure subgroup of Mod(N).

An isotopy class of curve c is in the canonical reduction system for a pure mapping class f if $f(c) = c$, and $f(d) \neq d$ whenever the geometric intersection number $i(c, d)$ is positive. Birman, Lubotzky and McCarthy in [7] introduced reduction systems, for the case of a nonorientable surface see [16]. We note that the centralizer of a pseudo-Anosov mapping class is virtually cyclic (see [10], for the case of nonorientable surfaces see ([2])).

Let G be a group, $H \leq G$ a subgroup and $f \in G$ an element of G. Then, $C(G)$, $C_G(H)$, $C_H(f)$ and $rk(G)$ denote the center of G, the centralizer of H in G, the centralizer of f in H and the rank of a group G, respectively. We notice that the maximal rank of a free abelian subgroup of $\mathrm{Mod}(N)$ is $n-2$ (see [14], [2], [5]). Also, one can see that any power of a Dehn twist is contained in an abelian subgroup of maximal rank.

2.3. **Relations.** Lemma 4.6. in [6] gives the correspondence between Dehn twist relations and half-twist relations, via a 2-sheeted branched cover over of surfaces $p : S \to D_n$, where S is an orientable surface with genus $\frac{n-1}{2}$ (if n is odd) and one boundary component (an orientable surface with genus $\frac{n-2}{2}$ and two boundary components, if n is even) and D_n is a disc with n marked points (the marked points are the branch points). The deck transformation is an involution ι switching the two sheets. If we blow up the cover $p : S \to D_n$, the resulting covering $p_B : \widetilde{N} \to N$ is a 2-sheeted branched cover over N with n marked points by $\widetilde{N}$ a nonorientable surface with genus $1+n$ (if n is odd) and one boundary component (a nonorientable surface with genus n and two boundary components, if n is even). Since $p : S \to D_n$ is fully ramified, the same is true for $p_B : \widetilde{N} \to N$. Therefore, by Theorem 2.1 and Theorem 1.1 in [3], both coverings have the Birman–Hilden property. Hence, we have $\mathrm{LMod}(N) \cong \mathrm{SMod}(\widetilde{N})/\langle\iota\rangle$. Here, $\mathrm{LMod}(N)$ is a finite index subgroup of $\mathrm{Mod}(N)$ formed by mapping classes of N lifting to diffeomorphisms of $\widetilde{N}$ and $\mathrm{SMod}(\widetilde{N})$ is the subgroup of $\mathrm{Mod}(\widetilde{N})$ formed by fiber preserving (or symmetric) mapping classes of $\mathrm{Mod}(\widetilde{N})$. We note that $\mathrm{LMod}(N)$ in Lemma 4.6 in [6] is $\mathrm{Mod}(D_n)$. However, since $n \geq 4$, by Theorem 3.2 in [4], $\mathrm{LMod}(N)$ is not isomorphic to $\mathrm{Mod}(N)$. On the other hand, using Lemma 2.3 in [4], we can see that $\mathrm{LMod}(N)$ is the subgroup of $\mathrm{Mod}(N)$ generated by the half-twists σ_{a_i} for $i = 1, 2, \ldots, n-1$. Now, we state the following analogue of Lemma 4.6 of [6] in our setting, whose proof is omitted since it is almost identical.

Lemma 2.2. *Powers of half-twists σ_a^j and σ_b^k satisfy a relation in $\mathrm{Mod}(N)$ if and only if the corresponding powers of Dehn twists $t_{\tilde{a}}^j$ and $t_{\tilde{b}}^k$ satisfy the same relation in in $\mathrm{SMod}(\widetilde{N})$ (and hence in $\mathrm{Mod}(\widetilde{N})$).*

We notice also that Stukow's results on Dehn twists on nonorientable surfaces demonstrate that Dehn twists share the same properties as those on orientable surfaces ([15], Propositions 4.6–4.8).

Combining Lemma 2.2 with Dehn twists relations, we have Lemmas 2.3-2.5 on nonorientable surfaces for j and k nonzero and a and b nontrivial:

Lemma 2.3. *$\sigma_a^j = \sigma_b^k$ if and only if $a = b$ and $j = k$.*

Lemma 2.4. *$[\sigma_a^j, \sigma_b^k] = 1$ if and only if $i(a, b) = 0$.*

Lemma 2.5. *$\sigma_a^j\sigma_b^k\sigma_a^j = \sigma_b^k\sigma_a^j\sigma_b^k$ if and only if $\langle a, b\rangle$ is simple pair and $j = k = \pm 1$.*

2.4. **Half-twists preserving.** Theorem 2.6 below is proved in [12]. It is also given implicitly in [10], for the case of nonorientable surfaces see [2], [5], [9].

Theorem 2.6. *Let Γ be a finite index subgroup consisting of pure elements in $\mathrm{Mod}(N)$. Suppose that $f \in \Gamma$ has canonical reduction system $\mathcal{C}$. Then $C(C_\Gamma(f)) \cong \mathbb{Z}^{c+p}$ where c is the number of curves in $\mathcal{C}$ and p is the number of pseudo-Anosov components of $f_{\mathcal{C}}$, where $C(C_\Gamma(f))$ is the center of the centralizer of f in Γ.*

The following group theoretical lemma is proved by Bell-Margalit in [6].

Lemma 2.7. *Let $\psi : \Gamma \to \Gamma'$ be an injective homomorphism. Suppose that $rk\Gamma' = rk\Gamma + r < \infty$ for some integer $r \geq 0$. Let $G < \Gamma$ be an abelian subgroup of maximal rank, and let $f \in G$. Then $rkC(C_{\Gamma'}(\psi(f))) \leq rkC(C_\Gamma(f)) + r$.*

Let Φ be an automorphism of $\mathrm{Mod}(N)$. We define $\Gamma' = \Gamma(m)$ and $\Gamma = \Phi^{-1}(\Gamma') \cap \Gamma'$.

Lemma 2.8. *Φ takes a power of a half-twist to a power of a half-twist.*

Proof. Let c be a 2-separating curve in N. So, $f = \sigma_c^k \in \Gamma$, and it belongs to a maximal rank free abelian subgroup of $\mathrm{Mod}(N)$. We have $rkC(C_\Gamma(f)) = 1$, by Theorem 2.6. It follows from Lemma 2.7 that $C(C_{\Gamma'}(\Phi(f)))$ has rank at most 1. Then, by Theorem 2.6, we obtain that $c + p \leq 1$ for a canonical reduction system of $\Phi(f)$. If $p = 1$, then $\Phi(f)$ is pseudo-Anosov, contradicting the fact that $C_\Gamma(f)$ (and so the centralizer of $\Phi(f)$) contains a free abelian group of rank 2. The case $c = p = 0$ is not possible, because then $\Phi(f)$ is the identity. Therefore, $p = 0$ and $c = 1$. Then, D has at most one two-sided curve, where D denotes the canonical reduction system for $\Phi(f)$. Now, assume that D has no two-sided curve. Say $D = \{d\}$, where d is a one-sided curve. Then, N_d is connected and the restriction of $\Phi(f)$ is either the identity or pseudo-Anosov. Since $\Phi(f)$ is not pseudo-Anosov, the restriction of $\Phi(f)$ would be the identity. However, if the restriction of $\Phi(f)$ is the identity, then $\Phi(f)$ must be power of Dehn twist about d which is not possible since d is one-sided. Then, there is a nontrivial two-sided curve d on N such that $\Phi(f) = t_d^m$ for some m.

We will show that d is a 2-separating curve. We consider a maximal collection of disjoint 2-separating curves on N, $\{c = c_1, \ldots, c_{\lfloor \frac{n}{2} \rfloor}\}$. The half-twists σ_{c_i} give a basis for a free abelian group of rank $\lfloor \frac{n}{2} \rfloor$, all of whose generators are conjugate in $\mathrm{Mod}(N)$. Since Φ is an automorphism, by Lemma 2.4, $\Phi(\sigma_{c_i}^k)$ is a set of $\lfloor \frac{n}{2} \rfloor$ powers of Dehn twists about disjoint curves bounding the same number of marked points; that is, all these curves are 2-separating curves. ◻

Proposition 2.9. *Then Φ is a half-twist preserving.*

Proof. Let c be a 2-separating curve. Then, by Lemma 2.8 we have $\Phi(\sigma_c^m) = t_{c'}^n$, where c' is a 2-separating curve. Since $[\sigma_c, \sigma_c^m] = 1$, we have $[\Phi(\sigma_c), \Phi(\sigma_c^m)] = [\Phi(\sigma_c), t_{c'}^n] = 1$. Hence, $\Phi(\sigma_c)(c') = c'$. We consider $N_{c'} = N_1 \cup N_2$, where N_1 denotes the disc with two marked points and N_2 is a Möbius strip with $n-2$ marked points. As $\Phi(\sigma_c)$ fixes c', there exist well-defined restrictions of $\Phi(\sigma_c)$ denoted by f_1 and f_2 to N_1 and N_2, respectively. These restrictions must be finite order mapping classes because $\Phi(\sigma_c^m) = t_{c'}^n$. Since N_1 is a disk with two marked points, f_1 is a power of a half-twist. Now, we will show that f_2 is the identity. To show this, let us take a 2-separating curve a disjoint from c $(n \geq 4)$. Then, by Lemma 2.8 we have $\Phi(\sigma_a^k) = t_{a'}^s$, where a' is a 2-separating curve. Moreover, by commutativity, $\Phi(\sigma_c)$ fixes the curve a', so f_2 fixes it. If N_2' is the complement of the interior of a' on N_2, then f_2 which is restricted to N_2' is the identity. It follows that f_2 is identity. Then, we obtain that $\Phi(\sigma_c) = \sigma_{c'}^m$.

Let a be a 2-separating curve such that $\langle a, c\rangle$ is a simple pair. Since σ_a and σ_c are conjugate, we have $\Phi(\sigma_a) = \sigma_{a'}^m$. Then, we have $\sigma_{c'}^m \sigma_{a'}^m \sigma_{c'}^m = \sigma_{a'}^m \sigma_{c'}^m \sigma_{a'}^m$. By Lemma 2.5, $m = \pm 1$.

This finishes the proof of the proposition. □

3. The Proof of Theorem 1.1

In this section, we closely follow the proof of Theorem 4.1 in [2].

Let $\Phi : \mathrm{Mod}(N) \to \mathrm{Mod}(N)$ be an automorphism. Let D denote the maximal chain of arcs from Figure 4. We will abuse notation and denote by the same Figure 4 the corresponding chain of half-twists. By Theorem 2.1, the half-twists σ_{a_i}, $i = 1, \cdots, n-1$ and a puncture slide, say v, generate the mapping class group $\mathrm{Mod}(N)$.

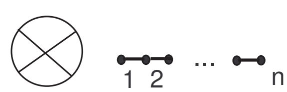

Figure 4

The image of D is again a maximal chain by the results of the previous section. Let $\Phi(\sigma_{a_i}) = \sigma_{a_i'}$ for each i, where a_i' is the unique geodesic in the homotopy class of an arc corresponding to the half-twist $\Phi(\sigma_{a_i})$. Let μ be the tubular neighborhood of the chain D. Then μ is a disc with n-marked points. Let us choose an orientation on μ and on every 2-separating a_i corresponding to each of arcs a_i such that the intersection $\{a_i, a_j\}$ $(i \leq j)$ is compatible with the orientation. Let μ' be a tubular neighborhood of the chain $\{a_1', \cdots, a_{n-1}'\}$. Now, choose an orientation of μ' and a diffeomorphism of a_1 onto a_1'. We can extend this diffeomorphism to all D by following the orientations, and so to a neighborhood of D. This diffeomorphism extends to a diffeomorphism of the surface.

Let $\phi : N \to N$ be one such a diffeomorphism. Composing Φ with ϕ_*^{-1}, we can suppose that Φ fixes σ_{a_i} for all i. Let $v' = \Phi(v)$, where v is a puncture slide. v' commutes with all σ_{a_i} for $i \geq 2$, because v commutes with all σ_{a_i} for $i \geq 2$. Let A be a two-sided curve as in Figure 5. Then, because a power of the Dehn twist t_A commutes with σ_{a_i} 's, $i \geq 2$, v' also commutes with t_A. Now, we may isotope v' to fix A pointwisely. Here, v' cannot interchange the two sides of a tubular neighborhood of the curve A. Thus, v' can be assumed to be identity on a tubular neighborhood of the curve A. Then, v' induces a diffeomorphism on $N \setminus A$ the disjoint union of disc with $(n-1)$ marked points and Möbius strip with one marked point.

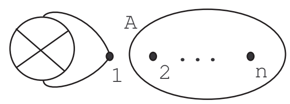

FIGURE 5. The curve A on N

v' commutes with all σ_{a_i} 's, $i \geq 2$, with t_A, and the center of the mapping class group of disc with $(n-1)$ marked points is $\langle t_A \rangle$. Because of this, we may assume that v' is identity on the disc with $(n-1)$ marked points, after an isotopy. Thus, it is supported on Möbius strip with one marked point. On the other hand, the mapping class group of Möbius strip with one marked point is generated by v so that $v^2 = t_A$ (see [13].) Any element of the mapping class group of Möbius strip with one marked point has the form v^k for some k. In particular, $\Phi(v) = v^k$ for some k. Since $\Phi(v^2) = v^2$, we see that $v^2 = (v^k)^2 = v^k v^k = v^k v^{(k-2)} v^2 = v^{(2k-2)} v^2$. So, $v^{(2k-2)} = id$. Then, $k = 1$.

In conclusion, by composing Φ with an inner automorphism (the automorphism ϕ_*^{-1} induced from the diffeomorphism ϕ^{-1} of the surface obtained from Φ in a unique way up to isotopy) we obtain that Φ is inner.

This concludes the proof of the theorem.

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