Research report

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August 2022 - July 2024

Introduction. Most of the work done during my stay at the MPIM concerned the structure of discrete groups acting on $\mathrm{RCD}(K,N)$ spaces. The class $\mathrm{RCD}(K,N)$ consists of metric measure spaces having "Ricci curvature $\geq K$ and dimension $\leq N$" in a synthetic sense; a Riemannian manifold equipped with its usual volume measure is an $\mathrm{RCD}(K,N)$ space if and only if it has both Ricci curvature $\geq K$ and dimension $\leq N$. Roughly speaking, an $\mathrm{RCD}(K,N)$ space is a metric measure space (X,d,μ) satisfying two independent properties:

- The entropy functional $\mathrm{Ent}_\mu : \mathbb{P}_2(X) \to \mathbb{R}$, defined on the space of probability measures having finite second moment, is more concave than the corresponding entropy functional in the N-dimensional space form of constant sectional curvature $K/(N-1)$ (and consequently constant Ricci curvature K).
- The Sobolev space $H^{1,2}(X)$ is a Hilbert space.

For a precise definition see [AGMR15] (which builds upon [CC97, LV09, S06a, S06b, AGS14]). The main advantage of working in this general and technical setting is that it is closed under some natural constructions.

- If M is an $(N-1)$-dimensional Riemannian manifold of Ricci curvature $\geq N-2$, then the cone and the suspension of M are in general not Riemannian manifolds, however, they are $\mathrm{RCD}(0,N)$ and $\mathrm{RCD}(N-1,N)$ spaces respectively (when equipped with suitable measures).
- If M is an N-dimensional Riemannian manifold of Ricci curvature $\geq K$ and $G \leq \mathrm{Iso}(X)$ is a compact group of isometries, then M/G is in general not a Riemannian manifold, but it is an $\mathrm{RCD}(K,N)$ space when equipped with an appropriate measure (the same also holds if G is discrete instead of compact).
- If M_i is a sequence of N-dimensional Riemannian manifolds of Ricci curvature $\geq K$, and the sequence M_i converges in the measured Gromov–Hausdorff sense to a metric measure space then such space is an $\mathrm{RCD}(K,N)$ space but rarely a Riemannian manifold.

C-abelianicity. The classic Bonnet–Myers theorem states that if a compact Riemannian manifold has strictly positive Ricci curvature then its fundamental group is finite. This is just a particular instance of the phenomenon

$$[\text{positive curvature}] \Rightarrow [\text{small } \pi_1].$$

A deeper instance of this phenomenon is that the fundamental group of a compact $\mathrm{RCD}(0,N)$ space contains an abelian subgroup of finite index. In the setting of Riemannian manifolds of non-negative sectional curvature it was conjectured by Fukaya–Yamaguchi that this index can be bounded by a number depending only on the dimension. This conjecture has been established under a non-collapsing condition on the universal cover [MRW08], and together with Santos-Rodríguez we generalized this result to $\mathrm{RCD}(0,N)$ spaces [SZ23].

Definition 1. *Let (X,d,μ,p) be a pointed $RCD(K,N)$ space. Its essential dimension is defined to be the supremum of $k \in \mathbb{N}$ for which there is $x \in X$ and a sequence $\lambda_i \to \infty$ such that $(\lambda_i X, x)$ converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k, 0)$. Its collapsing* *volume is defined as*

$$vol^*_{K,N}(X,p) := \inf d_{pGH}((X,p),(Y,q))$$

where d_{pGH} denotes the pointed Gromov–Hausdorff distance and the infimum is taken among pointed $RCD(K,N)$ spaces (Y,d_Y,μ_Y,q) of essential dimension strictly less than the one of (X,d,μ).

Theorem 2. *For each $N \leq 1$, $D > 0$, $\nu > 0$, there is $C > 0$ such that the following holds. Assume (X,d,μ) is an $RCD(0,N)$ space of diameter $\leq D$ and there is $p \in \tilde{X}$ in the universal cover such that $vol^*_{0,N}(\tilde{X},p) \geq \nu$. Then there is an abelian subgroup $\Gamma \leq \pi_1(X)$ of index $\leq C$.*

Assuming that the fundamental group is finite, we were able to establish a similar result under an arbitrary lower curvature bound.

Theorem 3. *For each $K \in \mathbb{R}$, $N \leq 1$, $D > 0$, $\nu > 0$, there is $C > 0$ such that the following holds. Assume (X,d,μ) is an $RCD(K,N)$ space of diameter $\leq D$ and there is $p \in \tilde{X}$ in the universal cover such that $vol^*_{K,N}(\tilde{X},p) \geq \nu$. If the group $\pi_1(X)$ is finite, then there is an abelian subgroup $\Gamma \leq \pi_1(X)$ of index $\leq C$.*

The proofs of both theorems above are similar to the corresponding ones for smooth manifolds with sectional curvature lower bounds. However, in order to establish Theorem 3 we required a diameter bound on the universal cover $\tilde{X}$. This generalizes a result of Kapovitch–Wilking [KW11] with an entirely different proof based on the description of approximate groups by Breuillard–Green–Tao [BGT12].

Theorem 4. *For each $K \in \mathbb{R}$, $N \leq 1$, $D > 0$, there is $\tilde{D} > 0$ such that if an $RCD(K,N)$ space of diameter $\leq D$ has finite fundamental group, then its universal cover has diameter $\leq \tilde{D}$.*

Anderson finiteness. A key tool to prove Theorems 2 and 3 is the principle that if a sequence of N-dimensional Riemannian manifolds of Ricci curvature $\geq K$ doesn't collapse, then their isometry groups do not admit non-trivial small subgroups. Santos-Rodríguez and I generalized this result to RCD(K,N) spaces of arbitrary rectifiable dimension. This was later refined in [Z23] as the ensuing result.

Theorem 5. *For each $K \in \mathbb{R}$, $N \leq 1$, $\nu > 0$, there is $\varepsilon > 0$ such that the following holds. Let (X,d,μ,p) be a pointed $RCD(K,N)$ space and $\Gamma \leq Iso(X)$ a discrete group of measure preserving isometries such that the quotient space satisfies $vol^*_{K,N}(X/\Gamma,[p]) \geq \nu$. Then*

$$\{g \in \Gamma \mid d_p(g, Id_X) \leq \varepsilon\} = \{Id_X\}.$$

In the above theorem, d_p denotes a natural metric in the group of isometries that induces the compact-open topology. As a consequence of this result, we deduce an extension to RCD(K,N) spaces of a finiteness result by Anderson [A90].

Theorem 6. *For each $K \in \mathbb{R}$, $N \leq 1$, $D > 0$, $\nu > 0$, the class of $RCD(K,N)$ spaces of diameter $\leq D$ and collapsing volume $\geq \nu$ contains finitely isomorphism classes of fundamental groups.*

Margulis Lemma. Together with Qin Deng, Jaime Santos-Rodríguez, and Xinrui Zhao, we fully described the fundamental groups of small RCD(K, N) spaces [DSZZ23].

Theorem 7. *For each RCD(K,N) space (X, d, μ) of diameter $\leq \varepsilon(K,N)$, there is a subgroup $\Gamma \leq \pi_1(X)$ of index $[\pi_1(X), \Gamma] \leq C(K,N)$ with $\Gamma = \langle u_1, \ldots, u_N \rangle$ and $[u_i, u_j] \in \langle u_{j+1}, \ldots, u_N \rangle$ for each i, j.*

This result was originally proven for smooth Riemannian manifolds by Kapovitch–Wilking and for most of our proof, we just follow their steps. However, the original proof makes heavy use of the fact that if one controls the derivative of a diffeomorphism $f : M \to M$ at a point $p \in M$, then one gets for free control on the local behavior of f in a neighborhood of p. In the non-smooth setting, this elementary fact of calculus is missing, so in order to prove Theorem 7, we establish a highly technical replacement of smoothness for gradient flows of harmonic functions [DSZZ23, Theorem 1.2], building on top of previous work of Deng.

It was also proven in [KW11] that in the setting of Theorem 7, if X is a smooth Riemannian manifold, and $\langle u_i, \ldots, u_N \rangle / \langle u_{i+1}, \ldots, u_N \rangle = \mathbb{Z}$ for all i, then X is homeomorphic to an infranilmanifold. Together with Xingu Zhu, we generalized this result to RCD(K, N) spaces as well [ZZ24], this time with a completely different proof, building upon the results of my PhD thesis, which were inspired on a short basis construction by Breuillard–Green–Tao [BGT12].

Papers

Papers written entirely during my stay: [DSZZ23, SZ23, Z22, Z23, ZZ24]

Papers significantly edited during my stay: [Z24b, Z24a].

Lectures

Hauptseminar Differentialgeometrie Topic: Curves and surfaces in $\mathbb{R}^3$. After this course, two Bachelor students wrote their theses under my direction [Ki24, Ko24].

Mathematicians from Bonn

Jaime Santos-Rodríguez: Postdoctoral researcher at MPIM. [DSZZ23, SZ23].
Xingyu Zhu: Postdoctoral researcher at University of Bonn. [ZZ24].
Cameron Rudd: Postdoctoral researcher at MPIM.
Andrew Ng: PhD student at University of Bonn.

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