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CHIRAL ANOMALY VIA VERTEX ALGEBROIDS

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1. Introduction

Suppose that $q : X \to S$ is a circle bundle equipped with a relative orientation and $\mathcal{E}$ is a complex vector bundle on X. These data give rise to an $\mathcal{O}^*$-gerbe $\operatorname{Det}(X/S;\mathcal{E})$ on S. A construction of $\operatorname{Det}(X/S;\mathcal{E})$ is given in Section 6 of [BKTV], where it is denoted $\mathcal{D}et(q_*\mathcal{E})$. Let $C_1(X/S;\mathcal{E})$ denote the image of the class of $\operatorname{Det}(X/S;\mathcal{E})$ under the canonical map $H^2(S;\mathcal{O}_S^*) \to H^2(S;\Omega^1_S \to \Omega^{2,cl}_S) \cong H^3_{dR}(S)$.

In [BKTV] a generalization of $C_1(X/S;\mathcal{E})$ to arbitrary relative dimension (i.e. to the case when $q : X \to S$ is a proper relatively oriented submersion of relative dimension $d = \dim X/S$ not necessarily equal to one) is constructed so that $C_1(X/S;\mathcal{E}) \in H^{d+2}_{dR}(S)$ and shown to satisfy a Riemann-Roch type formula

$$C_1(X/S;\mathcal{E}) = \int_{X/S} \left[\operatorname{ch}(\mathcal{E})\operatorname{Td}(\mathcal{T}_{X/S})\right]_{2d+2}, \tag{1.0.1}$$

where $\mathcal{T}_{X/S}$ is the complexified relative tangent bundle. It is conjectured in loc. cit. that the class $C_1(X/S;\mathcal{E})$ (denoted also by $C_1(q_*\mathcal{E})$) is the image of the class of a naturally defined "determinental d-gerbe" generalizing $\operatorname{Det}(X/S;\mathcal{E})$. The construction of [BKTV] can be loosely described as transgression of the canonical trace density (a cyclic cocycle) on $\mathcal{D}_{X/S,\mathcal{E}}$ (the algebra of relative differential operators acting on $\mathcal{E}$) with (1.0.1) obtained as a consequence of the results of [BNT].

In the present note we "animate", in the words of A. Beilinson, (1.0.1) in the case of relative dimension one. In this case the formula (1.0.1) simplifies to

$$C_1(X/S;\mathcal{E}) = \int_{X/S} \operatorname{ch}_2(\mathcal{E}). \tag{1.0.2}$$

The latter is "animated" as a relationship between certain stacks on S and X respectively. The one on S, representing $C_1(X/S;\mathcal{E})$, is the de Rham avatar of the determinental gerbe. The one on X representing $\operatorname{ch}_2(\mathcal{E})$ is the stack of vertex extensions of the Atiyah algebra of $\mathcal{E}$

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treated in [B]. The main ingredient is the construction of the functor $\int_{X/S}$ (see Section 5) which associates to vertex algebroid on X a certain Lie algebroid on S. In fact, throughout the note we work with an arbitrary transitive Lie algebroid together with an invariant symmetric pairing on the kernel of the anchor.

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2. Generalities

2.1. **Notation.** For a map $q\colon X \to S$ of manifolds and a sheaf $\mathcal{A} \to \mathcal{T}_X$ over $\mathcal{T}_X$ we denote by $\mathcal{A}_{/S}$ the sheaf defined by the pull-back square

$$\begin{array}{ccc} \mathcal{A}_{/S} & \longrightarrow & q^{-1}\mathcal{T}_S \\ \big\downarrow & & \big\downarrow \\ \mathcal{A} & \longrightarrow & q^*T_S \end{array}$$

where the bottom horizontal map is the composition $\mathcal{A} \to \mathcal{T}_X \xrightarrow{dq} q^*T_S$. The above construction applied to the structure map $\mathcal{A} \to \mathcal{T}_X$ produces a sheaf over $(\mathcal{T}_X)_{/S}$.

In the same situation we denote by $q_+\mathcal{A}$, the sheaf on S defined by the pull-back square

$$\begin{array}{ccc} q_+\mathcal{A} & \longrightarrow & \mathcal{T}_S \\ \big\downarrow & & \big\downarrow \\ q_*\mathcal{A}_{/S} & \longrightarrow & q_*q^{-1}\mathcal{T}_S \end{array} \tag{2.1.1}$$

3. Lie algebroids

3.1. **Notation.** Suppose that k is a sheaf of commutative algebras on X, $\mathcal{R}$ is a commutative k-algebra. By an $\mathcal{R}$-algebroid we mean a sheaf of $\mathcal{R}$-modules $\mathcal{A}$ together with an $\mathcal{R}$-linear map $\sigma\colon \mathcal{A} \to \mathrm{Der}_{\mathcal{R}/k}(\mathcal{R})$ called the *anchor map*, and an operation $[\ ,\ ] : \mathcal{A} \otimes_k \mathcal{A} \to \mathcal{A}$, which satisfy

- the operation $[\ ,\ ]$ endows $\mathcal{A}$ with a structure of a k-Lie algebra;
- the anchor map is a morphism of Lie algebras
- the Leibniz rule $[a, rb] = \sigma(a)(r)b + r[a, b]$, $a, b \in \mathcal{A}$, $r \in \mathcal{R}$ holds.

We denote by $\mathfrak{g}(\mathcal{A})$ the kernel of the anchor map; it is a sheaf of $\mathcal{R}$-Lie algebras (i.e. a Lie algebra in the category of $\mathcal{R}$-modules).

We will usually refer to an $\mathcal{O}_X$-Lie algebroid as a *Lie algebroid on* X.

For a Lie algebroid $\mathcal{A}$ on X, locally free of finite rank over $\mathcal{O}_X$ let $\Omega^1_{\mathcal{A}} := \mathcal{A}^\vee$, $\Omega^i_{\mathcal{A}} := \bigwedge^i \Omega^1_{\mathcal{A}}$. Clearly, $\Omega^i_{\mathcal{T}_X} = \Omega^i_X$, the sheaf of differential forms of degree i.

The composition $\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{\sigma^\vee} \Omega^1_{\mathcal{A}}$ extends uniquely to a square-zero derivation of dergree one of the graded algebra $\Omega^\bullet_{\mathcal{A}}$; it will be denoted by $d_{\mathcal{A}}$ or just d.

3.2. **Direct image.** Suppose that $\mathcal{A}$ is a Lie algebroid on X and $q\colon X \to S$ is a map of manifolds the sheaf $\mathcal{A}_{/S}$ has a canonical structure of a $q^{-1}\mathcal{O}_S$-Lie algebroid with the structure map $\mathcal{A}_{/S} \to q^{-1}\mathcal{T}_S$ as the anchor of $\mathcal{A}_{/S}$. The bracket is given by

$$[(a_1,\xi_1),(a_2,\xi_2)] = ([a_1,a_2],[\xi_1,\xi_2]),$$

where $a_i \in \mathcal{A}$, $\xi_i \in \mathcal{T}_S$, $i = 1,2$, and $dq(\sigma(a_i)) = \xi_i$.

The sheaf $q_+\mathcal{A}$ defined by (2.1.1) has a canonical structure of a Lie algebroid on S with the structure map $q_+\mathcal{A} \to \mathcal{T}_S$ as the anchor of $\mathcal{A}_{/S}$. Note that $\mathfrak{g}(q_+\mathcal{A}) = q_*\mathfrak{g}(\mathcal{A}_{/S})$.

Suppose in addition that

- $\mathcal{A}$ is transitive so that there is a short exact sequence

$$0 \to \mathfrak{g}(\mathcal{A}) \to \mathcal{A} \to \mathcal{T}_X \to 0,$$

- the map $q\colon X \to S$ is a submersion so that there is a short exact sequence

$$0 \to \mathcal{T}_{X/S} \to \mathcal{T}_X \xrightarrow{dq} q^*\mathcal{T}_S \to 0.$$

In this situation the $q^{-1}\mathcal{O}_S$-Lie algebroid $\mathcal{A}_{/S}$ is transitive, i.e. the map $\mathcal{A}_{/S} \to q^{-1}\mathcal{T}_S$ is surjective (because the composition $\mathcal{A} \to \mathcal{T}_X \xrightarrow{dq} q^*\mathcal{T}_S$ is). The projection $\mathcal{A}_{/S} \to \mathcal{A}$ restricts to an isomorphism $\mathfrak{g}(\mathcal{A}_{/S}) \to \ker(\mathcal{A} \to q^*\mathcal{T}_S)$. Since the composition $\ker(\mathcal{A} \to q^*\mathcal{T}_S) \to \mathcal{T}_X \xrightarrow{dq} q^*\mathcal{T}_S$ is equal to zero it follows that the composition $\ker(\mathcal{A} \to q^*\mathcal{T}_S) \to \mathcal{A} \to \mathcal{T}_X$ factors through $\mathcal{T}_{X/S} = \ker(\mathcal{T}_X \xrightarrow{dq} q^*\mathcal{T}_S)$. Since the diagram

$$\begin{array}{ccc} \ker(\mathcal{A} \to q^*\mathcal{T}_S) & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{T}_{X/S} & \longrightarrow & \mathcal{T}_X \end{array}$$

is cartesian it follows that the sequence

$$0 \to \mathfrak{g}(\mathcal{A}) \to \ker(\mathcal{A} \to q^*\mathcal{T}_S) \to \mathcal{T}_{X/S} \to 0$$

is exact. Thus, there is a short exact sequence

$$0 \to \mathfrak{g}(\mathcal{A}) \to \mathfrak{g}(\mathcal{A}_{/S}) \to \mathcal{T}_{X/S} \to 0.$$

The Lie agebroid structure on $\mathcal{A}$ restricts to one on $\mathfrak{g}(\mathcal{A}_{/S})$ with the anchor given by the composition $\mathfrak{g}(\mathcal{A}_{/S}) \to \mathcal{T}_{X/S} \to \mathcal{T}_X$.

If the Kodaira-Spencer map $\mathcal{T}_S \to \mathbf{R}^1 q_*\mathfrak{g}(\mathcal{A}_{/S})$, i.e. the composition $\mathcal{T}_S \to q_* q^{-1}\mathcal{T}_S \to \mathbf{R}^1 q_*\mathfrak{g}(\mathcal{A}_{/S})$, is equal to zero, then $q_+\mathcal{A}$ is transitive and there is an exact sequence

$$0 \to q_*\mathfrak{g}(\mathcal{A}) \to q_*\mathfrak{g}(\mathcal{A}_{/S}) \to q_*\mathcal{T}_{X/S} \to \mathbf{R}^1 q_*\mathfrak{g}(\mathcal{A}).$$

Vanishing of $\mathbf{R}^i q_*\mathfrak{g}(\mathcal{A})$, $i \geq 1$, holds in the C^∞ setting and in the algebraic (analytic) setting if the map q is affine (Stein), and $\mathcal{A}$ is a vector bundle as will be the case in examples of interest. In this case the sequence

$$0 \to q_*\mathfrak{g}(\mathcal{A}) \to \mathfrak{g}(q_+\mathcal{A}) \to q_*\mathcal{T}_{X/S} \to 0$$

is exact.

3.3. **$\mathcal{O}$-extensions of Lie algebroids.** Suppose that $\mathcal{B}$ is a transitive Lie algebroid on S and let $\mathfrak{g} = \mathfrak{g}(\mathcal{B})$ denote the kernel of the anchor map so that $\mathfrak{g}$ is a sheaf of $\mathcal{O}_S$-Lie algebras and there is an exact sequence

$$0 \to \mathfrak{g} \to \mathcal{B} \to \mathcal{T}_S \to 0$$

An *$\mathcal{O}_S$-extension of* $\mathcal{B}$ is a triple $(\widetilde{\mathcal{B}}, \mathfrak{c}, \sigma)$ which consists of a Lie algebroid $\widetilde{\mathcal{B}}$, a central section $\mathfrak{c} \in \Gamma(S; \widetilde{\mathcal{B}})$ and an isomorphism of Lie algebroids $\sigma : \widetilde{\mathcal{B}}/\mathcal{O}_S \cdot \mathfrak{c} \cong \mathcal{B}$. These data give rise to the associated short exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{\mathfrak{c}\cdot} \widetilde{\mathcal{B}} \xrightarrow{\sigma} \mathcal{B} \to 0 \tag{3.3.1}$$

The $\mathcal{O}_S$-extensions of $\mathcal{B}$ form a Picard stack under the operation of Baer sum of extensions which we denote $\mathcal{O}_S\mathcal{E}xt(\mathcal{B})$.

Since $\mathfrak{c}$ is central, it follows from the Leibniz rule that

- the (adjoint) action of $\widetilde{\mathcal{B}}$ on $\mathcal{O}_S \cdot \mathfrak{c} \cong \mathcal{O}_S$ factors through $\mathcal{T}_S$ and the latter action coincides with the Lie derivative action of vector fields on functions;
- $\mathcal{O}_S \cdot \mathfrak{c}$ is central in the $\mathcal{O}_S$-Lie algebra $\mathfrak{g}(\widetilde{\mathcal{B}})$;
- hence, the (adjoint) action of $\widetilde{\mathcal{B}}$ on $\mathfrak{g}(\widetilde{\mathcal{B}})$ factors through $\mathcal{B}$ and preserves the inclusion $\mathcal{O}_S \cdot \mathfrak{c} \hookrightarrow \mathfrak{g}(\widetilde{\mathcal{B}})$, and
- the induced action of $\mathcal{B}$ on $\mathfrak{g}$ coincides with the adjoint action.

3.4. **Central extensions.** Suppose that $\mathcal{B}$ is a Lie algebroid on S.

A $\mathcal{B}$-Lie algebra is a Lie algebra in $\mathcal{B}$-modules, i.e. an $\mathcal{O}_S$-Lie algebra equipped with a structure of a $\mathcal{B}$-module with respect to which the bracket is a morphism of $\mathcal{B}$-modules.

Example 1. For any Lie algebroid $\mathcal{B}$ the kernel of the anchor map $\mathfrak{g}(\mathcal{B})$ is a $\mathcal{B}$-Lie algebra.

A central extension of a $\mathcal{B}$-Lie algebra $\mathfrak{g}$ by $\mathcal{O}_S$ is a triple $(\widetilde{\mathfrak{g}}, \mathfrak{c}, \sigma)$, which consists of

(1) a $\mathcal{B}$-Lie algebra $\widetilde{\mathfrak{g}}$,
(2) a $\mathcal{B}$-invariant central section $\mathfrak{c} \in \Gamma(S; \widetilde{\mathfrak{g}})$ (i.e. $\mathcal{B}\mathfrak{c} = 0$), which implies that $\mathcal{O}_S \cdot \mathfrak{c}$ is a $\mathcal{B}$-submodule of $\widetilde{\mathfrak{g}}$,
(3) and an isomorphism of $\mathcal{B}$-Lie algebras $\sigma\colon \widetilde{\mathfrak{g}}/\mathcal{O}_S \cdot \mathfrak{c} \to \mathfrak{g}$.

Suppose that $\widetilde{\mathcal{B}}$ is an $\mathcal{O}_S$-extension of $\mathcal{B}$. Passing to the kernels of the respective anchor maps one obtains the central extension of $\mathcal{O}_S$-Lie algebras

$$0 \to \mathcal{O}_S \to \mathfrak{g}(\widetilde{\mathcal{B}}) \to \mathfrak{g} \to 0$$

The (adjoint) action of $\widetilde{\mathcal{B}}$ on $\mathfrak{g}(\widetilde{\mathcal{B}})$ factors through $\mathcal{B}$ and preserves the inclusion $\mathcal{O}_S \cdot \mathfrak{c} \hookrightarrow \mathfrak{g}(\widetilde{\mathcal{B}})$; the induced action of $\mathcal{B}$ on $\mathfrak{g}$ (respectively, $\mathcal{O}_S$) coincides with the adjoint action (respectively, the anchor). In other words, $\mathfrak{g}(\widetilde{\mathcal{B}})$ is a central extension of $\mathfrak{g}$ by $\mathcal{O}_S$ in $\mathcal{B}$-Lie algebras.

Let $\mathfrak{c}\mathcal{E}x\mathcal{T}_{\mathcal{B}}(\mathfrak{g}, \mathcal{O}_S)$ denote the Picard stack of central extension as above. "Restriction to the kernel of the anchor" is a morphism of Picard stacks

$$\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B}) \to \mathfrak{c}\mathcal{E}x\mathcal{T}_{\mathcal{B}}(\mathfrak{g}, \mathcal{O}_S)\colon \widetilde{\mathcal{B}} \mapsto \mathfrak{g}(\widetilde{\mathcal{B}}). \tag{3.4.1}$$

For $\widetilde{\mathfrak{g}} \in \mathfrak{c}\mathcal{E}x\mathcal{T}_{\mathcal{B}}(\mathfrak{g}, \mathcal{O}_S)(S)$ we denote by $\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\widetilde{\mathfrak{g}}}$ the corresponding "fiber" over $\widetilde{\mathfrak{g}}$ and by $\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\mathbf{0}}$ the fiber over the trivial (split) extension.

Note that $\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\mathbf{0}}$ is a Picard substack of $\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})$ and, if $\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\widetilde{\mathfrak{g}}}$ is locally non-empty, it is a (2-)torsor under $\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{A})_{\mathbf{0}}$.

3.5. **Classification of $\mathcal{O}_S$-extensions.** Suppose that $\mathcal{B}$ is locally free of finite rank over $\mathcal{O}_S$, i.e. a vector bundle. Then, there is a canonical equivalence of Picard stacks

$$\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B}) \xrightarrow{\cong} (\Omega^1_{\mathcal{B}} \to \Omega^{2,cl}_{\mathcal{B}}) - \text{tors}. \tag{3.5.1}$$

Recall that a $(\Omega^1_{\mathcal{B}} \to \Omega^{2,cl}_{\mathcal{B}})$-torsor is a pair $(\mathcal{P}, c)$ which consists of a $\Omega^1_{\mathcal{B}}$-torsor $\mathcal{P}$ and a map $c\colon \mathcal{P} \to \Omega^{2,cl}_{\mathcal{B}}$ which satisfies $c(p+\alpha) = c(p)+d\alpha$. The functor (3.5.1) associates to $\widetilde{\mathcal{B}}$ the $\Omega^1_{\mathcal{B}}$-torsor of (locally defined) splittings of $\sigma\colon \widetilde{\mathcal{B}} \to \mathcal{B}$. For a splitting ∇ as above, $c(\nabla)$ is defined by the equation $c(\nabla)(b_1, b_2) \cdot \mathfrak{c} = [\nabla(b_1), \nabla(b_2)] - \nabla([b_1, b_2])$.

3.6. **Picard-Lie algebroids.** An important particular case of the above story arises when $\mathcal{B} = \mathcal{T}_S$, i.e. $\mathfrak{g}$ is trivial. The $\mathcal{O}_S$-extensions of $\mathcal{T}_S$ are also known as *Picard-Lie algebroids* and we denote the stack of such by $\mathcal{PLA}_S$ so that the equivalence 3.5.1 becomes

$$\mathcal{PLA}_S \xrightarrow{\cong} (\Omega^1_S \to \Omega^{2,cl}_S) - \text{tors}.$$

For general $\mathcal{B}$ there is a canonical equivalence

$$\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\mathbf{0}} \to \mathcal{PLA}_S$$

given by $\widetilde{\mathcal{B}} \mapsto \widetilde{\mathcal{B}}/\mathfrak{g}$, where $\mathfrak{g}$ is embedded into $\widetilde{\mathcal{B}}$ using the splitting $\mathfrak{g} \to \widetilde{\mathfrak{g}}$.

The action of $\mathcal{PLA}_S$ on $\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\widetilde{\mathfrak{g}}}$ is given by the pairing

$$(\cdot) + (\cdot) : \mathcal{PLA}_S \times \mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\widetilde{\mathfrak{g}}} \to \mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\widetilde{\mathfrak{g}}}$$

For $\widetilde{\mathcal{T}} \in \mathcal{PLA}_S$ and $\widetilde{\mathcal{A}} \in \mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{A})_{\widetilde{\mathfrak{g}}}$ the algebroid $\widetilde{\mathcal{T}} + \widetilde{\mathcal{A}}$ is defined by the push-out square

$$\begin{array}{ccc} \mathcal{O}_S \times \mathcal{O}_S & \longrightarrow & \widetilde{\mathcal{T}} \times_{\mathcal{T}_S} \widetilde{\mathcal{A}} \\ {\scriptstyle +}\downarrow & & \downarrow \\ \mathcal{O}_S & \longrightarrow & \widetilde{\mathcal{T}} + \widetilde{\mathcal{A}} \end{array}$$

For $\widetilde{\mathcal{A}}_i \in \mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\widetilde{\mathfrak{g}}}$, $i = 1, 2$, let $\widetilde{\mathcal{A}}_2 - \widetilde{\mathcal{A}}_1$ denote the quotient of $\widetilde{\mathcal{A}}_2 \times_{\mathcal{A}} \widetilde{\mathcal{A}}_1$ by the diagonally embedded copy of $\widetilde{\mathfrak{g}}$. Then, $\widetilde{\mathcal{A}}_2 - \widetilde{\mathcal{A}}_1$ has a natural structure of a Picard-Lie algebroid, and there is a canonical isomorphism $(\widetilde{\mathcal{A}}_2 - \widetilde{\mathcal{A}}_1) + \widetilde{\mathcal{A}}_1 \cong \widetilde{\mathcal{A}}_2$ in $\mathcal{O}_S\mathcal{E}x\mathcal{T}(\mathcal{B})_{\widetilde{\mathfrak{g}}}$.

4. Vertex algebroids

Vertex algebroids were introduced in the work of V. Gorbunov, F. Malikov and V. Schechtman ([GMS]). In this section we will recall the results of [B] and simultaneously adapt them to the case of regular algebroids. Regular algebroids generalize transitive algegroids and were considered in [CSX].

4.1. **Notation and terminology.** A Lie algebroid $\mathcal{A}$ on a manifold X is called *regular* if the image of the anchor map $\mathcal{A} \to \mathcal{T}_X$ is a subbundle of $\mathcal{T}_X$. In other words, $\text{im}(\mathcal{A} \to \mathcal{T}_X)$ is locally free and so is $\text{coker}(\mathcal{A} \to \mathcal{T}_X)$. Note that the image of the anchor map is involutive, i.e. closed under the Lie bracket.

Throughout this section we work with a fixed involutive distribution (i.e. a sub-bundle of $\mathcal{T}_X$) $\mathcal{F}$. Thus, $\mathcal{F}$ is a Lie algebroid with the anchor given by the inclusion map.

The example we will be interested in later is $\mathcal{F} = \mathcal{T}_{X/S}$, where $X \to S$ is a submersion. In this situation it is traditional to write $\Omega^i_{X/S}$ in place of $\Omega^i_{\mathcal{F}}$.

A Lie algebroid $\mathcal{A}$ on X is called *$\mathcal{F}$-transitive* if $\operatorname{im}(\mathcal{A} \to \mathcal{T}_X) = \mathcal{F}$. If $\mathcal{A}$ is a transitive Lie algebroid on X, then $\mathcal{A}_{\mathcal{F}} := \mathcal{A} \times_{\mathcal{T}_X} \mathcal{F}$ is an $\mathcal{F}$-transitive Lie algebroid in a canonical way.

4.2. **Courant algebroids.** We refer the reader to [B] for definitions, notations and details.

A Courant algebroid $\mathcal{Q}$ is called *$\mathcal{F}$-transitive* if the associated Lie algebroid $\overline{\mathcal{Q}}$ is $\mathcal{F}$-transitive and the canonical map $\Omega^1_X \to \mathcal{Q}$ factors through $\Omega^1_{\mathcal{F}}$. An $\mathcal{F}$-transitive algebroid is *$\mathcal{F}$-exact* if the anchor map $\overline{\mathcal{Q}} \to \mathcal{T}_X$ is an isomorphism onto $\mathcal{F}$.

For an $\mathcal{F}$-transitive Courant algebroid $\mathcal{Q}$ the sequence

$$0 \to \Omega^1_{\mathcal{F}} \to \mathcal{Q} \to \overline{\mathcal{Q}} \to 0$$

is exact. In particular, an $\mathcal{F}$-exact Courant algebroid is an extension of $\mathcal{F}$ by $\Omega^1_{\mathcal{F}}$ with additional structure.

Suppose that $\mathcal{Q}$ is a transitive Courant algebroid on X. Then, the sheaf $\mathcal{Q}_{\mathcal{F}}$ defined by the push-out square

$$\begin{array}{ccc} \Omega^1_X & \longrightarrow & \mathcal{Q} \times_{\mathcal{T}_X} \mathcal{F} \\ \big\downarrow & & \big\downarrow \\ \Omega^1_{\mathcal{F}} & \longrightarrow & \mathcal{Q}_{\mathcal{F}} \end{array}$$

has a canonical structure of an $\mathcal{F}$-transitive Courant algebroid. Clearly, $\overline{\mathcal{Q}_{\mathcal{F}}} = \overline{\mathcal{Q}}_{\mathcal{F}}$. If $\mathcal{Q}$ is exact, then $\mathcal{Q}_{\mathcal{F}}$ is $\mathcal{F}$-exact.

Replacing "Courant" by "vertex" in the above we obtain analogous notions for vertex algebroids.

Let $\mathcal{ECA}_{\mathcal{F}}$ denote the Picard stack of $\mathcal{F}$-exact Courant algebroids. The constructions and results of [B], 3.8 yield a canonical equivalence of Picard stacks $\mathcal{ECA}_{\mathcal{F}} \cong (\Omega^2_{\mathcal{F}} \to \Omega^{3,cl}_{\mathcal{F}}) - \text{tors}$.

4.3. **Vertex extensions.** Suppose that $\mathcal{A}$ is an $\mathcal{F}$-transitive Lie algebroid. Let $\mathfrak{g} := \mathfrak{g}(\mathcal{A})$. Suppose that $\langle\ ,\ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $\mathfrak{g} = \mathfrak{g}(\mathcal{A})$. These data determine the vertex algebroid (with the trivial anchor map) $\widehat{\mathfrak{g}}$; it is defined by the pull-back diagram of

$\mathfrak{g}$-modules

$$\begin{array}{ccc} \widehat{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \\ \big\downarrow & & \big\downarrow{\scriptstyle \langle\ ,\ \rangle} \\ \mathcal{A}^\vee & \longrightarrow & \mathfrak{g}^\vee \end{array}$$

In particular, there is an exact sequence

$$0 \to \Omega^1_{\mathcal{F}} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0,$$

see [B], 3.2 for details.

Let $\mathcal{VEXT}_{\mathcal{F}}(\mathcal{A})_{\langle\ ,\ \rangle}$ denote the stack of vertex extensions of $\mathcal{A}$ which induce the given pairing $\langle\ ,\ \rangle$ on $\mathfrak{g}$. For any such extension $\widehat{\mathcal{A}}$ there is a canonical isomorphism $\mathfrak{g}(\widehat{\mathcal{A}}) \cong \widehat{\mathfrak{g}}$ and all of the above objects fit into the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \big\downarrow & & \big\downarrow & & & & \\ & & \Omega^1_{\mathcal{F}} & = & \Omega^1_{\mathcal{F}} & & & & \\ & & \big\downarrow & & \big\downarrow & & & & \\ 0 & \longrightarrow & \widehat{\mathfrak{g}} & \longrightarrow & \widehat{\mathcal{A}} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

with exact rows and columns.

The stack $\mathcal{VEXT}_{\mathcal{F}}(\mathcal{A})_{\langle\ ,\ \rangle}$ is locally non-empty if and only if

(P) locally on X the algebroid $\mathcal{A}$ admits a connection along $\mathcal{F}$ with exact Pontryagin form in $\Omega^\bullet_{\mathcal{F}}$.

The condition **(P)** is certainly implied by the Poincaré Lemma, which holds in the C^∞ and in the analytic setting; it also holds for Atiyah algebras as these admit flat connections locally on X. If $\mathcal{A}$ satisfies the condition **(P)**, then $\mathcal{VEXT}_{\mathcal{F}}(\mathcal{A})_{\langle\ ,\ \rangle}$ is a (2-)torsor under $\mathcal{ECA}_{\mathcal{F}}$. The same applies to the stack $\mathcal{CEXT}_{\mathcal{F}}(\mathcal{A})_{\langle\ ,\ \rangle}$ of Courant extensions. In what follows we will implicitly assume that the algebroid under consideration satisfies **(P)**.

The same applies to Courant extensions

Torsors under $\mathcal{ECA}_{\mathcal{F}}$ are classified by $H^2(X; \Omega^2_{\mathcal{F}} \to \Omega^{3,cl}_{\mathcal{F}})$. The class of $\mathcal{VEXT}_{\mathcal{F}}(\mathcal{A})_{\langle\ ,\ \rangle}$ (respectively, $\mathcal{CEXT}_{\mathcal{F}}(\mathcal{A})_{\langle\ ,\ \rangle}$) is equal to $\mathrm{ch}_2(\mathcal{F})$ −

$\frac{1}{2}\pi(\mathcal{A}, \langle\ ,\ \rangle)$, (respectively, $-\frac{1}{2}\pi(\mathcal{A}, \langle\ ,\ \rangle)$), where $\pi(\mathcal{A}, \langle\ ,\ \rangle)$ denotes the (first) Pontryagin class of the pair $(\mathcal{A}, \langle\ ,\ \rangle)$.

4.4. **Special case: rank one distributions.** Suppose that $\mathcal{F}$ is a rank one distribution on X. Then, $\Omega^i_{\mathcal{F}} = 0$ for $i \geq 2$.

Lemma 2. *Every $\mathcal{F}$-exact Courant algebroid admits a unique flat connection along $\mathcal{F}$.*

Corollary 3. *Suppose that $\mathcal{F}$ is a rank one distribution on X, $\mathcal{A}$ is an $\mathcal{F}$-transitive Lie algebroid and $\langle\ ,\ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $\mathfrak{g}(\mathcal{A})$. Then, the category $\mathcal{VEXT}_{\mathcal{F}}(\mathcal{A})_{\langle\ ,\ \rangle}(X)$ is equivalent to the category with one object and no non-trivial morphisms, ditto for $\mathcal{CEXT}_{\mathcal{F}}(\mathcal{A})_{\langle\ ,\ \rangle}(X)$*

5. Fiber integration for vertex algebroids

In this section $q\colon X \to S$ is a proper submersion of C^∞-manifolds of relative dimension one equipped with a relative orientation. In this setting we will associate to a vertex algebroid $\mathcal{V}$ on X a Lie algebroid $\int_{X/S}\mathcal{V}$ on S.

We refer the reader to [B] terminology and notation.

Remark 4. The construction described below applies equally well to Courant algebroids (with significantly simpler verifications of properties). Thus, we will use the Courant version of the results of this section without further discussion.

5.1. **The underlying sheaf on S.** Suppose that $\mathcal{V}$ is a vertex algebroid on X with anchor $\pi\colon \mathcal{V} \to \mathcal{T}_X$ and derivation $\partial\colon \mathcal{O}_X \to \mathcal{Q}$. Recall that the latter factors as $\mathcal{O}_X \to \Omega^1_X \to \mathcal{Q}$.

The sheaf $\int_{X/S}\mathcal{V}$ is defined by the push-out square

$$\begin{array}{ccc} q_*\Omega^1_X & \longrightarrow & q_+\mathcal{V} \\ {\scriptstyle \int_{X/S}}\big\downarrow & & \big\downarrow \\ \mathcal{O}_S & \longrightarrow & \int_{X/S}\mathcal{V} \end{array}$$

We are going to equip $\int_{X/S}\mathcal{V}$ with a structure of a Lie algebroid on S.

5.2. **The $\mathcal{O}_S$-module structure.** A (locally defined) section of $q_+\mathcal{V}$ is a pair (v, ξ), $v \in \mathcal{V}$, $\xi \in \mathcal{T}_S$, such that $dq(\pi(v)) = q^*(\xi)$. For $f \in \mathcal{O}_S$ let

$$f * (v, \xi) := (q^*(f) * v, f\xi).$$

This formula defines a pairing

$$\mathcal{O}_S \otimes_{\mathbb{C}} q_+\mathcal{V} \to q_+\mathcal{V}. \tag{5.2.1}$$

Lemma 5. *The pairing* (5.2.1) *descends to a pairing*

$$\mathcal{O}_S \otimes_{\mathbb{C}} \int_{X/S} \mathcal{V} \to \int_{X/S} \mathcal{V}. \tag{5.2.2}$$

Proof. A section of $\int_{X/S} \mathcal{V}$ is represented by a pair $(f, (v, \xi))$, $f \in \mathcal{O}_S$, $(v, \xi) \in q_+\mathcal{V}$. Two pairs are considered equivalent if their (componentwise) difference is a sum of sections of the form $(\int_{X/S} \alpha, (-\overline{\alpha}, 0))$, where $\overline{\alpha}$ denotes the image of $\alpha \in q_*\Omega^1_X$ in $\mathcal{V}$. The calculation

$$\begin{aligned} f * (\int_{X/S} \alpha, (-\overline{\alpha}, 0)) &= (\int_{X/S} q^*(f)\alpha, (-q^*(f) * \overline{\alpha}, 0)) \\ &= (\int_{X/S} q^*(f)\alpha, (-\overline{q^*(f)\alpha}, 0)) \end{aligned}$$

shows that the action of $\mathcal{O}_S$ preserves the kernel of the projection $\mathcal{O}_S \oplus q_+\mathcal{V} \to \int_{X/S} \mathcal{V}$, hence descends to a pairing (5.2.2). □

Note that the pairing (5.2.1) is not associative.

Proposition 6. *The induced paring* (5.2.2) *is associative, i.e. it defines a structure of an $\mathcal{O}_S$-module on $\int_{X/S} \mathcal{V}$.*

Proof. for $f, g \in \mathcal{O}_S$, $(v, \xi) \in q_+\mathcal{V}$ the associator is given by

$$\begin{aligned} f * (g * (v, \xi)) - (fg) * (v, \xi) &= (q^*(f) * (q^*(g) * v) - q^*(fg) * v, 0) \\ &= (\pi(v)(q^*(f)) * \partial(q^*(g)) + \pi(v)(q^*(g)) * \partial(q^*(f)), 0) \\ &= (q^*(dq(\pi(v)(f)) * \partial(q^*(g)) + q^*(dq(\pi(v)(g)) * \partial(q^*(f)), 0). \end{aligned}$$

The above calculation shows that the associator factors though the composition $\Omega^1_S \xrightarrow{(dq)^t} q_*\Omega^1_X \to q_+\mathcal{V}$. Since the composition $\Omega^1_S \xrightarrow{(dq)^t} q_*\Omega^1_X \xrightarrow{\int_{X/S}} \mathcal{O}_S$ is equal to zero, it follows that the induced pairing $\mathcal{O}_S \otimes_{\mathbb{C}} \int_{X/S} \mathcal{V} \to \int_{X/S} \mathcal{V}$ is associative. □

In what follows we will regard $\int_{X/S} \mathcal{V}$ as an $\mathcal{O}_S$-module with respect to the above structure. It is clear that the map $\mathcal{O}_S \to \int_{X/S} \mathcal{V}$ is $\mathcal{O}_S$-linear.

5.3. **The Lie bracket.** Let

$$[\cdot,\cdot]\colon q_+\mathcal{V}\otimes_{\mathbb{C}} q_+\mathcal{V}\to q_+\mathcal{V} \tag{5.3.1}$$

denote the pairing defined by

$$[(v_1,\xi_1),(v_2,\xi_2)] = ([v_1,v_2],[\xi_1,\xi_2]).$$

Lemma 7. *The pairing* (5.3.1) *defines a Leibniz bracket on* $q_+\mathcal{V}$ *whose symmetrization is given by*

$$[(v_1,\xi_1),(v_2,\xi_2)] + [(v_2,\xi_2),(v_1,\xi_1)] = (\partial(\langle v_1,v_2\rangle),0). \tag{5.3.2}$$

Proof. Direct calculation left to the reader. □

Let

$$[\cdot,\cdot]\colon (\mathcal{O}_S\oplus q_+\mathcal{V})\otimes_{\mathbb{C}}(\mathcal{O}_S\oplus q_+\mathcal{V})\to\mathcal{O}_S\oplus q_+\mathcal{V} \tag{5.3.3}$$

denote the pairing defined by

$$\begin{aligned}[(f_1,(v_1,\xi_1)),(f_2,(v_2,\xi_2))] &= (\xi_1(f_2)-\xi_2(f_1),[(v_1,\xi_1),(v_2,\xi_2)])\\ &= (\xi_1(f_2)-\xi_2(f_1),([v_1,v_2],[\xi_1,\xi_2])),\end{aligned}$$

where $f_i\in\mathcal{O}_S$, $(v_i,\xi_i)\in q_+\mathcal{V}$, $i=1,2$.

Proposition 8. *The pairing* (5.3.3) *descends to an operation*

$$[\cdot,\cdot]\colon \int_{X/S}\mathcal{V}\otimes_{\mathbb{C}}\int_{X/S}\mathcal{V}\to\int_{X/S}\mathcal{V} \tag{5.3.4}$$

which is skew symmetric and satisfies the Jacobi identity.

Proof. For $\alpha\in q_*\Omega^1_X$, $f\in\mathcal{O}_S$, $(v,\xi)\in q_+\mathcal{V}$

$$\begin{aligned}&[(f,(v,\xi)),(\int_{X/S}\alpha,(-\overline{\alpha},0))] =\\ &\qquad (\xi(\int_{X/S}\alpha),([v,-\overline{\alpha}],0)) = (\xi(\int_{X/S}\alpha),(-\overline{L_{\pi(v)}\alpha},0))\end{aligned}$$

and

$$\begin{aligned}&[(\int_{X/S}\alpha,(-\overline{\alpha},0)),(f,(v,\xi))]\\ &\quad = (0,(\partial\langle v,-\overline{\alpha}\rangle,0)) - [(f,(v,\xi)),(\int_{X/S}\alpha,(-\overline{\alpha},0))]\\ &\quad = (0,(-\overline{d\iota_{\pi(v)}\alpha},0)) - [(f,(v,\xi)),(\int_{X/S}\alpha,(-\overline{\alpha},0))].\end{aligned}$$

Therefore, the bracket defined by (5.3.3) descends to the operation

$$[\cdot,\cdot]\colon \int_{X/S}\mathcal{V}\otimes_{\mathbb{C}}\int_{X/S}\mathcal{V}\to\int_{X/S}\mathcal{V}.$$

It is skew symmetric because of (5.3.2) and satisfies the Jacobi identity because the bracket on $q_+\mathcal{V}$ does. □

5.4. **The Lie algebroid structure.**

Lemma 9. *The composition*

$$\mathcal{O}_S\oplus q_+\mathcal{V}\to q_+\mathcal{V}\to\mathcal{T}_S$$

factors through $\int_{X/S}\mathcal{V}$*. The induced map*

$$\pi\colon \int_{X/S}\mathcal{V}\to\mathcal{T}_S \tag{5.4.1}$$

is an $\mathcal{O}_S$*-linear map of Lie algebras with respect to the Lie bracket* (5.3.4) *on* $\int_{X/S}\mathcal{V}$ *and satisfies the Leibniz identity .*

Proof. The Leibniz identity for $\int_{X/S}\mathcal{V}$ follows from the Leibniz identity for $\mathcal{V}$. □

Proposition 6, Proposition 8 and Lemma 9 imply the following theorem.

Theorem 10. *The pairings* (5.2.2) *and* (5.3.4) *and the map* (5.4.1) *endow the sheaf* $\int_{X/S}\mathcal{V}$ *with a structure of a Lie algebroid on* S*.*

5.5. **The central section.** It follows immediately from the definition of the bracket on $\int_{X/S}\mathcal{V}$ that the image of $1\in\Gamma(S;\mathcal{O}_S)$ under the map $\mathcal{O}_S\to\int_{X/S}\mathcal{V}$ is central with respect to the bracket. We will denote this section by $\mathfrak{c}$. Let $\overline{\int_{X/S}\mathcal{V}}=\int_{X/S}\mathcal{V}/\mathcal{O}_S\cdot\mathfrak{c}$.

5.6. **The associated Lie algebroid.** Recall, that $\overline{\mathcal{V}}:=\mathcal{V}/\mathcal{O}_X*\partial\mathcal{O}_X$ is a Lie algebroid in a canonical manner.

The projection $\mathcal{V}\to\overline{\mathcal{V}}$ induces the map $q_+\mathcal{V}\to q_+\overline{\mathcal{V}}$ which factors through $\overline{\int_{X/S}\mathcal{V}}$. The induced map

$$\overline{\int_{X/S}\mathcal{V}}\to q_+\overline{\mathcal{V}}$$

is easily seen to be an isomorphism.

5.7. **$\mathcal{T}_{X/S}$-transitive algebroids.** Note that, if $\mathcal{A}$ is a Lie algebroid on X whose anchor map factors though $\mathcal{T}_{X/S}$, then the anchor map of $q_+\mathcal{A}$ is trivial, i.e. it is a sheaf of $\mathcal{O}_S$-Lie algebras and $q_+\mathcal{A} = q_*\mathcal{A}$.

If $\mathcal{V}$ is a $\mathcal{T}_{X/S}$-transitive vertex algebroid on X then the map $\Omega^1_X \to \mathcal{V}$ factors through $\Omega^1_{X/S}$. So does the integration map, giving rise to the map $\int_{X/S} \colon q_*\Omega^1_{X/S} \to \mathcal{O}_S$. Therefore, for $\mathcal{V}$ as above the construction of $\int_{X/S}\mathcal{V}$ can be carried out entirely in terms of $\Omega^1_{X/S}$.

Suppose that $\mathcal{A}$ is a $\mathcal{T}_{X/S}$-transitive Lie algebroid on X and $\langle\ ,\ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $\mathfrak{g}(\mathcal{A})$. Let $\widehat{\mathcal{A}}_{\langle\ ,\ \rangle}$ denote the unique vertex extension of $\mathcal{A}$ furnished by Corollary 3. The exact sequence

$$0 \to \Omega^1_{X/S} \to \widehat{\mathcal{A}}_{\langle\ ,\ \rangle} \to \mathcal{A} \to 0$$

gives rise to the exact sequence of $\mathcal{O}_S$-Lie algebras

$$0 \to \mathcal{O}_S \xrightarrow{\cdot\mathfrak{c}} \int_{X/S} \widehat{\mathcal{A}}_{\langle\ ,\ \rangle} \to q_+\mathcal{A} \to 0. \tag{5.7.1}$$

We refer to the above canonically defined central extension as the Kač-Moody-Virasoro extension and to its restriction to $q_*\mathfrak{g}(\mathcal{A})$ as the Kač-Moody extension. The Virasoro extension corresponds to the case $\mathcal{A} = \mathcal{T}_{X/S}$. This terminology is justified by the example $q \colon S^1 \to \mathrm{pt}$.

6. Fiber integration for vertex extensions

We now specialize the construction of 5 of to the case of transitive vertex algebroids.

6.1. **The fiber integration functors.** Suppose that $\mathcal{V}$ is a transitive vertex algebroid on X, so that the sequence

$$0 \to \Omega^1_X \to \mathcal{V} \to \overline{\mathcal{V}} \to 0$$

is exact. It then follows that $\int_{X/S}\mathcal{V}$ is a transitive Lie algebroid on S and the sequence

$$0 \to \mathcal{O}_S \xrightarrow{\cdot\mathfrak{c}} \int_{X/S} \mathcal{V} \to q_+\overline{\mathcal{V}} \to 0$$

is exact, i.e. $\int_{X/S}\mathcal{V}$ is an $\mathcal{O}_S$-extension of $q_+\overline{\mathcal{V}}$.

Suppose that $\mathcal{A}$ is a transitive Lie algebroid on X. Since the construction of Section 5 is obviously functorial and local on S it defines a morphisms of stacks on S

$$\int_{X/S} \colon q_*\mathcal{VEXT}_X(\mathcal{A}) \to \mathcal{O}_S\mathcal{E}x\mathcal{T}(q_+\mathcal{A}). \tag{6.1.1}$$

In a completely analogous fashion the fiber integration functor gives rise to the morphism of stacks

$$\int_{X/S} \colon q_*\mathcal{CEXT}_X(\mathcal{A}) \to \mathcal{O}_S\mathcal{E}xt(q_+\mathcal{A}). \tag{6.1.2}$$

6.2. **Special case: ECA.** We are going to modify slightly the definition of the functor $\int_{X/S}$ in the case of exact Courant algebroids.

Suppose that $\mathcal{Q}$ is an exact Courant algebroid on X. Then, the $\mathcal{T}_{X/S}$-exact Courant algebroid $\mathcal{Q}_{\mathcal{T}_{X/S}}$ admits a unique connection, i.e. a unique $\mathcal{O}_X$-linear lagrangian splitting. The canonical connection on $\mathcal{Q}_{\mathcal{T}_{X/S}}$ is flat, i.e. it is a morphism of Leibniz algebras. It gives rise to the canonical splitting of the $\mathcal{O}_S$-extension $\int_{X/S}\mathcal{Q}$ of $q_+\mathcal{T}_X$ over $q_*\mathcal{T}_{X/S} = \mathfrak{g}(q_+\mathcal{T}_X)$, i.e. a canonical isomorphism of $\mathfrak{g}(\int_{X/S}\mathcal{Q})$ with the trivial central extension.

Lemma 11. *The canonical splitting of $\mathfrak{g}(\int_{X/S}\mathcal{Q})$ is a morphism of $q_+\mathcal{T}_X$-modules.*

Proof. Let $\nabla\colon \mathcal{T}_{X/S} \to \mathcal{Q}_{\mathcal{T}_{X/S}}$ denote the unique flat connection. Let $\widetilde{\nabla}\colon \mathcal{T}_{X/S} \to \mathcal{Q}\times_{\mathcal{T}_X}\mathcal{T}_{X/S}$ denote an isotropic lift of ∇.

For $a \in \mathcal{Q}_{/S}$, $\xi \in \mathcal{T}_{X/S}$ let $[\mathrm{ad}(a), \widetilde{\nabla}](\xi) = [a, \widetilde{\nabla}(\xi)] - \widetilde{\nabla}([\overline{a}, \xi])$, where $\overline{a}$ denotes the image of a in $(\mathcal{T}_X)_{/S}$. The latter formula defines a map $[\mathrm{ad}(a), \widetilde{\nabla}]\colon \mathcal{T}_{X/S} \to \Omega^1_X$ which easily seen to be $\mathcal{O}_X$-linear.

Let $\overline{\nabla}$ denote the splitting of $\mathfrak{g}(\int_{X/S}\mathcal{Q})$ induced by ∇, i.e. the canonical splitting. Then, for $b \in q_+\mathcal{T}_X$, $\xi \in \mathcal{T}_{X/S}$ and a lift $\widetilde{b} \in q_+\mathcal{Q}$ of b $[\overline{\mathrm{ad}}(b), \overline{\nabla}](\xi) = \int_{X/S}[\mathrm{ad}(\widetilde{b}), \widetilde{\nabla}](\xi)$, where $\overline{\mathrm{ad}}$ denotes the action of $q_+\mathcal{T}_X$ induced by the adjoint action of $\int_{X/S}\mathcal{Q}$.

An easy calculation using the isotropy of $\widetilde{\nabla}$ shows that the map $(\xi, \eta) \mapsto \iota_\eta[\mathrm{ad}(a), \widetilde{\nabla}](\xi)$ is skew-symmetric. Therefore, the composition $\mathcal{T}_{X/S} \xrightarrow{[\mathrm{ad}(a),\widetilde{\nabla}]} \Omega^1_X \to \Omega^1_{X/S}$ is equal to zero for reasons of rank, i.e. $[\mathrm{ad}(a), \widetilde{\nabla}]$ takes values in the kernel of the integration map $\int_{X/S}\colon q_*\Omega^1_X \to \mathcal{O}_S$. □

Corollary 12. *The functor*

$$\int_{X/S} \colon q_*\mathcal{ECA}_X \to \mathcal{O}_S\mathcal{E}xt(q_+\mathcal{T}_X) \tag{6.2.1}$$

takes values in $\mathcal{O}_S\mathcal{E}xt(q_+\mathcal{T}_X)_{\mathbf{0}}$.

As explained in 3.6, $\mathcal{O}_S\mathcal{E}x\tau(q_+\mathcal{T}_X)_{\mathbf{0}}$ is canonically equivalent to $\mathcal{PLA}_S$. Using this equivalence, we will regard (6.2.1) as a morphism

$$\int_{X/S} : q_*\mathcal{ECA}_X \to \mathcal{PLA}_S \tag{6.2.2}$$

Lemma 13. *The morphism* (6.2.2) *is a morphism of Picard stacks.*

Under the standard equivalences $\mathcal{ECA}_X \cong (\Omega^2_X \to \Omega^{3,cl}_X) - \text{tors}$ and $\mathcal{PLA}_S \cong (\Omega^1_S \to \Omega^{2,cl}_S) - \text{tors}$ it corresponds to the morphism of complexes

$$\int_{X/S} : q_*(\Omega^2_X \to \Omega^{3,cl}_X) \to (\Omega^1_S \to \Omega^{2,cl}_S)$$

given by integration of differential forms along the fibers of q.

6.3. **Fiber integration and linear algebra.** The morphisms (6.1.1) (respectively, (6.1.2)) and (6.2.2) are compatible with the respective actions of $\mathcal{ECA}_X$ on $\mathcal{VEXT}_X(\mathcal{A})$ (respectively, $\mathcal{VEXT}_X(\mathcal{A})$) and of $\mathcal{PLA}_S$ on $\mathcal{O}_S\mathcal{E}x\tau(q_+\mathcal{A})$ in the sense that the diagram

$$\begin{array}{ccc} q_*\mathcal{ECA}_X \times q_*\mathcal{VEXT}_X(\mathcal{A}) & \xrightarrow{+} & q_*\mathcal{VEXT}_X(\mathcal{A}) \\ {\scriptstyle \int_{X/S}\times\int_{X/S}}\downarrow & & \downarrow{\scriptstyle \int_{X/S}} \\ \mathcal{PLA}_S \times \mathcal{O}_S\mathcal{E}x\tau(q_+\mathcal{A}) & \xrightarrow{+} & \mathcal{O}_S\mathcal{E}x\tau(q_+\mathcal{A}) \end{array} \tag{6.3.1}$$

commutes up to a natural transformation with suitable properties.

6.4. **Fiber integration for the torsor of vertex extensions.** Suppose that $\mathcal{A}$ is a transitive Lie algebroid on X and $\langle\ ,\ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $\mathfrak{g}(\mathcal{A})$. Let $\mathfrak{g} = \mathfrak{g}(q_+\mathcal{A}) = q_*\mathfrak{g}(\mathcal{A}_{/S})$.

Proposition 14. *The composition*

$$q_*\mathcal{VEXT}_X(\mathcal{A})_{\langle\ ,\ \rangle} \xrightarrow{\int_{X/S}} \mathcal{O}_S\mathcal{E}x\tau(q_+\mathcal{A}) \xrightarrow{(3.4.1)} \mathfrak{c}\mathcal{E}x\tau_{q_+\mathcal{A}}(\mathfrak{g}, \mathcal{O}_S)$$

is essentially constant.

Proof. For $\widehat{\mathcal{A}} \in \mathcal{VEXT}_X(\mathcal{A})_{\langle\ ,\ \rangle}(X)$, $\widehat{\mathcal{A}}_{\mathcal{T}_{X/S}}$ is a vertex extension of $\mathfrak{g}(\mathcal{A}_{/S}) = \mathcal{A} \times_{\mathcal{T}_X} \mathcal{T}_{X/S}$, a $\mathcal{T}_{X/S}$-transitive algebroid with the induced pairing on $\mathfrak{g}(\mathcal{A})$ equal to $\langle\ ,\ \rangle$. According to Corollary 3 any two such are uniquely isomorphic. The commutativity of (6.3.1) implies that for $\mathcal{Q} \in \mathcal{ECA}_X(X)$ the induced isomorphism $\mathfrak{g}(\int_{X/S}\widehat{\mathcal{A}}) = \int_{X/S}\widehat{\mathcal{A}}_{\mathcal{T}_{X/S}} \cong \int_{X/S}(\mathcal{Q}+\widehat{\mathcal{A}})_{\mathcal{T}_{X/S}} = \mathfrak{g}(\int_{X/S}(\mathcal{Q}+\widehat{\mathcal{A}}))$ is an isomorphism of $q_+\mathcal{A}$-modules. Therefore, all $\widehat{\mathcal{A}} \in \mathcal{VEXT}_X(\mathcal{A})_{\langle\ ,\ \rangle}(X)$ give rise to the "same" K.-M.-V. extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ in $q_+\mathcal{A}$-Lie algebras. □

Proposition 15. *The morphism* (6.1.1) *restricts to the morphism*

$$\int_{X/S} : q_*\mathcal{VEXT}_X(\mathcal{A})_{\langle\ ,\ \rangle} \to \mathcal{O}_S\mathcal{E}x\mathcal{T}(q_+\mathcal{A})_{\widetilde{\mathfrak{g}}} \tag{6.4.1}$$

compatible with respective actions of $q_*\mathcal{ECA}_X$ *and* $\mathcal{PLA}_S$*, and* (6.2.2).

Equivalently, the $\mathcal{PLA}_S$-torsor $\mathcal{O}_S\mathcal{E}x\mathcal{T}(q_+\mathcal{A})_{\widetilde{\mathfrak{g}}}$ is obtained from the $q_*\mathcal{ECA}_X$-torsor $q_*\mathcal{VEXT}_X(\mathcal{A})_{\langle\ ,\ \rangle}$ by "change of the structure group" along the map (6.2.2).

6.5. **Characteristic classes.** For $(\mathcal{A}, \langle\ ,\ \rangle)$ let $C_1(X/S; (\mathcal{A}, \langle\ ,\ \rangle))$ denote the characteristic class of the $\mathcal{PLA}_S$-torsor $\mathcal{O}_S\mathcal{E}x\mathcal{T}(q_+\mathcal{A})_{\widetilde{\mathfrak{g}}}$ in $H^2(S; \Omega^1_S \to \Omega^{2,cl}_S) \cong H^3_{dR}(S)$.

Theorem 16. $C_1(X/S; (\mathcal{A}, \langle\ ,\ \rangle)) = -\dfrac{1}{2}\displaystyle\int_{X/S} \text{п}(\mathcal{A}, \langle\ ,\ \rangle)$.

Proof. It follows from the preceding discussion that $C_1(X/S; (\mathcal{A}, \langle\ ,\ \rangle))$ is the image of the class of the $\mathcal{ECA}_X$-torsor $\mathcal{VEXT}_X(\mathcal{A})_{\langle\ ,\ \rangle}$ in $H^2(X; \Omega^2_X \to \Omega^{3,cl}_X) \cong H^4_{dR}(S)$ under the map

$$\int_{X/S} : H^4_{dR}(X) \to H^3_{dR}(S).$$

The characteristic class of $\mathcal{VEXT}_X(\mathcal{A})_{\langle\ ,\ \rangle}$ is equal to $\mathrm{ch}_2(\mathcal{T}_X) - \frac{1}{2}\text{п}(\mathcal{A}, \langle\ ,\ \rangle)$ ([B]).

The exact sequence

$$0 \to \mathcal{T}_{X/S} \to \mathcal{T}_X \to q^*\mathcal{T}_S \to 0$$

and the additivity of the Chern character show that

$$\mathrm{ch}_2(\mathcal{T}_X) = \mathrm{ch}_2(\mathcal{T}_{X/S}) + q^*\,\mathrm{ch}_2(\mathcal{T}_S).$$

Since $\mathcal{T}_{X/S}$ is a complexification of a real line bundle it follows that $\mathrm{ch}_2(\mathcal{T}_{X/S}) = 0$, and $\int_{X/S} q^*\,\mathrm{ch}_2(\mathcal{T}_S) = 0$ (by the projection formula). Hence, we arrive at the desired formula. □

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