Report on scientific activities at MPIM-Bonn

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1 Research Summary

During my stay at Bonn, I worked on the following two topics:

1. *Certain families of K3 surfaces and their modularity*: We studied K3 surfaces and their modularity/automorphy using geometric reduction process. This is a joint work with Adrian Clingher (University of Missouri, St. Louis, USA), Andreas Malmendier (Utah State University, USA), and Noriko Yui (Queen's University, Canada).

2. *Paley-like quasi-random graphs arising from polynomials*: In this paper, we provide new constructions of families of quasi-random graphs that behave like Paley graphs but are neither Cayley graphs nor Cayley sum graphs. In particular, we give a sufficient condition for the clique number of quasi-random graphs of n vertices to be at least $(1 - o(1)) \log_{3.008} n$. Such a condition applies to many classical quasi-random graphs, including Paley graphs and Paley sum graphs, as well as some new Paley-like graphs we construct. This is a joint work with Kyle Yip (University of Britisch Columbia, Canada) and Semin Yoo (Institute for Basic Science, South Korea).

2 Detailed description of research activities

1. We considered various families of K3 surfaces and their modularity using geometric reduction process. First, we consider a 4-parameter family of K3 surfaces with Picard number 16 considered by Matsumoto, Sasaki, and Yoshida [5]. Using the Dolgachev-Oortland invariant, we obtain 3-parameter family of interest which can be also written as the Jacobian of a hyperelliptic curve with 3 parameters [3]. We give an additional condition on the hyperelliptic curve so that we can obtain a 2 parameter family which reduces the 3-dimensional moduli space to a Humbert surface. This enables us to consider three different models with the algebraic correspondences between them and the same Galois representation, which is proved in [1]. One of the models of the 2 parameter family is a double sextic family of K3 surfaces of Picard rank 18 with two parameters:
$$y^2 = z_1 z_2 (z_1 - z_2)(z_1 - z_3)(a z_2 - z_3)(b z_2 - z_3)$$

The family turns out to be of hypergeometric type: The K3 surfaces of the family have Picard-Fuchs differential equations with periods which are either hypergeometric functions or the Heun functions. We will show, at some special points, these K3 surfaces are modular with associated Galois representations of dimension 3 or 4, coming from the structure of transcendental lattices, based on the theorem of Livné [4]. This is an on-going work which was *mostly* initiated during my visit in MPIM with Noriko Yui.

2. Let $f \in \mathbb{F}_q[x, y]$ be a polynomial. We define a graph $X_{f,q}$ on $\mathbb{F}_q$, such that two distinct vertices $a, b \in \mathbb{F}_q$ are adjacent if and only if $f(a, b)$ is a square in $\mathbb{F}_q$. To ensure that the graph $X_{f,q}$ is undirected, we assume that for $a, b \in \mathbb{F}_q$, $f(a, b)$ is a square if and only if $f(b, a)$ is a square. Furthermore, some conditions on the polynomial f have to be imposed so that $X_{f,q}$ could behave like Paley graphs. The graphs of the form $X_{f,q}$ encompass a broad spectrum of well-known graphs as their special cases. For example, Paley graphs correspond to the case $f(x, y) = x - y$ for $q \equiv 1 \pmod 4$, and Paley sum graphs are realized when $f(x, y) = x+y$. In this paper, we showed that various families consisted with graphs of the form $X_{f,q}$ are quasi-random in the sense of [2]. Moreover, we proved that, for a homogeneous admissible polynomial $f \in \mathbb{F}_q[x, y]$ with odd degree d, we have the lower bound of the clique number $\omega(X_{f,q}) \geq (1 - o(1)) \log_{3.712} q$ as $q \to \infty$. The project already started prior to my visit at MPIM Bonn, but many parts of the editing process took place in MPIM Bonn.

3 List of the papers

1. A. Clingher, S. Kim, A. Malmendier, and N. Yui, Certain families of K3 surfaces and their modularity, arXiv

2. S. Kim, K. Yip, and S. Yoo, Paley-like quasi-random graphs arising from polynomials, arXiv:2405.09319.

4 Lectures and Courses

1. *Certain families of K3 surfaces and their modularity*, Oberseminar on analytic number theory, University of Göttingen, Germany, June 10, 2024.

2. *Certain families of K3 surfaces and their modularity*, Number theory lunch seminar, MPIM Bonn, Germany, July 9, 2024.

5 Other scientific activities

1. Served as referees for two research articles for two different journals.

2. Served as a reviewer for Mathematical Reviews.

3. Attended seminars and conferences at MPIM and discussed with academics at MPIM.

4. Discussed mathematics with several external visitors, Andreas Malmendier (Utah State University, USA) and Matthias Schüt (Leibniz University Hannover, Germany).

References

[1] N. Braeger, A. Clingher, A. Malmendier, and S. Spatig. Isogenies of certain K3 surfaces of rank 18. *Res. Math. Sci.*, 8(4):Paper No. 57, 60, 2021.

[2] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[3] A. Clingher, A. Malmendier, and T. Shaska. Six line configurations and string dualities. *Comm. Math. Phys.*, 371(1):159–196, 2019.

[4] R. Livné. Motivic orthogonal two-dimensional representations of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$). *Israel J. Math.*, 92(1-3):149–156, 1995.

[5] K. Matsumoto, T. Sasaki, and M. Yoshida. The monodromy of the period map of a 4-parameter family of $K3$ surfaces and the hypergeometric function of type $(3, 6)$. *Internat. J. Math.*, 3(1):164, 1992.