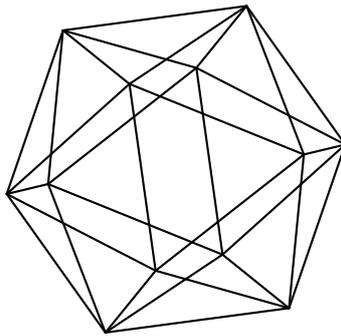# Max-Planck-Institut für Mathematik Bonn

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De Concini-Kac-Procesi conjecture

by

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Introduction

Schur–Weyl duality philosophy in the Hecke algebra setting

In this monograph we consider a phenomenon which occurs in the study of certain classes and categories of representations of semisimple Lie algebras, groups of Lie type, and the related quantum groups. This phenomenon is similar to the classical Schur–Weyl duality. However, the relevant classes of representations are quite different from the finite-dimensional irreducible representations of the general linear group or, more generally, of complex semisimple Lie groups which appear in the Schur–Weyl setting.

All examples of the above mentioned type are realizations of the following quite general construction a homological version of which was suggested in [109] (in fact, in the quantum group case the construction presented below requires some technical modifications; we shall not discuss them in the introduction). Let A be an associative algebra over a unital ring $\mathbf{k}$, $B \subset A$ a subalgebra with a character $\chi : B \to \mathbf{k}$. Denote by $\mathbf{k}_\chi$ the corresponding rank one representation of B. Let $Q_\chi = A \otimes_B \mathbf{k}_\chi$ be the induced representation of A.

Let $\mathrm{Hk}(A, B, \chi) = \mathrm{End}_A(Q_\chi)^{opp}$ be the algebra of A–endomorphisms of Q_χ with the opposite multiplication. One says that the algebra $\mathrm{Hk}(A, B, \chi)$ is obtained from A by a quantum constrained reduction with respect to the subalgebra B. $\mathrm{Hk}(A, B, \chi)$ is an algebra of Hecke type. Indeed, if A is the group algebra of a Chevalley group over a finite field, B the group algebra of a Borel subgroup in it, and χ is the trivial complex representation of the Borel subgroup one obtains the Iwahori–Hecke algebra this way (see [56]).

For any representation V of A the algebra $\mathrm{Hk}(A, B, \chi)$ naturally acts in the space

$$V_\chi = \mathrm{Hom}_A(Q_\chi, V) \simeq \mathrm{Hom}_B(\mathbf{k}_\chi, V)$$

by compositions of homomorphisms and for any $\mathrm{Hk}(A, B, \chi)$–module W $Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} W$ is a left A–module. Let $\mathrm{Hk}(A, B, \chi) - \mathrm{mod}$ be the category of left $\mathrm{Hk}(A, B, \chi)$–modules and $A - \mathrm{mod}_B^\chi$ the category of left A–modules of the form $Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} W$, where $W \in \mathrm{Hk}(A, B, \chi) - \mathrm{mod}$, with morphisms induced by morphisms of left $\mathrm{Hk}(A, B, \chi)$–modules. The point is that in many important examples $A - \mathrm{mod}_B^\chi$ is a full subcategory in the category of left A–modules, and the functors $\mathrm{Hom}_A(Q_\chi, \cdot) : A - \mathrm{mod}_B^\chi \to \mathrm{Hk}(A, B, \chi) - \mathrm{mod}$ and $Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} \cdot :$ $\mathrm{Hk}(A, B, \chi) - \mathrm{mod} \to A - \mathrm{mod}_B^\chi$ yield mutually inverse Schur–Weyl type equivalences of the categories,

$$A - \mathrm{mod}_B^\chi \simeq \mathrm{Hk}(A, B, \chi) - \mathrm{mod}. \tag{1}$$

In these cases for $W \in \mathrm{Hk}(A, B, \chi) - \mathrm{mod}$ one has

$$(Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} W)_\chi = \mathrm{Hom}_A(Q_\chi, Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} W) \simeq \mathrm{Hom}_B(\mathbf{k}_\chi, Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} W) \simeq W,$$

and for $W, W' \in \mathrm{Hk}(A, B, \chi) - \mathrm{mod}$ by the formula above

$$\mathrm{Hom}_A(Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} W', Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} W) \simeq \mathrm{Hom}_{\mathrm{Hk}(A,B,\chi)}(W', \mathrm{Hom}_A(Q_\chi, Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} W)) \simeq$$

$$= \mathrm{Hom}_{Hk(A,B,\chi)}(W', W).$$

At first sight the category $A - \mathrm{mod}_B^\chi$ looks a bit exotic. But it turns out that in many situations it has alternative descriptions in terms of the algebra A, its subalgebra B and the character χ only, and actually such categories and the related algebras of Hecke type played a very important, if not central, role in representation theory for at least last sixty years.

An important example of equivalences of type (1) was considered in [72]. In this paper Kostant showed that algebraic analogues of the principal series representations, called irreducible Whittaker modules, for a complex semisimple Lie algebra $\mathfrak{g}$ are in one–to–one correspondence with the one–dimensional representations of the center $Z(U(\mathfrak{g}))$ of the enveloping algebra $U(\mathfrak{g})$

In the situation considered in [72] the algebra $Z(U(\mathfrak{g}))$ is isomorphic to $\mathrm{Hk}(A, B, \chi)$ with $A = U(\mathfrak{g})$, $B = U(\mathfrak{n}_-)$, where $\mathfrak{n}_-$ is a nilradical of $\mathfrak{g}$, and χ being a non–singular character of $U(\mathfrak{n}_-)$, i.e. it does not vanish on all simple root vectors in $\mathfrak{n}_-$. The category $A - \mathrm{mod}_B^\chi$ in this case can be described as the category of $\mathfrak{g}$–modules on which $x - \chi(x)$ acts locally nilpotently for any $x \in \mathfrak{n}_-$.

This correspondence was generalized in [89] and in the Appendix to [97] to a more general categorical setting and the categorical equivalence established in the Appendix to [97] is called the Skryabin equivalence.

Similar equivalences were obtained in [97] in the case of semisimple Lie algebras over fields of prime characteristic and in [116] in case of quantum groups associated to complex semisimple Lie algebras for generic values of the deformation parameter. Various approaches to the proofs of the above mentioned statements have been developed in [42, 120].

An analogous construction appears also in the case of finite groups of Lie type (see [19], Chapter 10) and of finite Chevalley groups (see [63, 64]). In the latter case the corresponding modules Q_χ are called generalized Gelfand–Graev representations.

Note that in general the problem of classification of $\mathrm{Hk}(A, B, \chi)$–modules is usually very difficult. Sometimes it is easier to classify irreducible objects in the category $A - \mathrm{mod}_B^\chi$ and then to translate the result to the category $\mathrm{Hk}(A, B, \chi) - \mathrm{mod}$ (see [76, 77] for the case of algebras $\mathrm{Hk}(A, B, \chi)$ considered in [97]).

A strategy for establishing Schur–Weyl type equivalences in the Hecke algebra setting for Lie algebras and quantum groups, and its relation to Zhelobenko operators

In all cases considered in [72, 89, 97, 116] the algebras A and B, the characters χ and the appropriate categories $A - \mathrm{mod}_B^\chi$ and $\mathrm{Hk}(A, B, \chi) - \mathrm{mod}$ of representations of A and of $\mathrm{Hk}(A, B, \chi)$ are relatively easy to define. It is much more difficult to obtain alternative descriptions of the category $A - \mathrm{mod}_B^\chi$. However, one should note that the approach to this problem in all papers mentioned above is slightly different: all those papers start with the description of a category of A–modules in intrinsic terms using the algebra A, its subalgebra B and the character χ. And then one proves that this category is equivalent to the category $\mathrm{Hk}(A, B, \chi) - \mathrm{mod}$, the equivalence being established using the functors $\mathrm{Hom}_A(Q_\chi, \cdot)$ and $Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} \cdot$. Finally one deduces that this category actually coincides with $A - \mathrm{mod}_B^\chi$.

In the Lie algebra case the most simple proofs of statements of this kind were proposed in the Appendix to [97] in the zero characteristic case and in [120] in the prime characteristic case. But the phenomenon behind these proofs is already manifest in [72]. Namely, in the case of Lie algebras over fields of zero characteristic one always has $A = U(\mathfrak{g})$ and $B = U(\mathfrak{m})$ for some reductive Lie algebra $\mathfrak{g}$ and a nilpotent Lie subalgebra $\mathfrak{m} \subset \mathfrak{g}$, and the above mentioned phenomenon amounts to introducing a second $U(\mathfrak{m})$–module structure on Q_χ by tensoring with the one–dimensional representation $\mathbf{k}_{-\chi}$ and to demonstrating that for $\mathbf{k} = \mathbb{C}$ a certain "classical limit" of the $U(\mathfrak{m})$–module $Q_\chi \otimes \mathbf{k}_{-\chi}$ is isomorphic to the algebra of regular functions $\mathbb{C}[\mathcal{C}]$ on a closed algebraic variety $\mathcal{C}$, and the "classical limit" of the $U(\mathfrak{m})$–action on $Q_\chi \otimes \mathbf{k}_{-\chi}$ is induced by a free action of the complex unipotent algebraic group M corresponding to the Lie algebra $\mathfrak{m}$ on $\mathcal{C}$. The "classical limits" here are understood in the sense of taking associate graded objects with respect to suitable filtrations.

The action

$$M \times \mathcal{C} \to \mathcal{C}$$

has a global cross-section $\Sigma \subset \mathcal{C}$, called a Slodowy slice, so that the action map

$$M \times \Sigma \to \mathcal{C} \tag{2}$$

is an isomorphism of varieties, and

$$\mathbb{C}[\mathcal{C}] \simeq \mathbb{C}[\Sigma] \otimes \mathbb{C}[M]. \tag{3}$$

The space $W_0 = \mathbb{C}[\Sigma] \simeq \mathbb{C}[\mathcal{C}]^M$ can be regarded as a "classical limit" of $\mathrm{Hk}(A, B, \chi)$ which is called a W–algebra in this case. We can also write $\mathbb{C}[\mathcal{C}] \simeq W_0[M]$, where $W_0[M]$ is the algebra of regular functions on M with values in W_0. In fact W_0 carries the natural structure of a Poisson algebra. It is called a Poisson W–algebra.

Let $A - \mathrm{mod}_B^\chi$ be the category of left A–modules V for which the $U(\mathfrak{m})$–action on $V \otimes \mathbf{k}_{-\chi}$ is locally nilpotent. In the Appendix to [97] it is shown that if one equips $V \in A - \mathrm{mod}_B^\chi$ with a second $U(\mathfrak{m})$–module structure by tensoring with $\mathbf{k}_{-\chi}$ then, as a $U(\mathfrak{m})$–module, $V \otimes \mathbf{k}_{-\chi}$ is isomorphic to $\mathrm{hom}_{\mathbf{k}}(U(\mathfrak{m}), V_\chi)$,

$$V \otimes \mathbf{k}_{-\chi} \simeq \mathrm{hom}_{\mathbf{k}}(U(\mathfrak{m}), V_\chi) \simeq V_\chi[M], \tag{4}$$

where $\mathrm{hom}_{\mathbf{k}}$ stands for the space of homomorphisms vanishing on some power of the augmentation ideal of $U(\mathfrak{m})$, $V_\chi = \mathrm{Hom}_{U(\mathfrak{m})}(\mathbf{k}_\chi, V)$ is called the space of Whittaker vectors in V, and, as above, the latter isomorphism holds

if $\mathbf{k} = \mathbb{C}$. In the Appendix to [97] it is shown that isomorphisms (4) directly imply an equivalence between the category of left A–modules V for which the $U(\mathfrak{m})$–action on $V \otimes \mathbf{k}_{-\chi}$ is locally nilpotent and the corresponding category $A - \mathrm{mod}_B^\chi$ introduced before formula (1).

Isomorphisms of type (2) occur in the quantum group setting as well (see [113, 114, 116]), and the same idea is applied in [116] to establish similar categorical equivalences in the quantum group case for generic values of the deformation parameter.

In [115, 119] it was observed that an isomorphism of type (2) gives rise to a natural projection operator $\Pi : \mathbb{C}[\Sigma] \to \mathbb{C}[\mathcal{C}]^M \simeq \mathbb{C}[\Sigma] = W_0$. Namely, according to (2) any $x \in \mathcal{C}$ can be uniquely represented in the form

$$x = n(x) \circ \sigma(x), n(x) \in M, \sigma(x) \in \Sigma. \tag{5}$$

If for $f \in \mathbb{C}[\mathcal{C}]$ we define $\Pi f \in \mathbb{C}[\mathcal{C}]$ by

$$(\Pi f)(x) = f(n^{-1}(x) \circ x) = f(\sigma(x)) \tag{6}$$

then Πf is an M–invariant function, and any M–invariant regular function on $\mathcal{C}$ can be obtained this way. Moreover, by the definition $\Pi^2 = \Pi$, i.e. Π is a projection onto $\mathbb{C}[\mathcal{C}]^M$.

In the quantum group setting considered in [113, 114, 116] the "classical limiting" variety $\mathcal{C}$ is always a closed subvariety in a complex semisimple algebraic Lie group G, Σ is an analogue of a Slodowy slice for G introduced in [113, 118], and M is a unipotent subgroup of G, where the "classical limit" simply corresponds now to the $q = 1$ specialization of the deformation parameter q. The peculiarity of the quantum group case is that every element of M can be uniquely represented as an ordered product of elements of some one–parameter subgroups $M_i \subset G$, $i = 1, \ldots, c$ corresponding to roots, i.e. $M = M_1 \ldots M_c$. If we denote by t_i the parameter in M_i and by $X_i(t_i)$ the element of M_i corresponding to the value $t_i \in \mathbb{C}$ of the parameter then factorizing $n(x)$ in (5) as follows

$$n(x) = X_1(t_1(x)) \ldots X_c(t_c(x)) \tag{7}$$

one can express the operator Π as a composition of operators Π_i,

$$(\Pi_i f)(x) = f(X_i(-t_i(x)) \circ x), \tag{8}$$

$$\Pi f = \Pi_1 \ldots \Pi_c f. \tag{9}$$

$t_i(x)$ here can be regarded as regular functions on $\mathcal{C} \subset G$.

The first miracle of the quantum group case is that there are explicit formulas for the functions $t_i(x)$ in (7) expressing them in terms of matrix elements of finite-dimensional irreducible representations of G. These formulas were obtained in [119].

The main objective of this book is to obtain quantum group counterparts of these formulas. This provides a description of quantum group analogues of W–algebras, called q-W–algebras, as images of operators Π_c^q which are quantum analogues of Π, or more precisely, of operators Π_c introduced by formula (3.5.21). This description implies that q-W–algebras belong to the class of the so-called Mickelsson algebras (see e.g. [133, 136, 137, 138, 140] and [141], Ch. 4).

Magically, the classical formulas for $t_i(x)$ and formulas (8) can be directly extrapolated to the quantum case, so the operator Π_c^q is given in a factorized form similar to (9). Note that no operators similar to Π_c^q can be defined in the Lie algebra setting discussed above.

Using the quantum group analogues B_{jk} of the functions $t_i(x)$ one can also construct natural bases in modules V from the corresponding category $A - \mathrm{mod}_B^\chi$ and establish isomorphisms similar to (4) in the case when the deformation parameter is not a root of unity. Recall that in the Lie algebra case with $\mathbf{k} = \mathbb{C}$ for any $V \in A - \mathrm{mod}_B^\chi$ the Skryabin equivalence provides an isomorphism $V \simeq Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} V_\chi$. If we denote by V_χ^0 the "classical limit", i.e. the $q = 1$ specialization, of V_χ then recalling that the "classical limit" of Q_χ is $\mathbb{C}[\mathcal{C}]$ and the "classical limit" of $\mathrm{Hk}(A,B,\chi)$ is $\mathbb{C}[\Sigma]$ we infer from (3) that the "classical limit" of $Q_\chi \otimes_{\mathrm{Hk}(A,B,\chi)} V_\chi \simeq V$ is

$$\mathbb{C}[\mathcal{C}] \otimes_{\mathbb{C}[\Sigma]} V_\chi^0 \simeq (\mathbb{C}[\Sigma] \otimes \mathbb{C}[M]) \otimes_{\mathbb{C}[\Sigma]} V_\chi^0 \simeq \mathbb{C}[M] \otimes V_\chi^0.$$

These isomorphisms together with (5) and (7) give a hint how to construct natural bases in modules from the category $V \in A - \mathrm{mod}_B^\chi$ in the quantum group case. Namely, if V is such a module it is natural to expect that if one picks up a linear basis v_p, $p \in \mathbb{N}$ in the space of Whittaker vectors V_χ then the elements of V given by properly defined ordered monomials in B_{jk} applied to v_p, $p \in \mathbb{N}$ form a linear basis in V. We show that this is indeed the case. These bases are key ingredients for an alternative proof of the Skryabin equivalence for quantum groups.

Operators conceptually similar to Π_c^q appeared in the literature a long time ago as the projection operators onto subspaces of singular vectors in some modules over a complex finite-dimensional semisimple Lie algebra $\mathfrak{g}$ the action of a nilradical $\mathfrak{n}_- \subset \mathfrak{g}$ on which is locally nilpotent. The first example of such operators, called extremal projection operators, for $\mathfrak{g} = \mathfrak{sl}_2$ was explicitly constructed in [78]. In papers [3, 4, 5] the results of [78] were generalized to the case of arbitrary complex semisimple Lie algebras, and explicit formulas for extremal projection operators were obtained. A summary of these results can be found in [128]. Later, using a certain completion of an extension of the universal enveloping algebra of $\mathfrak{g}$, Zhelobenko observed in [133] that the existence of extremal projection operators is an almost trivial fact. In [133] he also introduced a family of operators which are analogues to our operators P_i. These operators are called now Zhelobenko operators. Properties of extremal projection operators and of the Zhelobenko operators have been extensively studied in [133]–[140], and the results obtained in these papers were summarized in book [141].

In our terminology the situation considered in these works corresponds to the case when $A = U(\mathfrak{g})$, $B = U(\mathfrak{n}_-)$ and χ is the trivial character of $U(\mathfrak{n}_-)$. As observed in [115], in this case $\mathcal{C} = \mathfrak{b}_-$, the Borel subalgebra $\mathfrak{b}_- \subset \mathfrak{g}$ containing $\mathfrak{n}_-$, and the action of the unipotent group N_- corresponding to $\mathfrak{n}_-$ on $\mathcal{C}$ is induced by the adjoint action of a Lie group G with the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}$. This action is not free but is gives rise to a birational equivalence

$$N_- \times \mathfrak{h} \to \mathfrak{b}_-,$$

where $\mathfrak{h} = \mathfrak{b}_-/\mathfrak{n}_-$ is a Cartan subalgebra. In [115] it is shown that using this birational equivalence one can still define operators similar to Π_i and Π acting on a certain localization of the algebra of regular functions $\mathbb{C}[\mathfrak{b}_-]$ and these operators are "classical limits" of the Zhelobenko and of the extremal projection operators, respectively.

Kac–Weisfeiler and De Concini–Kac–Procesi conjectures

Remarkably, as observed in [120], the arguments from the Appendix to [97] are applicable to obtain alternative descriptions of the corresponding categories $A - \mathrm{mod}_B^\chi$ from [97] for Lie algebras over fields of prime characteristic.

Along the same line, formulas for the quantum group analogues B_{jk} of the functions $t_i(x)$ and for the operator Π_c^q can be specialized to the case when q is a primitive odd m-th root of unity ε subject to a few other conditions depending on the Cartan matrix of the corresponding semisimple Lie algebra $\mathfrak{g}$. This provides technical tools for the proof of a root of unity version of the Skryabin equivalence for quantum groups. Similarly to the case of generic q one can construct bases in modules V from the corresponding category $A - \mathrm{mod}_B^\chi$. In case when q is a root of unity all such irreducible modules are finite-dimensional, and if one picks up a linear basis v_p, $p = 1, \ldots, n$ in the space of Whittaker vectors V_χ then the elements of V given by applied to v_p, $p = 1, \ldots, n$ properly defined ordered monomials in c variables B_{jk} powers of which are truncated at the degree m form a linear basis in V. In particular, the dimension of V is divisible by m^c. It turns out that any finite-dimensional module over the standard quantum group $U_\varepsilon(\mathfrak{g})$, where ε is a primitive odd m-th root of unity subject to the extra conditions mentioned in the beginning of this paragraph, belongs to one of the categories $A - \mathrm{mod}_B^\chi$ with appropriate A, B and χ, so its dimension is divisible by $b = m^c$. Moreover, the number b is equal to the number from the De Concini–Kac–Procesi conjecture on dimensions of irreducible modules over quantum groups at roots of unity suggested in [25]. Thus our result confirms this conjecture. Due to its importance we are going to discuss the De Concini–Kac–Procesi conjecture in more detail.

It is very well known that the number of simple modules for a finite-dimensional algebra is finite. However, often it is very difficult to classify such representations. In some important particular examples even dimensions of simple modules over finite-dimensional algebras are not known.

One of the important examples of that kind is representation theory of semisimple Lie algebras over algebraically closed fields of prime characteristic. Let $\mathfrak{g}'$ be the Lie algebra of a semisimple algebraic group G' over an algebraically closed field $\mathbf{k}$ of characteristic $p > 0$. Let $x \mapsto x^{[p]}$ be the p-th power map of $\mathfrak{g}'$ into itself. The structure of the enveloping algebra of $\mathfrak{g}'$ is quite different from the zero characteristic case. Namely, the elements $x^p - x^{[p]}$, $x \in \mathfrak{g}'$ are central. For any linear form θ on $\mathfrak{g}'$, let U_θ be the quotient of the enveloping algebra of $\mathfrak{g}'$ by the ideal generated by the central elements $x^p - x^{[p]} - \theta(x)^p$ with $x \in \mathfrak{g}'$. Then U_θ is a finite-dimensional algebra. Kac and Weisfeiler proved that any simple $\mathfrak{g}'$-module can be regarded as a module over U_θ for a unique θ as above (this explains why all simple $\mathfrak{g}'$–modules are finite-dimensional). The Kac–Weisfeiler conjecture formulated in [61] and proved in [98] says that if the G'–coadjoint orbit of θ has dimension $\dim \mathcal{O}_\theta$ and p is good for the root system of G' then $p^{\frac{\dim \mathcal{O}_\theta}{2}}$ divides the dimension of every finite-dimensional U_θ–module.

One can identify θ with an element of $\mathfrak{g}'$ via the Killing form and reduce the proof of the Kac–Weisfeiler conjecture to the case of nilpotent θ. In that case Premet defines in [98] a subalgebra $U_\theta(\mathfrak{m}_\theta) \subset U_\theta$ generated

by a Lie subalgebra $\mathfrak{m}_\theta \subset \mathfrak{g}'$ such that $U_\theta(\mathfrak{m}_\theta)$ has dimension $p^{\frac{\dim \mathcal{O}_\theta}{2}}$ and every finite-dimensional U_θ–module is $U_\theta(\mathfrak{m}_\theta)$–free. Verification of the latter fact uses the theory of support varieties (see [39, 40, 41, 99]). Namely, according to the theory of support varieties, in order to prove that a U_θ–module is $U_\theta(\mathfrak{m}_\theta)$–free one should check that it is free over every subalgebra $U_\theta(x)$ generated in $U_\theta(\mathfrak{m}_\theta)$ by a single element $x \in \mathfrak{m}_\theta$.

There is a more elementary and straightforward proof of the Kac–Weisfeiler conjecture given in [96]. The simplest proof of this conjecture follows from the results of [120] on a prime characteristic version of the Skryabin equivalence which we already discussed above. A proof of the conjecture for $p > h$, where h is the Coxeter number of the corresponding root system, using localization of $\mathcal{D}$–modules was presented in [7].

Another important example of finite-dimensional algebras is related to the theory of quantum groups at roots of unity. Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra. A remarkable property of the standard Drinfeld-Jimbo quantum group $U_\varepsilon(\mathfrak{g})$ associated to $\mathfrak{g}$, where ε is a primitive m-th root of unity, is that its center contains a large commutative subalgebra. In this book we consider the simply connected version of $U_\varepsilon(\mathfrak{g})$ and the case when m is odd. In this case the large commutative subalgebra is isomorphic to the algebra Z_G of regular functions on (a finite covering of a big cell in) the connected, simply connected complex algebraic group corresponding to $\mathfrak{g}$.

Consider finite-dimensional representations of $U_\varepsilon(\mathfrak{g})$, on which Z_G acts according to non–trivial characters η_g given by evaluation of regular functions at various points $g \in G$. Note that all irreducible representations of $U_\varepsilon(\mathfrak{g})$ are of that kind, and every such representation is a representation of the algebra $U_{\eta_g} = U_\varepsilon(\mathfrak{g})/U_\varepsilon(\mathfrak{g})\mathrm{Ker}\,\eta_g$ for some η_g. In [25] De Concini, Kac and Procesi showed that if g_1 and g_2 are two conjugate elements of G then the algebras $U_{\eta_{g_1}}$ and $U_{\eta_{g_2}}$ are isomorphic. Moreover in [25] De Concini, Kac and Procesi formulated the following conjecture.

De Concini–Kac–Procesi conjecture. *The dimension of any finite-dimensional representation of the algebra U_{η_g} is divisible by $b = m^{\frac{1}{2}\dim \mathcal{O}_g}$, where $\mathcal{O}_g$ is the conjugacy class of g.*

This conjecture is the quantum group counterpart of the Kac–Weisfeiler conjecture for semisimple Lie algebras over fields of prime characteristic.

As it is shown in [24] it suffices to verify the De Concini–Kac–Procesi conjecture in case of exceptional elements $g \in G$ (an element $g \in G$ is called exceptional if the centralizer in G of its semisimple part has a finite center). However, the De Concini–Kac–Procesi conjecture is related to the geometry of the group G which is more complicated than the geometry of the linear space $\mathfrak{g}'$ in case of the Kac–Weisfeiler conjecture.

The De Concini–Kac–Procesi conjecture is known to be true for the conjugacy classes of regular elements (see [26]), for the subregular unipotent conjugacy classes in type A_n when m is a power of a prime number (see [14]), for all conjugacy classes in A_n when m is a prime number (see [16]), for the conjugacy classes $\mathcal{O}_g$ of $g \in SL_n$ when the conjugacy class of the unipotent part of g is spherical (see [15]), and for spherical conjugacy classes (see [13]). In [73] a proof of the De Concini–Kac–Procesi conjecture using localization of quantum $\mathcal{D}$–modules was outlined in case of unipotent conjugacy classes. In contract to many papers quoted above the strategy of the proof of the De Concini–Kac–Procesi conjecture developed in this book does not use the reduction to the case of exceptional elements, and all conjugacy classes are treated uniformly.

Namely, following Premet's philosophy we use certain subalgebras $U_{\eta_g}(\mathfrak{m}_-) \subset U_{\eta_g}$ introduced in [117]. These subalgebras have non–trivial characters $\chi : U_{\eta_g}(\mathfrak{m}_-) \to \mathbb{C}$. In terms of the previously introduced notation, we show that for $A = U_{\eta_g}$, $B = U_{\eta_g}(\mathfrak{m}_-)$ and an appropriate χ the category $A - \mathrm{mod}_B^\chi$ can be identified with the category of finite-dimensional representations of U_{η_g} and equivalence (1) holds if $\mathrm{Hk}(A, B, \chi) - \mathrm{mod}$ is the category of finite-dimensional representations of the corresponding algebra $\mathrm{Hk}(A, B, \chi)$.

As observed in [117] every finite-dimensional U_{η_g}–module is also equipped with an action of the algebra $U_{\eta_1}(\mathfrak{m}_-)$ corresponding to the trivial character η_1 of Z_G given by the evaluation at the identity element of G. In the setting of quantum groups at roots of unity this action is a counterpart of the second $U(\mathfrak{m})$–module structure on objects V of the category $A - \mathrm{mod}_B^\chi$ which appeared in (4) in the case of Lie algebras over fields of zero characteristic.

Since the De Concini–Kac–Procesi conjecture is related to the structure of the set of conjugacy classes in G it is natural to look at transversal slices to the set of conjugacy classes. It turns out that the definition of the subalgebras $U_{\eta_g}(\mathfrak{m}_-)$ is related to the existence of some special transversal slices Σ_s to conjugacy classes in G. These slices Σ_s associated to (conjugacy classes of) elements s in the Weyl group of $\mathfrak{g}$ were introduced by the author in [113]. The slices Σ_s play the role of Slodowy slices in algebraic group theory. In the particular case of elliptic Weyl group elements these slices were also introduced later by He and Lusztig in paper [52] within a different framework.

A remarkable property of a slice Σ_s observed in [117] is that if g is conjugate to an element in Σ_s then the dimension of the corresponding subalgebra $U_{\eta_g}(\mathfrak{m}_-) \subset U_{\eta_g}$ is equal to $m^{\frac{1}{2}\mathrm{codim}\,\Sigma_s}$. The dimension of the algebra $U_{\eta_1}(\mathfrak{m}_-)$ is also equal to $m^{\frac{1}{2}\mathrm{codim}\,\Sigma_s}$. If $g \in \Sigma_s$ (in fact g may belong to a larger variety) then the corresponding

subalgebras $U_{\eta_g}(\mathfrak{m}_-)$ and $U_{\eta_1}(\mathfrak{m}_-)$ can be explicitly described in terms of quantum group analogues of root vectors. Note that one can also define analogues $U_h^s(\mathfrak{m}_-)$ of subalgebras $U_{\eta_g}(\mathfrak{m}_-)$ in the standard Drinfeld–Jimbo quantum group $U_h(\mathfrak{g})$ over the ring of formal power series $\mathbb{C}[[h]]$ (see [114]).

In [114], Theorem 5.2 it is shown that for every conjugacy class $\mathcal{O}$ in G one can find a transversal slice Σ_s such that $\mathcal{O}$ intersects Σ_s and $\dim \mathcal{O} = \mathrm{codim}\ \Sigma_s$. Using this result we showed in [117] that for every element $g \in G$ one can find a a subalgebra $U_{\eta_g}(\mathfrak{m}_-)$ in U_{η_g} of dimension $m^{\frac{1}{2}\dim \mathcal{O}_g}$ with a non–trivial character χ. The dimension of the corresponding algebra $U_{\eta_1}(\mathfrak{m}_-)$ is also equal to $m^{\frac{1}{2}\dim \mathcal{O}_g}$.

Following the strategy outlined in the beginning of this section we show that if m satisfies a certain condition then every finite-dimensional U_{η_g}–module is free over $U_{\eta_1}(\mathfrak{m}_-)$. Thus the dimension of every such module is divisible by $m^{\frac{1}{2}\dim \mathcal{O}_g}$. This establishes the De Concini–Kac–Procesi conjecture.

Note that in the case of restricted representations of a small quantum group similar results were obtained in [34]. The situation in [34] is rather similar to the case of the trivial character $\eta = \eta_1$ in our setting.

We also show that the rank of every finite-dimensional U_{η_g}–module V over $U_{\eta_1}(\mathfrak{m}_-)$ is equal to the dimension of the space V_χ and that U_{η_g} is the algebra of matrices of size $m^{\frac{1}{2}\dim \mathcal{O}_g}$ over the corresponding q-W–algebra $\mathrm{Hk}(A, B, \chi) = \mathrm{Hk}(U_{\eta_g}, U_{\eta_g}(\mathfrak{m}_-), \chi)$ which has dimension $m^{\dim \Sigma_s}$. In case of Lie algebras over fields of prime characteristic similar results were obtained in [97].

Note that the support variety technique used in [98] to prove the Kac–Weisfeiler conjecture can not be transferred to the case of quantum groups straightforwardly. The notion of the support variety is still available in case of quantum groups (see [34, 47, 93]). But in practical applications it is much less efficient since in the case of quantum groups there is no any underlying linear space.

The structure of the book

In conclusion we would like to make a few remarks on the structure of the book. It consists of six chapters. In this introduction we have given a very superficial and incomplete review of the content of the book which rather aims to provide the reader with a general guide outlining the main ideas and the strategy of the main proofs. More technical comments are given in the beginning of each chapter.

In Chapters 1 and 2 we summarize results from [113, 114, 116, 118, 119] on the algebraic group analogues Σ_s of the Slodowy slices and the related results on quantum groups and on the subalgebras $U_h^s(\mathfrak{m}_-) \subset U_h(\mathfrak{g})$. Chapter 1 also contains some results on combinatorics of Weyl groups and on root systems required for the definition of the slices Σ_s, and Chapter 2 contains some advanced results on quantum groups required later for the study of q-W–algebras.

In Chapter 3, following [114, 116], we recall the definition of q-W–algebras and the description of their classical Poisson counterparts given in [119] in terms of the Zhelobenko type operators Π_{jk} and Π. The main purpose of this chapter is to bring this description to a form suitable for quantization. Formulas (3.5.9), (3.5.19) and (3.5.21) obtained in this chapter for Π_{jk} and Π_c have direct quantum analogues (4.2.27), (4.6.1) and (4.7.3) obtained in Chapter 4 for P_{jk} and Π_c^q. The main result of Chapter 4 (Theorem 4.7.2) is the description of the q-W–algebra as the image of the operator Π_c^q.

In Chapter 5 we prove a version of the Skryabin equivalence of type (1) for equivariant modules over quantum groups established in [116]. The new proof of this equivalence in Theorem 5.2.1 is based on Corollary 4.6.8 which allows to construct some nice bases in modules from the category $A - \mathrm{mod}_B^\chi$ (see the discussion in the introduction above). Theorem 5.2.1 also gives precise values of ε of the deformation parameter q for which the categorical equivalence holds while in [116] it was established for generic ε only.

Finally in Chapter 6 we apply the results of Chapter 4 to the study of representations of quantum groups at roots of unity and prove the De Concini–Kac–Procesi conjecture. The strategy of this proof has already been discussed above.

Citations in the main text are reduced to a minimum. References to proofs which are omitted in the body of the text and some historic remarks are given in the bibliographic comments after each chapter.

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Contents

Chapter 1

Algebraic group analogues of Slodowy slices

The q-W–algebras are non-commutative deformations of algebras of regular functions on certain algebraic varieties in algebraic groups transversal to conjugacy classes. In this book these varieties play a role similar to that of the Slodowy slices in the theory of W–algebras and of generalized Gelfand-Geraev representations of semisimple Lie algebras. In this chapter we define these varieties and study their properties. We also develop the relevant Weyl group combinatorics.

1.1 Notation

Fix the notation used throughout the book.

In this book we denote by $\mathbb{N}$ the set of non-negative integer numbers, $\mathbb{N} = \{0, 1, \ldots\}$.

Let $G_{\mathbf{k}}$ be a connected finite–dimensional semisimple algebraic group over an algebraically closed field $\mathbf{k}$. Denote by $\mathfrak{g}_{\mathbf{k}}$ the Lie algebra of $G_{\mathbf{k}}$. Let $H_{\mathbf{k}} \subset G_{\mathbf{k}}$ be a maximal torus in $G_{\mathbf{k}}$, $\mathfrak{h}_{\mathbf{k}} \subset \mathfrak{g}_{\mathbf{k}}$ the corresponding Cartan subalgebra.

If the characteristic exponent of $\mathbf{k}$ is p, i.e. $p = \mathrm{char}\mathbf{k}$ if $\mathrm{char}\mathbf{k} > 0$ and $p = 1$ if $\mathrm{char}\mathbf{k} = 0$, we also write $G_{\mathbf{k}} = G_p$, $H_{\mathbf{k}} = H_p$ and if $\mathbf{k} = \mathbb{C}$ we write $G_{\mathbb{C}} = G$, $H_{\mathbb{C}} = H$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}$, $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}$. Note that G is also a connected finite-dimensional complex semisimple Lie group.

Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$, Q the corresponding root lattice, and P the weight lattice. Let $\Gamma = \{\alpha_i| \ i = 1, \ldots, l\}$, $l = rank(\mathfrak{g})$ be a system of simple roots, $\Delta_+ = \{\beta_1, \ldots, \beta_D\}$ the set of positive roots, $Q_+ = \mathbb{N}\Delta_+$, P_+ the set of the corresponding integral dominant weights, $\omega_1, \ldots, \omega_l$ the fundamental weights. Let also $H_1, \ldots, H_l$ be the set of simple root generators of $\mathfrak{h}$.

Denote by a_{ij} the corresponding Cartan matrix, and let $d_1, \ldots, d_l$, $d_i \in \{1, 2, 3\}$, $i = 1, \ldots, l$ be coprime positive integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. There exists a unique canonical non–degenerate invariant symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ such that $(H_i, H_j) = d_j^{-1} a_{ij}$. It induces an isomorphism of vector spaces $\mathfrak{h} \simeq \mathfrak{h}^*$ under which $\alpha_i \in \mathfrak{h}^*$ corresponds to $d_i H_i \in \mathfrak{h}$. We denote by $h^\vee$ the element of $\mathfrak{h}$ that corresponds to $h \in \mathfrak{h}^*$ under this isomorphism. For a root $\alpha \in \Delta$ the element $\alpha^\vee \in \mathfrak{h}$ is called the corresponding coroot. The induced bilinear form on $\mathfrak{h}^*$ is given by $(\alpha_i, \alpha_j) = b_{ij}$.

Let W be the Weyl group of the root system Δ. W is the subgroup of $GL(\mathfrak{h})$ generated by the fundamental reflections $s_1, \ldots, s_l$,

$$s_i(h) = h - \alpha_i(h)H_i, \quad h \in \mathfrak{h}.$$

The action of W preserves the bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}$.

For any root $\alpha \in \Delta$ we also denote by s_α the corresponding reflection.

For every element $w \in W$ one can introduce the set $\Delta_w = \{\alpha \in \Delta_+ : w(\alpha) \in -\Delta_+\}$, and the number of the elements in the set Δ_w is equal to the length $l(w)$ of the element w with respect to the system Γ of simple roots in Δ_+. We also write $\Delta_- = -\Delta_+$.

Let $\mathfrak{h}_{\mathbb{R}}$ be the real form of $\mathfrak{h}$, the real linear span of simple coroots in $\mathfrak{h}$. The set of roots Δ is a subset of the dual space $\mathfrak{h}_{\mathbb{R}}^*$.

For any $w \in W$ we denote by $\mathfrak{h}^w$ the fixed point space of w in $\mathfrak{h}$, $\mathfrak{h}^w = \{x \in \mathfrak{h}|wx = x\}$, and by $\mathfrak{h}_{\mathbb{R}}^w$ the fixed point space of w in $\mathfrak{h}_{\mathbb{R}}$, $\mathfrak{h}_{\mathbb{R}}^w = \{x \in \mathfrak{h}_{\mathbb{R}}|wx = x\} = \mathfrak{h}^w \cap \mathfrak{h}_{\mathbb{R}}$.

One can define Δ as the root system of the pair (G, H), $\Delta = \Delta(G, H)$, and if, for some algebraically closed field $\mathbf{k}$, G and $G_{\mathbf{k}}$ have the same root system then one can also define Δ as the root system of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$, $\Delta = \Delta(G_{\mathbf{k}}, H_{\mathbf{k}})$.

Similarly, one can define W as the root system of the pair (G, H), $W = W(G, H)$, and if, for some algebraically closed field $\mathbf{k}$, G and $G_{\mathbf{k}}$ have the same root system then one can also define W as the Weyl group of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$, $W = W(G_{\mathbf{k}}, H_{\mathbf{k}})$.

For an arbitrary algebraically closed field $\mathbf{k}$, we denote by $\dot{w}$ a representative of $w \in W = W(G_{\mathbf{k}}, H_{\mathbf{k}})$ in the normalizer of $H_{\mathbf{k}}$ in $G_{\mathbf{k}}$ with respect to the conjugation action. If $\mathbf{k} = \mathbb{C}$ we simply denote this representative by w. It can also be regarded as a representative of w in the normalizer $N_G(\mathfrak{h})$ of $\mathfrak{h}$ in G with respect to the adjoint action. For $w \in W, g \in G$ we write $w(g) = wgw^{-1}$.

Let B_+ be the Borel subgroup of G corresponding to Δ_+ and B_- the opposite Borel subgroup of G, $N_{\pm}$ their unipotent radicals, respectively.

We denote by $\mathfrak{b}_{\pm}$ and $\mathfrak{n}_{\pm}$ the Lie subalgebras of $\mathfrak{g}$ corresponding to $B_{\pm}$ and $N_{\pm}$, respectively.

We identify $\mathfrak{g}$ and its dual by means of the canonical bilinear form $(\cdot, \cdot)$. Then the coadjoint action of G on $\mathfrak{g}^*$ is naturally identified with the adjoint one. Using the canonical bilinear form we shall also identify $\mathfrak{n}_+{}^* \simeq \mathfrak{n}_-$, $\mathfrak{b}_+{}^* \simeq \mathfrak{b}_-$, $\mathfrak{h} \simeq \mathfrak{h}^*$.

Let $\mathfrak{g}_{\beta}$ be the root subspace corresponding to a root $\beta \in \Delta$, $\mathfrak{g}_{\beta} = \{x \in \mathfrak{g}|[h, x] = \beta(h)x \text{ for every } h \in \mathfrak{h}\}$. $\mathfrak{g}_{\beta} \subset \mathfrak{g}$ is a one–dimensional subspace. It is well known that for $\alpha \neq -\beta$ the root subspaces $\mathfrak{g}_{\alpha}$ and $\mathfrak{g}_{\beta}$ are orthogonal with respect to the canonical invariant bilinear form $(\cdot, \cdot)$. Moreover $\mathfrak{g}_{\alpha}$ and $\mathfrak{g}_{-\alpha}$ are non–degenerately paired by this form.

Let $X_{\alpha} \in \mathfrak{g}$ be a non–zero root vector corresponding to a root $\alpha \in \Delta$. Root vectors $X_{\alpha} \in \mathfrak{g}_{\alpha}$ satisfy the following relations:

$$[X_{\alpha}, X_{-\alpha}] = (X_{\alpha}, X_{-\alpha})\alpha^{\vee}.$$

1.2   Systems of positive roots associated to Weyl group elements

Algebraic group analogues of the Slodowy slices are associated to (conjugacy classes) in the Weyl group. In this section we recall the relevant combinatorics of the Weyl group and of root systems. We start by defining systems of positive roots associated to Weyl group elements which play the key role in the definition of the algebraic group analogues of the Slodowy slices.

Let s be an element of the Weyl group W and denote by $\mathfrak{h}'$ the orthogonal complement in $\mathfrak{h}$, with respect to the canonical bilinear form on $\mathfrak{g}$, to the subspace $\mathfrak{h}^s = \{h \in \mathfrak{h}|sh = h\}$ fixed by the natural action of s on $\mathfrak{h}$, $\mathfrak{h}' = (\mathfrak{h}^s)^{\perp}$, so that $\mathfrak{h}'^{\perp} = \mathfrak{h}^s$. Let $\mathfrak{h}'^*$ be the image of $\mathfrak{h}'$ in $\mathfrak{h}^*$ under the identification $\mathfrak{h}^* \simeq \mathfrak{h}$ induced by the canonical bilinear form on $\mathfrak{g}$. Thus $\mathfrak{h}'^*$ embeds into $\mathfrak{h}$ thanks to the direct vector space decomposition $\mathfrak{h} = \mathfrak{h}^s + \mathfrak{h}'$. By Theorem C in [18] s can be represented as a product of two involutions,

$$s = s^1 s^2, \tag{1.2.1}$$

where $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \ldots, \gamma_n$ and $\gamma_{n+1}, \ldots, \gamma_{l'}$ are positive and mutually orthogonal, and the roots $\gamma_1, \ldots, \gamma_{l'}$ form a linear basis of $\mathfrak{h}'^*$.

The Weyl group elements naturally act on $\mathfrak{h}_{\mathbb{R}}$ as orthogonal transformations with respect to the scalar product induced by the symmetric bilinear form of $\mathfrak{g}$, and one can define the real forms $\mathfrak{h}_{\mathbb{R}}^s = \mathfrak{h}^s \cap \mathfrak{h}_{\mathbb{R}}, \mathfrak{h}'_{\mathbb{R}} = \mathfrak{h}' \cap \mathfrak{h}_{\mathbb{R}}$.

Let $f_1, \ldots, f_{l'}$ be the vectors of unit length in the directions of $\gamma_1, \ldots \gamma_{l'}$, and $\widehat{f}_1, \ldots, \widehat{f}_{l'}$ the basis of $\mathfrak{h}'_{\mathbb{R}}$ dual to $f_1, \ldots, f_{l'}$. Let O be the $l' \times l'$ symmetric matrix with real entries $O_{ij} = (f_i, f_j)$. $I - O$ is also a symmetric real matrix, and hence it is diagonalizable and has real eigenvalues.

The following proposition gives a recipe for constructing a spectral decomposition for the action of the orthogonal transformation s on $\mathfrak{h}_{\mathbb{R}}$.

Proposition 1.2.1. *Let λ be a (real) eigenvalue of the symmetric matrix $I - O$, and $u \in \mathbb{R}^{l'}$ a corresponding non–zero real eigenvector with components u_i, $i = 1, \ldots, l'$. Let $a_u, b_u \in \mathfrak{h}_{\mathbb{R}}$ be defined by*

$$a_u = \sum_{i=1}^{n} u_i \widehat{f}_i, \ b_u = \sum_{i=n+1}^{l'} u_i \widehat{f}_i. \tag{1.2.2}$$

(i) If λ is an eigenvalue of $I - O$ then $-\lambda$ is also an eigenvalue of $I - O$, and $\lambda = \pm 1$ are not eigenvalues of $I - O$.

(ii) If $\lambda \neq 0$ then the angle θ between a_u and b_u satisfies $\cos\theta = \lambda$, the plane $\mathfrak{h}_\lambda \subset \mathfrak{h}_\mathbb{R}$ spanned by a_u and b_u is invariant with respect to the involutions s^i, $i = 1, 2$, s^1 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by b_u, and s^2 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by a_u. If $\lambda > 0$ the orthogonal transformation $s = s^1 s^2$ acts on $\mathfrak{h}_\lambda$ as a rotation through the angle 2θ.

(iii) If $\lambda \neq \mu$ are two positive eigenvalues of $I - O$ then the planes $\mathfrak{h}_\lambda$ and $\mathfrak{h}_\mu$ are mutually orthogonal.

(iv) Let $\lambda \neq 0$ be an eigenvalue of $I - O$ of multiplicity greater than 1, and $u^k \in \mathbb{R}^{l'}$, $k = 1, \ldots,$ mult λ a basis of the eigenspace corresponding to λ. If the basis u^k is orthonormal with respect to the standard scalar product on $\mathbb{R}^{l'}$ then the corresponding planes $\mathfrak{h}_\lambda^k$ defined with the help of u^k, $k = 1, \ldots,$ mult λ are mutually orthogonal.

(v) If $\lambda = 0$ is an eigenvalue of $I - O$, then there is a basis $u^k \in \mathbb{R}^{l'}$, $k = 1, \ldots,$ mult 0 of the eigenspace corresponding to 0 orthonormal with respect to the standard scalar product on $\mathbb{R}^{l'}$ and such that the corresponding non–zero elements a_{u^k}, b_{u^k} are all mutually orthogonal. Moreover, $s^1 a_{u^k} = -a_{u^k}$, $s^2 a_{u^k} = a_{u^k}$, $s^1 b_{u^k} = b_{u^k}$, $s^2 b_{u^k} = -b_{u^k}$ for non–zero elements a_{u^k}, b_{u^k}. In particular, for non–zero elements a_{u^k}, b_{u^k} we have $s a_{u^k} = -a_{u^k}$, $s b_{u^k} = -b_{u^k}$, and non–zero elements a_{u^k}, b_{u^k} is a basis of the subspace of $\mathfrak{h}_\mathbb{R}$ on which s acts by multiplication by -1.

Proof. Firstly, we study some general properties of eigenvalues and eigenvectors of the matrix O. By definition the matrix O can be written in a block form,

$$O = \begin{pmatrix} I_n & A \\ A^\top & I_{l'-n} \end{pmatrix}, \tag{1.2.3}$$

where A is an $n \times (l' - n)$ matrix, $A^\top$ is the transpose to A, I_n and $I_{l'-n}$ are the unit matrices of sizes n and $l' - n$. O^{-1} is also symmetric and has a similar block form,

$$O^{-1} = \begin{pmatrix} B & C \\ C^\top & D \end{pmatrix}, \; B = B^\top, \; D = D^\top, \tag{1.2.4}$$

with the entries $O_{ij}^{-1} = (\widehat{f}_i, \widehat{f}_j)$.

For any vector $u \in \mathbb{R}^{l'}$ we introduce its $\mathbb{R}^n$ and $\mathbb{R}^{l'-n}$ components $\widetilde{u}$ and $\widetilde{\widetilde{u}}$ in a similar way,

$$u = \begin{pmatrix} \widetilde{u} \\ \widetilde{\widetilde{u}} \end{pmatrix}. \tag{1.2.5}$$

We shall consider both $\widetilde{u}$ and $\widetilde{\widetilde{u}}$ as elements of $\mathbb{R}^{l'}$ using natural embeddings $\mathbb{R}^n, \mathbb{R}^{l'-n} \subset \mathbb{R}^{l'}$ associated to decomposition (1.2.5).

If u is a non–zero eigenvector of $I - O$ corresponding to an eigenvalue λ then the equation $(I - O)u = \lambda u$ gives

$$-A\widetilde{\widetilde{u}} = \lambda\widetilde{u}, \; -A^\top\widetilde{u} = \lambda\widetilde{\widetilde{u}}. \tag{1.2.6}$$

From these equations we deduce that

$$\begin{pmatrix} -\widetilde{u} \\ \widetilde{\widetilde{u}} \end{pmatrix}$$

is a non–zero eigenvector of $I - O$ corresponding to the eigenvalue $-\lambda$ This proves the first claim in (i).

$\lambda = 1$ is not an eigenvalue of $I - O$ since the matrix O is invertible. Therefore $\lambda = -1$ is also not an eigenvalue of $I - O$ by the first part of part (i) which is already proved. This justifies (i).

Since $O^{-1}O = I$ one has

$$BA + C = 0, \; C^\top + DA^\top = 0. \tag{1.2.7}$$

Multiplying the first and the second equations in (1.2.6) from the left by B and D, respectively, and using (1.2.7) we obtain that

$$C\widetilde{\widetilde{u}} = \lambda B\widetilde{u}, \; C^\top\widetilde{u} = \lambda D\widetilde{\widetilde{u}}. \tag{1.2.8}$$

Now if u^1 and u^2 are two non–zero eigenvectors of $I - O$ corresponding to an eigenvalue λ then by (1.2.4) we have

$$(a_{u^1}, a_{u^2}) = \sum_{i,j=1}^n u_i^1 u_j^2 (\widehat{f}_i, \widehat{f}_j) = \sum_{i,j=1}^n u_i^1 u_j^2 B_{ij} = \widetilde{u}^1 \cdot B\widetilde{u}^2, \tag{1.2.9}$$

where $\cdot$ stands for the standard scalar product in $\mathbb{R}^{l'}$.

Similarly,

$$(b_{u^1}, b_{u^2}) = D\widetilde{u}^1 \cdot \widetilde{u}^2, (a_{u^1}, b_{u^2}) = \widetilde{u}^1 \cdot C\widetilde{u}^2 \tag{1.2.10}$$

From (1.2.8), (1.2.9) and the first identity in (1.2.10) we also obtain that if $\lambda \neq 0$ then

$$(a_{u^1}, a_{u^2}) = \widetilde{u}^1 \cdot B\widetilde{u}^2 = \frac{1}{\lambda}\widetilde{u}^1 \cdot C\widetilde{u}^2 = \frac{1}{\lambda}C^\top\widetilde{u}^1 \cdot \widetilde{u}^2 = D\widetilde{u}^1 \cdot \widetilde{u}^2 = (b_{u^1}, b_{u^2}). \tag{1.2.11}$$

Similarly, for any real eigenvalue λ we have

$$(a_{u^1}, b_{u^2}) = \widetilde{u}^1 \cdot C\widetilde{u}^2 = \lambda(a_{u^1}, a_{u^2}), \ (b_{u^1}, a_{u^2}) = \widetilde{u}^1 \cdot C^\top\widetilde{u}^2 = \lambda(a_{u^1}, a_{u^2}). \tag{1.2.12}$$

Therefore if $\lambda \neq 0$, taking into account (1.2.11), we obtain for $u^1 = u^2 = u$

$$\lambda = \frac{(a_u, b_u)}{(a_u, a_u)} = \frac{(a_u, b_u)}{\sqrt{(b_u, b_u)}\sqrt{(a_u, a_u)}} = \cos\theta,$$

which justifies the first claim in (ii). Note that $(b_u, b_u) = (a_u, a_u) \neq 0$ for otherwise $a_u = b_u = 0$, and hence $u = 0$ which contradicts the choice of u.

Let again u be a non–zero eigenvector of $I - O$ corresponding to an eigenvalue λ. For $i = 1, \ldots, n$ by the definition of the matrices B and C we have

$$(\widehat{f}_i, \lambda a_u - b_u) = \lambda(B\widetilde{u})_i - (C\widetilde{u})_i = 0,$$

where at the last step we used the first identity in (1.2.8). From the last identity we deduce that $\lambda a_u - b_u$ is a linear combination of $f_{n+1}, \ldots, f_{l'}$, and hence

$$s^2(\lambda a_u - b_u) = -(\lambda a_u - b_u).$$

However, by the definition of a_u, $s^2 a_u = a_u$. Therefore

$$s^2 b_u = 2\lambda a_u - b_u. \tag{1.2.13}$$

Let $\lambda \neq 0$. Then recalling that by (1.2.11) $(a_u, a_u) = (b_u, b_u)$ we conclude that $\lambda a_u = \cos(\theta)a_u$ is the orthogonal projection of b_u onto the line spanned by a_u and that $s^2 b_u$ is obtained from b_u by the reflection in the line spanned by a_u as shown at Figure 1.
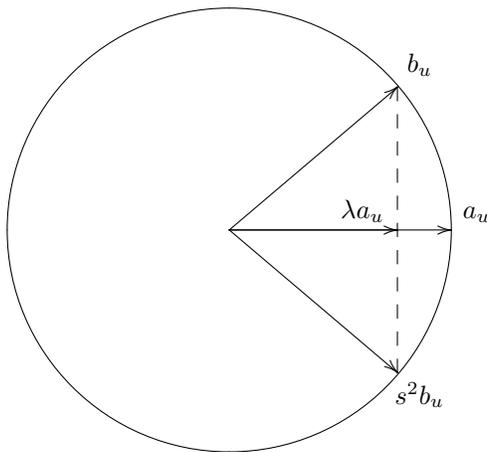
Fig. 1

Similarly, $s^1 b_u = b_u$, $s^1 a_u$ is obtained from a_u by the reflection in the line spanned by b_u.

Thus the plane $\mathfrak{h}_\lambda \subset \mathfrak{h}_\mathbb{R}$ spanned by a_u and b_u is invariant with respect to the involutions s^i, $i = 1, 2$, s^1 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by b_u, and s^2 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by a_u.

Since the angle between a_u and b_u is θ, for $\lambda > 0$ the orthogonal transformation $s = s^1 s^2$ acts on $\mathfrak{h}_\lambda$ as a rotation through the angle 2θ which completes the proof of (ii).

From the general theory of orthogonal transformations it follows that if $\lambda \neq \mu$ are two positive eigenvalues of $I - O$, $\lambda, \mu \neq 1$ then the planes $\mathfrak{h}_\lambda$ and $\mathfrak{h}_\mu$ are mutually orthogonal which confirms part (iii).

Now if u^1 and u^2 are two non–zero eigenvectors of $I - O$ corresponding to an eigenvalue $\lambda \neq 0$ then by part (i) $\lambda \neq \pm 1$, and (1.2.9), (1.2.11), (1.2.12) and the identity $O^{-1} u^2 = \frac{1}{1-\lambda} u^2$ yield

$$(a_{u^1} + b_{u^1}, a_{u^2} + b_{u^2}) = 2(a_{u^1}, a_{u^2})(\lambda + 1) = u^1 \cdot O^{-1} u^2 = \frac{1}{1-\lambda} u^1 \cdot u^2.$$

Thus if $\lambda \neq 0$ and $u^{1,2}$ are mutually orthogonal, a_{u^1}, a_{u^2} are also mutually orthogonal, and from (1.2.11) and (1.2.12) we obtain that b_{u^1} and b_{u^2}, a_{u^1} and b_{u^2}, a_{u^2} and b_{u^1} are mutually orthogonal. Therefore the planes spanned by a_{u^1}, b_{u^1} and by a_{u^2}, b_{u^2} are mutually orthogonal. Part (iv) immediately follows from this property.

It remains to prove part (v). If $\lambda = 0$ is an eigenvalue of $I - O$ then $\widetilde{u}$ and $\widetilde{\widetilde{u}}$ are the components of an eigenvector u of $I - O$ with eigenvalue 0 if and only if $A\widetilde{u} = 0$ and $A^\top \widetilde{u} = 0$. Therefore using the usual orthogonalization procedure one can construct a basis $u^k \in \mathbb{R}^{l'}$, $k = 1, \ldots,$ mult 0 of the eigenspace corresponding to 0 orthonormal with respect to the standard scalar product on $\mathbb{R}^{l'}$ and such that the components $\widetilde{u}^k$ and $\widetilde{\widetilde{u}}^k$ $k = 1, \ldots,$ mult 0 are all mutually orthogonal.

By (1.2.13) $s^2 b_{u^k} = -b_{u^k}$. Also by the definition of a_{u^k} $s^2 a_{u^k} = a_{u^k}$. Similarly, $s^1 a_{u^k} = -a_{u^k}$ and $s^1 b_{u^k} = b_{u^k}$.

Now using the definition of eigenvectors in the form $Ou^k = u^k$ and (1.2.8) with $u = u^k$ we deduce that for the basis u^k the following relations hold: $B\widetilde{u}^k = \widetilde{u}^k$, $D\widetilde{\widetilde{u}}^k = \widetilde{\widetilde{u}}^k$.

From these relations we obtain for $k \neq l$ by (1.2.9)

$$(a_{u^k}, a_{u^l}) = \widetilde{u}^k \cdot B\widetilde{u}^l = \widetilde{u}^k \cdot \widetilde{u}^l = 0$$

and by (1.2.10)

$$(b_{u^k}, b_{u^l}) = D\widetilde{\widetilde{u}}^k \cdot \widetilde{\widetilde{u}}^l = \widetilde{\widetilde{u}}^k \cdot \widetilde{\widetilde{u}}^l = 0.$$

By (1.2.12) we always have

$$(a_{u^k}, b_{u^l}) = \lambda(a_{u^k}, a_{u^l}) = 0.$$

This completes the proof of part (v).

$\square$

Using the previous proposition we can decompose $\mathfrak{h}_\mathbb{R}$ into a direct orthogonal sum of s–invariant subspaces,

$$\mathfrak{h}_\mathbb{R} = \bigoplus_{i=0}^{K} \mathfrak{h}_i, \tag{1.2.14}$$

where each of the subspaces $\mathfrak{h}_i \subset \mathfrak{h}_\mathbb{R}$, $i = 1, \ldots, K$ is invariant with respect to both involutions s^i, $i = 1, 2$ in the decomposition $s = s^1 s^2$, and there are the following three possibilities for each $\mathfrak{h}_i$: $\mathfrak{h}_i$ is two–dimensional ($\mathfrak{h}_i = \mathfrak{h}_\lambda^k$ for an eigenvalue $0 < \lambda < 1$ of the matrix $I - O$, and $k = 1, \ldots,$ mult λ) and the Weyl group element s acts on it as rotation with angle θ_i, $0 < \theta_i < \pi$ or $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\lambda = 0$, $k = 1, \ldots,$ mult λ has dimension 1 and s acts on it by multiplication by -1 or $\mathfrak{h}_i$ coincides with the linear subspace of $\mathfrak{h}_\mathbb{R}$ fixed by the action of s. Note that since s has finite order, we have $\theta_i = \frac{2\pi n_i}{m_i}$, $n_i, m_i \in \{1, 2, \ldots\}$.

Since the number of roots in the root system Δ is finite one can always choose elements $h_i \in \mathfrak{h}_i$, $i = 0, \ldots, K$, such that $h_i(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s–invariant subspace $\mathfrak{h}_i$ with respect to the natural pairing between $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$.

Now we consider certain s–invariant subsets of roots Δ_i, $i = 0, \ldots, K$, defined as follows

$$\Delta_i = \{\alpha \in \Delta : h_j(\alpha) = 0, j > i, \ h_i(\alpha) \neq 0\}, \tag{1.2.15}$$

where we formally assume that $h_{K+1} = 0$. Note that for some indexes i the subsets Δ_i are empty, and that the definition of these subsets depends on the order of the terms in direct sum (1.2.14).

Now consider the nonempty s–invariant subsets of roots Δ_{i_k}, $k = 0, \ldots, M$. For convenience we assume that indexes i_k are labeled in such a way that $i_j < i_k$ if and only if $j < k$.

Observe also that the root system Δ is the disjoint union of the subsets Δ_{i_k},

$$\Delta = \bigcup_{k=0}^{M} \Delta_{i_k}. \tag{1.2.16}$$

Now assume that

$$|h_{i_k}(\alpha)| > |\sum_{l \le j < k} h_{i_j}(\alpha)|, \text{ for any } \alpha \in \Delta_{i_k}, \ k = 0, \dots, M, \ l < k. \tag{1.2.17}$$

Condition (1.2.17) can be always fulfilled by suitable rescalings of the elements h_{i_k}.
    Consider the element

$$\bar{h} = \sum_{k=0}^{M} h_{i_k} \in \mathfrak{h}_{\mathbb{R}}. \tag{1.2.18}$$

From definition (1.2.15) of the sets Δ_i we obtain that for $\alpha \in \Delta_{i_k}$

$$\bar{h}(\alpha) = \sum_{j \le k} h_{i_j}(\alpha) = h_{i_k}(\alpha) + \sum_{j < k} h_{i_j}(\alpha) \tag{1.2.19}$$

Now condition (1.2.17), the previous identity and the inequality $|x + y| \ge ||x| - |y||$ imply that for $\alpha \in \Delta_{i_k}$ we have

$$|\bar{h}(\alpha)| \ge ||h_{i_k}(\alpha)| - |\sum_{j < k} h_{i_j}(\alpha)|| > 0.$$

Since Δ is the disjoint union of the subsets Δ_{i_k}, $\Delta = \bigcup_{k=0}^{M} \Delta_{i_k}$, the last inequality ensures that $\bar{h}$ belongs to a Weyl chamber of the root system Δ.
    Denote by Δ_+^s the subset of positive roots with respect to the Weyl chamber containing $\bar{h}$,

$$\Delta_+^s = \{\alpha \in \Delta | \alpha(\bar{h}) > 0\}. \tag{1.2.20}$$

From condition (1.2.17) and formula (1.2.19) we also obtain that a root $\alpha \in \Delta_{i_k}$ is positive if and only if

$$h_{i_k}(\alpha) > 0. \tag{1.2.21}$$

We denote by $(\Delta_{i_k})_+$ the set of positive roots contained in Δ_{i_k}, $(\Delta_{i_k})_+ = \Delta_+^s \cap \Delta_{i_k}$.
    We also define other s–invariant subsets of roots $\overline{\Delta}_{i_k}$, $k = 0, \dots, M$,

$$\overline{\Delta}_{i_k} = \bigcup_{i_j \le i_k} \Delta_{i_j}. \tag{1.2.22}$$

According to this definition we have a chain of strict inclusions

$$\overline{\Delta}_{i_M} \supset \overline{\Delta}_{i_{M-1}} \supset \dots \supset \overline{\Delta}_{i_0}, \tag{1.2.23}$$

such that $\overline{\Delta}_{i_M} = \Delta$, $\overline{\Delta}_{i_0} = \Delta_{i_0}$, and $\overline{\Delta}_{i_k} \setminus \overline{\Delta}_{i_{k-1}} = \Delta_{i_k}$.
    The following lemma shows that the subsets of roots $\overline{\Delta}_{i_k} \subset \Delta$ are root systems of some standard Levi subalgebras in $\mathfrak{g}$.

Lemma 1.2.2. *Let Γ^s be the set of simple roots in Δ_+^s. Then $\Gamma^s \cap \overline{\Delta}_{i_k}$ is a set of simple roots in $\overline{\Delta}_{i_k}$.*

Proof. Indeed, let $\alpha \in \overline{\Delta}_{i_k} \cap \Delta_+^s$, $\alpha = \sum_{i=1}^{l} n_i \alpha_i$, where $n_i \in \{0, 1, 2, \dots\}$ and $\Gamma^s = \{\alpha_1, \dots, \alpha_l\}$. Assume that α does not belong to the linear span of roots from $\Gamma^s \cap \overline{\Delta}_{i_k}$ and $t > i_k$ is maximal possible such that for some $\alpha_q \in \Delta_t$ one has $n_q > 0$. Then by (1.2.15) and (1.2.21) $h_t(\alpha) = \sum_{i=1}^{l} n_i h_t(\alpha_i) = \sum_{\alpha_i \in \Delta_t} n_i h_t(\alpha_i) > 0$, and by the choice of t $h_r(\alpha) = 0$ for $r > t$. Therefore $\alpha \in \Delta_t$, and hence $\alpha \notin \overline{\Delta}_{i_k}$. Thus we arrive at a contradiction. $\qquad \square$

1.3 Algebraic group analogues of Slodowy slices

In this section we define analogues of the Slodowy slices for algebraic groups. As this construction is important for semisimple algebraic groups over arbitrary algebraically closed fields, especially for the study of the Lusztig partition in Sections 1.4 and 1.5, we work over an arbitrary algebraically closed field $\mathbf{k}$ in this and in the next two sections. Applications to quantum groups only require $\mathbf{k} = \mathbb{C}$.

Let $s \in W = W(G_{\mathbf{k}}, H_{\mathbf{k}})$ be a Weyl group element, Δ_+^s a system of positive roots associated to (the conjugacy class of) s in the previous section, Γ^s the set of simple roots in Δ_+^s. We shall assume in this section that in sum (1.2.14) $\mathfrak{h}_0$ is the linear subspace $\mathfrak{h}_{\mathbb{R}}^s = \mathfrak{h}^s \cap \mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s pointwise. If $\mathfrak{h}_{\mathbb{R}}^s$ is trivial it will be still convenient to keep the same notation with $\mathfrak{h}_0 = 0$ and include it into sum (1.2.14).

According to this convention we always have that $\Delta_0 = \{\alpha \in \Delta : s\alpha = \alpha\}$ is the set of roots fixed by the action of s elementwise, and Δ_0 may be empty. In this case it will be also convenient to add the empty set Δ_0 to union (1.2.16), so that we shall always have $i_0 = 0$ in (1.2.16) and $\Delta_{i_0} = \Delta_0$.

Denote by $P_{\mathbf{k}}$ the parabolic subgroup of $G_{\mathbf{k}}$ containing the Borel subgroup $B_{\mathbf{k}, -}^s$ corresponding to $-\Delta_+^s$ and associated to the subset $-\Gamma_0^s$ of the set of simple roots in $-\Gamma^s$, where $\Gamma_0^s = \Gamma^s \cap \Delta_0$. Let $N_{\mathbf{k}}$ and $L_{\mathbf{k}}$ the unipotent radical and the Levi factor of $P_{\mathbf{k}}$, respectively, and $\overline{N}_{\mathbf{k}}$ the opposite unipotent radical.

Note that we have a natural inclusion $P_{\mathbf{k}} \supset N_{\mathbf{k}}$, and by Lemma 1.2.2 Δ_0 is the root system of the reductive algebraic group $L_{\mathbf{k}}$, while, by the definition of $P_{\mathbf{k}}$, its unipotent radical $N_{\mathbf{k}}$ is generated by the one-parameter subgroups corresponding to the roots from the set $(-\Delta_+^s) \setminus \Delta_0$.

As in Section 1.1, denote a representative for the Weyl group element s in $G_{\mathbf{k}}$ by $\dot{s}$. Let $Z_{\mathbf{k}}$ be the connected subgroup of $G_{\mathbf{k}}$ generated by the semisimple part of the standard Levi subgroup $L_{\mathbf{k}}$ and by the identity component $H_{\mathbf{k}}^0$ of centralizer of $\dot{s}$ in $H_{\mathbf{k}}$.

Let $N_{\mathbf{k},s} = \{v \in N_{\mathbf{k}} | \dot{s}v\dot{s}^{-1} \in \overline{N}_{\mathbf{k}}\}$. Observe that $N_{\mathbf{k},s} \subset N_{\mathbf{k}}$ is the algebraic subgroup generated by the one–parameter subgroups corresponding to the roots from the set $-\Delta_s^s$, where $\Delta_s^s = \{\alpha \in \Delta_+^s : s\alpha \in -\Delta_+^s\}$, and the cardinality of the set $-\Delta_s^s$ is equal to $l(s)$, where $l(s)$ is the length of the Weyl group element $s \in W$ with respect to the system of simple roots in Δ_+^s (see e.g. [17], §2.2, 8.4). Therefore $\dim N_{\mathbf{k},s} = l(s)$.

Denote by $\mathfrak{p}_{\mathbf{k}}$, $\mathfrak{n}_{\mathbf{k}}$, $\overline{\mathfrak{n}}_{\mathbf{k}}$, $\mathfrak{l}_{\mathbf{k}}$, and $\mathfrak{z}_{\mathbf{k}}$ the Lie subalgebras of $\mathfrak{g}_{\mathbf{k}}$ corresponding to $P_{\mathbf{k}}$, $N_{\mathbf{k}}$, $\overline{N}_{\mathbf{k}}$, $L_{\mathbf{k}}$, and $Z_{\mathbf{k}}$, respectively. Note that the Lie subalgebra corresponding to $H_{\mathbf{k}}^0$ is the fixed point subspace $\mathfrak{h}_{\mathbf{k}}^s$ for the action of s on $\mathfrak{h}_{\mathbf{k}}$.

As in Section 1.1, when $\mathbf{k} = \mathbb{C}$ we drop the subscript $\mathbf{k}$ in the symbols above, so that $P_{\mathbb{C}} = P$, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}$, $N_{\mathbb{C},s} = N_s$, etc. Recall that in this case, according to our convention, we also write $\dot{s} = s$.

In the proofs below we shall frequently use the following lemma which is a direct consequence of Proposition 8.1.1, Corollary 8.1.2 and Lemma 8.2.2 in [124].

Lemma 1.3.1. *Let $\Psi, \Psi_i \subset \Delta$, $i = 1, \ldots, m$ be additively closed subsets of roots such that*

$$\Psi = \bigcup_{i=1}^{m} \Psi_i$$

is a disjoint union and Ψ does not contain opposite roots. For any additively closed subset $\Xi \subset \Delta$ which does not contain opposite roots, denote by $N_\Xi \subset G_{\mathbf{k}}$ the algebraic subgroup generated by the one–parameter subgroups of $G_{\mathbf{k}}$ corresponding to the roots from Ξ. Then multiplication in $G_{\mathbf{k}}$ yields an isomorphism of varieties

$$N_\Psi \simeq N_{\Psi_1} \ldots N_{\Psi_m} \simeq N_{\Psi_1} \times \ldots \times N_{\Psi_m}.$$

Part (i) of the following Lemma is Corollary in Section 7.4 of [55], part (ii) is Lemma 2.3.3 and 1.6.10(4) in [124], and part (iii) is Theorem 5.3.2(iii) in [124].

Lemma 1.3.2. *(i) Let $G_{\mathbf{k}}^1$ and $G_{\mathbf{k}}^2$ be algebraic subgroups in an algebraic group $A_{\mathbf{k}}$ over an algebraically closed field $\mathbf{k}$, and $G_{\mathbf{k}}^1$ normalizes $G_{\mathbf{k}}^2$. Then $G_{\mathbf{k}}^1 G_{\mathbf{k}}^2$ is an algebraic subgroup of $A_{\mathbf{k}}$ and multiplication in $A_{\mathbf{k}}$ defines an isomorphism of varieties, $G_{\mathbf{k}}^1 \times G_{\mathbf{k}}^2 \to G_{\mathbf{k}}^1 G_{\mathbf{k}}^2 = G_{\mathbf{k}}^2 G_{\mathbf{k}}^1$.*

(ii) Let $A_{\mathbf{k}}$ be an algebraic group over an algebraically closed field $\mathbf{k}$. Then any orbit $\mathcal{O}$ for a regular algebraic group action of $A_{\mathbf{k}}$ on a variety V over $\mathbf{k}$ is open in its closure. Thus $\mathcal{O}$ has the natural structure of an algebraic variety.

(iii) Let $\phi : X \to Y$ be an equivariant morphism of homogeneous spaces for an algebraic group $A_{\mathbf{k}}$ over an algebraically closed field $\mathbf{k}$. Then ϕ is an isomorphism if and only if it is bijective and induces a bijection of the tangent spaces at the points $x \in X$ and $y \in Y$.

To formulate our main statement in this section we need the following lemma.

Lemma 1.3.3. *Let $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}} \subset G_{\mathbf{k}}$ be the image in $G_{\mathbf{k}}$ of $N_{\mathbf{k}} \times Z_{\mathbf{k}} \times N_{\mathbf{k}}$ under the map*

$$N_{\mathbf{k}} \times Z_{\mathbf{k}} \times N_{\mathbf{k}} \to G_{\mathbf{k}}, (n, z, n') \mapsto nz\dot{s}n' \tag{1.3.1}$$

induced by the group multiplication in $G_{\mathbf{k}}$. Then
 (i) $Z_{\mathbf{k}}N_{\mathbf{k}}$ is an algebraic subgroup of $G_{\mathbf{k}}$ and multiplication defines an isomorphism of varieties, $Z_{\mathbf{k}} \times N_{\mathbf{k}} \to Z_{\mathbf{k}}N_{\mathbf{k}} = N_{\mathbf{k}}Z_{\mathbf{k}}$;
 (ii) $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}$ is an algebraic subvariety of $G_{\mathbf{k}}$, and the map

$$N_{\mathbf{k}} \times Z_{\mathbf{k}} \times N_{\mathbf{k},s} \to N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k},s} = N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}, (n, z, n') \mapsto nz\dot{s}n' \tag{1.3.2}$$

induced by the group multiplication in $G_{\mathbf{k}}$ is an isomorphism of algebraic varieties;
 (iii) Moreover,

$$N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}} = N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k},s} = N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}. \tag{1.3.3}$$

Proof. The proof of part (i) follows from Lemma 1.3.2 (i) with $A_{\mathbf{k}} = G_{\mathbf{k}}$, $G_{\mathbf{k}}^1 = Z_{\mathbf{k}}$ and $G_{\mathbf{k}}^2 = N_{\mathbf{k}}$.

 (ii) The proof is parallel to that of Lemma 8.3.6 in [124] where a similar statement is justified in the case of Bruhat cells.

 Using the isomorphism of part (i) and definition (1.3.1) one can describe the set $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}$ as the orbit of the element $\dot{s}$ in $G_{\mathbf{k}}$ for the following regular action of $N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k}}$ on $G_{\mathbf{k}}$:

$$(N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k}}) \times G_{\mathbf{k}} \to G_{\mathbf{k}}, ((nz, n'), g) \mapsto nzgn'^{-1}.$$

Now the first claim in part (ii) follows from Lemma 1.3.2 (ii) with $A_{\mathbf{k}} = N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k}}$, $V = G_{\mathbf{k}}$, $\mathcal{O} = N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}$.

 To establish isomorphism (1.3.2) we introduce the algebraic subgroup $N'_{\mathbf{k},s} = N_{\mathbf{k}} \cap \dot{s}^{-1}N_{\mathbf{k}}\dot{s} \subset N_{\mathbf{k}}$. Applying Lemma 1.3.1 for $\Psi_1 = -\Delta_s^s$ and $\Psi_2 = -(\Delta_+^s \setminus (\Delta_s^s \cup \Delta_0))$, so that $N_{\Psi_1} = N_{\mathbf{k},s}$ and $N_{\Psi_2} = N'_{\mathbf{k},s}$, one has an isomorphism of varieties

$$N_{\mathbf{k}} = N'_{\mathbf{k},s}N_{\mathbf{k},s} \simeq N'_{\mathbf{k},s} \times N_{\mathbf{k},s}, \tag{1.3.4}$$

induced by the group multiplication in $G_{\mathbf{k}}$. Hence

$$N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}} = N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N'_{\mathbf{k},s}N_{\mathbf{k},s} = N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k},s}, \tag{1.3.5}$$

as $Z_{\mathbf{k}}$ normalizes $N_{\mathbf{k}}$. This implies that morphism of varieties (1.3.2) is bijective. So in fact one can view $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}$ as the orbit of the element $\dot{s}$ in $G_{\mathbf{k}}$ for the following regular action of the algebraic group $N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k},s}$ on $G_{\mathbf{k}}$:

$$(N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k},s}) \times G_{\mathbf{k}} \to G_{\mathbf{k}}, ((nz, n'), g) \mapsto nzgn'^{-1},$$

and the action map gives rise to a bijective equivariant morphism

$$N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k},s} \to N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k},s}, (nz, n') \mapsto nz\dot{s}n'^{-1} \tag{1.3.6}$$

of $N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k},s}$ and $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k},s}$ viewed as homogeneous spaces for $N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k},s}$.

 One verifies straightforwardly that this morphism induces an isomorphism of the tangent spaces at the points $(1, 1)$ and $\dot{s}$. Therefore by Lemma 1.3.2 (iii) with $X = N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k},s}$, $Y = N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k},s}$, $A_{\mathbf{k}} = N_{\mathbf{k}}Z_{\mathbf{k}} \times N_{\mathbf{k},s}$ morphism (1.3.6) is an isomorphism of varieties. Composing it with the algebraic group isomorphism $N_{\mathbf{k}}Z_{\mathbf{k}} \simeq N_{\mathbf{k}} \times Z_{\mathbf{k}}$ established in part (i) and with the map taking inverse on $N_{\mathbf{k},s}$ we deduce that (1.3.2) is an isomorphism of varieties.

 (iii) The first identity in (1.3.3) was established in part (ii), the second one follows from the fact that s fixes the root system Δ_0 of $Z_{\mathbf{k}}$, and hence $\dot{s}Z_{\mathbf{k}}\dot{s}^{-1} = Z_{\mathbf{k}}$. $\qquad\square$

 Now we can state the main proposition in this section in which we define transversal slices to conjugacy classes in algebraic groups.

Proposition 1.3.4. *(i) $Z_{\mathbf{k}}N_{\mathbf{k},s}$ is an algebraic subgroup in $G_{\mathbf{k}}$, with the variety structure inherited from $Z_{\mathbf{k}} \times N_{\mathbf{k},s}$ via the bijection $Z_{\mathbf{k}}N_{\mathbf{k},s} \simeq Z_{\mathbf{k}} \times N_{\mathbf{k},s}$ induced by the multiplication map in $G_{\mathbf{k}}$. $\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} = Z_{\mathbf{k}}\dot{s}N_{\mathbf{k},s}$ with the variety structure induced from $Z_{\mathbf{k}}N_{\mathbf{k},s}$ is a closed subvariety of $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}$ and of $G_{\mathbf{k}}$, and the conjugation map*

$$N_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} \to N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}, (n, g) \mapsto ngn^{-1}, \tag{1.3.7}$$

is an isomorphism of varieties;
 (ii) The variety $\Sigma_{\mathbf{k},s} = \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} = Z_{\mathbf{k}}\dot{s}N_{\mathbf{k},s}$ is a transversal slice to the set of conjugacy classes in $G_{\mathbf{k}}$;
 (iii) Assume that $\mathbf{k} = \mathbb{C}$ and that $G_{\mathbf{k}} = G$ is simply-connected. Then $NsZN$ is a closed subvariety of G.

Proof. (i) Note first that $Z_{\mathbf{k}}N_{\mathbf{k},s}$ is an algebraic subgroup in $G_{\mathbf{k}}$. Indeed, by the definition s fixes the root system Δ_0 of $Z_{\mathbf{k}}$, so $\dot{s}Z_{\mathbf{k}}\dot{s}^{-1} = Z_{\mathbf{k}}$, and also $Z_{\mathbf{k}}$ normalizes $N_{\mathbf{k}}$ and $\overline{N}_{\mathbf{k}}$. Hence for any $n_s \in N_{\mathbf{k},s}$ and any $z \in Z_{\mathbf{k}}$ one has $zn_sz^{-1} \in N_{\mathbf{k}}$ and $\dot{s}zn_sz^{-1}\dot{s}^{-1} = (\dot{s}z\dot{s}^{-1})(\dot{s}n_s\dot{s}^{-1})(\dot{s}z\dot{s}^{-1})^{-1} \in \overline{N}_{\mathbf{k}}$ as $\dot{s}z\dot{s}^{-1} \in Z_{\mathbf{k}}$ and $\dot{s}n_s\dot{s}^{-1} \in \overline{N}_{\mathbf{k}}$ by the definition of $N_{\mathbf{k},s}$. We deduce that $Z_{\mathbf{k}}$ normalizes $N_{\mathbf{k},s}$ in $G_{\mathbf{k}}$. Thus by Lemma 1.3.2 (i) with $A_{\mathbf{k}} = G_{\mathbf{k}}$, $G_{\mathbf{k}}^1 = Z_{\mathbf{k}}$ and $G_{\mathbf{k}}^2 = N_{\mathbf{k},s}$, $Z_{\mathbf{k}}N_{\mathbf{k},s}$ is an algebraic subgroup in $G_{\mathbf{k}}$, with the variety structure inherited from $Z_{\mathbf{k}} \times N_{\mathbf{k},s}$ via the bijection $Z_{\mathbf{k}}N_{\mathbf{k},s} \simeq Z_{\mathbf{k}} \times N_{\mathbf{k},s}$ induced by the multiplication map in $G_{\mathbf{k}}$.

$\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} \simeq Z_{\mathbf{k}}N_{\mathbf{k},s}$ is a closed subvariety of $G_{\mathbf{k}}$ as the left multiplication by $\dot{s}$ in $G_{\mathbf{k}}$ is a morphism of varieties. $\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} \simeq Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}$ is a closed subvariety of $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}$ due to the closed embedding

$$Z_{\mathbf{k}} \times N_{\mathbf{k},s} \hookrightarrow N_{\mathbf{k}} \times Z_{\mathbf{k}} \times N_{\mathbf{k},s}, (z,n) \mapsto (1,z,n)$$

and isomorphism of varieties (1.3.2).

We show that map (1.3.7) is an isomorphism of varieties. By the definition this map is a morphism of varieties. We shall define the inverse map and show that it is also a morphism of varieties.

Observe that map (1.3.7) is bijective if and only if for any given $k_s \in N_{\mathbf{k},s}, u \in N_{\mathbf{k}}$ and $z \in Z_{\mathbf{k}}$ the equation

$$u\dot{s}k_s = nz'\dot{s}n_sn^{-1} \tag{1.3.8}$$

has a unique solution $n \in N_{\mathbf{k}}, n_s \in N_{\mathbf{k},s}, z' \in Z_{\mathbf{k}}$. In this case the solution defines the inverse map to morphism of varieties (1.3.7). We shall construct this map explicitly. From this construction it will be clear that the inverse map is also a morphism of varieties.

First observe that any element $u\dot{s}k_s$ is uniquely conjugated by $k_s \in N_{\mathbf{k},s}$ to $v z\dot{s} \in N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}, v = k_su$, and hence we can assume that $k_s = 1$ in (1.3.8),

$$vz\dot{s} = nz'\dot{s}n_sn^{-1}. \tag{1.3.9}$$

Using isomorphism (1.3.2) we deduce that the corresponding map

$$N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k},s} \to N_{\mathbf{k},s} \times N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}, u\dot{s}k_s \mapsto (k_s, k_su z\dot{s}) \tag{1.3.10}$$

is an isomorphism of varieties.

Now we show that for any given $v \in N_{\mathbf{k}}$ and $z \in Z_{\mathbf{k}}$ equation (1.3.9) has a unique solution $n \in N_{\mathbf{k}}, n_s \in N_{\mathbf{k},s}, z' \in Z_{\mathbf{k}}$ which is expressed in terms of $vz \in N_{\mathbf{k}}Z_{\mathbf{k}}$ using only a composition of morphisms of algebraic varieties: the isomorphism of varieties $N_{\mathbf{k}}Z_{\mathbf{k}} \simeq N_{\mathbf{k}} \times Z_{\mathbf{k}}$, algebraic factorization of elements of $N_{\mathbf{k}}$ as products of elements from one–parameter subgroups, Chevalley commutation relations between one–parameter subgroups of $G_{\mathbf{k}}$, conjugation by elements of $Z_{\mathbf{k}}$ and by the element $\dot{s} \in G_{\mathbf{k}}$. This implies that the corresponding map $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s} \to N_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}$, $vz\dot{s} \mapsto (n, z'\dot{s}n_s)$ is an injective morphism of varieties, and hence the composition of this map with isomorphism (1.3.10) is the inverse to (1.3.7) morphism of varieties.

In order to construct the unique solution to equation equation (1.3.9) we shall use two inductive constructions.

The first induction is over certain $\dot{s}$–invariant reductive subgroups in $G_{\mathbf{k}}$ that we are going to define. Using inclusions (1.2.23) of s–invariant sets of roots and Lemma 1.2.2 we can define the corresponding standard Levi subgroups G_k with root systems $\overline{\Delta}_{i_k}$, $k = 0, \ldots, M$. Note that according to the convention introduced in the beginning of this section we always have $\overline{\Delta}_{i_0} = \Delta_0$, so that $G_0 = L_{\mathbf{k}}$. In the case when Δ_0 is empty this reduces to $G_0 = H_{\mathbf{k}}$.

We shall use induction over the reductive subgroups which appear in the chain of strict inclusions

$$G_{\mathbf{k}} = G_M \supset G_{M-1} \supset \ldots \supset G_0 = L_{\mathbf{k}} \tag{1.3.11}$$

corresponding to inclusions (1.2.23).

Note that by Lemma 1.2.2 G_{k-1} is also the Levi factor of the parabolic subgroup $P_{k-1} \subset G_k$ containing the Borel subgroup $B_{\mathbf{k},-}^s \cap G_k \subset G_k$ and associated to the set of simple roots $-\Gamma^s \cap \overline{\Delta}_{i_{k-1}}$. Let N_{k-1} be the unipotent radical of P_{k-1}. We also denote by $\overline{N}_{k-1}$ the unipotent radical of the opposite parabolic subgroup.

By these definitions the group multiplication in $G_{\mathbf{k}}$ yields the following isomorphisms of subvarieties $\overline{G}_k \subset G_k$,

$$\overline{G}_k := P_{k-1}\overline{N}_{k-1} = N_{k-1}G_{k-1}\overline{N}_{k-1} \simeq P_{k-1} \times \overline{N}_{k-1} \simeq N_{k-1} \times G_{k-1} \times \overline{N}_{k-1}, \tag{1.3.12}$$

Note that by the definition N_{k-1} is generated by the one–parameter subgroups corresponding to the roots from the set $-\Delta_{i_k} \cap \Delta_+^s$, and by (1.2.16) one can represent $-\Delta_+^s \setminus \Delta_0$ as the following disjoint union

$$-\Delta_+^s \setminus \Delta_0 = \bigcup_{k=0, i_k \neq 0}^{M} (-\Delta_{i_k} \cap \Delta_+^s). \tag{1.3.13}$$

Since $N_{\mathbf{k}}$ is generated by the one–parameter subgroups corresponding to the roots from the set $-\Delta_+^s \setminus \Delta_0$, Lemma 1.3.1 and decomposition (1.3.13) imply that the group multiplication in $G_{\mathbf{k}}$ gives rise to the following isomorphisms of algebraic varieties

$$N_{\mathbf{k}} = N_{M-1}N_{M-2}\ldots N_0 \simeq N_{M-1} \times N_{M-2} \times \ldots \times N_0, \tag{1.3.14}$$

$$N_{\mathbf{k}} = N_0 N_1 \ldots N_{M-1} \simeq N_0 \times N_1 \times \ldots \times N_{M-1},$$

and similar isomorphisms

$$\overline{N}_{\mathbf{k}} = \overline{N}_{M-1}\overline{N}_{M-2}\ldots\overline{N}_0 \simeq \overline{N}_{M-1} \times \overline{N}_{M-2} \times \ldots \times \overline{N}_0, \tag{1.3.15}$$

$$\overline{N}_{\mathbf{k}} = \overline{N}_0\overline{N}_1\ldots\overline{N}_{M-1} \simeq \overline{N}_0 \times \overline{N}_1 \times \ldots \times \overline{N}_{M-1}.$$

Note that, since the subsets of roots $\overline{\Delta}_{i_k}$ are s–invariant, the subgroups G_k are invariant with respect to the action of $\dot{s}$ on $G_{\mathbf{k}}$ by conjugations.

Applying isomorphisms (1.3.12), (1.3.14) and (1.3.15) successively we also obtain the following isomorphisms of subvarieties $\overline{G}^k \subset G_{\mathbf{k}}$,

$$\overline{G}^k := N^k G_k \overline{N}^k \simeq N^k \times G_k \times \overline{N}^k, \quad N^k := N_{M-1}N_{M-2}\ldots N_k \simeq N_{M-1} \times N_{M-2} \times \ldots \times N_k, \tag{1.3.16}$$

$$\overline{N}^k := \overline{N}_{M-1}\overline{N}_{M-2}\ldots\overline{N}_k \simeq \overline{N}_{M-1} \times \overline{N}_{M-2} \times \ldots \times \overline{N}_k.$$

Note that $N^k G_k$ is a parabolic subgroup in $G_{\mathbf{k}}$, N^k is its unipotent radical, and $\overline{N}^k$ the opposite unipotent radical.

Induction 1.

The first induction we are going to use in order to prove that equation (1.3.9) has a unique solution is over the reductive subgroups G_k starting with $k = 0$. At the same time we shall also show that the solution is given by a map $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s} \to N_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}$ which is a morphism of varieties.

First we rewrite equation (1.3.9) in a slightly different form,

$$vz\dot{s}n\dot{s}^{-1} = nz'\dot{s}n_s\dot{s}^{-1}. \tag{1.3.17}$$

To establish the base of induction we first observe that both the l.h.s. and the r.h.s. of equation (1.3.17) belong to the subvariety $\overline{G}^0 \subset G_{\mathbf{k}}$.

Observe that the $G_0 = L_{\mathbf{k}}$–component of equation (1.3.17) with respect to decomposition (1.3.16) for $k = 0$ is reduced to

$$z = z'. \tag{1.3.18}$$

Indeed, using (1.3.4) we can write

$$\dot{s}N_{\mathbf{k}}\dot{s}^{-1} = \dot{s}N'_{\mathbf{k},s}\dot{s}^{-1}\dot{s}N_{\mathbf{k},s}\dot{s}^{-1} \subset N_{\mathbf{k}}\overline{N}_{\mathbf{k}}. \tag{1.3.19}$$

If $n = mm_s$ is the decomposition of n corresponding to the decomposition $N_{\mathbf{k}} = N'_{\mathbf{k},s}N_{\mathbf{k},s}$ then recalling that $Z_{\mathbf{k}}$ normalizes both $N_{\mathbf{k}}$ and $\overline{N}_{\mathbf{k}}$ we deduce that the decompositions of the r.h.s. and of the l.h.s. of equation (1.3.17) corresponding to the decomposition $\overline{G}^0 = N_{\mathbf{k}}L_{\mathbf{k}}\overline{N}_{\mathbf{k}}$ take the form

$$(vz\dot{s}m\dot{s}^{-1}z^{-1})z(\dot{s}m_s\dot{s}^{-1}) = nz'(\dot{s}n_s\dot{s}^{-1}),$$

where $vz\dot{s}m\dot{s}^{-1}z^{-1}, n \in N_{\mathbf{k}}$ and $\dot{s}m_s\dot{s}^{-1}, \dot{s}n_s\dot{s}^{-1} \in \overline{N}_{\mathbf{k}}$, $z, z' \in Z_{\mathbf{k}} \subset L_{\mathbf{k}}$. This implies (1.3.18) and establishes the base of induction.

Note also that the map $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s} \to Z_{\mathbf{k}}$, $vz\dot{s} \mapsto z$ is a morphism of varieties.

Now let

$$n = n_0\ldots n_{M-2}n_{M-1}, v = v_{M-1}\ldots v_1 v_0, \ n_s = n_{s0}\ldots n_{sM-2}n_{sM-1} \tag{1.3.20}$$

be the decompositions of the elements n, v, n_s corresponding to decompositions (1.3.14) and assume that n_j and n_{sj} have already been uniquely defined for $j < k-1$ and are given by injective maps $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s} \to N_j$ and $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s} \to N_j \cap N_{\mathbf{k},s}$, $j < k-1$, which are morphisms of varieties. We shall show that using equation (1.3.17) one can find n_{k-1} and n_{sk-1} in a unique way using some injective morphisms of varieties $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s} \to N_{k-1}$ and $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s} \to N_{k-1} \cap N_{\mathbf{k},s}$.

Observe that both the l.h.s. and the r.h.s. of equation (1.3.17) belong to the subvariety $\overline{G}^k \subset G$ and that the G_k–component of equation (1.3.17) with respect to decomposition (1.3.16) is reduced to

$$v_{k-1}(v)_{k-1}z\dot{s}(n)_{k-1}n_{k-1}\dot{s}^{-1} = (n)_{k-1}n_{k-1}z\dot{s}(n_s)_{k-1}n_{sk-1}\dot{s}^{-1}, \qquad (1.3.21)$$

where $(n_s)_{k-1} = n_{s0}\ldots n_{sk-2} \in G_{k-1}$, $(n)_{k-1} = n_0\ldots n_{k-2} \in G_{k-1}$, $(v)_{k-1} = v_{k-2}\ldots v_0 \in G_{k-1}$ and $(n)_{k-1}$, $(n_s)_{k-1}$, $(u)_{k-1}$, v_{k-1}, z are already known. This follows, similarly to the case $k = 0$, from decompositions (1.3.20), the facts that that G_{k-1} normalizes both N^{k-1} and $\overline{N}^{k-1}$, and that the subgroups G_k are invariant with respect to the action of $\dot{s}$ on $G_{\mathbf{k}}$ by conjugations.

The same properties imply that after multiplying by $(n)_{k-1}^{-1}$ from the left, equation (1.3.21) takes the form

$$wz_0\dot{s}n_{k-1}\dot{s}^{-1} = n_{k-1}z_0'\dot{s}n_{sk-1}\dot{s}^{-1}, \qquad (1.3.22)$$

with some known $w = (n)_{k-1}^{-1}v_{k-1}(n)_{k-1} \in N_{k-1}$, $z_0 = (n)_{k-1}^{-1}(v)_{k-1}z\dot{s}(n)_{k-1}\dot{s}^{-1}$, $z_0' = z\dot{s}(n_s)_{k-1}\dot{s}^{-1}$, $z_0, z_0' \in G_{k-1}$, and the compatibility of the equation of type (1.3.21) with k replaced by $k-1$ implies that $z_0 = z_0'$. Therefore (1.3.22) takes the form

$$wz_0\dot{s}\bar{n}\dot{s}^{-1} = \bar{n}z_0\dot{s}\bar{n}_s\dot{s}^{-1} \qquad (1.3.23)$$

where we renamed the unknowns $\bar{n} = n_{k-1}, \bar{n}_s = n_{sk-1}$ to simplify the notation.

Let $\bar{n} = \bar{m}\bar{m}_s$ be the decomposition of the element $\bar{n}$ corresponding to the factorization

$$N_{k-1} = (N_{k-1} \cap N_{\mathbf{k},s}')(N_{k-1} \cap N_{\mathbf{k},s})$$

which follows from isomorphism of varieties (1.3.4). In terms of this factorization equation (1.3.23) can be rewritten as follows

$$wz_0\dot{s}\bar{m}\dot{s}^{-1}\dot{s}\bar{m}_s\dot{s}^{-1} = \bar{n}z_0\dot{s}\bar{n}_s\dot{s}^{-1}, \qquad (1.3.24)$$

and the $\overline{N}_{k-1}$–component of the last equation with respect to factorization (1.3.12) is

$$\dot{s}\bar{m}_s\dot{s}^{-1} = \dot{s}\bar{n}_s\dot{s}^{-1}.$$

From this relation we obtain that

$$\bar{m}_s = \bar{n}_s, \qquad (1.3.25)$$

and hence (1.3.24) yields

$$wz_0\dot{s}\bar{m}\dot{s}^{-1}z_0^{-1} = \bar{n}. \qquad (1.3.26)$$

Now we show that the last equation defines $\bar{n}$ in a unique way.

First observe that $\bar{n} \in N_{k-1}$, and N_{k-1} is generated by one–parameter subgroups corresponding to the roots from the set $(\Delta_{i_k})_- = -(\Delta_{i_k})_+$, where $(\Delta_{i_k})_+ = \Delta_{i_k} \cap \Delta_+^s$.

By the definition of the set Δ_s^s each s–orbit in the s–invariant set Δ_{i_k} contains a unique element from $\Delta_s^s \cap \Delta_{i_k}$. This observation implies that the set $(\Delta_{i_k})_-$ is the disjoint union of the subsets $\Delta_{i_k}^p = \{\alpha \in -(\Delta_{i_k})_+ : s^{-1}\alpha,\ldots,s^{-(p-1)}\alpha \in -(\Delta_{i_k})_+, s^{-p}\alpha \in (\Delta_{i_k})_+\}$, $p = 1,\ldots,D_k+1$,

$$(\Delta_{i_k})_- = \bigcup_{p=1}^{D_k+1} \Delta_{i_k}^p. \qquad (1.3.27)$$

Here D_k is chosen in such a way that $\Delta_{i_k}^{D_k+1} \subset -\Delta_s^s \cap \Delta_{i_k}$, and

$$\Delta_{i_k}^{D_k} = \Delta'_{i_k}^{D_k} \cup \Delta''_{i_k}^{D_k} \text{ (disjoint union)}, \qquad (1.3.28)$$

where $\Delta'_{i_k}^{D_k} = \Delta_{i_k}^{D_k} \cap -\Delta_s^s$, and $\Delta''_{i_k}^{D_k} = \Delta_{i_k}^{D_k} \setminus \Delta'_{i_k}^{D_k}$. The set $\Delta_{i_k}^{D_k+1}$ may be empty.

In the case when $\mathfrak{h}_{i_k}$ is a plane, the orthogonal projections of the roots from the subsets $\Delta_{i_k}^p$ onto $\mathfrak{h}_{i_k}$ are contained in the interior of the sectors labeled $\Delta_{i_k}^p$ at Figure 2. All those sectors belong to the lower half plane and have the same central angles equal to θ_{i_k}, except for the last sector labeled by $\Delta_{i_k}^{D_k+1}$, which can possibly have a smaller angle.
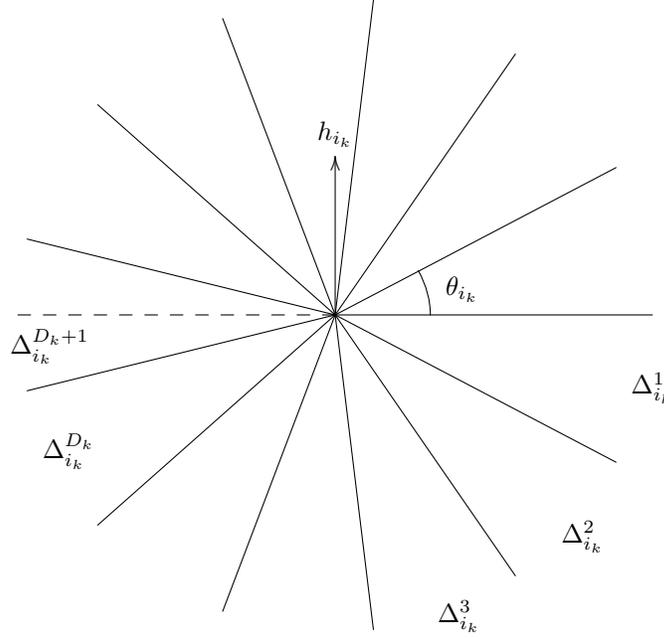
Fig. 2

The vector h_{i_k} is directed upwards at the picture, and the orthogonal projections of elements from $-(\Delta_{i_k})_+$ onto $\mathfrak{h}_{i_k}$ are contained in the lower half plane. The element $s \in W$ acts on the plane $\mathfrak{h}_{i_k}$ by clockwise rotation by the angle θ_{i_k}.

Now consider the unipotent subgroups N_{k-1}^p, $p = 1, \ldots, D_k + 1$, N'_{k-1}, N''_{k-1} generated by the one–dimensional subgroups corresponding to the roots from the sets $\Delta_{i_k}^p$, $\Delta'^{D_k}_{i_k}$, $\Delta''^{D_k}_{i_k}$, respectively.

By Lemma 1.3.1 applied to disjoint unions (1.3.27), (1.3.28) and $(\Delta_{i_k})_- \cap (-\Delta_s^s) = \Delta'^{D_k}_{i_k} \cup \Delta^{D_k+1}_{i_{k-1}}$ we have isomorphisms of varieties

$$N_{k-1} = N_{k-1}^1 N_{k-1}^2 \ldots N_{k-1}^{D_k+1} \simeq N_{k-1}^1 \times N_{k-1}^2 \times \ldots \times N_{k-1}^{D_k+1}, N_{k-1}^{D_k} = N''_{k-1} N'_{k-1}, N_{k-1} \cap N_{\mathbf{k},s} = N'_{k-1} N_{k-1}^{D_k+1}.$$
(1.3.29)

Let
$$\bar{n} = \bar{n}^1 \ldots \bar{n}^{D_k+1}, \ \bar{n}^{D_k} = \bar{n}'' \bar{n}', \ \bar{m} = \bar{n}^1 \ldots \bar{n}^{D_k-1} \bar{n}'', \ w = w^1 w^2 \ldots w^{D_k+1}, \bar{m}_s = \bar{n}' \bar{n}^{D_k+1}$$
(1.3.30)

be the corresponding decomposition of elements $\bar{n}$, $\bar{m}$, w and $\bar{m}_s$, respectively.

Induction 2.

We claim that the components $\bar{n}^p$, $p = 1, \ldots, D_k + 1$ can be uniquely calculated by induction starting with $\bar{n}^1$. Indeed, substituting decompositions (1.3.30) into (1.3.26) we obtain

$$w^1 w^2 \ldots w^{D_k+1} z_0 \dot{s} \bar{n}^1 \ldots \bar{n}^{D_k-1} \bar{n}'' \dot{s}^{-1} z_0^{-1} = \bar{n}^1 \ldots \bar{n}^{D_k+1}.$$

Now comparing the N_{k-1}^p–components of the last equation, with respect to the first factorization in (1.3.29), and using the fact that $\dot{s} N_{k-1}^p \dot{s}^{-1} \subset N_{k-1}^{p+1}$, $p = 1, \ldots, D_k-1$, $\dot{s} N''_{k-1} \dot{s}^{-1} \subset N_{k-1}^{D_k+1}$, and that z_0 normalizes the subgroups N_{k-1}^p, $p = 1, \ldots, D_k + 1$, N'_{k-1}, N''_{k-1}, we obtain

$$\bar{n}^1 = w^1, \bar{n}^p = (w^p \ldots w^{D_k+1} z_0 \dot{s} \bar{n}^1 \ldots \bar{n}^{p-1} \dot{s}^{-1} z_0^{-1})_p, p = 2, \ldots, D_k,$$
(1.3.31)

$$\bar{n}^{D_k+1} = (w^{D_k+1} z_0 \dot{s} \bar{n}^1 \ldots \bar{n}^{D_k-1} \bar{n}'' \dot{s}^{-1} z_0^{-1})_{D_k+1},$$
(1.3.32)

where $\bar{n}''$ is defined from the factorization $\bar{n}^{D_k} = \bar{n}'' \bar{n}'$, and the subscript $(\ldots)_p$ stands for the N_{k-1}^p–component with respect to the first factorization in (1.3.29). From formulas (1.3.31), (1.3.32) one can recursively find the components $\bar{n}^p$ starting from $\bar{n}^1 = w^1$, and finally one can find $\bar{n}_s$ using (1.3.25).

Note that by the construction and by the induction hypothesis the maps $N_{\mathbf{k}} Z_{\mathbf{k}} \dot{s} \to N_{k-1}$, $vz\dot{s} \mapsto n_{k-1} = \bar{n} = \bar{n}^1 \ldots \bar{n}^{D_k+1}$ and $N_{\mathbf{k}} Z_{\mathbf{k}} \dot{s} \to N_{k-1} \cap N_{\mathbf{k},s}$, $vz\dot{s} \mapsto n_{sk-1} = \bar{n}_s = \bar{m}_s = \bar{n}' \bar{n}^{D_k+1}$ are injective morphisms of varieties. This establishes the induction step.

Finally, using isomorphisms of varieties (1.3.14) and Lemma 1.3.3 we conclude that the map $N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s} \to N_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}$, $vz\dot{s} \mapsto (n, z\dot{s}n_s)$, $n = n_0 \ldots n_{M-2}n_{M-1}$, $n_s = n_{s0} \ldots n_{sM-2}n_{sM-1}$, is an injective morphism of varieties. Thus its composition with isomorphism of varieties (1.3.10) is the injective morphism inverse to morphism (1.3.7). This establishes isomorphism (1.3.7) and completes the proof of part (i).

(ii) Next we have to show that the variety $\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} \subset G_{\mathbf{k}}$ is a transversal slice to the set of conjugacy classes in $G_{\mathbf{k}}$, i.e. that the differential of the conjugation map

$$\gamma : G_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} \to G_{\mathbf{k}} \tag{1.3.33}$$

is surjective.

Note that the set of smooth points of map (1.3.33) is stable under the $G_{\mathbf{k}}$–action by left translations on the first factor of $G_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}$. Therefore it suffices to show that the differential of map (1.3.33) is surjective at points $(1, szn_s)$, $n_s \in N_{\mathbf{k},s}$, $z \in Z_{\mathbf{k}}$.

In terms of the left trivialization of the tangent bundle $TG_{\mathbf{k}}$ and the induced trivialization of $T(\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s})$ the differential of map (1.3.33) at points $(1, \dot{s}zn_s)$ takes the form

$$d\gamma_{(1,\dot{s}zn_s)} : (x, (n, w)) \to -(Id - \operatorname{Ad}(\dot{s}zn_s)^{-1})x + n + w, \tag{1.3.34}$$
$$x \in \mathfrak{g}_{\mathbf{k}} \simeq T_1(G_{\mathbf{k}}), (n, w) \in \mathfrak{n}_{\mathbf{k},s} + \mathfrak{z}_{\mathbf{k}} \simeq T_{\dot{s}zn_s}(\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}),$$

where $\mathfrak{n}_{\mathbf{k},s} \subset \mathfrak{g}_{\mathbf{k}}$ is the Lie algebra of $N_{\mathbf{k},s}$.

In order to show that the image of map (1.3.34) coincides with $T_{\dot{s}zn_s}G_{\mathbf{k}} \simeq \mathfrak{g}_{\mathbf{k}}$ we shall need a direct sum decomposition of the Lie algebra $\mathfrak{g}_{\mathbf{k}}$ as a vector space,

$$\mathfrak{g}_{\mathbf{k}} = \mathfrak{n}_{\mathbf{k}} + \mathfrak{z}_{\mathbf{k}} + \overline{\mathfrak{n}}_{\mathbf{k}} + \mathfrak{h}'_{\mathbf{k}}, \tag{1.3.35}$$

where $\mathfrak{h}'_{\mathbf{k}}$ is a complementary s–invariant subspace to $\mathfrak{h}^s_{\mathbf{k}}$ in $\mathfrak{h}_{\mathbf{k}}$.

We shall use isomorphism (1.3.7), $\alpha : N_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} \to N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} = N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k}}$. By the definition α is the restriction of the map γ to the subvariety $N_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} \subset G_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}$. Observe that in terms of the left trivialization of the tangent bundle $TG_{\mathbf{k}}$ the differential of the map α at points $(1, \dot{s}zn_s) \in N_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}$, $n_s \in N_{\mathbf{k},s}, z \in Z_{\mathbf{k}}$ is given by

$$d\alpha_{(1,szn_s)} : (x, (n, w)) \to -(Id - \operatorname{Ad}(\dot{s}zn_s)^{-1})x + n + w, \tag{1.3.36}$$
$$x \in \mathfrak{n} \simeq T_1(N_{\mathbf{k}}), (n, w) \in \mathfrak{n}_{\mathbf{k},s} + \mathfrak{z}_{\mathbf{k}} \simeq T_{\dot{s}zn_s}(\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}).$$

Recall that the conjugation map $\alpha : N_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} \to N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}$ is an isomorphism of varieties, and hence its differential is a vector space isomorphism of the corresponding tangent spaces at all points. Using the left trivialization of the tangent bundle $TG_{\mathbf{k}}$ the tangent space $T_{\dot{s}zn_s}(N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s})$ can be identified with $\mathfrak{n}_{\mathbf{k},s} + \mathfrak{z}_{\mathbf{k}} + \operatorname{Ad}(\dot{s}zn_s)^{-1}\mathfrak{n}_{\mathbf{k}}$, $T_{\dot{s}zn_sz}(N_{\mathbf{k}}\dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}) \simeq \mathfrak{n}_{\mathbf{k},s} + \mathfrak{z}_{\mathbf{k}} + \operatorname{Ad}(\dot{s}zn_s)^{-1}\mathfrak{n}_{\mathbf{k}}$. Therefore using (1.3.36) and the fact that $d\alpha_{(1,\dot{s}zn_s)}$ is a vector space isomorphism we deduce that

$$(Id - \operatorname{Ad}(\dot{s}zn_s)^{-1})\mathfrak{n}_{\mathbf{k}} + \mathfrak{n}_{\mathbf{k},s} + \mathfrak{z}_{\mathbf{k}} = \operatorname{Ad}(\dot{s}zn_s)^{-1}\mathfrak{n}_{\mathbf{k}} + \mathfrak{n}_{\mathbf{k},s} + \mathfrak{z}_{\mathbf{k}}. \tag{1.3.37}$$

Now observe that by the definition the vector subspace $(Id - \operatorname{Ad}(\dot{s}zn_s)^{-1})\mathfrak{n}_{\mathbf{k}} \subset \mathfrak{g}_{\mathbf{k}}$ is contained in the image of $d\alpha_{(1,\dot{s}zn_s)}$, and by (1.3.37) the vector subspace $\operatorname{Ad}(\dot{s}zn_s)^{-1}\mathfrak{n}_{\mathbf{k}} \subset \mathfrak{g}_{\mathbf{k}}$ is also contained in the image of $d\alpha_{(1,\dot{s}zn_s)}$. Since $\mathfrak{n}_{\mathbf{k}} = (Id - \operatorname{Ad}(\dot{s}zn_s)^{-1})\mathfrak{n}_{\mathbf{k}} + \operatorname{Ad}(\dot{s}zn_s)^{-1}\mathfrak{n}_{\mathbf{k}}$, we deduce that $\mathfrak{n}_{\mathbf{k}}$ is contained in the image of $d\alpha_{(1,\dot{s}zn_s)}$, and hence in the image of $d\gamma_{(1,\dot{s}zn_s)}$,

$$\mathfrak{n}_{\mathbf{k}} \subset \operatorname{Im} d\gamma_{(1,\dot{s}zn_s)}. \tag{1.3.38}$$

Next observe that similarly to (1.3.7) one can show that the conjugation map

$$N_{\mathbf{k}} \times N_{\mathbf{k},s^{-1}}Z_{\mathbf{k}}\dot{s} \to N_{\mathbf{k}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}} = N_{\mathbf{k},s^{-1}}Z_{\mathbf{k}}\dot{s}N_{\mathbf{k}}, N_{\mathbf{k},s^{-1}} = \{n \in N_{\mathbf{k}} : \dot{s}^{-1}n\dot{s} \in \overline{N}_{\mathbf{k}}\} \tag{1.3.39}$$

is an isomorphism of varieties.

Interchanging the roles of $N_{\mathbf{k}}$ and $\overline{N}_{\mathbf{k}}$ in (1.3.39) we immediately obtain that the conjugation map $\overline{\alpha} : \overline{N}_{\mathbf{k}} \times \overline{N}_{\mathbf{k},s^{-1}}Z_{\mathbf{k}}\dot{s} \to \overline{N}_{\mathbf{k},s^{-1}}Z_{\mathbf{k}}\dot{s}\overline{N}_{\mathbf{k}}$, where $\overline{N}_{\mathbf{k},s^{-1}} = \dot{s}N_{\mathbf{k},s}\dot{s}^{-1}$, is an isomorphism of varieties. Observe also that by the definition the map $\overline{\alpha}$ is the restriction of γ to the subvariety $\overline{N}_{\mathbf{k}} \times \overline{N}_{\mathbf{k},s^{-1}}Z_{\mathbf{k}}\dot{s} = \overline{N}_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s} \subset G_{\mathbf{k}} \times \dot{s}Z_{\mathbf{k}}N_{\mathbf{k},s}$.

Using the differential of the map $\overline{\alpha}$ we immediately infer, similarly to inclusion (1.3.38), that

$$\overline{\mathfrak{n}}_{\mathbf{k}} \subset \operatorname{Im} d\gamma_{(1,\dot{s}zn_s)}. \tag{1.3.40}$$

Now observe that $\mathfrak{h}_{\mathbf{k}}$ normalizes $\mathfrak{n}_{\mathbf{k}}$, and $Z_{\mathbf{k}}$ is a subgroup of $L_{\mathbf{k}}$ the adjoint action of which has the property that for any $x' \in \mathfrak{h}'_{\mathbf{k}}$, $l \in L_{\mathbf{k}}$ one has $(\mathrm{Ad}(l)x')_{\mathfrak{h}'_{\mathbf{k}}} = x'$, where for any element $y \in \mathfrak{g}_{\mathbf{k}}$ we denote by $(y)_{\mathfrak{h}'_{\mathbf{k}}}$ the $\mathfrak{h}'_{\mathbf{k}}$–component of y with respect to decomposition (1.3.35). Therefore we have for any $x' \in \mathfrak{h}'_{\mathbf{k}}$

$$\left((Id - \mathrm{Ad}(\dot{s}zn_s)^{-1})x'\right)_{\mathfrak{h}'_{\mathbf{k}}} = (Id - \mathrm{Ad}\dot{s}^{-1})x'. \tag{1.3.41}$$

Since by the definition of $\mathfrak{h}'_{\mathbf{k}}$ the operator $\mathrm{Ad}\dot{s}^{-1}$ has no fixed points in it, from formula (1.3.41) it follows that the operator $\mathfrak{h}'_{\mathbf{k}} \to \mathfrak{h}'_{\mathbf{k}}$, $x' \mapsto (Id - \mathrm{Ad}\dot{s}^{-1})x'$ is invertible, so that $\mathfrak{h}'_{\mathbf{k}}$ is contained in the image of $(Id - \mathrm{Ad}(\dot{s}zn_s)^{-1})$, and hence, by formula (1.3.34), in the image of $d\gamma_{(1,\dot{s}zn_s)}$. Recalling also inclusions (1.3.38) and (1.3.40) and taking into account the obvious inclusion $\mathfrak{z}_{\mathbf{k}} \subset \mathrm{Im}\, d\gamma_{(1,\dot{s}zn_s)}$ and decomposition (1.3.35) we deduce that the image of the map $d\gamma_{(1,\dot{s}zn_s)}$ coincides with $\mathfrak{g}_{\mathbf{k}} \simeq T_{\dot{s}zn_s}G_{\mathbf{k}}$. Therefore the differential of the map γ is surjective at all points. This completes the proof of part (ii).

(iii) Finally we show that $NsZN$ is a closed subvariety in G provided that G is simply–connected.

Recall that by Lemma 1.3.3 (ii) $NsZN \simeq NsZN_s \simeq NZsN_s \simeq N \times Z \times N_s$ is a subvariety of G, and observe that multiplication by s^{-1} from the right induces an isomorphism of varieties, $NZsN_s \simeq NZsN_s s^{-1}$. By the definition of N_s, $NZsN_s s^{-1}$ is also a subset of $NZ\overline{N}$.

Lemma 1.3.5. *(i) $NZ\overline{N}$ and $\overline{N}ZN$ are subvarieties of G with the variety structure induced from $N \times Z \times \overline{N}$ using the bijective maps*

$$N \times Z \times \overline{N} \to NZ\overline{N}, (n, z, \bar{n}) \mapsto nz\bar{n}, \tag{1.3.42}$$

$$\overline{N} \times Z \times N \to \overline{N}ZN, (\bar{n}, z, n) \mapsto \bar{n}zn,$$

induced by the multiplication in G.

(ii) The varieties $NZ\overline{N}$ and $\overline{N}ZN$ are closed in G.

Proof. (i) We prove the statement for $NZ\overline{N}$. The other case is treated in a similar way.

By Lemma 1.3.2 (i) with $A_{\mathbf{k}} = G$, $G^1_{\mathbf{k}} = Z$ and $G^2_{\mathbf{k}} = N$, $NZ \subset G$ is an algebraic subgroup with the variety structure induced from $N \times Z$ by the multiplication map $N \times Z \to NZ$ in G.

One can describe the set $NZ\overline{N}$ as the orbit of the element 1 in G for the following regular action of $NZ \times \overline{N}$ on G:

$$(NZ \times \overline{N}) \times G \to G, ((nz, \bar{n}), g) \mapsto nzg\bar{n}^{-1}. \tag{1.3.43}$$

Thus $NZ\overline{N}$ has the natural structure of an algebraic variety by part (ii) of Lemma 1.3.2 with $A_{\mathbf{k}} = NZ \times \overline{N}$, $V = G_{\mathbf{k}}$, $\mathcal{O} = NZ\overline{N}$.

By definition (1.3.43) with $g = 1$ gives rise to a bijective $NZ \times \overline{N}$-equivariant morphism of $NZ \times \overline{N}$ and $NZ\overline{N}$ viewed as homogeneous spaces for $NZ \times \overline{N}$,

$$NZ \times \overline{N} \to NZ\overline{N}, (nz, \bar{n}) \mapsto nz\bar{n}^{-1}. \tag{1.3.44}$$

One verifies straightforwardly that this morphism induces an isomorphism of the tangent spaces at the points $(1, 1)$ and 1. Therefore by Lemma 1.3.2 (iii) with $X = NZ \times \overline{N}$, $Y = NZ\overline{N}$, $A_{\mathbf{k}} = NZ \times \overline{N}$ morphism (1.3.44) is an isomorphism of varieties. Composing it with the algebraic group isomorphism $NZ \simeq N \times Z$ and with the map taking the inverse in $\overline{N}$ we deduce that the first map in (1.3.42) is an isomorphism of varieties.

(ii) We shall prove the statement for $NZ\overline{N}$. The other case is treated in a similar way. We shall consider the case when Δ_0 is not empty. The other case can be considered in a similar way.

Recall that by the definition $\mathfrak{h}'_{\mathbb{R}}$ and $\mathfrak{h}_0$ are annihilators of each other with respect to the restriction of the bilinear form on $\mathfrak{g}$ to $\mathfrak{h}_{\mathbb{R}}$. As before, let $\mathfrak{h}'^*_{\mathbb{R}}$ and $\mathfrak{h}^*_0$ be the images of $\mathfrak{h}'_{\mathbb{R}}$ and $\mathfrak{h}_0$, respectively, under the isomorphism $\mathfrak{h}_{\mathbb{R}} \simeq \mathfrak{h}^*_{\mathbb{R}}$ induced by the bilinear form on $\mathfrak{g}$.

Introduce the element

$$\bar{h}_0 = \sum_{k=1}^{M} h_{i_k} \in \mathfrak{h}_{\mathbb{R}}. \tag{1.3.45}$$

By the definition of Δ^s_+ for any $x \in \mathfrak{h}^*_0$ one has $\bar{h}_0(x) = 0$ and a root $\alpha \in \Delta \setminus \Delta_0$ belongs to Δ^s_+ if and only if $\bar{h}_0(\alpha) > 0$,

$$\Delta^s_+ \setminus \Delta_0 = \{\alpha \in \Delta | \bar{h}_0(\alpha) > 0\}, \ \bar{h}_0(x) = 0, \ x \in \mathfrak{h}^*_0. \tag{1.3.46}$$

Let $\bar{h}^*_0 \in \mathfrak{h}^*_{\mathbb{R}}$ be the image in $\mathfrak{h}_{\mathbb{R}}$ of the element $\bar{h}_0 \in \mathfrak{h}'_{\mathbb{R}}$. Since $\bar{h}_0 \in \mathfrak{h}'_{\mathbb{R}}$ we actually have $\bar{h}^*_0 \in \mathfrak{h}'^*_{\mathbb{R}}$.

Let $\alpha_1, \ldots, \alpha_p$ be the simple roots in Γ^s which do not belong to Δ_0, $\omega_1, \ldots, \omega_p$ the corresponding fundamental weights. $\mathfrak{h}'^*_{\mathbb{R}}$ is a linear subspace in the real linear span Π of $\omega_1, \ldots, \omega_p$ as Π is the annihilator of the subspace of

$\mathfrak{h}_{\mathbb{R}}^*$ spanned by the roots from Δ_0 which is contained in $\mathfrak{h}_0^*$. The subset Π_+ of Π which consists of x satisfying the condition $(x, \alpha) > 0$, $\alpha \in \Delta_+^s \setminus \Delta_0$ is open in Π and by definition $\bar{h}_0^* \in \Pi_+ \cap \mathfrak{h}_{\mathbb{R}}'^*$. Therefore the intersection $\Pi_+ \cap \mathfrak{h}_{\mathbb{R}}'^*$ is not empty and open in $\mathfrak{h}_{\mathbb{R}}'^*$.

The roots $\gamma_1, \ldots, \gamma_{l'}$ form a linear basis of $\mathfrak{h}_{\mathbb{R}}'^*$. They also span a $\mathbb{Z}$–sublattice Q' in the $\mathbb{Z}$–lattice generated by $\omega_1, \ldots, \omega_p$ as every root is a linear combination of fundamental weights with integer coefficients and $\gamma_1, \ldots, \gamma_{l'}$ form a linear basis of $\mathfrak{h}_{\mathbb{R}}'^* \subset \Pi$. Linear combinations of elements of Q' with rational coefficients are dense in $\mathfrak{h}_{\mathbb{R}}'^*$, and, in particular, in the open set $\Pi_+ \cap \mathfrak{h}_{\mathbb{R}}'^*$. Since the subset Π_+ of Π consists of x satisfying the condition $(x, \alpha) > 0$, $\alpha \in \Delta_+^s \setminus \Delta_0$, there is a linear basis of $\mathfrak{h}_{\mathbb{R}}'^*$ which consists of linear combinations of $\omega_1, \ldots, \omega_p$ with positive rational coefficients. Multiplying the elements of this basis by appropriate positive integer numbers we obtain a linear basis Ω_i, $i = 1, \ldots, l'$ of $\mathfrak{h}_{\mathbb{R}}'^*$ which consists of integral dominant weights of the form $\Omega_i = \sum_{j=1}^p g_{ij} \omega_j$, $g_{ij} \in \mathbb{Z}, g_{ij} > 0$.

Recall that by the definition $\mathfrak{h}^s = \mathfrak{h}'^\perp$ is the orthogonal compliment to $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the restriction of the symmetric bilinear form on $\mathfrak{g}$ to $\mathfrak{h}$. $\mathfrak{h}^s$ is the complexification of $\mathfrak{h}_0$, and hence we deduce that an element $x \in \mathfrak{h}$ belongs to $\mathfrak{h}^s = \mathfrak{h}'^\perp$ if and only if $\Omega_i(x) = 0$, $i = 1, \ldots, l'$,

$$x \in \mathfrak{h}^s = \mathfrak{h}'^\perp \iff \Omega_i(x) = 0, i = 1, \ldots, l'. \tag{1.3.47}$$

Let B_+^s be the Borel subgroup of G corresponding to the system Δ_+^s of positive roots, B_-^s the opposite Borel subgroup , $N_\pm^s$, their unipotent radicals, respectively.

Next observe that since G is simply–connected, for each integral dominant weight λ with respect to the system Δ_+^s of positive roots there exists an irreducible finite–dimensional representation of G with highest weight λ.

Let V_{Ω_i}, $i = 1, \ldots, l'$ be the irreducible finite–dimensional representation of G with highest weight Ω_i with respect to the system Δ_+^s of positive roots. Denote by v_{Ω_i} a nonzero highest weight vector in V_{Ω_i} and by $(\cdot, \cdot)$ the contravariant form on V_{Ω_i} normalized in such a way that $(v_{\Omega_i}, v_{\Omega_i}) = 1$. The matrix element $g \mapsto (v_{\Omega_i}, g v_{\Omega_i})$, $g \in G$ can be regarded as a regular function on G whose restriction to the big dense cell $N_-^s H N_+^s$ is given by the character Ω_i of H,

$$(v_{\Omega_i}, n_- h n_+ v_{\Omega_i}) = (v_{\Omega_i}, h v_{\Omega_i}) = \Omega_i(h), n_- \in N_-^s, h \in H, n_+ \in N_+^s. \tag{1.3.48}$$

Each fundamental weight ω_i can be regarded as a regular function $g \mapsto (v_{\omega_j}, g v_{\omega_j})$ on G defined similarly to the regular function $g \mapsto (v_{\Omega_i}, g v_{\Omega_i})$ with V_{Ω_i} replaced by the irreducible finite–dimensional representation V_{ω_i} with highest weight ω_i. By the definition of Ω_i the function $(v_{\Omega_i}, g v_{\Omega_i})$ can be expressed as a product of functions $(v_{\omega_j}, g v_{\omega_j})$,

$$(v_{\Omega_i}, g v_{\Omega_i}) = \prod_{j=1}^p (v_{\omega_j}, g v_{\omega_j})^{g_{ij}}, g \in G. \tag{1.3.49}$$

Consider the closed subvariety in G defined by the equations $(v_{\Omega_i}, g v_{\Omega_i}) = 1$, $i = 1, \ldots, l'$, $g \in G$. According to the Bruhat decomposition every element $g \in G$ belongs to $g \in B_-^s w B_+^s$ for some $w \in W$. In this case g can be written in the form $g = n_- w h n_+$ for some $n_\pm \in N_\pm^s$, $h \in H$. Now by (1.3.48) and (1.3.49) we have $(v_{\Omega_i}, g v_{\Omega_i}) = \Omega_i(h)(v_{\Omega_i}, w v_{\Omega_i}) = \Omega_i(h) \prod_{j=1}^p (v_{\omega_j}, w v_{\omega_j})^{g_{ij}}$. As different weight spaces of V_{ω_j} are orthogonal with respect to the contravariant form, the right hand side of the last identity is not zero for all $i = 1, \ldots, l'$ if and only if w fixes all weights ω_i, $i = 1, \ldots, p$, i.e. if and only if w belongs to the Weyl group of the root subsystem Δ_0. Since Δ_0 is the root system of the Levi factor L, and $(v_{\omega_i}, v_{\omega_i}) = 1$, one has $(v_{\Omega_i}, w v_{\Omega_i}) \neq 0$, $i = 1, \ldots, l'$ if and only if $g \in \overline{N} L N$, and in that case $(v_{\Omega_i}, g v_{\Omega_i}) = \Omega_i(h)$, where $g = n_- w h n_+$ for some $n_\pm \in N_\pm^s$, $h \in H$, and w is an element of the Weyl group of the root subsystem Δ_0.

As we already proved in (1.3.47), an element $x \in \mathfrak{h}$ belongs to $\mathfrak{h}^s$ if and only if $\Omega_i(x) = 0$, $i = 1, \ldots, l'$. Therefore the conditions $(v_{\Omega_i}, g v_{\Omega_i}) = \Omega_i(h) = 1$, $i = 1, \ldots, l'$ are equivalent to the fact that h belongs to a subgroup $H^{0'}$ of H with Lie algebra $\mathfrak{h}^s$. Hence the equations $(v_{\Omega_i}, g v_{\Omega_i}) = 1$, $i = 1, \ldots, l'$ hold if and only if $g \in N Z' \overline{N}$, where $Z' \subset L$ is a subgroup of L with the same Lie algebra as Z. Thus $N Z' \overline{N}$ is a closed subvariety of G. Its connected component containing the identity element of G is obviously $N Z \overline{N}$. Thus $N Z \overline{N}$ is a closed subvariety of G.

$\square$

Now recall that $N Z s N_s \simeq N Z s N_s s^{-1}$ is a subset of $N Z \overline{N}$ as by the definition of N_s, $s N_s s^{-1} \subset \overline{N}$. In fact $s N_s s^{-1}$ is the algebraic subgroup in $\overline{N}$ generated by the one–parameter subgroups corresponding to the roots from the set $\{\alpha \in \Delta_+^s : s^{-1}\alpha \in -\Delta_+^s\}$. So by isomorphisms (1.3.2) and (1.3.42), and by Lemma 1.3.5 (i), $N Z s N_s s^{-1} \simeq N Z s N_s$ is a closed subvariety of $N Z \overline{N}$ due to the closed embedding

$$N \times Z \times s N_s s^{-1} \hookrightarrow N \times Z \times \overline{N}.$$

Also Lemma 1.3.5 (ii) immediately implies that the closed subvariety $NZsN_ss^{-1} \subset NZ\overline{N}$ is also closed in G, and hence the variety $NsZN = NZsN_s \simeq NZsN_ss^{-1}s \simeq NZsN_ss^{-1}$ is closed in G.

$\square$

The subvarieties $\Sigma_{\mathbf{k},s} \subset G_{\mathbf{k}}$ are analogues of the Slodowy slices in algebraic group theory.

Remark 1.3.6. *In fact, in the construction of the inverse map to morphism (1.3.7) suggested in the proof of the previous proposition we only used relations in $G_{\mathbf{k}}$ arising from the corresponding Chevalley group over $\mathbb{Z}$. Therefore isomorphisms similar to (1.3.7) hold in the case when $G_{\mathbf{k}}$ is replaced with the Chevalley group (or even group scheme) over an arbitrary ring.*

1.4   The Lusztig partition

In this section, as before, $G_{\mathbf{k}}$ is a connected finite–dimensional semisimple algebraic group over an algebraically closed field $\mathbf{k}$. At the same time we shall also consider connected finite–dimensional semisimple algebraic groups G_p of the same type as $G_{\mathbf{k}}$, i.e. with the same root system, over algebraically closed fields for all characteristic exponents p.

In the next section we shall show that for every conjugacy class $\mathcal{O}$ in $G_{\mathbf{k}}$ one can find a subvariety $\Sigma_{s,\mathbf{k}} \subset G_{\mathbf{k}}$ such that $\mathcal{O}$ intersects $\Sigma_{s,\mathbf{k}}$ and dim $\mathcal{O} = \text{codim } \Sigma_{s,\mathbf{k}}$. It turns out that there is a remarkable partition of the group $G_{\mathbf{k}}$ introduced in [83] the strata of which are unions of conjugacy classes of the same dimension. For each stratum of this partition there is a Weyl group element s such that all conjugacy classes $\mathcal{O}$ from that stratum intersect $\Sigma_{s,\mathbf{k}}$, and dim $\mathcal{O} = \text{codim } \Sigma_{s,\mathbf{k}}$. This, in particular, determines the dimensions of the conjugacy classes in each stratum. In this section, which is rather descriptive, we recall the definition of this partition called the Lusztig partition. The main property of this partition gives an affirmative answer to an old question about intersection of the conjugacy classes in $G_{\mathbf{k}}$ with Bruhat cells. The exposition in this section mainly follows paper [83] to which we refer the reader for technical details.

Note that all objects introduced in this section, except for the map ϕ_{G_p} and the Lusztig partition itself, only depend on the Weyl group of $G_{\mathbf{k}}$ or of G_p, on the characteristic of $\mathbf{k}$ and on p. The fundamental reason behind this phenomenon is that in the definitions of these objects only representation theory of the Weyl group and the sets of unipotent conjugacy classes in $G_{\mathbf{k}}$ or in G_p are used. These sets only depend on the characteristic of $\mathbf{k}$, on p and on the Weyl group. The reader may always assume that $G_{\mathbf{k}}$ and all G_p have the same root datum.

For any Weyl group W let $\widehat{W}$ be the set of isomorphism classes of irreducible representations of W over $\mathbb{Q}$. For any $E \in \widehat{W}$ let b_E be the smallest nonnegative integer such that E appears with non–zero multiplicity in the b_E-th symmetric power of the reflection representation of W. If this multiplicity is equal to 1 then one says that E is good. If $W' \subset W$ are two Weyl groups, and $E \in \widehat{W'}$ is good then there is a unique $\widetilde{E} \in \widehat{W}$ such that $\widetilde{E}$ appears in the decomposition of the induced representation $\text{Ind}_{W'}^{W}E$, $b_{\widetilde{E}} = b_E$, and $\widetilde{E}$ is good. The representation $\widetilde{E}$ is called j-induced from E, $\widetilde{E} = j_{W'}^{W}E$.

Let $g \in G_p$, and $g = g_sg_u$ its decomposition as a product of the semisimple part g_s and the unipotent part g_u. Let $C = Z_{G_p}(g_s)^\circ$ be the identity component of the centralizer of g_s in G_p. C is a reductive subgroup of G_p of the same rank as G_p. Let H_p be a maximal torus of C. H_p is also a maximal torus in G_p, and hence one has a natural imbedding

$$W' = N_C(H_p)/H_p \to N_{G_p}(H_p)/H_p = W,$$

where $N_C(H_p), N_{G_p}(H_p)$ stand for the normalizers of H_p in C and in G_p, respectively, W' is the Weyl group of C and W is the Weyl group of G_p.

Let E be the irreducible representation of W' associated with the help of the Springer correspondence to the conjugacy class of g_u and the trivial local system on it. Then E is good, and let $\widetilde{E}$ be the j-induced representation of W. This gives a well-defined map $\phi_{G_p} : G_p \to \widehat{W}$. The fibers of this map are called the strata of G_p. By definition the map ϕ_{G_p} is constant on each conjugacy class in G_p. Therefore the strata are unions of conjugacy classes.

Moreover, by 2.4 in [83] we have the following formula for the dimension of the centralizer $Z_{G_p}(g)$ of any element $g \in G_p$ in G_p:

$$\dim Z_{G_p}(g) = \text{rank } G_p + 2b_{\phi_{G_p}(g)}, \tag{1.4.1}$$

where rank G_p is the rank of G_p.

It turns out that the image $\mathcal{R}(W)$ of ϕ_{G_p} only depends on W and not on p or the underlying root datum of G_p. It can be described as follows. Let $\mathcal{N}(G_p)$ be the unipotent variety of G_p and $\underline{\mathcal{N}}(G_p)$ the set of unipotent conjugacy

classes in G_p. Let $\mathcal{X}^p(W)$ be the set of irreducible representations of W associated by the Springer correspondence to unipotent classes in $\underline{\mathcal{N}}(G_p)$ and the trivial local systems on them. We shall identify $\mathcal{X}^p(W)$ and $\underline{\mathcal{N}}(G_p)$. Let $f_p : \underline{\mathcal{N}}(G_p) \to \mathcal{X}^p(W)$ be the corresponding bijective map.

Proposition 1.4.1. ([83], Sections 1 and 2) *We have*

$$\mathcal{R}(W) = \mathcal{X}^1(W) \cup_{r \text{ prime}} \mathcal{X}^r(W).$$

If G_p is of type A_n, $(n \geq 1)$ or E_6 then $\mathcal{R}(W) = \mathcal{X}^1(W)$.
If G_p is of type B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, F_4 or E_7 then $\mathcal{R}(W) = \mathcal{X}^2(W)$.
If G_p is of type G_2 then $\mathcal{R}(W) = \mathcal{X}^3(W)$.
If G_p is of type E_8 then $\mathcal{R}(W) = \mathcal{X}^2(W) \cup \mathcal{X}^3(W)$, and $\mathcal{X}^2(W) \cap \mathcal{X}^3(W) = \mathcal{X}^1(W)$.

The description of the set $\mathcal{R}(W)$ given in Proposition 1.4.1 and the bijections $\underline{\mathcal{N}}(G_p) \to \mathcal{X}^p(W)$ yield certain maps between sets $\underline{\mathcal{N}}(G_p)$ which preserve dimensions of conjugacy classes by (1.4.1). For instance, one always has an inclusion $\mathcal{X}^1(W) \subset \mathcal{X}^r(W)$ for any $r \geq 2$. The corresponding inclusion $\underline{\mathcal{N}}(G_1) \subset \underline{\mathcal{N}}(G_p)$ coincides with the Spaltenstein map $\pi_p^{G_1} : \underline{\mathcal{N}}(G_1) \to \underline{\mathcal{N}}(G_p)$ which is a bijection for good p (see [122], Théorème III.5.2).

Now we introduce an alternative description of the strata in terms of intersections of conjugacy classes with Bruhat cells. Fix a system of positive roots in Δ. Note that Δ can be regarded as the root system of the pair (G_p, H_p), $\Delta = \Delta(G_p, H_p)$. Let B_p be the Borel subgroup in G_p associated to the corresponding system of negative roots, $H_p \subset B_p$ the maximal torus, and l the corresponding length function on W. Denote by $\underline{W}$ the set of conjugacy classes in W. For each $w \in W = N_{G_p}(H_p)/H_p$ one can pick up a representative $\dot{w} \in G_p$. If p is the characteristic exponent of $\mathbf{k}$, we write $B_p = B_{\mathbf{k}}$, $\underline{\mathcal{N}}(G_p) = \underline{\mathcal{N}}(G_{\mathbf{k}})$, etc.

Let $\mathcal{C}$ be a conjugacy class in W. Pick up a representative $w \in \mathcal{C}$ of minimal possible length with respect to l. By Theorem 0.4 in [86] there is a unique conjugacy class $\mathcal{O} \in \underline{\mathcal{N}}(G_1)$ of minimal possible dimension which intersects the Bruhat cell $B_1 \dot{w} B_1$ and does not depend on the choice of the minimal possible length representative w in $\mathcal{C}$. We denote this class by $\Phi_1^{G_1}(\mathcal{C})$.

As shown in Section 1.1 in [86], one can always find a representative $w \in \mathcal{C}$ of minimal possible length with respect to l which is elliptic in a parabolic Weyl subgroup $W' \subset W$, i.e. w acts without fixed points in the reflection representation of W'. Indeed, by Theorem 3.2.12 in [44] there is a parabolic subgroup $W' \subset W$ such that $\mathcal{C} \cap W'$ is an elliptic conjugacy class in W', i.e. every element in it is elliptic in W'. By Lemma 3.1.14 in [44] if $w \in \mathcal{C} \cap W'$ is of minimal possible length in its conjugacy class in W' with respect to the restriction of l to W' then it is also of minimal possible length in $\mathcal{C}$ with respect to l.

Let $P_p' \subset G_p$ be the parabolic subgroup which contains B_p and corresponds to W', and M_p' the semi-simple part of the Levi factor of P_p', so that W' is the Weyl group of M_p'. Let $\Phi_p^{G_1}(\mathcal{C})$ be the unipotent class in G_p containing the class $\pi_p^{M_p'} \Phi_1^{M_p'}(\mathcal{C})$. This class only depends on the conjugacy class $\mathcal{C}$, and hence one has a map $\Phi_p^{G_1} : \underline{W} \to \underline{\mathcal{N}}(G_p)$ which is in fact surjective by 4.5(a) in [86].

Let $\mathcal{C} \in \underline{W}$, and $m_{\mathcal{C}}$ the dimension of the fixed point space for the action of any $w \in \mathcal{C}$ in the reflection representation. Then by Theorem 0.2 in [85] for any $\gamma \in \underline{\mathcal{N}}(G_p)$ there is a unique $\mathcal{C}_0 \in (\Phi_p^{G_1})^{-1}(\gamma)$ such that the function $m_{\mathcal{C}} : (\Phi_p^{G_1})^{-1}(\gamma) \to \mathbb{N}$ reaches its minimum at $\mathcal{C}_0$. We denote $\mathcal{C}_0$ by $\Psi_p^{G_1}(\gamma)$. Thus one obtains an injective map $\Psi_p^{G_1} : \underline{\mathcal{N}}(G_p) \to \underline{W}$.

Now using identifications $f_p : \underline{\mathcal{N}}(G_p) \to \mathcal{X}^p(W)$ one can define the union $\widehat{\underline{\mathcal{N}}}(W)$ of the sets $\underline{\mathcal{N}}(G_1)$ and of $\underline{\mathcal{N}}(G_r)$ over all prime r as the union $\mathcal{X}^1(W) \cup_{r \text{ prime}} \mathcal{X}^r(W) = \mathcal{R}(W)$. Thus we have a bijection

$$F : \widehat{\underline{\mathcal{N}}}(W) = \underline{\mathcal{N}}(G_1) \cup_{r \text{ prime}} \underline{\mathcal{N}}(G_r) \to \mathcal{X}^1(W) \cup_{r \text{ prime}} \mathcal{X}^r(W) = \mathcal{R}(W). \tag{1.4.2}$$

Using maps $\Phi_p^{G_1}$ one can also define a surjective map $\Phi^W : \underline{W} \to \widehat{\underline{\mathcal{N}}}(W)$ as described in Section 4.1 in [83]. As it is observed in Section 4.1 of [83] (see also 0.4 in [85]), if $\Phi_r^{G_1}(\mathcal{C}) \in \underline{\mathcal{N}}(G_1)$ for all $r > 1$ then $\Phi_r^{G_1}(\mathcal{C})$ is independent of r, and one puts $\Phi^W(\mathcal{C}) = \Phi_r^{G_1}(\mathcal{C})$ for any $r > 1$, and if $\Phi_r^{G_1}(\mathcal{C}) \notin \underline{\mathcal{N}}(G_1)$ for some $r > 1$ then r is unique, and one defines $\Phi^W(\mathcal{C}) = \Phi_r^{G_1}(\mathcal{C})$. By Proposition 1.4.1 and formula (1.4.2) Φ^W introduced this way is well defined.

By the definition there is a right-sided injective inverse Ψ^W to Φ^W such that if $\gamma \in \underline{\mathcal{N}}(G_1)$ then $\Psi^W(\gamma) = \Psi_1^{G_1}(\gamma)$, and if $\gamma \notin \underline{\mathcal{N}}(G_1)$, and $\gamma \in \underline{\mathcal{N}}(G_r)$ then $\Psi^W(\gamma) = \Psi_r^{G_1}(\gamma)$.

Denote by $C(W)$ the image of $\widehat{\underline{\mathcal{N}}}(W)$ in $\underline{W}$ under the map Ψ^W, $C(W) = \Psi^W(\widehat{\underline{\mathcal{N}}}(W))$. We shall identify $C(W)$, $\widehat{\underline{\mathcal{N}}}(W)$ and $\mathcal{R}(W)$.

Now the strata of the Lusztig partition can be described geometrically as follows. Let $\mathcal{C} \in C(W)$. Pick up a representative $w \in \mathcal{C}$ of minimal possible length with respect to l. Denote by $\underline{G}_p$ the set of conjugacy classes in G_p,

and by $G'_{\mathcal{C}}$ the set of all conjugacy classes in G_p which intersect the Bruhat cell $B_p \dot{w} B_p$. This definition does not depend on the choice of the minimal possible length representative w. Let

$$d_{\mathcal{C}} = \min_{\gamma \in G'_{\mathcal{C}}} \dim \gamma.$$

Then the stratum $G_{\mathcal{C}} = \phi_{G_p}^{-1}(F(\Phi^W(\mathcal{C})))$ can be described as follows (see Theorem 5.2, [83]),

$$G_{\mathcal{C}} = \bigcup_{\gamma \in G'_{\mathcal{C}}, \ \dim \gamma = d_{\mathcal{C}}} \gamma. \tag{1.4.3}$$

Thus we have a disjoint union

$$G_p = \bigcup_{\mathcal{C} \in C(W)} G_{\mathcal{C}}.$$

Note that by the definition of the stratum, for good p, if $\mathcal{C} \in \mathrm{Im}(\Psi_1^{G_1})$ then $G_{\mathcal{C}}$ contains a unique unipotent class, and if $\mathcal{C} \notin \mathrm{Im}(\Psi_1^{G_1})$ then $G_{\mathcal{C}}$ does not contain unipotent classes.

The maps introduced above are summarized in the following diagram

$$
\begin{array}{ccccccc}
\mathcal{X}^1(W) & \xleftarrow{\ f_1\ } & \underline{\mathcal{N}}(G_1) & & & & \\
\downarrow \iota & & \downarrow \pi^{G_1} & & & & \\
G_{\mathbf{k}} & \xrightarrow{\ \phi_{G_{\mathbf{k}}}\ } & \mathcal{R}(W) & \xleftarrow{\ F\ } & \widehat{\mathcal{N}}(W) & \overset{\Phi^W}{\underset{\Psi^W}{\rightleftarrows}} & W,
\end{array}
\tag{1.4.4}
$$

where ι is an inclusion, bijections f_1 and F are induced by the Springer correspondence with the trivial local data, and the inclusion π^{G_1} is induced by the Spaltenstein map.

For exceptional groups the maps f_1 and F can be described explicitly using tables in [123], the maps Φ^W and Ψ^W can be described using the tables in Section 2 in [85], and the maps ι and π^{G_1} can be described explicitly using the tables of unipotent classes in [75], Chapter 22 or [123] (note that the labeling for unipotent classes in bad characteristics in [75] differs from that in [123]). The dimensions of the conjugacy classes in the strata in $G_{\mathbf{k}}$ can be obtained using dimension tables of centralizers of unipotent elements in case when a stratum contains a unipotent class (see [19, 75]), the tables for dimensions of the centralizers of unipotent elements in bad characteristic when a stratum does not contain a unipotent class (see [75]) or formula (1.4.1) and the tables of the values of the b–invariant b_E for representations of Weyl groups (see [19, 44]). Note that formula (1.4.1) implies that if $\mathcal{O}$ is any conjugacy class in $G_{\mathcal{C}}$, $\mathcal{O} \in G_{\mathcal{C}}$ then

$$\dim \mathcal{O} = \dim \Phi^W(\mathcal{C}). \tag{1.4.5}$$

In case of classical groups all those maps and dimensions are described in terms of partitions (see [19, 45, 75, 84, 85, 86, 122]). In case of classical matrix groups the strata can also be described explicitly (see [83]). We recall this description below. By (1.4.5) the dimensions of the conjugacy classes in every stratum of $G_{\mathbf{k}}$ are equal to the dimension of the corresponding conjugacy class in $\widehat{\mathcal{N}}(W)$. The dimensions of centralizers of unipotent elements in arbitrary characteristic can be found in [54, 75].

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ is a partition we denote by $\lambda^* = (\lambda_1^* \geq \lambda_2^* \geq \ldots \geq \lambda_m^*)$ the corresponding dual partition. It is defined by the property that $\lambda_1^* = m$ and $\lambda_i^* - \lambda_{i+1}^* = l_i(\lambda)$, where $l_i(\lambda)$ is the number of times i appears in the partition λ. We also denote by $\tau(\lambda)$ the length of λ, $\tau(\lambda) = m$. If a partition μ is obtained from λ by adding a number of zeroes, we shall identify λ and μ.

$\mathbf{A_n}$

$G_{\mathbf{k}}$ is of type $\mathrm{SL}(V)$ where V is a vector space of dimension $n + 1 \geq 1$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \geq 1$. W is the group of permutations of $n+1$ elements. All sets in (1.4.4), except for $G_{\mathbf{k}}$, are identified with the set of partitions of $n+1$, and under this identification all the maps, except for $\phi_{G_{\mathbf{k}}}$, are the identity maps.

To describe $\phi_{G_{\mathbf{k}}}$ for $G_{\mathbf{k}} = \mathrm{SL}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G_{\mathbf{k}}$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$ and let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_m^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$. Then $\phi_{G_{\mathbf{k}}}(g)$ is the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_m$ given by $\lambda(g)_j = \sum_{x \in \mathbf{k}^*} \lambda_j^x$.

If g is any element in the stratum G_λ corresponding to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$, $\lambda_m \geq 1$, then

$$\dim Z_{G_\mathbf{k}}(g) = n + 2\sum_{i=1}^{m}(i-1)\lambda_i. \tag{1.4.6}$$

The element of $\underline{W}$ which corresponds to λ is the Coxeter class in the Weyl subgroup of the type

$$A_{\lambda_1-1} + A_{\lambda_2-1} + \ldots + A_{\lambda_m-1}. \tag{1.4.7}$$

The summands in diagram (1.4.7) are called blocks. Blocks of type A_0 are called trivial.

$\mathbf{C_n}$

$G_\mathbf{k}$ is of type $\mathrm{Sp}(V)$ where V is a symplectic space of dimension $2n$, $n \geq 2$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent p. W is the group of permutations of the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$ which also commute with the involution $\varepsilon_i \mapsto -\varepsilon_i$. Each element $s \in W$ can be expressed as a product of disjoint cycles of the form

$$\varepsilon_{k_1} \to \pm\varepsilon_{k_2} \to \pm\varepsilon_{k_3} \to \ldots \to \pm\varepsilon_{k_r} \to \pm\varepsilon_{k_1}.$$

This cycle is of length r; it is called positive if $s^r(\varepsilon_{k_1}) = \varepsilon_{k_1}$ and negative if $s^r(\varepsilon_{k_1}) = -\varepsilon_{k_1}$. The lengths of the cycles together with their signs give a set of positive or negative integers called the signed cycle-type of s. To each positive cycle of s of length r there corresponds a pair of orbits $X, -X$, $X \neq -X$, $|X| = r$, for the action of the group $\langle s \rangle$ generated by s on the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$, and to each negative cycle of s of length r there corresponds a negative orbit X, $X = -X$, $|X| = 2r$, for the action of $\langle s \rangle$ on E. We call orbits of the former type positive and orbits of the latter type negative. A positive cycle of length 1 is called trivial. It corresponds to a pair of fixed points for the action of $\langle s \rangle$ on E.

Elements of $\underline{W}$ are parametrized by pairs of partitions (λ, μ) satisfying the following conditions.

- The parts of λ are even (for any $w \in \mathcal{C} \in \underline{W}$ they are the numbers of elements in the negative orbits X, $X = -X$, in E for the action of the group $\langle w \rangle$ generated by w);

- μ consists of pairs of equal parts (they are the numbers of elements in the positive $\langle w \rangle$–orbits X in E; these orbits appear in pairs $X, -X$, $X \neq -X$);

- $\sum \lambda_i + \sum \mu_j = 2n$.

We denote this set of pairs of partitions by $\mathcal{A}_{2n}^1$.

An element of $\underline{W}$ which corresponds to a pair (λ, μ), $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m)$ and $\mu = (\mu_1 = \mu_2 \leq \ldots \leq \mu_{2k-1} = \mu_{2k})$ is the Coxeter class in the Weyl subgroup of the type

$$C_{\frac{\lambda_1}{2}} + C_{\frac{\lambda_2}{2}} + \ldots + C_{\frac{\lambda_m}{2}} + A_{\mu_1-1} + A_{\mu_3-1} + \ldots + A_{\mu_{2k-1}-1}. \tag{1.4.8}$$

If the characteristic exponent of $\mathbf{k}$ is not equal to 2, elements of $\mathcal{N}(G_\mathbf{k})$ are parametrized by partitions λ of $2n$ for which $l_j(\lambda)$ is even for odd j. We denote this set of partitions by $\mathcal{T}_{2n}$. In case of $G_\mathbf{k} = \mathrm{Sp}(V)$ the parts of λ are just the sizes of the Jordan blocks in V of the unipotent elements from the conjugacy class corresponding to λ.

In this case $\widehat{\mathcal{N}}(W) = \mathcal{N}(G_2)$, and G_2 is of type $\mathrm{Sp}(V)$ where V is a symplectic space of dimension $2n$ over an algebraically closed field of characteristic 2. Elements of $\mathcal{N}(G_2)$ are parametrized by pairs (λ, ε), where $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m) \in \mathcal{T}_{2n}$, and $\varepsilon : \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \to \{0, 1, \omega\}$ is a function such that

$$\varepsilon(k) = \begin{cases} \omega & \text{if } k \text{ is odd;} \\ 1 & \text{if } k = 0; \\ 1 & \text{if } k > 0 \text{ is even, } l_k(\lambda) \text{ is odd;} \\ 0 \text{ or } 1 & \text{if } k > 0 \text{ is even, } l_k(\lambda) \text{ is even.} \end{cases} \tag{1.4.9}$$

We denote the set of such pairs (λ, ε) by $\mathcal{T}_{2n}^2$.

Elements of $\widehat{W}$ are parametrized by pairs of partitions (α, β) written in non–decreasing order, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{\tau(\alpha)}$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{\tau(\beta)}$, and such that $\sum \alpha_i + \sum \beta_i = n$. By adding zeroes we can assume that the length $\tau(\alpha)$ of α is related to the length of β by $\tau(\alpha) = \tau(\beta) + 1$. The set of such pairs is denoted by $X_{n,1}$.

The maps f_1, F can be described as follows. Let $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}) \in \mathcal{T}_{2n}$, and assume that $\lambda_1 = 0$. If $f_1(\lambda) = ((c'_1, c'_3, \ldots, c'_{2m+1}), (c'_2, c'_4, \ldots, c'_{2m}))$ then the parts c'_i are defined by induction starting from $c'_1 = 0$,

$$
\begin{array}{ll}
c'_i = \frac{\lambda_i}{2} & \text{if } \lambda_i \text{ is even and } c'_{i-1} \text{ is already defined;} \\
c'_i = \frac{\lambda_i + 1}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c'_{i-1} \text{ is already defined;} \\
c'_{i+1} = \frac{\lambda_i - 1}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c'_i \text{ is already defined.}
\end{array}
$$

The image of f_1 consists of all pairs $((c'_1, c'_3, \ldots, c'_{2m+1}), (c'_2, c'_4, \ldots, c'_{2m})) \in X_{n,1}$ such that $c'_i \leq c'_{i+1} + 1$ for all i.

If $F(\lambda, \varepsilon) = ((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m}))$ then the parts c_i are defined by induction starting from $c_1 = 0$,

$$
\begin{array}{ll}
c_i = \frac{\lambda_i}{2} & \text{if } \lambda_i \text{ is even, } \varepsilon(\lambda_i) = 1 \text{ and } c_{i-1} \text{ is already defined;} \\
c_i = \frac{\lambda_i + 1}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_{i-1} \text{ is already defined;} \\
c_{i+1} = \frac{\lambda_i - 1}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_i \text{ is already defined;} \\
c_i = \frac{\lambda_i + 2}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = \varepsilon(\lambda_{i+1}) = 0 \text{ and } c_{i-1} \text{ is already defined;} \\
c_{i+1} = \frac{\lambda_i - 2}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = \varepsilon(\lambda_{i+1}) = 0 \text{ and } c_i \text{ is already defined.}
\end{array}
$$

The image $\mathcal{R}(W)$ of F consists of all pairs $((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m})) \in X_{n,1}$ such that $c_i \leq c_{i+1} + 2$ for all i.

The map Φ^W is defined by $\Phi^W(\lambda, \mu) = (\nu, \varepsilon)$, where the set of parts of ν is just the union of the sets of parts of λ and μ, and

$$
\varepsilon(k) = \left\{
\begin{array}{ll}
1 & \text{if } k \in 2\mathbb{N} \text{ is a part of } \lambda; \\
0 & \text{if } k \in 2\mathbb{N} \text{ is not a part of } \lambda; \\
\omega & \text{if } k \text{ is odd.}
\end{array}
\right.
$$

The map Ψ^W associates to each pair (ν, ε) a unique point (λ, μ) in the preimage $(\Phi^W)^{-1}(\nu, \varepsilon)$ such that the number of parts of μ is minimal possible. This point is defined by the conditions

$$
l_k(\lambda) = \left\{
\begin{array}{ll}
0 & \text{if } k \text{ is odd or } k \text{ is even, } l_k(\nu) \geq 2 \text{ is even and } \varepsilon(k) = 0; \\
l_k(\nu) & \text{otherwise,}
\end{array}
\right.
$$

$$
l_k(\mu) = \left\{
\begin{array}{ll}
l_k(\nu) & \text{if } k \text{ is odd or } k \text{ is even, } l_k(\nu) \geq 2 \text{ is even and } \varepsilon(k) = 0; \\
0 & \text{otherwise.}
\end{array}
\right.
$$

The map π^{G_1} is given by $\pi^{G_1}(\lambda) = (\lambda, \varepsilon')$, where

$$
\varepsilon'(k) = \left\{
\begin{array}{ll}
\omega & \text{if } k \text{ is odd;} \\
1 & \text{if } k \text{ is even.}
\end{array}
\right. \tag{1.4.10}
$$

The map π^{G_1} is injective and its image consists of pairs $(\lambda, \varepsilon') \in \mathcal{T}_{2n}^2$, where ε' satisfies conditions (1.4.10).

To describe $\phi_{G_{\mathbf{k}}}$ for $G_{\mathbf{k}} = \mathrm{Sp}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G_{\mathbf{k}}$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For any $x \in \mathbf{k}^*$ with $x^2 = 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$, where $((\lambda_1^x \geq \lambda_3^x \geq \ldots \geq \lambda_{2m+1}^x), (\lambda_2^x \geq \lambda_4^x \geq \ldots \geq \lambda_{2m}^x))$ is the pair of partitions such that the corresponding irreducible representation of the Weyl group of type $B_{\dim V_x/2}$ is the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{Sp}(V_x)$ and to the trivial local data.

Let $\lambda(g)$ be the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_{2m+1}$ given by $\lambda(g)_j = \sum_x \lambda_j^x$, where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Now $\phi_{G_{\mathbf{k}}}(g)$ is the pair of partitions $((\lambda(g)_1 \geq \lambda(g)_3 \geq \ldots \geq \lambda(g)_{2m+1}), (\lambda(g)_2 \geq \lambda(g)_4 \geq \ldots \geq \lambda(g)_{2m}))$.

If g is any element in the stratum $G_{(\lambda, \varepsilon)}$ corresponding to a pair $(\lambda, \varepsilon) \in \mathcal{T}_{2n}^2$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ then

$$
\dim Z_{G_{\mathbf{k}}}(g) = n + \sum_{i=1}^m (i-1)\lambda_i + \frac{1}{2}|\{i : \lambda_i \text{ is odd}\}| + |\{i : \lambda_i \text{ is even and } \varepsilon(\lambda_i) = 0\}|. \tag{1.4.11}
$$

$\mathbf{B_n}$

$G_{\mathbf{k}}$ is of type $SO(V)$ where V is a vector space of dimension $2n + 1$, $n \geq 2$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent p. If $p \neq 2$ then V is equipped with a non–degenerate symmetric bilinear form. If $p = 2$ then V is equipped with a bilinear form $(\cdot, \cdot)$ and a non–zero quadratic form Q such that

$$(x, y) = Q(x + y) - Q(x) - Q(y), \ x, y \in V_2,$$

and the restriction of Q to the null space $V^{\perp} = \{x \in V_2 : (x, y) = 0 \ \forall \ y \in V\}$ of $(\cdot, \cdot)$ has zero kernel.

W is the same as in case of C_n. Therefore we can use the description of the set $\underline{W}$ in terms of pairs of partitions introduced in the case of C_n.

An element of $\underline{W}$ which corresponds to a pair (λ, μ), $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ and $\mu = (\mu_1 = \mu_2 \geq \ldots \geq \mu_{2k-1} = \mu_{2k})$ is the class represented by the sum of the blocks in the following diagram (we use the notation of [18], Section 7 for the conjugacy classes in W)

$$A_{\mu_1 - 1} + A_{\mu_3 - 1} + \ldots + A_{\mu_{2k-1} - 1} +$$
$$+ D_{\frac{\lambda_1 + \lambda_2}{2}}(a_{\frac{\lambda_2}{2} - 1}) + D_{\frac{\lambda_3 + \lambda_4}{2}}(a_{\frac{\lambda_4}{2} - 1}) + \ldots + D_{\frac{\lambda_{m-2} + \lambda_{m-1}}{2}}(a_{\frac{\lambda_{m-1}}{2} - 1}) + B_{\frac{\lambda_m}{2}} \ (m \text{ is odd}), \qquad (1.4.12)$$
$$A_{\mu_1 - 1} + A_{\mu_3 - 1} + \ldots + A_{\mu_{2k-1} - 1} +$$
$$+ D_{\frac{\lambda_1 + \lambda_2}{2}}(a_{\frac{\lambda_2}{2} - 1}) + D_{\frac{\lambda_3 + \lambda_4}{2}}(a_{\frac{\lambda_4}{2} - 1}) + \ldots + D_{\frac{\lambda_{m-1} + \lambda_m}{2}}(a_{\frac{\lambda_m}{2} - 1}) \ (m \text{ is even}),$$

where it is assumed that $D_k(a_0) = D_k$.

If the characteristic of $\mathbf{k}$ is not equal to 2, the elements of $\mathcal{N}(G_{\mathbf{k}})$ are parametrized by partitions λ of $2n + 1$ for which $l_j(\lambda)$ is even for even j. We denote this set of partitions by $\mathcal{Q}_{2n+1}$. In case of $G_{\mathbf{k}} = SO(V)$ the parts of λ are just the sizes of the Jordan blocks in V of the unipotent elements from the conjugacy class corresponding to λ.

In this case $\widehat{\mathcal{N}}(W) = \mathcal{N}(G_2)$, and G_2 is of type $SO(V)$. In fact G_2 is isomorphic to a group of type $Sp(V')$, $\dim V' = 2n$ (see e.g. Section 8.1 in [122]), and hence $\mathcal{N}(G_2) \simeq \mathcal{T}_{2n}^2$.

We also have $\widehat{W} \simeq X_{n,1}$, and the map F is the same as in case of C_n.

The map f_1 can be described as follows. Let $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}) \in \mathcal{Q}_{2n+1}$. If

$$f_1(\lambda) = ((c'_1, c'_3, \ldots, c'_{2m+1}), (c'_2, c'_4, \ldots, c'_{2m}))$$

then the parts c'_i are defined by induction starting from c'_1,

$$
\begin{array}{ll}
c'_i = \frac{\lambda_i - 1}{2} + i - 1 - 2\left[\frac{i-1}{2}\right] & \text{if } \lambda_i \text{ is odd and } c'_{i-1} \text{ is already defined;} \\
c'_i = \frac{\lambda_i}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is even and } c'_{i-1} \text{ is already defined;} \qquad (1.4.13) \\
c'_{i+1} = \frac{\lambda_i}{2} & \text{if } \lambda_i = \lambda_{i+1} \text{ is even and } c'_i \text{ is already defined.}
\end{array}
$$

The image of f_1 consists of all pairs $((c'_1, c'_3, \ldots, c'_{2m+1}), (c'_2, c'_4, \ldots, c'_{2m})) \in X_{n,1}$ such that $c'_i \leq c'_{i+1}$ for all odd i and $c'_i \leq c'_{i+1} + 2$ for all even i.

The image $\mathcal{R}(W)$ of F consists of all pairs $((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m})) \in X_{n,1}$ such that $c_i \leq c_{i+1} + 2$ for all i.

The maps Φ^W and Ψ^W are the same as in case of C_n.

The map π^{G_1} is given by $\pi^{G_1}(\lambda) = (\nu, \varepsilon')$, $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}) \in \mathcal{Q}_{2n+1}$, where

$$
\nu_i = \begin{cases}
\lambda_i - 1 & \text{if } \lambda_i \text{ and } i \text{ are odd and } \lambda_{i-1} < \lambda_i; \\
\lambda_i + 1 & \text{if } \lambda_i \text{ is odd, } i \text{ is even and } \lambda_i < \lambda_{i+1}; \\
\lambda_i & \text{otherwise,}
\end{cases}
$$

and

$$
\varepsilon'(k) = \begin{cases}
\omega & \text{if } k \text{ is odd;} \\
0 & \text{if } k \text{ is even, there exists even } \lambda_i = k \text{ with even } i \text{ such that } \lambda_{i-1} < \lambda_i; \\
1 & \text{otherwise.}
\end{cases}
$$

The map π^{G_1} is injective and its image consists of pairs $(\nu, \varepsilon) \in \mathcal{T}_{2n}^2$ such that $\varepsilon(k) \neq 0$ if ν_k^* is odd and for each even i such that ν_i^* is even we have $\nu_{i-1}^* = \nu_i^*$, i.e. $i - 1$ does not appear in the partition ν. Here $\nu_1^* \geq \nu_2^* \geq \ldots \geq \nu_m^*$ is the partition dual to ν.

To describe $\phi_{G_{\mathbf{k}}}$ for $G_{\mathbf{k}} = SO(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G_{\mathbf{k}}$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For any $x \in \mathbf{k}^*$ with $x^2 = 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$, where $((\lambda_1^x \geq \lambda_3^x \geq \ldots \geq \lambda_{2m+1}^x), (\lambda_2^x \geq \lambda_4^x \geq \ldots \geq \lambda_{2m}^x))$ is the pair of partitions such that the corresponding irreducible representation of the Weyl group of type $B_{(\dim V_x - 1)/2}$ (if $x \neq -1$ or $p = 2$) or $D_{\dim V_x/2}$ (if $x = -1$ or $p \neq 2$) is the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$ and to the trivial local data.

Let $\lambda(g)$ be the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_{2m+1}$ given by $\lambda(g)_j = \sum_x \lambda_j^x$, where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Now $\phi_{G_\mathbf{k}}(g)$ is the pair of partitions $((\lambda(g)_1 \geq \lambda(g)_3 \geq \ldots \geq \lambda(g)_{2m+1}), (\lambda(g)_2 \geq \lambda(g)_4 \geq \ldots \geq \lambda(g)_{2m}))$.

If g is any element in the stratum $G_{(\lambda,\varepsilon)}$ corresponding to a pair $(\lambda, \varepsilon) \in \mathcal{T}_{2n}^2$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ then the dimension of the centralizer of g in $G_\mathbf{k}$ is given by formula (1.4.11),

$$\dim Z_{G_\mathbf{k}}(g) = n + \sum_{i=1}^m (i-1)\lambda_i + \frac{1}{2}|\{i : \lambda_i \text{ is odd}\}| + |\{i : \lambda_i \text{ is even and } \varepsilon(\lambda_i) = 0\}|. \tag{1.4.14}$$

$\mathbf{D_n}$

$G_\mathbf{k}$ is of type $\mathrm{SO}(V)$ where V is a vector space of dimension $2n$, $n \geq 3$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent p. If $p \neq 2$ V is equipped with a non–degenerate symmetric bilinear form. If $p = 2$ V is equipped with a non–degenerate bilinear form $(\cdot, \cdot)$ and a non–zero quadratic form Q such that

$$(x, y) = Q(x + y) - Q(x) - Q(y), \ x, y \in V_2.$$

We remind that in this case $\mathrm{SO}(V)$ is the connected component containing the identity of the group of linear automorphisms of V preserving the quadratic, and hence the bilinear, form.

W is the group of even permutations of the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$ which also commute with the involution $\varepsilon_i \mapsto -\varepsilon_i$. W can be regarded as a subgroup in the Weyl group W' of type C_n.

Let $\widetilde{W}$ be the set of W'–conjugacy classes in W. Elements of $\widetilde{W}$ are parametrized by pairs of partitions (λ, μ) satisfying the following conditions.

- The parts of λ are even (for any $w \in \mathcal{C} \in \widetilde{W}$ they are the numbers of elements in the negative orbits X, $X = -X$, in E for the action of the group $\langle w \rangle$ generated by w);

- The number of parts of λ is even;

- μ consists of pairs of equal parts (they are the numbers of elements in the positive $\langle w \rangle$–orbits X in E; these orbits appear in pairs $X, -X, X \neq -X$);

- $\sum \lambda_i + \sum \mu_j = 2n$.

We denote this set of pairs of partitions by $\mathcal{A}_{2n}^0$.

To each pair $(-, \mu)$, where all parts of μ are even, there correspond two conjugacy classes in W. To all other elements of $\mathcal{A}_{2n}^0$ there corresponds a unique conjugacy class in W.

An element of $\widetilde{W}$ which corresponds to a pair (λ, μ), $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ and $\mu = (\mu_1 = \mu_2 \geq \ldots \geq \mu_{2k-1} = \mu_{2k})$ is the class represented by the sum of the blocks in the following diagram (we use the notation of [18], Section 7)

$$A_{\mu_1 - 1} + A_{\mu_3 - 1} + \ldots + A_{\mu_{2k-1} - 1} + D_{\frac{\lambda_1 + \lambda_2}{2}}(a_{\frac{\lambda_2}{2} - 1}) + D_{\frac{\lambda_3 + \lambda_4}{2}}(a_{\frac{\lambda_4}{2} - 1}) + \ldots + D_{\frac{\lambda_{m-1} + \lambda_m}{2}}(a_{\frac{\lambda_m}{2} - 1}). \tag{1.4.15}$$

Consider now the case $p \neq 2$. Let $G_\mathbf{k}'$ be the extension of $G_\mathbf{k}$ by the Dynkin graph automorphism of order 2. Then $G_\mathbf{k}'$ is of type $\mathrm{O}(V)$. Denote by $\widetilde{\mathcal{N}}(G_\mathbf{k})$ the set of unipotent classes of $G_\mathbf{k}'$. Note that they are all contained in $G_\mathbf{k}$. The elements of $\widetilde{\mathcal{N}}(G_\mathbf{k})$ are parametrized by partitions λ of $2n$ for which $l_j(\lambda)$ is even for even j. Note that the number of parts of such partitions is even. We denote this set of partitions by $\mathcal{Q}_{2n}$. In case when $G_\mathbf{k} = \mathrm{SO}(V)$ the parts of λ are just the sizes of the Jordan blocks in V of the unipotent elements from the conjugacy class corresponding to λ. If λ has only even parts then λ corresponds to two unipotent classes in $G_\mathbf{k}$ of the same dimension. In all other cases there is a unique unipotent class in $G_\mathbf{k}$ which corresponds to λ.

One has $\widehat{\mathcal{N}}(W) = \underline{\mathcal{N}}(G_2)$, and G_2 is of type $\mathrm{SO}(V_2)$.

Let G_2' be the extension of G_2 by the Dynkin graph automorphism of order 2. Then G_2' is of type $\mathrm{O}(V)$. Denote by $\widetilde{\mathcal{N}}(G_2)$ the set of unipotent classes of G_2' contained in G_2. Since the bilinear form $(\cdot, \cdot)$ is also alternating in

characteristic 2 there is a natural injective homomorphism from $O(V)$ to $Sp(V)$, $\dim V = 2n$, and $\underline{\widetilde{\mathcal{N}}}(G_2) \simeq \widetilde{\mathcal{T}}_{2n}^2$, where $\widetilde{\mathcal{T}}_{2n}^2$ is the set of elements $(\lambda, \varepsilon) \in \mathcal{T}_{2n}^2$ such that λ has an even number of parts (see I.2.6 in [122]).

Let $\widetilde{\widehat{W}}$ be the set of orbits of irreducible characters of W under the action of W'. Elements of $\widetilde{\widehat{W}}$ are parametrized by unordered pairs of partitions (α, β) written in non–decreasing order, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{\tau(\alpha)}$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{\tau(\beta)}$, and such that $\sum \alpha_i + \sum \beta_i = n$. By adding zeroes we can assume that the length of α is equal to the length of β. The set of such pairs is denoted by $Y_{n,0}$.

Instead of the maps in (1.4.4) we shall describe the following maps

$$
\begin{array}{ccccccc}
\widetilde{\mathcal{X}}^1(W) & \xleftarrow{\widetilde{f}_1} & \widetilde{\mathcal{N}}(G_1) & & & & \\
\downarrow \iota & & \downarrow \widetilde{\pi}^{G_1} & & & & \\
G_{\mathbf{k}} \xrightarrow{\widetilde{\phi}_{G_{\mathbf{k}}}} \widetilde{\mathcal{R}}(W) & \xleftarrow{\widetilde{F}} & \widetilde{\mathcal{N}}(G_2) & \overset{\widetilde{\Phi}^W}{\underset{\widetilde{\Psi}^W}{\rightleftarrows}} & \underline{\widetilde{W}}, & &
\end{array}
\tag{1.4.16}
$$

where $\widetilde{f}_1$ and $\widetilde{F}$ are induced by the restrictions of the maps f_1 and F for G_1', G_2' to $\widetilde{\mathcal{N}}(G_1)$, $\widetilde{\mathcal{N}}(G_2)$, respectively, $\widetilde{\mathcal{X}}^1(W)$ and $\widetilde{\mathcal{R}}(W)$ are their images, $\widetilde{\phi}_{G_{\mathbf{k}}}, \widetilde{\Psi}^W, \widetilde{\Phi}^W$ and $\widetilde{\pi}^{G_1}$ are also induced by the corresponding maps for G_1', G_2' and W'.

The map $\widetilde{f}_1$ is defined by the same algorithm as in the case of B_n (see (1.4.13)). The image of $\widetilde{f}_1$ consists of all pairs $((c_1', c_3', \ldots, c_{2m+1}'), (c_2', c_4', \ldots, c_{2m}')) \in Y_{n,0}$ such that $c_i' \leq c_{i+1}'$ for all odd i and $c_i' \leq c_{i+1}' + 2$ for all even i.

If $(\lambda, \varepsilon) \in \widetilde{\mathcal{T}}_{2n}^2$, $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m})$ and $\widetilde{F}(\lambda, \varepsilon) = ((c_1, c_3, \ldots, c_{2m-1}), (c_2, c_4, \ldots, c_{2m}))$ then the parts c_i are defined by induction starting from c_1,

$$
\begin{array}{ll}
c_i = \frac{\lambda_i - 2}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i \text{ is even, } \varepsilon(\lambda_i) = 1 \text{ and } c_{i-1} \text{ is already defined;} \\
c_i = \frac{\lambda_i - 1}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_{i-1} \text{ is already defined;} \\
c_{i+1} = \frac{\lambda_i - 3}{2} + 2i - 4\left[\frac{i}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_i \text{ is already defined;} \\
c_i = \frac{\lambda_i}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = 0 \text{ and } c_{i-1} \text{ is already defined;} \\
c_{i+1} = \frac{\lambda_i}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = 0 \text{ and } c_i \text{ is already defined.}
\end{array}
$$

The image $\widetilde{\mathcal{R}}(W)$ of $\widetilde{F}$ consists of all pairs $((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m})) \in Y_{n,0}$ such that $c_i \leq c_{i+1}$ for all odd i and $c_i \leq c_{i+1} + 4$ for all even i.

The maps $\widetilde{\Phi}^W$ and $\widetilde{\Psi}^W$ are defined by the same formulas as in case of C_n.

The map $\widetilde{\pi}^{G_1}$ is given by $\widetilde{\pi}^{G_1}(\lambda) = (\nu, \varepsilon')$, $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m}) \in \mathcal{Q}_{2n}$, where

$$
\nu_i = \begin{cases}
\lambda_i - 1 & \text{if } \lambda_i \text{ is odd, } i \text{ is even and } \lambda_{i-1} < \lambda_i; \\
\lambda_i + 1 & \text{if } \lambda_i \text{ and } i \text{ are odd, and } \lambda_i < \lambda_{i+1}; \\
\lambda_i & \text{otherwise,}
\end{cases}
$$

and

$$
\varepsilon'(k) = \begin{cases}
\omega & \text{if } k \text{ is odd;} \\
0 & \text{if } k \text{ is even, there exists even } \lambda_i = k \text{ with odd } i \text{ such that } \lambda_{i-1} < \lambda_i; \\
1 & \text{otherwise.}
\end{cases}
$$

The map $\widetilde{\pi}^{G_1}$ is injective and its image consists of pairs $(\nu, \varepsilon) \in \widetilde{\mathcal{T}}_{2n}^2$ such that $\varepsilon(k) \neq 0$ if ν_k^* is odd and for each even i such that ν_i^* is even we have $\nu_{i-1}^* = \nu_i^*$, i.e. $i-1$ does not appear in the partition ν. Here $\nu_1^* \geq \nu_2^* \geq \ldots \geq \nu_m^*$ is the partition dual to ν.

To describe $\widetilde{\phi}_{G_{\mathbf{k}}}$ for $G_{\mathbf{k}} = SO(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G_{\mathbf{k}}$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m}^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For any $x \in \mathbf{k}^*$ with $x^2 = 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m}^x$ be the sequence in $\mathbb{N}$, where $((\lambda_1^x \geq \lambda_3^x \geq \ldots \geq \lambda_{2m-1}^x), (\lambda_2^x \geq \lambda_4^x \geq \ldots \geq \lambda_{2m}^x))$ is the pair of partitions such that the corresponding irreducible representation of the Weyl group of type $D_{\dim V_x/2}$ is the Springer representation attached to the unipotent element $x^{-1}g \in SO(V_x)$ and to the trivial local data.

Let $\lambda(g)$ be the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_{2m+1}$ given by $\lambda(g)_j = \sum_x \lambda_j^x$, where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Now $\widetilde{\phi}_{G_{\mathbf{k}}}(g)$ is the pair of partitions $((\lambda(g)_1 \geq \lambda(g)_3 \geq \ldots \geq \lambda(g)_{2m-1}), (\lambda(g)_2 \geq \lambda(g)_4 \geq \ldots \geq \lambda(g)_{2m}))$.

The preimage $\widetilde{\phi}_{G_{\mathbf{k}}}^{-1}(\lambda, \mu)$ is a stratum in $G_{\mathbf{k}}$ in all cases except for the one when the pair (λ, μ) is of the form $((\lambda_1 \geq \lambda_3 \geq \ldots \geq \lambda_{2m-1}), (\lambda_1 \geq \lambda_3 \geq \ldots \geq \lambda_{2m-1}))$. In that case $\widetilde{\phi}_{G_{\mathbf{k}}}^{-1}(\lambda, \mu)$ is a union of two strata, and the conjugacy classes in each of them have the same dimension.

If g is any element in the stratum $G_{(\lambda, \varepsilon)}$ corresponding to a pair $(\lambda, \varepsilon) \in \widetilde{\mathcal{T}}_{2n}^2$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ then the dimension of the centralizer of g in $G_{\mathbf{k}}$ is given by the following formula

$$\dim Z_{G_{\mathbf{k}}}(g) = n + \sum_{i=1}^{m} (i-1)\lambda_i - \frac{1}{2}|\{i : \lambda_i \text{ is odd}\}| - |\{i : \lambda_i \text{ is even and } \varepsilon(\lambda_i) = 1\}|. \qquad (1.4.17)$$

1.5   The strict transversality condition

Recall that the definition of Δ_+^s, and hence of $\Sigma_{\mathbf{k},s}$, depends on the choice of ordering of terms in decomposition (1.2.14). In this section for every conjugacy class $\mathcal{C} \in C(W)$ we define a variety $\Sigma_{\mathbf{k},s}$, $s \in \mathcal{C}$ such that every conjugacy class $\mathcal{O} \in G_{\mathcal{C}}$ intersects $\Sigma_{\mathbf{k},s}$ and

$$\dim \mathcal{O} = \operatorname{codim} \Sigma_{\mathbf{k},s}. \qquad (1.5.1)$$

It turns out that in order to fulfill condition (1.5.1) the subspaces $\mathfrak{h}_i$ in (1.2.14) should be ordered in such a way that $\mathfrak{h}_0 \subset \mathfrak{h}_{\mathbb{R}}$ is the subspace fixed by the action of s, and if $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\mathfrak{h}_j = \mathfrak{h}_\mu^l$ and $0 \leq \lambda < \mu < 1$ then $i < j$, where λ and μ are eigenvalues of the corresponding matrix $I - O$ for s. In the case of exceptional root systems this is verified using a computer program, and in the case of classical root systems this is confirmed by explicit computation based on a technical lemma. In order to formulate this lemma we recall realizations of classical irreducible root systems.

Let V be a real Euclidean n–dimensional vector space with an orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$. The root systems of types A_{n-1}, B_n, C_n and D_n can be realized in V as follows.

$\mathbf{A_n}$

The roots are $\varepsilon_i - \varepsilon_j$, $1 \leq i, j \leq n$, $i \neq j$, $\mathfrak{h}_{\mathbb{R}}$ is the hyperplane in V consisting of the points the sum of whose coordinates is zero.

$\mathbf{B_n}$

The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$, $\pm\varepsilon_i$, $1 \leq i \leq n$, $\mathfrak{h}_{\mathbb{R}} = V$.

$\mathbf{C_n}$

The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$, $\pm 2\varepsilon_i$, $1 \leq i \leq n$, $\mathfrak{h}_{\mathbb{R}} = V$.

$\mathbf{D_n}$

The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$, $\mathfrak{h}_{\mathbb{R}} = V$.

In all these cases the corresponding Weyl group W is a subgroup of the Weyl group of type C_n acting on the elements of the basis $\varepsilon_1, \ldots, \varepsilon_n$ by permuting the basis vectors and changing the sign of an arbitrary subset of them.

Now we formulate the main lemma.

Lemma 1.5.1. *Let s be an element of the Weyl group of type C_n operating on the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$ as indicated in Section 1.4, where $\varepsilon_1, \ldots, \varepsilon_n$ is the basis of V introduced above. Assume that s has either only one nontrivial cycle of length $k/2$ (k is even), which is negative, or only one nontrivial cycle of length k, which is positive, $1 < k \leq n$. Let Δ be a root system of type A_{n-1}, B_n, C_n or D_n realized in V as above.*

(i) If s has only one nontrivial cycle of length $k/2$, which is negative, then k is even, the spectrum of s in the complexification $V_{\mathbb{C}}$ of V is $\epsilon_r = \exp(\frac{2\pi i(k-2r+1)}{k})$, $r = 1, \ldots, k/2$, and possibly $\epsilon_0 = 1$, all eigenvalues are simple except for possibly 1.

(ii) If s only has one nontrivial cycle of length k, which is positive, then the spectrum of s in the complexification of V is $\epsilon_r = \exp(\frac{2\pi i(k-r)}{k})$, $r = 1, \ldots, k-1$, and $\epsilon_0 = 1$, all eigenvalues are simple except for possibly 1.

In both cases we denote by V_r the invariant subspace in V which corresponds to $\epsilon_r = \exp(\frac{2\pi i([k/2]+1-r)}{k})$, $r = 1, \ldots, \left[\frac{k}{2}\right]$ or $\epsilon_0 = 1$ in case of a positive nontrivial cycle and to $\epsilon_r = \exp(\frac{2\pi i(2\left[\frac{k/2+1}{2}\right]+1-2r)}{k})$, $r = 1, \ldots, \left[\frac{k/2+1}{2}\right]$ or $\epsilon_0 = 1$ in case of a negative cycle. For $r \neq 0$ the space V_r is spanned by the real and the imaginary parts of a nonzero eigenvector of s in $V_\mathbb{C}$ corresponding to ϵ_r, and V_0 is the subspace of fixed points of s in V.

V_r is two–dimensional if $\epsilon_r \neq \pm 1$, one–dimensional if $\epsilon_r = -1$ or may have arbitrary dimension if $\epsilon_r = 1$.

Let Δ_+^s be a system of positive roots associated to s and defined as in Section 1.2, where we use the decomposition

$$V = \bigoplus_i V_i \tag{1.5.2}$$

as (1.2.14) in the definition of Δ_+^s. Denote by $\Delta_i \subset \Delta$ the corresponding subsets of roots defined as in (1.2.15).

Let Δ_0^s be the root subsystem fixed by the action of s and $\underline{l}(s)$ the number of positive roots which become negative under the action of s.

(iii) If s has only one nontrivial cycle of length k, which is positive, we have

1. if $\Delta = A_{n-1}$ then $\Delta_0^s = A_{n-k-1}$, $\underline{l}(s) = 2n - k - 1$;

2. if $\Delta = B_n$ (resp. C_n) then $\Delta_0^s = B_{n-k}$ (resp. C_{n-k}), $\underline{l}(s) = 4n - 2k$ for odd k and $\underline{l}(s) = 4n - 2k + 1$ for even k;

3. if $\Delta = D_n$ then $\Delta_0^s = D_{n-k}$, $\underline{l}(s) = 4n - 2k - 2$ for odd k and $\underline{l}(s) = 4n - 2k - 1$ for even k.

(iv) If s has only one nontrivial cycle of length $\frac{k}{2}$, which is negative, we have

1. if $\Delta = B_n(C_n)$ then $\Delta_0^s = B_{n-k/2}(C_{n-k/2})$, $\underline{l}(s) = 2n - k/2$;

2. if $\Delta = D_n$ then $\Delta_0^s = D_{n-k/2}$, $\underline{l}(s) = 2n - k/2 - 1$.

(v) If s has only one nontrivial cycle of length k, which is positive, Δ is of type B_n, C_n or D_n, and k is even then $\Delta = \Delta_{k/2} \cup \Delta_{k/2-1} \cup \Delta_0^s$ (disjoint union), and all roots in $\Delta_{k/2-1}$ are orthogonal to the fixed point subspace for the action of s on V.

(vi) In all other cases $\Delta = \Delta_{i_{\max}} \cup \Delta_0^s$ (disjoint union), where $i_{\max}$ is the maximal possible index i which appears in decomposition (1.5.2).

Proof. The proof is similar in all cases. We only give details in the most complicated case when s has only one nontrivial cycle, which is positive, Δ is of type $B_n(C_n)$, and k is even. Without loss of generality one can assume that s corresponds to the cycle of the form

$$\varepsilon_1 \to \varepsilon_2 \to \varepsilon_4 \to \varepsilon_6 \to \cdots \to \varepsilon_{k-2} \to \varepsilon_k \to \varepsilon_{k-1} \to \varepsilon_{k-3} \to \cdots \to \varepsilon_3 \to \varepsilon_1 \ (k > 2), \ \varepsilon_1 \to \varepsilon_2 \to \varepsilon_1 \ (k = 2).$$

From this definition one easily sees that $\Delta_0^s = B_{n-k}(C_{n-k}) = \Delta \cap V'$, where $V' \subset V$ is the subspace generated by $\varepsilon_{k+1}, \ldots, \varepsilon_n$. Computing the eigenvalues of s in $V_\mathbb{C}$ is a standard exercise in linear algebra. The eigenvalues are expressed in terms of the exponents of the root system of type A_{k-1} (see [17], Ch. 10).

The invariant subspace V_r is spanned by the real and the imaginary parts of a nonzero eigenvector of s in $V_\mathbb{C}$ corresponding to the eigenvalue ϵ_r. If $\epsilon_r \neq \pm 1$ then V_r is two–dimensional, and for $\epsilon_r = -1$ V_r is one–dimensional. In the former case V_r will be regarded as the real form of a complex plane with the orthonormal basis $1, i$. Under this convention the orthogonal projection operator onto V_r acts on the basic vectors ε_j as follows

$$\varepsilon_{2j+1} \mapsto c\epsilon_r^j, \ j = 0, \ldots, \frac{k}{2} - 1, \varepsilon_{2j} \mapsto c\epsilon_r^{-j}, \ j = 1, \ldots, \frac{k}{2}, \tag{1.5.3}$$

where $c = \sqrt{\frac{2}{k}}$. Consider the case when $k > 2$; the case $k = 2$ can be analyzed in a similar way.

To compute $\underline{l}(s)$ using the definition of Δ_+^s given in Section 1.2 one should first look at all roots which have nonzero projections onto $V_{k/2}$ on which s acts by rotation with the angle $\frac{2\pi}{k}$.

From (1.5.3) we deduce that the roots which are not fixed by s and have zero orthogonal projections onto $V_{k/2}$ are $\pm(\varepsilon_j + \varepsilon_{k-j+1})$, $j = 1, \ldots \frac{k}{2}$. The number of those roots is equal to k, and they all have nonzero orthogonal projections onto $V_{k/2-1}$. From (1.5.3) we also obtain that all the other roots which are not fixed by s have nonzero orthogonal projections onto $V_{k/2}$, hence $|\Delta_{k/2-1}| = k$. The number of roots fixed by s is $2(n-k)^2$ since it is equal

to the number of roots in $\Delta_0 = \Delta_0^s = B_{n-k}(C_{n-k})$. Hence $\Delta = \Delta_{k/2} \cup \Delta_{k/2-1} \cup \Delta_0$ (disjoint union), the number of roots in $\Delta_{k/2}$ is $|\Delta| - |\Delta_0| - |\Delta_{k/2-1}| = 2n^2 - 2(n-k)^2 - k = 4nk - 2k^2 - k$, $|\Delta_{k/2}| = 4nk - 2k^2 - k$.

Now using the symmetry of the root system Δ as a subset of V and the fact that s acts as rotation by the angles $\frac{2\pi}{k}$ and $\frac{4\pi}{k}$ in $V_{k/2}$ and $V_{k/2-1}$, respectively, we deduce that the number of positive roots in $\Delta_{k/2}$ ($\Delta_{k/2-1}$) which become negative under the action of s is equal to the number of roots in $\Delta_{k/2}$ ($\Delta_{k/2-1}$) divided by the order of s in $V_{k/2}(V_{k/2-1})$. Therefore

$$\underline{l}(s) = \frac{|\Delta_{k/2}|}{k} + \frac{|\Delta_{k/2-1}|}{k/2} = \frac{4nk - 2k^2 - k}{k} + \frac{k}{k/2} = 4n - 2k + 1.$$

This completes the proof in the considered case.   □

Now we are in a position to prove the main statement of this section.

Theorem 1.5.2. *Let $G_{\mathbf{k}}$ be a connected semisimple algebraic group over an algebraically closed filed $\mathbf{k}$, and $\mathcal{O} \in \widehat{\underline{N}}(W)$. Let $H_{\mathbf{k}}$ be a maximal torus of $G_{\mathbf{k}}$, W the Weyl group of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$, and $s \in W$ an element from the conjugacy class $\Psi^W(\mathcal{O})$. Let Δ be the root system of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$ and Δ_+^s a system of positive roots in Δ associated to s and defined in Section 1.2 with the help of decomposition (1.2.14), where the subspaces $\mathfrak{h}_i$ are ordered in such a way that $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s, and if $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\mathfrak{h}_j = \mathfrak{h}_\mu^l$ and $0 \leq \lambda < \mu < 1$ then $i < j$. In the case of exceptional root systems we assume, in addition, that Δ_+^s is chosen as in the tables in Appendix 2, so that $s = s^1 s^2$ is defined by the data from columns three and four in the tables in Appendix 2. Then all conjugacy classes in the stratum $G_{\mathcal{O}} = \phi_{G_{\mathbf{k}}}^{-1}(F(\mathcal{O}))$ intersect the corresponding variety $\Sigma_{\mathbf{k},s}$ at some points of the subvariety $\dot{s} H_{\mathbf{k}}^0 N_{\mathbf{k},s}$, where $H_{\mathbf{k}}^0 \subset H_{\mathbf{k}}$ is the identity component of the centralizer of $\dot{s}$ in $H_{\mathbf{k}}$. Moreover, if $\mathcal{O} \in \underline{N}(G_p) \subset \widehat{\underline{N}}(W)$ for some p, then for any $g \in G_{\mathcal{O}}$*

$$\dim Z_{G_{\mathbf{k}}}(g) = \mathrm{codim}_{G_p} \mathcal{O} = \dim \Sigma_{\mathbf{k},s}. \tag{1.5.4}$$

Proof. First note that the first identity in (1.5.4) for the codimensions is equivalent to identity (1.4.5) for the corresponding dimensions. We shall divide the proof the remaining identity into several lemmas. First we compute the dimension of the slice $\Sigma_{\mathbf{k},s}$, $s \in \Psi^W(\mathcal{O})$ and justify that for any $g \in G_{\mathcal{O}}$ the last equality in (1.5.4) holds.

Lemma 1.5.3. *Assume that the conditions of Theorem 1.5.2 are satisfied. Then for any $g \in G_{\mathcal{O}}$, where $\mathcal{O} \in \underline{N}(G_p) \subset \widehat{\underline{N}}(W)$ for some p, equality (1.5.4) holds, i.e.*

$$\dim Z_{G_{\mathbf{k}}}(g) = \dim \Sigma_{\mathbf{k},s} = \mathrm{codim}_{G_p} \mathcal{O}.$$

Proof. Observe that by the definition of the slice $\Sigma_{\mathbf{k},s}$

$$\dim \Sigma_{\mathbf{k},s} = l(s) + |\Delta_0| + \dim \mathfrak{h}_0,$$

where $l(s)$ is the length of s with respect to the system of simple roots in Δ_+^s. Hence to compute $\dim \Sigma_{\mathbf{k},s}$ we have to find all numbers in the right hand side of the last equality.

Consider the case of classical groups when each Weyl group element is a product of cycles in a permutation group. In this case identity (1.5.4) is proved by a straightforward calculation using Lemma 1.5.1.

Let $G_{\mathbf{k}}$ be of type A_n and s a representative in the conjugacy class of the Weyl group which corresponds to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$. Recall that s is the product of the cycles which correspond to the parts of λ. The particular ordering of the invariant subspaces $\mathfrak{h}_i$ in the formulation of this theorem implies that the length $l(s)$ equal to the number Δ_s^s of positive roots in Δ_+^s which become negative under the action of s should be computed by successive application of Lemma 1.5.1 to the cycles s_i of s, which correspond to λ_i placed in a non–increasing order. We claim that according to this observation one has

$$l(s) = \sum \underline{l}(s_k), \ \underline{l}(s_k) = 2(n - \sum_{i=1}^{k-1} \lambda_i) - \lambda_k + 1, \tag{1.5.5}$$

where the first sum in (1.5.5) is taken over k for which $\lambda_k > 1$, and we keep the notation of Lemma 1.5.1.

Indeed, recall that according to formula (1.2.16) we have the following disjoint union decomposition

$$\Delta = \bigcup_{k=0}^{M} \Delta_{i_k}$$

which implies another disjoint union decomposition

$$\Delta_s^s = \bigcup_{k=0, i_k \neq 0}^{M} (\Delta_{i_k} \cap \Delta_s^s).$$

Thus the length $l(s)$ equal to the cardinality of the set Δ_s^s can be found as the sum of the cardinalities of the sets $\Delta_{i_k} \cap \Delta_s^s$,

$$l(s) = |\Delta_s^s| = \bigcup_{k=0, i_k \neq 0}^{M} |\Delta_{i_k} \cap \Delta_s^s|. \tag{1.5.6}$$

We find the cardinalities of the sets $\Delta_{i_k} \cap \Delta_s^s$ by successive application of Lemma 1.5.1 to the cycles s_i of s, which correspond to λ_i placed in a non–increasing order.

According to our convention for the ordering of terms in sum (1.2.14) used to define Δ_+^s, the cycle s_1 of s which corresponds to the maximal part λ_1 is the only cycle of s non-trivially acting on $\mathfrak{h}_{i_M}$ by Lemma 1.5.1 (vi). Applying part (iii) 1. of Lemma 1.5.1 to the cycle s_1 we obtain $|\Delta_{i_M} \cap \Delta_s^s| = \underline{l}(s_1) = 2n - \lambda_1 + 1$ and $\Delta_0^{s_1} = A_{n-\lambda_1} = \Delta \setminus \Delta_{i_M}$. The remaining cycles $s_2, \ldots, s_m$ of s corresponding to $\lambda_2 \geq \ldots \geq \lambda_m$ act on $\Delta_0^{s_1}$ and leave the set Δ_{i_M} and its subset $\Delta_s^s \cap \Delta_{i_M}$ invariant, so we can apply Lemma 1.5.1 to s_2 acting on $\Delta_0^{s_1}$ to get $\left|\Delta_{i_{M-1}} \cap \Delta_s^s\right| = \underline{l}(s_2) = 2(n - \lambda_1) - \lambda_2 + 1$ and $\Delta_0^{s_2} = A_{n-\lambda_1-\lambda_2} = \Delta \setminus \left(\Delta_{i_M} \cup \Delta_{i_{M-1}}\right)$. Iterating this procedure and using (1.5.6) we obtain (1.5.5).

The number of roots fixed by s can be represented in a similar form,

$$|\Delta_0| = \sum \underline{l}(s_k), \ \underline{l}(s_k) = 2(n - \sum_{i=1}^{k-1} \lambda_i) - \lambda_k + 1, \tag{1.5.7}$$

where the sum in (1.5.7) is taken over k for which $\lambda_k = 1$.

Finally the dimension of the fixed point space $\mathfrak{h}_0$ of s in $\mathfrak{h}$ is $m - 1$, $\dim \mathfrak{h}_0 = m - 1$.

Recall now that

$$\dim \Sigma_{\mathbf{k},s} = l(s) + |\Delta_0| + \dim \mathfrak{h}_0, \tag{1.5.8}$$

and hence

$$\dim \Sigma_{\mathbf{k},s} = \sum_{k=1}^{m} \underline{l}(s_k) + m - 1 = \sum_{k=1}^{m} \left(2(n - \sum_{i=1}^{k-1} \lambda_i) - \lambda_k + 1\right) + m - 1.$$

Exchanging the order of summation and simplifying this expression we obtain that

$$\dim \Sigma_{\mathbf{k},s} = n + 2 \sum_{i=1}^{m} (i - 1)\lambda_i$$

which coincides with (1.4.6).

The computations of $\dim \Sigma_{\mathbf{k},s}$ in case of B_n and of C_n are similar. If $(\nu, \varepsilon) \in \mathcal{T}_{2n}^2$, $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m)$, corresponds to $\mathcal{O} \in \widehat{\underline{\mathcal{N}}}(W) = \underline{\mathcal{N}}(G_2)$ then $\Psi^W(\nu, \varepsilon) = (\lambda, \mu) \in \mathcal{A}_{2n}^1 \simeq \underline{W}$ is defined in Section 1.4, part $\mathbf{C_n}$. λ consists of even parts ν_i of ν for which $\varepsilon(\nu_i) = 1$, and μ consists of all odd parts of ν and of even parts ν_i of ν for which $\varepsilon(\nu_i) = 0$, the last two types of parts appear in pairs of equal parts. Let s be a representative in the conjugacy class $\Psi^W(\nu, \varepsilon)$. Then each part λ_i corresponds to a negative cycle of s of length $\frac{\lambda_i}{2}$, and each pair $\mu_i = \mu_{i+1}$ of equal parts of μ corresponds to a positive cycle of s of length μ_i. We order the cycles s_k of s associated to the (pairs of equal) parts of the partition ν in a way compatible with a non–increasing ordering of the parts of the partition $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m)$, i.e. if we denote by s_k the cycle that corresponds to an even part ν_k of ν for which $\varepsilon(\nu_k) = 1$ or to a pair $\nu_k = \nu_{k+1}$ of odd parts of ν or of even parts of ν for which $\varepsilon(\nu_k) = 0$ then $s_k \geq s_l$ if $\nu_k \geq \nu_l$.

Similarly to the case of A_n, by the definition of Δ_+^s and by Lemma 1.5.1 applied iteratively to the cycles s_k in the order defined in the previous paragraph, the length $l(s)$ of s is the sum of the following terms $\underline{l}(s_k)$.

To each even part ν_k of ν for which $\varepsilon(\nu_k) = 1$ we associate the term

$$\underline{l}(s_k) = 2(n - \sum_{i=1}^{k-1} \frac{\nu_i}{2}) - \frac{\nu_k}{2};$$

to each pair of odd parts $\nu_k = \nu_{k+1} > 1$ we associate the term

$$\underline{l}(s_k) = 4(n - \sum_{i=1}^{k-1} \frac{\nu_i}{2}) - 2\nu_k = \left(2(n - \sum_{i=1}^{k-1} \frac{\nu_i}{2}) - \frac{\nu_k}{2}\right) + \left(2(n - \sum_{i=1}^{k} \frac{\nu_i}{2}) - \frac{\nu_{k+1}}{2}\right);$$

note that the sum of these terms over all pairs $\nu_k = \nu_{k+1} = 1$ gives the number $|\Delta_0|$ of the roots fixed by s;

to each pair of even parts $\nu_k = \nu_{k+1}$ for which $\varepsilon(\nu_k) = 0$ we associate the term

$$\underline{l}(s_k) = 4(n - \sum_{i=1}^{k-1} \frac{\nu_i}{2}) - 2\nu_k + 1 = \left(2(n - \sum_{i=1}^{k-1} \frac{\nu_i}{2}) - \frac{\nu_k}{2} + \frac{1}{2}\right) + \left(2(n - \sum_{i=1}^{k} \frac{\nu_i}{2}) - \frac{\nu_{k+1}}{2} + \frac{1}{2}\right).$$

The dimension of the fixed point space $\mathfrak{h}_0$ of s in $\mathfrak{h}_\mathbb{R}$ is equal to a half of the sum of the number of all even parts ν_k for which $\varepsilon(\nu_k) = 0$ and of the number of all odd parts ν_k,

$$\dim \mathfrak{h}_0 = \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + \frac{1}{2}|\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}|. \tag{1.5.9}$$

Finally substituting all the computed contributions into formula (1.5.8) we obtain

$$\dim \Sigma_{\mathbf{k},s} = \sum_{k=1}^{m} \left(2(n - \sum_{i=1}^{k-1} \frac{\nu_i}{2}) - \frac{\nu_k}{2}\right) + \frac{1}{2}|\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}| +$$
$$+ \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + \frac{1}{2}|\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}|.$$

Exchanging the order of summation and simplifying this expression we obtain that

$$\dim \Sigma_{\mathbf{k},s} = n + \sum_{i=1}^{m} (i-1)\nu_i + \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + |\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}| \tag{1.5.10}$$

which coincides with (1.4.11) or (1.4.14).

In case of D_n the number $\dim \Sigma_{\mathbf{k},s}$ can be easily obtained if we observe that the map $\widetilde{\Psi}^W$ is defined by the same formula as Ψ^W in case of C_n. In case when $\widetilde{\Psi}^W(\nu, \varepsilon) = (-, \mu)$, where all parts of μ are even, there are two conjugacy classes in W which correspond to $\widetilde{\Psi}^W(\nu, \varepsilon)$. However, the numbers $l(s)$, $|\Delta_0|$ and $\dim \mathfrak{h}_0$ are the same in both cases. They only depend on $\widetilde{\Psi}^W(\nu, \varepsilon)$ in all cases. Let $s \in W$ be a representative from the conjugacy class $\widetilde{\Psi}^W(\nu, \varepsilon)$, $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m)$.

From Lemma 1.5.1 we deduce that in the case of D_n the contributions of the cycles s_k of s to the formula for $\dim \Sigma_{\mathbf{k},s}$ can be obtained from the corresponding contributions in case of C_n in the following way: for each pair of odd parts $\nu_k = \nu_{k+1}$ and for each pair of even parts $\nu_k = \nu_{k+1}$ with $\varepsilon(\nu_k) = 0$ the corresponding contribution $\underline{l}(s_k)$ to $l(s)$ should be reduced by 2 and for each even part ν_k of ν with $\varepsilon(\nu_k) = 1$ the corresponding contribution $\underline{l}(s_k)$ to $l(s)$ should be reduced by 1. This observation and formula (1.5.10) yield

$$\dim \Sigma_{\mathbf{k},s} = n + \sum_{i=1}^{m} (i-1)\nu_i + \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + |\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}| -$$
$$- |\{i : \nu_i \text{ is odd}\}| - |\{i : \nu_i \text{ is even}\}| =$$
$$= n + \sum_{i=1}^{m} (i-1)\nu_i - \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| - |\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 1\}|$$

which coincides with (1.4.17).

In case of root systems of exceptional types $\dim \Sigma_{\mathbf{k},s}$ can be found in the tables in Appendix 2. According to those tables equality (1.5.4) holds in all cases.

$\square$

Now we show that all conjugacy classes in the stratum $G_\mathcal{O} = \phi_{G_{\mathbf{k}}}^{-1}(F(\mathcal{O}))$ intersect the corresponding variety $\Sigma_{\mathbf{k},s}$, $s \in \Psi^W(\mathcal{O})$. The strategy of the proof is as follows. We are going to use characterization (1.4.3) the stratum $G_\mathcal{O}$.

Let Δ_+^s be the system of the positive roots introduced in the statement of Theorem 1.5.2,

$$\mathfrak{h}_{\mathbb{R}} = \bigoplus_{i=0}^{K} \mathfrak{h}_i \tag{1.5.11}$$

the corresponding decomposition of $\mathfrak{h}_{\mathbb{R}}$ and $h_i \in \mathfrak{h}_i$ the corresponding elements of the subspaces $\mathfrak{h}_i$.

We are going to rearrange the terms in the sum (1.5.11) and define a system of positive roots Δ_+^1, using the rearranged sum as described in Section 1.2, in such a way that s is elliptic in a standard parabolic subgroup $W_1 \subset W$ with respect to the system of simple roots in Δ_+^1.

Let w be a minimal length representative in the conjugacy class of s in W_1 with respect to the system of simple reflections in W_1.

Let B_1 be the Borel subgroup in $G_{\mathbf{k}}$ corresponding to $-\Delta_+^1$, L_1 the standard Levi subgroup with the Weyl group W_1. Denote by $B_2 = B_1 \cap L_1$ the Borel subgroup in L_1. One can always find a representative $\dot{w} \in L_1$ of w.

By characterization (1.4.3) any conjugacy class in $G_{\mathcal{O}}$ intersects $B_1 \dot{w} B_1$. In Lemma 1.5.4 we show that in fact any conjugacy class in $G_{\mathcal{O}}$ intersects $B_2 \dot{w} B_2 \subset B_1 \dot{w} B_1$.

Next, in Lemmas 1.5.5 and 1.5.6 we prove that in fact one can take $w = s$. In Lemma 1.5.7 we verify that $B_2 \dot{s} B_2 \subset N_{\mathbf{k}} \dot{s} H_{\mathbf{k}} N_{\mathbf{k}}$, and hence any conjugacy class in $G_{\mathcal{O}}$ intersects $N_{\mathbf{k}} \dot{s} H_{\mathbf{k}} N_{\mathbf{k}} \supset B_2 \dot{s} B_2$.

We prove that any element of $N_{\mathbf{k}} \dot{s} H_{\mathbf{k}} N_{\mathbf{k}}$ can be conjugated to an element of $N_{\mathbf{k}} \dot{s} H_{\mathbf{k}}^0 N_{\mathbf{k}} \subset N_{\mathbf{k}} \dot{s} Z_{\mathbf{k}} N_{\mathbf{k}}$. By Proposition 1.3.4 (i) this implies that any conjugacy class in $G_{\mathcal{O}}$ also intersects $\Sigma_{\mathbf{k},s}$. Finally we verify that in fact any conjugacy class $G_{\mathcal{O}}$ intersects $\Sigma_{\mathbf{k},s}$ at some point of $\dot{s} H_{\mathbf{k}}^0 N_{\mathbf{k},s}$.

We start realizing this program by defining Δ_+^1. Recall that $\mathfrak{h}_0$ is the subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s. If $\mathfrak{h}_0 = 0$, let $\mathfrak{h}_i' = \mathfrak{h}_i$ and $h_i' = h_i$, $i = 0, \ldots, K$. Otherwise let $\mathfrak{h}_K' = \mathfrak{h}_0$, $\mathfrak{h}_i' = \mathfrak{h}_{i+1}$, $h_i' = h_{i+1}$, $i = 0, \ldots, K-1$ and choose an element $h_K' \in \mathfrak{h}_K'$ such that $h_K'(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s–invariant subspace $\mathfrak{h}_K'$ with respect to the natural pairing between $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$.

By a suitable rescaling of h_K' we can assume that conditions (1.2.17) are satisfied for the elements h_i' and roots α from sets Δ_i' defined as in (1.2.22) with h_j, h_i replaced by h_j', h_i'. Indeed, observe that

$$\Delta_i' = \{\alpha \in \Delta : h_j'(\alpha) = 0, j > i, \ h_i'(\alpha) \neq 0\} \subset \{\alpha \in \Delta : h_j(\alpha) = 0, j > i+1, \ h_{i+1}(\alpha) \neq 0\} = \Delta_{i+1}, i = 0, \ldots, K-1$$

by the definition of the elements h'. Thus, since (1.2.17) is satisfied for h_i, $i = 0, \ldots, K$, it is also satisfied for $h_i' = h_i$ if $i < K$. By a suitable rescaling of h_K' we can assume that (1.2.17) is satisfied for h_K' as well.

Let Δ_+^1 be a system of positive roots in $\Delta = \Delta(G_{\mathbf{k}}, H_{\mathbf{k}})$ which corresponds to the Weyl chamber containing the element $\bar{h}' = \sum_{i=0}^{K} h_i'$. By Lemma 1.2.2 the set of roots annihilating $\mathfrak{h}_0$ is the root system of a standard Levi subgroup $L_1 \subset G_{\mathbf{k}}$ with respect to the system of simple roots in Δ_+^1. We denote the root system of the pair $(L_1, H_{\mathbf{k}})$ by $\Delta(L_1, H_{\mathbf{k}})$.

Using formula (1.2.1) and recalling that the roots $\gamma_1, \ldots \gamma_{l'}$ form a linear basis of $\mathfrak{h}'^*$, i.e. for $i = 1, \ldots, l'$ γ_i annihilate $\mathfrak{h}_0$, we deduce that for $i = 1, \ldots, l'$ $\gamma_i \in \Delta(L_1, H_{\mathbf{k}})$, and hence s belongs to the Weyl group $W_1 \subset W$ of the root system $\Delta(L_1, H_{\mathbf{k}})$. Note that, as L_1 is a standard Levi subgroup in $G_{\mathbf{k}}$, W_1 is a parabolic subgroup in W with respect to the system of simple roots in Δ_+^1. Since $\gamma_1, \ldots \gamma_{l'}$ form a linear basis of $\mathfrak{h}'^*$, the linear span of roots from $\Delta(L_1, H_{\mathbf{k}})$ coincides with $\mathfrak{h}'^*$, and hence the element s is elliptic in W_1 as s acts without fixed points on $\mathfrak{h}'^*$.

Let w be a minimal length representative in the conjugacy class of s in W_1 with respect to the system of simple roots in $\Delta(L_1, H_{\mathbf{k}})_+ = \Delta_+^1 \cap \Delta(L_1, H_{\mathbf{k}})$. By Lemma 3.1.14 in [44] if $w \in \Psi^W(\mathcal{O}) \cap W_1$ is of minimal possible length with respect to the system of simple reflections in W_1 then it is also of minimal possible length with respect to the system of simple reflections in W, where in both cases the simple reflections are the reflections with respect to the simple roots in Δ_+^1. Note that w is elliptic in W_1 as well.

Let B_1 be the Borel subgroup in $G_{\mathbf{k}}$ corresponding to $-\Delta_+^1$, $P_1 \supset B_1$ the parabolic subgroup of $G_{\mathbf{k}}$ corresponding to W_1. Thus L_1 is the Levi factor of P_1.

Denote by $B_2 = B_1 \cap L_1$ the Borel subgroup in L_1. One can always find a representative $\dot{w} \in L_1$ of w.

Lemma 1.5.4. *Any conjugacy class in $G_{\mathcal{O}}$ intersects $B_2 \dot{w} B_2 \subset B_1 \dot{w} B_1$.*

Proof. By characterization (1.4.3) the stratum $G_{\mathcal{O}}$ consists of all conjugacy classes of minimal possible dimension which intersect the Bruhat cell $B_1 \dot{w} B_1$. Denote by U_1 the unipotent radical of P_1. Then by the definition of parabolic subgroups one can always find a one parameter subgroup $\rho : \mathbf{k}^* \to Z_{G_{\mathbf{k}}}(L_1)$ such that

$$\lim_{t \to 0} \rho(t) n \rho(t^{-1}) = 1 \tag{1.5.12}$$

for any $n \in U_1$.

Let $\gamma \in G_{\mathcal{O}}$ be a conjugacy class which intersects $B_1 \dot{w} B_1$ at point $b \dot{w} b'$, $b, b' \in B_1$ such that $b \dot{w} b' \notin B_2 \dot{w} B_2$. Since by definitions of B_2 and U_1 we have $B_1 = B_2 U_1$ there are unique factorizations $b = un$, $b' = u'n'$, $u, u' \in B_2$, $n, n' \in U_1$. By (1.5.12) we have

$$\lim_{t \to 0} \rho(t) b \dot{w} b' \rho(t^{-1}) = \lim_{t \to 0} u \rho(t) n \rho(t^{-1}) \dot{w} u' \rho(t) n' \rho(t^{-1}) = u \dot{w} u' \in B_2 \dot{w} B_2,$$

and hence the closure of γ contains a conjugacy class γ' which intersects $B_1 \dot{w} B_1$ at some point of $B_2 \dot{w} B_2 \subset B_1 \dot{w} B_1$. In particular, $\dim \gamma > \dim \gamma'$. This is impossible by characterization (1.4.3) of $G_{\mathcal{O}}$ as γ has minimal possible dimension among the conjugacy classes intersecting $B_1 \dot{w} B_1$. Hence γ intersects $B_1 \dot{w} B_1$ at some point of $B_2 \dot{w} B_2 \subset B_1 \dot{w} B_1$.

$\square$

Lemma 1.5.5. *Let $G_{\mathbf{k}}$ be a connected semisimple algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic good for $G_{\mathbf{k}}$. Let $H_{\mathbf{k}}$ be a maximal torus of $G_{\mathbf{k}}$, W the Weyl group of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$, and $s \in W$ an elliptic element. Denote by $\mathcal{O}_s$ the conjugacy class of s in W. Then $\Phi^W(\mathcal{O}_s) \subset \underline{\mathcal{N}}(G_{\mathbf{k}})$.*

Proof. The statement of this lemma is a consequence of the fact that s is elliptic. Indeed, it suffices to consider the case when $G_{\mathbf{k}}$ is simple.

In case when $G_{\mathbf{k}}$ is of type A_n this is obvious since $\widehat{\underline{\mathcal{N}}}(W)$ consists of the unipotent classes of $G_{\mathbf{k}}$, $\widehat{\underline{\mathcal{N}}}(W) = \underline{\mathcal{N}}(G_{\mathbf{k}})$. In fact in this case $\mathcal{O}_s$ is the Coxeter class, and $\Phi^W(\mathcal{O}_s)$ is the class of regular unipotent elements.

If $G_{\mathbf{k}}$ is of type B_n, C_n or D_n, formula (1.5.9) implies that if $\Phi^W(\mathcal{O}_s)$ corresponds to $(\nu, \varepsilon) \in \mathcal{T}_{2n}^2(\widetilde{\mathcal{T}}_{2n}^2)$ then ν has no odd parts and no even parts ν_i with $\varepsilon(\nu_i) = 0$. According to the description given in the previous section the map $\pi^{G_1}(\widetilde{\pi}^{G_1})$ is injective and its image consists of pairs $(\nu, \varepsilon) \in \mathcal{T}_{2n}^2(\widetilde{\mathcal{T}}_{2n}^2)$ such that $\varepsilon(k) \neq 0$ if ν_k^* is odd and for each even i such that ν_i^* is even we have $\nu_{i-1}^* = \nu_i^*$, i.e. $i - 1$ does not appear in the partition ν. We deduce that $\Phi^W(\mathcal{O}_s)$ is contained in the image of $\pi^{G_1}(\widetilde{\pi}^{G_1})$, i.e. $\Phi^W(\mathcal{O}_s) \in \underline{\mathcal{N}}(G_{\mathbf{k}})$ is a unipotent class in $G_{\mathbf{k}}$.

In case when $G_{\mathbf{k}}$ is of exceptional type this can be checked by examining the tables in Appendix 2.

$\square$

Now we show that in fact one can always take $w = s$.

Lemma 1.5.6. *The element s is of minimal length in its conjugacy class in W_1 with respect to the system of simple roots in $\Delta(L_1, H_{\mathbf{k}})_+ = \Delta_+^1 \cap \Delta(L_1, H_{\mathbf{k}})$.*

Proof. First observe that the formulation of this lemma only uses root systems which can be described independently of $\mathbf{k}$. Therefore we can assume in the proof that the characteristic of $\mathbf{k}$ is good for $G_{\mathbf{k}}$.

Let M_1 be the semisimple part of L_1 and $\mathcal{O}_n = \Phi^{W_1}(\mathcal{O}_w) \subset M_1$, where $\mathcal{O}_w$ is the conjugacy class of w in the Weyl group $W_1 = W(L_1, H)$. By the previous lemma applied to the group M_1 and the elliptic element $w \in W_1$ we have $\mathcal{O}_n \in \underline{\mathcal{N}}(M_1)$.

Therefore $\mathcal{O}_n$ is the unipotent class of minimal possible dimension which intersects $B_2 \dot{w} B_2$. By Theorem 0.7 in [86] the codimension of $\mathcal{O}_n$ in M_1 is equal to $l_1(w)$, where l_1 is the length function in W_1 with respect to the system of simple roots in $\Delta(L_1, H_{\mathbf{k}})_+ = \Delta_+^1 \cap \Delta(L_1, H_{\mathbf{k}})$,

$$\mathrm{codim}_{M_1} \mathcal{O}_n = l_1(w). \tag{1.5.13}$$

Now we show that s has minimal length in the Weyl group W_1 with respect to the system of simple roots in the set of positive roots $\Delta(L_1, H_{\mathbf{k}})_+$.

Indeed, let $\Sigma'_{\mathbf{k},s}$ be the variety in M_1 associated to $s \in W_1$ and defined similarly to $\Sigma_{\mathbf{k},s} \subset G_{\mathbf{k}}$, where we use $\Delta(L_1, H_{\mathbf{k}})_+$ as the system of positive roots in the definition of $\Sigma'_{\mathbf{k},s}$.

Formula (1.5.4) confirmed in Lemma 1.5.3 is applicable to the slice $\Sigma'_{\mathbf{k},s}$ and yields

$$\mathrm{codim}_{M_1} \mathcal{O}_n = \dim \Sigma'_{\mathbf{k},s}.$$

Formula (1.5.8) and the fact that s is elliptic in W_1 imply that

$$\dim \Sigma'_{\mathbf{k},s} = l_1(s),$$

From the last two formulas we infer

$$\mathrm{codim}_{M_1} \mathcal{O}_n = \dim \Sigma'_{\mathbf{k},s} = l_1(s).$$

The last formula and (1.5.13) yield $l_1(w) = l_1(s)$, and hence s has minimal possible length in its conjugacy class in W_1 with respect to the system of simple roots in $\Delta(L_1, H_{\mathbf{k}})_+$.

$\square$

Now we can assume that $s = w$.

Lemma 1.5.7. *Any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $\Sigma_{\mathbf{k},s}$ at some point of $\dot{s} H_{\mathbf{k}}^0 N_{\mathbf{k},s}$.*

Proof. Let $\alpha \in \Delta \cap \mathfrak{h}'^*$. Then $\alpha \in \Delta_i \cap \mathfrak{h}'^*$ for some $i > 0$. Observe that by the definition of the subspaces $\mathfrak{h}'_k$ and by the choice of the elements h_k, $k = 0, \ldots, K$

$$\Delta_i \cap \mathfrak{h}'^* = \{\beta \in \Delta : h_j(\beta) = 0, j > i, h_i(\beta) \neq 0, h_0(\beta) = 0\} = \{\beta \in \Delta : h'_j(\beta) = 0, j > i - 1, h'_{i-1}(\beta) \neq 0, \} = \Delta'_{i-1},$$

and hence by (1.2.21) $\alpha \in (\Delta_i)_+ \cap \mathfrak{h}'^*$ if and only if $h'_{i-1}(\alpha) = h_i(\alpha) > 0$, i.e. $\alpha \in (\Delta_i)_+ \cap \mathfrak{h}'^*$ if and only if $\alpha \in \Delta_+^1 \cap \Delta'_i$. Therefore if we denote $B_3 = B_{\mathbf{k}} \cap L_1$, where $B_{\mathbf{k}}$ is the Borel subgroup corresponding to $-\Delta_+^s$, then $B_3 = B_1 \cap L_1 = B_2$.

By Lemma 1.5.4 applied to the element $s \in W$ any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $B_2 \dot{s} B_2$, and hence it also intersects $B_3 \dot{s} B_3 \subset B \dot{s} B$ as $B_3 = B_2$. But by the definition of L_1 s acts on the root system of the pair $(L_1, H_{\mathbf{k}})$ without fixed points. Since $B_3 = B \cap L_1$ and s fixes all the roots of the pair $(Z_{\mathbf{k}} H_{\mathbf{k}}, H_{\mathbf{k}})$ we have an inclusion $B_3 \subset H_{\mathbf{k}} N_{\mathbf{k}}$, where $N_{\mathbf{k}}$ is the unipotent radical of the parabolic subgroup $P_{\mathbf{k}}$ associated to s in the beginning of this section. Hence $B_3 \dot{s} B_3 \subset N_{\mathbf{k}} \dot{s} H_{\mathbf{k}} N_{\mathbf{k}}$.

Recall that we denote by $H_{\mathbf{k}}^0$ the identity component of the centralizer $H_{\mathbf{k}}^s$ of $\dot{s}$ in $H_{\mathbf{k}}$. Next we show that all elements of $N_{\mathbf{k}} \dot{s} H_{\mathbf{k}} N_{\mathbf{k}}$ are conjugate to elements of $N_{\mathbf{k}} \dot{s} H_{\mathbf{k}}^0 N_{\mathbf{k}} \subset N_{\mathbf{k}} \dot{s} Z_{\mathbf{k}} N_{\mathbf{k}}$ by elements of $H_{\mathbf{k}}$.

Observe that by parts (2) and (3) of the proof of the Theorem in Section 18.3 of [55] the set of elements $h \dot{s} h^{-1} \dot{s}^{-1}$, $h \in H_{\mathbf{k}}$ is a closed subgroup $H'_{\mathbf{k}}$ of $H_{\mathbf{k}}$, and the map

$$H_{\mathbf{k}} \to H'_{\mathbf{k}}, h \mapsto h \dot{s} h^{-1} \dot{s}^{-1}$$

is a surjectice group homomorphism with the kernel being $H_{\mathbf{k}}^s$. Thus $\dim H_{\mathbf{k}} = \dim H_{\mathbf{k}}^s + \dim H'_{\mathbf{k}}$.

Now consider the group homomorphism

$$\phi : H_{\mathbf{k}}^s \times H'_{\mathbf{k}} \to H_{\mathbf{k}}$$

induced by the group multiplication in $H_{\mathbf{k}}$. By Corollary B in Section 18.1 of [55] its differential at the identity element of $H_{\mathbf{k}}^s \times H'_{\mathbf{k}}$ is a linear isomorphism of the corresponding tangent spaces, i.e. of the Lie algebras of $H_{\mathbf{k}}^s \times H'_{\mathbf{k}}$ and of $H_{\mathbf{k}}$. Since ϕ is a group homomorphism, this differential is an isomorphism of the corresponding tangent spaces at all points of $H_{\mathbf{k}}^s \times H'_{\mathbf{k}}$. Thus its image must have the dimension equal to $\dim H_{\mathbf{k}}^s + \dim H'_{\mathbf{k}} = \dim H_{\mathbf{k}}$.

By 4.4, Proposition B in [55] the image of the homomorphism ϕ is a closed subgroup in $H_{\mathbf{k}}$. The identity $\dim H_{\mathbf{k}}^s + \dim H'_{\mathbf{k}} = \dim H_{\mathbf{k}}$ implies that this image contains the identity component of $H_{\mathbf{k}}$. But $H_{\mathbf{k}}$ is irreducible, and hence the image of ϕ coincides with $H_{\mathbf{k}}$, i.e. ϕ is surjective.

For the same reason the image of $H_{\mathbf{k}}^0 \times H'_{\mathbf{k}}$ in $H_{\mathbf{k}}$ under ϕ coincides with $H_{\mathbf{k}}$, i.e. the restriction of ϕ to $H_{\mathbf{k}}^0 \times H'_{\mathbf{k}}$ is surjective.

This implies that for any $h \in H_{\mathbf{k}}$ there exist $h_0 \in H_{\mathbf{k}}^0$ and $h' \dot{s} h'^{-1} \dot{s}^{-1} \in H'_{\mathbf{k}}$ $(h' \in H_{\mathbf{k}})$ such that

$$h = h_0 h' \dot{s} h'^{-1} \dot{s}^{-1} = h' h_0 \dot{s} h'^{-1} \dot{s}^{-1},$$

or

$$h'^{-1} h \dot{s} h' = h_0 \dot{s}. \tag{1.5.14}$$

Since $H_{\mathbf{k}}$ normalizes $N_{\mathbf{k}}$, we deduce from the last identity that any element $n h \dot{s} n' \in N_{\mathbf{k}} H_{\mathbf{k}} \dot{s} N_{\mathbf{k}} = N_{\mathbf{k}} \dot{s} H_{\mathbf{k}} N_{\mathbf{k}}$, $n, n' \in N_{\mathbf{k}}$, $h \in H_{\mathbf{k}}$, can be conjugated by an element $h' \in H_{\mathbf{k}}$ to an element $h'^{-1} n h' h_0 \dot{s} h'^{-1} n' h' \in N_{\mathbf{k}} H_{\mathbf{k}}^0 \dot{s} N_{\mathbf{k}} = N_{\mathbf{k}} \dot{s} H_{\mathbf{k}}^0 N_{\mathbf{k}}$, where $h_0 \in H_{\mathbf{k}}^0$ and h' are related to h by (1.5.14).

Finally observe that $N_{\mathbf{k}} \dot{s} H_{\mathbf{k}}^0 N_{\mathbf{k}} \subset N_{\mathbf{k}} \dot{s} Z_{\mathbf{k}} N_{\mathbf{k}}$, and hence any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $N_{\mathbf{k}} \dot{s} H_{\mathbf{k}}^0 N_{\mathbf{k}} \subset N_{\mathbf{k}} \dot{s} Z_{\mathbf{k}} N_{\mathbf{k}}$. By Proposition 1.3.4 (i) γ also intersects $\Sigma_{\mathbf{k},s}$. Formula (1.3.18) implies that the $Z_{\mathbf{k}}$–component of any element from $N_{\mathbf{k}} \dot{s} Z_{\mathbf{k}} N_{\mathbf{k}}$ is equal to the $Z_{\mathbf{k}}$–component in $\Sigma_{\mathbf{k},s} = \dot{s} Z_{\mathbf{k}} N_{\mathbf{k},s}$ of its image under the isomorphism $N_{\mathbf{k}} \dot{s} Z_{\mathbf{k}} N_{\mathbf{k}} \simeq N_{\mathbf{k}} \times \dot{s} Z_{\mathbf{k}} N_{\mathbf{k},s}$. Therefore any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $\Sigma_{\mathbf{k},s}$ at some point of $\dot{s} H_{\mathbf{k}}^0 N_{\mathbf{k},s}$. This completes the proof.

$\square$

By the previous lemma the statement of this theorem holds.

$\square$

1.6 Some normal orderings of positive root systems associated to Weyl group elements

For the purpose of quantization we shall need a certain normal ordering on the root system Δ_+^s.

An ordering of a set of positive roots Δ_+ is called *normal* if it satisfies the following property.

Property N. *For any three roots α, β, γ such that $\gamma = \alpha + \beta$ we have either $\alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$.*

If

$$\beta_1, \ldots, \beta_D$$

is a normal ordering of Δ_+, any part of it of the form

$$\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}$$

is called a *segment* and denoted by $[\beta_h, \beta_{h+b}]$.

If $A, B \subset \Delta_+$ are two subsets, we write $A < B$ if for any $\theta \in A$, and $\vartheta \in B$ one has $\theta < \vartheta$ with respect to a normal ordering $<$ on Δ_+.

Proposition 1.6.1. ([133], §3, Proposition 2) *Let $\alpha_1, \ldots, \alpha_l$ be the simple roots in Δ_+, $s_1, \ldots, s_l$ the corresponding simple reflections. Let $\overline{w}$ be the element of W of maximal length with respect to the system $s_1, \ldots, s_l$ of simple reflections. For any reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$ of $\overline{w}$ the ordering*

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \ldots, \beta_D = s_{i_1} \ldots s_{i_{D-1}}\alpha_{i_D} \tag{1.6.1}$$

is a normal ordering in Δ_+, and there is a one–to–one correspondence between normal orderings of Δ_+ and reduced decompositions of $\overline{w}$.

From this proposition and from properties of Coxeter groups it follows that any two normal orderings in Δ_+ can be reduced to each other by the so–called elementary transpositions. The elementary transpositions for rank 2 root systems are inversions of the following normal orderings (or the inverse normal orderings):

$$
\begin{array}{ll}
\alpha, \ \beta & A_1 + A_1 \\[2mm]
\alpha, \ \alpha + \beta, \ \beta & A_2 \\[2mm]
\alpha, \ \alpha + \beta, \ \alpha + 2\beta, \ \beta & B_2 \\[2mm]
\alpha, \ \alpha + \beta, \ 2\alpha + 3\beta, \ \alpha + 2\beta, \ \alpha + 3\beta, \ \beta & G_2
\end{array}
\tag{1.6.2}
$$

where it is assumed that $(\alpha, \alpha) \geq (\beta, \beta)$. Moreover, for any rank 2 root system there exist two normal orderings one of which is contained in the list (1.6.2) and the other is the inverse ordering.

In general an elementary transposition of a normal ordering in a set of positive roots Δ_+ is the inversion of an ordered segment of form (1.6.2) (or of a segment with the inverse ordering) in the ordered set Δ_+, where $\alpha - \beta \notin \Delta$.

We shall need the following property of normal orderings.

Lemma 1.6.2. *Any root α in any normal ordering of a system of positive roots Δ_+ cannot be represented as a linear combination of roots strictly greater xor strictly less than α with non-negative integer coefficients.*

Proof. Let $\overline{w} = s_{i_1} \ldots s_{i_D}$ be the reduced decomposition of the longest element $\overline{w}$ of W corresponding to a normal ordering of Δ_+, so that the ordering has the form

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \ldots, \beta_D = s_{i_1} \ldots s_{i_{D-1}}\alpha_{i_D}.$$

Let $\alpha = \beta_k$ for some k and define $v = s_{i_k} \ldots s_{i_1}$. As this decomposition of v is reduced, i.e. has minimal possible length, being a part of the reduced decomposition of $\overline{w}^{-1}$, we have

$$v\alpha = v\beta_k = s_{i_k} \ldots s_{i_1}s_{i_1} \ldots s_{i_{k-1}}\alpha_{i_k} = -\alpha_{i_k} \in \Delta_-,$$

and for $n > k$

$$v\beta_n = s_{i_k} \ldots s_{i_1}s_{i_1} \ldots s_{i_k}s_{i_{k+1}} \ldots s_{i_{n-1}}\alpha_{i_n} = s_{i_{k+1}} \ldots s_{i_{n-1}}\alpha_{i_n} \in \Delta_+.$$

Therefore if we suppose that $\alpha = \beta_k = \sum_{n=k+1}^{D} c_n \beta_n$, $c_n \in \mathbb{N}$ then

$$-\alpha_{i_k} = v\beta_k = v\alpha = \sum_{n=k+1}^{D} c_n s_{i_{k+1}} \ldots s_{i_{n-1}} \alpha_{i_n} \in \Delta_+$$

Thus we arrive at a contradiction, and hence the presentation $\alpha = \beta_k = \sum_{n=k+1}^{D} c_n \beta_n$, $c_n \in \mathbb{N}$ is impossible.

The case of similar decompositions of α as linear combinations of roots strictly less than α with non-negative integer coefficients is treated in a similar way. $\qquad\square$

We recall some results of §3 and §4 in [133] on normal orderings and reduced decompositions. For $w \in W$ a decomposition $w = uv$, $u, v \in W$, is called *reduced* if $l(w) = l(u) + l(v)$, where $l(\cdot)$ is the length function with respect to the system of simple roots in Δ_+. In this case $\Delta_w = \Delta_v \cup v^{-1}\Delta_u$.

Recall that for $w \in W$, a decomposition $w = s_{i_1} \ldots s_{i_n}$ is called reduced if it has minimal possible length equal to $l(w)$. Obviously, if $w = uv$, $u, v, w \in W$ is a reduced decomposition then the product of reduced decompositions of u and v is a reduced decomposition of w.

An ordering of the set $\Delta_w = \{\alpha \in \Delta_+ | w\alpha \in -\Delta_+\}$ is called *normal* if it coincides with an initial segment of some normal ordering of Δ_+.

Lemma 1.6.3. ([133], §4, Theorem 1) *Let $w \in W$. Then the following statements are true.*

(i) Any simple root in Δ_w can be moved to the first position in a normal ordering of Δ_w by a composition of elementary transpositions.

(ii) Any two normal orderings of the set Δ_w can be obtained from each other by elementary transpositions within Δ_w.

(iii) Any two reduced decompositions of w can be obtained from each other using braid group relations in W.

(iv) If $w = s_{j_b} \ldots s_{j_1}$ is a reduced decomposition of w then $\beta_1 = \alpha_{j_1}, \beta_2 = s_{j_1}\alpha_{j_2} \ldots, \beta_b = s_{j_1} \ldots s_{j_{b-1}}\alpha_{j_b}$ is a normal order of the set Δ_w, and

$$w = s_{\beta_1} \ldots s_{\beta_b}. \tag{1.6.3}$$

Moreover, there is a one-to-one correspondence between the reduced decompositions of w, the presentations of w of the form (1.6.3), and the normal orderings of the set Δ_w. In particular, any reduced decomposition of w is an initial part of a reduced decomposition of the longest element $\overline{w} \in W$.

We shall also need the following generalization of the previous lemma.

Lemma 1.6.4. *Let $\alpha_1, \ldots, \alpha_l$ be the simple roots in Δ_+, $s_1, \ldots, s_l$ the corresponding simple reflections, $\overline{w} = s_{i_1} \ldots s_{i_D}$ a reduced decomposition of the longest element $\overline{w}$ of W with respect to the system $s_1, \ldots, s_l$ of its generators, and*

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \ldots, \beta_D = s_{i_1} \ldots s_{i_{D-1}}\alpha_{i_D} \tag{1.6.4}$$

the corresponding normal ordering in Δ_+.

Let $\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}$ be a segment of this normal ordering, $w = s_{i_h} \ldots s_{i_{h+b}}$ the corresponding part of the reduced decomposition of $\overline{w}$.

Let $w = s_{j_h} \ldots s_{j_{h+b}}$ be another reduced expression of w. Then the following statements are true.

(i)

$$\overline{w} = s_{i_1} \ldots s_{i_{h-1}} s_{j_h} \ldots s_{j_{h+b}} s_{i_{h+b+1}} \ldots s_{i_D} \tag{1.6.5}$$

is a reduced expression of $\overline{w}$.

(ii) The normal ordering of Δ_+ corresponding to reduced decomposition (1.6.5) has the form

$$\beta_1, \beta_2, \beta_{h-1}, \beta_{h_1}, \ldots, \beta_{h_b}, \beta_{h+b+1} \ldots, \beta_D, \tag{1.6.6}$$

where the segment $\beta_{h_1}, \ldots, \beta_{h_b}$ is obtained from $\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}$ by applying elementary transpositions to normal ordering (1.6.4) such that each such transposition does not change the positions of all positive roots, except for $\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}$.

(iii) Let $u = s_{i_1} \ldots s_{i_{h-1}}$. Then $u\Delta_{w^{-1}} = \{\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}\}$, and the segments $\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}$ and $\beta_{h_1}, \ldots, \beta_{h_b}$ correspond to the following two decompositions of the element $uw^{-1}u^{-1}$,

$$uw^{-1}u^{-1} = s_{\beta_h} \ldots s_{\beta_{h+b}} = s_{\beta_{h_1}} \ldots s_{\beta_{h_b}}. \tag{1.6.7}$$

Moreover, if a segment $\beta_{h_1}, \ldots, \beta_{h_b}$ *is obtained from* $\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}$ *by applying elementary transpositions to normal ordering (1.6.4) such that each such transposition does not change the positions of all positive roots, except for* $\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}$, *then the corresponding ordering (1.6.6) is normal, and there is a one-to-one correspondence between reduced decompositions of* w, *decompositions of the element* $uw^{-1}u^{-1}$ *of the form (1.6.7), segments obtained from* $\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}$ *by applying elementary transpositions within the segment, and normal orderings of* Δ_+ *of type (1.6.6).*

Proof. For part (i) we observe that the decompositions $u = s_{i_1} \ldots s_{i_{h-1}}$, $v = s_{i_{h+1}} \ldots s_{i_D}$ are reduced being parts of a reduced decomposition of $\overline{w}$. Thus the decomposition $\overline{w} = uwv$ given by (1.6.5) must be reduced as its length is equal to the length of $\overline{w}$, i.e. to the cardinality of the set Δ_+.

To justify (ii) we observe that, according to the definition of the normal ordering given in (1.6.4), after replacing the part $w = s_{i_h} \ldots s_{i_{h+b}}$ of the corresponding reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$ with $w = s_{j_h} \ldots s_{j_{h+b}}$, the roots at the first $h - 1$ and the last $D - (h + b)$ positions will remain at the same places in new ordering (1.6.6) as for the roots at these positions we still have the same expressions,

$$\beta_k = s_{i_1} \ldots s_{i_{k-1}} \alpha_{i_k}, \ 1 \le k < h,$$

$$\beta_k = uw s_{i_{h+1}} \ldots s_{i_{k-1}} \alpha_{i_k}, \ D \ge k > h + b.$$

Let $\beta_1' = \alpha_{i_h}, \beta_2' = s_{i_h} \alpha_{i_{h+1}} \ldots, \beta_b' = s_{i_h} \ldots s_{i_{h+b-1}} \alpha_{i_{h+b}}$ be the normal ordering of $\Delta_{w^{-1}}$ associated to the reduced decomposition $w = s_{i_h} \ldots s_{i_{h+b}}$ according to Lemma 1.6.3 (iv). Then the segment $\beta_h, \beta_{h+1}, \ldots, \beta_{h+b}$ coincides with $u\beta_1' = u\alpha_{i_h}, u\beta_2' = us_{i_h}\alpha_{i_{h+1}} \ldots, u\beta_b' = us_{i_h} \ldots s_{i_{h+b-1}}\alpha_{i_{h+b}}$. As a set this segment copincides with $u\Delta_{w^{-1}}$.

According to definition (1.6.1) in the new normal ordering (1.6.5) the roots from the set $u\Delta_{w^{-1}}$ forming the segment $\beta_{h_1}, \ldots, \beta_{h_b}$ are placed as follows $u\alpha_{j_h}, us_{j_h}\alpha_{j_{h+1}} \ldots, us_{j_h} \ldots s_{j_{h+b-1}}\alpha_{j_{h+b}}$. Note that according to Lemma 1.6.3 (iv) the segment $\alpha_{j_h}, s_{j_h}\alpha_{j_{h+1}} \ldots, s_{j_h} \ldots s_{j_{h+b-1}}\alpha_{j_{h+b}}$ is the normal ordering of the set $\Delta_{w^{-1}}$ which is obtained from the segment $\beta_1' = \alpha_{i_h}, \beta_2' = s_{i_h}\alpha_{i_{h+1}} \ldots, \beta_b' = s_{i_h} \ldots s_{i_{h+b-1}}\alpha_{i_{h+b}}$ by applying elementary transpositions by Lemma 1.6.3 (ii). Therefore the segment

$$\beta_{h_1}, \ldots, \beta_{h_b} = u\alpha_{j_h}, us_{j_h}\alpha_{j_{h+1}} \ldots, us_{j_h} \ldots s_{j_{h+b-1}}\alpha_{j_{h+b}}$$

is obtained from the segment $\beta_h, \beta_{h+1}, \ldots, \beta_{h+b} = u\beta_1', u\beta_2' \ldots, u\beta_b'$ by applying elementary transpositions. At the same time part (iv) of Lemma 1.6.3 implies statement (iii) of this lemma. This completes the proof.

$\square$

To formulate the main statement of this section we shall consider a special type of factorizations of Weyl group elements introduced in (1.2.1). This type is described in the following lemma.

Lemma 1.6.5. *Let* $s \in W$ *be an element of the Weyl group* W. *There there is a decomposition* $s = s^1 s^2$ *of the form (1.2.1) such that any root* $\alpha \in \Delta$ *fixed by the action of* s^2 *is also fixed by the action of* s, *i.e.*

$$s^2 \alpha = \alpha \Rightarrow \alpha \in \Delta_0. \tag{1.6.8}$$

Proof. Let $s = s^1 s^2$ be any presentation of the form (1.2.1) of a Weyl group element $s \in W$. In the proof we shall use a decomposition (1.2.14) such that in sum (1.2.14) $\mathfrak{h}_0$ is the linear subspace $\mathfrak{h}_{\mathbb{R}}^s = \mathfrak{h}^s \cap \mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s pointwise. It will be convenient to put $\mathfrak{h}_0 = \mathfrak{h}_{\mathbb{R}}^s$ even in the case when $\mathfrak{h}_{\mathbb{R}}^s$ is trivial. According to this convention $\Delta_0 = \{\alpha \in \Delta : s\alpha = \alpha\}$ is always the set of roots fixed by the action of s, and Δ_0 may be empty.

We shall also assume that the one–dimensional subspaces $\mathfrak{h}_i$ in sum (1.2.14) on which s^1 acts by multiplication by -1 and s^2 acts trivially, if there are any non–trivial subspaces of this type, are placed immediately after $\mathfrak{h}_0$ in (1.2.14), i.e. they are labeled by indexes forming a set of the type $\{1, 2, \ldots, t\}$.

Recall that in Section 1.2 we denoted by Δ_{i_k}, $k = 0, \ldots, M$ the non–empty sets in the collection Δ_i, $i = 0, \ldots, K$, and that they are labeled in such a way that $i_j < i_k$ if and only if $j < k$.

Suppose that the direct sum $\bigoplus_{k=0, i_k > 0}^u \mathfrak{h}_{i_k}$ of the subspaces $\mathfrak{h}_{i_k}$, which correspond to the non–empty sets Δ_{i_k}, $k = 0, \ldots, M$ and on which s^1 acts by multiplication by -1 and s^2 acts trivially, is not trivial. Since the one–dimensional subspaces $\mathfrak{h}_i$ on which s^1 acts by multiplication by -1 and s^2 acts trivially are placed immediately after $\mathfrak{h}_0$ in sum (1.2.14), the roots from the union $\bigcup_{k=0, i_k > 0}^u \Delta_{i_k}$ must be orthogonal to all subspaces $\mathfrak{h}_{i_k}, i_k > 0$ on which s^1 does not act by multiplication by -1 and to all roots from the set $\gamma_{n+1}, \ldots \gamma_{l'}$ as s^2 acts trivially on $\bigoplus_{k=0}^u \mathfrak{h}_{i_k}$. Pick up a root $\gamma \in \bigcup_{k=0, i_k > 0}^u \Delta_{i_k}$. Then γ is orthogonal to the roots $\gamma_{n+1}, \ldots \gamma_{l'}$. Therefore, by the

choice of γ, $s_0^1 := s^1 s_\gamma$ is an involution the dimension of the fixed point space of which is equal to the dimension of the fixed point space of the involution s^1 plus one, and $s_0^2 := s_\gamma s^2$ is another involution the dimension of the fixed point space of which is equal to the dimension of the fixed point space of the involution s^2 minus one,

$$\dim \mathfrak{h}^{s_0^1} = \dim \mathfrak{h}^{s^1} + 1, \dim \mathfrak{h}^{s_0^2} = \dim \mathfrak{h}^{s^2} - 1.$$

By the construction we also have a decomposition $s = s_0^1 s_0^2$.

Now we can construct decomposition (1.2.14) using the new factorization $s = s_0^1 s_0^2$ of s and the conventions stated in the first two paragraphs of this proof. After that we can also apply the algorithm described in the previous paragraph. Iterating these two steps we shall eventually arrive at the situation when the direct sum $\bigoplus_{k=0,i_k>0}^u \mathfrak{h}_{i_k}$ of the subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1 and s^2 acts trivially is trivial.

This property also implies that any root fixed by the action of s^2 is fixed by the action of s as well,

$$s^2\alpha = \alpha \Rightarrow \alpha \in \Delta_0,$$

which is property (1.6.8).

Indeed, if $s^2\alpha = \alpha$ then by the definition of the subspaces $\mathfrak{h}_{i_k} \subset \mathfrak{h}_{\mathbb{R}}$, α annihilates any s-invariant subspace $\mathfrak{h}_{i_k} \subset \mathfrak{h}_{\mathbb{R}}$ on which s^2 acts in a non–trivial way. Therefore α may not vanish identically only on the direct sum $\bigoplus_{k=0,i_k>0}^u \mathfrak{h}_{i_k}$ of the subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1 and s^2 acts trivially or on $\mathfrak{h}_0$. But the sum $\bigoplus_{k=0,i_k>0}^u \mathfrak{h}_{i_k}$ is trivial by the construction. Therefore α does not vanish identically only on $\mathfrak{h}_0$ (if $\mathfrak{h}_0$ is not trivial) which implies $\alpha \in \Delta_0$.

Note that condition (1.6.8) is a property of the corresponding decomposition (1.2.1) and that this condition does not depend on decomposition (1.2.14) used in the arguments in the previous paragraph. This completes the proof.

$\square$

From now on we shall make the following assumptions.

1. We shall only consider decompositions (1.2.1) which satisfy property (1.6.8).

2. It will be convenient to use the convention adopted in the proof of the previous lemma and assume that in sum (1.2.14) $\mathfrak{h}_0$ is the linear subspace $\mathfrak{h}_{\mathbb{R}}^s = \mathfrak{h}^s \cap \mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s pointwise, and it will be convenient to put $\mathfrak{h}_0 = \mathfrak{h}_{\mathbb{R}}^s$ even in the case when $\mathfrak{h}_{\mathbb{R}}^s$ is trivial. According to this convention $\Delta_0 = \{\alpha \in \Delta : s\alpha = \alpha\}$ is always the set of roots fixed by the action of s, and Δ_0 may be empty.

3. As in the proof of the previous lemma, we shall also always assume that the one–dimensional subspaces $\mathfrak{h}_i$ in sum (1.2.14) on which s^1 acts by multiplication by -1 and s^2 acts trivially, if there are any non–trivial subspaces of this type, are placed immediately after $\mathfrak{h}_0$ in (1.2.14), i.e. they are labeled by indexes forming a set of the type $\{1, 2, \ldots, t\}$.

4. According to the arguments in the proof of the previous lemma, under assumptions 1-3 we can suppose that the direct sum $\bigoplus_{k=0,i_k>0}^u \mathfrak{h}_{i_k}$ of the subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1, s^2 acts trivially, and which correspond to the non–empty sets in the collection Δ_i, $i = 0, \ldots, K$, is trivial.

Proposition 1.6.6. *Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$, $s = s^1 s^2$ a presentation (1.2.1) for s, i.e. s^1, s^2 are involutions, $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \ldots, \gamma_n$ and $\gamma_{n+1}, \ldots, \gamma_{l'}$ are positive and mutually orthogonal.*

Let Δ_+^s be a system of positive roots defined by (1.2.20) using the presentation $s = s^1 s^2$ and under assumptions 1-4. For any $w \in W$ denote $\Delta_w^s = \{\alpha \in \Delta_+^s : w\alpha \in -\Delta_+^s\}$. Then the following statements are true.

(i) The decomposition $s = s^1 s^2$ is reduced in the sense that $l(s) = l(s^2) + l(s^1)$, where $l(\cdot)$ is the length function in W with respect to the system of simple roots in Δ_+^s, and $\Delta_s^s = \Delta_{s^2}^s \cup s^2(\Delta_{s^1}^s)$, $\Delta_{s^{-1}}^s = \Delta_{s^1}^s \cup s^1(\Delta_{s^2}^s)$ (disjoint unions).

(ii) If $\alpha \in \Delta_{s^1}^s$ (resp. $\alpha \in \Delta_{s^2}^s$, $\alpha \in \Delta_s^s$, or $\alpha \in \Delta_{s^{-1}}^s$), $\beta \in \Delta_0$, and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta_{s^1}^s$ (resp. $\alpha + \beta \in \Delta_{s^2}^s$, $\alpha + \beta \in \Delta_s^s$, or $\alpha + \beta \in \Delta_{s^{-1}}^s$).

(iii) $s^2(\Delta_{s^1}^s) \subset \Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_{s^2}^s \cup \Delta_0)$.

(iv) There is a normal ordering of the root system Δ_+^s of the following form

$$\beta_1^1, \ldots, \beta_t^1, \beta_{t+1}^1, \ldots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \ldots, \gamma_2, \ldots, \gamma_3, \ldots, \gamma_n, \beta_{t+p+1}^1, \ldots, \beta_{l(s^1)}^1, \ldots, \tag{1.6.9}$$

$$\beta_1^2, \ldots, \beta_{t'}^2, \gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}, \beta_{t'+\frac{q+l'-n}{2}+1}^2, \ldots, \beta_{t'+q}^2, \beta_{t'+q+1}^2, \ldots, \beta_{l(s^2)}^2, \beta_1^0, \ldots, \beta_{D_0}^0,$$

where

$$\Delta_{s^1}^s = \{\beta_1^1, \ldots, \beta_{l(s^1)}^1\} = \{\beta_1^1, \ldots, \beta_t^1, \beta_{t+1}^1, \ldots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \ldots, \gamma_2, \ldots, \gamma_3, \ldots, \gamma_n, \beta_{t+p+1}^1, \ldots, \beta_{l(s^1)}^1\},$$

$$\Delta_{s^1}^{-1} := \{\alpha \in \Delta_+^s | s^1\alpha = -\alpha\} = \{\beta_{t+1}^1, \ldots, \beta_{t+p}^1\} = \{\beta_{t+1}^1, \ldots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \ldots, \gamma_2, \ldots, \gamma_3, \ldots, \gamma_n\},$$

where $p = |\Delta_{s^1}^{-1}|$,

$$t = |[\beta_1^1, \beta_t^1]| = |[\beta_{t+p+1}^1, \beta_{l(s^1)}^1]| = \frac{l(s^1) - p}{2}, \tag{1.6.10}$$

so that

$$2t = |\Delta_{s^1}^s \setminus \Delta_{s^1}^{-1}| = l(s^1) - |\Delta_{s^1}^{-1}|,$$

$$|[\beta_{t+1}^1, \beta_{t+\frac{p-n}{2}}^1]| = \frac{p-n}{2}, \quad |[\gamma_1, \ldots, \gamma_n]| = \frac{p+n}{2}; \tag{1.6.11}$$

$$\Delta_{s^2}^s := \{\beta_1^2, \ldots, \beta_{l(s^2)}^2\} = \{\beta_1^2, \ldots, \beta_{t'}^2, \gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}, \beta_{t'+\frac{q+l'-n}{2}+1}^2, \ldots, \beta_{t'+q}^2, \beta_{t'+q+1}^2, \ldots, \beta_{l(s^2)}^2\},$$

$$\Delta_{s^2}^{-1} := \{\alpha \in \Delta_+^s | s^2\alpha = -\alpha\} = \{\beta_{t'+1}^2, \ldots, \beta_{t'+q}^2\} = \{\gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}, \beta_{t'+\frac{q+l'-n}{2}+1}^2, \ldots, \beta_{t'+q}^2\},$$

where $q = |\Delta_{s^2}^{-1}|$,

$$t' = |[\beta_1^2, \beta_{t'}^2]| = |[\beta_{t'+q+1}^2, \beta_{l(s^2)}^2]| = \frac{l(s^2) - q}{2}, \tag{1.6.12}$$

so that

$$2t' = |\Delta_{s^2}^s \setminus \Delta_{s^2}^{-1}| = l(s^2) - |\Delta_{s^2}^{-1}|,$$

$$|[\beta_{t'+\frac{q+l'-n}{2}+1}^2, \ldots, \beta_{t'+q}^2]| = \frac{q - l' + n}{2}, \quad |[\gamma_{n+1}, \ldots, \gamma_{l'}]| = \frac{q + l' - n}{2}; \tag{1.6.13}$$

$$(\Delta_0)_+ := \{\beta_1^0, \ldots, \beta_{D_0}^0\} = \{\alpha \in \Delta_+^s | s(\alpha) = \alpha\} = \Delta_+^s \cap \Delta_0.$$

Moreover, normal ordering (1.6.9) has the following properties.

(v) The length of the ordered segment $\Delta_{\mathfrak{m}_+} \subset \Delta$ in normal ordering (1.6.9),

$$\Delta_{\mathfrak{m}_+} := \{\gamma_1, \ldots, \gamma_2, \ldots, \gamma_3, \ldots, \gamma_n, \beta_{t+p+1}^1, \ldots, \beta_{l(s^1)}^1, \ldots, \beta_1^2, \ldots, \beta_{t'}^2, \gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}\}, \tag{1.6.14}$$

is equal to

$$|\Delta_{\mathfrak{m}_+}| = D - \left(\frac{l(s) - l'}{2} + D_0\right), \tag{1.6.15}$$

where D is the number of roots in Δ_+^s, and D_0 is the number of positive roots fixed by the action of s.

(vi) For any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ cannot be represented as a linear combination $\sum_{k=1}^t c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_t} < \beta$.

(vii) The roots from the set $s^2(\Delta_{s^1}^s)$ form a segment in normal ordering (1.6.9) preceding the segment formed by the roots from the set $\Delta_{s^2}^s$ which is, in turn, followed by the final segment $(\Delta_0)_+ = \Delta_0 \cap \Delta_+^s$. Thus the roots from the set $\Delta_s^s = \Delta_{s^2}^s \cup s^2(\Delta_{s^1}^s)$ form a segment in Δ_+^s which contains $\Delta_{s^2}^s$.
The roots from the set $\Delta_{s^1}^s$ form an initial segment which does not intersect the final segment $\Delta_s^s \cup (\Delta_0)_+$.

(viii) For any $\alpha \in (\Delta_{i_k})_+$, $i_k > 0$ such that $s\alpha \in (\Delta_{i_k})_+$ one has $s\alpha > \alpha$, and if $\beta, \gamma \in \Delta_{i_j} \cup \{0\}$, $j < k$ and $s\alpha + \beta, \alpha + \gamma \in \Delta$ then $s\alpha + \beta, \alpha + \gamma \in \Delta_+^s$ and $s\alpha + \beta > \alpha + \gamma$.

In particular, for any $\alpha \in \Delta_+^s$, $\alpha \notin \Delta_0$ and any $\alpha_0 \in \Delta_0$ such that $s\alpha \in \Delta_+^s$ one has $s\alpha > \alpha$ and if $s\alpha + \alpha_0 \in \Delta$ then $s\alpha + \alpha_0 \in \Delta_+^s$ and $s\alpha + \alpha_0 > \alpha$.

(ix) If $\alpha, \beta \in \Delta_{i_k} \cap \Delta_+^s$, $i_k > 0$, $\alpha \leq \beta$ and $s\beta \in \Delta_+^s$ then $\mathfrak{h}_{i_k}$ is two–dimensional, and the orthogonal projection of $s\beta$ onto $\mathfrak{h}_{i_k}$ is obtained by a clockwise rotation with a non–zero angle and by a rescaling with a positive coefficient from the orthogonal projection of α onto $\mathfrak{h}_{i_k}$.

Remark 1.6.7. *According to the conditions 1-4 imposed before Proposition 1.6.6 in the decomposition $s = s^1 s^2$ we always have $s^2 \neq id$ provided $s \neq id$ but in the case when s is an involution we have $s = s^2$ and $s^1 = id$. Hence in this case $\Delta_{s^1}^s = \emptyset$, conditions 1-4 imply $\Delta_+^s = \Delta_{s^2}^s \cup (\Delta_0)_+$ (disjoint union), so that normal ordering (1.6.9) has the form*

$$\beta_1^2, \ldots, \beta_{t'}^2, \gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}, \beta_{t' + \frac{q+l'-n}{2}+1}^2, \ldots, \beta_{t'+q}^2, \beta_{t'+q+1}^2, \ldots, \beta_{l(s^2)}^2, \beta_1^0, \ldots, \beta_{D_0}^0,$$

and

$$\Delta_{\mathfrak{m}_+} = \{\beta_1^2, \ldots, \beta_{t'}^2, \gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}\}.$$

Remark 1.6.8. *The most important properties of ordering (1.6.9), which we shall need later, are (v) and (vi). As $\Delta_{\mathfrak{m}_+}$ is a segment, it is additively closed, and hence the roots subspaces in $\mathfrak{g}$ corresponding to the roots from it generate a Lie subalgebra $\mathfrak{m}_+ \subset \mathfrak{n}$ of dimension $D - \left(\frac{l(s)-l'}{2} + D_0\right)$. We shall define a quantum group counterpart of its enveloping algebra $U(\mathfrak{m}_+)$, and condition (vi) will ensure that the quantum group counterpart of $U(\mathfrak{m}_+)$ has a non–trivial character which does not vanish on suitably chosen quantum root vectors corresponding to the roots $\gamma_1, \ldots, \gamma_l$. This subalgebra of the quantum group and the non–trivial character will be used to define q-W–algebras in the framework of the Hecke algebra philosophy outlined in the Introduction. Formula (1.6.15) for the dimension of $\mathfrak{m}_+$ will play a crucial role in the proof of the De Concini-Kac-Procesi conjecture. More precisely, we shall obtain and use the following relation*

$$2\mathrm{dim}\ \mathfrak{m}_+ + \mathrm{dim}\ \Sigma_s = \mathrm{dim}\ G,$$

where $\Sigma_s \subset G$ is the transversal slice defined in part (ii) of Proposition 1.3.4 with the help of the same system of positive roots Δ_+^s as ordering (1.6.9).

Proof. Firstly we describe the set $(\Delta_{i_k})_+ = \Delta_{i_k} \cap \Delta_+^s$ for $i_k > 0$ and some its subsets. Suppose that the corresponding s–invariant subspace $\mathfrak{h}_{i_k}$ is a two–dimensional plane. The case when $\mathfrak{h}_{i_k}$ is an invariant line on which s^2 acts by reflection and s^1 acts trivially can be treated in a similar way. The plane $\mathfrak{h}_{i_k}$ is shown at Figure 3.
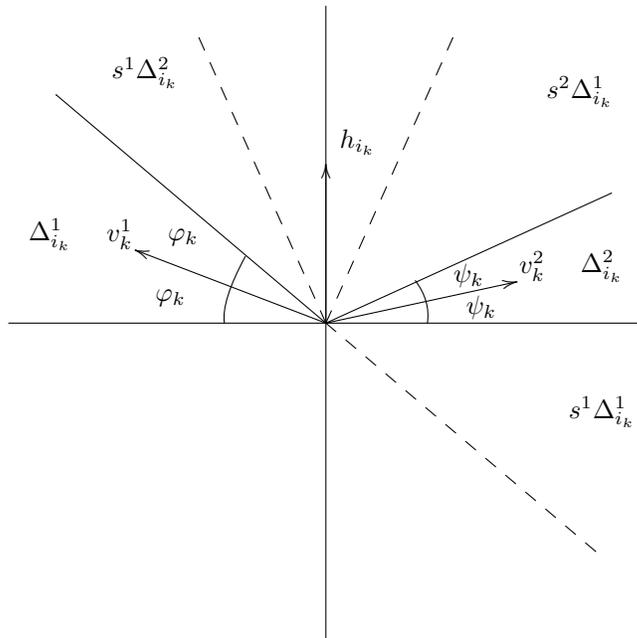

Fig. 3

The vector h_{i_k} is directed upwards at the picture. By (1.2.21) a root $\alpha \in \Delta_{i_k}$ belongs to the set $(\Delta_{i_k})_+$ if and only if $h_{i_k}(\alpha) > 0$. Identifying $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$ with the help of the bilinear form one can deduce that $\alpha \in \Delta_{i_k}$ is in Δ_+^s if and only if its orthogonal, with respect to the bilinear form, projection onto $\mathfrak{h}_{i_k}$ is contained in the upper–half plane shown at Figure 3.

The involutions s^1 and s^2 act in $\mathfrak{h}_{i_k}$ as reflections with respect to the lines orthogonal to the vectors labeled by v_k^1 and v_k^2, respectively, at Figure 3, the angle between v_k^1 and v_k^2 being equal to $\pi - \theta_{i_k}/2$. The nonzero projections of the roots from the set $\{\gamma_1, \ldots \gamma_n\} \cap \Delta_{i_k}$ onto the plane $\mathfrak{h}_{i_k}$ are proportional to the vector v_k^1, and the nonzero projections of the roots from the set $\{\gamma_{n+1}, \ldots, \gamma_{l'}\} \cap \Delta_{i_k}$ onto the plane $\mathfrak{h}_{i_k}$ are proportional to the vector v_k^2. The element s acts on $\mathfrak{h}_{i_k}$ by clockwise rotation with the angle $\theta_{i_k} = 2(\varphi_k + \psi_k)$.

Let $i_k > 0$ and $\Delta_{i_k}^{(r)}$ the subset of roots in $(\Delta_{i_k})_+$ orthogonal projections of which onto $\mathfrak{h}_{i_k}$ are directed along a ray $r \subset \mathfrak{h}_{i_k}$ starting at the origin. We call $\Delta_{i_k}^{(r)}$ the family corresponding to the ray r. Below we shall only consider rays r which correspond to nonempty sets of the form $\Delta_{i_k}^{(r)}$, so that $(\Delta_{i_k})_+ = \bigcup_{j=1}^{M_k} \Delta_{i_k}^{(r_j)}$ (disjoint union) of non-empty sets).

Lemma 1.6.9. *Suppose that $i_k > 0$. The following statements are true.*
(i) Each $\Delta_{i_k}^{(r)}$ is an additively closed set of roots.
(ii) Let $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ be two families corresponding to rays r_1 and r_2, and $\delta_1 \in \Delta_{i_k}^{(r_1)}$, $\delta_2 \in \Delta_{i_k}^{(r_2)}$ two roots such that $\delta_1 + \delta_2 = \delta_3 \in \Delta$. Then the rays r_1 and r_2 form an angle strictly less than π, and $\delta_3 \in \Delta_{i_k}^{(r_3)}$, where $\Delta_{i_k}^{(r_3)}$ is the family corresponding to a ray r_3 such that r_3 lies inside of the angle formed by r_1 and r_2.
(iii) Let $0 \leq j < k$, and $\delta_1 \in \Delta_{i_j}$, $\delta_2 \in \Delta_{i_k}^{(r)}$ two roots such that $\delta_1 + \delta_2 = \delta_3 \in \Delta$. Then $\delta_3 \in \Delta_{i_k}^{(r)}$.

Proof. All statements are simple consequences of the fact that the sum of the orthogonal projections of any two roots onto $\mathfrak{h}_{i_k}$ is equal to the orthogonal projection of the sum.

For part (i) we observe that the orthogonal projections of any two roots α, β from $\Delta_{i_k}^{(r)}$ onto $\mathfrak{h}_{i_k}$ have the same direction therefore the orthogonal projection of the sum $\alpha + \beta$ onto $\mathfrak{h}_{i_k}$ has the same direction as the orthogonal projections of α and β, and hence $\alpha + \beta \in \Delta_{i_k}^{(r)}$.

For (ii) it suffices to observe that the sum of the orthogonal projections of δ_1 and δ_2 onto $\mathfrak{h}_{i_k}$ is equal to the orthogonal projection of the sum, and the sum of the orthogonal projections of δ_1 and δ_2 onto $\mathfrak{h}_{i_k}$ lies inside of the angle formed by r_1 and r_2. The rays r_1 and r_2 form an angle strictly less than π since by (1.2.21) the orthogonal projections of all roots from $(\Delta_{i_k})_+$ onto $\mathfrak{h}_{i_k}$ belong to an open half plane.

Part (iii) follows from the fact that δ_1 has zero orthogonal projection onto $\mathfrak{h}_{i_k}$. $\qquad\square$

Now we prove properties (i), (ii) and (iii) in the statement of the proposition. Recall that by Theorem C in [18] the roots $\gamma_1, \ldots, \gamma_{l'}$ form a linear basis in the annihilator $\mathfrak{h}_\mathbb{R}'^*$ of $\mathfrak{h}_0$ with respect to the pairing between $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$. Therefore $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$ and $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$ fix all roots from $\Delta_0 \subset \mathfrak{h}_0$.

Thus, from (1.2.16) with the convention $\mathfrak{h}_0 = \mathfrak{h}^s$ adopted in this section, we have disjoint union decompositions

$$\Delta_{s^1}^s = \bigcup_{k=0, i_k>0}^M \Delta_{i_k}^1, \Delta_{s^2}^s = \bigcup_{k=0, i_k>0}^M \Delta_{i_k}^2, \Delta_s^s = \bigcup_{k=0, i_k>0}^M \Delta_{i_k}^s, \tag{1.6.16}$$

where $\Delta_{i_k}^1 = \Delta_{i_k} \cap \Delta_{s^1}^s$, $\Delta_{i_k}^2 = \Delta_{i_k} \cap \Delta_{s^2}^s$, $\Delta_{i_k}^s = \Delta_{i_k} \cap \Delta_s^s$.

Lemma 1.6.10. *If $\mathfrak{h}_{i_k}$ is a plane then the sets $\Delta_{i_k}^1$ and $\Delta_{i_k}^2$ have empty intersection and are the unions of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the sectors $\Delta_{i_k}^1$ (resp. $\Delta_{i_k}^2$). The set $\Delta_{i_k}^s$ is the union of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the union of the non-overlapping sectors $s^2 \Delta_{i_k}^1$ and $\Delta_{i_k}^2$.*

If $\mathfrak{h}_{i_k}$ is an invariant line on which s acts by multiplication by -1 the set $\Delta_{i_k}^1$ is empty and $\Delta_{i_k}^s = \Delta_{i_k}^2$. This set is the set $\Delta_{i_k}^{(r)} = (\Delta_{i_k})_+$, where r is the positive semi-axis in $\mathfrak{h}_{i_k}$.

Proof. Consider the case when $\mathfrak{h}_{i_k}$ is a plane. At Figure 3 the elements from the sets $\Delta_{i_k}^1$ and $\Delta_{i_k}^2$ are orthogonally projected onto the interiors of the sectors in the plane $\mathfrak{h}_{i_k}$ labeled by $\Delta_{i_k}^1$ and $\Delta_{i_k}^2$, respectively. Therefore the sets $\Delta_{i_k}^1$ and $\Delta_{i_k}^2$ have empty intersection and are the unions of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the sectors $\Delta_{i_k}^1$ (resp. $\Delta_{i_k}^2$).

In the case when $\mathfrak{h}_{i_k}$ is a plane the element s acts on $\mathfrak{h}_{i_k}$ by clockwise rotation with the angle $\theta_{i_k} = 2(\varphi_k + \psi_k)$ (see Figure 3). Therefore the set $\Delta_{i_k}^s$ consists of the roots the orthogonal projections of which onto $\mathfrak{h}_{i_k}$ belong to the union of the non-overlapping sectors labeled $s^2\Delta_{i_k}^1$ and $\Delta_{i_k}^2$ at Figure 3.

If $\mathfrak{h}_{i_k}$ is one–dimensional, we recall that by the assumption 4 made before this proposition there are no one–dimensional subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1. Thus in this case the set $\Delta_{i_k}^1$ is empty, and hence $\Delta_{i_k}^s = \Delta_{i_k}^2$. This set is the set $\Delta_{i_k}^{(r)} = (\Delta_{i_k})_+$, where r is the positive semi-axis in $\mathfrak{h}_{i_k}$. $\qquad\square$

By the previous lemma $\Delta_{i_k}^s = \Delta_{i_k}^2 \cup s^2\Delta_{i_k}^1$ (disjoint union), and hence by (1.6.16) $\Delta_s^s = \bigcup_{k=1}^M \Delta_{i_k}^s = \bigcup_{k=1}^M \left(\Delta_{i_k}^2 \cup s^2\Delta_{i_k}^1\right) = \left(\bigcup_{k=1}^M \Delta_{i_k}^2\right) \bigcup s^2 \left(\bigcup_{k=1}^M \Delta_{i_k}^1\right) = \Delta_{s^2}^s \cup s^2\Delta_{s^1}^s$ (disjoint union). In particular, by the results of §3 in [133] the decomposition $s = s^1 s^2$ is reduced in the sense that $l(s) = l(s^1) + l(s^2)$, as $l(s)$, $l(s^1)$ and $l(s^2)$ are equal to the cardinalities of the sets Δ_s^s, $\Delta_{s^1}^s$ and $\Delta_{s^2}^s$, respectively. Similarly, $\Delta_{s^{-1}}^s = \Delta_{s^1}^s \cup s^1(\Delta_{s^2}^s)$ (disjoint union). This proves (i).

Part (ii) immediately follows from Lemma 1.6.9 (iii) with $i_j = 0$, Lemma 1.6.10 and decompositions (1.6.16) and similar results for s^{-1}.

To justify (iii) we recall that by (1.6.16) for any root $\alpha \in \Delta_{s^1}^s$ one has $\alpha \notin \Delta_0$. In fact, in this case $\alpha \in \Delta_{i_k}^1$, where $\mathfrak{h}_{i_k}$ is a two–dimensional plane, as by assumption 4 there are no one–dimensional subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1. Hence using Figure 3 we deduce that the orthogonal projection of $s^2\alpha$ onto $\mathfrak{h}_{i_k}$ belongs to the interior of the sector labeled $s^2\Delta_{i_k}^1$ which belongs to the upper half plane and does not overlap with the sectors $\Delta_{i_k}^1$ and $\Delta_{i_k}^2$. Obviously $s^2\alpha \notin \Delta_0$. This implies $s^2\alpha \in \Delta_+^s \setminus \left(\Delta_{s^1}^s \cup \Delta_{s^2}^s \cup \Delta_0\right)$ which proves (iii).

Next we construct in several steps normal ordering (1.6.9) satisfying properties (iv)-(ix).

Step 1.

First we construct an auxiliary normal ordering on Δ_+^s satisfying properties summarized in Lemmas 1.6.11 and 1.6.12 below. We do it by induction over the sets $(\overline{\Delta}_{i_k})_+ = \overline{\Delta}_{i_k} \cap \Delta_+^s$, $k = 0, \ldots, M$, where the sets $\overline{\Delta}_{i_k}$ are defined by (1.2.22). Note that by Lemma 1.2.2 each $\overline{\Delta}_{i_k}$ is the root system of a standard Levi subalgebra in $\mathfrak{g}$.

For the base of the induction, consider the set $(\overline{\Delta}_{i_0})_+ = (\Delta_{i_0})_+$. If $i_0 = 0$ or $\mathfrak{h}_{i_0}$ is one–dimensional then we fix an arbitrary normal order on $(\overline{\Delta}_{i_0})_+ = (\Delta_{i_0})_+$.

If $\mathfrak{h}_{i_0}$ is two–dimensional then we choose a normal ordering in $(\Delta_{i_0})_+$ in the following way. First fix an initial arbitrary normal ordering on $(\Delta_{i_0})_+$. Since by Lemma 1.6.9 each set $\Delta_{i_0}^{(r)}$ is additively closed we obtain an induced ordering for $\Delta_{i_0}^{(r)}$ which satisfies the defining property for the normal ordering.

Now using these induced orderings on the sets $\Delta_{i_0}^{(r)}$ we define an auxiliary normal ordering on $(\Delta_{i_0})_+$ such that on the sets $\Delta_{i_0}^{(r)}$ it coincides with the induced normal ordering defined in the previous paragraph, and if $\Delta_{i_0}^{(r_1)}$ and $\Delta_{i_0}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in $\mathfrak{h}_{i_0}$ then for any $\alpha \in \Delta_{i_0}^{(r_1)}$ and $\beta \in \Delta_{i_0}^{(r_2)}$ one has $\alpha < \beta$. By Lemma 1.6.9 the two conditions imposed on the auxiliary normal ordering in $(\Delta_{i_0})_+$ are compatible and define it in a unique way for the given initial normal ordering on $(\Delta_{i_0})_+$. Since s acts by a clockwise rotation on $\mathfrak{h}_{i_0}$ we have $s(\Delta_{i_0}^{(r)}) = \Delta_{i_0}^{(s(r))}$ for $s(r)$ in the upper–half plane, and hence the new normal ordering satisfies the condition that for any $\alpha \in (\Delta_{i_0})_+$ such that $s\alpha \in (\Delta_{i_0})_+$ one has $s\alpha > \alpha$.

Now assume that an auxiliary normal ordering has already been constructed for the set $\overline{\Delta}_{i_{k-1}}$ and define it for the set $\overline{\Delta}_{i_k}$.

By Lemma 1.2.2 $\overline{\Delta}_{i_{k-1}}$ is generated by some subset of simple roots of the set of simple roots of $(\overline{\Delta}_{i_k})_+$. Therefore there exists an initial normal ordering on $(\overline{\Delta}_{i_k})_+$ in which the roots from the set $(\overline{\Delta}_{i_k})_+ \setminus (\overline{\Delta}_{i_{k-1}})_+ = (\Delta_{i_k})_+$ form an initial segment and the remaining roots from $(\overline{\Delta}_{i_{k-1}})_+$ are ordered according to the previously defined auxiliary normal ordering. As in case of the induction base this initial normal ordering gives rise to an induced ordering on each set $\Delta_{i_k}^{(r)}$.

Now using these induced orderings on the sets $\Delta_{i_k}^{(r)}$ we define an auxiliary normal ordering on $(\overline{\Delta}_{i_k})_+$. We impose the following conditions on it. Firstly we require that the roots from the set $(\Delta_{i_k})_+$ form an initial segment and the remaining roots from $(\overline{\Delta}_{i_{k-1}})_+$ are ordered according to the previously defined auxiliary normal ordering. Secondly, on the sets $\Delta_{i_k}^{(r)}$ the auxiliary normal ordering coincides with the induced normal ordering which is already defined, and if $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ are two families corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in $\mathfrak{h}_{i_k}$ then for any $\alpha \in \Delta_{i_k}^{(r_1)}$ and $\beta \in \Delta_{i_k}^{(r_2)}$ one has $\alpha < \beta$. By Lemma 1.6.9 the conditions imposed on the auxiliary normal ordering in $(\Delta_{i_k})_+$ are compatible and define it in a unique way.

Now we can proceed in a similar way by induction which yields an auxiliary normal ordering on Δ_+^s of the following form:

$$(\Delta_{i_M})_+, \ldots, (\Delta_{i_0})_+, \tag{1.6.17}$$

where $(\Delta_{i_k})_+$, $k = 0, \ldots, M$ are disjoint segments placed in the auxiliary normal ordering as in (1.6.17), and for each k with $i_k > 0$ the segment $(\Delta_{i_k})_+$ is the disjoint union of the disjoint segments $\Delta_{i_k}^{(r_j)}$, $j = 1, \ldots, M_k$ which are placed in the auxiliary normal ordering in the following way

$$\Delta_{i_k}^{(r_1)}, \ldots, \Delta_{i_k}^{(r_{M_k})}, \tag{1.6.18}$$

where for $i < j$ the ray r_j lies on the right from r_i in $\mathfrak{h}_{i_k}$.

Since s acts by a clockwise rotation on two-dimensional $\mathfrak{h}_{i_k}$ we have $s(\Delta_{i_k}^{(r)}) = \Delta_{i_k}^{(s(r))}$ for $s(r)$ in the upper–half plane in $\mathfrak{h}_{i_k}$, and hence the new normal ordering satisfies the condition that for any $\alpha \in (\Delta_{i_k})_+$ such that $s\alpha \in (\Delta_{i_k})_+$ one has $s\alpha > \alpha$.

Note also that the roots from $\overline{\Delta}_{i_{k-1}}$ have zero orthogonal projections onto $\mathfrak{h}_{i_k}$. Therefore if $\alpha \in \Delta_{i_k}^{(r)} \subset (\Delta_{i_k})_+$, $\beta, \gamma \in \overline{\Delta}_{i_{k-1}}$ are such that $s\alpha \in \Delta_{i_k}^{(s(r))} \subset (\Delta_{i_k})_+$, $s\alpha + \beta, \alpha + \gamma \in \Delta$ then by (1.2.15) and (1.2.21) $s\alpha + \beta \in \Delta_{i_k}^{(s(r))} \subset \Delta_+^s \cap (\Delta_{i_k})_+, \alpha + \gamma \in \Delta_{i_k}^{(r)} \subset \Delta_+^s \cap (\Delta_{i_k})_+$ and $s\alpha + \beta > \alpha + \gamma$ as $s(\Delta_{i_k}^{(r)}) = \Delta_{i_k}^{(s(r))}$ for $s(r)$ in the upper–half plane. These properties of the new normal ordering are summarized in the following lemma.

Lemma 1.6.11. *(i) Suppose that $i_k > 0$. Then for two-dimensional $\mathfrak{h}_{i_k}$ one has $s(\Delta_{i_k}^{(r)}) = \Delta_{i_k}^{(s(r))}$ if $s(r)$ in the upper–half plane, and $\Delta_{i_k}^{(r)} < s(\Delta_{i_k}^{(r)})$.*

(ii) In particular, for any $\alpha \in \Delta_{i_k}^{(r)}$ such that $s\alpha \in (\Delta_{i_k})_+$ one has $s\alpha > \alpha$ and if $\beta, \gamma \in \overline{\Delta}_{i_{k-1}} \cup \{0\}$, $s\alpha + \beta, \alpha + \gamma \in \Delta$ then $\alpha + \gamma \in \Delta_{i_k}^{(r)}$, $s\alpha + \beta \in \Delta_{i_k}^{(s(r))}$ and $s\alpha + \beta > \alpha + \gamma$.

Observe that, according to the definition of the auxiliary normal ordering of Δ_+^s we have the following properties of this normal ordering which can be seen from (1.6.17) and (1.6.18).

Lemma 1.6.12. *The auxiliary normal ordering of $\Delta_+^s = \bigcup_{k=0}^M (\Delta_{i_k})_+$ (disjoint union) has the following properties.*

(i) For any $k = 0, \ldots, M$ the roots from the disjoint sets $(\Delta_{i_k})_+$ form segments, and for any $k = 0, \ldots, M-1$ one has $(\Delta_{i_k})_+ > (\Delta_{i_{k+1}})_+$;

(ii) For any $k = 0, \ldots, M$ with $i_k > 0$ the roots from the sets $\Delta_{i_k}^{(r_j)}$, $j = 1, \ldots, M_k$ form segments, $(\Delta_{i_k})_+ = \bigcup_{j=1}^{M_k} \Delta_{i_k}^{(r_j)}$ (disjoint union), and the roots from the set $(\Delta_0)_+ = \Delta_0 \cap \Delta_+^s$, if it is non-empty, form a final segment;

(iii) For any $k = 0, \ldots, M$ such that the corresponding $\mathfrak{h}_{i_k}$ is two–dimensional and for any two segments $\Delta_{i_k}^{(r_1)}$ and $\Delta_{i_k}^{(r_2)}$ corresponding to rays r_1 and r_2 such that r_2 lies on the right from r_1 in $\mathfrak{h}_{i_k}$ one has $\Delta_{i_k}^{(r_1)} < \Delta_{i_k}^{(r_2)}$.

Step 2.

Now we modify the auxiliary normal ordering in Δ_+^s in such a way that the roots from the set $\Delta_{s^1}^s$ will form an initial segment, the roots from the set $s^1(\Delta_{s^2}^s)$ will follow immediately after it, the roots from the set $s^2(\Delta_{s^1}^s)$ will form a segment preceding the segment formed by the roots from the set $\Delta_{s^2}^s$ which will, in turn, be followed by the final segment $(\Delta_0)_+ = \Delta_0 \cap \Delta_+^s$.

We shall do it with the help of the following lemma.

Lemma 1.6.13. *Assume that Δ_+^s is equipped with an arbitrary normal ordering such that the roots from a set $\Delta_{i_k}^{(r)} = \{\delta_1, \ldots, \delta_a\}$ for some ray $r \subset \mathfrak{h}_{i_k}$ form a segment $\delta_1, \ldots, \delta_a$. Suppose also that for some natural p such that $0 \le p < k$, $i_p \ne 0$ the roots from a set $\Delta_{i_p}^{(t)} = \{\xi_1, \ldots, \xi_b\}$ for some ray $t \subset \mathfrak{h}_{i_p}$ form a segment $\xi_1, \ldots, \xi_b$ and that $\delta_1, \ldots, \delta_a, \xi_1, \ldots, \xi_b$ is also a segment. Then applying elementary transpositions within the segment $\delta_1, \ldots, \delta_a, \xi_1, \ldots, \xi_b$ one can reduce it to the form $\xi_{i_1}, \ldots, \xi_{i_b}, \delta_{j_1}, \ldots, \delta_{j_a}$.*

Proof. The proof is by induction. First consider the segment $\delta_1, \ldots, \delta_a, \xi_1$.

Since the orthogonal projections of the roots from the set Δ_{i_p} onto $\mathfrak{h}_{i_k}$ are equal to zero, for any $\varsigma \in \Delta_{i_p}^{(t)}$ and $\upsilon \in \Delta_{i_k}^{(r)}$ such that $\varsigma + \upsilon \in \Delta$ we have $\varsigma + \upsilon \in \Delta_{i_k}^{(r)}$. Assume now that ς and υ are contained in an ordered segment of form (1.6.2) or in a segment with the inverse ordering. By the previous observation this segment contains no other roots from $\Delta_{i_p}^{(t)}$, and ς is the first or the last element in that segment. For the same reason the other roots

in that segment must also belong to $\Delta_{i_k}^{(r)}$. Therefore applying an elementary transposition, if necessarily, one can move ς to the first position in that segment.

Applying this procedure iteratively to the segment $\delta_1, \ldots, \delta_a, \xi_1$ we can reduce it to the form $\xi_1, \delta_{k_1}, \ldots, \delta_{k_a}$ by applying elementary transpositions within the segment $\delta_1, \ldots, \delta_a, \xi_1$.

Now we can apply the same procedure to the segment $\delta_{k_1}, \ldots, \delta_{k_a}, \xi_2$ to reduce the segment $\xi_1, \delta_{k_1}, \ldots, \delta_{k_a}, \xi_2$ to the form $\xi_1, \xi_2, \delta_{l_1}, \ldots, \delta_{l_a}$ by applying elementary transpositions within the segment $\delta_{k_1}, \ldots, \delta_{k_a}, \xi_2$.

Iterating this procedure we obtain the statement of the lemma.

$\square$

Now observe that according to Lemma 1.6.12 (i) the roots from each of the sets $(\Delta_{i_k})_+$ form a segment in the auxiliary normal ordering of Δ_+^s, and by Lemma 1.6.12 (ii) the roots from the sets $\Delta_{i_k}^{(r)}$ form segments inside $(\Delta_{i_k})_+$. As we observed in Lemma 1.6.10, the sets $\Delta_{i_k}^1$ (resp. $\Delta_{i_k}^2$) are the unions of the sets $\Delta_{i_k}^{(r)}$ with r belonging to the sectors $\Delta_{i_k}^1$ (resp. $\Delta_{i_k}^2$) at Figure 3, and hence by Lemma 1.6.12 (i) and (ii) (see also (1.6.17), (1.6.18)) the roots from the sets $\Delta_{i_k}^1$ and $\Delta_{i_k}^2$ form an initial and a final segment, respectively, inside $(\Delta_{i_k})_+$.

Therefore we can apply Lemma 1.6.13 to move all roots from the segments $\Delta_{i_k}^1$, $k = 0, \ldots, M$ to the left and to move all roots from the segments $\Delta_{i_k}^2$, $k = 0, \ldots, M$ to the right to positions preceding the final segment formed by the roots from $(\Delta_0)_+$. After this modification the roots from the set $\Delta_{s^1}^s$ form an initial segment.

Now using similar arguments the roots from the sets $s^2\Delta_{i_k}^1$, $k = 0, \ldots, M$ forming segments by Lemma 1.6.10 and by Lemma 1.6.12 (i) and (ii) (see also (1.6.17), (1.6.18)) as well can be moved to the right to positions preceding the final segment formed by the roots from the set $\Delta_{s^2}^s \cup (\Delta_0)_+$. Note that by parts (i) and (iii) $\Delta_{s^1}^s \cap s^2(\Delta_{s^1}^s) = \{\emptyset\}$ and $\Delta_{s^2}^s \cap s^2(\Delta_{s^1}^s) = \{\emptyset\}$. So the last modification does not affect the positions of the roots in $\Delta_{s^1}^s$ and in $\Delta_{s^2}^s$ and after applying it the roots from the set $s^2(\Delta_{s^1}^s)$ will form a segment preceding the segment formed by the roots from the set $\Delta_{s^2}^s$ which is, in turn, be followed by the final segment $(\Delta_0)_+ = \Delta_0 \cap \Delta_+^s$. Thus the roots from the set $\Delta_s^s = \Delta_{s^2}^s \cup s^2(\Delta_{s^1}^s)$ form a segment in Δ_+^s which contains $\Delta_{s^2}^s$. By construction the segment $\Delta_{s^1}^s$ does not intersect the final segment $\Delta_s^s \cup (\Delta_0)_+$. This proves that property (vii) holds for the ordering constructed at Step 2.

Thus at Step 2 we obtain a normal ordering on Δ_+^s of the following form:

$$\Delta_{s^1}^s, (\Delta_{i_M})_+', \ldots, (\Delta_{i_0})_+', s^2(\Delta_{s^1}^s), \Delta_{s^2}^s, (\Delta_0)_+, \tag{1.6.19}$$

where $(\Delta_{i_k})_+' = (\Delta_{i_k})_+ \setminus (\Delta_s^s \cup \Delta_{s^1}^s \cup (\Delta_0)_+)$, $k = 0, \ldots, M$ are disjoint segments placed in the normal ordering as in (1.6.19), and for each k with $i_k > 0$ the segment $(\Delta_{i_k})_+'$ is the disjoint union of the disjoint segments $\Delta_{i_k}'^{(r_j)} = \Delta_{i_k}^{(r_j)} \setminus (\Delta_s^s \cup \Delta_{s^1}^s) \cup (\Delta_0)_+)$, $j = 1, \ldots, M_k$,

$$(\Delta_{i_k})_+' = \bigcup_{j=1}^{M_k} \Delta_{i_k}'^{(r_j)}, k = 0, \ldots, M, \tag{1.6.20}$$

which are placed in the normal ordering in the following way

$$\Delta_{i_k}'^{(r_1)}, \ldots, \Delta_{i_k}'^{(r_{M_k})}, \tag{1.6.21}$$

where for $i < j$ the ray r_j lies on the right from r_i in $\mathfrak{h}_{i_k}$.

The segment $\Delta_{s^1}^s$ (resp. $\Delta_{s^2}^s$) is the disjoint union of the sets $\Delta_{i_k}^1$ (resp. $\Delta_{i_k}^2$), $k = 0, \ldots, M$,

$$\Delta_{s^1}^s = \bigcup_{k=0}^{M} \Delta_{i_k}^1, \Delta_{s^2}^s = \bigcup_{k=0}^{M} \Delta_{i_k}^2 \tag{1.6.22}$$

which are placed in the normal ordering obtained at Step 2 in the following way

$$\Delta_{i_M}^1, \ldots, \Delta_{i_0}^1, \text{ (resp. } \Delta_{i_M}^2, \ldots, \Delta_{i_0}^2), \tag{1.6.23}$$

and the segment $s^2(\Delta_{s^1}^s)$ is the disjoint union of the segments $s^2(\Delta_{i_k}^1)$, $k = 0, \ldots, M$

$$s^2(\Delta_{s^1}^s) = \bigcup_{k=0}^{M} s^2(\Delta_{i_k}^1) \tag{1.6.24}$$

which are placed in the normal ordering obtained at Step 2 in the following way

$$s^2(\Delta^1_{i_M}), \ldots, s^2(\Delta^1_{i_0}). \tag{1.6.25}$$

Note that $\Delta^1_{i_k}$ (resp. $\Delta^2_{i_k}$) is the disjoint union of the sets $\Delta^{(r)}_{i_k}$ with r belonging to the sectors $\Delta^1_{i_k}$ (resp. $\Delta^2_{i_k}$) at Figure 3 (we assume that $\Delta^1_{i_k}$ is the positive semiaxis if $\mathfrak{h}_{i_k}$ is one–dimensional; for $\Delta^2_{i_k}$ this situation does not occur),

$$\Delta^1_{i_k} = \bigcup_{r \in \Delta^1_{i_k} \text{ at Figure 3}} \Delta^{(r)}_{i_k}, \Delta^2_{i_k} = \bigcup_{r \in \Delta^2_{i_k} \text{ at Figure 3}} \Delta^{(r)}_{i_k}, \tag{1.6.26}$$

so that for $k = 0, \ldots, M$

$$\Delta'^{(r)}_{i_k} = \Delta^{(r)}_{i_k} \setminus (\Delta^s_s \cup \Delta^{s^1}_s \cup (\Delta_0)_+) = \left\{ \begin{array}{ll} \Delta^{(r)}_{i_k} & \text{if } r \notin \Delta^1_{i_k} \cup s^2(\Delta^1_{i_k}) \cup \Delta^2_{i_k} \text{ at Figure 3} \\ \emptyset & \text{otherwise} \end{array} \right. , \tag{1.6.27}$$

The segments $s^2(\Delta^1_{i_k})$, $k = 0, \ldots, M$ are non–empty for two–dimensional $\mathfrak{h}_{i_k}$ only. They can be represented as the following disjoint unions of segments

$$s^2(\Delta^1_{i_k}) = \bigcup_{r \in s^2(\Delta^1_{i_k}) \text{ at Figure 3}} \Delta^{(r)}_{i_k}, \tag{1.6.28}$$

which are placed in the normal ordering in such a way that

$$\Delta^{(r)}_{i_k} < \Delta^{(t)}_{i_k}, \tag{1.6.29}$$

where for any $r, t \in s^2(\Delta^1_{i_k})$ at Figure 3, the ray t lies on the right from r in $\mathfrak{h}_{i_k}$.

The empty sets which may formally appear in the description of the ordering in formulas (1.6.19), (1.6.21), (1.6.23), (1.6.25), (1.6.28), (1.6.29) should be omitted.

Note that these formulas imply that the ordering defined at this step has the form

$$\Delta^1, \Delta^2, \ldots, \Delta^{R-1}, \Delta^R, \tag{1.6.30}$$

where Δ^j, $j = 1, \ldots, R$ are disjoint segments, $\Delta^1 = \Delta^s_{s^1}$, $\Delta^{R-1} = \Delta^s_{s^2}$, $\Delta^R = (\Delta_0)_+$, and for $j = 2, \ldots, R-2$ one has $\Delta^j = \Delta^{(r)}_{i_k}$ for some k and r.

In formula (1.6.30) we could partition further the segments $\Delta^1 = \Delta^s_{s^1}$ and $\Delta^{R-1} = \Delta^s_{s^2}$ as in (1.6.26). We shall not do that since at the next step these segments will be reordered.

Note that according to the algorithm given in Lemma 1.6.13 for each fixed k the order of the segments formed by the roots from the sets $\Delta^{(r)}_{i_k}$ is preserved after applying that lemma. Therefore the new normal ordering obtained this way still satisfies the properties stated in Lemma 1.6.11 and 1.6.12 (ii), (iii). In particular, by Lemma 1.6.11 (i) for any j such that $\Delta^j \not\subset \Delta^s_s \cup \Delta_0$ one has

$$\Delta^j < s(\Delta^j). \tag{1.6.31}$$

Step 3.

Now we can apply elementary transpositions to bring the initial segment formed by the roots from $\Delta^s_{s^1} = \{\beta^1_1, \ldots, \beta^1_{l(s^1)}\}$ and the segment formed by the roots from $\Delta^s_{s^2} = \{\beta^2_1, \ldots, \beta^2_{l(s^2)}\}$ and preceding the final segment $(\Delta_0)_+$ to the form described in (1.6.9).

For this purpose we shall use the following lemma.

Lemma 1.6.14. ([101], Theorem A; [18], Lemma 5; [125], Proposition 3.3) *Let $z \in W = W(\mathfrak{g}, \mathfrak{h})$ be an involution. Fix a system of positive roots Δ_+ in $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, and let $s_1, \ldots, s_l$ be the corresponding simple reflections in W. Then the following statements are true.*

(i) There is a Levi subalagbra of $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and semisimple part $\mathfrak{m}_z$, which has Cartan subalgebra $\mathfrak{h}_z \subset \mathfrak{h}$, such that z is the longest element of the Weyl group $W(\mathfrak{m}_z, \mathfrak{h}_z)$ with respect to the system of simple roots in $\Delta_+ \cap \Delta(\mathfrak{m}_z, \mathfrak{h}_z)$, and z acts by multiplication by -1 on the Cartan subalgebra $\mathfrak{h}_z$.

(ii) z can be expressed as a product of $\dim \mathfrak{h}_z$ reflections from the Weyl group $W(\mathfrak{m}_z, \mathfrak{h}_z)$ of the pair $(\mathfrak{m}_z, \mathfrak{h}_z)$, with respect to mutually orthogonal roots.

(iii) There is a reduced decomposition of z, with respect to the system of simple roots in Δ_+, of the form

$$z = xyx^{-1}, \tag{1.6.32}$$

where $x \in W$, and $y \in W'$ is the longest element in a standard parabolic subgroup W' of W such that $W(\mathfrak{m}_z, \mathfrak{h}_z) = xW'x^{-1}$, and if $y = s_{j_1} \ldots s_{j_q}$ is a reduced decomposition of y in W' and $x = s_{i_1} \ldots s_{i_t}$ is a reduced decomposition of x then

$$z = s_{i_1} \ldots s_{i_t} s_{j_1} \ldots s_{j_p} s_{i_t} \ldots s_{i_1} \tag{1.6.33}$$

is a reduced decomposition of z, so that the corresponding normal ordering of the set $\Delta_{z^{-1}} = \Delta_z$ has the form

$$\beta_1, \ldots, \beta_t, \beta_{t+1}, \ldots, \beta_{t+p}, \beta_{t+p+1}, \ldots, \beta_{2t+p},$$

where $z\beta_j = -\beta_j$, $j = t+1, \ldots, t+p$, and $\{\beta_{t+1}, \ldots, \beta_{t+p}\} = \Delta_+(\mathfrak{m}_z, \mathfrak{h}_z) = \Delta(\mathfrak{m}_z, \mathfrak{h}_z) \cap \Delta_+$.

Let $s_{i_1} \ldots s_{i_{l(s^1)}}$ be the initial part of the reduced decomposition of $\overline{w}$ corresponding to the normal ordering of Δ_+^s obtained at Step 2, so that $(s^1)^{-1} = s^1 = s_{i_1} \ldots s_{i_{l(s^1)}} = s_{\beta^1_{l(s^1)}} \ldots s_{\beta^1_1}$. By parts (i) and (iii) of Lemma 1.6.14 and by parts (ii) and (iii) of Lemma 1.6.4 for $z = w = (s^1)^{-1} = s^1 = s_{i_1} \ldots s_{i_{l(s^1)}} = s_{\beta^1_{l(s^1)}} \ldots s_{\beta^1_1}$ and $u = 1$, using elementary transpositions within the segment $\beta^1_1, \ldots, \beta^1_{l(s^1)}$ one can reduce it to the form

$$\beta^1_1, \ldots, \beta^1_t, \beta^1_{t+1}, \ldots, \beta^1_{t+p}, \beta^1_{t+p+1}, \ldots, \beta^1_{l(s^1)}, \tag{1.6.34}$$

where $\beta^1_{t+1}, \ldots, \beta^1_{t+p}$ is a normal ordering of the system $\Delta_+^s(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) = \Delta(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) \cap \Delta_+^s = \Delta_{s^1}^{-1}$ of positive roots in the root system $\Delta(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ of the pair $(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$, so that p is the number of positive roots in $\Delta_+^s(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) = \Delta_{s^1}^{-1}$.

Note that by (1.6.33) one has $t = l(s^1) - (t+p)$, i. e. there are equal numbers of roots on the left and on the right from the segment $\beta^1_{t+1}, \ldots, \beta^1_{t+p}$ in the segment

$$\beta^1_1, \ldots, \beta^1_t, \beta^1_{t+1}, \ldots, \beta^1_{t+p}, \beta^1_{t+p+1}, \ldots, \beta^1_{l(s^1)},$$

and

$$t = \frac{l(s^1) - p}{2}. \tag{1.6.35}$$

This proves (1.6.10).

Observe that in the case of the involution s^1, $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$ is the expression mentioned in part (ii) of Lemma 1.6.14, and the roots $\gamma_1, \ldots, \gamma_n$ span the Cartan subalgebra $\mathfrak{h}_{s^1}$.

Now according to the results of Appendix 1, applying elementary transpositions we can reduce the ordering $\beta^1_{t+1}, \ldots, \beta^1_{t+p}$ to the form compatible with the decomposition $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, i.e., we can bring it to the form

$$\beta^1_{t+1}, \ldots, \beta^1_{t+\frac{p-n}{2}}, \gamma_1, \ldots, \gamma_2, \ldots, \gamma_3, \ldots, \gamma_n, \tag{1.6.36}$$

where in the last formula we relabeled the roots β^1_i in such a way that after reordering $\beta^1_i < \beta^1_j$ if and only if $i < j$, and, according to the definition of normal orderings compatible with decompositions of Weyl group involutions given in Appendix 1, the new ordering has the property that for any two positive roots $\alpha, \beta \in \Delta_+^s(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ such that $\gamma_1 \leq \alpha < \beta$ the sum $\alpha + \beta$ cannot be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$. Formula (1.6.11) follows from the definition given in Appendix 1 (see formula (A1.1)).

Note that by Lemma 1.6.4 the elementary transpositions applied to obtain ordering (1.6.36) do not affect the positions of the roots which do not belong to the set $\Delta_+^s(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) = \Delta_{s^1}^{-1}$.

Now let $\overline{w} = s_{i_1} \ldots s_{i_D}$ be the reduced decomposition $\overline{w}$ corresponding to the normal ordering of Δ_+^s obtained so far, $w = s_{i_{h+1}} \ldots s_{i_{h+l(s^2)}}$ the product of all simple reflections in the reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$ such that for $k = 1, \ldots, l(s^2)$ the position of $s_{i_{h+k}}$ in the reduced decomposition of $\overline{w}$ is the same as the position of the root β^2_k in the segment $\beta^2_1, \ldots, \beta^2_{l(s^2)}$ of the normal ordering of Δ_+^s. We recall that $\{\beta^2_1, \ldots, \beta^2_{l(s^2)}\} = \Delta_{s^2}^s$.

Now, similarly to the case of the segment $\Delta_{s^1}^s$, we can apply parts (i) and (iii) of Lemma 1.6.14 and parts (ii) and (iii) of Lemma 1.6.4 with $z = w = s_{i_{h+1}} \ldots s_{i_{h+l(s^2)}}$, $u = s_{i_1} \ldots s_{i_h}$ to the segment $\{\beta^2_1, \ldots, \beta^2_{l(s^2)}\} = \Delta_{s^2}^s$. In the notation of Lemmas 1.6.4 and 1.6.14, $s^2 = (s^2)^{-1} = uwu^{-1} = uw^{-1}u^{-1} = s_{\beta^2_{l(s^2)}} \ldots s_{\beta^2_1}$. Using also the normal ordering of the system of positive roots $\Delta_+^s(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) = \Delta(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \cap \Delta_+^s$ in the root system $\Delta(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ of the

pair $(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ inverse to that compatible with the decomposition $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$ we finally obtain the following normal ordering of the set Δ^s_+

$$\beta^1_1, \ldots, \beta^1_t, \beta^1_{t+1}, \ldots, \beta^1_{t+\frac{p-n}{2}}, \gamma_1, \ldots, \gamma_2, \ldots, \gamma_3, \ldots, \gamma_n, \beta^1_{t+p+1}, \ldots, \beta^1_{l(s^1)}, \ldots, \tag{1.6.37}$$

$$\beta^2_1, \ldots, \beta^2_{t'}, \gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}, \beta^2_{t'+\frac{q+l'-n}{2}+1}, \ldots, \beta^2_{t'+q}, \beta^2_{t'+q+1}, \ldots, \beta^2_{l(s^2)}, \beta^0_1, \ldots, \beta^0_{D_0},$$

where

$$\gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}, \beta^2_{t'+\frac{q+l'-n}{2}+1}, \ldots, \beta^2_{t'+q}$$

is the normal ordering of the system of positive roots $\Delta^s_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ inverse to that compatible with the decomposition $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$. By construction normal ordering (1.6.37) has the required form (1.6.9). Formulas (1.6.13) and (1.6.12) are established similarly to (1.6.11) and (1.6.10), respectively. This completes the proof of part (iv).

We claim that normal ordering (1.6.37) has properties (v)-(ix) listed in the statement of this proposition.

Firstly we recall that according to Lemma 1.6.4 the elementary transpositions used at Step 3 do not affect the positions of the roots which do not belong to $\Delta^s_{s^1}$ and $\Delta^s_{s^2}$. We state this property for future references as a lemma.

Lemma 1.6.15. *The elementary transpositions used at Step 3 to bring the segments $\Delta^s_{s^1}$ and $\Delta^s_{s^2}$ to the form required in (1.6.9) do not affect the positions of the roots which do not belong to $\Delta^s_{s^1}$ and $\Delta^s_{s^2}$. Thus the normal ordering obtained at Step 3 has properties described in (1.6.19)-(1.6.31), except for (1.6.23).*

From this lemma we deduce that the normal ordering constructed at Step 3 still has form (1.6.19), and hence property (vii) holds for normal ordering (1.6.37) as it holds for the ordering constructed at Step 2.

For for $\alpha \in \Delta^{(r)}_{i_k}$, $i_k > 0$ we still have $\alpha \in \Delta^{(s(r))}_{i_k}$ if $s\alpha \in \Delta^s_+$ by the definition of the sets $\Delta^{(r)}_{i_k}$. We claim that Lemma 1.6.11 still holds for the normal ordering obtained at Step 3. We shall need a slightly more detailed version of it which implies property (viii).

Lemma 1.6.16. *(i) Suppose that $i_k > 0$. Then for two-dimensional $\mathfrak{h}_{i_k}$ one has $s(\Delta^{(r)}_{i_k}) = \Delta^{(s(r))}_{i_k}$ if $s(r)$ in the upper–half plane, and $\Delta^{(r)}_{i_k} < s(\Delta^{(r)}_{i_k})$.*

(ii) In particular, for any $\alpha \in \Delta^{(r)}_{i_k}$ such that $s\alpha \in (\Delta_{i_k})_+$ one has $s\alpha > \alpha$ and if $\beta, \gamma \in \overline{\Delta}_{i_{k-1}} \cup \{0\}$, $s\alpha + \beta, \alpha + \gamma \in \Delta$ then $\alpha + \gamma \in \Delta^{(r)}_{i_k}$, $s\alpha + \beta \in \Delta^{(s(r))}_{i_k}$ and $s\alpha + \beta > \alpha + \gamma$.

(iii)

$$s(\Delta^s_{s^1}) \subset \Delta^s_+, \; s(\Delta^s_{s^1}) > \Delta^s_{s^1}. \tag{1.6.38}$$

Proof. First observe that by Proposition 1.6.6 (iii), which has been already proved, $s^2(\Delta^s_{s^1}) \subset \Delta^s_+ \setminus (\Delta^s_{s^1} \cup \Delta^s_{s^2} \cup \Delta_0)$. Since $\Delta^s_{s^1} = s^1(-\Delta^s_{s^1})$ we deduce that $s(\Delta^s_{s^1}) = s^1 s^2(\Delta^s_{s^1}) \subset \Delta^s_+$, and $s(\Delta^s_{s^1}) = s^1 s^2(\Delta^s_{s^1}) > \Delta^s_{s^1}$. This proves (1.6.38).

Note that by Lemma 1.6.15 the normal ordering obtained at Step 3 still has properties described in (1.6.19)-(1.6.27), except for (1.6.23). By (1.6.22), (1.6.24) and (1.6.26) formula (1.6.38) also implies that part (i), and hence (ii), hold for r which belong to the sector $\Delta^1_{i_k}$ at Figure 3.

If r belongs to the union of the sectors $\Delta^2_{i_k}$ and $s^2(\Delta^1_{i_k})$ at Figure 3 then $\Delta^{(r)}_{i_k} \subset \Delta^s_s$, so $s(\Delta^{(r)}_{i_k}) \subset \Delta^s_-$, and properties (i) and (ii) are void.

If $r \notin \Delta^1_{i_k} \cup \Delta^2_{i_k} \cup s^1(\Delta^2_{i_k})$ at Figure 3 then $\Delta^{(r)}_{i_k} = \Delta'^{(r)}_{i_k} \subset (\Delta_{i_k})'_+$, and properties (i) and (ii) hold for the normal ordering defined at Step 2 which may only differ by positions of roots in $\Delta^s_{s^1}$ and $\Delta^s_{s^2}$ from the normal ordering defined at Step 3 by Lemma 1.6.15. In particular, $\Delta^s_{s^2} > \Delta^{(r)}_{i_k}$ with respect to both the normal ordering defined at Step 2 and the normal ordering defined at Step 3. Therefore (i) and (ii) hold for the normal ordering defined at Step 3 as well. This completes the proof.

$\square$

Parts (i) and (ii) of the previous lemma imply property (viii) since when the corresponding $\mathfrak{h}_{i_k}$ is one–dimensional this property is void.

If $\alpha, \beta \in \Delta_{i_k} \cap \Delta^s_+$, $\alpha \leq \beta$ and $s\beta \in \Delta^s_+$ then $\mathfrak{h}_{i_k}$ must be two dimensional, as otherwise $s\beta \in \Delta^s_-$. We show that the orthogonal projection of $s\beta$ onto $\mathfrak{h}_{i_k}$ is obtained by a clockwise rotation with a non–zero angle and by a rescaling with a positive coefficient from the orthogonal projection of α onto $\mathfrak{h}_{i_k}$.

Indeed, observe that by Lemma 1.6.15 the elementary transpositions which we used at Step 3 to bring the segments formed by the roots from the sets $\Delta^s_{s^1}$ and $\Delta^s_{s^2}$ to the form required in normal ordering (1.6.9) do not

affect the positions of the other roots and after this rearrangement the orthogonal projections of the roots from $\Delta^1_{i_k}$ (resp. $\Delta^2_{i_k}$) onto $\mathfrak{h}_{i_k}$ still belong to the sectors labeled $\Delta^1_{i_k}$ (resp. $\Delta^2_{i_k}$) at Figure 3.

Therefore by Lemma 1.6.12 (iii) if $\alpha, \beta \in \Delta_{i_k} \cap \Delta^s_+$, $\beta \in \Delta^1_{i_k}$ and $\alpha \leq \beta$ then the orthogonal projection of α onto $\mathfrak{h}_{i_k}$ belongs to the sector labeled $\Delta^1_{i_k}$ at Figure 3. On the other hand since $s\beta \in \Delta^s_+$ the orthogonal projection of $s\beta$ onto $\mathfrak{h}_{i_k}$ belongs to the upper half plane and does not belong to the sector labeled $\Delta^1_{i_k}$ at Figure 3 as s acts on $\mathfrak{h}_{i_k}$ by clockwise rotation by the angle $\theta_{i_k} = 2(\varphi_k + \psi_k) > 2\varphi_k$. Thus the orthogonal projection of $s\beta$ onto $\mathfrak{h}_{i_k}$ is obtained by a clockwise rotation with a non–zero angle and by a rescaling with a positive coefficient from the orthogonal projection of α onto $\mathfrak{h}_{i_k}$.

The case when $\beta \in (\Delta_{i_k})_+$, $\beta \notin \Delta^1_{i_k}$, $s\beta \in (\Delta_{i_k})_+$ is treated in a similar way with the help of Lemma 1.6.12 (iii). This proves (ix).

By the definition of normal ordering (1.6.9) the number of roots in the segment $\Delta_{\mathfrak{m}_+}$,

$$\Delta_{\mathfrak{m}_+} = \{\gamma_1, \ldots, \gamma_2, \ldots, \gamma_3, \ldots, \gamma_n, \beta^1_{t+p+1}, \ldots, \beta^1_{l(s^1)}, \ldots, \beta^2_1, \ldots, \beta^2_{t'}, \gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}\}$$

is

$$|\Delta_{\mathfrak{m}_+}| = D - |[\beta^1_1, \beta^1_{t+\frac{p-n}{2}}]| - |[\beta^2_{t'+\frac{q+l'-n}{2}+1}, \beta^2_{l(s^2)}]| - |[\beta^0_1, \beta^0_{D_0}]| =$$

$$= D - D_0 - |[\beta^1_1, \beta^1_t]| - |[\beta^1_{t+1}, \beta^1_{t+\frac{p-n}{2}}]| - |[\beta^2_{t'+\frac{q+l'-n}{2}+1}, \beta^2_{t'+q}]| - |[\beta^2_{t'+q+1}, \beta^2_{l(s^2)}]|,$$

where $D_0 = |[\beta^0_1, \beta^0_{D_0}]|$ is the number of positive roots fixed by the action of s. By formulas (1.6.10), (1.6.11), (1.6.12) and (1.6.13) the last expression takes the form

$$|\Delta_{\mathfrak{m}_+}| = D - D_0 - t - \frac{p-n}{2} - t' - \frac{q-(l'-n)}{2} =$$

$$= D - D_0 - \frac{l(s^1)-p}{2} - \frac{l(s^2)-q}{2} - \frac{p-n}{2} - \frac{q-(l'-n)}{2} = D - \left(\frac{l(s)-l'}{2} + D_0\right),$$

where $l(s) = l(s^1) + l(s^2)$ is the length of s. This establishes (v).

Now let $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$, be any two roots such that $\alpha < \beta$. We shall show that the sum $\alpha + \beta$ cannot be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$.

Suppose that such a decomposition exists, $\alpha + \beta = \sum_{k=1}^q c_k \gamma_{i_k}$. Obviously at least one of the roots α, β must belong to the set $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) \cap \Delta_{\mathfrak{m}_+}$ or to the set $\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \cap \Delta_{\mathfrak{m}_+}$ for otherwise the set of roots γ_{i_k} such that $\alpha < \gamma_{i_k} < \beta$ is empty because by the definition of $\Delta_{\mathfrak{m}_+}$, $(\Delta_{\mathfrak{m}_+} \setminus (\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) \cup \Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}))) \cap \{\gamma_1, \ldots, \gamma_{l'}\} = \emptyset$.

Suppose that $\alpha \in \Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) \cap \Delta_{\mathfrak{m}_+}$. The other cases are considered in a similar way.

If $\beta \notin \Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \cap \Delta_{\mathfrak{m}_+}$ then $\alpha + \beta = \sum_{k=1}^q c_k \gamma_{i_k}$, and $\gamma_{i_k} \leq \gamma_n$. In particular, since $\alpha \in \mathfrak{h}_{s^1}$ and $\gamma_{i_k} \in \mathfrak{h}_{s^1}$ if $\gamma_{i_k} \leq \gamma_n$, we have $\beta = \sum_{k=1}^q c_k \gamma_{i_k} - \alpha \in \mathfrak{h}_{s^1}$. This is impossible by the definition of the ordering of the set $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ compatible with the decomposition $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$.

If $\beta \in \Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \cap \Delta_{\mathfrak{m}_+}$ then $\alpha + \beta = \sum_{k=1}^q c_k \gamma_{i_k} = \sum_{i_k \leq n} c_k \gamma_{i_k} + \sum_{i_k > n} c_k \gamma_{i_k}$. This implies

$$\alpha - \sum_{i_k \leq n} c_k \gamma_{i_k} = \sum_{i_k > n} c_k \gamma_{i_k} - \beta.$$

The l.h.s. of the last formula is an element of $\mathfrak{h}_{s^1}$ and the r.h.s. is an element $\mathfrak{h}_{s^2}$. Since $\mathfrak{h}' = \mathfrak{h}_{s^1} + \mathfrak{h}_{s^2}$ is a direct vector space decomposition we infer that

$$\alpha = \sum_{i_k \leq n, \alpha < \gamma_{i_k}} c_k \gamma_{i_k}$$

and

$$\beta = \sum_{i_k > n, \gamma_{i_k} < \beta} c_k \gamma_{i_k}.$$

But this is impossible by Lemma 1.6.2. Therefore the sum $\alpha + \beta$, $\alpha < \beta$, $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ cannot be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$. This confirms (vi) and completes the proof of the proposition.

$\square$

We shall also need another system of positive roots associated to (the conjugacy class of) the Weyl group element s. In order to define it we need to recall the definition of a circular normal ordering of the root system Δ.

Let $\beta_1, \beta_2, \ldots, \beta_D$ be a normal ordering of a positive root system Δ_+. Then one can introduce the corresponding circular normal ordering of the root system Δ where the roots in Δ are located on a circle in the following way

$$\beta_2 \qquad\qquad\qquad\qquad \beta_D$$

$$\beta_1 \qquad\qquad\qquad\qquad -\beta_1$$

$$\text{-}\beta_D \qquad\qquad\qquad\qquad -\beta_2$$
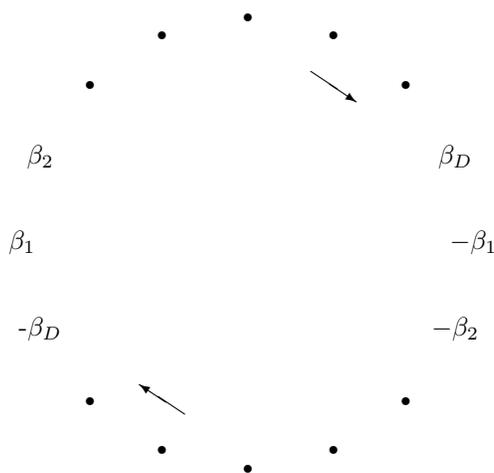
Fig. 4

Let $\alpha, \beta \in \Delta$. One says that the segment $[\alpha, \beta]$ of the circle is minimal if it does not contain the opposite roots $-\alpha$ and $-\beta$ and the root β follows after α on the circle above, the circle being oriented clockwise. In that case one also says that $\alpha < \beta$ in the sense of the circular normal ordering,

$$\alpha < \beta \Leftrightarrow \text{the segment } [\alpha, \beta] \text{ of the circle is minimal.} \tag{1.6.39}$$

Later we shall need the following property of minimal segments which is a direct consequence of Proposition 3.3 in [69].

Lemma 1.6.17. *Let $[\alpha, \beta]$ be a minimal segment in a circular normal ordering of a root system Δ. Then if $\alpha + \beta$ is a root we have*

$$\alpha < \alpha + \beta < \beta.$$

This lemma immediately implies the following property of minimal segments.

Lemma 1.6.18. *Let $G_{\mathbf{k}}$ be a semisimple connected algebraic group, $\mathfrak{g}_{\mathbf{k}}$ its Lie algebra, $\mathfrak{h}_{\mathbf{k}} \subset \mathfrak{g}_{\mathbf{k}}$ a Cartan subalgebra, Δ the root system of the pair $(\mathfrak{g}_{\mathbf{k}}, \mathfrak{h}_{\mathbf{k}})$. For any minimal segment $[\alpha, \beta] \subset \Delta$ with respect to any circular normal ordering of Δ, the linear subspace of $\mathfrak{g}_{\mathbf{k}}$ spanned by the root vectors corresponding to the roots from $[\alpha, \beta]$ is an algebraic Lie subalgebra $\mathfrak{n}_{[\alpha,\beta]}$ of $\mathfrak{g}_{\mathbf{k}}$. We denote the subgroup of $G_{\mathbf{k}}$ corresponding to this subalgebra by $N_{[\alpha,\beta]} \subset G_{\mathbf{k}}$.*

Note that any segment in a circular normal ordering of Δ of length equal to the number of positive roots is a system of positive roots.

Now consider the circular normal ordering of Δ corresponding to the system of positive roots Δ_+^s and to its normal ordering introduced in Proposition 1.6.6. The minimal segment which consists of the roots α satisfying $\gamma_1 \leq \alpha < -\gamma_1$ is a system of positive roots in Δ as its length is equal to the number of positive roots and it is closed under addition of roots by Lemma 1.6.17.

Definition 1.6.19. *Let $\Delta_+ = \{\alpha \in \Delta : \gamma_1 \leq \alpha < -\gamma_1\}$, where the inequalities are with respect to the circular normal ordering of Δ corresponding to the system of positive roots Δ_+^s and to its normal ordering (1.6.9). The system of positive roots Δ_+ equipped with the normal ordering induced by the circular normal ordering is called the normally ordered system of positive roots associated to the (conjugacy class of) the Weyl group element $s \in W$.*

Note that for the root system Δ_+ introduced in Definition 1.6.19 $\Delta_+^s \cap \Delta_+ = \Delta_+^s \setminus \Delta_{s^1}^-$, where $\Delta_{s^1}^- = \{\beta_1^1, \ldots, \beta_{t+\frac{p-n}{2}}^1\} = \Delta_+^s \cap \Delta_-$. Therefore

$$\Delta_+ = (\Delta_+^s \cap \Delta_+) \cup \{-\beta_1^1, \ldots, -\beta_{t+\frac{p-n}{2}}^1\} = (\Delta_+^s \cap \Delta_+) \cup (-\Delta_{s^1}^-).$$

By Lemma 1.6.3 (iv) this implies that $w_s \Delta_+ = \Delta_+^s$, where $w_s = s_{\beta_1^1} \ldots, s_{\beta_{t+\frac{p-n}{2}}^1}$, and if $\overline{w} = s_{i_1} \ldots s_{i_{t+\frac{p-n}{2}}} \ldots s_{i_D}$ is the reduced decomposition corresponding to normal ordering (1.6.9) then $w_s = s_{i_{t+\frac{p-n}{2}}} \ldots s_{i_1}$.

We have the following property of the length of s with respect to the sets of simple roots in Δ_+^s.

Proposition 1.6.20. *For all systems of positive roots Δ_+^s the lengths $l(s)$ of s with respect to the sets of simple roots in Δ_+^s are the same, and they are equal to the length of s with respect to the set of simple roots in any system of positive roots Δ_+ associated to s.*

Proof. The first statement is a consequence of the definition of Δ_+^s.

To prove the second assertion we recall that a root $\alpha \in \Delta_{i_k}$ belongs to the set $(\Delta_{i_k})_+ = (\Delta_{i_k})_+ \cap \Delta_+^s$ if and only if $h_{i_k}(\alpha) > 0$. Identifying $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$ with the help of the bilinear form one can deduce that $\alpha \in \Delta_{i_k}$ is in Δ_+^s if and only if its orthogonal projection onto $\mathfrak{h}_{i_k}$ is contained in the upper–half plane shown at Figure 3.

According to the definition of the set Δ_+,

$$\Delta_{i_k} \cap \Delta_+ = ((\Delta_{i_k})_+ \setminus \{\alpha \in (\Delta_{i_k})_+ \cap \Delta_{s^1}^s : \alpha < \gamma_1\}) \cup \{-\alpha : \alpha \in (\Delta_{i_k})_+ \cap \Delta_{s^1}^s, \alpha < \gamma_1\}, \qquad (1.6.40)$$

i.e $\Delta_{i_k} \cap \Delta_+$ is obtained from $(\Delta_{i_k})_+$ by removing some roots the orthogonal projections of which onto $\mathfrak{h}_{i_k}$ belong to the sector labeled $\Delta_{i_k}^1$ at Figure 3 and by adding the opposite negative roots the orthogonal projections of which onto $\mathfrak{h}_{i_k}$ belong to the sector labeled $s^1\Delta_{i_k}^1$ at Figure 3.

Recall that the involutions s^1 and s^2 act in $\mathfrak{h}_{i_k}$ as reflections with respect to the lines orthogonal to the vectors labeled by v_k^1 and v_k^2, respectively, at Figure 3, the angle between v_k^1 and v_k^2 being equal to $\pi - \theta_{i_k}/2$. Therefore the element s acts on $\mathfrak{h}_{i_k}$ by clockwise rotation with the angle $\theta_{i_k} = 2(\varphi_k + \psi_k)$, and hence the set $\Delta_s^s \cap \Delta_{i_k}$ consists of the roots the orthogonal projections of which onto $\mathfrak{h}_{i_k}$ belong to the union of the sectors labeled $s^2\Delta_{i_k}^1$ and $\Delta_{i_k}^2$ at Figure 3. Together with the description of the set $\Delta_{i_k} \cap \Delta_+$ given in (1.6.40) it implies that the number of the roots in the set $\Delta_s \cap \Delta_{i_k} = \{\alpha \in \Delta_{i_k} \cap \Delta_+ : s\alpha \in \Delta_-\}$, where $\Delta_s = \{\alpha \in \Delta_+ : s\alpha \in \Delta_-\}$, is equal to the number of roots in the set $\Delta_s^s \cap \Delta_{i_k}$. From this observation we deduce that the length $l(s)$ of s with respect to the system of simple roots in Δ_+^s is the same as the length of s with respect to the system of simple roots in Δ_+, as both of them are equal to the cardinality to the set $\bigcup_{k=0}^M \Delta_s^s \cap \Delta_{i_k}$ (disjoint union) which is the same as the cardinality of the set $\bigcup_{k=0}^M \Delta_s \cap \Delta_{i_k}$ (disjoint union).

$\square$

We shall also need a family of systems of positive roots in Δ related to the circular normal ordering associated to (1.6.9). According to Lemma 1.6.15 (see, in particular, formula (1.6.30)) for $j = 1, \ldots R$ the minimal segment

$$\Delta^j, \Delta^{j+1}, \ldots, -\Delta^{j-1} \qquad (1.6.41)$$

of the circular normal ordering associated to (1.6.9) is a system of positive roots in Δ as its length is equal to the number of positive roots and it is closed under addition of roots by Lemma 1.6.17. Denote this system of positive roots by Δ_+^j. Note that by this definition $\Delta_+^1 = \Delta_+^s$.

Lemma 1.6.21. *Let $\Gamma^k \subset \Delta_+^k$, $k = 1, \ldots, R$ be the set of simple roots. Then $\Gamma^k \cap \Delta_0$ is a set of simple roots in $(\Delta_0)_+ \subset \Delta_+^k$, so the set of roots $\Delta_+^k \cup \Delta_0$ is parabolic.*

Proof. We show first that for any $\alpha, \beta \in \Delta_+^k$ such that α or β is not an element of Δ_0 one has $\alpha + \beta \notin \mathfrak{h}_0$.

Indeed if $\alpha, \beta \in \Delta_+^k$ are such that α or β does not belong to Δ_0 then $\alpha \in \Delta_{i_p}$, $\beta \in \Delta_{i_q}$ for some i_p, i_q and $i_p > 0$ or $i_q > 0$. If $\alpha + \beta \in \mathfrak{h}_0$ then the orthogonal projections of $\alpha + \beta$ onto $\mathfrak{h}_{i_v}$ with any $i_v > 0$ must be equal to zero. The definition of the sets Δ_{i_k} implies now that $p = q$, $i_p = i_q > 0$ and either $\alpha \in \Delta_{i_p}^{(r)}$, $\beta \in -\Delta_{i_p}^{(r)}$ or $\alpha \in -\Delta_{i_p}^{(r)}$, $\beta \in \Delta_{i_p}^{(r)}$ for some ray r. But this is impossible as by the definition of Δ_+^k and by the definition of the sets Δ^j in (1.6.30) the set Δ_+^k only contains elements of one of the sets $\Delta_{i_p}^{(r)}$ xor $-\Delta_{i_p}^{(r)}$. Thus $\alpha + \beta \notin \mathfrak{h}_0$.

Now if $\Gamma^k = \{\alpha_1, \ldots, \alpha_l\}$ and $\Gamma^k \cap \Delta_0 = \{\alpha_1, \ldots, \alpha_u\}$ with $u \leq l$ then by the first part of the proof for $\alpha = \sum_{i=1}^l n_i\alpha_i \in \Delta_+^k$ with $n_i > 0$ for some $i > u$ one has $\alpha \notin \mathfrak{h}_0$. In particular, $\alpha \notin \Delta_0 \subset \mathfrak{h}_0$. Thus any element of $(\Delta_0)_+ \subset \Delta_+^k$ is a linear combination of the roots from the set $\Gamma^k \cap \Delta_0$ with non-negative integer coefficients, i.e. $\Gamma^k \cap \Delta_0$ is a set of sinmple roots in $(\Delta_0)_+$. This completes the proof.

$\square$

The following lemma will be useful for defining Zhelobenko type operators for q-W–algebras both in the classical and quantum case.

Lemma 1.6.22. *For $j = 1, \ldots, R-1$ the following statements are true.*

(i) $s(\Delta^j) \subset \Delta_+^{j+1}$.

(ii) $\Delta^j \cap \mathbb{N}s(\Delta_+^j \cup \Delta_0) = \{\emptyset\}$.

(iii) $\Delta^j \cap \mathbb{N}s(\Delta_s^s \cup \Delta_0) = \{\emptyset\}$.

Proof. (i) For j such that $\Delta^j \not\subset \Delta^s_s \cup \Delta_0$ one has by (1.6.31) $\Delta^j < s(\Delta^j) \subset \Delta^s_+$, and hence (i) holds in this case by the definition of Δ^{j+1}_+.

If j such that $\Delta^j \subset s^2\Delta^s_{s1}$ then

$$s(\Delta^j) \subset s^1 s^2 s^2 \Delta^s_{s1} = s^1 \Delta^s_{s1} = -\Delta^s_{s1}.$$

Observing that $\Delta^s_{s1} < s^2\Delta^s_{s1}$ by Proposition 1.6.6 (vii) we deduce again that (i) holds by the definition of Δ^{j+1}_+.

Finally, if $j = R - 1$ then $\Delta^{R-1} = \Delta^s_{s2}$, and

$$s(\Delta^j) = s^1 s^2 \Delta^s_{s2} = -s^1 \Delta^s_{s2} \subset -(\Delta^s_- \setminus \Delta_0) \subset -(\Delta^s_- \setminus \Delta_0) \cup (\Delta_0)_+ = \Delta^R_+.$$

This completes the proof of part (i).

(ii) First observe that by Lemma 1.6.21 the set of roots $\Delta^j_+ \cup \Delta_0$ is parabolic. Therefore to establish (ii) it suffices to show that $\Delta^j \cap s(\Delta^j_+ \cup \Delta_0) = \{\emptyset\}$.

Indeed, if $\alpha \in (\Delta^j_+ \cup \Delta_0) \cap \Delta^s_+$ is such that $s\alpha \in \Delta^s_+$ then $\alpha \in \Delta^k \not\subset \Delta^s_s$, $k \geq j$, so $s\Delta^k \subset \Delta^s_+$, and by (1.6.31) $s\alpha \in s\Delta^k > \Delta^k \geq \Delta^j$ for $k \neq R$, $s\alpha \in \Delta^R > \Delta^j$ for $k = R$, and hence (i) holds in this case.

If $\alpha \in \Delta^j_+ \cup \Delta_0$ is such that $s\alpha \in \Delta^s_-$ then (ii) obviously holds as $\Delta^j_+ \subset \Delta^s_+$.

If $\alpha \in (\Delta^j_+ \cup \Delta_0) \cap \Delta^s_-$ is such that $s\alpha \in \Delta^s_+$ then $\alpha \in -\Delta^s_s = -(\Delta^s_{s2} \cup s^2\Delta^s_{s1})$.

Assume that $s\alpha \in \Delta^j$. If $\alpha \in -s^2\Delta^s_{s1}$ then $s\alpha \in \Delta^s_{s1} = \Delta^1$, so $j = 1$ and $\Delta^j_+ = \Delta^s_+$. Thus we arrive at a contradiction as by the assumption $\alpha \in -s^2\Delta^s_{s1} \subset \Delta^s_-$.

If $\alpha \in -\Delta^s_{s2}$ then $\alpha \in \Delta^R_+$ by the definition of Δ^R_+, and $\alpha \notin \Delta^j_+$ for any $j < R$. Thus we again arrive at a contradiction as by the assumption $\alpha \in \Delta^j_+$ for $j < R$. This completes the proof.

(iii) is obvious as $s(\Delta^s_s \cup \Delta_0) \subset \Delta^s_- \cup \Delta_0$, the set of roots $\Delta^s_- \cup \Delta_0$ is parabolic, and $\Delta^j \subset \Delta^s_+ \setminus \Delta_0$.

$\square$

The relative positions of the systems of positive rots Δ^s_+, Δ_+ and of the minimal segments introduced in Proposition 1.6.6 are shown at the following picture where all the segments are placed on a circle according to the circular normal ordering of roots corresponding to normal ordering (1.6.9) of Δ^s_+.
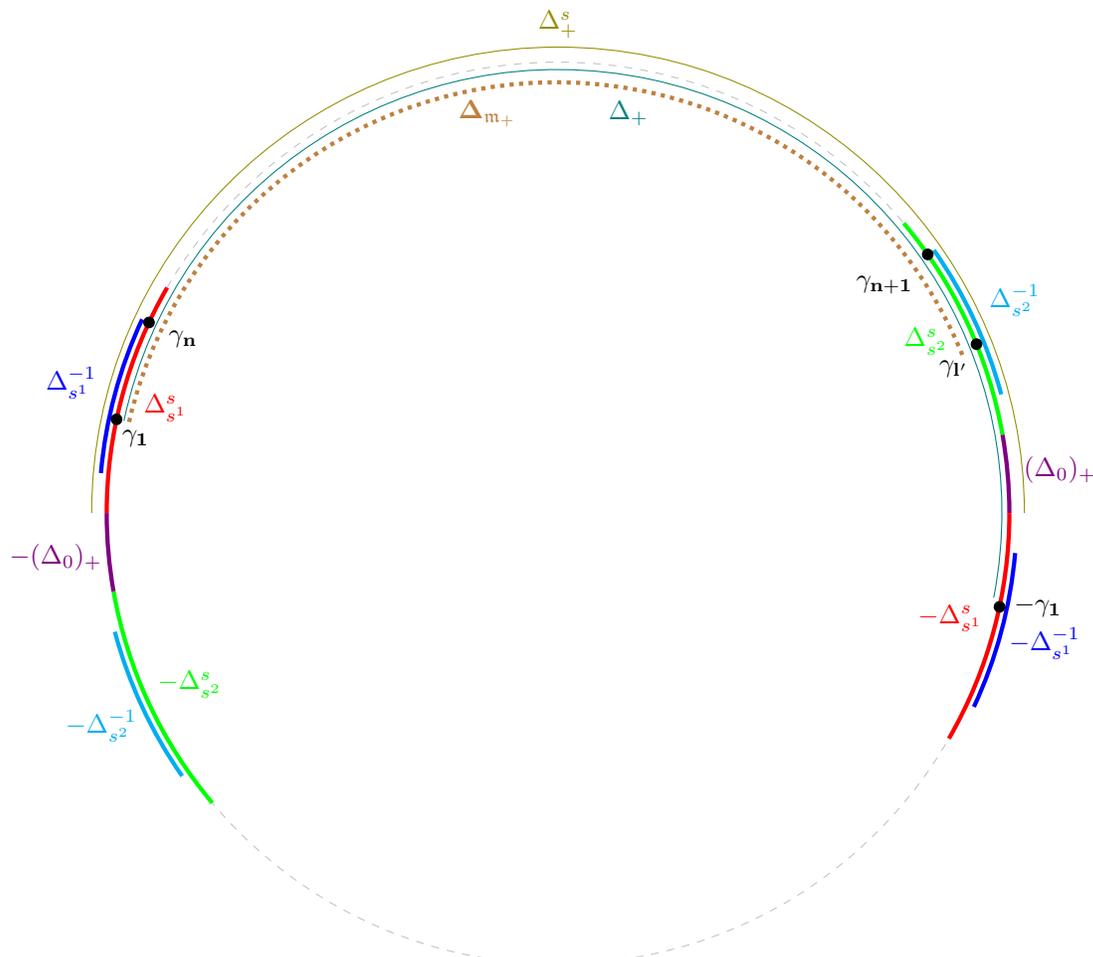
Fig. 5

The reader may find this picture useful in combination with Lemma 1.6.17 when adding roots or commuting roots vectors. This picture can be also useful for deriving some formulas containing q-commutators of quantum root vectors as explained in the next chapter.

1.7 Bibliographic comments

A uniform classification of conjugacy classes of Weyl group elements from which one can obtain presentation (1.2.1) was suggested in [18].

The definition of systems of positive roots Δ_+^s associated to (conjugacy classes of) Weyl group elements was suggested in [113]. It is based on a deep generalization of the results by Coxeter and Steinberg on the properties of the Coxeter elements. In our notation this corresponds to the case when $\gamma_1, \ldots, \gamma_{l'}$ is a set of simple roots in Δ, so that according to (1.2.1) s is a product of simple reflections, i.e. a Coxeter element. In this case there is a unique plane in $\mathfrak{h}_\mathbb{R}$, called a Coxeter plane, on which s acts by rotation by the angle $2\pi/h$, where h is the Coxeter number of $\mathfrak{g}$. This plane was introduced by Coxeter in book [21], and the pictures of root systems of Lie algebras of high ranks which one can find in many textbooks are obtained using orthogonal projections of roots onto these planes. The key observation is that all these projections are non–zero. Coxeter originally applied the above mentioned procedure to construct regular polytopes.

Later in paper [126] Steinberg proved interesting properties of Coxeter elements using the properties of the action of Coxeter elements on Coxeter planes.

The construction of the spectral decomposition for Weyl group elements in Proposition 1.2.1 suggested in [118] is a generalization of similar results on the properties of the Coxeter plane which can be found in [17], Section 10.4.

The slices $\Sigma_{\mathbf{k},s}$ introduced in [113] in the case $\mathbf{k} = \mathbb{C}$ are generalizations of the Steinberg cross-sections to the set of conjugacy classes of regular elements in $G_\mathbf{k}$ suggested in [127]. $\Sigma_{\mathbf{k},s}$ reduces to a Steinberg cross-section when

$\gamma_1, \ldots, \gamma_{l'}$ is a set of simple roots, i.e. when s is a Coxeter element. In this case isomorphism (1.3.7) is stated in [127] without proof. The first proof of this result appeared in [107]. However, that proof is not applicable for the root system of type E_6. The proof of isomorphism (1.3.7) given in Proposition 1.3.4 is a refined version of the proof of this result presented in [113].

Another construction of the slices $\Sigma_{\mathbf{k},s}$ in the case when s are elliptic can be found in [52].

A more general approach to the definition of the transverse slices to conjugacy classes in algebraic groups generalizing the definition given in this chapter and the results of [52] was given in [91], and the relevant Weyl combinatorics was developed in [90].

The closedness of the varieties $NZsN$ was justified in [116], Proposition 6.2.

In book [121] Slodowy proved Brieskorn's conjectures announced in [11] on the realization of simple singularities using the adjoint quotient of complex semisimple Lie algebras. Although a significant part of Slodowy's book is devoted to the study of the conjugation quotient for semisimple algebraic groups and to constructing some its resolutions, he ended up with a Lie algebra version of the construction of simple singularities and introduced transverse slices for the adjoint action for this purpose. These slices are called now the Slodowy slices. The slices Σ_s can be regarded as algebraic group analogues of the Slodowy slices.

The Lusztig partition was introduced in [83]. Its definition is related to the study of the properties of intersections of conjugacy classes in $G_{\mathbf{k}}$ with Bruhat cells established in [85, 86] where the map Φ^W from the set of Weyl group conjugacy classes to the set of unipotent classes and its one sided inverse Ψ^W, which we use in Section 1.4, are defined using these properties. These properties are also related to the generalized Springer correspondence. We only briefly discussed the relevant results in this book.

The study of intersections of conjugacy classes in $G_{\mathbf{k}}$ with Bruhat cells was initiated in [127]. Some results on these intersections were obtained in [35], and another map from nilpotent orbits in a complex semisimple Lie algebra to conjugacy classes in the Weyl group was defined in [65]. In [85] it is mentioned that this map is likely to coincide with the map Ψ^W introduced in [85]. The results of [131] imply that the restriction of Ψ^W to $\underline{\mathcal{N}}(G_1)$ coincides with the Kazhdan-Lusztig map defined in [65].

The main result of Theorem 1.5.2 on the dimensions of the slices $\Sigma_{\mathbf{k},s}$ is an experimental observation made in [118], Theorem 5.2. Other results of Section 1.5 can also be found in [118].

Note that the slices $\Sigma_{\mathbf{k},s}$ listed in Appendix 2 are slightly different from those from Appendix B to [118]. The corresponding slices in both sets have the same dimensions. But in this book the roots $\gamma_1, \ldots, \gamma_{l'}$ in the tables in Appendix 2 are chosen in such a way that the corresponding root systems Δ_+^s satisfy condition (1.6.8). The algorithm for constructing the slices Σ_s listed in the tables in Appendix B to [118] was modified to fulfill this condition. The description of the original algorithm can be found in [118].

The ordering of the s–invariant planes in $\mathfrak{h}_{\mathbb{R}}$ according to the angles of rotations by which s acts in the planes as in Theorem 1.5.2 was used in [53] to prove properties of minimal length elements in finite Coxeter groups.

Normal orderings of positive root systems of the form Δ_+^s described in Proposition 1.6.6 were firstly introduced in [114] where one can also find the construction of normal orderings of positive root systems compatible with Weyl group involutions from Appendix 1. Later the original definition was refined in [119]. Proposition 1.6.6 is a modified version of Proposition 5.1 in [114] and Proposition 2.2 in [119].

Circular orderings of root systems were defined in [68] to describe commutation relations between quantum group analogues of root vectors. In [116] this construction was used to modify the positive root systems Δ_+^s in order to construct positive root systems associated to (conjugacy classes of) Weyl group elements which appear in the end of Section 1.6.

Chapter 2

Quantum groups

In this chapter we recall some definitions and results on quantum groups required for the study of q-W–algebras. Besides the standard definitions and results related to quantum groups we shall need some rather non–standard realizations of the Drinfeld–Jimbo quantum group in terms of which q-W–algebras are defined. These realizations are related to the definition of the algebraic group analogues of the Slodowy slices in the previous section. We shall consider the Drinfeld–Jimbo quantum group $U_h(\mathfrak{g})$ defined over the ring of formal power series $\mathbb{C}[[h]]$, where h is an indeterminate, and some its specializations defined over smaller rings.

2.1 The definition of quantum groups

In this section we remind the definition of the standard Drinfeld-Jimbo quantum group $U_h(\mathfrak{g})$. We mainly follow the notation of [20].

Let V be a $\mathbb{C}[[h]]$–module equipped with the h–adic topology. This topology is characterized by requiring that $\{h^n V \mid n \geq 0\}$ is a base of the neighborhoods of 0 in V, and that translations in V are continuous. In this book all $\mathbb{C}[[h]]$–modules are supposed to be complete with respect to this topology.

A topological Hopf algebra over $\mathbb{C}[[h]]$ is a complete $\mathbb{C}[[h]]$–module equipped with a structure of $\mathbb{C}[[h]]$–Hopf algebra, the algebraic tensor products entering the axioms of the Hopf algebra are replaced by their completions in the h–adic topology.

The standard quantum group $U_h(\mathfrak{g})$ associated to a complex finite-dimensional semisimple Lie algebra $\mathfrak{g}$ is a topological Hopf algebra over $\mathbb{C}[[h]]$ topologically generated by elements H_i, X_i^+, X_i^-, $i = 1, \ldots, l$, subject to the following defining relations:

$$[H_i, H_j] = 0, \quad [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, \quad X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} (X_i^\pm)^{1-a_{ij}-r} X_j^\pm (X_i^\pm)^r = 0, \ i \neq j, \tag{2.1.1}$$

where

$$K_i = e^{d_i h H_i}, \quad e^h = q, \quad q_i = q^{d_i} = e^{d_i h},$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![n-m]_q!}, \ [n]_q! = [n]_q \ldots [1]_q, \ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

with comultiplication defined by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta_h(X_i^+) = X_i^+ \otimes K_i^{-1} + 1 \otimes X_i^+, \quad \Delta_h(X_i^-) = X_i^- \otimes 1 + K_i \otimes X_i^-,$$

antipode defined by

$$S_h(H_i) = -H_i, \quad S_h(X_i^+) = -X_i^+ K_i, \quad S_h(X_i^-) = -K_i^{-1} X_i^-,$$

and counit defined by

$$\varepsilon_h(H_i) = \varepsilon_h(X_i^\pm) = 0.$$

We shall also use the weight–type generators

$$Y_i = \sum_{j=1}^{l} d_i a_{ij}^{-1} H_j,$$

where a_{ij}^{-1} are the entries of the matrix inverse to the Cartan matrix a_{ij}.

Let $L_i^{\pm 1} = e^{\pm hY_i}$. These elements commute with the quantum simple root vectors $X_i^{\pm}$ as follows:

$$L_i X_j^{\pm} L_i^{-1} = q_i^{\pm \delta_{ij}} X_j^{\pm}. \tag{2.1.2}$$

We also obviously have

$$L_i L_j = L_j L_i. \tag{2.1.3}$$

The Hopf algebra $U_h(\mathfrak{g})$ is a quantization of the standard bialgebra structure on $\mathfrak{g}$ in the sense that $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(\mathfrak{g})$, $\Delta_h = \Delta \pmod h$, where Δ is the standard comultiplication on $U(\mathfrak{g})$, and

$$\frac{\Delta_h - \Delta_h^{opp}}{h} \pmod h = -\delta.$$

Here $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the standard cocycle on $\mathfrak{g}$, and $\Delta_h^{opp} = \sigma \Delta_h$, σ is the permutation in $U_h(\mathfrak{g})^{\otimes 2}$, $\sigma(x \otimes y) = y \otimes x$. Recall that

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) 2r_{\pm}, \quad r_{\pm} \in \mathfrak{g} \otimes \mathfrak{g},$$

$$r_{\pm} = \pm \frac{1}{2} \sum_{i=1}^{l} Y_i \otimes H_i \pm \sum_{\beta \in \Delta_+} (X_\beta, X_{-\beta})^{-1} X_{\pm \beta} \otimes X_{\mp \beta}. \tag{2.1.4}$$

Here $X_{\pm \beta} \in \mathfrak{g}_{\pm \beta}$ are non–zero root vectors of $\mathfrak{g}$. The element $r_{\pm} \in \mathfrak{g} \otimes \mathfrak{g}$ is called a classical r–matrix.

2.2 The braid group action

One can define a quantum group analogue of the braid group action on $\mathfrak{g}$. The material covered in this section can be found, e.g., in Section 8.1 of [20]. Let m_{ij}, $i \neq j$ be equal to $2, 3, 4, 6$ if $a_{ij}a_{ji}$ is equal to $0, 1, 2, 3$, respectively. The braid group $\mathcal{B}_\mathfrak{g}$ associated to $\mathfrak{g}$ has generators T_i, $i = 1, \ldots, l$, and defining relations

$$T_i T_j T_i T_j \ldots = T_j T_i T_j T_i \ldots$$

for all $i \neq j$, where there are m_{ij} T's on each side of the equation.

Recall that if $X_{\pm \alpha_i}$ are non–zero simple root vectors of $\mathfrak{g}$ then one can introduce an action of the braid group $\mathcal{B}_\mathfrak{g}$ by algebra automorphisms of $\mathfrak{g}$ defined on the standard generators as follows:

$$T_i(X_{\pm \alpha_i}) = -X_{\mp \alpha_i}, \ T_i(H_j) = H_j - a_{ji}H_i,$$

$$T_i(X_{\alpha_j}) = \frac{1}{(-a_{ij})!} \mathrm{ad}_{X_{\alpha_i}}^{-a_{ij}} X_{\alpha_j}, \ i \neq j, \tag{2.2.1}$$

$$T_i(X_{-\alpha_j}) = \frac{(-1)^{a_{ij}}}{(-a_{ij})!} \mathrm{ad}_{X_{-\alpha_i}}^{-a_{ij}} X_{-\alpha_j}, \ i \neq j.$$

Similarly, $\mathcal{B}_\mathfrak{g}$ acts by algebra automorphisms of $U_h(\mathfrak{g})$ as follows:

$$T_i(X_i^+) = -X_i^- e^{hd_i H_i}, \ T_i(X_i^-) = -e^{-hd_i H_i} X_i^+, \ T_i(H_j) = H_j - a_{ji}H_i,$$

$$T_i(X_j^+) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} (X_i^+)^{(-a_{ij}-r)} X_j^+ (X_i^+)^{(r)}, \ i \neq j, \tag{2.2.2}$$

$$T_i(X_j^-) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{r} (X_i^-)^{(r)} X_j^- (X_i^-)^{(-a_{ij}-r)}, \ i \neq j,$$

where

$$(X_i^+)^{(r)} = \frac{(X_i^+)^r}{[r]_{q_i}!}, \ (X_i^-)^{(r)} = \frac{(X_i^-)^r}{[r]_{q_i}!}, \ r \geq 0, \ i = 1, \ldots, l.$$

Recall that action (2.2.1) of the generators T_i is induced by the adjoint action of certain representatives of the Weyl group elements s_i in G. Similarly, action (2.2.2) is induced by conjugation by certain elements of a completion of $U_h(\mathfrak{g})$.

To define these elements consider the restricted dual $\mathbb{C}_h[G]$ of $U_h(\mathfrak{g})$, i.e. the algebra topologically generated by the matrix elements of finite rank representations of $U_h(\mathfrak{g})$, with the multiplication induced by the comultiplication on $U_h(\mathfrak{g})$. $\mathbb{C}_h[G]$ is naturally a Hopf algebra with the comultiplication induced by the multiplication on $U_h(\mathfrak{g})$ (see e.g. [20], p. 113).

The topological dual $\mathbb{C}_h[G]^*$ of the Hopf algebra $\mathbb{C}_h[G]$ is just an algebra, and there is a natural embedding of algebras $U_h(\mathfrak{g}) \hookrightarrow \mathbb{C}_h[G]^*$.

Now define a q-exponential by

$$\exp'_q(x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{x^k}{[k]_q!}. \tag{2.2.3}$$

Then the automorphism T_i in (2.2.2) is given by the conjugation of elements of $U_h(\mathfrak{g}) \hookrightarrow \mathbb{C}_h[G]^*$ in $\mathbb{C}_h[G]^*$ by the invertible element $T_i \in \mathbb{C}_h[G]^*$ (see [103])

$$T_i = \exp'_{q_i^{-1}}(-q_i^{-1}X_i^- K_i)\exp'_{q_i^{-1}}(X_i^+)\exp'_{q_i^{-1}}(-q_iX_i^- K_i^{-1})q_i^{\frac{H_i(H_i+1)}{2}} = \tag{2.2.4}$$

$$= \exp'_{q_i^{-1}}(q_i^{-1}X_i^+ K_i^{-1})\exp'_{q_i^{-1}}(-X_i^-)\exp'_{q_i^{-1}}(q_iX_i^+ K_i)q_i^{\frac{H_i(H_i+1)}{2}}.$$

Note that the right hand side of the previous formula indeed defines an endomorphism for every finite rank representation of $U_h(\mathfrak{g})$ as the elements $X_i^{\pm}$ act nilpotently on every such representation, and the action of the elements H_i is semisimple.

The inverse of T_i in (2.2.4) can be found using the identity

$$\exp'_q(x)\exp'_{q^{-1}}(-x) = 1$$

which implies

$$T_i^{-1} = q_i^{-\frac{H_i(H_i+1)}{2}}\exp'_{q_i}(q_iX_i^- K_i^{-1})\exp'_{q_i}(-X_i^+)\exp'_{q_i}(q_i^{-1}X_i^- K_i) = \tag{2.2.5}$$

$$= q_i^{-\frac{H_i(H_i+1)}{2}}\exp'_{q_i}(-q_iX_i^+ K_i)\exp'_{q_i}(X_i^-)\exp'_{q_i}(-q_i^{-1}X_i^+ K_i^{-1}).$$

From formula (2.2.5) we obtain the following relations in $\mathbb{C}_h[G]^*$

$$\exp'_{q_i}(-X_i^+) = \exp'_{q_i^{-1}}(-q_iX_i^- K_i^{-1})q_i^{\frac{H_i(H_i+1)}{2}}T_i^{-1}\exp'_{q_i^{-1}}(-q_i^{-1}X_i^- K_i) = \tag{2.2.6}$$

$$= \exp'_{q_i^{-1}}(-q_iX_i^- K_i^{-1})q_i^{\frac{H_i(H_i+1)}{2}}\exp'_{q_i^{-1}}(q_i^{-1}X_i^+)T_i^{-1}.$$

Let

$$\exp_q(x) = exp'_q(qx) = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!}. \tag{2.2.7}$$

The multiplication in $\mathbb{C}_h[G]$ induces a map of $\mathbb{C}[[h]]$–modules $\Delta_h : \mathbb{C}_h[G]^* \to (\mathbb{C}_h[G] \otimes \mathbb{C}_h[G])^*$, $(\Delta_h(f))(x \otimes y) = f(xy)$, $f \in \mathbb{C}_h[G]^*$, $x, y \in \mathbb{C}_h[G]$, the restriction of which to $U_h(\mathfrak{g})$ coincides with the comultiplication on $U_h(\mathfrak{g})$. With respect to this map we have (see [20], Proposition 8.2.6)

$$\Delta_h(T_i) = \theta_i T_i \otimes T_i = T_i \otimes T_i \bar{\theta}_i, \tag{2.2.8}$$

$$\theta_i = \exp_{q_i}[(1 - q_i^{-2})X_i^+ \otimes X_i^-], \bar{\theta}_i = \exp_{q_i}[(1 - q_i^{-2})K_i^{-1}X_i^- \otimes X_i^+ K_i],$$

$$\Delta_h(T_i^{-1}) = \bar{\theta}_i^{-1}T_i^{-1} \otimes T_i^{-1} = T_i^{-1} \otimes T_i^{-1}\theta_i^{-1}, \tag{2.2.9}$$

$$\theta_i^{-1} = \exp_{q_i^{-1}}[(1 - q_i^2)X_i^+ \otimes X_i^-], \bar{\theta}_i^{-1} = \exp_{q_i^{-1}}[(1 - q_i^2)K_i^{-1}X_i^- \otimes X_i^+ K_i], \tag{2.2.10}$$

where the right hand sides of these identities are well–defined automorphisms of tensor products of finite rank representations of $U_h(\mathfrak{g})$, so that they can be evaluated on products in $\mathbb{C}_h[G]$ of matrix elements of such representations.

Denote $q_\alpha = q^{d_i}$ if the positive root α is Weyl group conjugate to the simple root α_i. By Proposition 8.1.3 in [20], for a reduced decomposition $w = s_{i_1} \ldots s_{i_k}$, $T_w = T_{i_1} \ldots T_{i_k}$ only depends on w and (2.2.8) implies

$$\Delta_h(T_w) = \prod_{p=1}^{k} \theta_{\beta_p} T_w \otimes T_w = T_w \otimes T_w \prod_{p=1}^{k} \overline{\theta}_{\beta'_p}, \tag{2.2.11}$$

where in the products θ_{β_p} (resp. $\overline{\theta}_{\beta'_p}$) appears on the left from θ_{β_q} (resp. $\overline{\theta}_{\beta'_q}$) if $p < q$, and for $p = 1, \ldots, k$

$$\beta_p = s_{i_1} \ldots s_{i_{p-1}} \alpha_{i_p}, \; \beta'_p = s_{i_k} \ldots s_{i_{p+1}} \alpha_{i_p},$$

$$X^{\pm}_{\beta_p} = T_{i_1} \ldots T_{i_{p-1}} X^{\pm}_{i_p}, \; \overline{X}^{\pm}_{\beta'_p} = T_{i_k}^{-1} \ldots T_{i_{p+1}}^{-1} X^{\pm}_{i_p}, \; K_{\beta'_p} = T_{i_k}^{-1} \ldots T_{i_{p+1}}^{-1} K_{i_p},$$

$$\theta_{\beta_p} = \exp_{q_{\beta_p}}[(1 - q_{\beta_p}^{-2}) X^{+}_{\beta_p} \otimes X^{-}_{\beta_p}], \; \overline{\theta}_{\beta'_p} = \exp_{q_{\beta'_p}}[(1 - q_{\beta'_p}^{-2}) K_{\beta'_p}^{-1} \overline{X}^{-}_{\beta'_p} \otimes \overline{X}^{+}_{\beta'_p} K_{\beta'_p}].$$

Note that for a reduced decomposition $w = s_{i_1} \ldots s_{i_k}$ one has $T_{i_1}^{-1} \ldots T_{i_k}^{-1} = (T_{i_k} \ldots T_{i_1})^{-1} = T_{w^{-1}}^{-1}$, and $T_{w^{-1}}$ only depends on w. Therefore $\overline{T}_w = T_{i_1}^{-1} \ldots T_{i_k}^{-1} = T_{w^{-1}}^{-1}$ only depends on w and (2.2.9) yields

$$\Delta_h(\overline{T}_w) = \prod_{p=1}^{k} \overline{\theta}'_{\beta_p} \overline{T}_w \otimes \overline{T}_w = \overline{T}_w \otimes \overline{T}_w \prod_{p=1}^{k} \theta'_{\beta'_p}, \tag{2.2.12}$$

where in the products $\overline{\theta}'_{\beta_p}$ (resp. $\theta'_{\beta'_p}$) appears on the left from $\overline{\theta}'_{\beta_q}$ (resp. $\theta'_{\beta'_q}$) if $p < q$, and for $p = 1, \ldots, k$

$$\overline{X}^{\pm}_{\beta_p} = T_{i_1}^{-1} \ldots T_{i_{p-1}}^{-1} X^{\pm}_{i_p}, \; X^{\pm}_{\beta'_p}{}' = T_{i_k} \ldots T_{i_{p+1}} X^{\pm}_{i_p}, \; \overline{K}_{\beta_p} = T_{i_1}^{-1} \ldots T_{i_{p-1}}^{-1} K_{i_p},$$

$$\theta'_{\beta'_p} = \exp_{q_{\beta'_p}^{-1}}[(1 - q_{\beta'_p}^{2}) X^{+}_{\beta'_p}{}' \otimes X^{-}_{\beta'_p}{}'], \; \overline{\theta}'_{\beta_p} = \exp_{q_{\beta_p}^{-1}}[(1 - q_{\beta_p}^{2}) \overline{K}_{\beta_p}^{-1} \overline{X}^{-}_{\beta_p} \otimes \overline{X}^{+}_{\beta_p} \overline{K}_{\beta_p}].$$

If $w\alpha_i = \alpha_j$ for some i and j then by Proposition 8.1.6. in [20]

$$T_w X^{\pm}_i = X^{\pm}_j, \; \overline{T}_w X^{\pm}_i = X^{\pm}_j. \tag{2.2.13}$$

2.3 Quantum root vectors

In this section we recall the construction of analogues of root vectors for $U_h(\mathfrak{g})$ in terms of the braid group action on $U_h(\mathfrak{g})$. Recall that for any reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$ of the longest element $\overline{w}$ of the Weyl group W of $\mathfrak{g}$ the ordering

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \ldots, \beta_D = s_{i_1} \ldots s_{i_{D-1}} \alpha_{i_D}$$

is a normal ordering in Δ_+, and there is a one–to–one correspondence between normal orderings of Δ_+ and reduced decompositions of $\overline{w}$.

Fix a reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$ of $\overline{w}$ and define the corresponding quantum root vectors in $U_h(\mathfrak{g})$ by

$$X^{\pm}_{\beta_k} = T_{i_1} \ldots T_{i_{k-1}} X^{\pm}_{i_k}. \tag{2.3.1}$$

We also define

$$K_{\beta_k} = T_{i_1} \ldots T_{i_{k-1}} K_{i_k}.$$

Note that one can construct root vectors in the Lie algebra $\mathfrak{g}$ in a similar way. Namely, the root vectors $X_{\pm\beta_k} \in \mathfrak{g}_{\pm\beta_k}$ of $\mathfrak{g}$ can be defined by

$$X_{\pm\beta_k} = T_{i_1} \ldots T_{i_{k-1}} X_{\pm\alpha_{i_k}}, \tag{2.3.2}$$

where $X_{\pm\alpha_{i_k}}$ are as in (2.2.1).

The root vectors $X_\beta^\pm$ satisfy the following relations:

$$X_\alpha^\pm X_\beta^\pm - q^{(\alpha,\beta)} X_\beta^\pm X_\alpha^\pm = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C(k_1,\ldots,k_n)(X_{\delta_1}^\pm)^{k_1}(X_{\delta_2}^\pm)^{k_2}\ldots(X_{\delta_n}^\pm)^{k_n} =$$

$$= \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C'(k_1,\ldots,k_n)(X_{\delta_1}^\pm)^{(k_1)}(X_{\delta_2}^\pm)^{(k_2)}\ldots(X_{\delta_n}^\pm)^{(k_n)}, \quad \alpha<\beta, \tag{2.3.3}$$

where for $\alpha \in \Delta_+$ we put $(X_\alpha^\pm)^{(k)} = \frac{(X_\alpha^\pm)^k}{[k]_{q_\alpha}!}$, $k \geq 0$, $q_\alpha = q^{d_i}$ if the positive root α is Weyl group conjugate to the simple root α_i, $C'(k_1,\ldots,k_n) \in \mathbb{C}[q,q^{-1}]$, $C(k_1,\ldots,k_n) \in \mathcal{P}$, and $\mathcal{P}$ is the algebra defined by

$$\mathcal{P} = \begin{cases} \mathbb{C}[q,q^{-1}] & \text{if } \mathfrak{g} \text{ is simply-laced} \\ \mathbb{C}[q,q^{-1},\frac{1}{[2]_q}] & \text{if } \mathfrak{g} \text{ is of type } B_l, C_l \text{ or } F_4 \\ \mathbb{C}[q,q^{-1},\frac{1}{[2]_q},\frac{1}{[3]_q}] & \text{if } \mathfrak{g} \text{ is of type } G_2 \end{cases} . \tag{2.3.4}$$

The fact that the coefficients $C(k_1,\ldots,k_n)$ in the right hand sides of formulas (2.3.3) belong to the algebra $\mathcal{P}$ was noted in [34], Lemma 1.1.1.

If $X = c_1 H_1 + \ldots + c_l H_l$ for some $c_1,\ldots,c_l \in \mathbb{R}$ then from commutation relations (2.1.1) and from the fact that the action of the braid group on the elements H_i coincides with the action of the Weyl group on the corresponding simple root generators of $\mathfrak{h}$ we obtain

$$e^{hX} X_\alpha^\pm e^{-hX} = q^{\pm\alpha(X)} X_\alpha^\pm, \tag{2.3.5}$$

where in the expression $\alpha(X)$ X is regarded as an element of $\mathfrak{h}$ under the natural identification of the elements $H_1,\ldots,H_l$ with the simple root generators of $\mathfrak{h}$.

In particular, (2.3.5) implies that for any $k \in \mathbb{N}$

$$(X_i^\pm K_i)^k = q_i^{\pm k(k-1)}(X_i^\pm)^k K_i^k. \tag{2.3.6}$$

Note that by construction

$$X_\beta^+ \pmod h = X_\beta \in \mathfrak{g}_\beta,$$
$$X_\beta^- \pmod h = X_{-\beta} \in \mathfrak{g}_{-\beta} \tag{2.3.7}$$

are root vectors of $\mathfrak{g}$.

Define an algebra automorphism τ of $U_h(\mathfrak{g})$ by

$$\tau(X_i^\pm) = X_i^\mp, \tau(H_i) = -H_i, \tau(h) = -h.$$

It satisfies the relations $T_i^{-1} = \tau T_i \tau$ and hence for any $\alpha \in \Delta_+$

$$\tau(X_\alpha^\pm) = \overline{X}_\alpha^\mp,$$

where

$$\overline{X}_{\beta_k}^\pm = T_{i_1}^{-1}\ldots T_{i_{k-1}}^{-1} X_{i_k}^\pm.$$

By applying τ to relations (2.3.3) one can obtain the following relations for the root vectors $\overline{X}_\beta^\pm$

$$\overline{X}_\alpha^\pm \overline{X}_\beta^\pm - q^{-(\alpha,\beta)} \overline{X}_\beta^\pm \overline{X}_\alpha^\pm = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} D(k_1,\ldots,k_n)(\overline{X}_{\delta_1}^\pm)^{k_1}(\overline{X}_{\delta_2}^\pm)^{k_2}\ldots(\overline{X}_{\delta_n}^\pm)^{k_n} =$$

$$= \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} D'(k_1,\ldots,k_n)(\overline{X}_{\delta_1}^\pm)^{(k_1)}(\overline{X}_{\delta_2}^\pm)^{(k_2)}\ldots(\overline{X}_{\delta_n}^\pm)^{(k_n)}, \quad \alpha<\beta, \tag{2.3.8}$$

where for $\alpha \in \Delta_+$ we put $(\overline{X}_\alpha^\pm)^{(k)} = \frac{(\overline{X}_\alpha^\pm)^k}{[k]_{q_\alpha}!}$, $k \geq 0$, $D(k_1,\ldots,k_n) \in \mathcal{P}$, $D'(k_1,\ldots,k_n) \in \mathbb{C}[q,q^{-1}]$.

One can also obtain commutation relations between positive and negative root vectors. These relations are known in some form. For completeness we give a proof of them using (2.3.3) and (2.3.8) only.

Lemma 2.3.1. *Let $[-\beta, \alpha]$, $\alpha, \beta \in \Delta_+$ be a minimal segment with respect to the circular normal ordering of Δ corresponding to a normal ordering $\beta_1, \ldots, \beta_D$ of Δ_+. Then*

$$X_\alpha^+ X_\beta^- - X_\beta^- X_\alpha^+ = \sum_{-\beta < \delta_1 < \ldots < \delta_n < \alpha} \bar{C}(k_1, \ldots, k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2} \ldots (X_{\delta_n})^{k_n} =$$

$$= \sum_{-\beta < \delta_1 < \ldots < \delta_n < \alpha} \bar{C}'(k_1, \ldots, k_n)(X_{\delta_1})^{(k_1)}(X_{\delta_2})^{(k_2)} \ldots (X_{\delta_n})^{(k_n)}, \qquad (2.3.9)$$

where the inequalities for roots in the sum are with respect to the circular normal ordering in Δ corresponding to the normal ordering $\beta_1, \ldots, \beta_D$ of Δ_+, $X_\delta = X_\delta^+$ for $\delta \in \Delta_+$ and $X_\delta = X_\delta^-$ for $\delta \in \Delta_-$, $\bar{C}'(k_1, \ldots, k_n) \in U_q(H)$, $U_q(H)$ is the $\mathbb{C}[q, q^{-1}]$–subalgebra of $U_h(\mathfrak{g})$ generated by $K_i^{\pm 1}$, $i = 1, \ldots, l$, $\bar{C}(k_1, \ldots, k_n) \in \mathcal{P}'$, where $\mathcal{P}'$ is the $\mathcal{P}$–algebra generated by $U_q(H)$.

Also

$$\overline{X}_\alpha^+ \overline{X}_\beta^- - \overline{X}_\beta^- \overline{X}_\alpha^+ = \sum_{-\beta < \delta_1 < \ldots < \delta_n < \alpha} \bar{D}(k_1, \ldots, k_n)(\overline{X}_{\delta_1})^{k_1}(\overline{X}_{\delta_2})^{k_2} \ldots (\overline{X}_{\delta_n})^{k_n} =$$

$$= \sum_{-\beta < \delta_1 < \ldots < \delta_n < \alpha} \bar{D}'(k_1, \ldots, k_n)(\overline{X}_{\delta_1})^{(k_1)}(\overline{X}_{\delta_2})^{(k_2)} \ldots (\overline{X}_{\delta_n})^{(k_n)}, \ \alpha < \beta, \qquad (2.3.10)$$

where the inequalities for roots in the sum are with respect to the circular normal ordering in Δ corresponding to the normal ordering $\beta_1, \ldots, \beta_D$ of Δ_+, $\overline{X}_\delta = \overline{X}_\delta^+$ for $\delta \in \Delta_+$ and $\overline{X}_\delta = \overline{X}_\delta^-$ for $\delta \in \Delta_-$, $\bar{D}'(k_1, \ldots, k_n) \in U_q(H)$, $\bar{D}(k_1, \ldots, k_n) \in \mathcal{P}'$.

Proof. The proof is by induction over the length of the positive part of the segment $[-\beta, \alpha]$, i.e. over the cardinality of the set $[-\beta, \alpha] \cap \Delta_+$. We shall consider the first identity in the case when $\alpha < \beta$. The others are proved in a similar way.

Let $\overline{w} = s_{i_1} \ldots s_{i_D}$ be the reduced decomposition of the longest element of the Weyl group corresponding to the normal ordering $\beta_1, \ldots, \beta_D$ of Δ_+.

First assume that $\alpha = \beta_1 = \alpha_{i_1}$. Let $\beta = \beta_m = s_{i_1} \ldots s_{i_{m-1}} \alpha_{i_m}$. Then $s_{i_1}^{-1} \beta_2, \ldots, s_{i_1}^{-1} \beta_D, \alpha_{i_1}$ is another normal ordering of Δ_+, and by (2.3.3) for this normal ordering

$$T_{i_1}^{-1}(X_\alpha^+ X_\beta^- - X_\beta^- X_\alpha^+) = T_{i_1}^{-1}(X_{i_1}^+ X_{\beta_m}^- - X_{\beta_m}^- X_{i_1}^+) =$$

$$= K_{i_1}^{-1}(-X_{i_1}^- X_{s_{i_1}^{-1}\beta_m}^- + q^{-(\alpha_{i_1}, s_{i_1}^{-1}\beta_m)} X_{s_{i_1}^{-1}\beta_m}^- X_{i_1}^-) =$$

$$= \sum_{-s_{i_1}^{-1}\beta_m < \delta_1 < \ldots < \delta_n < -\alpha_{i_1}} K_{i_1}^{-1} C(k_1, \ldots, k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2} \ldots (X_{\delta_n})^{k_n},$$

where $C(k_1, \ldots, k_n) \in \mathcal{P}$, the inequalities for the roots in the sum are with respect to the circular normal ordering of Δ associated to the ordering $s_{i_1}^{-1} \beta_2, \ldots, s_{i_1}^{-1} \beta_D, \alpha_{i_1}$ of Δ_+, and the quantum root vectors are defined using the ordering $s_{i_1}^{-1} \beta_2, \ldots, s_{i_1}^{-1} \beta_D, \alpha_{i_1}$ of Δ_+.

Now applying T_{i_1} to the last identity we get

$$X_\alpha^+ X_\beta^- - X_\beta^- X_\alpha^+ = \sum_{-\beta_m < \delta_1 < \ldots < \delta_n \leq -\beta_D} K_{i_1} C(k_1, \ldots, k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2} \ldots (X_{\delta_n})^{k_n} =$$

$$= \sum_{-\beta < \delta_1 < \ldots < \delta_n < \alpha} K_{i_1} C(k_1, \ldots, k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2} \ldots (X_{\delta_n})^{k_n},$$

where the inequalities for the roots in the sum are with respect to the circular normal ordering associated to the original ordering $\beta_1, \ldots, \beta_D$ of Δ_+ and the quantum root vectors are defined using the ordering $\beta_1, \ldots, \beta_D$ of Δ_+. This establishes the base of the induction.

Now assume that the identity in question is proved for all normal orderings of Δ_+ and for all $\alpha = \beta_k$ with $k < n$ for some $n > 0$ and for all possible β such that $[-\beta, \alpha]$, $\alpha, \beta \in \Delta_+$ is a minimal segment.

Let $\alpha = \beta_n = s_{i_1} \ldots s_{i_{n-1}} \alpha_{i_n}$, $\beta = \beta_m = s_{i_1} \ldots s_{i_{m-1}} \alpha_{i_m}$, $n < m$. Then $s_{i_1}^{-1} \beta_2, \ldots, s_{i_1}^{-1} \beta_D, \alpha_{i_1}$ is another normal ordering of Δ_+, and by the induction hypothesis for this normal ordering with $s_{i_1}^{-1} \alpha = s_{i_2} \ldots s_{i_{n-1}} \alpha_{i_n}$,

$s_{i_1}^{-1}\beta = \beta_m = s_{i_2}\ldots s_{i_{m-1}}\alpha_{i_m}$ we have

$$X^+_{s_{i_1}^{-1}\alpha} X^-_{s_{i_1}^{-1}\beta} - X^-_{s_{i_1}^{-1}\beta} X^+_{s_{i_1}^{-1}\alpha} =$$

$$= \sum_{-s_{i_1}^{-1}\beta_m < \delta'_1 < \ldots < \delta'_n < s_{i_1}^{-1}\alpha} \bar{C}(k'_1,\ldots,k'_n)(X_{\delta'_1})^{k'_1}(X_{\delta'_2})^{k'_2}\ldots(X_{\delta'_n})^{k'_n},$$

where $\bar{C}(k'_1,\ldots,k'_n) \in \mathcal{P}'$, the inequalities for the roots in the sum are with respect to the circular normal ordering in Δ associated to the ordering $s_{i_1}^{-1}\beta_2,\ldots,s_{i_1}^{-1}\beta_D, \alpha_{i_1}$ of Δ_+ and the quantum root vectors are defined using the ordering $s_{i_1}^{-1}\beta_2,\ldots,s_{i_1}^{-1}\beta_D, \alpha_{i_1}$ of Δ_+.

Now applying T_{i_1} to the last identity we get

$$X^+_\alpha X^-_\beta - X^-_\beta X^+_\alpha = \sum_{-\beta < \delta_1 < \ldots < \delta_n < \alpha} K_{i_1}^{k'_q}\bar{C}''(k_1,\ldots,k_n)(X_{\delta_1})^{k_1}(X_{\delta_2})^{k_2}\ldots(X_{\delta_n})^{k_n},$$

where k'_q is such that $\delta'_q = -\alpha_{i_1}$ in the previous formula, the inequalities for the roots in the sum are with respect to the circular normal ordering associated to the original ordering $\beta_1,\ldots,\beta_D$ of Δ_+, the quantum root vectors are defined using the ordering $\beta_1,\ldots,\beta_D$ of Δ_+, and $\bar{C}''(k_1,\ldots,k_n) \in \mathcal{P}'$. This establishes the induction step and completes the proof.

$\square$

Define an algebra antiautomorphism ω of $U_h(\mathfrak{g})$ by

$$\omega(X^\pm_i) = X^\mp_i, \omega(H_i) = H_i, \omega(h) = -h. \tag{2.3.11}$$

It commutes with the braid group action and for any $\alpha \in \Delta_+$ satisfies

$$\omega(X^+_\alpha) = X^-_\alpha. \tag{2.3.12}$$

Define also an algebra antiautomorphism ω_0 of $U_h(\mathfrak{g})$ by

$$\omega_0(X^\pm_i) = X^\pm_i, \omega_0(H_i) = -H_i, \omega_0(h) = -h. \tag{2.3.13}$$

It satisfies

$$\omega\omega_0 = \omega_0\omega,$$

$$\omega_0(T_i X^\pm_j) = (-1)^{a_{ij}} q_i^{\mp a_{ij}} T_i(\omega_0 X^\pm_j) = (-1)^{a_{ij}} q_i^{\mp a_{ij}} T_i(X^\pm_j), i \neq j,$$

$$\omega_0(T_i X^+_i) = q_i^{-2} T_i(\omega_0 X^+_i) = q_i^{-2} T_i(X^+_i), \omega_0(T_i X^-_i) = q_i^2 T_i(\omega_0 X^-_i) = q_i^2 T_i(X^-_i).$$

As a consequence we obtain that if X is a homogeneous polynomial in quantum simple root vectors then

$$T_i(\omega_0 X) = c_X \omega_0(T_i X),$$

where $c_X = \epsilon p$, $\epsilon = \pm 1$, $p \in q^{\mathbb{Z}}$, and hence

$$\omega_0(X^+_\alpha) = c_\alpha X^+_\alpha, \tag{2.3.14}$$

where $c_\alpha = \epsilon_\alpha p_\alpha$, $\epsilon_\alpha = \pm 1$, $p_\alpha \in q^{\mathbb{Z}}$. We also have

$$\omega_0(X^-_\alpha) = \omega_0\omega(X^+_\alpha) = \omega\omega_0(X^+_\alpha) = \omega(c_\alpha X^+_\alpha) = c_\alpha^{-1}\omega(X^+_\alpha) = c_\alpha^{-1} X^-_\alpha \tag{2.3.15}$$

2.4 Some subalgebras in quantum groups and their Poincaré–Birkhoff–Witt bases

Now we shall explicitly describe a topological $\mathbb{C}[[h]]$–basis for $U_h(\mathfrak{g})$. We shall also recall the definition of some rational forms of $U_h(\mathfrak{g})$ and of their bases.

Denote by $U_h(\mathfrak{n}_+), U_h(\mathfrak{n}_-)$ and $U_h(\mathfrak{h})$ the $\mathbb{C}[[h]]$–subalgebras of $U_h(\mathfrak{g})$ topologically generated by the X^+_i, by the X^-_i and by the H_i, respectively. For any $\alpha \in \Delta_+$ one has $X^\pm_\alpha \in U_h(\mathfrak{n}_\pm)$. From the definition of the quantum root vectors it also follows that $[H_i, X^\pm_\alpha] = \pm\alpha(H_i)X^\pm_\alpha$, $i = 1,\ldots,l$. Therefore using the uniqueness of the presentation of any positive root as a sum of simple roots we immediately deduce the following property of the quantum root vectors.

Proposition 2.4.1. *For $\beta = \sum_{i=1}^{l} m_i \alpha_i$, $m_i \in \mathbb{N}$ $X_{\beta}^{\pm}$ is a polynomial in the noncommutative variables $X_i^{\pm}$ homogeneous in each $X_i^{\pm}$ of degree m_i.*

Denote by $U_q^{res}(\mathfrak{g})$ the subalgebra in $U_h(\mathfrak{g})$ generated over $\mathbb{C}[q, q^{-1}]$ by the elements

$$K_i^{\pm 1}, \ (X_i^{\pm})^{(k)}, \ i = 1, \ldots, l, k \geq 1.$$

The elements

$$\begin{bmatrix} K_i; c \\ r \end{bmatrix}_{q_i} = \prod_{s=1}^{r} \frac{K_i q_i^{c+1-s} - K_i^{-1} q_i^{s-1-c}}{q_i^s - q_i^{-s}} \ , \ i = 1, \ldots, l, \ c \in \mathbb{Z}, \ r \in \mathbb{N} \tag{2.4.1}$$

belong to $U_q^{res}(\mathfrak{g})$. Denote by $U_q^{res}(H)$ the subalgebra of $U_q^{res}(\mathfrak{g})$ generated by those elements and by $K_i^{\pm 1}$, $i = 1, \ldots, l$.

The subalgebras $U_q(H)$, $U_q^{res}(\mathfrak{g})$ and $U_q^{res}(H)$ of $U_h(\mathfrak{g})$ are invariant under the braid group action.

Let $U_{\mathcal{P}}(\mathfrak{n}_+), U_{\mathcal{P}}(\mathfrak{n}_-)$, (resp. $U_q^{res}(\mathfrak{n}_+)$, and $U_q^{res}(\mathfrak{n}_-)$) be the $\mathcal{P}$ (resp. $\mathbb{C}[q, q^{-1}]$)–subalgebras of $U_h(\mathfrak{g})$ (resp. $U_q^{res}(\mathfrak{g})$) generated by the X_i^+, by the X_i^-, $i = 1, \ldots, l$, (resp. by the $(X_i^+)^{(r)}$, and by the $(X_i^-)^{(r)}$, $i = 1, \ldots, l$, $r \geq 0$), respectively. Denote also by $U_h(\mathfrak{b}_{\pm})$ (resp. $U_q^{res}(\mathfrak{b}_{\pm})$) the $\mathbb{C}[[h]]$ (resp. $\mathbb{C}[q, q^{-1}]$)–subalgebras of $U_h(\mathfrak{g})$ (resp. $U_q^{res}(\mathfrak{g})$) generated by $U_h(\mathfrak{n}_{\pm})$ and by $U_h(\mathfrak{h})$ (resp. by $U_q^{res}(\mathfrak{n}_{\pm})$ and by $U_q^{res}(H)$).

Using the root vectors $X_{\beta}^{\pm}$, and the elements $(X_{\beta}^{\pm})^{(r)}$ we can construct bases for these subalgebras. Namely, let $(X^{\pm})^{\mathbf{r}} = (X_{\beta_1}^{\pm})^{r_1} \ldots (X_{\beta_D}^{\pm})^{r_D}$, $(X^{\pm})^{(\mathbf{r})} = (X_{\beta_1}^{\pm})^{(r_1)} \ldots (X_{\beta_D}^{\pm})^{(r_D)}$, $\mathbf{r} = (r_1, \ldots r_D) \in \mathbb{N}^D$, $H^{\mathbf{k}} = H_1^{k_1} \ldots H_l^{k_l}$, $\mathbf{k} = (k_1, \ldots, k_l) \in \mathbb{N}^l$.

Commutation relations (2.3.3), (2.3.8), (2.3.9) and (2.3.10) between quantum root vectors imply the following lemma.

Lemma 2.4.2. *(i) The elements $(X^+)^{\mathbf{r}}$, $(X^-)^{\mathbf{t}}$, $(X^+)^{(\mathbf{r})}$, and $(X^-)^{(\mathbf{t})}$ for $\mathbf{r}$, $\mathbf{t} \in \mathbb{N}^D$ form bases of $U_{\mathcal{P}}(\mathfrak{n}_+)$, $U_{\mathcal{P}}(\mathfrak{n}_-)$, $U_q^{res}(\mathfrak{n}_+)$, and $U_q^{res}(\mathfrak{n}_-)$, respectively.*

(ii) The elements $(X^+)^{\mathbf{r}}$, $(X^-)^{\mathbf{t}}$ and $H^{\mathbf{k}}$ form topological bases of $U_h(\mathfrak{n}_+), U_h(\mathfrak{n}_-)$ and $U_h(\mathfrak{h})$, respectively.

(iii) The multiplication defines an isomorphisms of $\mathbb{C}[q, q^{-1}]$–modules:

$$U_q^{res}(\mathfrak{n}_+) \otimes U_q^{res}(H) \otimes U_q^{res}(\mathfrak{n}_-) \to U_q^{res}(\mathfrak{g}), \tag{2.4.2}$$

$$U_q^{res}(\mathfrak{n}_-) \otimes U_q^{res}(H) \otimes U_q^{res}(\mathfrak{n}_+) \to U_q^{res}(\mathfrak{g}),$$

and of complete $\mathbb{C}[[h]]$–modules

$$U_h(\mathfrak{n}_+) \otimes U_h(\mathfrak{h}) \otimes U_h(\mathfrak{n}_-) \to U_h(\mathfrak{g}),$$

$$U_h(\mathfrak{n}_-) \otimes U_h(\mathfrak{h}) \otimes U_h(\mathfrak{n}_+) \to U_h(\mathfrak{g}),$$

where the tensor products in the left hand side are completed in the h–adic topology.

(iv) Let $[\alpha, \beta] = \{\beta_p, \ldots, \beta_q\}$ be a minimal segment in Δ_+, $U_{\mathcal{P}}([\alpha, \beta])$, $U_{\mathcal{P}}([-\alpha, -\beta])$ (resp. $U_q^{res}([\alpha, \beta])$, $U_q^{res}([-\alpha, -\beta])$) the $\mathcal{P}$ (resp. $\mathbb{C}[q, q^{-1}]$)–subalgebras of $U_h(\mathfrak{g})$ generated by the X_γ^+ and by the X_γ^-, $\gamma \in [\alpha, \beta]$ (resp. by the $(X_\gamma^+)^{(r)}$ and by the $(X_\gamma^-)^{(r)}$, $\gamma \in [\alpha, \beta]$, $r \geq 0$), respectively. Then $U_{\mathcal{P}}([\alpha, \beta]) \subset U_{\mathcal{P}}(\mathfrak{n}_+)$, $U_{\mathcal{P}}([-\alpha, -\beta]) \subset U_{\mathcal{P}}(\mathfrak{n}_-)$ (resp. $U_q^{res}([\alpha, \beta]) \subset U_q^{res}(\mathfrak{n}_+)$, $U_q^{res}([-\alpha, -\beta]) \subset U_q^{res}(\mathfrak{n}_-)$), and the elements $(X_{\beta_p}^{\pm})^{r_p} \ldots (X_{\beta_q}^{\pm})^{r_q}$ (resp. $(X_{\beta_p}^{\pm})^{(r_p)} \ldots (X_{\beta_q}^{\pm})^{(r_q)}$), $r_i \in \mathbb{N}$ form bases of $U_{\mathcal{P}}([\alpha, \beta])$, $U_{\mathcal{P}}([-\alpha, -\beta])$ (resp. $U_q^{res}([\alpha, \beta])$, $U_q^{res}([-\alpha, -\beta])$).

(v) Let $\overline{U}_{\mathcal{P}}([\alpha, \beta])$, $\overline{U}_{\mathcal{P}}([-\alpha, -\beta])$ (resp. $\overline{U}_q^{res}([\alpha, \beta])$, $\overline{U}_q^{res}([-\alpha, -\beta])$) be the $\mathcal{P}$ (resp. $\mathbb{C}[q, q^{-1}]$)–subalgebras of $U_h(\mathfrak{g})$ generated by the $\overline{X}_\gamma^+$ and by the $\overline{X}_\gamma^-$, $\gamma \in [\alpha, \beta]$ (resp. by the $(\overline{X}_\gamma^+)^{(r)}$ and by the $(\overline{X}_\gamma^-)^{(r)}$, $\gamma \in [\alpha, \beta]$, $r \geq 0$), respectively. Then $\overline{U}_{\mathcal{P}}([\alpha, \beta]) \subset U_{\mathcal{P}}(\mathfrak{n}_+)$, $\overline{U}_{\mathcal{P}}([-\alpha, -\beta]) \subset U_{\mathcal{P}}(\mathfrak{n}_-)$ (resp. $\overline{U}_q^{res}([\alpha, \beta]) \subset U_q^{res}(\mathfrak{n}_+)$, $\overline{U}_q^{res}([-\alpha, -\beta]) \subset U_q^{res}(\mathfrak{n}_-)$), and the elements $(\overline{X}_{\beta_p}^{\pm})^{r_p} \ldots (\overline{X}_{\beta_q}^{\pm})^{r_q}$ (resp. $(\overline{X}_{\beta_p}^{\pm})^{(r_p)} \ldots (\overline{X}_{\beta_q}^{\pm})^{(r_q)}$), $r_i \in \mathbb{N}$ form bases of $\overline{U}_{\mathcal{P}}([\alpha, \beta])$, $\overline{U}_{\mathcal{P}}([-\alpha, -\beta])$ (resp. $\overline{U}_q^{res}([\alpha, \beta])$, $\overline{U}_q^{res}([-\alpha, -\beta])$).

(vi) Let $[\alpha, -\beta] = \{\beta_p, \ldots, \beta_q\}$, $\alpha, \beta \in \Delta_+$ be a minimal segment in Δ, $U_{\mathcal{P}'}([\alpha, -\beta])$, $U_{\mathcal{P}'}([-\alpha, \beta])$ (resp. $U_{U_q^{res}(H)}^{res}([\alpha, -\beta])$, $U_{U_q^{res}(H)}^{res}([-\alpha, \beta])$) the $\mathcal{P}'$ (resp. $U_q^{res}(H)$)–subalgebras of $U_h(\mathfrak{g})$ generated by the X_γ (resp. $(X_\gamma)^{(r)}$), where $\gamma \in [\alpha, -\beta]$ or $\gamma \in [-\alpha, \beta]$, respectively, and $X_\gamma = X_\gamma^+$ if $\gamma \in \Delta_+$, $X_\gamma = X_\gamma^-$ if $\gamma \in \Delta_-$. Then the elements $(X_{\beta_p})^{r_p} \ldots (X_{\beta_q})^{r_q}$ (resp. $(X_{\beta_p})^{(r_p)} \ldots (X_{\beta_q})^{(r_q)}$), $r_i \in \mathbb{N}$ form bases of $U_{\mathcal{P}'}([\alpha, -\beta])$ (resp. $U_{U_q^{res}(H)}^{res}([\alpha, -\beta])$), and the elements $(X_{-\beta_p})^{r_p} \ldots (X_{-\beta_q})^{r_q}$ (resp. $(X_{-\beta_p})^{(r_p)} \ldots (X_{-\beta_q})^{(r_q)}$), $r_i \in \mathbb{N}$ form bases of $U_{\mathcal{P}'}([-\alpha, \beta])$ (resp. $U_{U_q^{res}(H)}^{res}([-\alpha, \beta])$).

(vii) Let $\overline{U}_{\mathcal{P}'}([\alpha, -\beta])$, $\overline{U}_{\mathcal{P}'}([-\alpha, \beta])$ (resp. $\overline{U}^{res}_{U^{res}_q(H)}([\alpha, -\beta])$, $\overline{U}^{res}_{U^{res}_q(H)}([-\alpha, \beta])$) be the $\mathcal{P}'$ (resp. $U^{res}_q(H)$)– subalgebras of $U_h(\mathfrak{g})$ generated by the $\overline{X}_\gamma$ (resp. by the $(\overline{X}_\gamma)^{(r)}$), where $\gamma \in [\alpha, -\beta]$ or $\gamma \in [-\alpha, \beta]$, respectively, and $\overline{X}_\gamma = \overline{X}^+_\gamma$ if $\gamma \in \Delta_+$, $\overline{X}_\gamma = \overline{X}^-_\gamma$ if $\gamma \in \Delta_-$. Then the elements $(\overline{X}_{\beta_p})^{r_p} \dots (\overline{X}_{\beta_q})^{r_q}$ (resp. $(\overline{X}_{\beta_p})^{(r_p)} \dots (\overline{X}_{\beta_q})^{(r_q)}$), $r_i \in \mathbb{N}$ form bases of $\overline{U}_{\mathcal{P}'}([\alpha, -\beta])$ (resp. $\overline{U}^{res}_{U^{res}_q(H)}([\alpha, -\beta])$), and the elements $(\overline{X}_{-\beta_p})^{r_p} \dots (\overline{X}_{-\beta_q})^{r_q}$ (resp. $(\overline{X}_{-\beta_p})^{(r_p)} \dots (\overline{X}_{-\beta_q})^{(r_q)}$), $r_i \in \mathbb{N}$ form bases of $\overline{U}_{\mathcal{P}'}([-\alpha, \beta])$ (resp. $\overline{U}^{res}_{U^{res}_q(H)}([-\alpha, \beta])$).

Proof. The first four statements of this lemma are just Propositions 8.1.7, 9.1.3 and 9.3.3 in [20], and Proposition 40.2.1, Corollary 40.2.2, and Proposition 41.1.4 in [81] from which statement (v) also follows. The proofs of the other claims are similar to each other. Consider, for instance, the case of the algebra $U^{res}_{U^{res}_q(H)}([\alpha, -\beta])$.

Let $[\alpha, -\beta] = \{\beta_p, \dots, \beta_q\}$, $\alpha, \beta \in \Delta_+$ be a minimal segment in Δ, $U^{res}_{U^{res}_q(H)}([\alpha, -\beta])$ the $U^{res}_q(H)$–subalgebra of $U_h(\mathfrak{g})$ generated by the $(X_\gamma)^{(r)}$, where $\gamma \in [\alpha, -\beta]$, and $X_\gamma = X^+_\gamma$ if $\gamma \in \Delta_+$, $X_\gamma = X^-_\gamma$ if $\gamma \in \Delta_-$. We show that the elements $(X_{\beta_p})^{(r_p)} \dots (X_{\beta_q})^{(r_q)}$, $r_i \in \mathbb{N}$ form a basis of $U^{res}_{U^{res}_q(H)}([\alpha, -\beta])$.

Consider the algebra $U^{res}_q(\mathfrak{g}) \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}(q)$. If $x \in U^{res}_{U^{res}_q(H)}([\alpha, -\beta]) \subset U_q(\mathfrak{g})$ then using commutation relations (2.3.3) and (2.3.9) one can represent x as a $U^{res}_q(H) \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}(q)$–linear combination of the elements $(X_{\beta_p})^{(r_p)} \dots (X_{\beta_q})^{(r_q)}$, $r_i \in \mathbb{N}$. We can also consider x as an element of $U^{res}_q(\mathfrak{g}) \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}(q)$ and by the Poincaré–Birkhoff–Witt theorem for $U^{res}_q(\mathfrak{g}) \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}(q)$ (see Proposition 9.1.3 in [20]) the above mentioned presentation of x is unique. Now by the Poincaré–Birkhoff–Witt theorem for $U^{res}_q(\mathfrak{g})$ (see Proposition 9.3.3 in [20] or parts (iii) and (iv) of this lemma) the coefficients in this presentation must belong to $U^{res}_q(H)$. This completes the proof in the considered case.

$\square$

A basis for $U^{res}_q(H)$ is a little bit more difficult to describe. We do not need its explicit description.

Remark 2.4.3. *The antiautomorphisms ω and ω_0 give rise to antiautomorphisms of $U_q(\mathfrak{g})$ and $U^{res}_q(\mathfrak{g})$ which we denote by the same letters. Applying the antiautomorphism ω_0 to the elements of the bases constructed in Lemma 2.4.2 and using (2.3.14), (2.3.15) we obtain other bases of similar types where the order of the quantum root vectors in the products defining the elements of the bases is reversed.*

For any minimal segment $[\alpha, \beta] \subset \Delta_+$, let $U^{res}_{U^{res}_q(H)}([\pm\alpha, \pm\beta])$ (resp. $\overline{U}^{res}_{U^{res}_q(H)}([\pm\alpha, \pm\beta])$) be the subalgebra in $U^{res}_q(\mathfrak{g})$ generated by $U^{res}_q([\pm\alpha, \pm\beta])$ (resp. by $\overline{U}^{res}_q([\pm\alpha, \pm\beta])$) and by $U^{res}_q(H)$. Note that by this definition

$$U^{res}_q(\mathfrak{b}_\pm) = U^{res}_{U^{res}_q(H)}([\pm\beta_1, \pm\beta_D]) = \overline{U}^{res}_{U^{res}_q(H)}([\pm\beta_1, \pm\beta_D]), \tag{2.4.3}$$

and that

$$U^{res}_q(\mathfrak{n}_\pm) = U^{res}_q([\pm\beta_1, \pm\beta_D]) = \overline{U}^{res}_q([\pm\beta_1, \pm\beta_D]). \tag{2.4.4}$$

Using Remark 2.4.3 we obtain from parts (iii)-(vii) of Lemma 2.4.2 the following corollary which is a quantum group counterpart of the properties of algebraic groups stated in Lemma 3.4.4 below.

Corollary 2.4.4. *For any two subalgebras $A, B \subset C$ of an algebra C denote by $AB \subset C$ the image of the map $A \otimes B \to C$ induced by the multiplication in C.*

Let $[\alpha, \beta] \subset \Delta$ be any minimal segment, such that $[\alpha, \beta] = [\alpha, \gamma] \cup [\delta, \beta]$ (disjoint union of minimal segments). Then the following statements are true.
(i)

$$U^{res}_{U^{res}_q(H)}([\alpha, \beta]) = U^{res}_{U^{res}_q(H)}([\alpha, \gamma])U^{res}_{U^{res}_q(H)}([\delta, \beta]) = U^{res}_{U^{res}_q(H)}([\delta, \beta])U^{res}_{U^{res}_q(H)}([\alpha, \gamma]),$$

and

$$\overline{U}^{res}_{U^{res}_q(H)}([\alpha, \beta]) = \overline{U}^{res}_{U^{res}_q(H)}([\alpha, \gamma])\overline{U}^{res}_{U^{res}_q(H)}([\delta, \beta]) = \overline{U}^{res}_{U^{res}_q(H)}([\delta, \beta])\overline{U}^{res}_{U^{res}_q(H)}([\alpha, \gamma]).$$

(ii) If $[\alpha, \beta] \subset \Delta_+$ or $[\alpha, \beta] \subset \Delta_-$, the multiplication in $U^{res}_q(\mathfrak{g})$ defines isomorphisms of $\mathbb{C}[q, q^{-1}]$–modules

$$U^{res}_q([\alpha, \beta]) \otimes U^{res}_q(H) \to U^{res}_q([\alpha, \beta])U^{res}_q(H) = U^{res}_{U^{res}_q(H)}([\alpha, \beta]),$$

$$U^{res}_q(H) \otimes U^{res}_q([\alpha, \beta]) \to U^{res}_q(H)U^{res}_q([\alpha, \beta]) = U^{res}_{U^{res}_q(H)}([\alpha, \beta]),$$

$$\overline{U}^{res}_q([\alpha, \beta]) \otimes U^{res}_q(H) \to \overline{U}^{res}_q([\alpha, \beta])U^{res}_q(H) = \overline{U}^{res}_{U^{res}_q(H)}([\alpha, \beta]),$$

$$U_q^{res}(H) \otimes \overline{U}_q^{res}([\alpha, \beta]) \to U_q^{res}(H)\overline{U}_q^{res}([\alpha, \beta]) = U_{U_q^{res}(H)}^{res}([\alpha, \beta]).$$

(iii) If $[\alpha, \beta] \subset \Delta_+$ or $[\alpha, \beta] \subset \Delta_-$, the multiplication in $U_q^{res}(\mathfrak{g})$ defines isomorphisms of $\mathbb{C}[q, q^{-1}]$–modules

$$U_q^{res}([\alpha, \gamma]) \otimes U_q^{res}([\delta, \beta]) \to U_q^{res}([\alpha, \gamma])U_q^{res}([\delta, \beta]) = U_q^{res}([\alpha, \beta]),$$

$$U_q^{res}([\delta, \beta]) \otimes U_q^{res}([\alpha, \gamma]) \to U_q^{res}([\delta, \beta])U_q^{res}([\alpha, \gamma]) = U_q^{res}([\alpha, \beta]),$$

and

$$\overline{U}_q^{res}([\alpha, \gamma]) \otimes \overline{U}_q^{res}([\delta, \beta]) \to \overline{U}_q^{res}([\alpha, \gamma])\overline{U}_q^{res}([\delta, \beta]) = \overline{U}_q^{res}([\alpha, \beta]),$$

$$\overline{U}_q^{res}([\delta, \beta]) \otimes \overline{U}_q^{res}([\alpha, \gamma]) \to \overline{U}_q^{res}([\delta, \beta])\overline{U}_q^{res}([\alpha, \gamma]) = \overline{U}_q^{res}([\alpha, \beta]).$$

Remark 2.4.5. *By part (a) of the Proposition in Section 2.2 of [27], for any minimal segment $[\alpha, \beta] \subset \Delta$ the algebras $U_{U_q^{res}(H)}^{res}([\alpha, \beta])$, $\overline{U}_{U_q^{res}(H)}^{res}([\alpha, \beta])$, and other algebras of similar type which are defined in parts (iv)-(vii) of Lemma 2.4.2, only depend on the Weyl group element $s_\alpha \dots s_\beta$, where the product of reflections is taken over all roots contained in the segment $[\alpha, \beta]$ in the order on $[\alpha, \beta]$ induced by the circular normal ordering of Δ. We shall not need this result in this book.*

We shall also need the following simple lemma.

Lemma 2.4.6. *Let $\overline{w} \in W$ be the longest element of the Weyl group. Then $T_{\overline{w}}U_q^{res}(\mathfrak{n}_\pm) \subset U_q^{res}(\mathfrak{b}_\mp)$.*

Proof. Let $\overline{w} = s_{i_1} \dots s_{i_D}$ be a reduced decomposition of the longest element $\overline{w}$ of the Weyl group W of $\mathfrak{g}$,

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \dots, \beta_D = s_{i_1} \dots s_{i_{D-1}}\alpha_{i_D}$$

the corresponding normal ordering in Δ_+, and

$$\overline{X}_{\beta_k}^\pm = T_{i_1}^{-1} \dots T_{i_{k-1}}^{-1}X_{i_k}^\pm, \; k = 1, \dots, D$$

the quantum root vectors.

Consider the inverse reduced decomposition of $\overline{w}$, $\overline{w} = s_{i_D} \dots s_{i_1}$. Since $T_{\overline{w}}$ only depends on $\overline{w}$, we have for $k = 1, \dots, D, r \in \mathbb{N}$, using commutation relations (2.3.6),

$$T_{\overline{w}}(\overline{X}_{\beta_k}^+)^{(r)} = T_{i_D} \dots T_{i_1}T_{i_1}^{-1} \dots T_{i_{k-1}}^{-1}(X_{i_k}^+)^{(r)} = T_{i_D} \dots T_{i_k}(X_{i_k}^+)^{(r)} =$$

$$= \frac{1}{[r]_{q_{i_k}}!}(-T_{i_D} \dots T_{i_{k-1}}(X_{i_k}^-K_{i_k}))^r = \frac{(-1)^r}{[r]_{q_{i_k}}!}T_{i_D} \dots T_{i_{k-1}}((X_{i_k}^-)^rK_{i_k}^r) =$$

$$= (-1)^rq_{i_k}^{-r(r-1)}T_{i_D} \dots T_{i_{k-1}}(X_{i_k}^-)^{(r)}T_{i_D} \dots T_{i_{k-1}}K_{i_k}^r \in U_q^{res}(\mathfrak{b}_-),$$

where to justify the last inclusion we used the fact that $T_{i_D} \dots T_{i_{k-1}}(X_{i_k}^-)^{(r)} \in U_q^{res}(\mathfrak{n}_-)$ by part (i) of Lemma 2.4.2.

Since the elements $(\overline{X}_{\beta_k}^+)^{(r)}, k = 1, \dots, D, r \in \mathbb{N}$ generate the algebra $U_q^{res}(\mathfrak{n}_+)$ by Lemma 2.4.2 (v) and by (2.4.4), we deduce that $T_{\overline{w}}U_q^{res}(\mathfrak{n}_+) \subset U_q^{res}(\mathfrak{b}_-)$. The other inclusion is established in a similar way. This completes the proof. $\square$

In conclusion we note that the algebras $U_q^{res}(\mathfrak{g}) \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}(q)$ and $U_q^{res}(\mathfrak{g})$ are graded by the elements of the root lattice Q. Indeed, one can assign weight $-r_1\beta_1 - \dots - r_D\beta_D + k_1\beta_1 + \dots + k_D\beta_D$ to each element

$$(X_{\beta_1}^-)^{(r_1)} \dots (X_{\beta_D}^-)^{(r_D)}X(X_{\beta_1}^+)^{(k_1)} \dots (X_{\beta_D}^+)^{(k_D)},$$

$$\mathrm{wt}((X_{\beta_1}^-)^{(r_1)} \dots (X_{\beta_D}^-)^{(r_D)}X(X_{\beta_1}^+)^{(k_1)} \dots (X_{\beta_D}^+)^{(k_D)}) = -r_1\beta_1 - \dots - r_D\beta_D + k_1\beta_1 + \dots + k_D\beta_D,$$

where $X \in U_q^{res}(H)$, use parts (i) and (ii) of Lemma 2.4.2 to extend the definition of the weight to $U_q^{res}(\mathfrak{g})$ and observe that the relations in $U_q^{res}(\mathfrak{g})$ following from (2.1.1) are homogeneous with respect to this grading. Then one can naturally extend this grading to $U_q^{res}(\mathfrak{g}) \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}(q)$.

For $\mu \in Q$ we introduce the $\mathbb{C}[q, q^{-1}]$–submodule $(U_q^{res}(\mathfrak{g}))_\mu \subset U_q^{res}(\mathfrak{g}), (U_q^{res}(\mathfrak{g}))_\mu = \{x \in U_q^{res}(\mathfrak{g}) : \mathrm{wt}(x) = \mu\}$, and call it the weight subspace of weight μ of $U_q^{res}(\mathfrak{g})$. Alternatively, one has

$$(U_q^{res}(\mathfrak{g}))_\mu = \{x \in U_q^{res}(\mathfrak{g}) : [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h} \subset U_h(\mathfrak{g})\}, \tag{2.4.5}$$

where the commutator is defined in $U_h(\mathfrak{g})$, and $\mathfrak{h}$ is regarded as a complex vector subspace of $U_h(\mathfrak{g})$ spanned by the elements H_i, $i = 1, \ldots, l$.

By parts (i) and (iii) of Lemma 2.4.2 we have the following direct sum of $\mathbb{C}[q, q^{-1}]$–modules

$$U_q^{res}(\mathfrak{g}) = \bigoplus_{\mu \in Q} (U_q^{res}(\mathfrak{g}))_\mu.$$

From the definition of the braid group action on $U_h(\mathfrak{g})$ and from (2.4.5) it follows that for all $w \in W$ one has $T_w((U_q^{res}(\mathfrak{g}))_\mu) = (U_q^{res}(\mathfrak{g}))_{w\mu}$.

The algebra $U_h(\mathfrak{g})$ is, of course, not graded by the elements of the root lattice. But one can still define weights of the elements of its topological basis introduced in parts (ii) and (iii) of Lemma 2.4.2,

$$\mathrm{wt}((X_{\beta_1}^-)^{r_1} \ldots (X_{\beta_D}^-)^{r_D} X (X_{\beta_1}^+)^{k_1} \ldots (X_{\beta_D}^+)^{k_D}) = -r_1\beta_1 - \ldots - r_D\beta_D + k_1\beta_1 + \ldots + k_D\beta_D,$$

where $X \in U_h(\mathfrak{h})$.

2.5 The universal R–matrix

$U_h(\mathfrak{g})$ is a quasitriangular Hopf algebra, i.e. there exists an invertible element $\mathcal{R} \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ (completed tensor product), called a universal R–matrix, such that

$$\Delta_h^{opp}(a) = \mathcal{R}\Delta_h(a)\mathcal{R}^{-1} \text{ for all } a \in U_h(\mathfrak{g}), \tag{2.5.1}$$

and

$$(\Delta_h \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23},$$
$$\tag{2.5.2}$$
$$(id \otimes \Delta_h)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12},$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$, $\mathcal{R}_{13} = (\sigma \otimes id)\mathcal{R}_{23}$, and $\sigma(x \otimes y) = y \otimes x$, $x, y \in U_h(\mathfrak{g})$.

From (2.5.1) and (2.5.2) it follows that $\mathcal{R}$ satisfies the quantum Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2.5.3}$$

For every quasitriangular Hopf algebra we also have

$$(S \otimes id)\mathcal{R} = (id \otimes S^{-1})\mathcal{R} = \mathcal{R}^{-1}, \tag{2.5.4}$$

and

$$(S \otimes S)\mathcal{R} = \mathcal{R}, \tag{2.5.5}$$

where S is the antipode.

An explicit expression for $\mathcal{R}$ may be written by making use of the q–exponential

$$\exp_q(x) = \exp_q'(qx) = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!}$$

in terms of which the element $\mathcal{R}$ takes the form

$$\mathcal{R} = \prod_{\beta} \exp_{q_\beta}[(1 - q_\beta^{-2})X_\beta^- \otimes X_\beta^+]\exp\left[h\sum_{i=1}^{l}(Y_i \otimes H_i)\right] = \prod_{\beta} \theta_\beta \exp\left[h\sum_{i=1}^{l}(Y_i \otimes H_i)\right], \tag{2.5.6}$$

where the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that the α–term appears to the left of the β–term if $\alpha < \beta$ with respect to the normal ordering

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \ldots, \beta_D = s_{i_1} \ldots s_{i_{D-1}}\alpha_{i_D}$$

of Δ_+ which is used in the definition of the quantum root vectors $X_\beta^\pm$.

One can calculate the action of the comultiplication on the root vectors $X_{\beta_k}^\pm$ in terms of the universal R–matrix. For instance for $\Delta_h(X_{\beta_k}^-)$ one has

$$\Delta_h(X_{\beta_k}^-) = \theta_{w_{k-1}}(X_{\beta_k}^- \otimes 1 + e^{h\beta^\vee} \otimes X_{\beta_k}^-)\theta_{w_{k-1}}^{-1}, \tag{2.5.7}$$

where for $w_{k-1} = s_{i_1} \ldots s_{i_{k-1}}$

$$\theta_{w_{k-1}} = \theta_{\beta_1} \ldots \theta_{\beta_{k-1}}.$$

The r–matrix $r_- = -\frac{1}{2}h^{-1}(\mathcal{R} - 1 \otimes 1) \pmod{h}$, which is the classical limit of $\mathcal{R}$, coincides with the classical r–matrix (2.1.4).

2.6 Realizations of quantum groups associated to Weyl group elements

q-W–algebras will be defined in terms of certain integral forms of non–standard realizations of quantum groups associated to Weyl group elements.

Let s be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, and $\mathfrak{h}'$ the orthogonal complement, with respect to the symmetric bilinear form, to the subspace of $\mathfrak{h}$ fixed by the natural action of s on $\mathfrak{h}$. Let $\mathfrak{h}'^*$ be the image of $\mathfrak{h}'$ in $\mathfrak{h}^*$ under the identification $\mathfrak{h}^* \simeq \mathfrak{h}$ induced by the symmetric bilinear form on $\mathfrak{g}$. The restriction of the natural action of s on $\mathfrak{h}^*$ to the subspace $\mathfrak{h}'^*$ has no fixed points. Therefore one can define the Cayley transform $\frac{1+s}{1-s}$ of the restriction of s to $\mathfrak{h}'^*$. Denote by $P_{\mathfrak{h}'^*}$ the orthogonal projection operator onto $\mathfrak{h}'^*$ in $\mathfrak{h}^*$, with respect to the bilinear form.

Let $\kappa \in \mathbb{Z}$ be an integer number and $U_h^s(\mathfrak{g})$ the topological algebra over $\mathbb{C}[[h]]$ topologically generated by elements $e_i, f_i, H_i, \ i = 1, \ldots l$ subject to the relations:

$$[H_i, H_j] = 0, \quad [H_i, e_j] = a_{ij} e_j, \quad [H_i, f_j] = -a_{ij} f_j, \quad e_i f_j - q^{c_{ij}} f_j e_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$c_{ij} = \kappa \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i, \alpha_j \right), \quad K_i = e^{d_i h H_i},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r q^{r c_{ij}} \left[\begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_{q_i} (e_i)^{1-a_{ij}-r} e_j (e_i)^r = 0, \ i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r q^{r c_{ij}} \left[\begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_{q_i} (f_i)^{1-a_{ij}-r} f_j (f_i)^r = 0, \ i \neq j.$$

(2.6.1)

Proposition 2.6.1. *For every solution* $n_{ij} \in \mathbb{C}, \ i, j = 1, \ldots, l$ *of equations*

$$d_j n_{ij} - d_i n_{ji} = c_{ij} \tag{2.6.2}$$

there exists an algebra isomorphism $\psi_{\{n_{ij}\}} : U_h^s(\mathfrak{g}) \to U_h(\mathfrak{g})$ *defined by the formulas:*

$$\psi_{\{n_{ij}\}}(e_i) = X_i^+ \prod_{p=1}^{l} L_p^{n_{ip}}, \quad \psi_{\{n_{ij}\}}(f_i) = \prod_{p=1}^{l} L_p^{-n_{ip}} X_i^-, \quad \psi_{\{n_{ij}\}}(H_i) = H_i.$$

Proof. The proof of this proposition is by direct verification of defining relations (2.6.1). The most nontrivial part is to verify the deformed quantum Serre relations, i.e. the last two relations in (2.6.1). For instance, the defining relations of $U_h(\mathfrak{g})$ imply the following relations for $\psi_{\{n_{ij}\}}(e_i)$,

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} q^{k(d_j n_{ij} - d_i n_{ji})} \psi_{\{n_{ij}\}}(e_i)^{1-a_{ij}-k} \psi_{\{n_{ij}\}}(e_j) \psi_{\{n_{ij}\}}(e_i)^k = 0,$$

for any $i \neq j$. Now using equation (2.6.2) we arrive to the quantum Serre relations for e_i in (2.6.1). $\square$

The general solution of equation (2.6.2) is given by

$$n_{ij} = \frac{1}{2d_j}(c_{ij} + s_{ij}), \tag{2.6.3}$$

where $s_{ij} = s_{ji}$.

We shall only use the solution for which $s_{ij} = 0$ for all $i, j = 1, \ldots l$. Then

$$n_{ij} = \frac{1}{2d_j} c_{ij} \tag{2.6.4}$$

From now on we assume that solution (2.6.4) is used to identify $U_h^s(\mathfrak{g})$ and $U_h(\mathfrak{g})$.

The algebra $U_h^s(\mathfrak{g})$ is called the realization of the quantum group $U_h(\mathfrak{g})$ corresponding to the element $s \in W$. Denote by $U_h^s(\mathfrak{n}_\pm)$ the subalgebra in $U_h^s(\mathfrak{g})$ generated by e_i (resp. f_i), $i = 1, \ldots, l$. Let $U_h^s(\mathfrak{h})$ be the subalgebra in $U_h^s(\mathfrak{g})$ generated by $H_i, \ i = 1, \ldots, l$.

We shall construct analogues of quantum root vectors for $U_h^s(\mathfrak{g})$. It is convenient to introduce an operator $K_s \in \mathrm{End}\,\mathfrak{h}$ defined by

$$K_s H_i = \sum_{j=1}^{l} \frac{n_{ij}}{d_i} Y_j. \tag{2.6.5}$$

From (2.6.4) we obtain that

$$K_s h = \frac{\kappa}{2} \frac{1+s}{1-s} P_{\mathfrak{h}'} h, h \in \mathfrak{h}. \tag{2.6.6}$$

Proposition 2.6.2. *Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. Let $U_h^s(\mathfrak{g})$ be the realization of the quantum group $U_h(\mathfrak{g})$ associated to s.*
For any normal ordering of the root system Δ_+ the elements

$$e_\beta(s) = \psi_{\{n_{ij}\}}^{-1}(X_\beta^+ e^{hK_s \beta^\vee}) \text{ and } f_\beta(s) = \psi_{\{n_{ij}\}}^{-1}(e^{-hK_s \beta^\vee} X_\beta^-), \ \beta \in \Delta_+$$

lie in the subalgebras $U_h^s(\mathfrak{n}_+)$ and $U_h^s(\mathfrak{n}_-)$, respectively.
The elements $f_\beta(s) \in U_h^s(\mathfrak{n}_-)$, $\beta \in \Delta_{\mathfrak{m}_+}$ defined with the help of the normal ordering (1.6.9) generate a subalgebra $U_h^s(\mathfrak{m}_-) \subset U_h^s(\mathfrak{g})$ such that

$$U_h^s(\mathfrak{m}_-)/h U_h^s(\mathfrak{m}_-) \simeq U(\mathfrak{m}_-),$$

where $\mathfrak{m}_-$ is the Lie subalgebra of $\mathfrak{g}$ generated by the root vectors $X_{-\alpha}$, $\alpha \in \Delta_{\mathfrak{m}_+}$.

Proof. Fix a normal ordering of the root system Δ_+. Let $\beta = \sum_{i=1}^{l} m_i \alpha_i \in \Delta_+$ be a positive root, $X_\beta^+ \in U_h(\mathfrak{g})$ the corresponding quantum root vector constructed with the help of the fixed normal ordering of Δ_+. Then $\beta^\vee = \sum_{i=1}^{l} m_i d_i H_i$, and so $K_s \beta^\vee = \sum_{i,j=1}^{l} m_i n_{ij} Y_j$. Now the proof of the first statement follows immediately from Proposition 2.4.1, commutation relations (2.1.2) and the definition of the isomorphism $\psi_{\{n_{ij}\}}$.
The second assertion is a consequence of (2.3.7). $\qquad\square$

Remark 2.6.3. *To simplify the notation we shall often write $e_\beta(s) = e_\beta$ and $f_\beta(s) = f_\beta$ if it does not cause any confusion.*

The realizations $U_h^s(\mathfrak{g})$ of the quantum group $U_h(\mathfrak{g})$ are related to quantizations of some nonstandard bialgebra structures on $\mathfrak{g}$. At the quantum level changing bialgebra structure corresponds to the so–called Drinfeld twist. The relevant class of such twists is described in the following proposition which is a combination of Propositions 4.2.13, 16.1.5, of formula (15) in §16.1 in [20], and of the results of §1 in [33].

Proposition 2.6.4. *Let $(A, \mu, \imath, \Delta, \varepsilon, S)$ be a Hopf algebra over a commutative ring with multiplication μ, unit $\imath$, comultiplication Δ, counit ε and antipode S.*
(i) Let $\mathcal{F}$ be an invertible element of $A \otimes A$ such that

$$(\varepsilon \otimes id)(\mathcal{F}) = (id \otimes \varepsilon)(\mathcal{F}) = 1, \tag{2.6.7}$$

and

$$\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}). \tag{2.6.8}$$

Then

$$v = \mu(id \otimes S)(\mathcal{F}) \tag{2.6.9}$$

is an invertible element of A with

$$v^{-1} = \mu(S \otimes id)(\mathcal{F}^{-1}).$$

Moreover, if we define $\Delta^{\mathcal{F}} : A \to A \otimes A$ and $S^{\mathcal{F}} : A \to A$ by

$$\Delta^{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \quad S^{\mathcal{F}}(a) = vS(a)v^{-1}, \tag{2.6.10}$$

then $(A, \mu, \imath, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ is a Hopf algebra denoted by $A^{\mathcal{F}}$ and called the twist of A by $\mathcal{F}$.
(ii) Suppose that A and $\mathcal{F}$ are as in part (i), but assume in addition that A is quasitriangular with universal R–matrix $\mathcal{R}$. Then $A^{\mathcal{F}}$ is quasitriangular with universal R–matrix

$$\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \tag{2.6.11}$$

where $\mathcal{F}_{21} = \sigma\mathcal{F}$.

(iii) If $\mathcal{F}, \mathcal{G} \in A \otimes A$ are invertible elements, $\mathcal{G}$ satisfies (2.6.7) and (2.6.8), $\mathcal{F}$ satisfies (2.6.7) and (2.6.8) with Δ replaced by $\Delta^{\mathcal{G}}$ then $\mathcal{F}\mathcal{G}$ satisfies (2.6.7) and (2.6.8) and the twist of A by $\mathcal{F}\mathcal{G}$ is the composition of the twists of A by $\mathcal{F}$ and by $\mathcal{G}$.

Let

$$\mathcal{F}_s = \exp(-h\sum_{i,j=1}^{l} \frac{n_{ij}}{d_i} Y_i \otimes Y_j) = \exp(-h\sum_{i=1}^{l} Y_i \otimes K_s H_i) \in U_h(\mathfrak{h}) \otimes U_h(\mathfrak{h}). \tag{2.6.12}$$

Then from the definition of the commutative Hopf subalgebra $U(\mathfrak{h})[[h]] \simeq U_h(\mathfrak{h}) \subset U_h(\mathfrak{g})$ it immediately follows that $\mathcal{F}_s \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ satisfies (2.6.7) and (2.6.8) (see also [20], Proposition 6.5.8), and according to Proposition 2.6.4 (i)

$$\Delta_s(a) = (\psi_{\{n_{ij}\}}^{-1} \otimes \psi_{\{n_{ij}\}}^{-1})\mathcal{F}_s\Delta_h(\psi_{\{n_{ij}\}}(a))\mathcal{F}_s^{-1}. \tag{2.6.13}$$

defines a comultiplication on $U_h^s(\mathfrak{g})$.

Let $P_{\mathfrak{h}'}$ be the orthogonal projection operator onto $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the bilinear form on $\mathfrak{h}$. From the formulas in part (i) of Proposition 2.6.4 it follows that on the generators the comultiplication Δ_s is explicitly given by the following formulas

$$\Delta_s(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_s(e_i) = e_i \otimes e^{-hd_i H_i} + e^{h\kappa d_i \frac{1+s}{1-s}P_{\mathfrak{h}'}H_i} \otimes e_i, \quad \Delta_s(f_i) = f_i \otimes 1 + e^{-h\kappa d_i \frac{1+s}{1-s}P_{\mathfrak{h}'}H_i+hd_i H_i} \otimes f_i,$$

the corresponding antipode $S_s(x)$ is given by

$$S_s(e_i) = -e^{-h\kappa d_i \frac{1+s}{1-s}P_{\mathfrak{h}'}H_i}e_i e^{hd_i H_i}, \quad S_s(f_i) = -e^{h\kappa d_i \frac{1+s}{1-s}P_{\mathfrak{h}'}H_i-hd_i H_i}f_i, \quad S_s(H_i) = -H_i, \tag{2.6.14}$$

and the corresponding counit ε_s is given by

$$\varepsilon_s(H_i) = \varepsilon_s(e_i) = \varepsilon_s(f_i) = 0.$$

We shall always assume that the algebra $U_h^s(\mathfrak{g})$ is equipped with this Hopf algebra structure.

Note that the Hopf algebra $U_h^s(\mathfrak{g})$ is a quantization of the bialgebra structure on $\mathfrak{g}$ defined by the cocycle

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x)2r_{\pm}^s, \quad r_{\pm}^s \in \mathfrak{g} \otimes \mathfrak{g}, \tag{2.6.15}$$

where $r_{\pm}^s = r_{\pm} + \frac{1}{2}\sum_{i=1}^{l} \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i \otimes Y_i$, and $r_{\pm}$ is given by (2.1.4).

By Proposition 2.6.4 (ii) $U_h^s(\mathfrak{g})$ is a quasitriangular topological Hopf algebra with the universal R–matrix $\mathcal{R}^s = (\psi_{\{n_{ij}\}}^{-1} \otimes \psi_{\{n_{ij}\}}^{-1})(\mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1})$,

$$\begin{aligned}
\mathcal{R}^s &= \prod_{\beta} \exp_{q_\beta}[(1-q_\beta^{-2})f_\beta \otimes e_\beta e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee}]\times \\
&\times \exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i) - \sum_{i=1}^{l}\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i \otimes Y_i)\right] =
\end{aligned} \tag{2.6.16}$$

$$\begin{aligned}
&= \exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i) - \sum_{i=1}^{l}\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i \otimes Y_i)\right] \times \\
&\times \prod_{\beta} \exp_{q_\beta}[(1-q_\beta^{-2})e^{h(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta^\vee}f_\beta \otimes e_\beta e^{h\beta^\vee}],
\end{aligned}$$

where the order of the terms in the product over the positive roots is such that the α–term appears to the left of the β–term if $\alpha < \beta$ in the normal ordering of Δ_+ with the help of which the quantum root vectors e_β, f_β are defined in Proposition 2.6.2.

Similarly to (2.2.11) one obtains that for a reduced decomposition $w = s_{i_1}\ldots s_{i_k}$ and $T_w = T_{i_1}\ldots T_{i_k}$ only depending on w one has from (2.2.11) and (2.6.13)

$$\Delta_s(T_w) = \prod_{p=1}^{k} \theta_{\beta_p}^s \mathcal{F}_s(T_w \otimes T_w)(\mathcal{F}_s^{-1})T_w \otimes T_w = \tag{2.6.17}$$

$$= T_w \otimes T_w(T_{w^{-1}} \otimes T_{w^{-1}})(\mathcal{F}_s)\mathcal{F}_s^{-1}\prod_{p=1}^{k} \bar{\theta}_{\beta_p'}^s,$$

where in the products $\theta_{\beta_p}^s$ (resp. $\overline{\theta}_{\beta_p'}^s$) appears on the left from $\theta_{\beta_q}^s$ (resp. $\overline{\theta}_{\beta_q'}^s$) if $p < q$, and for $p = 1, \ldots, k$

$$e_{\beta_p} = \psi_{\{n_{ij}\}}^{-1}(X_{\beta_p}^+ e^{hK_s \beta_p^\vee}), f_\beta = \psi_{\{n_{ij}\}}^{-1}(e^{-hK_s \beta_p^\vee} X_{\beta_p}^-), \beta_p = s_{i_1} \ldots s_{i_{p-1}} \alpha_{i_p},$$

$$X_{\beta_p}^\pm = T_{i_1} \ldots T_{i_{p-1}} X_{i_p}^\pm, \ \overline{X}_{\beta_p'}^\pm = T_{i_k}^{-1} \ldots T_{i_{p+1}}^{-1} X_{i_p}^\pm,$$

$$\overline{e}_{\beta_p'} = \psi_{\{n_{ij}\}}^{-1}(\overline{X}_{\beta_p'}^+ e^{hK_s \beta_p'^\vee}), \overline{f}_{\beta_p'} = \psi_{\{n_{ij}\}}^{-1}(e^{-hK_s \beta_p'^\vee} \overline{X}_{\beta_p'}^-), \ K_{\beta_p'} = T_{i_k}^{-1} \ldots T_{i_{p+1}}^{-1} K_{i_p}, \beta_p' = s_{i_k} \ldots s_{i_{p+1}} \alpha_{i_p}$$

$$\theta_{\beta_p}^s = \mathcal{F}_s \theta_{\beta_p} \mathcal{F}_s^{-1} = \exp_{q_{\beta_p}}[(1 - q_{\beta_p}^{-2})e_{\beta_p} e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta_p^\vee} \otimes f_{\beta_p}],$$

$$\overline{\theta}_{\beta_p'}^s = \mathcal{F}_s \overline{\theta}_{\beta_p'} \mathcal{F}_s^{-1} = \exp_{q_{\beta_p'}}[(1 - q_{\beta_p'}^{-2})K_{\beta_p'}^{-1} e^{h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta_p'^\vee} \overline{f}_{\beta_p'} \otimes \overline{e}_{\beta_p'} K_{\beta_p'}].$$

In the same way, for $\overline{T}_w = T_{i_1}^{-1} \ldots T_{i_k}^{-1}$ only depending on w one has from (2.2.12) and (2.6.13)

$$\Delta_s(\overline{T}_w) = \prod_{p=1}^k \overline{\theta}_{\beta_p}^s{}' \mathcal{F}_s(T_w \otimes T_w)(\mathcal{F}_s^{-1}) \overline{T}_w \otimes \overline{T}_w = \tag{2.6.18}$$

$$= \overline{T}_w \otimes \overline{T}_w (T_{w^{-1}} \otimes T_{w^{-1}})(\mathcal{F}_s)\mathcal{F}_s^{-1} \prod_{p=1}^k \theta_{\beta_p'}^s{}',$$

where in the products $\overline{\theta}_{\beta_p}^s{}'$ (resp. $\theta_{\beta_p'}^s{}'$) appears on the left from $\overline{\theta}_{\beta_q}^s{}'$ (resp. $\theta_{\beta_q'}^s{}'$) if $p < q$, and for $p = 1, \ldots, k$

$$\overline{e}_{\beta_p} = \psi_{\{n_{ij}\}}^{-1}(\overline{X}_{\beta_p}^+ e^{hK \beta_p^\vee}), \overline{f}_{\beta_p} = \psi_{\{n_{ij}\}}^{-1}(e^{-hK \beta_p^\vee} \overline{X}_{\beta_p}^-), \overline{K}_{\beta_p} = T_{i_1}^{-1} \ldots T_{i_{p-1}}^{-1} K_{i_p},$$

$$\overline{X}_{\beta_p}^\pm = T_{i_1}^{-1} \ldots T_{i_{p-1}}^{-1} X_{i_p}^\pm, \beta_p = s_{i_1} \ldots s_{i_{p-1}} \alpha_{i_p},$$

$$e_{\beta_p'}' = \psi_{\{n_{ij}\}}^{-1}(X_{\beta_p'}^+{}' e^{hK \beta_p'^\vee}), f_{\beta_p'}' = \psi_{\{n_{ij}\}}^{-1}(e^{-hK \beta_p'^\vee} X_{\beta_p'}^-{}'), \beta_p' = s_{i_k} \ldots s_{i_{p+1}} \alpha_{i_p}, X_{\beta_p'}^\pm{}' = T_{i_k} \ldots T_{i_{p+1}} X_{i_p}^\pm,$$

$$\theta_{\beta_p'}^s{}' = \mathcal{F}_s \theta_{\beta_p'}' \mathcal{F}_s^{-1} = \exp_{q_{\beta_p'}^{-1}}[(1 - q_{\beta_p'}^2)e_{\beta_p'}' e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta_p'^\vee} \otimes f_{\beta_p'}'], \tag{2.6.19}$$

$$\overline{\theta}_{\beta_p}^s{}' = \mathcal{F}_s \overline{\theta}_{\beta_p}' \mathcal{F}_s^{-1} = \exp_{q_{\beta_p}^{-1}}[(1 - q_{\beta_p}^2)\overline{K}_{\beta_p}^{-1} e^{h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta_p^\vee} \overline{f}_{\beta_p} \otimes \overline{e}_{\beta_p} \overline{K}_{\beta_p}].$$

2.7 The adjoint action

Next we discuss the properties of the adjoint action of a Hopf algebra on itself with respect to Drinfeld twists. Define the right adjoint action of a Hopf algebra $(A, \mu, \imath, \Delta, \varepsilon, S)$ on itself by the formula

$$\mathrm{Ad}x(z) = S(x^1)zx^2, \tag{2.7.1}$$

and the left adjoint action by

$$\mathrm{Ad}'x(z) = x^1 z S(x^2), \tag{2.7.2}$$

where we use the abbreviated Sweedler notation for the comultiplication $\Delta(x) = x^1 \otimes x^2$, $x, z \in A$.

Note that by Lemma 2.2 in [60]

$$\mathrm{Ad}x(wz) = \mathrm{Ad}x^1(w)\mathrm{Ad}x^2(z). \tag{2.7.3}$$

Proposition 2.7.1. *Let $(A, \mu, \imath, \Delta, \varepsilon, S)$ be a Hopf algebra, $\mathcal{F} \in A \otimes A$ an invertible element satisfying conditions (2.6.7) and (2.6.8), $(A, \mu, \imath, \Delta^\mathcal{F}, \varepsilon, S^\mathcal{F})$ the twist of A by $\mathcal{F}$. Denote by $\mathrm{Ad}^\mathcal{F}$ the right adjoint action of $(A, \mu, \imath, \Delta^\mathcal{F}, \varepsilon, S^\mathcal{F})$ on itself. Then for all $x, z \in A$*

$$\mathrm{Ad}^\mathcal{F} x(z) = \psi^\mathcal{F} \mathrm{Ad}x(\psi^\mathcal{F})^{-1}(z), \tag{2.7.4}$$

where $\psi^\mathcal{F} : A \to A$ is an invertible morphism of A, regarded as a module over the commutative ground ring, defined by

$$\psi^\mathcal{F}(z) = v \sum_i S(c_i)zd_i, \tag{2.7.5}$$

where v is given by (2.6.9), and $\mathcal{F}^{-1} = \sum_i c_i \otimes d_i$.

The inverse map $(\psi^{\mathcal{F}})^{-1}$ is given by the formula

$$(\psi^{\mathcal{F}})^{-1}(z) = \sum_i S(a_i)v^{-1}zb_i, \tag{2.7.6}$$

where $\mathcal{F} = \sum_i a_i \otimes b_i$.

Proof. Firstly we check that formula (2.7.6) defines the inverse to $\psi^{\mathcal{F}}$. Indeed,

$$(\psi^{\mathcal{F}})^{-1}\psi^{\mathcal{F}}(z) = \sum_j S(a_j)v^{-1}v\sum_i S(c_i)zd_ib_j = \sum_{i,j} S(c_ia_j)zd_ib_j = z,$$

as $1 \otimes 1 = \mathcal{F}^{-1}\mathcal{F} = \sum_{i,j} c_ia_j \otimes d_ib_j$.

Similarly,

$$\psi^{\mathcal{F}}(\psi^{\mathcal{F}})^{-1}(z) = v\sum_j S(c_j)\sum_i S(a_i)v^{-1}zb_id_j = \sum_{i,j} vS(a_ic_j)v^{-1}zb_id_j = vv^{-1}z = z,$$

as $1 \otimes 1 = \mathcal{F}\mathcal{F}^{-1} = \sum_{i,j} a_ic_j \otimes b_id_j$.

Now by (2.6.10), (2.7.1), (2.7.5) and (2.7.6) we have

$$\mathrm{Ad}^{\mathcal{F}}x(z) = vS(a_ix^1c_j)v^{-1}zb_ix^2d_j = vS(c_j)S(x^1)S(a_i)v^{-1}zb_ix^2d_j = \psi^{\mathcal{F}}\mathrm{Ad}x(\psi^{\mathcal{F}})^{-1}(z).$$

This justifies (2.7.4).

$\square$

Proposition 2.7.2. *Let $(A, \mu, \imath, \Delta, \varepsilon, S)$ be a Hopf algebra, $\mathcal{F} \in A \otimes A$ an invertible element satisfying conditions (2.6.7) and (2.6.8), $v \in A$ given by (2.6.9), and $\mathcal{F}^{-1} = \sum_i c_i \otimes d_i$.*

Then

$$\Delta v = \mathcal{F}^{-1}(v \otimes v)(S \otimes S)(\mathcal{F}_{21}^{-1}), \tag{2.7.7}$$

and for all $z \in A$

$$\Delta\psi^{\mathcal{F}}(z) = \mathcal{F}^{-1}(1 \otimes v)\sum_{i,j}(\psi^{\mathcal{F}}(\mathrm{Ad}(d_i^1c_j)(z^1)) \otimes S(c_i)z^2d_i^2d_j)\mathcal{F}, \tag{2.7.8}$$

where $\psi^{\mathcal{F}} : A \to A$ is defined by (2.7.5) and we use the Sweedler notation for the comultiplication.

Proof. First we prove identity (2.7.7). Denote as before $\mathcal{F} = \sum_i a_i \otimes b_i$. Then condition (2.6.8) can be written in the form

$$\sum_{i,j} a_ia_j^1 \otimes b_ia_j^2 \otimes b_j = \sum_{i,j} a_i \otimes a_jb_i^1 \otimes b_jb_i^2. \tag{2.7.9}$$

Applying the antipode and the comultiplication to the last factor in the tensor product we obtain

$$\sum_{i,j} a_ia_j^1 \otimes b_ia_j^2 \otimes (Sb_j)^1 \otimes (Sb_j)^2 = \sum_{i,j} a_i \otimes a_jb_i^1 \otimes (S(b_jb_i^2))^1 \otimes (S(b_jb_i^2))^2.$$

Multiplying the first and the third, and the second and the fourth tensor factors we deduce

$$\sum_{i,j} a_ia_j^1(Sb_j)^1 \otimes b_ia_j^2(Sb_j)^2 = \sum_{i,j} a_i(S(b_jb_i^2))^1 \otimes a_jb_i^1(S(b_jb_i^2))^2,$$

or, recalling that Δ is an algebra homomorphism, and S is an anticoautomorphism,

$$\sum_{i,j} a_i(a_jSb_j)^1 \otimes b_i(a_jSb_j)^2 = \sum_{i,j} a_iS((b_jb_i^2)^2) \otimes a_jb_i^1S((b_jb_i^2)^1).$$

Now, using the definition of v and the identity $\mathcal{F} = \sum_i a_i \otimes b_i$ in the left hand aside, and recalling that S is an algebra antiautomorphism and Δ is coassociative in the right hand side we have

$$\mathcal{F}\Delta v = \sum_{i,j} a_iS(b_j^2b_i^3) \otimes a_jb_i^1S(b_j^1b_i^2) = \sum_{i,j} a_iS(b_i^3)S(b_j^2) \otimes a_jb_i^1S(b_i^2)S(b_j^1).$$

Applying the identity $S(b_i^3) \otimes b_i^1 S(b_i^2) = S(b_i^2) \otimes \varepsilon(b_i^1) = S(b_i)$ following from the defining properties of the counit and of the antipode we get

$$\mathcal{F}\Delta v = \sum_{i,j} a_i S(b_i^2) S(b_j^2)) \otimes a_j \varepsilon(b_i^1) S(b_j^1) = \sum_{i,j} a_i S(b_i) S(b_j^2) \otimes a_j S(b_j^1) = (v \otimes 1) \sum_j S(b_j^2) \otimes a_j S(b_j^1), \quad (2.7.10)$$

where at the last step we also used the definition of v.

Now we simplify the sum in the right hand side of the last formula. Applying the antipode to the second and the third factor in the tensor product in (2.7.9) and multiplying the first and the second factors after that we have

$$\sum_{i,j} a_i a_j^1 S(a_j^2) S(b_i) \otimes S(b_j) = \sum_{i,j} a_i S(b_i^1) S(a_j) \otimes S(b_i^2) S(b_j).$$

Using the defining property of the antipode in the left hand side, and the formula $\mathcal{F} = \sum_i a_i \otimes b_i$ in the right hand side we can rewrite the previous identity in the form

$$\sum_{i,j} a_i \varepsilon(a_j) S(b_i) \otimes S(b_j) = \sum_i a_i S(b_i^1) \otimes S(b_i^2)(S \otimes S)(\mathcal{F}),$$

or by (2.6.7) and by the definition of v applied in the left hand side

$$v \otimes 1 = \sum_i a_i S(b_i^1) \otimes S(b_i^2)(S \otimes S)(\mathcal{F}),$$

which is, by swapping the tensor factors, equivalent to

$$1 \otimes v = \sum_i S(b_i^2) \otimes a_i S(b_i^1)(S \otimes S)(\mathcal{F}_{21}),$$

or

$$\sum_i S(b_i^2) \otimes a_i S(b_i^1) = (1 \otimes v)(S \otimes S)(\mathcal{F}_{21}^{-1}).$$

Substituting this expression into the right hand side of (2.7.10) and multiplying by $\mathcal{F}^{-1}$ from the left we obtain (2.7.7).

Formula (2.7.8) is established in a similar way. Firstly we rewrite (2.6.8) in the following form

$$(\Delta \otimes id)(\mathcal{F}^{-1})\mathcal{F}_{12}^{-1} = (id \otimes \Delta)(\mathcal{F}^{-1})\mathcal{F}_{23}^{-1}, \quad (2.7.11)$$

or explicitly, using the expression $\mathcal{F}^{-1} = \sum_i c_i \otimes d_i$,

$$\sum_{i,j} c_i^1 c_j \otimes c_i^2 d_j \otimes d_i = \sum_{i,j} c_i \otimes d_i^1 c_j \otimes d_i^2 d_j. \quad (2.7.12)$$

Applying the antipode to the first and to the second factor of the tensor product and the comultiplication to the last factor we obtain from this identity

$$\sum_{i,j} S(c_j) S(c_i^1) \otimes S(d_j) S(c_i^2) \otimes d_i^1 \otimes d_i^2 = \sum_{i,j} S(c_i) \otimes S(c_j) S(d_i^1) \otimes (d_i^2 d_j)^1 \otimes (d_i^2 d_j)^2,$$

or, recalling that $\mathcal{F}^{-1} = \sum_j c_j \otimes d_j$ and swapping the first and the second factors in the tensor product,

$$((S \otimes S)(\mathcal{F}_{21}^{-1}) \otimes 1 \otimes 1) \sum_{i,j} S(c_i^2) \otimes S(c_i^1) \otimes d_i^1 \otimes d_i^2 = \sum_{i,j} S(c_j) S(d_i^1) \otimes S(c_i) \otimes (d_i^2 d_j)^1 \otimes (d_i^2 d_j)^2.$$

Multiplying this identity by $z^1 \otimes z^2 \otimes 1 \otimes 1 = \Delta(z) \otimes 1 \otimes 1$ from the right we also have

$$((S \otimes S)(\mathcal{F}_{21}^{-1}) \otimes 1 \otimes 1) \sum_i S(c_i^2) z^1 \otimes S(c_i^1) z^2 \otimes d_i^1 \otimes d_i^2 = \sum_{i,j} S(c_j) S(d_i^1) z^1 \otimes S(c_i) z^2 \otimes (d_i^2 d_j)^1 \otimes (d_i^2 d_j)^2.$$

Now we multiply the first and the third, and the second and the fourth factors in the tensor product in this identity. Together with the coassociativity of the comultiplication in the right hand side this yields

$$(S \otimes S)(\mathcal{F}_{21}^{-1}) \sum_i S(c_i^2) z^1 d_i^1 \otimes S(c_i^1) z^2 d_i^2 = \sum_{i,j} S(c_j) S(d_i^1) z^1 d_i^2 d_j^1 \otimes S(c_i) z^2 d_i^3 d_j^2. \qquad (2.7.13)$$

Applying the identity

$$\Delta(\sum_i S(c_i) z d_i) = \sum_i S(c_i^2) z^1 d_i^1 \otimes S(c_i^1) z^2 d_i^2$$

in the left hand side of (2.7.13) we obtain

$$(S \otimes S)(\mathcal{F}_{21}^{-1}) \Delta(\sum_i S(c_i) z d_i) = \sum_{i,j} S(c_j) S(d_i^1) z^1 d_i^2 d_j^1 \otimes S(c_i) z^2 d_i^3 d_j^2$$

which implies together with (2.7.5) and (2.7.7)

$$\Delta \psi^{\mathcal{F}}(z) = \Delta(v) \Delta(\sum_i S(c_i) z d_i) = \mathcal{F}^{-1}(v \otimes v)(S \otimes S)(\mathcal{F}_{21}^{-1}) \Delta(\sum_i S(c_i) z d_i) = \qquad (2.7.14)$$

$$= \mathcal{F}^{-1}(v \otimes v) \sum_{i,j} S(c_j) S(d_i^1) z^1 d_i^2 d_j^1 \otimes S(c_i) z^2 d_i^3 d_j^2.$$

Now we bring the right hand side of this identity to the form indicated in (2.7.8). For this purpose we rewrite (2.7.12) multiplying it by $\mathcal{F}_{23}$ from the right, by applying the antipode to the first factor in the tensor product, and by swapping the right hand side and the left hand side,

$$\sum_j S(c_j) \otimes d_j^1 \otimes d_j^2 = \sum_{j,k} S(c_k) S(c_j^1) \otimes c_j^2 d_k \otimes d_j \mathcal{F}_{23}, \qquad (2.7.15)$$

where we also renamed some summation indexes.

Applying this identity to the expression in the right hand side of (2.7.14) and recalling (2.7.5) we finally obtain

$$\Delta \psi^{\mathcal{F}}(z) = \mathcal{F}^{-1}(v \otimes v) \sum_{i,j,k} S(c_k) S(c_j^1) S(d_i^1) z^1 d_i^2 c_j^2 d_k \otimes S(c_i) z^2 d_i^3 d_j \mathcal{F} =$$

$$= \mathcal{F}^{-1}(1 \otimes v) \sum_{i,j} (\psi^{\mathcal{F}}(\mathrm{Ad}(d_i^1 c_j)(z^1)) \otimes S(c_i) z^2 d_i^2 d_j) \mathcal{F}$$

which completes the proof.

$\square$

Apart from Drinfeld twists, there is another natural way for obtaining new comultiplications on a given Hopf algebra described in the following obvious proposition (see e.g. [67], Proposition 5.2).

Proposition 2.7.3. *Let* $(A, \mu, \imath, \Delta, \varepsilon, S)$ *be a Hopf algebra,* $v : A \to A$ *an algebra automorphism. Then* $(A, \mu, \imath, \Delta^v, \varepsilon, S^v)$ *is a Hopf algebra, where*

$$\Delta^v(x) = (v \otimes v)\Delta(v^{-1}x), \ S^v = vS(v^{-1}x), \ x \in A. \qquad (2.7.16)$$

If $(A, \mu, \imath, \Delta, \varepsilon, S)$ *is quasitriangular with universal R–matrix* $\mathcal{R}$ *then* $(A, \mu, \imath, \Delta^v, \varepsilon, S^v)$ *is quasitriangular with universal R–matrix*

$$\mathcal{R}^v = (v \otimes v)\mathcal{R}.$$

We denote the Hopf algebra $(A, \mu, \imath, \Delta^v, \varepsilon, S^v)$ by A^v and call it the twist of A by v. Denote by Ad^v the right adjoint action of A^v. One immediately has the following relation between the adjoint actions of A and of A^v.

Proposition 2.7.4. *Let* $(A, \mu, \imath, \Delta, \varepsilon, S)$ *be a Hopf algebra,* $v : A \to A$ *an algebra automorphism. Then*

$$\mathrm{Ad}^v x(z) = v(\mathrm{Ad} v^{-1}(x)(v^{-1}(z))), x, z \in A. \qquad (2.7.17)$$

Proof. By (2.7.16) we have

$$\mathrm{Ad}^\upsilon x(z) = \upsilon(S(\upsilon^{-1}\upsilon(\upsilon^{-1}x)^1))z\upsilon(\upsilon^{-1}x)^2 = \upsilon(S(\upsilon^{-1}x)^1\upsilon^{-1}(z)(\upsilon^{-1}x)^2) = \upsilon(\mathrm{Ad}\upsilon^{-1}(x)(\upsilon^{-1}(z))).$$

This completes the proof. □

Now we shall relate the two types of twists in the case when $A = U_h(\mathfrak{g})$. The following statements can be found in [67], Theorem 5.1 and Proposition 5.4.

Proposition 2.7.5. *Let $w = s_{i_1} \ldots s_{i_k}$ be any reduced decomposition of an element $w \in W$, $T_w = T_{i_1} \ldots T_{i_k}$. Denote $\theta_w = \prod_{p=1}^k \theta_{\beta_p}$, where θ_{β_p} are defined in (2.2.11), and in the product θ_{β_p} appears on the left from θ_{β_q} if $p < q$. Then the following statements are true.*

(i) The element $\theta_w^{-1} \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ satisfies (2.6.7) and (2.6.8), and $\Delta_h^{\theta_w^{-1}}$ is a comultiplication on $U_h(\mathfrak{g})$.

(ii) One has $\Delta_h^{\theta_w^{-1}} = \Delta_h^{T_w}$.

Proof. (i) The proof is by induction over the length of w. When $w = s_i$ is a simple reflection condition (2.6.7) for $\theta_w^{-1} = \theta_i^{-1}$ is clear from its definition and from the definition of the counit ε_h, and condition (2.6.8) is the statement of Proposition 4.2.4 in [81], up to some change of the notation and different conventions on the comultiplication. The second claim in (i) for $\theta_w^{-1} = \theta_i^{-1}$ follows from the first one and from part (i) of Proposition 2.6.4.

Now assume that the statement is true for all elements w of length k and for some $w = s_{i_1} \ldots s_{i_k}$ and $1 \leq i \leq l$ the element $w' = s_i w$ has length $k + 1$. Then by the definitions of θ_w and $\theta_{s_i w}$ one has

$$\theta_{s_i w} = \theta_i(T_i \otimes T_i)(\theta_w),$$

so

$$\theta_{s_i w}^{-1} = (T_i \otimes T_i)(\theta_w^{-1})\theta_i^{-1}. \tag{2.7.18}$$

Condition (2.6.7) for $\theta_{s_i w}^{-1}$ is again clear from its definition and from the definition of the counit ε_h.
To check condition (2.6.8) for $\theta_{s_i w}^{-1}$ we write using (2.7.18)

$$(\theta_{s_i w}^{-1})_{12}(\Delta_h \otimes id)(\theta_{s_i w}^{-1}) = (T_i \otimes T_i \otimes id)(\theta_w^{-1})_{12}(\theta_i^{-1})_{12}(\Delta_h \otimes id)(T_i \otimes T_i \theta_w^{-1} T_i^{-1} \otimes T_i^{-1})(\Delta_h \otimes id)(\theta_i^{-1}).$$

Observe that by (2.2.8) $\Delta_h(T_i) = \theta_i T_i \otimes T_i$, so $\Delta_h(T_i(x)) = \theta_i(T_i \otimes T_i)(\Delta_h(x))\theta_i^{-1}$, $x \in U_h(\mathfrak{g})$, and that T_i is an algebra automorphism. Hence the previous identity can be rewritten as follows

$$(\theta_{s_i w}^{-1})_{12}(\Delta_h \otimes id)(\theta_{s_i w}^{-1}) =$$

$$= (T_i \otimes T_i \otimes T_i)(\theta_w^{-1})_{12}(\theta_i^{-1})_{12}(\theta_i)_{12}(T_i \otimes T_i \otimes T_i)((\Delta_h \otimes id)(\theta_w^{-1}))(\theta_i^{-1})_{12}(\Delta_h \otimes id)(\theta_i^{-1}) =$$

$$= (T_i \otimes T_i \otimes T_i)(\theta_w^{-1})_{12}(T_i \otimes T_i \otimes T_i)((\Delta_h \otimes id)(\theta_w^{-1}))(\theta_i^{-1})_{12}(\Delta_h \otimes id)(\theta_i^{-1}) =$$

$$= (T_i \otimes T_i \otimes T_i)(\theta_w^{-1})_{12}(T_i \otimes T_i \otimes T_i)((\Delta_h \otimes id)(\theta_w^{-1}))(\theta_i^{-1})_{12}(\Delta_h \otimes id)(\theta_i^{-1}) =$$

$$= (T_i \otimes T_i \otimes T_i)((\theta_w^{-1})_{12}(\Delta_h \otimes id)(\theta_w^{-1}))(\theta_i^{-1})_{12}(\Delta_h \otimes id)(\theta_i^{-1}). \tag{2.7.19}$$

Now recall that by the induction assumption θ_i and θ_w satisfy (2.6.8), so that after applying (2.6.8) in the right hand side of the previous identity we have

$$(\theta_{s_i w}^{-1})_{12}(\Delta_h \otimes id)(\theta_{s_i w}^{-1}) = (T_i \otimes T_i \otimes T_i)((\theta_w^{-1})_{23}(id \otimes \Delta_h)(\theta_w^{-1}))(\theta_i^{-1})_{23}(id \otimes \Delta_h)(\theta_i^{-1}).$$

Finally repeating in the opposite order arguments similar to those which lead us to (2.7.19) we arrive at

$$(\theta_{s_i w}^{-1})_{12}(\Delta_h \otimes id)(\theta_{s_i w}^{-1}) =$$

$$= (T_i \otimes T_i)((\theta_w^{-1}))_{23}(\theta_i^{-1})_{23}(\theta_i)_{23}(T_i \otimes T_i \otimes T_i)((id \otimes \Delta_h)(\theta_w^{-1}))(\theta_i^{-1})_{23}(id \otimes \Delta_h)(\theta_i^{-1}) =$$

$$= (\theta_{s_i w}^{-1})_{23}(id \otimes \Delta_h)((T_i \otimes T_i)(\theta_w^{-1}))(id \otimes \Delta_h)(\theta_i^{-1}) = (\theta_{s_i w}^{-1})_{23}(id \otimes \Delta_h)(\theta_{s_i w}^{-1})$$

which confirms (2.6.8) for $\theta_{s_i w}^{-1}$ and establishes the induction step. Thus the first claim in (i) is proved.

The second claim in (i) for θ_w^{-1} with arbitrary w follows from the first one and from part (i) of Proposition 2.6.4.

(ii) Observe that by (2.2.11) $\Delta_h(T_w^{-1}) = T_w^{-1} \otimes T_w^{-1}\theta_w^{-1}$, and hence

$$\Delta_h(T_w^{-1}(x)) = (T_w^{-1} \otimes T_w^{-1})(\theta_w^{-1}\Delta_h(x)\theta_w), x \in U_h(\mathfrak{g}), \tag{2.7.20}$$

or, by the definitions of $\Delta_h^{T_w}$ and of $\Delta_h^{\theta_w^{-1}}$,

$$\Delta_h^{T_w}(x) = (\theta_w^{-1}\Delta_h(x)\theta_w) = \Delta_h^{\theta_w^{-1}}(x), x \in U_h(\mathfrak{g}).$$

This completes the proof.

$\square$

We shall need the following proposition which is a version of Proposition 2.7.5 for the algebras $U_h^s(\mathfrak{g})$. To simplify the notation we shall identify $U_h^s(\mathfrak{g})$ with $U_h(\mathfrak{g})$ as algebras using the isomorphisms from Proposition 2.6.1 with n_{ij} given by (2.6.4), and omit $\psi_{\{n_{ij}\}}$ in all formulas.

Proposition 2.7.6. *Let $w = s_{i_1} \ldots s_{i_k}$ be any reduced decomposition of an element $w \in W$, $T_w = T_{i_1} \ldots T_{i_k}$. Denote $\theta_w^s = \prod_{p=1}^k \theta_{\beta_p}^s$, where $\theta_{\beta_p}^s$ are defined in (2.6.17), and in the product $\theta_{\beta_p}^s$ appears on the left from $\theta_{\beta_q}^s$ if $p < q$. Then the following statements are true.*

(i) The element $\theta_w^{s\,-1} \in U_h^s(\mathfrak{g}) \otimes U_h^s(\mathfrak{g})$ satisfies (2.6.7) and (2.6.8), for $\varepsilon = \varepsilon_s$ and $\Delta = \Delta_s$, and $\Delta_s^{\theta_w^{s\,-1}}$ is a comultiplication on $U_h^s(\mathfrak{g})$.

(ii) One has $\Delta_s^{\theta_w^{s\,-1}} = \Delta_{w^{-1}sw}^{T_w}$.

Proof. (i) Firstly by (2.6.17)

$$\mathcal{F}_s\theta_w\mathcal{F}_s^{-1} = \theta_w^s. \tag{2.7.21}$$

From the definition of the commutative Hopf subalgebra $U(\mathfrak{h})[[h]] \simeq U_h(\mathfrak{h}) \subset U_h(\mathfrak{g}) \simeq U_h^s(\mathfrak{g})$ it immediately follows that $\mathcal{F}_s^{-1} \in U_h^s(\mathfrak{g}) \otimes U_h^s(\mathfrak{g})$ satisfies (2.6.7) and (2.6.8) for $\varepsilon = \varepsilon_s$ and $\Delta = \Delta_s$, so $\Delta_s^{\mathcal{F}_s^{-1}}$ is a comultiplication on $U_h^s(\mathfrak{g})$ according to Proposition 2.6.4 (i). In fact by formula (2.6.13)

$$\Delta_s^{\mathcal{F}_s^{-1}} = \Delta_h. \tag{2.7.22}$$

By Proposition 2.7.5 (i) and (ii) the element $\theta_w^{-1} \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ satisfies (2.6.7) and (2.6.8) with $\varepsilon = \varepsilon_h$ and $\Delta = \Delta_h$, $\Delta_h^{\theta_w^{-1}}$ is a comultiplication on $U_h(\mathfrak{g})$, and $\Delta_h^{\theta_w^{-1}} = \Delta_h^{T_w}$. Using this identity and (2.7.22) we obtain

$$(\Delta_s^{\mathcal{F}_s^{-1}})^{\theta_w^{-1}} = \Delta_h^{\theta_w^{-1}} = \Delta_h^{T_w}. \tag{2.7.23}$$

Finally note that on the subalgebra $U_h(\mathfrak{h}) \subset U_h(\mathfrak{g})$ one has $\Delta_h^{T_w} = \Delta_h$ and from the definition of the commutative Hopf subalgebra $U(\mathfrak{h})[[h]] \simeq U_h(\mathfrak{h}) \subset U_h(\mathfrak{g}) \simeq U_h^s(\mathfrak{g})$ it immediately follows that $\mathcal{F}_s \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ satisfies (2.6.7) and (2.6.8) for $\varepsilon = \varepsilon_h$ and $\Delta = \Delta_h^{T_w}$, so $(\Delta_h^{T_w})^{\mathcal{F}_s}$ is a comultiplication on $U_h(\mathfrak{g})$ according to Proposition 2.6.4 (i).

Using this observation together with (2.7.23) and Proposition 2.6.4 (iii) twice we deduce that $\mathcal{F}_s\theta_w^{-1}\mathcal{F}_s^{-1} = \theta_w^{s\,-1}$ satisfies (2.6.7) and (2.6.8) for $\varepsilon = \varepsilon_s$ and $\Delta = \Delta_s$, $\Delta_s^{\theta_w^{s\,-1}}$ is a comultiplication on $U_h^s(\mathfrak{g})$, and

$$\Delta_s^{\theta_w^{s\,-1}} = ((\Delta_s^{\mathcal{F}_s^{-1}})^{\theta_w^{-1}})^{\mathcal{F}_s} = (\Delta_h^{T_w})^{\mathcal{F}_s}. \tag{2.7.24}$$

(ii) Observe that by (2.6.18) and by the definition (2.6.12) of $\mathcal{F}_s$ one has $\Delta_s(T_w^{-1}) = T_w^{-1} \otimes T_w^{-1}(T_w \otimes T_w)(\mathcal{F}_s)\mathcal{F}_s^{-1}\theta_w^{s\,-1}$, and hence by the definition of $\Delta_s^{T_w}$ we obtain

$$\Delta_s^{T_w}(x) = (T_w \otimes T_w)(\mathcal{F}_s)\mathcal{F}_s^{-1}\theta_w^{s\,-1}\Delta_s(x)\theta_w^s\mathcal{F}_s(T_w \otimes T_w)(\mathcal{F}_s^{-1}), x \in U_h^s(\mathfrak{g}).$$

Using the definition of $\Delta_s^{T_w}$ and conjugating by $\mathcal{F}_s(T_w \otimes T_w)(\mathcal{F}_s^{-1})$ one can rewrite this identity as follows

$$\mathcal{F}_s(T_w \otimes T_w)(\mathcal{F}_s^{-1})(T_w \otimes T_w)(\Delta_s(T_w^{-1}x))(T_w \otimes T_w)(\mathcal{F}_s)\mathcal{F}_s^{-1} = \theta_w^{s\,-1}\Delta_s(x)\theta_w^s.$$

Using (2.6.13) and the definition of the braid group action on the generators of $U_h(\mathfrak{h})$ we can further transform the left hand side and obtain

$$(T_w \otimes T_w)((T_w^{-1} \otimes T_w^{-1})(\mathcal{F}_s)\mathcal{F}_s^{-1}\mathcal{F}_s\Delta_h(T_w^{-1}x)\mathcal{F}_s^{-1}\mathcal{F}_s(T_w^{-1} \otimes T_w^{-1})(\mathcal{F}_s^{-1})) = \theta_w^{s\,-1}\Delta_s(x)\theta_w^s,$$

or

$$(T_w \otimes T_w)((T_w^{-1} \otimes T_w^{-1})(\mathcal{F}_s)\Delta_h(T_w^{-1}x)(T_w^{-1} \otimes T_w^{-1})(\mathcal{F}_s^{-1})) = \theta_w^{s\,-1}\Delta_s(x)\theta_w^s. \tag{2.7.25}$$

By definition (2.6.12) $(T_w^{-1} \otimes T_w^{-1})(\mathcal{F}_s) = \mathcal{F}_{w^{-1}sw}$, and hence by (2.6.13)

$$(T_w^{-1} \otimes T_w^{-1})(\mathcal{F}_s)\Delta_h(T_w^{-1}x)(T_w^{-1} \otimes T_w^{-1})(\mathcal{F}_s^{-1}) = \mathcal{F}_{w^{-1}sw}\Delta_h(T_w^{-1}x)\mathcal{F}_{w^{-1}sw}^{-1} = \Delta_{w^{-1}sw}(T_w^{-1}x).$$

Using this expression in the left hand side of (2.7.25) and recalling the definitions of $\Delta_s^{\theta_w^{s\,-1}}$ and of $\Delta_{w^{-1}sw}^{T_w}$ we derive $\Delta_{w^{-1}sw}^{T_w} = \Delta_s^{\theta_w^{s\,-1}}$. This completes the proof.

$\square$

Rewriting the identity in part (ii) of the previous proposition in the form

$$\Delta_{w^{-1}sw} = (\Delta_s^{\theta_w^{s\,-1}})^{T_w^{-1}}$$

and applying formulas (2.7.4) and (2.7.17) we obtain the following corollary.

Corollary 2.7.7. *Denote by* Ad_s *and* Ad_s' *the right and the left adjoint action of the Hopf algebra* $U_h^s(\mathfrak{g})$, *respectively. Then*

$$\mathrm{Ad}_{w^{-1}sw}x = (\mathrm{Ad}_s^{\theta_w^{s\,-1}})^{T_w^{-1}}(x) = T_w^{-1}\psi^{\theta_w^{s\,-1}}\mathrm{Ad}_s(T_wx)(\psi^{\theta_w^{s\,-1}})^{-1}T_w, \tag{2.7.26}$$

where $\psi^{\theta_w^{s\,-1}}$ *is defined by (2.7.5) with the help of the antipode* S_s *and of the multiplication in the Hopf algebra* $U_h^s(\mathfrak{g})$.

From (2.7.8) and (2.6.17) we also obtain the following proposition which will be crucial for the definition of the Zhelobenko type operators.

Proposition 2.7.8. *Let* $w = s_{i_1} \ldots s_{i_k}$ *be any reduced decomposition of an element* $w \in W$, $T_w = T_{i_1} \ldots T_{i_k}$. *Then*

$$\Delta_{w^{-1}sw}(T_w^{-1}\psi^{\theta_w^{s\,-1}}(z)T_w) = T_w^{-1} \otimes T_w^{-1}(1 \otimes v)\sum_{i,j}(\psi^{\theta_w^{s\,-1}}(\mathrm{Ad}_s(d_i^1c_j)(z^1)) \otimes S_s(c_i)z^2d_i^2d_j)T_w \otimes T_w, \tag{2.7.27}$$

where $\theta_w^{s\,-1} = \sum_i a_i \otimes b_i$, $v = \sum_i a_iS_s(b_i)$, $\theta_w^s = \sum_i c_i \otimes d_i$, $\Delta_s z = z^1 \otimes z^2$, $\Delta_s d_i = d_i^1 \otimes d_i^2$.

Proof. From (2.7.8) with $\Delta = \Delta_s$, $\mathcal{F} = \theta_w^{s\,-1}$, $S = S_s$ we obtain

$$\Delta_s(\psi^{\theta_w^{s\,-1}}(z)) = \theta_w^s(1 \otimes v)\sum_{i,j}(\psi^{\theta_w^{s\,-1}}(\mathrm{Ad}_s(d_i^1c_j)(z^1)) \otimes S_s(c_i)z^2d_i^2d_j)\theta_w^{s\,-1}, \tag{2.7.28}$$

where $\theta_w^{s\,-1} = \sum_i a_i \otimes b_i$, $v = \sum_i a_iS_s(b_i)$, $\theta_w^s = \sum_i c_i \otimes d_i$, $\Delta_s z = z^1 \otimes z^2$.

By (2.6.17) and (2.6.6)

$$\Delta_s(T_w) = \prod_{p=1}^k \theta_{\beta_p}^s\mathcal{F}_s(T_w \otimes T_w)(\mathcal{F}_s^{-1})T_w \otimes T_w = \theta_w^s\mathcal{F}_s(T_w \otimes T_w)(\mathcal{F}_s^{-1})T_w \otimes T_w = \tag{2.7.29}$$

$$= \theta_w^sT_w \otimes T_w(T_w^{-1} \otimes T_w^{-1})(\mathcal{F}_s)\mathcal{F}_s^{-1} = \theta_w^sT_w \otimes T_w\mathcal{F}_{w^{-1}sw}\mathcal{F}_s^{-1}.$$

Now (2.7.28) and (2.7.29) imply

$$\Delta_s(T_w^{-1}\psi^{\theta_w^{s\,-1}}(z)T_w) = \Delta_s(T_w^{-1})\Delta_s(\psi^{\theta_w^{s\,-1}}(z))\Delta_s(T_w) =$$

$$= \mathcal{F}_s\mathcal{F}_{w^{-1}sw}^{-1}T_w^{-1} \otimes T_w^{-1}\theta_w^{s\,-1}\theta_w^s(1 \otimes v)\sum_{i,j}(\psi^{\theta_w^{s\,-1}}(\mathrm{Ad}_s(d_i^1c_j)(z^1)) \otimes S_s(c_i)z^2d_i^2d_j)\theta_w^{s\,-1}\theta_w^sT_w \otimes T_w\mathcal{F}_{w^{-1}sw}\mathcal{F}_s^{-1} =$$

$$= \mathcal{F}_s\mathcal{F}_{w^{-1}sw}^{-1}T_w^{-1} \otimes T_w^{-1}(1 \otimes v)\sum_{i,j}(\psi^{\theta_w^{s\,-1}}(\mathrm{Ad}_s(d_i^1c_j)(z^1)) \otimes S_s(c_i)z^2d_i^2d_j)T_w \otimes T_w\mathcal{F}_{w^{-1}sw}\mathcal{F}_s^{-1}.$$

Conjugating this formula by $\mathcal{F}_{w^{-1}sw}\mathcal{F}_s^{-1}$ and observing that by the definition of Δ_s one has

$$\mathcal{F}_{w^{-1}sw}\mathcal{F}_s^{-1}\Delta_s(\cdot)\mathcal{F}_s\mathcal{F}_{w^{-1}sw}^{-1} = \mathcal{F}_{w^{-1}sw}\mathcal{F}_s^{-1}\mathcal{F}_s\Delta_h(\cdot)\mathcal{F}_s^{-1}\mathcal{F}_s\mathcal{F}_{w^{-1}sw}^{-1} =$$

$$= \mathcal{F}_{w^{-1}sw}\Delta_h(\cdot)\mathcal{F}_{w^{-1}sw}^{-1} = \Delta_{w^{-1}sw}(\cdot)$$

we obtain

$$\Delta_{w^{-1}sw}(T_w^{-1}\psi^{\theta_w^{s\,-1}}(z)T_w) = T_w^{-1} \otimes T_w^{-1}(1 \otimes v)\sum_{i,j}(\psi^{\theta_w^{s\,-1}}(\mathrm{Ad}_s(d_i^1c_j)(z^1)) \otimes S_s(c_i)z^2d_i^2d_j)T_w \otimes T_w.$$

This completes the proof.

$\square$

2.8 Some forms and specializations of quantum groups

In order to define q-W–algebras we shall actually need not the algebras $U_h^s(\mathfrak{g})$ themselves but some their forms defined over certain rings. They are similar to the rational form and the restricted integral form for the standard quantum group $U_h(\mathfrak{g})$. The motivations of the definitions given below will be clear in Section 3.2. The results below are slight modifications of similar statements for $U_h(\mathfrak{g})$.

We start with a very important technical lemma which will play the key role in the definition of q-W–algebras. Below we keep the notation introduced in Section 1.2.

Let $s \in W$ be an element of the Weyl group. By formula (1.2.1), s can be represented as a product of two involutions, $s = s^1 s^2$, where $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, and the roots $\gamma_1, \ldots, \gamma_{l'}$ form a basis of a subspace $\mathfrak{h}'^* \subset \mathfrak{h}^*$ on which s acts without fixed points. We shall study the matrix elements of the Cayley transform of the restriction of s to $\mathfrak{h}'^*$ with respect to this basis.

Lemma 2.8.1. *Let $P_{\mathfrak{h}'^*}$ be the orthogonal projection operator onto $\mathfrak{h}'^*$ in $\mathfrak{h}^*$, with respect to the bilinear form. Then the matrix elements of the operator $\frac{1+s}{1-s} P_{\mathfrak{h}'^*}$ in the basis $\gamma_1, \ldots, \gamma_{l'}$ are of the form*

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \gamma_i, \gamma_j \right) = \varepsilon_{ij} (\gamma_i, \gamma_j), \tag{2.8.1}$$

where

$$\varepsilon_{ij} = \begin{cases} -1 & i < j \\ 0 & i = j \\ 1 & i > j \end{cases} .$$

Proof. First we calculate the matrix of the element s with respect to the basis $\gamma_1, \ldots, \gamma_{l'}$. We obtain this matrix in the form of the Gauss decomposition of the operator $s : \mathfrak{h}'^* \to \mathfrak{h}'^*$.

Let $z_i = s\gamma_i$. Recall that $s_{\gamma_i}(\gamma_j) = \gamma_j - A_{ij}\gamma_i$, $A_{ij} = (\gamma_i^\vee, \gamma_j)$. Define

$$y_i = s_{\gamma_1} \ldots s_{\gamma_{i-1}} \gamma_i. \tag{2.8.2}$$

Using this definition the elements z_i may be represented as

$$z_i = s_{\gamma_1} \ldots s_{\gamma_{l'}} \gamma_i = s_{\gamma_1} \ldots s_{\gamma_{l'-1}} (\gamma_i - A_{l'i}\gamma_{l'}) = s_{\gamma_1} \ldots s_{\gamma_{l'-1}} \gamma_i - A_{l'i} y_{l'} = \ldots = y_i - \sum_{k \geq i} A_{ki} y_k.$$

Using the matrix notation we can rewrite the last formula as follows

$$z_i = \sum_{k=1}^{l'} (I - V)_{ki} y_k, \quad \text{where } V_{ki} = \begin{cases} A_{ki} & k \geq i \\ 0 & k < i \end{cases} \tag{2.8.3}$$

To calculate the matrix of the operator $s : \mathfrak{h}'^* \to \mathfrak{h}'^*$ with respect to the basis $\gamma_1, \ldots, \gamma_{l'}$ we have to express the elements y_i via $\gamma_1, \ldots, \gamma_{l'}$. Applying the definition of reflections to (2.8.2) we obtain

$$y_i = s_{\gamma_1} \ldots s_{\gamma_{i-1}} \gamma_i = s_{\gamma_1} \ldots s_{\gamma_{i-2}} \gamma_i - A_{i-1i} y_{i-1} = \ldots = \gamma_i - \sum_{k < i} A_{ki} y_k.$$

Therefore

$$\gamma_i = \sum_{k=1}^{l'} (I + U)_{ki} y_k , \quad \text{where } U_{ki} = \begin{cases} A_{ki} & k < i \\ 0 & k \geq i \end{cases}$$

Thus

$$y_k = \sum_{k=1}^{l'} (I + U)_{jk}^{-1} \gamma_j. \tag{2.8.4}$$

Summarizing (2.8.4) and (2.8.3) we obtain

$$s\gamma_i = \sum_{k=1}^{l'} \left((I + U)^{-1} (I - V) \right)_{ki} \gamma_k. \tag{2.8.5}$$

This implies

$$\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i = \sum_{k=1}^{l'}\left(\frac{2I+U-V}{U+V}\right)_{ki}\gamma_k. \tag{2.8.6}$$

Observe that $(U+V)_{ki} = A_{ki}$ and $(2I+U-V)_{ij} = -A_{ij}\varepsilon_{ij}$. Substituting these expressions into (2.8.6) we get

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j\right) = -(A^{-1})_{kp}\varepsilon_{pi}A_{pi}(\gamma_j,\gamma_k) = \varepsilon_{ij}(\gamma_i,\gamma_j). \tag{2.8.7}$$

This completes the proof of the lemma.

$\square$

Let γ_i^*, $i = 1,\ldots,l'$ be the basis of $\mathfrak{h}'^*$ dual to γ_i, $i = 1,\ldots,l'$ with respect to the restriction of the bilinear form $(\cdot,\cdot)$ to $\mathfrak{h}'^*$. Since the numbers (γ_i,γ_j) are integer, each element γ_i^* has the form $\gamma_i^* = \sum_{j=1}^{l'} m_{ij}\gamma_j$, where $m_{ij} \in \mathbb{Q}$. Therefore by Lemma 2.8.1 and using for simple roots α_i the decomposition of the form $P_{\mathfrak{h}'^*}\alpha_i = \sum_{p=1}^{l'}(\alpha_i,\gamma_p)\gamma_p^* = \sum_{p,q=1}^{l'}(\alpha_i,\gamma_p)m_{pq}\gamma_q$ we deduce that the numbers

$$p_{ij} = \frac{1}{2d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j\right) = \tag{2.8.8}$$

$$= \frac{1}{2d_j}\sum_{k,l,p,q=1}^{l'}(\gamma_k,\alpha_i)(\gamma_l,\alpha_j)\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_p,\gamma_q\right)m_{kp}m_{lq}, \ i,j = 1,\ldots,l$$

are rational, $p_{ij} \in \mathbb{Q}$, as all factors in the products in the sum in the right hand side are rational. Denote by d an integer number divisible by all the denominators of the rational numbers p_{ij}, $i,j = 1,\ldots,l$.

Let $r \in \mathbb{N}$ be such that $a_{ij}^{-1} \in \frac{1}{r}\mathbb{Z}$, $i,j = 1,\ldots,l$. Let $U_q^s(\mathfrak{g})$ be the $\mathbb{C}(q^{\frac{1}{dr^2}})$-algebra generated by the elements $e_i, f_i, L_i^{\pm 1}, t_i^{\pm 1}$, $i = 1,\ldots,l$ with the same relations as the relations in $U_h^s(\mathfrak{g})$ for the generators denoted by the same symbols, where we assume that $t_i^{\pm 1} = \exp(\pm\frac{h\kappa}{d}Y_i)$. The coefficients of these relations indeed belong to $\mathbb{C}(q^{\frac{1}{dr^2}})$, where $q^{\frac{1}{dr^2}} = e^{h\frac{1}{dr^2}}$.

Let $U_q(\mathfrak{g})$ be the $\mathbb{C}(q^{\frac{1}{dr^2}})$-algebra generated by the elements $X_i^{\pm}, L_i^{\pm 1}, t_i^{\pm 1}$, $i = 1,\ldots,l$ subject to the same relations as the relations in $U_h(\mathfrak{g})$ for the generators denoted by the same symbols, where we assume that $t_i^{\pm 1} = \exp(\pm\frac{h\kappa}{d}Y_i)$. The coefficients of these relations indeed belong to $\mathbb{C}(q^{\frac{1}{dr^2}})$, where $q^{\frac{1}{dr^2}} = e^{h\frac{1}{dr^2}}$.

Note that by the choice of d we have $q^{c_{ij}} \in \mathbb{C}[q^{\frac{1}{d}},q^{-\frac{\kappa}{d}}]$.

The second form of $U_h^s(\mathfrak{g})$ is a subalgebra $U_{\mathcal{A}}^s(\mathfrak{g})$ in $U_q^s(\mathfrak{g})$ over the ring $\mathcal{A} = \mathcal{P}[q^{\frac{1}{dr^2}},q^{-\frac{1}{dr^2}}]$. $U_{\mathcal{A}}^s(\mathfrak{g})$ is the subalgebra in $U_q^s(\mathfrak{g})$ generated over $\mathcal{A}$ by the elements

$$L_i^{\pm 1}, t_i^{\pm 1}, \ \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \ e_i, \ f_i, \ i = 1,\ldots,l.$$

Denote also by $U_{\mathcal{A}}(\mathfrak{g})$ the subalgebra in $U_q(\mathfrak{g})$ generated over $\mathcal{A}$ by the elements

$$L_i^{\pm 1}, t_i^{\pm 1}, \ \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \ X_i^{\pm}, \ i = 1,\ldots,l.$$

For the solution $n_{ij} = \frac{1}{2d_j}c_{ij}$ to equations (2.6.2) the root vectors e_β, f_β belong to all the above introduced specializations of $U_h(\mathfrak{g})$.

For any normal ordering of Δ_+ we denote $e_\beta^{(k)} = \frac{e_\beta^k}{[k]_{q_\alpha}!}, f_\beta^{(k)} = \frac{f_\beta^k}{[k]_{q_\alpha}!}$, where e_β, f_β are the corresponding quantum root vectors from Proposition 2.6.2. Let $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ be the subalgebra in $U_q^s(\mathfrak{g})$ generated over $\mathcal{B} = \mathbb{C}[q^{\frac{1}{dr^2}},q^{-\frac{1}{dr^2}}]$ by the elements

$$L_i^{\pm 1}, t_i^{\pm 1}, \ e_i^{(k)}, \ f_i^{(k)}, \ i = 1,\ldots,l, k \geq 1.$$

Denote also by $U_{\mathcal{B}}^{res}(\mathfrak{g})$ the subalgebra in $U_q(\mathfrak{g})$ generated over $\mathcal{B}$ by the elements

$$L_i^{\pm 1}, t_i^{\pm 1}, \ (X_i^{\pm})^{(k)}, \ i = 1,\ldots,l, k \geq 1.$$

Let $\varepsilon \in \mathbb{C}^*$. Fix a root of ε of degree $r^2 d$, $\varepsilon^{\frac{1}{dr^2}}$ and if $\varepsilon = 1$ put $\varepsilon^{\frac{1}{r^2 d}} = 1$. Then we define the specialization $U_\varepsilon^s(\mathfrak{g})$ of $U_{\mathcal{A}}^s(\mathfrak{g})$, $U_\varepsilon^s(\mathfrak{g}) = U_{\mathcal{A}}^s(\mathfrak{g})/(q^{\frac{1}{dr^2}} - \varepsilon^{\frac{1}{dr^2}})U_{\mathcal{A}}^s(\mathfrak{g})$, and

$$U_\varepsilon^{s,res}(\mathfrak{g}) = U_{\mathcal{B}}^{s,res}(\mathfrak{g})/(q^{\frac{1}{dr^2}} - \varepsilon^{\frac{1}{dr^2}})U_{\mathcal{B}}^{s,res}(\mathfrak{g})$$

$U_q^s(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$, $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$, $U_\varepsilon^{s,res}(\mathfrak{g})$ and $U_\varepsilon^s(\mathfrak{g})$ are Hopf algebras with the comultiplication given on generators by the same formulas as in $U_h^s(\mathfrak{g})$ with $q^{\frac{1}{dr^2}} = e^{h\frac{1}{dr^2}}$.

The elements t_i and L_i are central in the algebra $U_1^s(\mathfrak{g})$, and the quotient of $U_1^s(\mathfrak{g})$ by the two–sided ideal generated by $t_i - 1$ and $L_i - 1$ is isomorphic to $U(\mathfrak{g})$. Note that none of the specializations of $U_h^s(\mathfrak{g})$ introduced above is quasitriangular. Similarly, the quotient of $U_1^{s,res}(\mathfrak{g})$ by the two–sided ideal generated by $t_i - 1$ and $L_i - 1$ is isomorphic to $U(\mathfrak{g})$, and the quotient of $U_1^{res}(\mathfrak{g})$ by the two–sided ideal generated by and $K_i - 1$ is isomorphic to $U(\mathfrak{g})$ (see [20], Proposition 9.3.10).

The algebra isomorphism $\psi_{\{n_{ij}\}}$ with $n_{ij} = \frac{1}{2d_j}c_{ij}$ induces isomorphisms of $U_h(\mathfrak{g})$ and $U_h^s(\mathfrak{g})$ and of the forms and specializations of $U_h(\mathfrak{g})$ and $U_h^s(\mathfrak{g})$ with the superscript s defined above and of their counterparts with the superscript s dropped. We shall always identify them using these isomorphisms. To simplify the notation we shall also write, if it does not cause confusion, $e_\beta = X_\beta^+ e^{hK_s\beta^\vee}$, $f_\beta = e^{-hK_s\beta^\vee}X_\beta^-$, $\beta \in \Delta_+$.

Note that the structure constants in the commutation relations for $U_{\mathcal{A}}^s(\mathfrak{g}) = U_{\mathcal{A}}(\mathfrak{g})$ actually belong to $\mathcal{P}[q^{\frac{\kappa}{d}}, q^{-\frac{\kappa}{d}}]$, so if κ is divisible by d the specialization $U_\varepsilon^s(\mathfrak{g})$ actually depends on $\varepsilon \in \mathbb{C}^*$ but not on its root $\varepsilon^{\frac{1}{dr^2}}$. As we shall see below one can define an action of the universal R-matrix $\mathcal{R}^s$ on tensor products of finite rank $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–modules. This action will play a crucial role in subsequent considerations.

$U_{\mathcal{A}}(\mathfrak{g})$, $U_{\mathcal{B}}^{res}(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$ and $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ can be also regarded as subalgebras of $U_h^s(\mathfrak{g}) \simeq U_h(\mathfrak{g})$, and $U_q^{res}(\mathfrak{g})$ can be regarded as a $\mathbb{C}[q, q^{-1}]$–subalgebra in $U_{\mathcal{B}}^{s,res}(\mathfrak{g}) \simeq U_q^{res}(\mathfrak{g})$.

Denote by $U_q^s(\mathfrak{n}_+), U_q^s(\mathfrak{n}_-)$ and $U_q^s(\mathfrak{h})$ the subalgebras of $U_q^s(\mathfrak{g})$ generated by the e_i, f_i and by the t_i, L_i, respectively, and let $U_q^s(\mathfrak{b}_\pm)$ be the subalgebra in $U_q^s(\mathfrak{g})$ generated by $U_q^s(\mathfrak{n}_\pm)$ and by $U_q^s(\mathfrak{h})$, $U_q^s(\mathfrak{b}_\pm) = U_q^s(\mathfrak{n}_\pm)U_q^s(\mathfrak{h})$.

Let $U_{\mathcal{A}}^s(\mathfrak{n}_+), U_{\mathcal{A}}^s(\mathfrak{n}_-)$ $(U_{\mathcal{B}}^{s,res}(\mathfrak{n}_+), U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-))$ be the subalgebras of $U_{\mathcal{A}}^s(\mathfrak{g})$ $(U_{\mathcal{B}}^{s,res}(\mathfrak{g}))$ generated by the e_i and by the f_i, $i = 1,\ldots,l$ (by the $e_i^{(r)}$ and by the $f_i^{(r)}$, $i = 1,\ldots,l$, $r \geq 0$), respectively, and $U_{\mathcal{A}}(\mathfrak{h}) = U_{\mathcal{A}}^s(\mathfrak{h})$ the subalgebra in $U_{\mathcal{A}}(\mathfrak{g}) = U_{\mathcal{A}}^s(\mathfrak{g})$ generated by $t_i, L_i, i = 1,\ldots,l$.

The elements

$$\begin{bmatrix} K_i; c \\ r \end{bmatrix}_{q_i} = \prod_{s=1}^{r} \frac{K_i q_i^{c+1-s} - K_i^{-1} q_i^{s-1-c}}{q_i^s - q_i^{-s}} \ , \ i = 1,\ldots,l, \ c \in \mathbb{Z}, \ r \in \mathbb{N}$$

belong to $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$. Denote by $U_{\mathcal{B}}^{s,res}(\mathfrak{h})$ the subalgebra of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ generated by those elements and by $t_i^{\pm 1}, L_i^{\pm 1}$, $i = 1,\ldots,l$.

Let $U_{\mathcal{B}}^{s,res}(\mathfrak{b}_\pm)$ be the subalgebra in $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ generated by $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_\pm)$ and by $U_{\mathcal{B}}^{s,res}(\mathfrak{h})$, $U_{\mathcal{B}}^{s,res}(\mathfrak{b}_\pm) = U_{\mathcal{B}}^{s,res}(\mathfrak{n}_\pm)U_{\mathcal{B}}^{s,res}(\mathfrak{h})$.

The algebra antiautomorphism ω of $U_h(\mathfrak{g})$ defined by (2.3.11) gives rise to an algebra antiautomorphism of $U_h^s(\mathfrak{g}) \simeq U_h(\mathfrak{g})$. we denote it by the same letter. By (2.3.12) for any $\alpha \in \Delta_+$ it satisfies

$$\omega(f_\alpha) = e_\alpha, \omega(e_\alpha) = f_\alpha. \tag{2.8.9}$$

The algebra antiautomorphism ω_0 of $U_h(\mathfrak{g})$ also induces an algebra antiautomorphism of $U_h^s(\mathfrak{g}) \simeq U_h(\mathfrak{g})$ which we denote by the same letter. From (2.3.13) we deduce that it satisfies

$$\omega_0(e_i) = e_i, \omega_0(f_i) = f_i.$$

Formula (2.3.14) implies

$$\omega_0(e_\alpha) = c_\alpha e_\alpha, \tag{2.8.10}$$

where $c_\alpha = \epsilon_\alpha p_\alpha$, $\epsilon_\alpha = \pm 1$, $p_\alpha \in q^{\mathbb{Z}}$. We also have

$$\omega_0(f_\alpha) = \omega_0 \omega(e_\alpha) = \omega \omega_0(e_\alpha) = \omega(c_\alpha e_\alpha) = c_\alpha^{-1} \omega(e_\alpha) = c_\alpha^{-1} f_\alpha \tag{2.8.11}$$

One can check straightforwardly that ω_0 is a coalgebra homomorphism of $U_h^s(\mathfrak{g})$.

The antiautomorphisms ω and ω_0 give rise to antiautomorphisms of $U_q^s(\mathfrak{g})$, $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ and $U_{\mathcal{A}}^s(\mathfrak{g})$ which we denote by the same letters.

Using the root vectors e_β and f_β we can construct bases for the algebras introduced above.

Lemma 2.8.2. *Fix a normal ordering of the system of positive roots Δ_+ and let e_β, f_β be the corresponding quantum root vectors defined in Proposition 2.6.2. Then the following statements are true.*

(i) The elements f_β satisfy the following commutation relations

$$f_\alpha f_\beta - q^{(\alpha,\beta)+\kappa(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}f_\beta f_\alpha = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C(p_1,\ldots,p_n)f_{\delta_1}^{(p_1)}f_{\delta_2}^{(p_2)}\ldots f_{\delta_n}^{(p_n)} = \qquad (2.8.12)$$

$$= \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C'(p_1,\ldots,p_n)f_{\delta_1}^{p_1}f_{\delta_2}^{p_2}\ldots f_{\delta_n}^{p_n}, \ \alpha<\beta,$$

where $C(p_1,\ldots,p_n) \in \mathcal{B}$, $C'(p_1,\ldots,p_n) \in \mathcal{A}$.

(ii) The elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1}\ldots e_{\beta_D}^{r_D}$, $f^{\mathbf{t}} = f_{\beta_D}^{t_D}\ldots f_{\beta_1}^{t_1}$, for $\mathbf{r} = (r_1,\ldots r_D)$, $\mathbf{t} = (t_1,\ldots t_D) \in \mathbb{N}^D$, form bases of $U_q^s(\mathfrak{n}_+), U_q^s(\mathfrak{n}_-)$, respectively.

(iii) The multiplication defines an isomorphism of $\mathbb{C}(q^{\frac{1}{dr^2}})$–modules:

$$U_q^s(\mathfrak{n}_-) \otimes U_q^s(\mathfrak{h}) \otimes U_q^s(\mathfrak{n}_+) \to U_q^s(\mathfrak{g}).$$

(iv) The elements $e^{\mathbf{r}}$, $f^{\mathbf{t}}$ (resp. $e^{(\mathbf{r})} = e_{\beta_1}^{(r_1)}\ldots e_{\beta_D}^{(r_D)}$, $f^{(\mathbf{t})} = f_{\beta_D}^{(t_D)}\ldots f_{\beta_1}^{(t_1)}$) for $\mathbf{r}$, $\mathbf{t} \in \mathbb{N}^D$ form bases of $U_{\mathcal{A}}^s(\mathfrak{n}_+), U_{\mathcal{A}}^s(\mathfrak{n}_-)$ (resp. $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_+), U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$), respectively.

(v) The multiplication defines an isomorphisms of $\mathcal{B}$–modules:

$$U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-) \otimes U_{\mathcal{B}}^{s,res}(\mathfrak{h}) \otimes U_{\mathcal{B}}^{s,res}(\mathfrak{n}_+) \to U_{\mathcal{B}}^{s,res}(\mathfrak{g}).$$

(vi) Let $[\alpha,\beta] = \{\beta_p,\ldots,\beta_q\}$ be a minimal segment in Δ_+, $U_{\mathcal{A}}^s([\alpha,\beta])$, $U_{\mathcal{A}}^s([-\alpha,-\beta])$ (resp. $U_{\mathcal{B}}^{s,res}([\alpha,\beta])$, $U_{\mathcal{B}}^{s,res}([-\alpha,-\beta])$) the $\mathcal{A}(\mathcal{B})$–subalgebras of $U_h^s(\mathfrak{g})$ generated by the e_γ and by the f_γ, $\gamma \in [\alpha,\beta]$ (resp. by the $(e_\gamma)^{(r)}$ and by the $(f_\gamma)^{(r)}$, $\gamma \in [\alpha,\beta]$, $r \in \mathbb{N}$), respectively. Then the elements $(e_{\beta_p})^{r_p}\ldots(e_{\beta_q})^{r_q}$, $(f_{\beta_q})^{r_q}\ldots(f_{\beta_p})^{r_p}$ (resp. $(e_{\beta_p})^{(r_p)}\ldots(e_{\beta_q})^{(r_q)}$, $(f_{\beta_q})^{(r_q)}\ldots(f_{\beta_p})^{(r_p)}$), $r_i \in \mathbb{N}$ form bases of $U_{\mathcal{A}}^s([\alpha,\beta])$, $U_{\mathcal{A}}^s([-\alpha,-\beta])$ (resp. $U_{\mathcal{B}}^{s,res}([\alpha,\beta])$, $U_{\mathcal{B}}^{s,res}([-\alpha,-\beta])$), respectively.

(vii) The elements $f^{\mathbf{t}}$ (resp. $f^{(\mathbf{t})} = f_{\beta_D}^{(t_D)}\ldots f_{\beta_1}^{(t_1)}$) for $\mathbf{t} \in \mathbb{N}^D$ with $t_i > 0$ for at least one $i \geq p$ form a basis in the right ideal Y_p of $U_{\mathcal{A}}^s(\mathfrak{n}_-)$ (resp. $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$) generated by f_γ, $\gamma \in [\beta_p,\beta_D]$ (resp. by $(f_\gamma)^{(r)}$, $\gamma \in [\beta_p,\beta_D]$, $r > 0$).

(viii) For any $n > 0$, $p = 1,\ldots,D-1$ one has

$$f_{\beta_p}^{(n)}U_{\mathcal{B}}^{s,res}([-\beta_{p+1},-\beta_D]) \subset \sum_{i=0}^{n-1}(U_{\mathcal{B}}^{s,res}([-\beta_{p+1},-\beta_D]))_0 f_{\beta_p}^{(i)} + U_{\mathcal{B}}^{s,res}([-\beta_{p+1},-\beta_D])f_{\beta_p}^{(n)},$$

where $(U_{\mathcal{B}}^{s,res}([-\beta_{p+1},-\beta_D]))_0 = Y_{p+1} \cap U_{\mathcal{B}}^{s,res}([-\beta_{p+1},-\beta_D])$.

(ix) Let $[\alpha,\beta] \subset \Delta_+$ or $[\alpha,\beta] \subset \Delta_-$ be any minimal segment, such that $[\alpha,\beta] = [\alpha,\gamma] \cup [\delta,\beta]$ (disjoint union of minimal segments). Then the multiplication in $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ defines isomorphisms of $\mathcal{B}$–modules

$$U_{\mathcal{B}}^{s,res}([\alpha,\gamma]) \otimes U_{\mathcal{B}}^{s,res}([\delta,\beta]) \to U_{\mathcal{B}}^{s,res}([\alpha,\gamma])U_{\mathcal{B}}^{s,res}([\delta,\beta]) = U_{\mathcal{B}}^{s,res}([\alpha,\beta]),$$

$$U_{\mathcal{B}}^{s,res}([\delta,\beta]) \otimes U_{\mathcal{B}}^{s,res}([\alpha,\gamma]) \to U_{\mathcal{B}}^{s,res}([\delta,\beta])U_{\mathcal{B}}^{s,res}([\alpha,\gamma]) = U_{\mathcal{B}}^{s,res}([\alpha,\beta]).$$

Proof. Commutation relations (2.8.12) follow from commutation relations (2.3.3), (2.1.2), (2.1.3), Proposition 2.4.1, the definition of the elements e_β, f_β and the definition of the isomorphism $\psi_{\{n_{ij}\}}$. This proves (i).

Statements (ii)-(vi) of this lemma follow straightforwardly from parts (i), (iii) and (iv) of Lemma 2.4.2 and Propositions 2.6.1 and 2.6.2.

For (vii), using commutation relations (2.8.12) we can represent any element of the right ideal of $U_{\mathcal{A}}^s(\mathfrak{n}_-)$ generated by f_γ, $\gamma \in [\beta_p,\beta_D]$ as an $\mathcal{A}$–linear combination of the elements $f^{\mathbf{t}}$ for $\mathbf{t} \in \mathbb{N}^D$ with $t_i > 0$ for at least one $i \geq p$. This presentation is unique by the Poincaré–Birkhoff–Witt decomposition for $U_{\mathcal{A}}^s(\mathfrak{n}_-)$ stated in (vi).

Note that a similar result holds for the algebra $U_q^s(\mathfrak{n}_-) = U_{\mathcal{A}}^s(\mathfrak{n}_-) \otimes_{\mathcal{A}} \mathbb{C}(q^{\frac{1}{dr^2}})$ for the same reasons.

We can apply it to represent any element of the right ideal Y_p of $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-) \subset U_q^s(\mathfrak{n}_-)$ generated by $(f_\gamma)^{(r)}$, $\gamma \in [\beta_p,\beta_D]$, $r > 0$ as a $\mathbb{C}(q^{\frac{1}{dr^2}})$–linear combination of the elements $f^{(\mathbf{t})}$ for $\mathbf{t} \in \mathbb{N}^D$ with $t_i > 0$ for at least one $i \geq p$. This presentation is unique and by the uniqueness of the Poincaré–Birkhoff–Witt decomposition for $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ stated in (vi) the coefficients in this decomposition belong to $\mathcal{B}$. This completes the proof of part (vii).

(viii) is justified using similar arguments, commutation relations (2.8.12) and induction over n.

(ix) follows from Corollary 2.4.4 (iii) and Propositions 2.6.1 and 2.6.2.

$\square$

A basis for $U_{\mathcal{B}}^{s,res}(\mathfrak{h})$ is a little bit more difficult to describe. We do not need its explicit description.

Remark 2.8.3. *Applying the antiautomorphism ω_0 to the elements of the bases constructed in Lemma 2.8.2 and using (2.8.10) and (2.8.11) we obtain other bases of similar types where the order of the quantum root vectors in the products defining the elements of the bases is reversed.*

By specializing the above constructed bases for $q^{\frac{1}{rd^2}} = \varepsilon^{\frac{1}{rd^2}}$ one can obtain similar bases and similar subalgebras for $U_{\varepsilon}^s(\mathfrak{g})$ and $U_{\varepsilon}^{s,res}(\mathfrak{g})$.

Similarly to the case of $U_q^{res}(\mathfrak{g})$, using parts (ii), (iii), (iv) and (v) of Lemma 2.8.2 one can introduce Q–gradings on the algebras $U_q^s(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$, $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$, $U_{\varepsilon}^{s,res}(\mathfrak{g})$ and $U_{\varepsilon}^s(\mathfrak{g})$. For instance, we have

$$U_{\mathcal{B}}^{s,res}(\mathfrak{g}) = \bigoplus_{\mu \in Q} (U_{\mathcal{B}}^{s,res}(\mathfrak{g}))_\mu,$$

where

$$(U_{\mathcal{B}}^{s,res}(\mathfrak{g}))_\mu = \{x \in U_{\mathcal{B}}^{s,res}(\mathfrak{g}) : [h,x] = \mu(h)x \text{ for all } h \in \mathfrak{h} \subset U_h(\mathfrak{g}) \simeq U_h^s(\mathfrak{g})\}.$$

Using formulas (2.5.7) and (2.6.13) one can also find that

$$\Delta_s(f_{\beta_k}) = \theta_{w_{k-1}}^s (e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_k^\vee + h\beta_k^\vee} \otimes f_{\beta_k} + f_{\beta_k} \otimes 1)(\theta_{w_{k-1}}^s)^{-1} = \tag{2.8.13}$$

$$= G_{\beta_k}^{-1} \otimes f_{\beta_k} + f_{\beta_k} \otimes 1 + \sum_i y_i \otimes x_i,$$

where

$$G_\beta = e^{h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee - h\beta^\vee}, \quad y_i = e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\gamma_{x_i}^\vee + h\gamma_{x_i}^\vee} \overline{y}_i,$$

$$\overline{y}_i \in U_{\mathcal{A}}^s([-\beta_{k+1}, -\beta_D]) \cap U_{\mathcal{B}}^{s,res}([-\beta_{k+1}, -\beta_D]),$$

$$x_i \in U_{\mathcal{A}}^s([-\beta_1, -\beta_{k-1}]) \cap U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_{k-1}]),$$

$\overline{y}_i, x_i$ belong to weight subspaces and have non-zero weights, γ_{x_i} is the weight of x_i (see [34], Corollary 4.3.2), for $w_{k-1} = s_{i_1} \ldots s_{i_{k-1}}$

$$\theta_{w_{k-1}}^s = \theta_{\beta_1}^s \ldots \theta_{\beta_{k-1}}^s, \quad \theta_{\beta_r}^s = \exp_{q_{\beta_r}}[(1 - q_{\beta_r}^{-2})e_{\beta_r} e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee} \otimes f_{\beta_r}], \tag{2.8.14}$$

and

$$(\theta_{w_{k-1}}^s)^{-1} = (\theta_{\beta_{k-1}}^s)^{-1} \ldots (\theta_{\beta_1}^s)^{-1}, \quad (\theta_{\beta_r}^s)^{-1} = \exp_{q_{\beta_r}^{-1}}[(1 - q_{\beta_r}^2)e_{\beta_r} e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee} \otimes f_{\beta_r}]. \tag{2.8.15}$$

From (2.8.13) we also obtain

$$\Delta_s(f_{\beta_k}^{(n)}) = \frac{1}{[n]_{q_{\beta_k}}!} \theta_{w_{k-1}}^s (G_{\beta_k}^{-1} \otimes f_{\beta_k} + f_{\beta_k} \otimes 1)^n (\theta_{w_{k-1}}^s)^{-1} = \tag{2.8.16}$$

$$= \theta_{w_{k-1}}^s (\sum_{k=0}^n q_{\beta_k}^{k(n-k)} G_{\beta_k}^{-k} f_{\beta_k}^{(n-k)} \otimes f_{\beta_k}^{(k)})(\theta_{w_{k-1}}^s)^{-1} =$$

$$= \sum_{k=0}^n q_{\beta_k}^{k(n-k)} G_{\beta_k}^{-k} f_{\beta_k}^{(n-k)} \otimes f_{\beta_k}^{(k)} + \sum_i y_i^{(n)} \otimes x_i^{(n)},$$

where

$$y_i^{(n)} = e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\gamma_{x_i^{(n)}}^\vee + h\gamma_{x_i^{(n)}}^\vee} \overline{y}_i^{(n)},$$

$\overline{y}_i^{(n)} \in I_k^>$, $x_i^{(n)} \in I_k^<$ belong to weight subspaces and have non-zero weights, $\gamma_{x_i^{(n)}}$ is the weight of $x_i^{(n)}$, $I_k^>$ is the ideal in $U_{\mathcal{B}}^{s,res}([-\beta_k, -\beta_D])$ generated by $f_{\beta_i}^{(p)}$, $i = k+1, \ldots, D$, $p > 0$, and $I_k^<$ is the ideal in $U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_k])$ generated by $f_{\beta_i}^{(p)}$, $i = 1, \ldots, k-1$, $p > 0$.

Similarly

$$\Delta_s(e_{\beta_k}) = \theta_{w_{k-1}}^s (e_{\beta_k} \otimes e^{-h\beta_k^\vee} + e^{h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_k^\vee} \otimes e_{\beta_k})(\theta_{w_{k-1}}^s)^{-1} = \tag{2.8.17}$$

$$= e_{\beta_k} \otimes e^{-h\beta_k^\vee} + e^{h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta_k^\vee} \otimes e_{\beta_k} + \sum_i x_i' \otimes y_i',$$

where

$$y_i' \in U_{\mathcal{A}}^s([\beta_{k+1}, \beta_D])U_{\mathcal{A}}^s(\mathfrak{h}) \cap U_{\mathcal{B}}^{s,res}([\beta_{k+1}, \beta_D])U_{\mathcal{B}}^{s,res}(\mathfrak{h}),$$

$$x_i' \in U_{\mathcal{A}}^s([\beta_1, \beta_{k-1}])U_{\mathcal{A}}^s(\mathfrak{h}) \cap U_{\mathcal{B}}^{s,res}([\beta_1, \beta_{k-1}])U_{\mathcal{B}}^{s,res}(\mathfrak{h}),$$

and y_i', x_i' belong to weight subspaces and have non-zero weights.

From these formulas we deduce

$$S_s(f_\beta) = -G_\beta f_\beta - \sum_i S_s(y_i)x_i = -G_\beta(f_\beta + \sum_i y_i S_s(x_i)), \qquad (2.8.18)$$

$$S_s^{-1}(f_\beta) = -(f_\beta + \sum_i S_s^{-1}(x_i)y_i)G_\beta = -f_\beta G_\beta - \sum_i x_i S_s^{-1}(y_i). \qquad (2.8.19)$$

We also have

$$\omega_0 S_s^{-1}(f_\beta) = -G_\beta(\omega_0 f_\beta + \sum_i \omega_0(y_i)\omega_0 S_s^{-1}(x_i)) =$$

$$= -G_\beta \omega_0 f_\beta - \sum_i \omega_0 S_s^{-1}(y_i)\omega_0(x_i),$$

$$\omega_0 f_\beta = -G_\beta^{-1}(\omega_0 S_s^{-1}(f_\beta) + \sum_i \omega_0 S_s^{-1}(y_i)).$$

As usual, one can define highest weight, Verma and finite-dimensional modules for all forms and specializations of the quantum group $U_h^s(\mathfrak{g})$ introduced above. We recall that by Propositions 6.5.5 and 6.5.7 in [20] $U_h(\mathfrak{g}) \simeq U(\mathfrak{g})[[h]]$, and this isomorphism of algebras restricts to the identity map on $U(\mathfrak{h})$ and induces a canonical isomorphism of the center of $U_h(\mathfrak{g})$ and of $Z(U(\mathfrak{g}))[[h]]$, where $Z(U(\mathfrak{g}))$ is the center of $U(\mathfrak{g})$. Therefore if V is a $U_h(\mathfrak{g})$–module free and of finite rank over $\mathbb{C}[[h]]$ then $V_1 = V/hV$ is a finite-dimensional $U(\mathfrak{g})$–module which is completely reducible, and its irreducible components are highest weight irreducible finite-dimensional representations of $U(\mathfrak{g})$. By Corollary 6.5.6 in [20], one has an isomorphism of $U_h(\mathfrak{g})$–modules, $V \simeq V_1[[h]]$, where the action of $U_h(\mathfrak{g})$ on $V_1[[h]]$ is defined using the algebra isomorphism $U_h(\mathfrak{g}) \simeq U(\mathfrak{g})[[h]]$.

Using the algebra isomorphism $U_h(\mathfrak{g}) \simeq U(\mathfrak{g})[[h]]$, which restricts to the identity map on $U(\mathfrak{h})$, one can define weight vectors in V by requiring that $v \in V$ has weight $\lambda \in P$ if $hv = \lambda(h)v$ for any $h \in \mathfrak{h} \subset U(\mathfrak{h}) \subset U(\mathfrak{g})[[h]] \simeq U_h(\mathfrak{g})$. For $\lambda \in P$ one also defines the corresponding weight subspace $(V)_\lambda = \{v \in V : hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$.

If V_1 is the highest weight irreducible representation of $U(\mathfrak{g})$ with highest weight $\lambda \in P_+$ we call the corresponding representation $V = V_\lambda$ the highest weight indecomposable representation of highest weight λ. V_λ is generated by a highest weight vector with respect to the action of $\mathfrak{h} \subset U_h(\mathfrak{h})$. All indecomposable $U_h(\mathfrak{g})$–modules free and of finite rank over $\mathbb{C}[[h]]$ can be obtained this way.

Let V be a topologically free finite rank $U_h(\mathfrak{g})$–module. Recall that there is a contravariant non–degenerate form $(\cdot, \cdot)$ on V such that $(u, xv) = (\omega(x)u, v)$ for any $u, v \in V$, $x \in U_h(\mathfrak{g})$. Different weight spaces are orthogonal with respect to this form.

The definition (2.3.11) of the antiautomorphism ω implies that the ω maps elements of negative weights to elements of positive weights. From this remark we obtain the following obvious corollary which will be often used later.

Lemma 2.8.4. *Let V be a topologically free finite rank $U_h(\mathfrak{g})$–module. Let $u, v \in V$, $x, y \in U_h(\mathfrak{g})$.*

(i) If u is a highest weight vector and $x \in U_h(\mathfrak{g})$ has a weight which is not non–negative then $(u, xyv) = 0$.

(ii) If v is a highest weight vector and $y \in U_h(\mathfrak{g})$ has a weight which is not non–positive then $(u, xyv) = 0$.

Recall that $U_{\mathcal{B}}^{s,res}(\mathfrak{g}) \simeq U_{\mathcal{B}}^{res}(\mathfrak{g})$ can be regarded as a subalgebra of $U_h^s(\mathfrak{g}) \simeq U_h(\mathfrak{g})$. Let V be a $U_h(\mathfrak{g})$–module topologically free and of finite rank over $\mathbb{C}[[h]]$ (for brevity we shall call such modules finite rank $U_h(\mathfrak{g})$–modules). Then a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–module V^{res} is called a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice in V if $V^{res} \otimes_{\mathcal{B}} \mathbb{C}[[h]] \simeq V$.

For any $U_h(\mathfrak{g})$–module V topologically free and of finite rank over $\mathbb{C}[[h]]$ a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice V^{res} exists. Indeed, by the above discussion it suffices to consider the case of indecomposable $V = V_\lambda$ generated by a highest weight vector v. In this case, similarly to the proof of Proposition 4.2 in [88] (see also Proposition 10.1.4 in [20]) one can show that $V_\lambda^{res} = U_{\mathcal{B}}^{s,res}(\mathfrak{g})v$ is a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice, V_λ^{res} is the direct sum of its intersections with the weight spaces of V_λ and each such intersection is a finitely generated free $\mathcal{B}$–module of finite rank. Moreover, using the

arguments from the proof of Proposition 4.2 in Section 4.9 of [88] one can see that the last two properties hold for any $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice in any topologically free $U_h(\mathfrak{g})$–module V of finite rank over $\mathbb{C}[[h]]$. Together cwith the results of Section 10.1 in [20] this yields the following statement.

Proposition 2.8.5. *(i) For any $U_h(\mathfrak{g})$–module V topologically free and of finite rank over $\mathbb{C}[[h]]$ V^{res} is the direct sum of its intersections with the weight spaces of V and each such intersection is a finitely generated free $\mathcal{B}$–module of finite rank*

$$V^{res} = \bigoplus_{\lambda \in P} (V^{res})_\lambda, (V^{res})_\lambda = V^{res} \cap (V)_\lambda.$$

(ii) The specialization of V^{res} at $q^{\frac{1}{dr2}} = 1$ is naturally a finite–dimensional $U(\mathfrak{g})$–module.

For any $U_h(\mathfrak{g})$–module V topologically free and of finite rank over $\mathbb{C}[[h]]$, one can equip the module V^{res} with a natural action of $\mathfrak{h}$, where $h \in \mathfrak{h}$ acts on the weight subspace of V^{res} of weight λ by multiplication by $\lambda(h)$.

From the explicit formulas and the results in [81], Section 5.2 it follows that the braid group elements given by (2.2.4) and (2.2.5) act in $U_h(\mathfrak{g})$–modules topologically free and of finite rank over $\mathbb{C}[[h]]$ and leave $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattices in them.

More generally, if T_w is the braid group element corresponding to $w \in W$ then $T_w(V)_\lambda = (V)_{w\lambda}$. This action satisfies the property $T_w x v = T_w(x) T_w v$ for any $x \in U_h(\mathfrak{g})$ and $v \in V$. The $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–submodule $V^{res} \subset V$ is invariant under the action of the elements T_w.

The following lemma will be useful for some calculations later.

Lemma 2.8.6. *Let V be a $U_h(\mathfrak{g})$–module V topologically free and of finite rank over $\mathbb{C}[[h]]$. Then the elements of the subalgebra $U_q^{res}(H)$ act on weight vectors of V by multiplication by elements of $\mathbb{C}[q, q^{-1}]$, and the elements of the subalgebra $U_{\mathcal{B}}^{s,res}(\mathfrak{h}) \subset U_{\mathcal{B}}^{s,res}(\mathfrak{g}) \subset U_h^s(\mathfrak{g}) \simeq U_h(\mathfrak{g})$ act on weight vectors of V by multiplication by elements of $\mathcal{B}$. Moreover,*

$$(V)_\lambda = \{v \in V : K_i v = q_i^{\lambda(H_i)} v, i = 1, \ldots, l\}, (V^{res})_\lambda = \{v \in V^{res} : K_i v = q_i^{\lambda(H_i)} v, i = 1, \ldots, l\}. \qquad (2.8.20)$$

Proof. By the definition of the action of $\mathfrak{h}$ on V the elements

$$\begin{bmatrix} K_i; c \\ r \end{bmatrix}_{q_i} = \prod_{s=1}^{r} \frac{K_i q_i^{c+1-s} - K_i^{-1} q_i^{s-1-c}}{q_i^s - q_i^{-s}} \ , \ i = 1, \ldots, l, \ c \in \mathbb{Z}, \ r \in \mathbb{N}$$

act on a vector of weight λ by multiplication by $\prod_{s=1}^{r} \frac{q_i^{\lambda(H_i)+c+1-s} - q_i^{-\lambda(H_i)+s-1-c}}{q_i^s - q_i^{-s}} = \prod_{s=1}^{r} (-[\lambda(H_i) + c + 1]_{q_i^{-s}})$, and $K_i^{\pm 1}$, $i = 1, \ldots, l$, act by multiplication by $q_i^{\pm \lambda(H_i)}$. Also for $v \in V$ the condition $K_i v = q_i^{\lambda(H_i)} v$, $i = 1, \ldots, l$ is equivalent to $hv = \lambda(h)v$ for any $h \in \mathfrak{h} \subset U_h(\mathfrak{g})$. This implies (2.8.20).

The case of the generators $t_i^{\pm 1}, L_i^{\pm 1}$, $i = 1, \ldots, l$ of the algebra $U_{\mathcal{B}}^{s,res}(\mathfrak{h})$ is considered in a similar way.

$\square$

The specialization of V_λ^{res} at $q^{\frac{1}{dr2}} = \varepsilon^{\frac{1}{dr2}}$ is a highest weight $U_\varepsilon^{s,res}(\mathfrak{g})$–module. In general this module is not irreducible even if V is indecomposable.

The R-matrix $\mathcal{R}^s$ acts in tensor products of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattices in $U_h(\mathfrak{g})$–modules topologically free and of finite rank over $\mathbb{C}[[h]]$. Namely, recalling (2.6.16) we can represent $\mathcal{R}^s$ in the form $\mathcal{R}^s = \mathcal{E}^s \mathcal{R}_0^s = \mathcal{R}_0^s \mathcal{E}_1^s$, where

$$\mathcal{R}_0^s = \exp\left[h\left(\sum_{i=1}^{l}(Y_i \otimes H_i) - \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i\right)\right], \qquad (2.8.21)$$

$$\mathcal{E}^s = \prod_\beta \left(\sum_{k=0}^{\infty} q_\beta^{\frac{k(k+1)}{2}} (1 - q_\beta^{-2})^k f_\beta^{(k)} \otimes e_\beta^k e^{-kh\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta^\vee}\right) =$$

$$= \prod_\beta \left(\sum_{k=0}^{\infty} q_\beta^{\frac{k(k+1)}{2}} (1 - q_\beta^{-2})^k f_\beta^k \otimes e_\beta^{(k)} e^{-kh\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta^\vee}\right),$$

$$\mathcal{E}_1^s = \prod_\beta \left(\sum_{k=0}^{\infty} q_\beta^{\frac{k(k+1)}{2}} (1 - q_\beta^{-2})^k e^{kh\left(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} - id\right)\beta^\vee} f_\beta^{(k)} \otimes e_\beta^k e^{kh\beta^\vee}\right) =$$

$$= \prod_\beta \left(\sum_{k=0}^{\infty} q_\beta^{\frac{k(k+1)}{2}} (1 - q_\beta^{-2})^k e^{kh\left(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} - id\right)\beta^\vee} f_\beta^k \otimes e_\beta^{(k)} e^{kh\beta^\vee}\right),$$

and for $\beta = \sum_{i=1}^{l} c_i \alpha_i$

$$e^{-kh\kappa \frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^{\vee}} = e^{-kh\sum_{i,j=1}^{l} \frac{c_{ij}}{d_j}c_i Y_j} \in U_{\mathcal{B}}^{s,res}(\mathfrak{h}) \cap U_{\mathcal{A}}^{s}(\mathfrak{h}).$$

We can define the action of $\mathcal{R}_0^s$ on tensor products of modules of the form V^{res}, where V is a highest weight finite rank $U_h(\mathfrak{g})$–module as follows. If V^{res}, W^{res} are two such modules and $v_\lambda \in V^{res}, w_\mu \in W^{res}$ are vectors of weights λ and μ then we define

$$\mathcal{R}_0^s v_\lambda \otimes w_\mu = q^{(\lambda,\mu)-(\kappa \frac{1+s}{1-s}P_{\mathfrak{h}'*}\mu,\lambda)} v_\lambda \otimes w_\mu,$$

and $q^{(\lambda,\mu)-(\kappa \frac{1+s}{1-s}P_{\mathfrak{h}'*}\mu,\lambda)} \in \mathcal{B}$.

Since for any $\beta \in \Delta_+$ and any module of the form V^{res}, where V is a finite rank $U_h(\mathfrak{g})$–module, $f_\beta^{(k)}$ and e_β^k belong to $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ and act as zero operators on V^{res} for k large enough, and for any k

$$e^{-kh\kappa \frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^{\vee}} \in U_{\mathcal{B}}^{s,res}(\mathfrak{h}) \cap U_{\mathcal{A}}^{s}(\mathfrak{h}),$$

$\mathcal{E}^s$ naturally acts on tensor products of such modules.

For two modules of the form V^{res}, W^{res}, where V and W are highest weight finite rank $U_h(\mathfrak{g})$–modules, we denote by R^{VW} the operator corresponding to the action of $\mathcal{R}^s$ in $V^{res} \otimes W^{res}$.

With this definition the identity

$$\Delta_s^{opp}(x) = \mathcal{R}^s \Delta_s(x) \mathcal{R}^{s-1}, x \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$$

still holds being evaluated in tensor products of modules of the form V^{res}, where V is a finite rank $U_h(\mathfrak{g})$–module.

We shall also need the following technical lemma regarding the action of the elements T_w and $\overline{T}_w$ on finite rank indecomposable modules.

Lemma 2.8.7. *Let V be a finite rank $U_h(\mathfrak{g})$–module. If T, T' are two elements of the braid group $\mathcal{B}_{\mathfrak{g}}$ which act as the same transformation on $\mathfrak{h} \subset U_h(\mathfrak{h})$ and v is a highest weight vector in V then $Tv = tT'v$, where t is a non–zero multiple of a power of q.*

Proof. Since v generates a highest weight indecomposable submodule $V_\lambda \subset V$ of highest weight λ equal to the weight of v, and V_λ is invariant under the braid group action, we can assume without loss of generality that $V = V_\lambda$.

First observe that similarly to the proof of Proposition 4.2 in [88] (see also Proposition 10.1.4 in [20]) one can show that $V' = U_q^{res}(\mathfrak{g})v$ is a $U_q^{res}(\mathfrak{g})$–lattice in V_λ in the sense that $V' \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}[[h]] \simeq V_\lambda$. This module coincides with the one defined in Section 4.1 in [88] (see also Proposition 10.1.4 in [20]).

It is well known that any element T of the braid group acts as an invertible linear automorphism of V' which can be specialized to any non–zero numeric value of q in the sense that for any $\varepsilon \in \mathbb{C}^*$ T gives rise to a linear automorphism of $U_q^{res}(\mathfrak{g})/(q-\varepsilon)U_q^{res}(\mathfrak{g})$–module $V'_\varepsilon = V'/(q-\varepsilon)V'$. It suffices to verify this statement when $T = T_i$ for $i = 1, \ldots, l$, and in this case it follows from the explicit formulas and the results in [81], Section 5.2.

Recall that elements of the braid group act as Weyl group elements on $\mathfrak{h} \subset U_h(\mathfrak{h})$. Assume that the action of T and T' on $\mathfrak{h} \subset U_h(\mathfrak{h})$ coincides with the action of a Weyl group element w. Since the $\mathbb{C}[q,q^{-1}]$–submodule of V' which consists of elements of weight $w\mu$ has rank one and Tv and $T'v$ must belong to this submodule, the relation $Tv = t(q)T'v$ must hold for some rational function $t(q)$ of q with poles or zeroes only at zero and infinity. Indeed, if $t(q_0) = 0$, $q_0 \neq 0, \infty$ then in V'_{q_0} we have $Tv = 0$, i.e. T does not induce an automorphism of V'_{q_0}, and if $t^{-1}(q'_0) = 0$, $q'_0 \neq 0, \infty$ then in $V'_{q'_0}$ we have $T'v = 0$, i.e. T' does not induce an automorphism of $V'_{q'_0}$. In both cases we arrive at a contradiction. Thus $t(q)$ must be a non–zero multiple of a power of q. $\square$

2.9 Bibliographic comments

The material presented in Sections 2.1, 2.2, 2.3, 2.4 and 2.5 is mostly standard and we refer to books [20, 58, 81] for more details and omitted proofs. Formula (2.2.4) can be found in [103].

Realizations of quantum groups associated to Weyl group elements were introduced in [108] in the case of Coxeter elements and in [114] in general.

The results of Section 2.7 are new, except for Propositions 2.7.3 and 2.7.5 the statements of which can be found in [67] as Proposition 5.2, Theorem 5.1 and Proposition 5.4.

Lemma 2.8.1 is a generalization of the result of Exercise 3 in Chapter V, §6, [10].

Specializations of quantum groups similar to those which appear in this book were considered in [116]. In this book we introduce slightly different specializations of quantum groups in order to use restricted specializations as well.

Chapter 3

q-W–algebras

In this chapter we introduce q–W–algebras and study the structure of their the quasi–classical versions, Poisson q-W–algebras. In the next chapter similar results will be obtained for q-W–algebras.

As we briefly mentioned in the introduction the naive definition of q-W–algebras as Hecke type algebras $Hk(A, B, \chi)$ requires some modification. In fact the main ingredient of the definition of q-W–algebras is the adjoint action of the quantum group on itself, and they are defined using a $\mathcal{B}$–subalgebra $\mathbb{C}^s_\mathcal{B}[G_*]$ of the quantum group the restriction of the adjoint action to which is locally finite. When q is specialized to $\varepsilon \in \mathbb{C}^*$ which is not a root of unity the algebra $\mathbb{C}^s_\mathcal{B}[G_*]$ becomes the locally finite part of the quantum group with respect to the adjoint action which was introduced and studied by Joseph.

The algebra $\mathbb{C}^s_\mathcal{B}[G_*]$ is a quantization of the algebra of regular functions on a Poisson manifold G_* which is isomorphic to G as a manifold and the Poisson structure of which is closely related to that of the Poisson–Lie group G^* dual to a quasitriangular Poisson–Lie group G.

After recalling basic facts on Poisson–Lie groups in Section 3.1 we introduce an algebra $\mathbb{C}[G^*]$ of functions on G^* in Section 3.2, its quantization $\mathbb{C}^s_\mathcal{B}[G^*] \subset U^s_h(\mathfrak{g})$ and the subalgebra $\mathbb{C}^s_\mathcal{B}[G_*] \subset \mathbb{C}^s_\mathcal{B}[G^*]$.

A special choice of the bialgebra structure entering the definitions of $\mathbb{C}[G^*]$, $\mathbb{C}^s_\mathcal{B}[G^*] \subset U^s_h(\mathfrak{g})$ and $\mathbb{C}^s_\mathcal{B}[G_*]$ is crucial for the definition of q-W–algebras. It depends on the choice of a Weyl group element $s \in W$ and ensures that one can define a subalgebra $\mathbb{C}^s_\mathcal{B}[M_+] \subset \mathbb{C}^s_\mathcal{B}[G^*]$ equipped with a non–trivial character, so that the q-W–algebra $W^s_\mathcal{B}(G)$ can be defined as the result of a quantum constrained reduction with respect to the subalgebra $\mathbb{C}^s_\mathcal{B}[M_+]$.

Next, in Section 3.4 we proceed with the study of the specialization $W^s(G)$ of the algebra $W^s_\mathcal{B}(G)$ at $q^{\frac{1}{rd^2}} = 1$. We recall that $W^s(G)$ is naturally a Poisson algebra which can be regarded as the algebra of regular functions on a reduced Poisson manifold which is also an algebraic variety. Poisson reduction works well for differential Poisson manifolds. Therefore it is easier firstly to describe the reduced Poisson structure on the algebra of C^∞–functions on the reduced Poisson manifold and then to recover the structure of the algebraic variety on it. This is done in Proposition 3.4.3 and Theorem 3.4.5.

In Section 3.5 we define a projection operator Π into the algebra $W^s(G)$. In Theorem 3.5.7, which is central in this chapter, we obtain a formula for the operator Π suitable for quantization. This formula plays the key role in the proof of Theorem 4.7.2 describing a localization of the algebra $W^s_\mathcal{B}(G)$ in terms of a quantum counterpart of the operator Π. Miraculously the formula for Π from Theorem 3.5.7 can be directly extrapolated to the quantum case.

3.1   Some facts on Poisson–Lie groups

In this section, following [20], Ch. 1, we recall some notions related to Poisson–Lie groups. These facts will be needed for the study of Poisson q-W–algebras.

Let G be a finite-dimensional Lie group equipped with a Poisson bracket, $\mathfrak{g}$ its Lie algebra. G is called a Poisson–Lie group if the multiplication $G \times G \to G$ is a Poisson map. A Poisson bracket satisfying this axiom is degenerate and, in particular, is identically zero at the unit element of the group. Linearizing this bracket at the unit element defines the structure of a Lie algebra in the space $T^*_e G \simeq \mathfrak{g}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called the tangent bialgebra of G.

Lie brackets in $\mathfrak{g}$ and $\mathfrak{g}^*$ satisfy the following compatibility condition:

Let $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ be the dual of the commutator map $[,]_* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$. Then δ is a 1-cocycle on $\mathfrak{g}$ (with respect to the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$).

Let c_{ij}^k, f_c^{ab} be the structure constants of $\mathfrak{g}, \mathfrak{g}^*$ with respect to the dual bases $\{e_i\}, \{e^i\}$ in $\mathfrak{g}, \mathfrak{g}^*$. The compatibility condition means that

$$c_{ab}^s f_s^{ik} \; - \; c_{as}^i f_b^{sk} \; + \; c_{as}^k f_b^{si} \; - \; c_{bs}^k f_a^{si} \; + \; c_{bs}^i f_a^{sk} \; = \; 0.$$

This condition is symmetric with respect to exchange of c and f. Thus if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra.

The following proposition shows that the category of finite-dimensional Lie bialgebras is isomorphic to the category of finite-dimensional connected simply connected Poisson–Lie groups.

Proposition 3.1.1. ([20], Theorem 1.3.2) *If G is a connected simply connected finite-dimensional Lie group, every bialgebra structure on $\mathfrak{g}$ is the tangent bialgebra of a unique Poisson structure on G which makes G into a Poisson–Lie group.*

Let G be a finite-dimensional Poisson–Lie group, $(\mathfrak{g}, \mathfrak{g}^*)$ the tangent bialgebra of G. The connected simply connected finite-dimensional Poisson–Lie group corresponding to the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ is called the dual Poisson–Lie group and denoted by G^*.

$(\mathfrak{g}, \mathfrak{g}^*)$ is called a factorizable Lie bialgebra if the following conditions are satisfied (see [100]):

1. *$\mathfrak{g}$ is equipped with a non–degenerate invariant scalar product $(\cdot, \cdot)$.*

   We shall always identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by means of this scalar product.

2. *The dual Lie bracket on $\mathfrak{g}^* \simeq \mathfrak{g}$ is given by*

$$[X, Y]_* = \frac{1}{2} \left([rX, Y] + [X, rY] \right), X, Y \in \mathfrak{g}, \tag{3.1.1}$$

   where $r \in \mathrm{End}\, \mathfrak{g}$ is a skew symmetric linear operator (classical r-matrix).

3. *r satisfies the modified classical Yang-Baxter identity:*

$$[rX, rY] - r\left([rX, Y] + [X, rY] \right) = -[X, Y], \; X, Y \in \mathfrak{g}. \tag{3.1.2}$$

Define operators $r_\pm \in \mathrm{End}\, \mathfrak{g}$ by

$$r_\pm = \frac{1}{2} \left(r \pm id \right).$$

We shall need some properties of the operators $r_\pm$. Denote by $\mathfrak{b}_\pm$ and $\mathfrak{n}_\mp$ the image and the kernel of the operator $r_\pm$:

$$\mathfrak{i}_\pm = Im\, r_\pm, \quad \mathfrak{k}_\mp = Ker\, r_\pm. \tag{3.1.3}$$

Proposition 3.1.2. ([8], Lemma 6.6; [104], Sect. 4) *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra. Then*
(i) $\mathfrak{i}_\pm \subset \mathfrak{g}$ is a Lie subalgebra, the subspace $\mathfrak{k}_\pm$ is a Lie ideal in $\mathfrak{i}_\pm$, $\mathfrak{i}_\pm^\perp = \mathfrak{k}_\pm$.
(ii) $\mathfrak{k}_\pm$ is an ideal in $\mathfrak{g}^*$.
(iii) $\mathfrak{i}_\pm$ is a Lie subalgebra in $\mathfrak{g}^*$. Moreover $\mathfrak{i}_\pm = \mathfrak{g}^*/\mathfrak{k}_\pm$.
(iv) $(\mathfrak{i}_\pm, \mathfrak{i}_\pm^*)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{i}_\pm, \mathfrak{i}_\pm^*) \simeq (\mathfrak{i}_\pm, \mathfrak{i}_\mp)$. The canonical paring between $\mathfrak{i}_\mp$ and $\mathfrak{i}_\pm$ is given by

$$(X_\mp, Y_\pm)_\pm = (X_\mp, r_\pm^{-1} Y_\pm), \; X_\mp \in \mathfrak{i}_\mp; \; Y_\pm \in \mathfrak{i}_\pm. \tag{3.1.4}$$

The classical Yang–Baxter equation implies that $r_\pm$, regarded as a mapping from $\mathfrak{g}^*$ into $\mathfrak{g}$, is a Lie algebra homomorphism (see [104], Proposition 7). Moreover, $r_+^* = -r_-$, and $r_+ - r_- = id$.

By Proposition 9 in [104], the mapping

$$\mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g} \quad : X \mapsto (X_+, \; X_-), \quad X_\pm \; = \; r_\pm X, \tag{3.1.5}$$

is a Lie algebra embedding (Here $\mathfrak{g} \oplus \mathfrak{g}$ is the Lie algebra direct sum of two copies of $\mathfrak{g}$). Thus we may identify $\mathfrak{g}^*$ with the image of this embedding in $\mathfrak{g} \oplus \mathfrak{g}$.

Naturally, embedding (3.1.5) extends to a group homomorphism

$$G^* \to G \times G, \quad L \mapsto (L_+, L_-).$$

In the situations considered later in this book this homomorphism will be always an embedding. In such situations we shall identify G^* with the corresponding subgroup in $G \times G$.

3.2   Quantization of algebraic Poisson–Lie groups and the definition of q-W–algebras

In this section we introduce the main object of this book, q-W–algebras. We start by defining the relevant Poisson–Lie groups and their quantizations. We consider algebras defined over the ring $\mathcal{B}$ since later in our construction the restricted specialization of the quantum group $U_h^s(\mathfrak{g})$ defined over $\mathcal{B}$ will play the key role.

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra. Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ and Δ_+ the system of positive roots associated to s. Observe that cocycle (2.6.15) equips $\mathfrak{g}$ with the structure of a factorizable Lie bialgebra, where the scalar product is given by the symmetric bilinear form. Using the identification $\mathrm{End}\,\mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}$ the corresponding r–matrix may be represented as

$$r^s = P_+ - P_- + \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'},$$

where P_+, P_- and $P_{\mathfrak{h}'}$ are the orthogonal projection operators onto the nilradical $\mathfrak{n}_+$ corresponding to Δ_+, the opposite nilradical $\mathfrak{n}_-$, and $\mathfrak{h}'$, respectively, in the direct sum

$$\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h}' + \mathfrak{h}'^{\perp} + \mathfrak{n}_-,$$

and $\mathfrak{h}'^{\perp} = \mathfrak{h}^s$ is the orthogonal complement to $\mathfrak{h}'$ in $\mathfrak{h}$ with respect to the symmetric bilinear form.

Let G be the connected simply–connected semisimple Poisson–Lie group with the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, G^* the dual Poisson–Lie group.

Observe that G is a connected simply–connected semisimple complex algebraic group (see e.g. §104, Theorem 12 in [132]). Note also that

$$r_+^s = P_+ + \frac{\kappa}{2}\frac{1+s}{1-s} P_{\mathfrak{h}'} + \frac{1}{2}P_{\mathfrak{h}}, \quad r_-^s = -P_- + \frac{\kappa}{2}\frac{1+s}{1-s} P_{\mathfrak{h}'} - \frac{1}{2}P_{\mathfrak{h}},$$

where $P_{\mathfrak{h}}$ is the orthogonal projection operator onto $\mathfrak{h} \subset \mathfrak{g}$ with respect to the symmetric bilinear form, and hence the subspaces $\mathfrak{i}_{\pm}$ and $\mathfrak{k}_{\pm}$ defined by (3.1.3) coincide with the Borel subalgebras $\mathfrak{b}_{\pm}$ in $\mathfrak{g}$ corresponding to $\Delta_{\pm}$ and their nilradicals $\mathfrak{n}_{\pm}$, respectively. Therefore every element $(L_+, L_-) \in G^* \subset G \times G$ may be uniquely written as

$$(L_+, L_-) = (n_+, n_-)(h_+, h_-), \tag{3.2.1}$$

where $n_{\pm} \in N_{\pm}$, $h_+ = \exp((\frac{\kappa}{2}\frac{1+s}{1-s}P_{\mathfrak{h}'} + \frac{1}{2}id)x)$, $h_- = \exp((\frac{\kappa}{2}\frac{1+s}{1-s}P_{\mathfrak{h}'} - \frac{1}{2}id)x)$, $x \in \mathfrak{h}$. In particular, G^* is a solvable subgroup in $G \times G$. In general G^* is not algebraic.

Our main object will be a certain algebra of functions on G^*, $\mathbb{C}[G^*]$. This algebra may be explicitly described as follows. Let π_V be a finite-dimensional representation of G. Then matrix elements of $\pi_V(L_{\pm})$ are well–defined functions on G^*, and $\mathbb{C}[G^*]$ is defined as the subalgebra in $C^{\infty}(G^*)$ generated by the matrix elements of $\pi_V(L_{\pm})$, where V runs through all finite–dimensional representations of G. The elements $L^{\pm,V} = \pi_V(L_{\pm})$ may be viewed as elements of the space $\mathbb{C}[G^*] \otimes \mathrm{End}V$. If we fix a basis in V, $L^{\pm,V}$ can be regarded as matrices with entries $L_{ij}^{\pm,V}$ being elements of $\mathbb{C}[G^*]$.

For every two finite–dimensional $\mathfrak{g}$–modules V and W we denote $r_{\pm}^{s\,VW} = (\pi_V \otimes \pi_W)r_{\pm}^s \in \mathrm{End}V \otimes \mathrm{End}W$, where $r_{\pm}^s$ is regarded as an element of $\mathfrak{g} \otimes \mathfrak{g}$.

Proposition 3.2.1. ([106], Section 2) $\mathbb{C}[G^*]$ *is a Poisson subalgebra in the Poisson algebra $C^{\infty}(G^*)$, the Poisson bracket on $\mathbb{C}[G^*]$ is given by*

$$\{L_1^{\pm,V}, L_2^{\pm,W}\} = -2[r_{\pm}^{s\,VW}, L_1^{\pm,V} L_2^{\pm,W}],$$

$$\{L_1^{-,V}, L_2^{+,W}\} = -2[r_{\pm}^{s\,VW}, L_1^{-,V} L_2^{+,W}], \tag{3.2.2}$$

where

$$L_1^{\pm,V} = L^{\pm,V} \otimes I_W, \quad L_2^{\pm,W} = I_V \otimes L^{\pm,W}, L_1^{\pm,V}, L_2^{\pm,W} \in \mathbb{C}[G^*] \otimes \mathrm{End}V \otimes \mathrm{End}W,$$

$I_X \in \mathrm{End}X$ *is the identity endomorphism of a vector space X, $r_{\pm}^{s\,VW}$ is regarded as an element of $\mathbb{C}[G^*] \otimes \mathrm{End}V \otimes \mathrm{End}W$ via the embedding*

$$\mathrm{End}V \otimes \mathrm{End}W \to \mathbb{C}[G^*] \otimes \mathrm{End}V \otimes \mathrm{End}W, a \otimes b \mapsto 1 \otimes a \otimes b,$$

in the left hand sides in (3.2.2) the Poisson brackets of $L_1^{\pm,V}, L_2^{\pm,W} \in \mathbb{C}[G^*] \otimes \mathrm{End}V \otimes \mathrm{End}W$ are taken with respect to the Poisson structure on $\mathbb{C}[G^*]$, and the composition endomorphisms is taken in $\mathrm{End}V \otimes \mathrm{End}W$, and the commutators in the right hand sides of (3.2.2) are taken in $\mathbb{C}[G^*] \otimes \mathrm{End}V \otimes \mathrm{End}W$.

Moreover, the map $\Delta : \mathbb{C}[G^*] \to \mathbb{C}[G^*] \otimes \mathbb{C}[G^*]$ dual to the multiplication in G^*,

$$\Delta(L_{ij}^{\pm,V}) = \sum_k L_{ik}^{\pm,V} \otimes L_{kj}^{\pm,V}, \tag{3.2.3}$$

is a homomorphism of Poisson algebras, and the map $S : \mathbb{C}[G^*] \to \mathbb{C}[G^*]$ dual to taking inverse in G^*,

$$S(L_{ij}^{\pm,V}) = ((L^{\pm,V})^{-1})_{ij}$$

is an antihomomorphism of Poisson algebras.

Remark 3.2.2. *Recall that a Poisson–Hopf algebra is a Poisson algebra which is also a Hopf algebra such that the comultiplication is a homomorphism of Poisson algebras and the antipode is an antihomomorphism of Poisson algebras. According to Proposition 3.2.1 $\mathbb{C}[G^*]$ is a Poisson–Hopf algebra.*

Now we define a quantization of the Poisson–Hopf algebra $\mathbb{C}[G^*]$. For any finite rank representation $\pi_V : U_{\mathcal{B}}^{s,res}(\mathfrak{g}) \to V^{res}$, where V is a finite rank representation of $U_h(\mathfrak{g})$, one can define an action of elements H_i, $i = 1, \ldots, l$ on V^{res} by requiring that H_i acts on weight vectors of weight λ by multiplication by $\lambda(H_i)$. Then from the definition of the R–matrix $\mathcal{R}^s$ and from formula (2.8.21) it follows that $^qL^{\pm,V}$ given by

$$^qL^{-,V} = (id \otimes \pi_V)\mathcal{R}_{21}^s{}^{-1} = (id \otimes \pi_V S^s)\mathcal{R}_{21}^s, \quad {}^qL^{+,V} = (id \otimes \pi_V)\mathcal{R}^s.$$

are well–defined invertible elements of $U_h^s(\mathfrak{g}) \otimes \mathrm{End}_{\mathcal{B}}(V^{res})$.

If we fix a basis in V^{res}, $^qL^{\pm,V}$ may be regarded as matrices with matrix elements $(^qL^{\pm,V})_{ij}$ being elements of $U_h^s(\mathfrak{g})$.

We denote by $\mathbb{C}_{\mathcal{B}}^s[G^*]$ the $\mathcal{B}$–Hopf subalgebra in $U_h^s(\mathfrak{g})$ generated by the matrix elements of $(^qL^{+,V})^{\pm 1}$ and of $(^qL^{-,V})^{\pm 1}$, where V runs through all finite rank representation of $U_h(\mathfrak{g})$.

Let $\mathbb{C}_{\mathcal{B}}^s[G]$ be the restricted Hopf algebra dual to $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ which is generated by the matrix elements of finite rank representations of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ of the form V^{res}, where V is a finite rank representation of $U_h^s(\mathfrak{g})$. By Proposition 2.8.5 (i) $\mathbb{C}_{\mathcal{B}}^s[G]$ is naturally $P \times P$–graded. Note that for different s the Hopf algebras $\mathbb{C}_{\mathcal{B}}^s[G]$ are isomorphic as coalgebras.

We shall use the following notation for elements of $\mathbb{C}_{\mathcal{B}}^s[G]$. Let V^{res} be a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice in a finite rank $U_h(\mathfrak{g})$–module V. Recall that there is a contravariant non–degenerate form $(\cdot, \cdot)$ on V such that $(u, xv) = (\omega(x)u, v)$ for any $u, v \in V$, $x \in U_h(\mathfrak{g})$. Assume that u is such that $(u, w) \in \mathcal{B}$ for any $w \in V^{res}$. Then $(u, \cdot)$ is an element of the dual module V^{res*}. Since V and V^{res} are of finite ranks and $(\cdot, \cdot)$ is non–degenerate all elements of V^{res*} can be obtained this way. Clearly, for any $v \in V^{res}$ $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}^s[G]$, and by the definition $\mathbb{C}_{\mathcal{B}}^s[G]$ is generated by such elements.

If V is a complex algebraic variety, we denote by $\mathbb{C}[V]$ the algebra of regular functions on V. By the definition of $\mathbb{C}_{\mathcal{B}}^s[G]$ we have the following lemma.

Lemma 3.2.3. *The quotient $\mathbb{C}_{\mathcal{B}}^s[G]/(q^{\frac{1}{dr^2}}-1)\mathbb{C}_{\mathcal{B}}^s[G]$ is isomorphic to $\mathbb{C}[G]$ as an algebra, $\mathbb{C}_{\mathcal{B}}^s[G]/(q^{\frac{1}{dr^2}}-1)\mathbb{C}_{\mathcal{B}}^s[G] \simeq \mathbb{C}[G]$.*

We also have the following description of $\mathbb{C}_{\mathcal{B}}^s[G^*]$.

Lemma 3.2.4. *$\mathbb{C}_{\mathcal{B}}^s[G^*]$ is the $\mathcal{B}$–subalgebra in $U_h^s(\mathfrak{g})$ generated by the elements $q^{\pm(Y_i - \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $q^{\pm(-Y_i - \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $i = 1, \ldots, l$, $\tilde{f}_\beta = (1 - q_\beta^{-2})f_\beta$, $\tilde{e}_\beta = (1 - q_\beta^2)e_\beta e^{h\beta^\vee}$, $\beta \in \Delta_+$.*

Proof. This lemma is analogous to the results obtained in Section 1.4 and Theorem 3.2 of [29], and in Proposition 4.2 and Theorem 4.6 in [28], in the case of the quantum group associated to the standard bialgebra structure.

We only briefly outline the main steps in the proof as the arguments used in the case of the standard bialgebra structure can be applied verbatim in the setting of this lemma.

Let ω_i, $i = 1, \ldots, l$ be the fundamental weights of $\mathfrak{g}$. Then from the definition of the elements $^qL^{\pm,V}$ and from formula (2.8.21) it follows that $q^{Y_i - \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i}$ is equal to the matrix element $(id \otimes g_i)\mathcal{R}^s$, where $g_i(\cdot) = (v_{\omega_i}, \cdot v_{\omega_i})$, and v_{ω_i} is a highest weight vector of $V_{\omega_i}^{res}$ normalized in such a way that $(v_{\omega_i}, v_{\omega_i}) = 1$. Other elements from the set $q^{\pm(Y_i - \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}, q^{\pm(-Y_i - \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $i = 1, \ldots, l$ are obtained in a similar way.

The fact that $\tilde{f}_\beta = (1 - q_\beta^{-2})f_\beta$, $\tilde{e}_\beta = (1 - q_\beta^2)e_\beta e^{h\beta^\vee}$, $\beta \in \Delta_+$ belong to $\mathbb{C}_{\mathcal{B}}^s[G^*]$ follows, e.g, from a slight modification of Theorem 3.2 in [48] (one should just replace the quantum R-matrix associated with the standard bialgebra structure with R-matrix (2.8.21)). This theorem along with the description of the generators $q^{\pm(Y_i - \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $q^{\pm(-Y_i - \kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}Y_i)}$, $i = 1, \ldots, l$ as matrix elements also imply that the elements listed in the statement of the lemma generate $\mathbb{C}_{\mathcal{B}}^s[G^*]$.

$\square$

We refer the reader to [29], Section 4 of [28], Section 3.6 in [48] or Section 2.8 of [130] for a more precise relation between the algebras $\mathbb{C}_{\mathcal{B}}^s[G]$ and $\mathbb{C}_{\mathcal{B}}^s[G^*]$. We shall not need these results in this book.

From the quantum Yang–Baxter equation (2.5.3) for $\mathcal{R}^s$ we get relations between ${}^qL^{\pm,V}$,

$$R^{VW\,q}L_2^{\pm,W\,q}L_1^{\pm,V} = {}^qL_1^{\pm,V\,q}L_2^{\pm,W}R^{VW}, \tag{3.2.4}$$

$$R^{VW\,q}L_2^{+,W\,q}L_1^{-,V} = {}^qL_1^{-,V\,q}L_2^{+,W}R^{VW}, \tag{3.2.5}$$

where by ${}^qL_1^{\pm,W}$, ${}^qL_2^{\pm,V}$ we understand the following elements of $U_h^s(\mathfrak{g}) \otimes \mathrm{End}_{\mathcal{B}}(V^{res}) \otimes \mathrm{End}_{\mathcal{B}}(W^{res})$,

$${}^qL_1^{\pm,V} = {}^qL^{\pm,V} \otimes I_{W^{res}}, \quad {}^qL_2^{\pm,W} = I_{V^{res}} \otimes {}^qL^{\pm,W},$$

I_X is identity endomorphism of X, and $R^{VW} \in \mathrm{End}_{\mathcal{B}}(V^{res}) \otimes \mathrm{End}_{\mathcal{B}}(W^{res})$ is regarded as an element of $U_h^s(\mathfrak{g}) \otimes \mathrm{End}_{\mathcal{B}}(V^{res}) \otimes \mathrm{End}_{\mathcal{B}}(W^{res})$ via the embedding

$$\mathrm{End}_{\mathcal{B}}(V^{res}) \otimes \mathrm{End}_{\mathcal{B}}(W^{res}) \to U_h^s(\mathfrak{g}) \otimes \mathrm{End}_{\mathcal{B}}(V^{res}) \otimes \mathrm{End}_{\mathcal{B}}(W^{res}), a \otimes b \mapsto 1 \otimes a \otimes b.$$

From (2.5.2) we can obtain the action of the comultiplication on ${}^qL^{\pm,V}$:

$$\Delta_s({}^qL_{ij}^{\pm,V}) = \sum_k {}^qL_{ik}^{\pm,V} \otimes {}^qL_{kj}^{\pm,V} \tag{3.2.6}$$

and the antipode,

$$S_s({}^qL_{ij}^{\pm,V}) = (({}^qL^{\pm,V})^{-1})_{ij}. \tag{3.2.7}$$

Since $\mathcal{R}^s = 1 \otimes 1 \pmod{h}$ relations (3.2.4) and (3.2.5) imply that the quotient algebra $\mathbb{C}_{\mathcal{B}}^s[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G^*]$ is commutative, and one can equip it with a Poisson structure given by

$$\{x_1, x_2\} = \frac{1}{dr^2}\frac{[a_1, a_2]}{q^{\frac{1}{dr^2}} - 1} \pmod{(q^{\frac{1}{dr^2}} - 1)}, \tag{3.2.8}$$

where $a_1, a_2 \in \mathbb{C}_{\mathcal{B}}^s[G^*]$ reduce to $x_1, x_2 \in \mathbb{C}_{\mathcal{B}}^s[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G^*]$ mod $(q^{\frac{1}{dr^2}} - 1)$.

Obviously, the comultiplication and the antipode in $\mathbb{C}_{\mathcal{B}}^s[G^*]$ induce a comultiplication and an antipode in $\mathbb{C}_{\mathcal{B}}^s[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G^*]$ compatible with the introduced Poisson structure, and the quotient $\mathbb{C}_{\mathcal{B}}^s[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G^*]$ becomes a Poisson–Hopf algebra.

Proposition 3.2.5. *The Poisson–Hopf algebra $\mathbb{C}_{\mathcal{B}}^s[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G^*]$ is isomorphic to $\mathbb{C}[G^*]$ as a Poisson–Hopf algebra.*

Proof. In the case when the underlying bialgebra structure is the standard one used in the definition of the Drinfeld–Jimbo quantum group this proposition is the Theorem in Section 12.1 in [30]. The arguments in the proof given there can be used verbatim to prove this proposition. Note, however, that our approach to the definition of the algebra $\mathbb{C}_{\mathcal{B}}^s[G^*]$, which follows the ideas of [36], §2, is different from the one adapted in [30]. The two approaches are equivalent. But in our framework one can give a more straightforward and shorter proof of this proposition which is presented below.

Denote by $p: \mathbb{C}_{\mathcal{B}}^s[G^*] \to \mathbb{C}_{\mathcal{B}}^s[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G^*] := \mathbb{C}[G^*]'$ the canonical projection. We also denote by the same letter the canonical projection $p: U_{\mathcal{B}}^{s,res}(\mathfrak{g}) \to U_{\mathcal{B}}^{s,res}(\mathfrak{g})/(q^{\frac{1}{dr^2}} - 1)U_{\mathcal{B}}^{s,res}(\mathfrak{g}) := U^{res}(\mathfrak{g})$.

If V is a free over $\mathbb{C}[[h]]$ of finite rank $U_h(\mathfrak{g})$–module, π_V the corresponding representation of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ in the space V^{res} equipped also with the natural action of $\mathfrak{h}$ with respect to which elements H_i, $i = 1, \ldots, l$ act on weight vectors of weight λ by multiplication by $\lambda(H_i)$, then $\overline{V} := V^{res}/(q^{\frac{1}{dr^2}} - 1)V^{res}$ is a finite–dimensional module over $U^{res}(\mathfrak{g})$ equipped also with the natural action of $\mathfrak{h}$ with respect to which elements H_i, $i = 1, \ldots, l$ act on weight vectors of weight λ by multiplication by $\lambda(H_i)$. We denote by $\pi_{\overline{V}}$ the corresponding representation of $U(\mathfrak{g})$.

Let $p_V : V^{res} \to \overline{V} = V^{res}/(q^{\frac{1}{dr^2}} - 1)V^{res}$ be the projection of V^{res} onto $\overline{V}$. Then any element $T \in \operatorname{End}_{\mathcal{B}}(V^{res})$ gives rise to an element $\nu(T) \in \operatorname{End}(\overline{V})$ defined by $\nu(T)\bar{v} = p_V T v$, where $v \in V^{res}$ is any representative of $\bar{v} \in \overline{V} = V^{res}/(q^{\frac{1}{dr^2}} - 1)V^{res}$. If $x \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ then according to this definition $\nu(\pi_V(x)) = \pi_{\overline{V}}(px)$.

Let

$$\tilde{L}^{\pm,V} = (p \otimes \nu)(^q L^{\pm,V}) \in \mathbb{C}[G^*]' \otimes \operatorname{End}\overline{V}. \tag{3.2.9}$$

Then the map

$$\imath : \mathbb{C}[G^*]' \to \mathbb{C}[G^*]$$

defined by

$$(\imath \otimes id)\tilde{L}^{\pm,V} = L^{\pm,\overline{V}} \tag{3.2.10}$$

is a well–defined algebra isomorphism. Indeed, consider, for instance, element $\tilde{L}^{+,V}$. From (2.6.16) it follows that

$$\begin{aligned}
\tilde{L}^{+,V} &= \prod_\beta \exp[p((1 - q_\beta^{-2})f_\beta) \otimes \pi_{\overline{V}}(X_\beta)] \times \\
&\quad \times (p \otimes id)\exp\left[\sum_{i=1}^l hH_i \otimes \pi_{\overline{V}}((\kappa\tfrac{1+s}{1-s}P_{\mathfrak{h}'} + id)Y_i)\right].
\end{aligned} \tag{3.2.11}$$

On the other hand (3.2.1) implies that every element L_+ may be uniquely represented in the form

$$L_+ = \prod_\beta \exp[b_\beta X_\beta]\exp\left[\sum_{i=1}^l b_i(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'} + id)Y_i\right], \quad b_i, b_\beta \in \mathbb{C},$$

and hence

$$L^{+,\overline{V}} = \prod_\beta \exp[b_\beta \pi_{\overline{V}}(X_\beta)]\exp\left[\sum_{i=1}^l b_i\pi_{\overline{V}}((\kappa\tfrac{1+s}{1-s}P_{\mathfrak{h}'} + id)Y_i)\right],$$

where the order of the terms in the product over the positive roots is the same as in formula (2.8.21) for the R–matrix. Comparing this with (3.2.11) and recalling the definition of $\imath$ we deduce that $\imath$ is an algebra isomorphism. We have to prove that $\imath$ is an isomorphism of Poisson–Hopf algebras.

Observe that $\mathcal{R}^s = 1 \otimes 1 - 2hr_-^s \pmod{h^2}$. Therefore from commutation relations (3.2.4), (3.2.5) it follows that $\mathbb{C}[G^*]'$ is a commutative algebra, and the Poisson brackets of matrix elements $\tilde{L}_{ij}^{\pm,\overline{V}}$ (see (3.2.8)) are given by (3.2.2), where $L^{\pm,\overline{V}}$ are replaced by $\tilde{L}^{\pm,\overline{V}}$. The factor $\frac{1}{dr^2}$ in formula (3.2.8) normalizes the Poisson bracket in such a way that bracket (3.2.8) is in agreement with (3.2.2).

From (3.2.6) we also obtain that the action of the comultiplication on the matrices $\tilde{L}^{\pm,\overline{V}}$ is given by (3.2.3), where $L^{\pm,\overline{V}}$ are replaced by $\tilde{L}^{\pm,\overline{V}}$. This completes the proof.

$\square$

We shall call the map $p : \mathbb{C}_{\mathcal{B}}^s[G^*] \to \mathbb{C}_{\mathcal{B}}^s[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G^*] = \mathbb{C}[G^*]$ the quasiclassical limit.

From the definition of the elements $^q L^{\pm,V}$ and from formula (3.2.6) it follows that the matrix elements of $^q L^{+,V\pm 1}$ (resp. $^q L^{-,V\pm 1}$) form a Hopf subalgebra $\mathbb{C}_{\mathcal{B}}^s[B_+] \subset \mathbb{C}_{\mathcal{B}}^s[G^*]$ (resp. $\mathbb{C}_{\mathcal{B}}^s[B_-] \subset \mathbb{C}_{\mathcal{B}}^s[G^*]$), and that $\mathbb{C}_{\mathcal{B}}^s[G^*]$ contains the subalgebra $\mathbb{C}_{\mathcal{B}}^s[N_+]$ generated by the elements $\tilde{f}_\beta = (1 - q_\beta^{-2})f_\beta$, $\beta \in \Delta_+$.

Next we define the algebra $\mathbb{C}_{\mathcal{B}}^s[G_*]$. For any finite rank representation V of $U_h(\mathfrak{g})$, let $^q L^V = {}^q L^{-,V^{-1}}\,{}^q L^{+,V} = (id \otimes \pi_V)\mathcal{R}_{21}^s \mathcal{R}^s$. Let $\mathbb{C}_{\mathcal{B}}^s[G_*]$ be the $\mathcal{B}$–subalgebra in $\mathbb{C}_{\mathcal{B}}^s[G^*]$ generated by the matrix entries of $^q L^V$, where V runs over all finite rank representations of $U_h^s(\mathfrak{g})$. From the definition of $\mathcal{R}^s$ we have

$$\begin{aligned}
\mathcal{R}_{21}^s \mathcal{R}^s &= \prod_\beta \sum_{k=0}^\infty q_\beta^{\frac{k(k+1)}{2}}[(1 - q_\beta^{-2})e_\beta^k e^{-hk\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee} \otimes f_\beta^{(k)}] \times \\
&\quad \times \exp\left[2h\sum_{i=1}^l Y_i \otimes H_i\right]\prod_\beta \sum_{k=0}^\infty q_\beta^{\frac{k(k+1)}{2}}[(1 - q_\beta^{-2})e^{hk\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee - hk\beta^\vee}f_\beta^k \otimes e_\beta^{(k)}q^{k\beta^\vee}].
\end{aligned} \tag{3.2.12}$$

Using this formula one immediately checks that actually $\mathbb{C}_{\mathcal{B}}^s[G_*] \subset U_{\mathcal{A}}^s(\mathfrak{g}) \cap \mathbb{C}_{\mathcal{B}}^s[G^*]$.

Define the right adjoint action Ad_s of $U_q^s(\mathfrak{g})$ on $U_q^s(\mathfrak{g})$ by formula (2.7.1) and the left adjoint action Ad_s' of $U_q^s(\mathfrak{g})$ on $U_q^s(\mathfrak{g})$ by formula (2.7.2).

Recall that $\mathbb{C}_{\mathcal{B}}^s[G]$ is the restricted Hopf algebra dual to $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ which is generated by the matrix elements of finite rank representations of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ of the form V^{res}, where V is a finite rank representation of $U_h^s(\mathfrak{g})$. The action Ad_s' induces a right adjoint action Ad_s of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ on $\mathbb{C}_{\mathcal{B}}^s[G]$ defined by

$$(\operatorname{Ad}_s x f)(w) = f(\operatorname{Ad}_s' x(w)), f \in \mathbb{C}_{\mathcal{B}}^s[G], x, w \in U_{\mathcal{B}}^{s,res}(\mathfrak{g}). \tag{3.2.13}$$

One can also equip finite rank representations of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ of the form V^{res}, where V is a finite rank representation of $U_h^s(\mathfrak{g})$, with a natural action of $\mathbb{C}_{\mathcal{B}}^s[G^*]$, for which the elements

$$q^{\pm(Y_i - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i)}, q^{\pm(-Y_i - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i)}, \; i = 1, \ldots l$$

act on a weight vector v_λ of weight λ by multiplication by the elements

$$q^{\pm((Y_i, \lambda^\vee) - \kappa(\frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i, \lambda^\vee))}, q^{\pm(-(Y_i, \lambda^\vee) - \kappa(\frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i, \lambda^\vee))} \in \mathcal{B}, \; i = 1, \ldots l,$$

respectively, and all the other generators of $\mathbb{C}_{\mathcal{B}}^s[G^*]$ belonging to $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ act in a natural way. Therefore adjoint action (3.2.13) can be extended to an action of $\mathbb{C}_{\mathcal{B}}^s[G^*]$, where elements $x \in \mathbb{C}_{\mathcal{B}}^s[G^*]$ act by the same formula (3.2.13).

For $\varepsilon \in \mathbb{C}^*$ we define $\mathbb{C}_\varepsilon[G] = \mathbb{C}_{\mathcal{B}}^s[G]/(q^{\frac{1}{dr^2}} - \varepsilon^{\frac{1}{dr^2}})\mathbb{C}_{\mathcal{B}}^s[G]$, $\mathbb{C}_\varepsilon^s[G_*] = \mathbb{C}_{\mathcal{B}}^s[G_*]/(q^{\frac{1}{dr^2}} - \varepsilon^{\frac{1}{dr^2}})\mathbb{C}_{\mathcal{B}}^s[G_*]$, where $\varepsilon^{\frac{1}{dr^2}}$ is a root of ε of degree $\frac{1}{dr^2}$.

Recall that by Lemma 3.2.3 $\mathbb{C}_{\mathcal{B}}^s[G]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G] \simeq \mathbb{C}[G]$, and denote the canonical projections $\mathbb{C}_{\mathcal{B}}^s[G] \to \mathbb{C}_{\mathcal{B}}^s[G]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G] \simeq \mathbb{C}[G]$, $\mathbb{C}_{\mathcal{B}}^s[G_*] \to \mathbb{C}_{\mathcal{B}}^s[G_*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G_*] := \mathbb{C}[G_*]$ by the same symbol p.

Proposition 3.2.6. *(i) The map*

$$\phi_{\mathcal{B}} : \mathbb{C}_{\mathcal{B}}^s[G] \to \mathbb{C}_{\mathcal{B}}^s[G_*], \phi_{\mathcal{B}}(f) = (id \otimes f)(\mathcal{R}_{21}^s \mathcal{R}^s) \qquad (3.2.14)$$

is an isomorphism of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ and $\mathbb{C}_{\mathcal{B}}^s[G^*]$–modules with respect to the adjoint actions Ad_s defined by (3.2.13) and (2.7.1), respectively. In particular, $\mathbb{C}_{\mathcal{B}}^s[G_*]$ is stable under the adjoint action of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ and $\mathbb{C}_{\mathcal{B}}^s[G^*]$.

(ii) Let $G_* \subset G$ be the image of $G^* \subset G \times G$ under the map $q : G^* \to G$, $q(L_+, L_-) = L_-^{-1} L_+$ (see (3.2.1) for the description of G^* in terms of $G \times G$). Then $G_* = B_- B_+$ is the big Bruhat cell in G, the algebra $\mathbb{C}[G_*]$ is generated by the restrictions of elements of $\mathbb{C}[G]$ to $G_* = B_- B_+$, the restriction map induces an isomorphism of algebras $\phi_1 : \mathbb{C}[G] \to \mathbb{C}[G_*]$, and $\phi_1 p = p\phi_{\mathcal{B}}$.

(iii) If ε is not a root of unity the algebra $\mathbb{C}_\varepsilon^s[G_*]$ can be identified with the Ad_s locally finite part $U_\varepsilon^s(\mathfrak{g})^{fin}$ of $U_\varepsilon^s(\mathfrak{g})$,

$$U_\varepsilon^s(\mathfrak{g})^{fin} = \{x \in U_\varepsilon^s(\mathfrak{g}) : \dim(\mathrm{Ad}_s U_\varepsilon^s(\mathfrak{g})(x)) < +\infty\},$$

where the adjoint action of the algebra $U_\varepsilon^s(\mathfrak{g})$ on itself is defined by formula (2.7.1).

Proof. Firstly, we prove (i) and (ii). From the definitions of the algebras $\mathbb{C}_{\mathcal{B}}^s[G]$ and $\mathbb{C}_{\mathcal{B}}^s[G_*]$ it follows that (3.2.14) is surjective.

Recall that by Lemma 3.2.3 we have the following algebra isomorphism $\mathbb{C}_1[G] = \mathbb{C}_{\mathcal{B}}^s[G]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G] \simeq \mathbb{C}[G]$.

By (3.2.1) $G_* = B_- B_+$ is the big Bruhat cell in G, and by the definition the algebra $\mathbb{C}[G_*] = \mathbb{C}_1[G_*] = \mathbb{C}_{\mathcal{B}}^s[G_*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G_*]$ is generated by the restrictions of elements of $\mathbb{C}[G] \simeq \mathbb{C}_{\mathcal{B}}^s[G]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G]$ to $G_* = B_- B_+$. As the big Bruhat cell is dense in G, we have an isomorphism of algebras $\phi_1 : \mathbb{C}[G] \to \mathbb{C}[G_*]$ induced by the restriction map.

Now observe that by the definition of map (3.2.14) one has $\phi_1 p = p\phi_{\mathcal{B}}$.

Let $h \in \mathbb{C}_{\mathcal{B}}^s[G]$, $h \neq 0$ be such that $\phi_{\mathcal{B}}(h) = 0$. Dividing by an appropriate power of $(q^{\frac{1}{dr^2}} - 1)$ we can assume without loss of generality that $p(h) \neq 0$. Then $\phi_1 p(h) = p\phi_{\mathcal{B}}(h) = 0$ which implies $p(h) = 0$ as ϕ_1 is an algebra isomorphism. Thus we arrive at a contradiction, and hence $\phi_{\mathcal{B}}$ is injective. We conclude that $\phi_{\mathcal{B}}$ an isomorphism of $\mathcal{B}$–modules.

By Proposition 11(ii) in [70], Section 10.1.3, (3.2.14) is a morphism of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ and $\mathbb{C}_{\mathcal{B}}^s[G^*]$–modules with respect to the adjoint actions Ad_s defined by (2.7.1) and (3.2.13), respectively.

(iii) It remains to establish the isomorphism $\mathbb{C}_\varepsilon^s[G_*] = U_\varepsilon^s(\mathfrak{g})^{fin}$ which can be done similarly to the proof of Proposition 1.7 in [71]. Let V_{ω_i}, $i = 1, \ldots, l$ be the finite rank representation of $U_h(\mathfrak{g})$ with highest weight ω_i, $i = 1, \ldots, l$. Below we use the notation introduced in the proof of Lemma 3.2.4. From formula (3.2.12) and from the definition of $^q L^V = (id \otimes \pi_V)\mathcal{R}_{21}^s \mathcal{R}^s$ it follows that the matrix element $(id \otimes g_i)\mathcal{R}_{21}^s \mathcal{R}^s$ of $^q L^{V_{\omega_i}}$, where $g_i(\cdot) = (v_i, \cdot v_i)$, and v_i is the highest weight vector of $V_{\omega_i}^{res}$ normalized in such a way that $(v_i, v_i) = 1$, coincides with L_i^2. This implies that L_i^2 are elements of the algebra $\mathbb{C}_\varepsilon[G_*] \subset U_\varepsilon^s(\mathfrak{g})$ as well.

Denote by $\mathfrak{H} \subset \mathbb{C}_\varepsilon^s[G_*] \subset U_\varepsilon^s(\mathfrak{g})$ the subalgebra generated by the elements $L_i^2 \in \mathbb{C}_\varepsilon^s[G_*]$, $i = 1, \ldots, l$. Similarly to Theorem 7.1.6 and Lemma 7.1.16 in [59] one can obtain that $U_\varepsilon^s(\mathfrak{g})^{fin} = \mathrm{Ad}_s U_\varepsilon^s(\mathfrak{g})\mathfrak{H}$. Since $\mathbb{C}_\varepsilon^s[G_*]$ is stable under the adjoint action we have an inclusion, $U_\varepsilon^s(\mathfrak{g})^{fin} \subset \mathbb{C}_\varepsilon^s[G_*]$.

On the other hand the adjoint action of $U^s_\varepsilon(\mathfrak{g})$ on $\mathbb{C}^s_\varepsilon[G]$ is locally finite by the definition of $\mathbb{C}^s_\varepsilon[G]$ and of the adjoint action. Using isomorphism (3.2.14) we deduce that the adjoint action of $U^s_\varepsilon(\mathfrak{g})$ on $\mathbb{C}^s_\varepsilon[G_*]$ is locally finite as well. Hence $\mathbb{C}^s_\varepsilon[G_*] \subset U^s_\varepsilon(\mathfrak{g})^{fin}$, and $\mathbb{C}^s_\varepsilon[G_*] = U^s_\varepsilon(\mathfrak{g})^{fin}$.

$\square$

We shall need the following property of the algebras $\mathbb{C}_\mathcal{B}[G^*]$, $\mathbb{C}^s_\mathcal{B}[G_*]$ and $\mathbb{C}^s_\mathcal{B}[G]$.

Proposition 3.2.7. *(i) Let $\mathbb{C}^s_\mathcal{B}[H] \subset \mathbb{C}^s_\mathcal{B}[G^*]$ be the subalgebra generated by the elements $q^{\pm(Y_i - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i)}$, $q^{\pm(-Y_i - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i)}$, $i = 1, \ldots, l$. Then $\mathbb{C}^s_\mathcal{B}[H]$ is $\mathcal{B}$–free with a countable basis V_i, $i \in \mathbb{N}$.*
(ii) The algebra $\mathbb{C}_\mathcal{B}[G^*]$ is free over $\mathcal{B}$, and the elements

$$\tilde{e}^{n_1}_{\beta_1} \ldots \tilde{e}^{n_D}_{\beta_D} V_i \tilde{f}^{k_D}_{\beta_D} \ldots \tilde{f}^{k_1}_{\beta_1}$$

with $n_j, k_j, i \in \mathbb{N}$, $j = 1, \ldots, D$ form a $\mathcal{B}$–basis in $\mathbb{C}^s_\mathcal{B}[G^*]$.
(iii) The algebra $\mathbb{C}^s_\mathcal{B}[N_+]$ is free over $\mathcal{B}$, and the elements $\tilde{f}^{k_D}_{\beta_D} \ldots \tilde{f}^{k_1}_{\beta_1}$ (or $\tilde{f}^{k_1}_{\beta_1} \tilde{f}^{k_2}_{\beta_2} \ldots \tilde{f}^{k_D}_{\beta_D}$) with $k_j, \in \mathbb{N}$ is a $\mathcal{B}$–basis of $\mathbb{C}^s_\mathcal{B}[N_+]$.
(iv) The subalgebra $\mathbb{C}^s_\mathcal{B}[G_*] \subset \mathbb{C}_\mathcal{B}[G^*]$, and the algebra $\mathbb{C}^s_\mathcal{B}[G]$ are free $\mathcal{B}$–modules.

Proof. (i) Recall that by Lemma 3.2.4 $\mathbb{C}^s_\mathcal{B}[G^*]$ is the $\mathcal{B}$–subalgebra in $U^s_h(\mathfrak{g})$ generated by the elements $q^{\pm(Y_i - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i)}$, $q^{\pm(-Y_i - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i)}$, $i = 1, \ldots, l$, $\tilde{f}_\beta = (1 - q^{-2}_\beta) f_\beta$, $\tilde{e}_\beta = (1 - q^2_\beta) e_\beta e^{h\beta^\vee}$, $\beta \in \Delta_+$. The subalgebra $\mathbb{C}^s_\mathcal{B}[H]$ of $\mathbb{C}^s_\mathcal{B}[G^*]$ generated by the elements $q^{\pm(Y_i - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i)}, q^{\pm(-Y_i - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i)}$, $i = 1, \ldots, l$ is in turn a subalgebra of the $\mathcal{B}$–subalgebra $U'_\mathcal{B}(\mathfrak{h}) \subset U^s_h(\mathfrak{g})$ generated by the elements $U_i = q^{\frac{1}{dr^2} Y_i}, U_i^{-1}$, $i = 1, \ldots, l$. The last algebra is obviously $\mathcal{B}$–free with a basis consisting of the products $U^{n_1}_1 \ldots U^{n_l}_l$, $n_1, \ldots, n_l \in \mathbb{Z}$. Since $\mathcal{B}$ is a principal ideal domain the subalgebra in $\mathbb{C}^s_\mathcal{B}[H] \subset U'_\mathcal{B}(\mathfrak{h})$ is also $\mathcal{B}$–free by Theorem 6.5 in [102]. We denote by V_i, $i \in \mathbb{N}$ the elements of some $\mathcal{B}$–basis of this subalgebra.
(ii) From Step 1 of the proof of the Theorem in Section 12.1 in [30] it follows that the elements

$$\tilde{e}^{n_1}_{\beta_1} \ldots \tilde{e}^{n_D}_{\beta_D} V_i \tilde{f}^{k_D}_{\beta_D} \ldots \tilde{f}^{k_1}_{\beta_1}$$

with $n_j, k_j, i \in \mathbb{N}$, $j = 1, \ldots, D$ form a $\mathcal{B}$–basis in $\mathbb{C}^s_\mathcal{B}[G^*]$.
(iii) By Step 1 of the proof of the Theorem in Section 12.1 in [30] the products $\tilde{f}^{k_D}_{\beta_D} \ldots \tilde{f}^{k_1}_{\beta_1}$ with $k_j, \in \mathbb{N}$ is a $\mathcal{B}$–basis of $\mathbb{C}^s_\mathcal{B}[N_+]$. Applying the algebra antiautomorphism ω_0 to these elements we obtain that $\tilde{f}^{k_1}_{\beta_1} \tilde{f}^{k_2}_{\beta_2} \ldots \tilde{f}^{k_D}_{\beta_D}$ with $k_j, \in \mathbb{N}$ is also a $\mathcal{B}$–basis of $\mathbb{C}^s_\mathcal{B}[N_+]$.
(iv) Since $\mathcal{B}$ is a principal ideal domain, and the algebra $\mathbb{C}_\mathcal{B}[G^*]$ is $\mathcal{B}$–free by part (ii), the subalgebra $\mathbb{C}^s_\mathcal{B}[G_*] \subset \mathbb{C}_\mathcal{B}[G^*]$ is also $\mathcal{B}$–free by Theorem 6.5 in [102], and isomorphism (3.2.14) implies that $\mathbb{C}^s_\mathcal{B}[G]$ is $\mathcal{B}$–free.

$\square$

Suppose that the positive root system Δ_+ and its ordering are associated to s as in Definition 1.6.19. Denote by $\mathbb{C}^s_\mathcal{B}[M_+]$ the subalgebra in $\mathbb{C}^s_\mathcal{B}[N_+]$ generated by the elements $\tilde{f}_\beta$, $\beta \in \Delta_{\mathfrak{m}_+}$.
By Lemma 1.6.18 the linear subspace of $\mathfrak{g}$ generated by the root vectors X_α (resp. $X_{-\alpha}$), $\alpha \in \Delta_{\mathfrak{m}_+}$ is in fact a Lie subalgebra $\mathfrak{m}_+ \subset \mathfrak{g}$ (resp. $\mathfrak{m}_- \subset \mathfrak{g}$). By definition $\Delta_{\mathfrak{m}_+} \subset \Delta_+$, and hence $\mathfrak{m}_\pm \subset \mathfrak{n}_\pm$.
Note that one can consider $\mathfrak{n}_+$ and $\mathfrak{m}_\pm$ as Lie subalgebras in $\mathfrak{g}^*$ via embeddings

$$\mathfrak{n}_+ \to \mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}, \; x \mapsto (x, 0),$$

$$\mathfrak{m}_+ \to \mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}, \; x \mapsto (x, 0),$$

$$\mathfrak{m}_- \to \mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}, \; x \mapsto (0, x),$$

where $\mathfrak{g}^*$ is regarded as a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ using the embedding (3.1.5).
Using these embeddings the algebraic subgroups $N_+, M_\pm \subset G$ corresponding to the algebraic Lie subalgebras $\mathfrak{n}_+, \mathfrak{m}_\pm \subset \mathfrak{g}$ can be regarded as Lie subgroups in G^* corresponding to the Lie subalgebras $\mathfrak{n}_+, \mathfrak{m}_\pm \subset \mathfrak{g}^*$.
From this observation and from Proposition 3.2.5 (see especially formula (3.2.10)) we obtain the following properties of the algebras $\mathbb{C}^s_\mathcal{B}[N_+]$ and $\mathbb{C}^s_\mathcal{B}[M_+]$.

Proposition 3.2.8. *The algebra $\mathbb{C}^s_\mathcal{B}[N_+]$ is a deformation of the algebra of regular functions on the subgroup $N_+ \subset G^*$, and $\mathbb{C}^s_\mathcal{B}[M_+]$ is a deformation of the algebra of regular functions on the subgroup $M_+ \subset G^*$ in the sense that $p(\mathbb{C}^s_\mathcal{B}[N_+]) \simeq \mathbb{C}[N_+]$ and $p(\mathbb{C}^s_\mathcal{B}[M_+]) \simeq \mathbb{C}[M_+]$. Thus $\mathbb{C}[N_+] \simeq \mathbb{C}^s_\mathcal{B}[N_+]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}^s_\mathcal{B}[N_+]$ and $\mathbb{C}[M_+] \simeq \mathbb{C}^s_\mathcal{B}[M_+]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}^s_\mathcal{B}[M_+]$ are naturally Poisson subalgebras of $\mathbb{C}[G^*] \simeq \mathbb{C}^s_\mathcal{B}[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}^s_\mathcal{B}[G^*]$.*

Note that M_- can also be regarded as a subgroup in G^* corresponding to the Lie subalgebra $\mathfrak{m}_- \subset \mathfrak{g}^*$.

The following proposition gives the most important property of the subalgebra $\mathbb{C}_{\mathcal{B}}^s[M_+]$ which plays the key role in the definition of q-W–algebras.

Proposition 3.2.9. *(i) The defining relations in the subalgebra $\mathbb{C}_{\mathcal{B}}^s[M_+]$ for the generators $\tilde{f}_\beta = (1 - q_\beta^{-2})f_\beta$, $\beta \in \Delta_{\mathfrak{m}_+} := \{\beta_1, \ldots, \beta_c\}$ are of the form*

$$\tilde{f}_\alpha \tilde{f}_\beta - q^{(\alpha,\beta)+\kappa(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}\tilde{f}_\beta \tilde{f}_\alpha = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} C(k_1,\ldots,k_n)\tilde{f}_{\delta_1}^{k_1}\tilde{f}_{\delta_2}^{k_2}\ldots\tilde{f}_{\delta_n}^{k_n}, \ \alpha < \beta, \tag{3.2.15}$$

where $C(k_1, \ldots, k_n) \in \mathcal{B}$.
   (ii) The products $\tilde{f}_{\beta_1}^{k_1}\tilde{f}_{\beta_2}^{k_2}\ldots\tilde{f}_{\beta_c}^{k_c}$ form a $\mathcal{B}$–basis of $\mathbb{C}_{\mathcal{B}}^s[M_+]$.
   (iii) If $\kappa = 1$ then for any $k_i \in \mathcal{B}$, $i = 1, \ldots, l'$ the map $\chi_q^s : \mathbb{C}_{\mathcal{B}}^s[M_+] \to \mathcal{B}$,

$$\chi_q^s(\tilde{f}_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \ldots, \gamma_{l'}\} \\ k_i & \beta = \gamma_i \end{cases}, \tag{3.2.16}$$

is a character of $\mathbb{C}_{\mathcal{B}}^s[M_+]$ vanishing on the r.h.s. and on the l.h.s. of relations (3.2.15).
   (iv) Assume that $\varepsilon^{2d_i} \neq 1$, and $\varepsilon^4 \neq 1$ if $\mathfrak{g}$ is of type G_2. Suppose also that there exists $n \in \mathbb{Z}$ such that $\varepsilon^{nd-1} = 1$. Let $\kappa = nd$. Then the algebra $\mathbb{C}_\varepsilon^s[M_+] = \mathbb{C}_{\mathcal{B}}^s[M_+]/(q^{\frac{1}{dr^2}} - \varepsilon^{\frac{1}{dr^2}})\mathbb{C}_{\mathcal{B}}^s[M_+]$, where $\varepsilon^{\frac{1}{dr^2}}$ is a root of ε of degree $\frac{1}{dr^2}$, is isomorphic to $U_\varepsilon^s(\mathfrak{m}_-)$.
   (v) Under the assumptions of part (iv), the elements $f^{\mathbf{r}} = f_{\beta_1}^{r_1}\ldots f_{\beta_c}^{r_c}$, $r_i \in \mathbb{N}$, $i = 1, \ldots d$ form a linear basis of $U_\varepsilon^s(\mathfrak{m}_-)$.
   (vi) Under the assumptions of part (iv), for any $c_i \in \mathbb{C}$, $i = 1, \ldots, l'$ the map $\chi_\varepsilon^s : U_\varepsilon^s(\mathfrak{m}_-) \to \mathbb{C}$,

$$\chi_\varepsilon^s(f_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \ldots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i \end{cases}, \tag{3.2.17}$$

is a character of $U_\varepsilon^s(\mathfrak{m}_-)$.

Proof. (ii) By Lemma 2.8.2 and Remark 2.8.3 any element of $\mathbb{C}_{\mathcal{B}}^s[M_+]$ can be uniquely represented as a $\mathbb{C}(q^{\frac{1}{dr^2}})$–linear combination of the elements $\tilde{f}_{\beta_1}^{k_1}\tilde{f}_{\beta_2}^{k_2}\ldots\tilde{f}_{\beta_c}^{k_c}$. By Proposition 3.2.7 (ii) the coefficients of this decomposition must belong to $\mathcal{B}$.

(i) From (2.8.12) we also obtain commutation relations (3.2.15) with $C(k_1, \ldots, k_n) \in \mathbb{C}(q^{\frac{1}{dr^2}})$. As we already proved the products $\tilde{f}_{\beta_1}^{k_1}\tilde{f}_{\beta_2}^{k_2}\ldots\tilde{f}_{\beta_c}^{k_c}$ form a $\mathcal{B}$–basis of $\mathbb{C}_{\mathcal{B}}^s[M_+]$. Therefore the coefficients $C(k_1, \ldots, k_n)$ in (3.2.15) belong to $\mathcal{B}$.

(iii) Assume that $\kappa = 1$. In order to prove that the map $\chi_q^s : \mathbb{C}_{\mathcal{B}}^s[M_+] \to \mathcal{B}$ defined by (3.2.16) is a character we show that all relations (3.2.15) for $\tilde{f}_\alpha, \tilde{f}_\beta$ with $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$, which are obviously defining relations in the subalgebra $\mathbb{C}_{\mathcal{B}}^s[M_+]$, belong to the kernel of χ_q^s. By the definition the only generators of $\mathbb{C}_{\mathcal{B}}^s[M_+]$ on which χ_q^s may not vanish are $\tilde{f}_{\gamma_i}$, $i = 1, \ldots, l'$. By part (vi) of Proposition 1.6.6 for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ cannot be represented as a linear combination $\sum_{k=1}^t c_k\gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_t} < \beta$. Hence for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the value of the map χ_q^s on the right hand side of the corresponding commutation relation (3.2.15) is equal to zero.

Therefore it suffices to prove that

$$\chi_q^s(\tilde{f}_{\gamma_i}\tilde{f}_{\gamma_j} - q^{(\gamma_i,\gamma_j)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j)}\tilde{f}_{\gamma_j}\tilde{f}_{\gamma_i}) = k_ik_j(1 - q^{(\gamma_i,\gamma_j)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j)}) = 0, \ i < j.$$

The last identity holds provided $(\gamma_i, \gamma_j) + (\frac{1+s}{1-s}P_{\mathfrak{h}'}^*\gamma_i, \gamma_j) = 0$ for $i < j$ which is indeed the case by Lemma 2.8.1.

(iv) Assume now that $\varepsilon^{2d_i} \neq 1$, and $\varepsilon^4 \neq 1$ if $\mathfrak{g}$ is of type G_2. Suppose also that there exists $n \in \mathbb{Z}$ such that $\varepsilon^{nd-1} = 1$. Let $\kappa = nd$.

Under these conditions imposed on ε the map $\mathbb{C}_\varepsilon^s[M_+] \to U_\varepsilon^s(\mathfrak{m}_-)$, $\tilde{f}_\alpha \mapsto (1 - q_\alpha^{-2})f_\alpha$, $\alpha \in \Delta_{\mathfrak{m}_+}$ is obviously an algebra isomorphism.

(v) By Lemma 2.8.2 and Remark 2.8.3 the elements $f^{\mathbf{r}} = f_{\beta_1}^{r_1}\ldots f_{\beta_c}^{r_c}$, $r_i \in \mathbb{N}$, $i = 1, \ldots d$ form a linear basis of $U_\varepsilon^s(\mathfrak{m}_-)$.

From (2.8.12) we obtain the following commutation relations

$$f_\alpha f_\beta - \varepsilon^{(\alpha,\beta)+nd(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}f_\beta f_\alpha = \sum_{\alpha<\delta_1<\ldots<\delta_n<\beta} D(k_1,\ldots,k_n)f_{\delta_1}^{k_1}f_{\delta_2}^{k_2}\ldots f_{\delta_n}^{k_n}, \ \alpha < \beta, \tag{3.2.18}$$

where $D(k_1, \ldots, k_n) \in \mathbb{C}$.

(vi) In order to show that the map $\chi_\varepsilon^s : U_\varepsilon^s(\mathfrak{m}_-) \to \mathbb{C}$ defined by (3.2.17) is a character we verify that all relations (3.2.18) for f_α, f_β with $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$, which are obviously defining relations in the subalgebra $U_\varepsilon^s(\mathfrak{m}_-)$, belong to the kernel of χ_ε^s. By the definition the only generators of $U_\varepsilon^s(\mathfrak{m}_-)$ on which χ_ε^s may not vanish are f_{γ_i}, $i = 1, \ldots, l'$. By part (vi) of Proposition 1.6.6 for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ cannot be represented as a linear combination $\sum_{k=1}^t c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_t} < \beta$. Hence for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the value of the map χ_ε^s on the right hand side of the corresponding commutation relation (3.2.15) is equal to zero.

Therefore it suffices to prove that

$$\chi_\varepsilon^s(f_{\gamma_i} f_{\gamma_j} - \varepsilon^{(\gamma_i, \gamma_j) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \gamma_i, \gamma_j)} f_{\gamma_j} f_{\gamma_j}) = c_i c_j (1 - \varepsilon^{(\gamma_i, \gamma_j) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \gamma_i, \gamma_j)}) = 0, \ i < j.$$

By Lemma 2.8.1 $(\frac{1+s}{1-s} P_{\mathfrak{h}'}^* \gamma_i, \gamma_j) = -(\gamma_i, \gamma_j)$ for $i < j$, and hence

$$\chi_\varepsilon^s(f_{\gamma_i} f_{\gamma_j} - \varepsilon^{(\gamma_i, \gamma_j) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \gamma_i, \gamma_j)} f_{\gamma_j} f_{\gamma_j}) = c_i c_j (1 - \varepsilon^{(\gamma_i, \gamma_j)(1 - nd)}) = 0$$

for $i < j$ as by the assumption $\varepsilon^{nd-1} = 1$. This completes the proof. $\qquad \square$

Now we are ready to define q-W–algebras. In the rest of this section we assume that $\kappa = 1$. Denote by $I_\mathcal{B}$ the left ideal in $\mathbb{C}_\mathcal{B}^s[G^*]$ generated by the kernel of χ_q^s, and by $\rho_{\chi_q^s}$ the canonical projection $\mathbb{C}_\mathcal{B}^s[G^*] \to \mathbb{C}_\mathcal{B}^s[G^*]/I_\mathcal{B} := Q'_\mathcal{B}$. Let $Q_\mathcal{B}$ be the image of $\mathbb{C}_\mathcal{B}^s[G_*] \subset \mathbb{C}_\mathcal{B}^s[G^*]$ under the projection $\rho_{\chi_q^s}$, $Q_\mathcal{B} = \rho_{\chi_q^s}(\mathbb{C}_\mathcal{B}^s[G_*])$.

We shall need the following properties of $Q'_\mathcal{B}$ and $Q_\mathcal{B}$.

Proposition 3.2.10. $Q'_\mathcal{B}$ and $Q_\mathcal{B}$ are free $\mathcal{B}$–modules.

Proof. Using the $\mathcal{B}$–basis

$$\tilde{e}_{\beta_1}^{n_1} \ldots \tilde{e}_{\beta_D}^{n_D} V_i \tilde{f}_{\beta_D}^{k_D} \ldots \tilde{f}_{\beta_1}^{k_1}$$

with $n_j, k_j, i \in \mathbb{N}$, $j = 1, \ldots, D$ of $\mathbb{C}_\mathcal{B}^s[G^*]$ from Proposition 3.2.7 (ii) and the definition of $\mathbb{C}_\mathcal{B}^s[G^*]/I_\mathcal{B}$ one immediately sees that the classes of the elements $\tilde{e}_{\beta_1}^{n_1} \ldots \tilde{e}_{\beta_D}^{n_D} V_i \tilde{f}_{\beta_D}^{k_D} \ldots \tilde{f}_{\beta_{c+1}}^{k_{c+1}}$ with $n_j, k_m, i \in \mathbb{N}, j = 1, \ldots, D, m = c+1, \ldots, D$ form a $\mathcal{B}$–basis in $\mathbb{C}_\mathcal{B}^s[G^*]/I_\mathcal{B}$.

Since $\mathcal{B}$ is a principal ideal domain the $\mathcal{B}$–submodule $Q_\mathcal{B} \subset \mathbb{C}_\mathcal{B}^s[G^*]/I_\mathcal{B}$ is $\mathcal{B}$–free. $\qquad \square$

Lemma 3.2.11. *The adjoint action of $\mathbb{C}_\mathcal{B}^s[M_+]$ on $\mathbb{C}_\mathcal{B}^s[G_*]$ induces an action on $Q'_\mathcal{B}$ and on $Q_\mathcal{B}$.*

Proof. Observe that we have an inclusion $[\mathbb{C}_\mathcal{B}^s[M_+], \mathrm{Ker}\chi_q^s] \subset \mathrm{Ker}\chi_q^s$. Using this inclusion, formula (2.7.1), the fact that $\Delta_s(\mathbb{C}_\mathcal{B}^s[M_+]) \subset \mathbb{C}_\mathcal{B}^s[B_+] \otimes \mathbb{C}_\mathcal{B}^s[M_+]$ (see formula (2.8.13)) we deduce that the adjoint action of $\mathbb{C}_\mathcal{B}^s[M_+]$ on $\mathbb{C}_\mathcal{B}^s[G_*]$ induces an action on $Q'_\mathcal{B}$ and on $Q_\mathcal{B}$. $\qquad \square$

We call the action of $\mathbb{C}_\mathcal{B}^s[M_+]$ on $Q'_\mathcal{B}$ and on $Q_\mathcal{B}$ the adjoint action as well and denote it by Ad.

Let $\mathcal{B}_{\varepsilon_s}$ be the trivial representation of $\mathbb{C}_\mathcal{B}^s[M_+]$ given by the counit. Consider the $\mathcal{B}$–submodule $W_\mathcal{B}^s(G)$ of Ad–invariants in $Q_\mathcal{B}$,

$$W_\mathcal{B}^s(G) = \mathrm{Hom}_{\mathbb{C}_\mathcal{B}^s[M_+]}(\mathcal{B}_{\varepsilon_s}, Q_\mathcal{B}). \tag{3.2.19}$$

Proposition 3.2.12. *(i) $W_\mathcal{B}^s(G)$ is isomorphic to the $\mathcal{B}$–submodule of all $v + I_\mathcal{B} \in Q_\mathcal{B}$ such that $mv \in I_\mathcal{B}$ (or $[m, v] \in I_\mathcal{B}$) in $\mathbb{C}_\mathcal{B}^s[G^*]$ for any $m \in I_\mathcal{B}$, where $v \in \mathbb{C}_\mathcal{B}^s[G^*]$ is any representative of $v + I_\mathcal{B} \in Q_\mathcal{B}$.*
(ii) Multiplication in $\mathbb{C}_\mathcal{B}^s[G^*]$ induces a multiplication on $W_\mathcal{B}^s(G)$.

Proof. (i) For the proof we shall firstly derive the formula for the adjoint action of the generators $\tilde{f}_\beta$. From (2.8.13) using linear independence of weight components and the fact that $\mathbb{C}_\mathcal{B}^s[B_+]$ is a Hopf algebra we obtain

$$\Delta_s(\tilde{f}_{\beta_k}) = G_{\beta_k}^{-1} \otimes \tilde{f}_{\beta_k} + \tilde{f}_{\beta_k} \otimes 1 + \sum_i \tilde{y}_i \otimes \tilde{x}_i, \tag{3.2.20}$$

where

$$G_\beta = e^{h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta^\vee - h\beta^\vee} \in \mathbb{C}_\mathcal{B}^s[B_+], \tilde{y}_i = e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \gamma_{x_i}^\vee + h\gamma_{x_i}^\vee} \overline{\tilde{y}}_i,$$

$$\overline{\tilde{y}}_i \in \mathbb{C}_{\mathcal{B}}([-\beta_{k+1}, -\beta_D]),$$
$$\tilde{x}_i \in \mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_{k-1}]),$$

$\overline{\tilde{y}}_i, \tilde{x}_i$ belong to weight components and have non-zero weights, γ_{x_i} is the weight of $\tilde{x}_i$,

$$\mathbb{C}_{\mathcal{B}}([-\beta_{k+1}, -\beta_D]) \text{ (resp. } \mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_{k-1}]))$$

is the subalgebra in $\mathbb{C}_{\mathcal{B}}[N_+]$ generated by $\tilde{f}_{\beta_{k+1}}, \ldots, \tilde{f}_{\beta_D}$ (resp. $\tilde{f}_{\beta_1}, \ldots, \tilde{f}_{\beta_{k-1}}$).

By (3.2.20) we also have

$$S_s(\tilde{f}_\beta) = -G_\beta \tilde{f}_\beta - \sum_i S_s(\tilde{y}_i)\tilde{x}_i.$$

Combining this formula with (3.2.20) and using (2.7.1) we deduce

$$\mathrm{Ad}_s \tilde{f}_{\beta_k} w = -G_{\beta_k}[\tilde{f}_{\beta_k}, w] - \sum_i S_s(\tilde{y}_i)[\tilde{x}_i, w]$$

The induced action of the elements $\tilde{f}_{\beta_k} \in \mathbb{C}_{\mathcal{B}}^s[M_+]$, $\beta_k \in \Delta_{\mathfrak{m}_+}$, on $Q'_{\mathcal{B}}$ takes the form

$$\mathrm{Ad}_s \tilde{f}_{\beta_k} v = -G_{\beta_k}(\tilde{f}_{\beta_k} - \chi_q^s(\tilde{f}_{\beta_k}))v - \sum_i S_s(\tilde{y}_i)(\tilde{x}_i - \chi_q^s(\tilde{x}_i))v, \beta_k \in \Delta_{\mathfrak{m}_+}. \tag{3.2.21}$$

We have to show that $W_{\mathcal{B}}^s(G)$ is isomorphic to the $\mathcal{B}$–submodule of all $v \in Q_{\mathcal{B}} \subset Q'_{\mathcal{B}}$ such that $mv = 0$ in $Q'_{\mathcal{B}}$ for any $m \in I_{\mathcal{B}}$.

The left ideal $I_{\mathcal{B}}$ is generated by elements $x - \chi_q^s(x)$, $x \in \mathbb{C}_{\mathcal{B}}^s[M_+]$. Therefore by (3.2.21) if for some $v \in Q_{\mathcal{B}}$ $mv = 0$ in $Q'_{\mathcal{B}}$ for any $m \in I_{\mathcal{B}}$ then v is invariant with respect to the adjoint action of all generators $\tilde{f}_{\beta_k}$ of $\mathbb{C}_{\mathcal{B}}^s[M_+]$, and hence $v \in W_{\mathcal{B}}^s(G)$.

Now assume that $v \in W_{\mathcal{B}}^s(G)$. We shall prove that $mv = 0$ in $Q'_{\mathcal{B}}$ for any $m \in I_{\mathcal{B}}$.

Since the left ideal $I_{\mathcal{B}}$ is generated by elements $x - \chi_q^s(x)$, $x \in \mathbb{C}_{\mathcal{B}}^s[M_+]$ it suffices to show that $(x - \chi_q^s(x))v = 0$ for any $x \in \mathbb{C}_{\mathcal{B}}^s[M_+]$. We shall prove this statement by induction using the subalgebras $\mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_k])$, $k = 1, \ldots, c$, so that $\mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_c]) = \mathbb{C}_{\mathcal{B}}^s[M_+]$.

Observe that β_1 is a simple root, and hence from (3.2.21) we obtain

$$0 = \mathrm{Ad}_s \tilde{f}_{\beta_1} v = -G_{\beta_1}(\tilde{f}_{\beta_1} - \chi_q^s(\tilde{f}_{\beta_1}))v.$$

Since the element $G_{\beta_1} \in \mathbb{C}_{\mathcal{B}}[N_+]$ is invertible this implies

$$(\tilde{f}_{\beta_1} - \chi_q^s(\tilde{f}_{\beta_1}))v = 0,$$

i.e. $(x - \chi_q^s(x))v = 0$ in $Q'_{\mathcal{B}}$ for any $x \in \mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_1])$ as the subalgebra $\mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_1])$ is generated by $\tilde{f}_{\beta_1}$.

Now suppose that for some $1 < k \le c$ $(x - \chi_q^s(x))v = 0$ in $Q'_{\mathcal{B}}$ for any $x \in \mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_{k-1}])$. Then from (3.2.21) we obtain

$$0 = \mathrm{Ad}_s \tilde{f}_{\beta_k} v = -G_{\beta_k}(\tilde{f}_{\beta_k} - \chi_q^s(\tilde{f}_{\beta_k}))v - \sum_i S_s(\tilde{y}_i)(\tilde{x}_i - \chi_q^s(\tilde{x}_i))v = -G_{\beta_k}(\tilde{f}_{\beta_k} - \chi_q^s(\tilde{f}_{\beta_k}))v$$

since $\tilde{x}_i \in \mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_{k-1}])$. The previous identity and the fact that the element $G_{\beta_k} \in \mathbb{C}_{\mathcal{B}}[N_+]$ is invertible yield

$$(\tilde{f}_{\beta_k} - \chi_q^s(\tilde{f}_{\beta_k}))v = 0.$$

Now observe that by Proposition 3.2.9 any element x of $\mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_k])$ can be uniquely represented in the form $x = \tilde{f}_{\beta_k} z + z'$, where $z, z' \in \mathbb{C}_{\mathcal{B}}([-\beta_1, -\beta_{k-1}])$. Therefore

$$xv = (\tilde{f}_{\beta_k} z + z')v = (\tilde{f}_{\beta_k} \chi_q^s(z) + \chi_q^s(z'))v = (\chi_q^s(\tilde{f}_{\beta_k})\chi_q^s(z) + \chi_q^s(z'))v = \chi_q^s(\tilde{f}_{\beta_k} z + z')v = \chi_q^s(x)v.$$

This establishes the induction step and proves the first claim of this proposition.

(ii) From the description of $W_{\mathcal{B}}^s(G)$ obtained in part (i) it follows that if $v_1, v_2 \in \mathbb{C}_{\mathcal{B}}^s[G^*]$ are any representatives of elements $v_1 + I_{\mathcal{B}}, v_2 + I_{\mathcal{B}} \in W_{\mathcal{B}}^s(G)$ then the formula

$$(v_1 + I_{\mathcal{B}})(v_2 + I_{\mathcal{B}}) = v_1 v_2 + I_q$$

defines a multiplication in $W_{\mathcal{B}}^s(G)$. This completes the proof.

$$\square$$

The $\mathcal{B}$–module $W_{\mathcal{B}}^s(G)$ equipped with the multiplication opposite to the one defined in the previous proposition is called the q-W–algebra associated to (the conjugacy class of) the Weyl group element $s \in W$.

In conclusion we obtain some results on the structure of $Q_{\mathcal{B}}$. Consider the Lie algebra $\mathfrak{L}_{\mathcal{B}}$ associated to the associative algebra $\mathbb{C}_{\mathcal{B}}^s[M_+]$, i.e. $\mathfrak{L}_{\mathcal{B}}$ is the Lie algebra which is isomorphic to $\mathbb{C}_{\mathcal{B}}^s[M_+]$ as a $\mathcal{B}$–module, and the Lie bracket in $\mathfrak{L}_{\mathcal{B}}$ is given by the usual commutator of elements in $\mathbb{C}_{\mathcal{B}}^s[M_+]$.

Note that since χ_q^s is a character of $\mathbb{C}_{\mathcal{B}}^s[M_+]$ the ideal $I_{\mathcal{B}}$ is stable under the action of $\mathbb{C}_{\mathcal{B}}^s[M_+]$ on $\mathbb{C}_{\mathcal{B}}^s[G^*]$ by commutators. Therefore one can define an action of the Lie algebra $\mathfrak{L}_{\mathcal{B}}$ on $Q_{\mathcal{B}}' = \mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}$:

$$m \cdot (x + I_{\mathcal{B}}) = \rho_{\chi_q^s}([m, x]). \tag{3.2.22}$$

where $x \in \mathbb{C}_{\mathcal{B}}^s[G^*]$ is any representative of $x + I_{\mathcal{B}} \in \mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}$ and $m \in \mathbb{C}_{\mathcal{B}}^s[M_+]$. The algebra $W_{\mathcal{B}}^s(G)$ can be described now as the intersection of the $\mathcal{B}$–module of invariants $(\mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}})^{\mathfrak{L}_{\mathcal{B}}}$ with respect to action (3.2.22) with $Q_{\mathcal{B}} \subset \mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}$,

$$W_{\mathcal{B}}^s(G) \simeq (\mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}})^{\mathfrak{L}_{\mathcal{B}}} \cap Q_{\mathcal{B}}. \tag{3.2.23}$$

Denote by $\mathcal{B}_{\chi_q^s}$ the rank one representation of the algebra $\mathbb{C}_{\mathcal{B}}^s[M_+]$ defined by the character χ_q^s. Using the description of the algebra $W_{\mathcal{B}}^s(G)$ in terms of action (3.2.22) and the isomorphism $\mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}} = \mathbb{C}_{\mathcal{B}}^s[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}^s[M_+]} \mathcal{B}_{\chi_q^s}$ one can also define the algebra $W_{\mathcal{B}}^s(G)$ as the intersection

$$W_{\mathcal{B}}^s(G) = \mathrm{Hom}_{\mathbb{C}_{\mathcal{B}}^s[M_+]}(\mathcal{B}_{\chi_q^s}, \mathbb{C}_{\mathcal{B}}^s[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}^s[M_+]} \mathcal{B}_{\chi_q^s}) \cap Q_{\mathcal{B}}.$$

Using Frobenius reciprocity we also have

$$\mathrm{Hom}_{\mathbb{C}_{\mathcal{B}}^s[M_+]}(\mathcal{B}_{\chi_q^s}, \mathbb{C}_{\mathcal{B}}^s[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}^s[M_+]} \mathcal{B}_{\chi_q^s}) = \mathrm{End}_{\mathbb{C}_{\mathcal{B}}^s[G^*]}(\mathbb{C}_{\mathcal{B}}^s[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}^s[M_+]} \mathcal{B}_{\chi_q^s}).$$

Hence the algebra $W_{\mathcal{B}}^s(G)$ acts on $\mathbb{C}_{\mathcal{B}}^s[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}^s[M_+]} \mathcal{B}_{\chi_q^s}$ from the right by operators commuting with the natural left $\mathbb{C}_{\mathcal{B}}^s[G^*]$–action on $\mathbb{C}_{\mathcal{B}}^s[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}^s[M_+]} \mathcal{B}_{\chi_q^s}$. By the definition of $W_{\mathcal{B}}^s(G)$ this action preserves $Q_{\mathcal{B}} \subset Q_{\mathcal{B}}' = \mathbb{C}_{\mathcal{B}}^s[G^*] \otimes_{\mathbb{C}_{\mathcal{B}}^s[M_+]} \mathcal{B}_{\chi_q^s} = \mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}$ and by the arguments of this paragraph it commutes with the natural left $\mathbb{C}_{\mathcal{B}}^s[G_*]$–action on $Q_{\mathcal{B}}$.

Thus $Q_{\mathcal{B}}$ is a $\mathbb{C}_{\mathcal{B}}^s[G_*]$–$W_{\mathcal{B}}^s(G)$ bimodule equipped also with the adjoint action of $\mathbb{C}_{\mathcal{B}}^s[M_+]$. By (2.7.3) the adjoint action satisfies

$$\mathrm{Ad}_s x(yv) = \mathrm{Ad}_s x^1(y)\mathrm{Ad}_s x^2(v), \ \ x \in \mathbb{C}_{\mathcal{B}}^s[M_+], \ y \in \mathbb{C}_{\mathcal{B}}^s[G_*], v \in Q_{\mathcal{B}}, \tag{3.2.24}$$

and $\Delta_s(x) = x^1 \otimes x^2$.

Denote by $1 \in Q_{\mathcal{B}}$ the image of the element $1 \in \mathbb{C}_{\mathcal{B}}^s[G_*]$ in the quotient $Q_{\mathcal{B}}$ under the canonical projection $\mathbb{C}_{\mathcal{B}}^s[G_*] \to Q_{\mathcal{B}}$. Obviously 1 is the generating vector for $Q_{\mathcal{B}}$ as a module over $\mathbb{C}_{\mathcal{B}}^s[G_*]$. Using formula (3.2.24) and recalling that $Q_{\mathcal{B}}$ is a $\mathbb{C}_{\mathcal{B}}^s[G_*]$–$W_{\mathcal{B}}^s(G)$ bimodule, for $x \in \mathbb{C}_{\mathcal{B}}^s[M_+], y \in \mathbb{C}_{\mathcal{B}}^s[G_*]$, and for a representative $w \in \mathbb{C}_{\varepsilon}^s[G_*]$ of an element $w + I_{\mathcal{B}} \in W_{\mathcal{B}}^s(G)$ we have

$$\mathrm{Ad}_s x(wy1) = \mathrm{Ad}_s x(yw1) = \mathrm{Ad}_s x_1(y)\mathrm{Ad}_s x_2(w1) =$$

$$= \mathrm{Ad}_s x_1(y)\varepsilon_s(x_2)w1 = \mathrm{Ad}_s x(y)w1 = w\mathrm{Ad}_s x(y1).$$

Since $Q_{\mathcal{B}}$ is generated by the vector 1 over $\mathbb{C}_{\mathcal{B}}^s[G_*]$ the last relation implies that the action of $W_{\mathcal{B}}^s(G)$ on $Q_{\mathcal{B}}$ commutes with the adjoint action.

We can summarize the results of the discussion above in the following proposition.

Proposition 3.2.13. *The $\mathcal{B}$–module $Q_{\mathcal{B}}$ is naturally equipped with the structure of a left $\mathbb{C}_{\mathcal{B}}^s[G_*]$–module, a right $\mathbb{C}_{\mathcal{B}}^s[M_+]$–module via the adjoint action and a right $W_{\mathcal{B}}^s(G)$–module in such a way that the left $\mathbb{C}_{\mathcal{B}}^s[G_*]$–action and the right $\mathbb{C}_{\mathcal{B}}^s[M_+]$–action commute with the right $W_{\mathcal{B}}^s(G)$–action and compatibility condition (3.2.24) is satisfied.*

Finally we remark that by specializing q to a particular value $\varepsilon \in \mathbb{C}$, $\varepsilon \neq 0$, one can define a complex associative algebra $\mathbb{C}_{\varepsilon}[G_*] = \mathbb{C}_{\mathcal{B}}^s[G_*]/(q^{\frac{1}{dr2}} - \varepsilon^{\frac{1}{dr2}})\mathbb{C}_{\mathcal{B}}^s[G_*]$, its subalgebra $\mathbb{C}_{\varepsilon}[M_+]$ with a nontrivial character χ_{ε}^s and the corresponding W–algebra

$$W_{\varepsilon}^s(G) = \mathrm{Hom}_{\mathbb{C}_{\varepsilon}^s[M_+]}(\mathbb{C}_{\varepsilon_s}, Q_{\varepsilon}), \tag{3.2.25}$$

where $\mathbb{C}_{\varepsilon_s}$ is the trivial representation of the algebra $\mathbb{C}_{\varepsilon}^s[M_+]$ induced by the counit, $Q_{\varepsilon} = Q_{\mathcal{B}}/Q_{\mathcal{B}}(q^{\frac{1}{dr2}} - \varepsilon^{\frac{1}{dr2}})$.

Obviously, for generic ε we have $W_{\varepsilon}^s(G) = W_{\mathcal{B}}^s(G)/(q^{\frac{1}{dr2}} - \varepsilon^{\frac{1}{dr2}})W_{\mathcal{B}}^s(G)$.

3.3 Poisson reduction

In this section we recall basic facts on Poisson reduction. They will be used in the next section to describe Poisson q-W–algebras as reduced Poisson algebras.

Let M, B, B' be Poisson manifolds. Two Poisson submersions

$$
\begin{array}{ccc}
 & M & \\
\pi' \swarrow & & \searrow \pi \\
B' & & B
\end{array}
$$

onto B' and B form a dual pair if the pullback $\pi'^* C^\infty(B')$ is the centralizer of $\pi^* C^\infty(B)$ in the Poisson algebra $C^\infty(M)$. In this case the sets $B_b' = \pi'\left(\pi^{-1}(b)\right)$, $b \in B$ are Poisson submanifolds in B' called reduced Poisson manifolds (see [129], §8).

Fix an element $b \in B$. Then the algebra of functions $C^\infty(B_b')$ may be described as follows. Let I_b be the ideal in $C^\infty(M)$ generated by elements $\pi^*(f)$, $f \in C^\infty(B)$, $f(b) = 0$. Denote $M_b = \pi^{-1}(b)$. Then the algebra $C^\infty(M_b)$ is simply the quotient of $C^\infty(M)$ by I_b. Denote by $P_b : C^\infty(M) \to C^\infty(M)/I_b = C^\infty(M_b)$ the canonical projection onto the quotient.

Lemma 3.3.1. *Suppose that the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$. Then one can define an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$ by*

$$
f \cdot \varphi = P_b(\{\pi^*(f), \tilde\varphi\}), \tag{3.3.1}
$$

where $f \in C^\infty(B)$, $\varphi \in C^\infty(M_b)$ and $\tilde\varphi \in C^\infty(M)$ is a representative of φ in $C^\infty(M)$ such that $P_b(\tilde\varphi) = \varphi$. Moreover, $C^\infty(B_b')$ is the subspace of invariants in $C^\infty(M_b)$ with respect to this action.

Proof. Let $\varphi \in C^\infty(M_b)$. Choose a representative $\tilde\varphi \in C^\infty(M)$ such that $P_b(\tilde\varphi) = \varphi$. Since the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$, $b \in B$ is a Poisson submanifold of B with the zero Poisson structure, and hence the right hand side of (3.3.1) only depends on φ but not on the representative $\tilde\varphi$.

Indeed, if $f, g \in C^\infty(B)$, $g(b) = 0$ and $h \in C^\infty(M)$, so that $h\pi^*(g) \in I_b$, then we have

$$
\{\pi^*(f), h\pi^*(g)\} = h\{\pi^*(f), \pi^*(g)\} + \pi^*(g)\{\pi^*(f), h\} = h\pi^*\{f, g\} + \pi^*(g)\{\pi^*(f), h\}. \tag{3.3.2}
$$

Next, $h\pi^*\{f, g\} \in I_b$ since the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$, and hence $\{f, g\}(b) = 0$. Clearly, by the definition of I_b, $\pi^*(g)\{\pi^*(f), h\} \in I_b$. We deduce that the right hand side of (3.3.2) belongs to I_b.

Therefore for any $f \in C^\infty(B)$ one has $\{f, I_b\} \subset I_b$, and hence formula (3.3.1) defines an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$.

Note that at the same time this proves that Hamiltonian vector fields of functions $\pi^*(f)$, $f \in C^\infty(B)$ are tangent to the submanifold M_b.

Using the definition of the dual pair we obtain that $\varphi = \pi'^*(\psi)$ for some $\psi \in C^\infty(B_b')$ if and only if $P_b(\{\pi^*(f), \tilde\varphi\}) = 0$ for every $f \in C^\infty(B)$. This implies that $C^\infty(B_b')$ is exactly the subspace of invariants in $C^\infty(M_b)$ with respect to action (3.3.1). $\qquad\square$

The algebra $C^\infty(B_b')$ is called a *reduced Poisson algebra*. We also denote it by $C^\infty(M_b)^{C^\infty(B)}$.

Remark 3.3.2. *Note that the description of the algebra $C^\infty(M_b)^{C^\infty(B)}$ obtained in Lemma 3.3.1 is independent of both the manifold B' and the projection π'. Observe also that the Hamiltonian vector fields of functions $\pi^*(f), f \in C^\infty(B)$ are tangent to M_b, and hence the reduced space B_b' may be identified with a cross–section of the action of the Poisson algebra $C^\infty(B)$ on M_b by Hamiltonian vector fields in the case when this action is free. In particular, in this case B_b' may be regarded as a submanifold in M_b.*

In the case when the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$, the Poisson structure on the algebra $C^\infty(B_b')$ can be explicitly described as follows. Let $\varphi_1, \varphi_2 \in C^\infty(M_b)^{C^\infty(B)}$. Choose representatives $\tilde\varphi_1, \tilde\varphi_2 \in C^\infty(M)$ such that $P_b(\tilde\varphi_1) = \varphi_1$, $P_b(\tilde\varphi_2) = \varphi_2$. Then

$$
\{\varphi_1, \varphi_2\} = \{\tilde\varphi_1, \tilde\varphi_2\} \mod I_b. \tag{3.3.3}
$$

By Lemma 3.3.1 the class in $C^\infty(M)/I_b = C^\infty(M_b)$ of the function in right hand side of this formula is $C^\infty(B)$–invariant and independent of the choice of the representatives $\tilde\varphi_1, \tilde\varphi_2 \in C^\infty(M)$.

An important example of dual pairs is provided by Poisson–Lie group actions. We say that a Lie group A with Lie algebra $\mathfrak{a}$ locally acts on a manifold M if there is a Lie algebra homomorphism $X \mapsto \widehat{X}$ from $\mathfrak{a}$ to the Lie algebra of vector fields on M. In this case, by the existence and uniqueness theorem for solutions to ordinary differential equations, this homomorphism can be integrated to a local group action of A on M in the sense that for every point $m \in M$ there exists an open neighborhood U of m and an open neighborhood V of the identity element in A such that there is a smooth map $V \times U \to U$, $(a, u) \mapsto a \circ u$, and if $a_1, a_2 \in V$ and $a_1 a_2 \in V$ then $(a_1 a_2) \circ u = a_1 \circ (a_2 \circ m)$. Moreover, if we denote by Θ_m the differential of the map $V \to U$, $a \to a \circ m$ at the identity of A then $\Theta_m(X) = \widehat{X}(m)$ for any $X \in \mathfrak{a}$, $m \in M$.

Note that in this case the Lie algebra $\mathfrak{a}$ acts on $C^\infty(M)$ by

$$X \circ \varphi = \widehat{X}\varphi, \ X \in \mathfrak{a}, \ \varphi \in C^\infty(M).$$

We shall denote the space of invariants for this action by $C^\infty(M)^A$.

Right local Lie group actions are defined in a similar way, $X \mapsto \widehat{X}$ being a Lie algebra anti-homomorphism in this case.

If all vector fields $\widehat{X}$, $X \in \mathfrak{a}$ are complete then by the existence and uniqueness theorem for solutions to ordinary differential equations $U = M$ and $V = A$, i.e. there is a Lie group action of A on M.

Recall that a (local) left Poisson–Lie group action of a Poisson–Lie group A on a Poisson manifold M is a (local) left Lie group action $A \times M \to M$ which is also a (locally defined) Poisson map (as usual, we suppose that $A \times M$ is equipped with the product Poisson structure). If in this situation the space M/A is a smooth manifold, there exists a unique Poisson structure on M/A such that the canonical projection $M \to M/A$ is a Poisson map (see Proposition 3.3.5 below for a more general statement).

Right Poisson–Lie group actions are defined in a similar way.

Note that the property for a map to be Poisson is a local property, so it makes sense to consider locally defined Poisson maps.

Denote by $\langle \cdot, \cdot \rangle$ the canonical paring between $\mathfrak{a}^*$ and $\mathfrak{a}$. A map $\mu : M \to A^*$ is called a moment map for a (local) left Poisson group action $M \times A \to M$ if

$$L_{\widehat{X}}\varphi = \langle \mu^*(\theta_{A^*}), X \rangle(\xi_\varphi), \tag{3.3.4}$$

where θ_{A^*} is the universal left–invariant Maurer–Cartan form on A^*, $X \in \mathfrak{a}$, $\widehat{X}$ is the corresponding vector field on M and ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(M)$.

Proposition 3.3.3. ([79], Theorem 4.9) *Let $A \times M \to M$ be a left (local) Poisson group action of a Poisson–Lie group A on a Poisson manifold M with moment map $\mu : M \to A^*$. Denote by Π_{A^*} the Poisson tensor of A^*. Then there exists a right invariant bivector field Λ on A^* such that $\Pi_\mu = \Pi_{A^*} + \Lambda$ is a Poisson tensor on A^* and the map $\mu : M \to A^*_\mu$ is Poisson, where A^*_μ is the manifold A^* equipped with the Poisson structure associated to Π_μ.*

From the definition of the moment map it follows that if M/A is a smooth manifold then the canonical projection $M \to M/A$ and the moment map $\mu : M \to A^*$ form a dual pair.

The main example of Poisson–Lie group actions is the so–called dressing action. The dressing action may be described as follows (see [105], §3; [80], Theorem 2.4; Example 4.3 in [79]; and formula (2.24) in [106]).

Proposition 3.3.4. *Let G be a connected simply connected Poisson–Lie group with factorizable tangent Lie bialgebra, G^* the dual group. Then there exists a unique left local Poisson–Lie group action*

$$G \times G^* \to G^*, \quad ((L_+, L_-), g) \mapsto g \circ (L_+, L_-),$$

such that the identity mapping $\mu : G^* \to G^*$ is the moment map for this action.

Moreover, let $q : G^* \to G$ be the map defined by

$$q(L_+, L_-) = L_-^{-1} L_+.$$

Then

$$q(g \circ (L_+, L_-)) = g L_- L_+^{-1} g^{-1}. \tag{3.3.5}$$

The notion of Poisson–Lie groups may be generalized as follows.

Let $(\mathfrak{a}, \mathfrak{a}^*)$ be the tangent Lie bialgebra of a Poisson–Lie group A. A connected Lie subgroup $K \subset A$ with Lie algebra $\mathfrak{k} \subset \mathfrak{a}$ is called admissible if the annihilator $\mathfrak{k}^\perp$ of $\mathfrak{k}$ in $\mathfrak{a}^*$ is a Lie subalgebra $\mathfrak{k}^\perp \subset \mathfrak{a}^*$.

Proposition 3.3.5. ([105], Theorem 6; [79], §2) *Let $A \times M \to M$ be a Poisson–Lie group action of a Poisson–Lie group A on a Poisson manifold M. If $K \subset A$ is an admissible subgroup of A then the space $C^\infty(M)^K$ of K-invariants in $C^\infty(M)$ is a Poisson subalgebra in $C^\infty(M)$.*

If M/K is a smooth manifold, there exists a Poisson structure on M/K such that the canonical projection $M \to M/K$ is a Poisson map, and $C^\infty(M/K) \simeq C^\infty(M)^K$ as Poisson algebras.

We shall need the following particular example of dual pairs arising from Poisson group actions.

Let $A \times M \to M$ be a left (local) Poisson group action of a Poisson–Lie group A on a manifold M. Suppose that this action possesses a moment map $\mu : M \to A^*$. Let K be an admissible subgroup in A. Denote by $\mathfrak{k}$ the Lie algebra of K. Assume that $\mathfrak{k}^\perp \subset \mathfrak{a}^*$ is a Lie subalgebra in $\mathfrak{a}^*$. Suppose also that there is a splitting $\mathfrak{a}^* = \mathfrak{t} + \mathfrak{k}^\perp$ (direct sum of vector spaces), and that $\mathfrak{t}$ is a Lie subalgebra in $\mathfrak{a}^*$. Then the vector space $\mathfrak{k}^*$ is naturally identified with $\mathfrak{t}$.

Assume that $A^* = TK^\perp$ as a manifold, where $K^\perp, T$ are the Lie subgroups of A^* corresponding to the Lie subalgebras $\mathfrak{k}^\perp, \mathfrak{t} \subset \mathfrak{a}^*$, respectively. For any $a^* = tk^\perp \in A^*$ with $k^\perp \in K^\perp$, $t \in T$ denote $\pi_{K^\perp}(a^*) = k^\perp$, $\pi_T(a^*) = t$. This defines maps $\pi_{K^\perp} : A^* \to K^\perp$, $\pi_T : A^* \to T$.

Proposition 3.3.6. *Suppose that for any $k^\perp \in K^\perp$ the transformation*

$$\mathfrak{t} \to \mathfrak{t}, \tag{3.3.6}$$
$$t \mapsto (\mathrm{Ad}(k^\perp)t)_\mathfrak{t},$$

where the subscript $\mathfrak{t}$ stands for the $\mathfrak{t}$–component with respect to the decomposition $\mathfrak{a}^* = \mathfrak{t} + \mathfrak{k}^\perp$, is invertible.

Define a map $\overline{\mu} : M \to T$ by

$$\overline{\mu} = \pi_T \mu.$$

Then

(i) $\overline{\mu}^*(C^\infty(T))$ is a Poisson subalgebra in $C^\infty(M)$, and hence one can equip T with a Poisson structure such that $\overline{\mu} : M \to T$ is a Poisson map;

(ii) The algebra $C^\infty(M)^K$ is the centralizer of $\overline{\mu}^*(C^\infty(T))$ in the Poisson algebra $C^\infty(M)$. In particular, if M/K is a smooth manifold the maps

$$\begin{array}{ccc} & M & \\ {}^{\pi}\swarrow & & \searrow^{\overline{\mu}} \\ M/K & & T \end{array}, \tag{3.3.7}$$

form a dual pair.

Proof. (i) We claim that multiplication in A^* gives rise to a right Poisson–Lie group action $A^*_\mu \times A^* \to A^*_\mu$. Indeed, for $g \in A^*$ denote by l_g, r_g the left (right) translation by g on A^*. By the definition of Π_μ

$$\Pi_\mu(gh) = \Pi_{A^*}(gh) + \Lambda(gh) = l_{g*}\Pi_{A^*}(h) + r_{h*}\Pi_{A^*}(g) + r_{h*}\Lambda(g) = l_{g*}\Pi_{A^*}(h) + r_{h*}\Pi_\mu(g), \tag{3.3.8}$$

where we used the fact that $\Pi_{A^*}(gh) = l_{g*}\Pi_{A^*}(h) + r_{h*}\Pi_{A^*}(g)$ as A^* is a Poisson–Lie group and that Λ is right invariant. By the definition of Poisson–Lie group actions and Poisson maps, identity (3.3.8) is equivalent to the fact that multiplication in A^* gives rise to a right Poisson–Lie group action $A^*_\mu \times A^* \to A^*_\mu$.

Since $\mathfrak{k} \subset \mathfrak{a}$ is a Lie subalgebra and $\mathfrak{k}^{\perp\perp} = \mathfrak{k}$ the subgroup $K^\perp \subset A^*$ is admissible. Therefore restricting the action $A^*_\mu \times A^* \to A^*_\mu$ to $K^\perp$ we deduce that $C^\infty(A^*_\mu)^{K^\perp}$ is a Poisson subalgebra in $C^\infty(A^*_\mu)$, where the action of $K^\perp$ is induced by the action of $K^\perp \subset A^*$ on A^* by right translations.

Now recall that $A^* = TK^\perp$ as a manifold, and hence $C^\infty(A^*_\mu)^{K^\perp} = C^\infty(T)$. Thus T naturally becomes a Poisson manifold and the map $\pi_T : A^*_\mu \to T$ becomes Poisson. We deduce that the map $\overline{\mu} = \pi_T \mu$ is Poisson as the composition of the Poisson maps $\mu : M \to A^*_\mu$ and $\pi_T : A^*_\mu \to T$.

(ii) By the definition of the moment map we have

$$L_{\widehat{X}}\varphi = \langle \mu^*(\theta_{A^*}), X \rangle(\xi_\varphi), \tag{3.3.9}$$

where $X \in \mathfrak{a}$, $\widehat{X}$ is the corresponding vector field on M and ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(M)$. Since $A^* = TK^\perp$, the pullback of the left–invariant Maurer–Cartan form $\mu^*(\theta_{A^*})$ may be represented as follows

$$\mu^*(\theta_{A^*}) = Ad(\pi_{K^\perp}\mu)^{-1}(\overline{\mu}^*\theta_T) + (\pi_{K^\perp}\mu)^*\theta_{K^\perp},$$

where $(\pi_{K^\perp}\mu)^*\theta_{K^\perp} \in \mathfrak{k}^\perp$.

Now let $X \in \mathfrak{k}$. Then $\langle(\pi_{K^\perp}\mu)^*\theta_{K^\perp}), X\rangle = 0$ and formula (3.3.9) takes the form

$$L_{\widehat{X}}\varphi = \langle Ad(\pi_{K^\perp}\mu)^{-1}(\overline{\mu}^*\theta_T), X\rangle(\xi_\varphi) =$$

$$= \langle Ad(\pi_{K^\perp}\mu)^{-1}(\theta_T), X\rangle(\overline{\mu}_*(\xi_\varphi)). \tag{3.3.10}$$

Since by the assumption transformation (3.3.6) is invertible, $L_{\widehat{X}}\varphi = \langle Ad(\pi_{K^\perp}\mu)^{-1}(\theta_T), X\rangle(\overline{\mu}_*(\xi_\varphi)) = 0$ for any $X \in \mathfrak{k}$ if and only if $\overline{\mu}_*(\xi_\varphi) = 0$. Thus the function $\varphi \in C^\infty(M)$ is K–invariant if and only if $\{\varphi, \overline{\mu}^*(\psi)\} = 0$ for any $\psi \in C^\infty(T)$. This completes the proof.

$\square$

From the previous proposition, from Lemma 3.3.1 and Remark 3.3.2 we immediately obtain the following corollary.

Corollary 3.3.7. *Suppose that the conditions of Proposition 3.3.6 are satisfied. Let $t \in T$ be such that the map $f \mapsto f(t)$ is a character of the Poisson algebra $C^\infty(T)$. Then the action of K on M induces a (local) action on $\overline{\mu}^{-1}(t)$ and a (local) action on $C^\infty(\overline{\mu}^{-1}(t))$ given by*

$$X \circ \varphi = \langle Ad(\pi_{K^\perp}\mu)^{-1}(\theta_T), X\rangle(\overline{\mu}_*(\xi_{\tilde\varphi})), X \in \mathfrak{k}, \varphi \in C^\infty(\overline{\mu}^{-1}(t)),$$

where $\tilde\varphi$ is any representative of $\varphi \in C^\infty(\overline{\mu}^{-1}(t)) = C^\infty(M)/I_t$ in $C^\infty(M)$. The algebra $C^\infty(\overline{\mu}^{-1}(t))^K$ of invariants with respect to this action is isomorphic to the reduced Poisson algebra $C^\infty(\overline{\mu}^{-1}(t))^{C^\infty(T)}$.

3.4   Poisson reduction and q-W–algebras

In this section we realize the quasiclassical limit of the algebra $W^s_{\mathcal{B}}(G)$ as the algebra of functions on a reduced Poisson manifold. In this section we always assume that $\kappa = 1$ and use the notation and conventions introduced in Section 3.2. In particular, we always assume that the system of positive roots Δ_+ is associated to the Weyl group element s as in Definition 1.6.19.

By Proposition 3.2.8 one can define a character χ^s of the Poisson algebra $\mathbb{C}[M_+]$ such that $\chi^s(p(x)) = \chi^s_q(x)$ mod $(q^{\frac{1}{dr^2}} - 1)$ for every $x \in \mathbb{C}^s_{\mathcal{B}}[M_+]$.

Recall that the image of the algebra $\mathbb{C}^s_{\mathcal{B}}[G_*]$ under the projection $p : \mathbb{C}^s_{\mathcal{B}}[G^*] \to \mathbb{C}^s_{\mathcal{B}}[G^*]/(1 - q^{\frac{1}{dr^2}})\mathbb{C}^s_{\mathcal{B}}[G^*]$ is a certain subalgebra of $\mathbb{C}[G^*]$ that we denoted by $\mathbb{C}[G_*]$ in Section 3.2. By part (ii) of Proposition 3.2.6 $\mathbb{C}[G_*] \simeq \mathbb{C}[G]$ as algebras. Let $I = p(I_{\mathcal{B}})$ be the ideal in $\mathbb{C}[G^*]$ generated by the kernel of χ^s. Then by formula (3.2.23) the Poisson algebra $W^s(G) = W^s_q(G)/(q^{\frac{1}{dr^2}} - 1)W^s_q(G)$ is the subspace of all $x + I \in Q_1$, $Q_1 = Q_{\mathcal{B}}/(1 - q^{\frac{1}{dr^2}})Q_{\mathcal{B}} \subset \mathbb{C}[G^*]/I$, such that $\{m, x\} \in I$ for any $m \in \mathbb{C}[M_-]$, and the Poisson bracket in $W^s(G)$ takes the form $\{(x + I), (y + I)\} = \{x, y\} + I, x + I, y + I \in W^s(G)$. Using the Poisson analogues of formulas (3.2.22) and (3.2.23) we can also write $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_+]} \cap Q_1$, where the action of the Poisson algebra $\mathbb{C}[M_+]$ on the space $\mathbb{C}[G^*]/I$ is defined as follows

$$x \cdot (v + I) = \rho_{\chi^s}(\{x, v\}), \tag{3.4.1}$$

$v \in \mathbb{C}[G^*]$ is any representative of $v + I \in \mathbb{C}[G^*]/I$ and $x \in \mathbb{C}[M_+]$.

One can describe the space of invariants $(\mathbb{C}[G^*]/I)^{\mathbb{C}[M_+]}$ with respect to this action by analyzing the related underlying manifolds and varieties. First observe that the algebra $(\mathbb{C}[G^*]/I)^{\mathbb{C}[M_+]}$ is a particular example of the reduced Poisson algebra introduced in Lemma 3.3.1 .

Indeed, recall that according to (3.2.1) any element $(L_+, L_-) \in G^*$ may be uniquely written as

$$(L_+, L_-) = (n_+, n_-)(h_+, h_-), \tag{3.4.2}$$

where $n_\pm \in N_\pm$, $h_+ = \exp((\frac{1}{1-s}P_{\mathfrak{h}'} + \frac{1}{2}P_{\mathfrak{h}'^\perp})x)$, $h_- = \exp((\frac{s}{1-s}P_{\mathfrak{h}'} - \frac{1}{2}P_{\mathfrak{h}'^\perp})x)$, $x \in \mathfrak{h}$.

Formula (3.2.1) and a decomposition of elements of N_+ into products of elements which belong to one–dimensional subgroups corresponding to roots also imply that any element L_+ can be represented in the form

$$L_+ = \prod_\beta \exp[b_\beta X_\beta] \times$$
$$\exp\left[\sum_{i=1}^l b_i(\frac{1}{1-s}P_{\mathfrak{h}'} + \frac{1}{2}P_{\mathfrak{h}'^\perp})H_i\right], \; b_i, b_\beta \in \mathbb{C}, \tag{3.4.3}$$

where the product over roots is taken according to the normal ordering associated to s.

Now define a surjective submersion $\mu_{M_-} : G^* \to M_+$ by

$$\mu_{M_-}(L_+, L_-) = m_+, \tag{3.4.4}$$

where for L_+ given by (3.4.3) m_+ is defined as follows

$$m_+ = \prod_{\beta \in \Delta_{\mathfrak{m}_+}} \exp[b_\beta X_\beta],$$

and the product over roots is taken according to the normal order in the segment $\Delta_{\mathfrak{m}_+}$.

Note that by definition $\mathbb{C}[M_+] = \{\varphi \in \mathbb{C}[G^*] : \varphi = \varphi(m_+)\}$. Therefore $\mathbb{C}[M_+]$ is generated by the pullbacks of regular functions on M_+ with respect to the map μ_{M_-}. Since $\mathbb{C}[M_+]$ is a Poisson subalgebra in $\mathbb{C}[G^*]$, and regular functions on M_+ are dense in $C^\infty(M_+)$ on every compact subset, we can equip the manifold M_+ with the Poisson structure in such a way that μ_{M_-} becomes a Poisson mapping.

Let $u \in M_+$ be the element defined by

$$u = \prod_{i=1}^{l'} \exp[t_i X_{\gamma_i}], t_i = k_i \ (\mathrm{mod}\ (q^{\frac{1}{dr^2}} - 1)), \tag{3.4.5}$$

where the product over roots is taken according to the normal order in the segment $\Delta_{\mathfrak{m}_+}$ and k_i, $i = 1, \ldots, l'$ are defined in formula (3.2.16).

By Proposition 3.2.5 (see formulas (3.2.9) and (3.2.10)) the elements $\tilde{L}^{\pm,V} = (p \otimes \nu)(^q L^{\pm,V})$ belong to the space $\mathbb{C}[G^*] \otimes \mathrm{End}\overline{V}$, and the map

$$\imath : \mathbb{C}_{\mathcal{B}}^s[G^*]/(q^{\frac{1}{dr^2}} - 1)\mathbb{C}_{\mathcal{B}}^s[G^*] \to \mathbb{C}[G^*], \ (\imath \otimes id)\tilde{L}^{\pm,V} = L^{\pm,\overline{V}}$$

is an isomorphism of Poisson–Hopf algebras. In particular, from (2.6.16) it follows that

$$\tilde{L}^{+,\overline{V}} = \prod_\beta \exp[p((1 - q_\beta^{-2})f_\beta) \otimes \pi_{\overline{V}}(X_\beta)] \times \\ (p \otimes id)\exp\left[\sum_{i=1}^{l} hH_i \otimes \pi_{\overline{V}}((\frac{2}{1-s}P_{\mathfrak{h}'} + P_{\mathfrak{h}'^\perp})Y_i)\right]. \tag{3.4.6}$$

From (3.4.6) and the definition of χ^s we obtain that $\chi^s(\varphi) = \varphi(u)$ for every $\varphi \in \mathbb{C}[M_+]$. χ^s naturally extends to a character of the Poisson algebra $C^\infty(M_+)$.

Now applying Lemma 3.3.1 we can define a reduced Poisson algebra $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$ as follows. Denote by I_u the ideal in $C^\infty(G^*)$ generated by elements $\mu_{M_-}^*\psi$, $\psi \in C^\infty(M_+)$, $\psi(u) = 0$. Let $P_u : C^\infty(G^*) \to C^\infty(G^*)/I_u = C^\infty(\mu_{M_-}^{-1}(u))$ be the canonical projection. Define the action of $C^\infty(M_+)$ on $C^\infty(\mu_{M_-}^{-1}(u))$ by

$$\psi \cdot \varphi = P_u(\{\mu_{M_-}^*\psi, \tilde\varphi\}), \tag{3.4.7}$$

where $\psi \in C^\infty(M_-)$, $\varphi \in C^\infty(\mu_{M_-}^{-1}(u))$ and $\tilde\varphi \in C^\infty(G^*)$ is a representative of φ such that $P_u\tilde\varphi = \varphi$. The reduced Poisson algebra $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$ is the algebra of $C^\infty(M_+)$–invariants in $C^\infty(\mu_{M_-}^{-1}(u))$ with respect to action (3.4.7). The reduced Poisson algebra is naturally equipped with a Poisson structure induced from $C^\infty(G^*)$ as described in (3.3.3).

Lemma 3.4.1. *Let $\overline{q(\mu_{M_-}^{-1}(u))}$ be the closure of $q(\mu_{M_-}^{-1}(u))$ in G with respect to Zariski topology. Then $Q_1 \simeq \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$, and the algebra $W^s(G)$ is isomorphic to the algebra of regular functions on $\overline{q(\mu_{M_-}^{-1}(u))}$ pullbacks of which under the map q are invariant with respect to the action (3.4.7) of $C^\infty(M_+)$ on $C^\infty(\mu_{M_-}^{-1}(u))$, i.e.*

$$W^s(G) = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)},$$

where $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$ is regarded as a subalgebra in $C^\infty(\mu_{M_-}^{-1}(u))$ using the map $q^* : C^\infty(q(\mu_{M_-}^{-1}(u))) \to C^\infty(\mu_{M_-}^{-1}(u))$ and the imbedding $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \subset C^\infty(q(\mu_{M_-}^{-1}(u)))$.

Proof. First observe that by the definition $\mu_{M_-}^{-1}(u)$ is a submanifold in G^* and that $I = \mathbb{C}[G^*] \cap I_u$. Therefore by the definition of the algebra $\mathbb{C}[G^*]$ and of the map μ_{M_-} the quotient $\mathbb{C}[G^*]/I$ is identified with the algebra of functions on $\mu_{M_-}^{-1}(u)$ generated by the restrictions of elements of $\mathbb{C}[G^*]$ to $\mu_{M_-}^{-1}(u)$.

Also by the definition $Q_1 \subset \mathbb{C}[G^*]/I$ is the algebra generated by the restrictions to $\mu_{M_-}^{-1}(u)$ of the pullbacks of elements from the algebra of regular functions $\mathbb{C}[G]$ under the map $q : G^* \to G$. Therefore $Q_1 = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$.

From these observations we deduce that $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]} \cap Q_1 = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]} \cap \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$.

Since $\mathbb{C}[M_-]$ is dense in $C^\infty(M_-)$ on every compact subset in M_- we have

$$C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)} \simeq C^\infty(\mu_{M_+}^{-1}(u))^{\mathbb{C}[M_-]}.$$

Now observe that action (3.4.7) of elements from $\mathbb{C}[M_-]$ coincides with action (3.4.1) when restricted to elements from $\mathbb{C}[G^*]/I$, and hence $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]} \cap \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] = C^\infty(\mu_{M_+}^{-1}(u))^{\mathbb{C}[M_-]} \cap \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] = C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_-)} \cap \mathbb{C}[\overline{q(\mu_{M_+}^{-1}(u))}]$. This completes the proof.

$\square$

We shall realize the algebra $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$ as the algebra of functions on a reduced Poisson manifold. In this construction we use the dressing action of the Poisson–Lie group G on G^*.

Consider the restriction of the (local) dressing action $G \times G^* \to G^*$ to the subgroup $M_- \subset G$. We shall describe the reduced Poisson algebra $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_-)}$ in terms of the dressing action.

Lemma 3.4.2. *The preimage $\mu_{M_-}^{-1}(u) \subset G^*$ is locally invariant under the (locally defined) dressing action of M_-, and the algebra $C^\infty(\mu_{M_-}^{-1}(u))^{M_-}$ is isomorphic to $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$.*

Proof. The proof will be based on Corollary 3.3.7.

First observe that according to part (iv) of Proposition 3.1.2 $(\mathfrak{i}_-, \mathfrak{i}_+) = (\mathfrak{b}_-, \mathfrak{b}_+)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$. Therefore B_- is a Poisson–Lie subgroup in G.

By Proposition 3.3.4, by part (iv) of Proposition 3.1.2, and by the definition of the moment map we have for any $X \in \mathfrak{b}_-$, $\varphi \in C^\infty(G^*)$

$$L_{\widehat{X}}\varphi(L_+, L_-) = (\theta_{G^*}(L_+, L_-), X)(\xi_\varphi) = (r_+^{-1}\mu_{B_-}^*(\theta_{B_+}), X)(\xi_\varphi), \tag{3.4.8}$$

where $\widehat{X}$ is the corresponding vector field on G^*, ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(G^*)$, and the map $\mu_{B_-} : G^* \to B_+$ is defined by $\mu_{B_-}(L_+, L_-) = L_+$. Now from Proposition 3.1.2 (iv) and the definition of the moment map it follows that μ_{B_-} is a moment map for the restriction of the dressing action to the subgroup B_-.

Next observe that the complementary subset to $\Delta_{\mathfrak{m}_+}$ in Δ_+ is a minimal segment $\Delta_{\mathfrak{m}_+}^0$. Now using Proposition 3.1.2 (iv) the subspace $\mathfrak{m}_-^\perp$ in $\mathfrak{b}_+$ can be identified with the linear subspace in $\mathfrak{b}_+$ spanned by the Cartan subalgebra $\mathfrak{h}$ and by the root subspaces corresponding to the roots from the minimal segment $\Delta_{\mathfrak{m}_+}^0$. Using the fact that the adjoint action of $\mathfrak{h}$ normalizes root subspaces and Lemma 1.6.17 we deduce that $\mathfrak{m}_-^\perp \subset \mathfrak{b}_+$ is a Lie subalgebra, and hence $M_- \subset B_-$ is an admissible subgroup.

Moreover, the dual group B_+ can be uniquely factorized as $B_+ = M_+ M_-^\perp$, where $M_-^\perp \subset B_+$ is the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{m}_-^\perp \subset \mathfrak{b}_+$, and $M_+ \subset B_+$ is the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{m}_+$.

Now observe that $\mathfrak{m}_-^\perp = \mathfrak{h} + \mathfrak{m}_{-0}^\perp \subset \mathfrak{b}_+$ (direct sum of vector spaces), where $\mathfrak{m}_{-0}^\perp$ is the Lie subalgebra generated by the root vectors corresponding to the roots from the minimal segment $\Delta_{\mathfrak{m}_+}^0$. The Lie subalgebra $\mathfrak{m}_+$ is generated by the root subspaces corresponding to the roots from the minimal segment $\Delta_{\mathfrak{m}_+}$. Since all root subspaces are invariant under the adjoint action of $\mathfrak{h}$ and the restriction of the adjoint action of the root vectors corresponding to the roots from the minimal segment $\Delta_{\mathfrak{m}_+}^0$ is nilpotent we deduce that for any $m_+ \in \mathfrak{m}_+$, $k^\perp \in M_-^\perp$, $k^\perp = hk_0^\perp$, $h \in H$, $k_0^\perp = \exp(x)$, $x \in \mathfrak{m}_{-0}^\perp$ one has

$$(\mathrm{Ad}(hk_0^\perp)(m_+))_{\mathfrak{m}_+} = \mathrm{Ad}h((\mathrm{Ad}k_0^\perp(m_+))_{\mathfrak{m}_+}) = \mathrm{Ad}h((\exp(\mathrm{ad}x)(m_+))_{\mathfrak{m}_+}) = \mathrm{Ad}h((\mathrm{Id} + V)(m_+)),$$

where the subscript $\mathfrak{m}_+$ stands for the $\mathfrak{m}_+$–component in the direct vector space decomposition $\mathfrak{b}_+ = \mathfrak{m}_+ + \mathfrak{m}_-^\perp$, and V is a linear nilpotent transformation of $\mathfrak{m}_+$.

The maps $\mathrm{Ad}h$ and $\mathrm{Id} + V$ are obviously invertible. Hence for any $k^\perp \in M_-^\perp$ the map

$$\mathfrak{m}_+ \to \mathfrak{m}_+, m_+ \mapsto (\mathrm{Ad}(hk_0^\perp)(m_+))_{\mathfrak{m}_+}$$

is invertible as well.

We conclude that all the conditions of Corollary 3.3.7 are satisfied with $A = B_-, K = M_-, A^* = B_+, T = M_+, K^\perp = M_-^\perp, \mu = \mu_{B_-}$. It follows that the preimage $\mu_{M_-}^{-1}(u) \subset G^*$ is locally stable under the (locally defined) dressing action of M_-, and the algebra $C^\infty(\mu_{M_-}^{-1}(u))^{M_-}$ is isomorphic to $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$. This completes the proof.

$\square$

Observe that by (3.3.5) under the map $q : G^* \to G$, $q(L_+, L_-) = L_-^{-1}L_+$ the dressing action becomes the action of G on itself by conjugations. Consider the restriction of this action to the subgroup M_+. Denote by $\pi_q : G \to G/M_-$ the canonical projection onto the quotient with respect to this action. We shall see that $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ is an algebraic variety and $\mathbb{C}[\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})] \simeq W^s(G)$. We shall also obtain an explicit description of the variety $\overline{q(\mu_{M_-}^{-1}(u))}$ and of the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$.

First we describe the image of the level surface $\mu_{M_-}^{-1}(u)$ of the map μ_{M_-} under the map q. Let $X_\alpha(t) = \exp(tX_\alpha) \in G$, $t \in \mathbb{C}$ be the one–parameter subgroup in G corresponding to root $\alpha \in \Delta$. Recall that for any $\alpha \in \Delta_+$ and any $t \neq 0$ the element

$$s_\alpha(t) = X_\alpha(t)X_{-\alpha}(-t^{-1})X_\alpha(t) \in G \qquad (3.4.9)$$

is a representative for the reflection s_α corresponding to the root α. Denote by $s \in G$ the following representative of the Weyl group element $s \in W$,

$$s = s_{\gamma_1}(t_1) \ldots s_{\gamma_{l'}}(t_{l'}), \qquad (3.4.10)$$

where the numbers t_i are defined in (3.4.5), and we assume that $t_i \neq 0$ for any i.

We shall also use the following representatives for s^1 and s^2

$$s^1 = s_{\gamma_1}(t_1) \ldots s_{\gamma_n}(t_n), \;\; s^2 = s_{\gamma_{n+1}}(t_{n+1}) \ldots s_{\gamma_{l'}}(t_{l'}).$$

The following Proposition is an improved version of Proposition 7.2 in [117] suitable for the purposes of quantization.

Proposition 3.4.3. *Let* $q : G^* \to G$ *be the map defined by*

$$q(L_+, L_-) = L_-^{-1}L_+.$$

Suppose that the numbers t_i *defined in (3.4.5) are not equal to zero for all* i. *Then*

$$q(\mu_{M_-}^{-1}(u)) \subset N_- s H^0 Z_+ M_- = N_- s H^0 M_- Z_+ = (N_- \cap N)Z_- s H^0 M_- Z_+ = \qquad (3.4.11)$$

$$= (N_- \cap N)Z_- s H^0 Z_+ M_- \subset NsZN,$$

where H^0 *is the connected subgroup of* H *corresponding to the Lie subalgebra* $\mathfrak{h}^s = \mathfrak{h}'^\perp \subset \mathfrak{h}$, $Z_\pm = Z \cap N_\pm$.

The closure $\overline{q(\mu_{M_-}^{-1}(u))}$ *of* $q(\mu_{M_-}^{-1}(u))$ *in* G *with respect to the Zariski topology is also contained in* $NsZN$.

Proof. Using definition (3.4.4) of the map μ_{M_-} we can describe the preimage $\mu_{M_-}^{-1}(u)$ as follows

$$\mu_{M_-}^{-1}(u) = \{(uyh_+, n_-h_-)|n_- \in N_-, h_\pm = e^{r_\pm^s x}, x \in \mathfrak{h}, y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}\}, \qquad (3.4.12)$$

where, as in Section 1.3, for any additively closed subset of roots $\Xi \subset \Delta$ which does not contain opposite roots we denote by N_Ξ the subgroup in G generated by the one–parameter subgroups corresponding to the roots from Ξ. Therefore

$$q(\mu_{M_-}^{-1}(u)) = \{h_-^{-1}n_-^{-1}uyh_+|n_- \in N_-, h_\pm = e^{r_\pm^s x}, x \in \mathfrak{h}, y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}\}. \qquad (3.4.13)$$

First we show that for any $y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}$ and $n_- \in N_-$ the element $n_-^{-1}uy$ belongs to $N_- s H^0 Z_+ M_-$. Fix the circular normal ordering on Δ corresponding to the normal ordering of Δ_+ associated to s as in Definition 1.6.19.

In the proof we shall frequently use the following lemma.

Lemma 3.4.4. *Let* $[\alpha, \beta] \subset \Delta$ *be a minimal segment and assume that* $[\alpha, \beta] = [\alpha, \gamma] \cup [\delta, \beta]$, *where the segments* $[\alpha, \gamma]$ *and* $[\delta, \beta]$ *are disjoint and minimal as well. Then any element* $m \in N_{[\alpha, \beta]}$ *can be uniquely factorized as* $m = g_1 g_2 = g_2' g_1'$, $g_1, g_1' \in N_{[\alpha, \gamma]}$, $g_2, g_2' \in N_{[\delta, \beta]}$. *Moreover, if* $\delta = \beta$ *then for any* $m' \in N_{[\alpha, \gamma]}$ *and any* $t \in \mathbb{C}$ *one has* $m'X_\beta(t) = X_\beta(t)m''$, *where* $m'' \in N_{[\alpha, \gamma]}$.

Proof. The proof is obtained by straightforward application of Lemmas 1.3.1 and 1.6.18.

<div align="right">□</div>

Since the roots $\gamma_1, \ldots, \gamma_n$ are mutually orthogonal the adjoint action of $s_{\gamma_i}(t_i)$, $i = 1, \ldots, n$ on each of the root subspaces $\mathfrak{g}_{\gamma_j}$, $j = 1, \ldots, n, j \neq i$ is given by multiplication by a non–zero constant. Therefore there are non–zero constants $c_1, \ldots, c_n$ such that $X_{\gamma_k}(c_k)s_{\gamma_1} \ldots s_{\gamma_{k-1}} = s_{\gamma_1} \ldots s_{\gamma_{k-1}}X_{\gamma_k}(-t_k^{-1})$, $k = 2, \ldots, n$, and we define $c_1 = -t_1^{-1}$.

Obviously we have

$$X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n) = X_{-\gamma_1}(-c_1) \ldots X_{-\gamma_n}(-c_n)X_{-\gamma_n}(c_n) \ldots X_{-\gamma_1}(c_1)X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n) =$$

$$= n_1 X_{-\gamma_n}(c_n) \ldots X_{-\gamma_1}(c_1)X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n), \quad n_1 = X_{-\gamma_1}(-c_1) \ldots X_{-\gamma_n}(-c_n) \in N_{\Delta_-^n},$$

where $\Delta_-^n = \{\alpha \in \Delta_- : -\gamma_1 \leq \alpha \leq -\gamma_n\}$.

Using the relation $X_{-\gamma_1}(-t_1^{-1})X_{\gamma_1}(t_1) = X_{\gamma_1}(-t_1)s_{\gamma_1}$ one can rewrite the last identity as follows

$$X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n) = n_1 X_{-\gamma_n}(c_n) \ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)s_{\gamma_1}X_{\gamma_2}(t_2) \ldots X_{\gamma_n}(t_n). \tag{3.4.14}$$

Now we can write

$$X_{-\gamma_n}(c_n) \ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1) =$$

$$= X_{-\gamma_n}(c_n) \ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)X_{-\gamma_2}(-c_2) \ldots X_{-\gamma_n}(-c_n)X_{-\gamma_n}(c_n) \ldots X_{-\gamma_2}(c_2).$$

The product $X_{-\gamma_n}(c_n) \ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)X_{-\gamma_2}(-c_2) \ldots X_{-\gamma_n}(-c_n)$ belongs to the subgroup of G generated by the one–parameter subgroups corresponding to roots from the set $\Delta^1 = \{\alpha \in \Delta : -\gamma_2 \leq \alpha \leq \gamma_1, s^1\alpha = -\alpha\}$. By Lemma 1.6.17 the minimal segment $\{\alpha \in \Delta : -\gamma_2 \leq \alpha \leq \gamma_1\}$ is additively closed and the set of roots on which s^1 acts by multiplication by -1 is also additively closed. Hence Δ^1 is additively closed, and $X_{-\gamma_n}(c_n) \ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)X_{-\gamma_2}(-c_2) \ldots X_{-\gamma_n}(-c_n) \in N_{\Delta^1}$.

Let $\Delta_+^1 := \Delta^1 \cap \Delta_+ = \{\alpha \in \Delta_+ : \alpha \leq \gamma_1, s^1\alpha = -\alpha\}$, and $\Delta_-^1 := \Delta^1 \cap \Delta_- = \{\alpha \in \Delta_- : -\gamma_2 \leq \alpha, s^1\alpha = -\alpha\}$. Then $\Delta^1 = \Delta_+^1 \cup \Delta_-^1$ (disjoint union), and, using Lemma 1.3.1, the element

$$X_{-\gamma_n}(c_n) \ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)X_{-\gamma_2}(-c_2) \ldots X_{-\gamma_n}(-c_n) \in N_{\Delta^1} = N_{\Delta_-^1}N_{\Delta_+^1}$$

can be uniquely factorized as follows

$$X_{-\gamma_n}(c_n) \ldots X_{-\gamma_2}(c_2)X_{\gamma_1}(-t_1)X_{-\gamma_2}(-c_2) \ldots X_{-\gamma_n}(-c_n) = n_2'x_1',$$

where $n_2' \in N_{\Delta_-^1}, x_1' \in N_{\Delta_+^1}$.

Substituting the last relation into (3.4.14) and using the definition of c_2 and the orthogonality of roots γ_1 and γ_2 we obtain

$$X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n) = n_2 x_1' X_{-\gamma_n}(c_n) \ldots X_{-\gamma_3}(c_3)s_{\gamma_1}X_{-\gamma_2}(-t_2^{-1})X_{\gamma_2}(t_2) \ldots X_{\gamma_n}(t_n),$$

where $n_2 = n_1 n_2' \in N_{\Delta_-^{s^1}}$, $\Delta_-^{s^1} := \{\alpha \in \Delta_- : s^1\alpha = -\alpha\}$.

Now we can use the relation $X_{-\gamma_2}(-t_2^{-1})X_{\gamma_2}(t_2) = X_{\gamma_2}(-t_2)s_{\gamma_2}$, the orthogonality of roots γ_1 and γ_2, and apply similar arguments to get

$$X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n) = n_2 x_1' X_{-\gamma_n}(c_n) \ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2)s_{\gamma_1}s_{\gamma_2}X_{\gamma_3}(t_3) \ldots X_{\gamma_n}(t_n), a_2 \neq 0. \tag{3.4.15}$$

We can also write

$$X_{-\gamma_n}(c_n) \ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2) =$$

$$= X_{-\gamma_n}(c_n) \ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2)X_{-\gamma_3}(-c_3) \ldots X_{-\gamma_n}(-c_n)X_{-\gamma_n}(c_n) \ldots X_{-\gamma_3}(c_3).$$

The product $X_{-\gamma_n}(c_n) \ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2)X_{-\gamma_3}(-c_3) \ldots X_{-\gamma_n}(-c_n)$ belongs to the subgroup of G generated by the one–parameter subgroups corresponding to the roots from the set $\Delta^2 = \{\alpha \in \Delta : -\gamma_3 \leq \alpha \leq \gamma_2, s^1\alpha = -\alpha\}$. By Lemma 1.6.17 the minimal segment $\{\alpha \in \Delta : -\gamma_3 \leq \alpha \leq \gamma_2\}$ is additively closed and the set of roots on which s^1 acts by multiplication by -1 is also additively closed. Hence Δ^2 is additively closed, and $X_{-\gamma_n}(c_n) \ldots X_{-\gamma_3}(c_3)X_{\gamma_2}(a_2)X_{-\gamma_3}(-c_3) \ldots X_{-\gamma_n}(-c_n) \in N_{\Delta^2}$.

Let $\Delta_+^2 := \Delta^2 \cap \Delta_+ = \{\alpha \in \Delta_+ : \alpha \leq \gamma_2, s^1\alpha = -\alpha\}$ and $\Delta_-^2 := \Delta^2 \cap \Delta_- = \{\alpha \in \Delta_- : s^1\alpha = -\alpha, -\gamma_3 \leq \alpha\}$.

Then $\Delta^2 = \Delta^2_+ \cup \Delta^2_-$ (disjoint union), and, using Lemma 1.3.1, the element

$$X_{-\gamma_n}(c_n) \ldots X_{-\gamma_3}(c_3) X_{\gamma_2}(a_2) X_{-\gamma_3}(-c_3) \ldots X_{-\gamma_n}(-c_n) \in N_{\Delta^2} = N_{\Delta^2_-} N_{\Delta^2_+}$$

can be represented as follows

$$X_{-\gamma_n}(c_n) \ldots X_{-\gamma_3}(c_3) X_{\gamma_2}(a_2) X_{-\gamma_3}(-c_3) \ldots X_{-\gamma_n}(-c_n) = n'_3 x''_2, \tag{3.4.16}$$

where $n'_3 \in N_{\Delta^2_-}, x''_2 \in N_{\Delta^2_+}$.

Substituting the last relation into (3.4.15) and using the definition of c_3 and the orthogonality of roots γ_1, γ_2 and γ_3 we obtain

$$X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n) = n_2 x'_1 n'_3 x''_2 X_{-\gamma_n}(c_n) \ldots X_{-\gamma_4}(c_4) s_{\gamma_1} s_{\gamma_2} X_{\gamma_3}(-t_3^{-1}) X_{\gamma_3}(t_3) \ldots X_{\gamma_n}(t_n). \tag{3.4.17}$$

Since $\Delta^1_+ \subset \Delta^2_+$, $N_{\Delta^1_+} \subset N_{\Delta^2_+}$, and we deduce that $x'_1 n'_3 x''_2 \in N_{\Delta^2}$. Therefore using Lemma 1.3.1 and the decomposition $N_{\Delta^2} = N_{\Delta^2_-} N_{\Delta^2_+}$ we get $x'_1 n'_3 x''_2 = n''_3 x'_2$, $x'_2 \in N_{\Delta^2_+}$, $n''_3 \in N_{\Delta^2_-}$. Now (3.4.17) takes the form

$$X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n) = n_3 x'_2 X_{-\gamma_n}(c_n) \ldots X_{-\gamma_4}(c_4) s_{\gamma_1} s_{\gamma_2} X_{-\gamma_3}(-t_3^{-1}) X_{\gamma_3}(t_3) \ldots X_{\gamma_n}(t_n),$$

where $n_3 = n_2 n''_3 \in N_{\Delta^{s_1}_-}$.

We can proceed in a similar way to obtain the following representation

$$X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n) = n \widetilde{x} s_{\gamma_1} \ldots s_{\gamma_n} = n \widetilde{x} s^1, \quad n \in N_{\Delta^{s_1}_-}, \widetilde{x} \in N_{\Delta^n_+}, \tag{3.4.18}$$

where $\Delta^n_+ := \{\alpha \in \Delta_+ : \alpha \le \gamma_n, s^1 \alpha = -\alpha\} = \{\alpha \in \Delta_+ : \gamma_1 \le \alpha \le \gamma_n\}$.

Note that s^1 acts by multiplication by -1 on the roots from Δ^n_+, so $\Delta^n_- := -\Delta^n_+ = s^1(\Delta^n_+)$. Therefore $N_{\Delta^n_-} = (s^1)^{-1} N_{\Delta^n_+} s^1 \subset N_{\Delta^{s_1}_-}$ and (3.4.18) can be rewritten in the following form

$$X_{\gamma_1}(t_1) \ldots X_{\gamma_n}(t_n) = n s^1 (s^1)^{-1} \widetilde{x} s^1 = n s^1 n', \quad n \in N_{\Delta^{s_1}_-}, n' = (s^1)^{-1} \widetilde{x} s^1 \in N_{\Delta^n_-}. \tag{3.4.19}$$

Similarly one has

$$X_{-\gamma_{n+1}}(t_{n+1}) \ldots X_{-\gamma_{l'}}(t_{l'}) = n'' s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}} n''' = n'' s^2 n''', \quad n'' \in N_{\Delta^{s_2}_-}, n''' \in N_{\Delta^{l'}_-}, \tag{3.4.20}$$

where $\Delta^{s_2}_- := \{\alpha \in \Delta_- : s^2 \alpha = -\alpha\}$, and $\Delta^{l'}_- := \{\alpha \in \Delta_- : -\gamma_{n+1} \le \alpha \le -\gamma_{l'}\}$.

Combining (3.4.19) and (3.4.20), using the definition of the circular normal ordering of the root system Δ associated to s, Lemmas 1.3.1, 1.6.18 and 3.4.4 one can obtain

$$n_-^{-1} u y = n_-^{-1} n s^1 n' n'' s^2 n''' y = k s^1 g s^2 n''' y, g \in N_{\Delta^s_- \setminus \Delta_0}, k \in N_-. \tag{3.4.21}$$

By Proposition 1.6.6 (iii)
$$s^2 \Delta^s_{s_1} \subset \Delta^s_+ \setminus (\Delta^s_{s_1} \cup \Delta^s_{s_2} \cup \Delta_0) \subset \Delta_{\mathfrak{m}_+}. \tag{3.4.22}$$

Now the minimal segment $\Delta^s_+ \setminus \Delta_0$ can be represented as the following disjoint union

$$\Delta^s_+ \setminus \Delta_0 = \left(\Delta^s_+ \setminus (\Delta^s_{s_1} \cup \Delta_0)\right) \cup \Delta^s_{s_1}. \tag{3.4.23}$$

Note that by the definition of $\Delta^s_{s_1}$ we have

$$s^1(\Delta^s_+ \setminus (\Delta^s_{s_1} \cup \Delta_0)) \subset \Delta^s_+ \setminus (\Delta^s_{s_1} \cup \Delta_0) \subset \Delta_+ \setminus \Delta_0. \tag{3.4.24}$$

Observe that the segments $\Delta^s_{s_1}$, $\Delta^s_+ \setminus (\Delta^s_{s_1} \cup \Delta_0)$ are minimal with respect to the circular normal ordering on Δ associated to s. Thus from (3.4.23) and Lemma 3.4.4 we deduce that the element $g \in N_{\Delta^s_- \setminus \Delta_0}$ from formula (3.4.21) can be uniquely decomposed as the product $g = g' g''$, where $g' \in N_{-(\Delta^s_+ \setminus (\Delta^s_{s_1} \cup \Delta_0))}$, and $g'' \in N_{-\Delta^s_{s_1}}$. By (3.4.22), (3.4.24) we have $(s^2)^{-1} g'' s^2 \in M_-$ and $s^1 g'(s^1)^{-1} \in N_-$, and (3.4.21) takes the form

$$n_-^{-1} u y = k s^1 g' g'' s^2 n''' y = k s^1 g'(s^1)^{-1} s(s^2)^{-1} g'' s^2 n''' y = k' s \widehat{n} y, \widehat{n} = \tag{3.4.25}$$

$$= (s^2)^{-1} g'' s^2 n''' \in M_-, k' = k s^1 g'(s^1)^{-1} \in N_-.$$

The element $\widehat{n}y$ belongs to the subgroup $N_{\Delta'}$, $\Delta' := \{\alpha \in \Delta : \gamma_{l'} < \alpha \le -\gamma_{l'}\}$. We have the following disjoint union of minimal segments (see Figure 5)

$$\Delta' = (\Delta' \cap \Delta_{s^2}^s) \cup (\Delta' \cap (-\Delta_{s^1}^s)) \cup (\Delta_0 \cap \Delta_+) \cup (-\Delta_{\mathfrak{m}_+}).$$

Therefore by Lemmas 1.3.1 and 1.6.18 we obtain $N_{\Delta'} = N_{\Delta' \cap \Delta_{s^2}^s} N_{\Delta' \cap (-\Delta_{s^1}^s)} N_{\Delta_0 \cap \Delta_+} M_-$. Note that by the definition of Z, $N_{\Delta_0 \cap \Delta_+} = Z_+ = Z \cap N_+$. Therefore we have a unique factorization $\widehat{n}y = y' z_+ m$, where $z_+ \in Z_+$, $y' \in N_{\Delta' \cap \Delta_{s^2}^s} N_{\Delta' \cap (-\Delta_{s^1}^s)}$, $m \in M_-$. The images of all roots from the set $(\Delta' \cap \Delta_{s^2}^s) \cup (\Delta' \cap (-\Delta_{s^1}^s)) = (\Delta' \setminus \Delta_0) \cap \Delta_+$ under the action of s belong to Δ_- by Proposition 1.6.6 (vii), (iii) and (i). Hence $sN_{\Delta' \cap \Delta_{s^2}^s} N_{\Delta' \cap (-\Delta_{s^1}^s)} s^{-1} \subset N_-$, and $sy's^{-1} \in N_-$. Thus substituting the factorization $\widehat{n}y = y' z_+ m$ into (3.4.25) we obtain

$$n_-^{-1} uy = k'' s z_+ m = k'' z_+' sm, m \in M_-, k'' = k' sy' s^{-1} \in N_-, z_+, z_+' = s z_+ s^{-1} \in Z_+. \tag{3.4.26}$$

Hence

$$n_-^{-1} uy \in N_- s Z_+ M_-. \tag{3.4.27}$$

Next we prove that for any $n_- \in N_-, x \in \mathfrak{h}$ and $y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}$ we have $h_-^{-1} n_-^{-1} uy h_+ \in N_- s H^0 Z_+ M_-$, where $h_\pm = e^{r_\pm^s x}$, i.e. $q(\mu_{M_-}^{-1}(u)) \subset N_- s H^0 Z_+ M_-$.

Let $H' \subset H$ be the subgroup corresponding to the Lie subalgebra $\mathfrak{h}' \subset \mathfrak{h}$. We obviously have $H = H' H^0$. From the definition of $r_\pm^s$ it follows that for any $h_0 \in H^0$ and $h' \in H'$ the elements $h_+ = h_0 h'$ and $h_- = h_0^{-1} s(h')$ are of the form $h_\pm = e^{r_\pm^s x}$ for some $x \in \mathfrak{h}$ and all elements $h_\pm$ are obtained in this way.

Next observe that the set $N_- s H^0 Z_+ M_-$ is invariant with respect to the following action of the subgroup of $H \times H$ formed by elements of the form $(h_+, h_-) = (h_0 h', h_0^{-1} s(h'))$:

$$(h_+, h_-) \circ L = h_-^{-1} L h_+, h = h_+ = h_0 h', h_- = h_0^{-1} s(h'). \tag{3.4.28}$$

Indeed, let $L = vskz_+ w$, $v \in N_-, w \in M_-, z_+ \in Z_+, k \in H^0$ be an element of $N_- s H^0 Z_+ M_-$. Then

$$(h_+, h_-) \circ L = h_-^{-1} v h_- h_-^{-1} s k h_+ h_+^{-1} z_+ w h_+ = h_-^{-1} v h_- s k h_0^2 h_+^{-1} z_+ w h_+ \tag{3.4.29}$$

since $s^{-1} h_-^{-1} s h_+ = h_0 h'^{-1} h_0 h' = h_0^2$. The right hand side of the last equality belongs to $N_- s H^0 Z_+ M_-$ because H normalizes N_-, M_- and Z_+.

Comparing action (3.4.28) with (3.4.13) and recalling that by (3.4.27) for any $n_- \in N_-$ and $y \in N_{\Delta_+ \setminus \Delta_{\mathfrak{m}_+}}$ one has $n_-^{-1} uy \in N_- s Z_+ M_- \subset N_- s H^0 Z_+ M_-$ we deduce

$$q(\mu_{M_-}^{-1}(u)) \subset N_- s H^0 Z_+ M_-. \tag{3.4.30}$$

Now we show that

$$N_- s Z_+ M_- = N_- s M_- Z_+ = (N_- \cap N) Z_- s M_- Z_+ = (N_- \cap N) Z_- s Z_+ M_- = (N_- \cap N) s Z_- Z_+ M_-. \tag{3.4.31}$$

First observe that we have the following disjoint union of minimal segments (see Figure 5)

$$\Delta_- = ((\Delta_-^s \setminus \Delta_0) \cap \Delta_-) \cup (\Delta_0 \cap \Delta_-) \cup (\Delta_{s^1}^s \cap \Delta_-).$$

Therefore by Lemmas 1.3.1 and 1.6.18 we obtain a unique factorization

$$N_- = N_{(\Delta_-^s \setminus \Delta_0) \cap \Delta_-} N_{\Delta_0 \cap \Delta_-} N_{\Delta_{s^1}^s \cap \Delta_-} = (N \cap N_-) Z_- (N_- \cap \overline{N})$$

as $N_{(\Delta_-^s \setminus \Delta_0) \cap \Delta_-} = N \cap N_-$, $N_{\Delta_0 \cap \Delta_-} = Z_- = Z \cap N_-$, $N_{\Delta_{s^1}^s \cap \Delta_-} = N_- \cap \overline{N}$.

Note that the elements of the subgroup $N_- \cap \overline{N} = N_{\Delta_{s^1}^s \cap \Delta_-}$ are transformed to M_- by the conjugation by s^{-1} as by part (iii) of Proposition 1.6.6

$$s^{-1}(\Delta_{s^1}^s) = s^2 s^1(\Delta_{s^1}^s) = -s^2(\Delta_{s^1}^s) \subset -(\Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_{s^2}^s \cup \Delta_0)) \subset -\Delta_{\mathfrak{m}_+}.$$

We deduce that any $k'' \in N_-$ can be uniquely represented in the form $k'' = \overline{n} z_- k'''$, $\overline{n} \in N \cap N_-$, $z_- \in Z_- = Z \cap N_-$, $k''' \in N_- \cap \overline{N}$, $s^{-1} k''' s \in M_-$. Therefore for any $m \in M_-, z_+ \in Z_+$, we have

$$k'' s m z_+ = \overline{n} z_- s m' z_+, \overline{n} \in N \cap N_-, m' = s^{-1} k''' s m \in M_-, z_- \in Z_-,$$

and hence $N_- s M_- Z_+ \subset (N \cap N_-) Z_- s M_- Z_+$. The opposite inclusion is obvious. Thus $N_- s M_- Z_+ = (N \cap N_-) Z_- s M_- Z_+$.

Similarly one obtains $N_- s Z_+ M_- = N_- s M_- Z_+ = (N \cap N_-) Z_- s Z_+ M_- = (N \cap N_-) Z_- s Z_+ M_- = (N \cap N_-) s Z_- Z_+ M_-$. This proves (3.4.31).

Since the conjugation action of H^0 normalizes N_-, $(N_- \cap N)$, $Z_\pm$, and H^0 is fixed by the conjugation by s, one immediately obtains from (3.4.31)

$$H^0 N_- s Z_+ M_- = N_- s H^0 Z_+ M_- = N_- s H^0 M_- Z_+ = (N_- \cap N) Z_- s H^0 M_- Z_+ =$$

$$= (N_- \cap N) Z_- s H^0 Z_+ M_- = (N \cap N_-) s Z_- H^0 Z_+ M_- \subset N s Z N,$$

where the last inclusion follows from the inclusions $Z_- H^0 Z_+ \subset Z$, $M_- \subset N$. This proves the identities in (3.4.11) and together with inclusion (3.4.30) establishes (3.4.11) completely.

Since by Proposition 1.3.4 $N s Z N$ is Zariski closed in G, the Zariski closure $\overline{q(\mu_{M_-}^{-1}(u))}$ of $q(\mu_{M_-}^{-1}(u)) \subset N s Z N$ in G is contained in $N s Z N$.

$\square$

Now we are in a position to describe the closure $\overline{q(\mu_{M_-}^{-1}(u))}$, the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ and the algebra $W^s(G)$.

Theorem 3.4.5. *Suppose that the numbers t_i defined in (3.4.5) are not equal to zero for all i. Let $N^s \subset N_- \cap N_s$ be the subgroup generated by one–parameter subgroups corresponding to the roots from the minimal segment $-\Delta^s$, where $\Delta^s = \Delta_s^s \setminus \Delta_{m_+} = \{\alpha \in \Delta_+^s \setminus \Delta_0 : \gamma_{l'} < \alpha\}$, $N^s = N_{-\Delta^s}$, and $M_-^s = M_- \cap N_s$. Then*

(i) the variety $\overline{q(\mu_{M_-}^{-1}(u))}$ is invariant under conjugations by elements of M_-, the conjugation action of M_- on $\overline{q(\mu_{M_-}^{-1}(u))}$ is free, and the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ is a smooth variety;

(ii) $\overline{q(\mu_{M_-}^{-1}(u))} = N_- s Z M_- = N_- s Z M_-^s$;

(iii) $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))}) \simeq N^s s Z M_-^s \simeq \Sigma_s = s Z N_s$, the conjugation action

$$M_- \times N^s s Z M_-^s \to N_- s Z M_- \tag{3.4.32}$$

is an isomorphism of varieties, and hence the algebra $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] = \mathbb{C}[N_- s Z M_-]$ is isomorphic to $\mathbb{C}[M_-] \otimes \mathbb{C}[N^s s Z M_-^s]$;

(iv) The algebra $W^s(G)$ is isomorphic to the algebra of regular functions on $N^s s Z M_-^s$, $W^s(G) \simeq \mathbb{C}[\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})] \simeq \mathbb{C}[N^s s Z M_-^s] \simeq \mathbb{C}[\Sigma_s] \simeq \mathbb{C}[N_- s Z M_-]^{M_-}$. Thus the algebra $W_{\mathcal{B}}^s(G)$ is a non–commutative deformation of the algebra of regular functions on the transversal slice $\Sigma_s \simeq N^s s Z M_-^s$.

Proof. (i) Firstly, as we observed in Lemma 3.4.2 the preimage $\mu_{M_-}^{-1}(u)$ is locally stable under the (locally defined) dressing action of M_-. On the other hand by Proposition 3.4.3 $q(\mu_{M_-}^{-1}(u)) \subset N s Z N$, so by Proposition 1.3.4 (i) and Proposition 3.3.4, $q(\mu_{M_-}^{-1}(u))$ is (locally) stable under the action of $M_- \subset N$ on $N s Z N$ by conjugations. Since the conjugation action of N on $N s Z N$ is free the (locally defined) conjugation action of M_- on $q(\mu_{M_-}^{-1}(u))$ is (locally) free as well.

Now recall that by Proposition 3.4.3 $\overline{q(\mu_{M_-}^{-1}(u))} \subset N s Z N$. Since by Proposition 1.3.4 (i) the conjugation action of N on $N s Z N$ is free and regular, $s Z N_s$ being a cross–section for this action, and $\overline{q(\mu_{M_-}^{-1}(u))}$ is closed, the local action of M_- on $q(\mu_{M_-}^{-1}(u)) \subset N s Z N$ by conjugations extends by continuity to the genuine regular action of $M_- \subset N$ on $\overline{q(\mu_{M_-}^{-1}(u))}$ which is free as well. Therefore the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ is a smooth variety.

(ii) We show now that the closure of $q(\mu_{M_-}^{-1}(u))$ contains $N_- s Z M_-$. Recall that by (3.4.11) $q(\mu_{M_-}^{-1}(u)) \subset N_- s H^0 Z_+ M_- = N_- H^0 Z_+ s M_-$. From (3.4.13) it follows that $q(\mu_{M_-}^{-1}(u))$ is closed with respect to the right multiplication by arbitrary elements from Z_+ and with respect to the left multiplication by arbitrary elements from N_-, as $Z_+ \subset N_{\Delta_+ \setminus \Delta_{m_+}}$, and $q(\mu_{M_-}^{-1}(u))$ is closed with respect to the right multiplication by arbitrary elements from $N_{\Delta_+ \setminus \Delta_{m_+}}$. $q(\mu_{M_-}^{-1}(u))$ is also closed with respect to the restriction of action (3.4.28) to the subgroup of $H \times H$ which consists of elements of the form (h_0, h_0^{-1}), $h_0 \in H^0$. Thus by (3.4.29), and since Z_+ normalizes M_- and s centralizes H^0, $q(\mu_{M_-}^{-1}(u))$ contains elements of the form $k s n$ for some $n \in M_- \subset N$ and arbitrary $k \in N_- H^0 Z_+$.

Now recall that $\bar{h}_0(\alpha) > 0$ for $\alpha \in \Delta_+^s \setminus \Delta_0$ and $\bar{h}_0(\alpha) = 0$ for $\alpha \in \Delta_0$ (see formula (1.3.45) and the discussion after it), and hence the $\mathbb{C}^*$–action on G induced by conjugations by the elements $h(t)$ from the one–parameter

subgroup generated by $\bar{h}_0 \in \mathfrak{h}'$ is contracting on N and fixes all elements of Z. Applying action (3.4.28) with $h = h(t)$ to the elements ksn with arbitrary $k \in N_-H^0Z_+$ we immediately deduce, with the help of (3.4.28), that the M_-–component n can be contracted to the identity element using the above defined contracting action, and hence the closure of $q(\mu_{M_-}^{-1}(u))$ contains the variety N_-Zs as $Z_- \subset N_-$, and the closure of $Z_-H^0Z_+$ is Z.

By (3.4.13) $q(\mu_{M_-}^{-1}(u))$ is closed with respect to the left multiplication by arbitrary elements from N_-. Recall also that $M_- \subset N$ freely acts on $\overline{q(\mu_{M_-}^{-1}(u))}$ by conjugations by part (i). Therefore $\overline{q(\mu_{M_-}^{-1}(u))}$ also contains the set $N_-ZsM_- = N_-sZM_-$.

Note that

$$N_-sH^0Z_+M_- \subset N_-ZsM_- \subset \overline{q(\mu_{M_-}^{-1}(u))} \subset \overline{N_-sH^0Z_+M_-},$$

where the last inclusion follows from (3.4.11), and all Zariski closures are taken in G. This implies, after taking Zariski closures in G, that

$$\overline{N_-ZsM_-} = \overline{q(\mu_{M_-}^{-1}(u))} = \overline{N_-sH^0Z_+M_-}. \tag{3.4.33}$$

Lemma 3.4.6. N_-ZsM_- *is a closed irreducible subvariety of G of dimension* $\dim \Sigma_s + \dim M_-$.

Proof. First note that $\Delta_- = ((\Delta_-^s \setminus \Delta_0) \cap \Delta_-) \cup (\Delta_+^s \cap \Delta_-) \cup (\Delta_- \cap \Delta_0)$ (disjoint union of minimal segments). Now by lemma 3.4.4 there is a unique factorization

$$N_- = N_{\Delta_-} = N_{(\Delta_-^s \setminus \Delta_0) \cap \Delta_-} N_{\Delta_+^s \cap \Delta_-} N_{\Delta_- \cap \Delta_0} = (N_- \cap N)N_{\Delta_+^s \cap \Delta_-} Z_-$$

as $N_{(\Delta_-^s \setminus \Delta_0) \cap \Delta_-} = N_- \cap N$ and $N_{\Delta_- \cap \Delta_0} = Z \cap N_- = Z_-$. We deduce

$$N_-ZsM_- = (N_- \cap N)N_{\Delta_+^s \cap \Delta_-} Z_-ZsM_- = (N_- \cap N)N_{\Delta_+^s \cap \Delta_-} ZsM_-. \tag{3.4.34}$$

From the definition of Δ_+ and Δ_+^s (see formula 1.6.9 and Definition 1.6.19) it follows that $\Delta_+^s \cap \Delta_- \subset \Delta_{s^1}^s$. Therefore

$$N_{\Delta_+^s \cap \Delta_-} \subset N_{\Delta_{s^1}^s}. \tag{3.4.35}$$

Recall that Z is generated by H^0 and by the one-parameter subgroups corresponding to the roots from Δ_0. Note that H^0 normalizes $N_{\Delta_{s^1}^s}$, and by Proposition 1.6.6 (ii), the one-parameter subgroups corresponding to the roots from Δ_0 also normalize $N_{\Delta_{s^1}^s}$. Therefore Z normalizes $N_{\Delta_{s^1}^s}$, and we deduce using (3.4.35)

$$N_{\Delta_+^s \cap \Delta_-} Z \subset N_{\Delta_{s^1}^s} Z = Z N_{\Delta_{s^1}^s} \tag{3.4.36}$$

By part (iii) of Proposition 1.6.6

$$s^{-1}(\Delta_{s^1}^s) = s^2 s^1(\Delta_{s^1}^s) = -s^2(\Delta_{s^1}^s) \subset -(\Delta_+^s \setminus (\Delta_{s^1}^s \cup \Delta_{s^2}^s \cup \Delta_0)) \subset -\Delta_{\mathfrak{m}_+},$$

and hence, recalling (3.4.36), we obtain

$$N_{\Delta_+^s \cap \Delta_-} Zs \subset Zss^{-1} N_{\Delta_{s^1}^s} s = ZsN_{s^{-1}(\Delta_{s^1}^s)} \subset ZsM_-.$$

Together with (3.4.34) this implies (compare with (3.4.11))

$$N_-ZsM_- = (N_- \cap N)ZsM_-.$$

Now consider the group $N_{(\Delta_-^s \setminus \Delta_0) \cap \Delta_-} = N_- \cap N$. By the definition of Δ_+ and Δ_+^s (see formula 1.6.9 and Definition 1.6.19) $(\Delta_-^s \setminus \Delta_0) \cap \Delta_- = ((-\Delta_{s^{-1}}^s) \cap \Delta_-) \cup (\Delta_-^s \setminus ((-\Delta_{s^{-1}}^s) \cup \Delta_0))$ (disjoint union of additively closed subsets of roots).

Now by lemma 1.3.1 there is a unique factorization

$$N_- \cap N = N_{(\Delta_-^s \setminus \Delta_0) \cap \Delta_-} = N_{(-\Delta_{s^{-1}}^s) \cap \Delta_-} N_{\Delta_-^s \setminus ((-\Delta_{s^{-1}}^s) \cup \Delta_0)}.$$

We deduce

$$N_-ZsM_- = (N_- \cap N)N_{\Delta_+^s \cap \Delta_-} Z_-ZsM_- = N_{(-\Delta_{s^{-1}}^s) \cap \Delta_-} N_{\Delta_-^s \setminus ((-\Delta_{s^{-1}}^s) \cup \Delta_0)} ZsM_-. \tag{3.4.37}$$

Recall that Z is generated by H^0 and by the one-parameter subgroups corresponding to the roots from Δ_0. Note that H^0 normalizes $N_{\Delta_-^s \setminus ((-\Delta_{s-1}^s) \cup \Delta_0)}$, and by Proposition 1.6.6 (ii), the one-parameter subgroups corresponding to the roots from Δ_0 also normalize $N_{\Delta_-^s \setminus ((-\Delta_{s-1}^s) \cup \Delta_0)}$. Therefore Z normalizes $N_{\Delta_-^s \setminus ((-\Delta_{s-1}^s) \cup \Delta_0)}$, and we deduce

$$N_- Z s M_- = N_{(-\Delta_{s-1}^s) \cap \Delta_-} Z N_{\Delta_-^s \setminus ((-\Delta_{s-1}^s) \cup \Delta_0)} s M_- \qquad (3.4.38)$$

By the definition of the sets Δ_{s-1}^s, Δ_s^s and Δ_0 we have

$$s^{-1}(\Delta_-^s \setminus ((-\Delta_{s-1}^s) \cup \Delta_0)) = \Delta_-^s \setminus ((-\Delta_s^s) \cup \Delta_0),$$

and hence, recalling (3.4.38), we obtain

$$N_- Z s M_- = N_{(-\Delta_{s-1}^s) \cap \Delta_-} Z s s^{-1} N_{\Delta_-^s \setminus ((-\Delta_{s-1}^s) \cup \Delta_0)} s M_- = N_{(-\Delta_{s-1}^s) \cap \Delta_-} Z s N_{s^{-1}(\Delta_-^s \setminus ((-\Delta_{s-1}^s) \cup \Delta_0))} M_- =$$

$$= N_{(-\Delta_{s-1}^s) \cap \Delta_-} Z s N_{\Delta_-^s \setminus ((-\Delta_s^s) \cup \Delta_0)} M_- = N_{(-\Delta_{s-1}^s) \cap \Delta_-} s Z N_{-[\beta_1^1, \gamma_{l'}]},$$

where $N_{\Delta_-^s \setminus ((-\Delta_s^s) \cup \Delta_0)} M_- = N_{-[\beta_1^1, \gamma_{l'}]} \subset N$, and $[\beta_1^1, \gamma_{l'}] = \{\alpha \in \Delta_+^s : \alpha \le \gamma_{l'}\}$ (we use the notation of Proposition 1.6.6 (iv)).

Note that $N_{(-\Delta_{s-1}^s) \cap \Delta_-} = N_{s^{-1}} \cap N_-$, where $N_{s^{-1}} = \{n \in N : s^{-1} n s \in \overline{N}\}$, so we have

$$N_- Z s M_- = (N_{s^{-1}} \cap N_-) s Z N_{-[\beta_1^1, \gamma_{l'}]}.$$

Finally observe that
$$s^{-1} N_- Z s M_- = s^{-1}(N_{s^{-1}} \cap N_-) s Z N_{-[\beta_1^1, \gamma_{l'}]} \subset \overline{N} Z N.$$

Therefore by Lemma 1.3.5 $s^{-1} N_- Z s M_-$ is a closed subvariety of $\overline{N} Z N$ as

$$\overline{N} Z N \simeq \overline{N} \times Z \times N,$$

$s^{-1}(N_{s^{-1}} \cap N_-) s \subset \overline{N}$, $N_{-[\beta_1^1, \gamma_{l'}]} \subset N$ are closed algebraic subgroups, and $s^{-1} N_- Z s M_-$ is the image in $\overline{N} Z N$ of $s^{-1}(N_{s^{-1}} \cap N_-) s \times Z \times N_{-[\beta_1^1, \gamma_{l'}]}$ under the product map. In particular,

$$N_- Z s M_- \simeq s^{-1}(N_{s^{-1}} \cap N_-) s \times Z \times N_{-[\beta_1^1, \gamma_{l'}]}. \qquad (3.4.39)$$

The variety $\overline{N} Z N$ is closed in G by Lemma 1.3.5, and hence $s^{-1} N_- Z s M_-$ is a closed subvariety of G. Thus $N_- Z s M_-$ is a closed subvariety of G.

From the isomorphism of varieties (3.4.39) using Theorem 1.5.4 in [124] it follows that $N_- Z s M_-$ is also irreducible as $s^{-1}(N_{s^{-1}} \cap N_-) s$, Z, and $N_{-[\beta_1^1, \gamma_{l'}]}$ are irreducible.

To find the dimension of $N_- Z s M_-$ we observe that by formula 1.6.9, by Definition 1.6.19 and by part (vii) of Proposition 1.6.6 $(-\Delta_{s-1}^s) \cap \Delta_- = (-\Delta_{s-1}^s) \setminus \{\alpha \in \Delta_-^s : \alpha < -\gamma_1\}$, and $-[\beta_1^1, \gamma_{l'}] = (-\Delta_{\mathfrak{m}_+}) \cup \{\alpha \in \Delta_-^s : \alpha < -\gamma_1\}$ (disjoint union). Note also that $\{\alpha \in \Delta_-^s : \alpha < -\gamma_1\} \subset -\Delta_{s-1}^s$ Thus

$$\dim N_{s^{-1}} \cap N_- = \dim N_{(-\Delta_{s-1}^s) \cap \Delta_-} = |(-\Delta_{s-1}^s) \setminus \{\alpha \in \Delta_-^s : \alpha < -\gamma_1\}| = |-\Delta_{s-1}^s| - |\{\alpha \in \Delta_-^s : \alpha < -\gamma_1\}|,$$

and

$$\dim N_{-[\beta_1^1, \gamma_{l'}]} = |\{\alpha \in \Delta_-^s : \alpha < -\gamma_1\}| + |-\Delta_{\mathfrak{m}_+}|$$

Now using isomorphism (3.4.39) we obtain

$$\dim N_- Z s M_- = \dim N_{s^{-1}} \cap N_- + \dim Z + \dim N_{-[\beta_1^1, \gamma_{l'}]} =$$

$$= |-\Delta_{s-1}^s| - |\{\alpha \in \Delta_-^s : \alpha < -\gamma_1\}| + |\{\alpha \in \Delta_-^s : \alpha < -\gamma_1\}| + |-\Delta_{\mathfrak{m}_+}| + \dim Z =$$

$$= |\Delta_{s-1}^s| + \dim Z + \dim M_- = \dim N_s + \dim Z + \dim M_- = \dim \Sigma_s + \dim M_-.$$

This completes the proof of the lemma.

$\square$

Now from (3.4.33) and Lemma 3.4.6 we infer

$$N_- ZsM_- = \overline{q(\mu_{M_-}^{-1}(u))} = \overline{N_- s H^0 Z_+ M_-}.$$

Now we show that $N_- ZsM_- = N_- ZsM_-^s$. Observe that $M_- = N_{-\Delta_{\mathfrak{m}_+}}$, and by the definition of $\Delta_{\mathfrak{m}_+}$ (see part (v) of Proposition 1.6.6), and by part (vii) of Proposition 1.6.6, $\Delta_{\mathfrak{m}_+} = \Delta_{\mathfrak{m}_+}^s \cup (\Delta_{\mathfrak{m}_+} \setminus \Delta_s^s)$ (disjoint union of minimal segments), where $\Delta_{\mathfrak{m}_+}^s = \Delta_{\mathfrak{m}_+} \cap \Delta_s^s$.

Using Lemma 3.4.4 we have a unique decomposition

$$M_- = N_{-(\Delta_{\mathfrak{m}_+} \setminus \Delta_s^s)} N_{-\Delta_{\mathfrak{m}_+}^s} = N_{-(\Delta_{\mathfrak{m}_+} \setminus \Delta_s^s)} M_-^s \tag{3.4.40}$$

as by the definition $M_-^s = M_- \cap N_s = N_{-\Delta_{\mathfrak{m}_+}^s}$. If $\alpha \in \Delta_{\mathfrak{m}_+} \setminus \Delta_s^s$ then $s\alpha \in \Delta_+^s$ by the definition of Δ_s^s, and by Proposition 1.6.6 (viii) we have $s\alpha > \alpha$, and if $s\alpha + \alpha_0 \in \Delta$ for $\alpha_0 \in \Delta_0$ then $s\alpha + \alpha_0 \in \Delta_+^s$ and $s\alpha + \alpha_0 > \alpha$, so in both cases $s\alpha, s\alpha + \alpha_0 \in \Delta_+^s$ and $s\alpha, s\alpha + \alpha_0 > \alpha$. This implies $s\alpha, s\alpha + \alpha_0 \notin \Delta_s^s$. In particular, from the definition (1.6.19) of the root system Δ_+ it follows that in both cases $s\alpha, s\alpha + \alpha_0 \in \Delta_+$. Observing also that Z is generated by the one–parameter subgroups corresponding to roots from Δ_0 and by the centralizer of s in H which normalizes all one–parameter subgroups corresponding to roots, we deduce $sN_{-(\Delta_{\mathfrak{m}_+} \setminus \Delta_s^s)}s^{-1} \subset N_-$ and $zsN_{-(\Delta_{\mathfrak{m}_+} \setminus \Delta_s^s)}s^{-1}z^{-1} \subset N_-$ for any $z \in Z$. Thus by (3.4.40)

$$N_- ZsM_- = N_- ZsN_{-(\Delta_{\mathfrak{m}_+} \setminus \Delta_s^s)}M_-^s = N_- ZsN_{-(\Delta_{\mathfrak{m}_+} \setminus \Delta_s^s)}s^{-1}sM_-^s \subset N_- sZM_-^s.$$

The opposite inclusion is obvious. So we obtain that $N_- ZsM_- = N_- ZsM_-^s$.

(iii) We show that $N^s ZsM_-^s$ is a cross–section for the free conjugation action of M_- on $N_- ZsM_- = \overline{q(\mu_{M_-}^{-1}(u))}$. Note that by the definition $N^s sZM_-^s \subset N_- ZsM_- = \overline{q(\mu_{M_-}^{-1}(u))}$.

From Figure 5 and Proposition 1.6.6 (i), (iii), (vii) we obtain $-\Delta_s^s = (-\Delta^s) \cup -(\Delta_{\mathfrak{m}_+}^s)$ (disjoint union of minimal segments). By the definition $N_s = N_{-\Delta_s^s}$, $N^s = N_{-\Delta^s}$, $M_-^s = N_{-\Delta_{\mathfrak{m}_+}^s}$. Therefore Lemma 3.4.4 implies the following unique factorization $N_s = M_-^s N^s$. Thus if $szn_s \in sZN_s$, $z \in Z$, $n_s \in N_s$ then n_s can be uniquely factorized as $n_s = m_s n_s$, $m_s \in M_-^s$, $n_s \in N^s$ and we have

$$szn_s = szm_s n_s.$$

Conjugating this element by n_s we deduce that szn_s is uniquely conjugated to the element

$$n_s szm_s \in N^s sZM_-^s$$

and hence $N^s sZM_-^s \simeq sZN_s = \Sigma_s$ is a cross–section for the conjugation action of N on $NsZN$ as well. At the same time by construction the bijection $N^s sZM_-^s \simeq sZN_s$ is an isomorphism of varieties, where the variety structure on $N^s sZM_-^s$ is induced from $N^s \times Z \times M_-^s$ using the product map

$$N^s \times Z \times M_-^s \to N^s sZM_-^s, \ (n_s, z, m_s) \mapsto n_s szm_s.$$

Observe that any two points of $N^s ZsM_-^s$ are not M_-–conjugate. Indeed, we have an inclusion $N^s ZsM_-^s \subset \overline{q(\mu_{M_-}^{-1}(u))}$, and two points of $\overline{q(\mu_{M_-}^{-1}(u))}$ can not be M_--conjugate if they are not N–conjugate in $NsZN \supset \overline{q(\mu_{M_-}^{-1}(u))}$ as $M_- \subset N$. But $N^s ZsM_-^s \simeq \Sigma_s$ is a cross–section for the conjugation action of N on $ZsZN$ by Proposition 1.3.7 (i). Thus any two points of $N^s ZsM_-^s$ are not N–conjugate, and hence they are not M_-–conjugate. Therefore the closed variety $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ must contain the closed variety $N^s ZsM_-^s \simeq \Sigma_s$.

From formula (1.6.15) for the cardinality $|\Delta_{\mathfrak{m}_+}|$ of the set $\Delta_{\mathfrak{m}_+}$ and from the definitions of $\overline{q(\mu_{M_-}^{-1}(u))}$ and of $N^s ZsM_-^s$ we deduce that the dimension of the quotient $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))})$ is equal to the dimension of the variety $N^s ZsM_-^s$,

$$\dim \pi_q(\overline{q(\mu_{M_-}^{-1}(u))}) = \dim G - 2\dim M_- = 2D + l - 2|\Delta_{\mathfrak{m}_+}| = 2D + l - 2\left(D - \frac{l(s) - l'}{2} - D_0\right) =$$

$$= l(s) + 2D_0 + l - l' = \dim N_s + \dim Z = \dim sZN_s = \dim \Sigma_s = \dim N^s ZsM_-^s.$$

Since π_q is a morphism of varieties and the conjugation action of M_- on $\overline{q(\mu_{M_-}^{-1}(u))} = N_-ZsM_-$ is free by part (i), $\pi_q^{-1}(N^sZsM_-^s)$ is a closed smooth subvariety of the smooth variety $N_-ZsM_- = \overline{q(\mu_{M_-}^{-1}(u))}$, and $\dim \pi_q^{-1}(N^sZsM_-^s) = \dim N^sZsM_-^s + \dim M_- = \dim \Sigma_s + \dim M_-$.

Now recall that by Lemma 3.4.6 $N_-ZsM_- = \overline{q(\mu_{M_-}^{-1}(u))}$ is irreducible and has the same dimension, $\dim N_-ZsM_- = \dim \Sigma_s + \dim M_- = \dim \pi_q^{-1}(N^sZsM_-^s)$. By [50], Ex. 1.10(c) the identity for the dimensions and the closed inclusion $\pi_q^{-1}(N^sZsM_-^s) \subset N_-ZsM_-$ imply $\pi_q^{-1}(N^sZsM_-^s) = N_-ZsM_-$.

Therefore $\pi_q(\overline{q(\mu_{M_-}^{-1}(u))}) \simeq N^sZsM_-^s$, $N^sZsM_-^s$ is a cross–section for the action of M_- on $\overline{q(\mu_{M_-}^{-1}(u))}$, and the conjugation action

$$M_- \times N^sZsM_-^s \to N_-ZsM_-$$

is an isomorphism of varieties. We conclude that the algebra $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$ is isomorphic to $\mathbb{C}[M_-] \otimes \mathbb{C}[N^sZsM_-^s]$, $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \simeq \mathbb{C}[M_-] \otimes \mathbb{C}[N^sZsM_-^s]$.

(iv) Now recall that by Lemma 3.4.1

$$W^s(G) = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)},$$

where $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$ is regarded as a subalgebra in $C^\infty(\mu_{M_-}^{-1}(u))$ using the map $q^* : C^\infty(q(\mu_{M_-}^{-1}(u))) \to C^\infty(\mu_{M_-}^{-1}(u))$ and the imbedding $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \subset C^\infty(q(\mu_{M_-}^{-1}(u)))$.

By Lemma 3.4.2 the algebra $C^\infty(\mu_{M_-}^{-1}(u))^{M_-}$ is isomorphic to $C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)}$, and hence

$$W^s(G) = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{C^\infty(M_+)} = \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{M_-}. \tag{3.4.41}$$

As we already proved the variety $\overline{q(\mu_{M_-}^{-1}(u))}$ is stable under the conjugation action of M_-, and the map $\pi_q : \overline{q(\mu_{M_+}^{-1}(u))} \to \pi_q \overline{q(\mu_{M_+}^{-1}(u))}$ is a morphism of varieties. Moreover, under the map $q : G^* \to G$ the local dressing action of M_- on G^* becomes the conjugation action on G. Therefore the map

$$\mathbb{C}[\pi_q \overline{q(\mu_{M_-}^{-1}(u))}] \to \mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \cap C^\infty(\mu_{M_-}^{-1}(u))^{M_-}, \quad \psi \mapsto \pi_q^* \psi \tag{3.4.42}$$

is an algebra isomorphism, where $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}]$ is regarded as a subalgebra in $C^\infty(\mu_{M_-}^{-1}(u))$ using the map

$$q^* : C^\infty(q(\mu_{M_+}^{-1}(u))) \to C^\infty(\mu_{M_-}^{-1}(u))$$

and the imbedding $\mathbb{C}[\overline{q(\mu_{M_-}^{-1}(u))}] \subset C^\infty(q(\mu_{M_-}^{-1}(u)))$.

Combining (3.4.41) and (3.4.42) we obtain that $W^s(G) \simeq \mathbb{C}[\pi_q \overline{q(\mu_{M_-}^{-1}(u))}] \simeq \mathbb{C}[N_-sZM_-]^{M_-}$. This completes the proof.

$\square$

3.5 Zhelobenko type operators for Poisson q-W–algebras

In this section we present the main result of this chapter, a formula for a projection operator $\Pi : \mathbb{C}[N_-ZsM_-] \to \mathbb{C}[N_-ZsM_-]^{M_-}$ onto the subspace of invariants $\mathbb{C}[N_-ZsM_-]^{M_-}$ which is isomorphic to $W^s(G)$ as an algebra according to Theorem 3.4.5 (iv). This formula has a direct quantum analogue which will be introduced in the next chapter.

The operator Π can be defined following the philosophy of [119] where a similar projection operator onto the subspace $\mathbb{C}[NZsN]^N \subset \mathbb{C}[NZsN]$ was defined and studied. More precisely, according to Theorem 3.4.5 any $g \in N_-ZsM_-$ can be uniquely represented in the form

$$g = nn_szsm_sn^{-1}, n \in M_-, n_s \in N^s, m_s \in M_-^s, z \in Z. \tag{3.5.1}$$

If for $f \in \mathbb{C}[N_-ZsM_-]$ we define $\Pi f \in \mathbb{C}[N_-ZsM_-]$ by

$$(\Pi f)(g) = f(n^{-1}gn) = f(n_szsm_s) \tag{3.5.2}$$

then Πf is an M_--invariant function, and any M_--invariant regular function on $N_- Z s M_-$ can be obtained this way. Moreover, by the definition $\Pi^2 = \Pi$, i.e. Π is a projection onto $\mathbb{C}[N_- Z s M_-]^{M_-}$.

To obtain an explicit formula for the operator Π we firstly find an explicit formula for n in terms of g in (3.5.1). Denote by ω the Chevalley anti-involution on $\mathfrak{g}$ which is induced by the antiautomorphism ω of $U_h^s(\mathfrak{g})$ on $U(\mathfrak{g}) \simeq U_h^s(\mathfrak{g})/h U_h^s(\mathfrak{g})$. We also denote the corresponding anti-involution of G by the same letter.

An explicit formula for Π suitable for quantization will be given in terms of matrix elements of finite-dimensional irreducible representations of G. Let $H \subset G$ be a maximal torus, $\Delta_+ \subset \Delta = \Delta(G, H)$ a system of positive roots, P_+ the corresponding set of integral dominant weights, B_+ the corresponding Borel subgroup, B_- the opposite Borel subgroup, $N_\pm$ the unipotent radicals of $B_\pm$, respectively. For any $\mu \in P_+$ we denote by v_μ a non-zero highest weight vector in the irreducible highest weight G-module V_μ of highest weight μ, and by $(\cdot, \cdot)$ the contravariant non-degenerate form on V_μ such that $(v, xw) = (\omega(x)v, w)$ for any $v, w \in V_\mu$, $x \in \mathfrak{g}$ and $(v_\mu, v_\mu) = 1$.

Let $\beta_1, \ldots, \beta_N$ be a normal order on Δ_+, $X_{\pm\beta_1}, \ldots, X_{\pm\beta_D} \in \mathfrak{g}$ the corresponding root vectors defined in (2.3.2). We shall need some special matrix elements of finite-dimensional irreducible representations of G. These matrix elements can be defined by specializing the results of Lemma 2.3 in [29] at $q = 1$. By this lemma there are integral dominant weights $\mu_p \in P_+$, $p = 1, \ldots, D$ and elements $v_p \in V_{\mu_p}$ such that

$$(v_p, X_{-\beta_D}^{(n_D)} \ldots X_{-\beta_1}^{(n_1)} v_{\mu_p}) = \left\{ \begin{array}{ll} 1 & \text{if } X_{-\beta_D}^{(n_D)} \ldots X_{-\beta_1}^{(n_1)} = X_{-\beta_p} \\ 0 & \text{otherwise} \end{array} \right. , \qquad (3.5.3)$$

where for $\alpha \in \Delta$, $k \in \mathbb{N}$ we define $X_\alpha^{(k)} = \frac{X_\alpha^k}{k!}$.

Since by this definition $(v_p, X_{-\beta_p} v_{\mu_p}) = (\omega(X_{-\beta_p})v_p, v_{\mu_p}) = 1$, $(v_{\mu_p}, v_{\mu_p}) = 1$, and the highest weight subspace in V_{μ_p} is one-dimensional, we deduce that $\omega(X_{-\beta_p})v_p = v_{\mu_p}$.

We shall need the following properties of the matrix elements $(v_p, \cdot v_{\mu_p})$.

Lemma 3.5.1. *(i) For any $1 \le q < p \le D$, any $y = X_{-\beta_D}^{(n_D)} \ldots X_{-\beta_p}^{(n_p)} \in U(\mathfrak{g})$, where $n_i \in \mathbb{N}$, $y \ne 1$, one has $\omega(y)v_q = 0$. In particular, for any $u \in U(\mathfrak{g})$*

$$(v_q, yuv_{\mu_q}) = 0. \qquad (3.5.4)$$

(ii) For any $1 \le p \le D$, any $y = X_{-\beta_D}^{(n_D)} \ldots X_{-\beta_p}^{(n_p)} \in U(\mathfrak{g})$, where $n_i \in \mathbb{N}$, $y \ne 1, X_{-\beta_p}$, one has $\omega(y)v_q = 0$. In particular, for any $u \in U(\mathfrak{g})$

$$(v_p, yuv_{\mu_p}) = 0. \qquad (3.5.5)$$

We shall later prove Lemma 4.2.2 which is a quantum group analogue of this Lemma. Lemma 3.5.1 follows from Lemma 4.2.2 by specializing at $q^{\frac{1}{dr^2}} = 1$.

Corollary 3.5.2. *(i) For any $1 \le q < p \le D$ and any $g \in N_{[-\beta_p, -\beta_D]}$ one has $\omega(g)v_q = v_q$.*

(ii) For any $1 \le p \le D$ and $X_{-\beta_p}(u) \in N_{\{-\beta_p\}}$, $u \in \mathbb{C}$ one has $\omega(X_{-\beta_p}(u))v_p = v_p + u v_{\mu_p}$.

(iii) For any $1 \le p \le D$ and $n_1 \in N_{[-\beta_{p+1}, -\beta_D]}$, $n_2 \in N_{[-\beta_{p-1}, -\beta_1]}$, $X_{-\beta_p}(u) \in N_{\{-\beta_p\}}$, $u \in \mathbb{C}$ one has $(v_p, n_1 X_{-\beta_p}(u) n_2 v_{\mu_p}) = u$.

Proof. Using the unique factorization $N_- = N_{\{-\beta_D\}} \ldots N_{\{-\beta_1\}}$ introduced in Lemma 1.3.1 and the exponential map in the one-parameter subgroups $N_{\{-\beta_D\}}, \ldots, N_{\{-\beta_1\}}$ one reduces the proof of parts (i) (ii) and (iii) of this corollary to the statements of parts (i) and (ii) of the previous lemma, and to the definition of the matrix elements $(v_p, \cdot v_{\mu_p})$, respectively. $\qquad \square$

Let $G^0 := N_- H N_+ \subset G$ be the big Bruhat cell. Recall that by Lemma 1.3.2 (iii) we have an isomorphism of verieties $N_- H N_+ \simeq N_- \times H \times N_+$ as $N_- H N_+$ is the orbit of the unit element in G for the following action

$$(N_- \times B_+) \times G \to G, (n_-, b_+) \circ g = n_- g b_+^{-1},$$

and $B_+ \simeq N_+ \times H$ as a variety Lemma 1.3.2 (i) (see also Lemma 8.3.6 in [124]).

The big Bruhat cell can be defined in G as the compliment of the common zero locus of the regular functions $(v_\mu, \cdot v_\mu)$, $\mu \in P_+$, $\mu \ne 0$. These functions together with the constant function equal to $1 = (v_0, \cdot v_0)$ form a multiplicatively closed set $\mathcal{S}$, and the localization of $\mathbb{C}[G]$ by this set is isomorphic to $\mathbb{C}[N_- H N_+]$ (see e.g. [132], §9 and §100).

For any complex algebraic variety V and any subset $X \subset V$ we denote by $\mathcal{V}_V(X) \subset \mathbb{C}[V]$ the vanishing ideal of X in $\mathbb{C}[V]$.

Lemma 3.5.3. *Let $\varphi_p \in \mathbb{C}(G)$ be the rational function on G defined by*

$$\varphi_p(g) = \frac{(v_p, g v_{\mu_p})}{(v_{\mu_p}, g v_{\mu_p})}.$$

Then the following statements are true.

(i) $\varphi_p \in \mathbb{C}[N_- H N_+]$. Moreover, there are unique factorizations $N_\pm = N_{\{\pm\beta_D\}} \ldots N_{\{\pm\beta_1\}}$ which induce the isomorphism of varieties

$$N_- H N_+ \simeq \mathbb{C}^D \times H \times \mathbb{C}^D \tag{3.5.6}$$

Under this isomorphism the function φ_p becomes the p-th coordinate function on $\mathbb{C}^D$ on the first factor in (3.5.6)

(ii) φ_p and $\mathcal{V}_{G^0}(N_{[-\beta_p,-\beta_D]} H N_+)$ generate $\mathcal{V}_{G^0}(N_{[-\beta_{p+1},-\beta_D]} H N_+)$.

(iii) φ_q, $q = 1, \ldots, p$ generate $\mathcal{V}_{G^0}(N_{[-\beta_{p+1},-\beta_D]} H N_+)$.

Proof. (i) $\varphi_p \in \mathbb{C}[N_- H N_+]$ as $(v_{\mu_p}, \cdot v_{\mu_p}) \in \mathcal{S}$, $\varphi_p \in \mathbb{C}[G][\mathcal{S}^{-1}]$, and the localization of $\mathbb{C}[G]$ by $\mathcal{S}$ is isomorphic to $\mathbb{C}[N_- H N_+]$.

By Lemma 1.3.1 there are unique factorizations $N_\pm = N_{\{\pm\beta_D\}} \ldots N_{\{\pm\beta_1\}}$ which imply the isomorphism of varieties (3.5.6) using the exponential maps for the one–parameter subgroups, where for convenience we inverse the order of factors in $\mathbb{C}^D$, i.e. the k-th factor in $N_{\{\pm\beta_D\}} \ldots N_{\{\pm\beta_1\}}$ corresponds to the $D - k + 1$ factor in $\mathbb{C}^D$.

In particular, if $n_- h n_+ \in N_- H N_+$, $n_- \in N_-$, $h \in H$, $n_+ \in N_+$ then $n_- = n_1 X_{-\beta_p}(u) n_2$, $n_1 \in N_{[-\beta_{p+1},-\beta_D]}$, $n_2 \in N_{[-\beta_{p-1},-\beta_1]}$, $X_{-\beta_p}(u) \in N_{\{-\beta_p\}}$, $u \in \mathbb{C}$. Since v_{μ_p} is a highest weight vector $(v_p, n_- h n_+ v_{\mu_p}) = \mu_p(h)(v_p, n_- v_{\mu_p})$ and $(v_{\mu_p}, n_- h n_+ v_{\mu_p}) = \mu_p(h) \neq 0$. By Corollary 3.5.2 (iii) one has $(v_p, n_- v_{\mu_p}) = (v_p, n_1 X_{-\beta_p}(u) n_2 v_{\mu_p}) = u$. Using these identities in the definition of φ_p we obtain

$$\varphi_p(n_- h n_+) = \frac{(v_p, n_- h n_+ v_{\mu_p})}{(v_{\mu_p}, n_- h n_+ v_{\mu_p})} = u.$$

Thus under isomorphism (3.5.6) φ_p becomes the p-th coordinate function on $\mathbb{C}^D$ on the first factor in (3.5.6). This proves part (i).

(ii) Under isomorphism (3.5.6) $N_{[-\beta_p,-\beta_D]} H N_+ \simeq \mathbb{C}^{D-p+1} \times H \times \mathbb{C}^D$, and φ_p becomes the first coordinate function on $\mathbb{C}^{D-p+1}$. We deduce that the ideal generated by φ_p and by $\mathcal{V}_{G^0}(N_{[-\beta_p,-\beta_D]} H N_+)$ is the vanishing ideal of $\mathcal{V}_{G^0}(N_{[-\beta_{p+1},-\beta_D]} H N_+) \simeq \mathbb{C}^{D-p} \times H \times \mathbb{C}^D$. This proves part (i).

Part (iii) follows from part (ii) using induction over p. $\square$

Next, we shall need to consider some normal orderings on Δ_+ associated to s in Definition 1.6.19. We label the roots in the initial segment $\Delta_{\mathfrak{m}_+} \subset \Delta_+$ of the system Δ_+ ordered as in Definition 1.6.19 as follows

$$\beta_{11}, \ldots, \beta_{1n_1}, \beta_{21}, \ldots, \beta_{2n_2}, \ldots, \beta_{R-11} \ldots \beta_{R-1n_{R-1}},$$

where $\{\beta_{11}, \ldots, \beta_{1n_1}\} = \Delta^1 \cap \Delta_{\mathfrak{m}_+}$, $\{\beta_{R-11} \ldots \beta_{R-1n_{R-1}}\} = \Delta^{R-1} \cap \Delta_{\mathfrak{m}_+}$, and for $1 < j < R-1$ $\{\beta_{j1}, \ldots, \beta_{jn_j}\} = \Delta^j$.

Let

$$s_{i_{11}} \ldots s_{i_{1n_1}} s_{i_{21}} \ldots s_{i_{2n_2}} \ldots s_{i_{R-11}} \ldots s_{i_{R-1n_{R-1}}}$$

be the corresponding initial part of the reduced decomposition of the longest element of the Weyl group associated to the normal ordering introduced in Definition 1.6.19.

Let $w_j = s_{i_{j1}} \ldots s_{i_{jn_j}}$. Note that since

$$\Delta_{(w_1 \ldots w_{j-1})^{-1}} = \Delta^1 \cap \Delta_{\mathfrak{m}_+} \cup \Delta^2 \cup \ldots \cup \Delta^{j-1} \tag{3.5.7}$$

for $j = 2, \ldots, R-1$ one has

$$(w_1 \ldots w_{j-1})^{-1} \Delta_+^j = \Delta_+ \tag{3.5.8}$$

by the definition of Δ_+^j.

Remark 3.5.4. *Note that each root system Δ_+^j inherits a normal ordering from the circular normal ordering associated to (1.6.9). Therefore for each $j = 2, \ldots, R-1$ the corresponding isomorphism (3.5.8) induces a normal ordering on Δ_+ with respect to which the images of the roots from the segment $\beta_{j1}, \ldots, \beta_{jn_j}$ form an initial segment. We denote the roots in this segment by $\delta_{j1}, \ldots, \delta_{jn_j}$, $\delta_{jk} = (w_1 \ldots w_{j-1})^{-1} \beta_{jk}$, $k = 1, \ldots, n_j$, and the remaining roots by δ_{jk}, $k = n_j + 1, \ldots D$ in the increasing order according to the values of k. To keep the notation uniform we also write $\delta_{1k} = \beta_{1k}$, $k = 1, \ldots, D$.*

For $j = 1, \ldots, R - 1$ define $\widehat{s}_j = (w_1 \ldots w_{j-1})^{-1} s(w_1 \ldots w_{j-1})$, where we assume that $w_0 = 1$, so $\widehat{s}_0 = s$.

For each $j = 1, \ldots, R - 1$, $k = 1, \ldots, n_j$ we define matrix elements $(v_{jk}, \cdot v_{\mu_{jk}})$ by condition (3.5.3), where for $j = 1$ the normal ordering on Δ_+ introduced in Definition 1.6.19 is used, and for $j > 1$ the normal ordering on Δ_+ induced by the normal ordering on Δ_+^j with the help of isomorphism (3.5.8) is used, and $\beta_p = \delta_{jk}$. Let

$$\varphi_{jk}(g) = \frac{(v_{jk}, g\widehat{s}_j^{-1} v_{\mu_{jk}})}{(v_{\mu_{jk}}, g\widehat{s}_j^{-1} v_{\mu_{jk}})}. \tag{3.5.9}$$

Let $G^j = N_- H N_+ \widehat{s}_j$, $j = 1, \ldots, R - 1$. Multiplication by $\widehat{s}_p^{-1}$ in G from the right yields the isomorphism of G^p with the open dense Bruhat cell $G^0 = N_- H N_+ \subset G$,

$$G^p \to G^0, g \mapsto g\widehat{s}_j^{-1}. \tag{3.5.10}$$

G^0 can be defined in G as the compliment of the common zero locus of the regular functions $(v_\mu, \cdot v_\mu)$, $\mu \in P_+$, $\mu \neq 0$. These functions together with the constant function equal to $1 = (v_0, \cdot v_0)$ form a multiplicatively closed set, and the localization of $\mathbb{C}[G]$ by this set is isomorphic to $\mathbb{C}[N_- H N_+]$ (see e.g. [132], §9 and §100).

Thus if we introduce the multiplicatively closed sets $\mathcal{S}_j = \{(v_\mu, \cdot \widehat{s}_j^{-1} v_\mu), \mu \in P_+\}$, $j = 1, \ldots, R - 1$ then

$$G^j = \{g \in G : (v_\mu, g\widehat{s}_j^{-1} v_\mu) \neq 0 \; \forall \mu \in P_+\}, \tag{3.5.11}$$

and

$$\mathbb{C}[G^j] = \mathbb{C}[G][\mathcal{S}_j^{-1}]. \tag{3.5.12}$$

By Lemma 1.6.21 the set of roots $(w_1 \ldots w_{j-1})^{-1}(-\Delta_+^j \cup \Delta_0)$ is parabolic. Let $P^j \subset G$ be the corresponding parabolic subgroup, N_-^j its unipotent radical, and $P_s \subset G$ the subgroup generated by the one–parameter subgroups corresponding to the roots from the additively closed set $(w_1 \ldots w_{j-1})^{-1}(-\Delta_s^s \cup \Delta_0)$. The semisimple part L^j of P^j is $(w_1 \ldots w_{j-1})^{-1} L w_1 \ldots w_{j-1}$ as $-(-\Delta_+^j \cup \Delta_0) \cap (-\Delta_+^j \cup \Delta_0) = \Delta_0$.

Lemma 3.5.5. *(i) For any $j = 1, \ldots, R - 1$, $k = 1, \ldots, n_j$, and $g = n_- z\widehat{s}_j n \in N_{[-\delta_{jk}, -\delta_{jD}]}\widehat{s}_j G$, where $n_- \in N_{[-\delta_{jk}, -\delta_{jD}]}$, $z \in L^j$, and $n \in N_-^j$ or $n \in N_{(w_1 \ldots w_{j-1})^{-1}(-\Delta_s^s)}$ one has*

$$(v_{jk}, g\widehat{s}_j^{-1} v_{\mu_{jk}}) = u(v_{\mu_{jk}}, zv_{\mu_{jk}}), \tag{3.5.13}$$

and

$$(v_\mu, g\widehat{s}_j^{-1} v_\mu) = (v_\mu, zv_\mu), \tag{3.5.14}$$

where $u \in \mathbb{C}$ is defined from the unique factorizations $n_- = X_{-\delta_{jk}}(u)n' = n'' X_{-\delta_{jk}}(u)$, $n', n'' \in N_{[-\delta_{jk+1}, -\delta_{jD}]}$ with the help of Lemma 3.4.4. Thus

$$\varphi_{jk}(g) = u. \tag{3.5.15}$$

(ii) Let $Z_\pm^j = (w_1 \ldots w_{j-1})^{-1} Z_\pm w_1 \ldots w_{j-1}$, $M_-^j = N_-^j \cap (w_1 \ldots w_{j-1})^{-1} M_- w_1 \ldots w_{j-1}$, $j = 1, \ldots, R - 1$. Then for any $j = 1, \ldots, R - 1$, $k = 1, \ldots, n_j$ one has $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j N_-^j \subset G^j$. Moreover, for $k = 1, \ldots, n_j$ the function $\varphi_{jk} \in \mathbb{C}[G^j]$ and the ideal $J^{jk^{loc}} := \mathcal{V}_{G^j}(N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j)$ generate $J^{jk+1^{loc}} := \mathcal{V}_{G^j}(N_{[-\delta_{jk+1}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j)$.

(iii) For any $j = 1, \ldots, R - 1$, $k = 1, \ldots, n_j$ the functions φ_{jm}, $m = 1, \ldots, k$ and the ideal $J^{j1^{loc}}$ generate $J^{jk+1^{loc}}$.

Proof. By Corollary 3.5.2 (i) and (ii) we have

$$(v_{jk}, g\widehat{s}_j^{-1} v_{\mu_{jk}}) = (v_{jk}, n'' X_{-\delta_{jk}}(u)z\widehat{s}_j n\widehat{s}_j^{-1} v_{\mu_{jk}}) = (v_{jk}, z\widehat{s}_j n\widehat{s}_j^{-1} v_{\mu_{jk}}) + u(v_{\mu_{jk}}, z\widehat{s}_j n\widehat{s}_j^{-1} v_{\mu_{jk}}). \tag{3.5.16}$$

If $n \in N_-^j$ the weight components of the vector $z\widehat{s}_j n\widehat{s}_j^{-1} v_{\mu_{jk}}$ belong to $-(w_1 \ldots w_{j-1})^{-1} s(\Delta_0 \cup \Delta_+^j) + \mu_{jk}$ by the definition of P^j, and by Lemma (1.6.22) (i) this set empty intersection with $-(w_1 \ldots w_{j-1})^{-1} \Delta^j + \mu_{jk}$. On the other hand the weight of v_{jk} is equal to $\mu_{jk} - \delta_{jk} \in -(w_1 \ldots w_{j-1})^{-1} \Delta^j + \mu_{jk}$. Since different weight subspaces are orthogonal with respect to the contravariant form, the first term in (3.5.16) vanishes in this case.

The same conclusion can be obtained if $n \in N_{(w_1 \ldots w_{j-1})^{-1}(-\Delta_s^s)}$ with the help of part (ii) of Lemma 1.6.22.

In both cases the μ_{jk}–weight component of $z\widehat{s}_j n\widehat{s}_j^{-1} v_{\mu_{jk}}$ can give a nontrivial contribution to the right hand side of (3.5.16). If $n \in N_-^j$ we note that the set of roots $-(w_1 \ldots w_{j-1})^{-1}s(\Delta_0 \cup \Delta_+^j)$ is parabolic and $-(w_1 \ldots w_{j-1})^{-1}s(\Delta_0 \cup \Delta_+^j) \cup (w_1 \ldots w_{j-1})^{-1}s(\Delta_0 \cup \Delta_+^j) = (w_1 \ldots w_{j-1})^{-1}\Delta_0$. Therefore the μ_{jk}–weight component of $z\widehat{s}_j n\widehat{s}_j^{-1} v_{\mu_{jk}}$ is equal to the μ_{jk}–weight component of $zv_{\mu_{jk}}$, and we obtain (3.5.13).

The case when $n \in N_{(w_1 \ldots w_{j-1})^{-1}(-\Delta_s^s)}$ can be treated in a similar way noting that $L^j N_{(w_1 \ldots w_{j-1})^{-1}(-\Delta_s^s)} \subset (w_1 \ldots w_{j-1})^{-1}(P)$, and the semisimple part of the parabolic subgroup $(w_1 \ldots w_{j-1})^{-1}(P)$ is L^j.

Formula (3.5.14) is established using the same arguments.

Formula (3.5.15) follows from the definition of φ_{jk}, formula (3.5.13) and (3.5.14) with $\mu = \mu_{jk}$.

(ii) Note that $Z_\pm^j = (w_1 \ldots w_{j-1})^{-1}Z_\pm w_1 \ldots w_{j-1} \subset N_\pm$ as $\Delta_{(w_1 \ldots w_{j-1})^{-1}} \cap \Delta_0 = \{\emptyset\}$ by (3.5.7). Thus for $g = n_- z_- h z_+ \widehat{s}_j n \in N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j N_-^j$, where $n_- \in N_{[-\delta_{jk}, -\delta_{jD}]}$, $z_\pm \in Z_\pm^j$, $h \in H$ and $n \in N_-^j$ one has from (3.5.14)

$$(v_\mu, g\widehat{s}_j^{-1} v_\mu) = (v_\mu, z_- h z_+ v_\mu) = (v_\mu, h v_\mu) \neq 0.$$

From (3.5.11) it follows that $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j N_-^j \subset G^j$.

By the definition $\varphi_{jk} \in \mathbb{C}[G^j] \simeq \mathbb{C}[G][\mathcal{S}_j^{-1}]$.

Now let I be the ideal in $\mathbb{C}[G^j]$ generated by the function φ_{jk} and the ideal $J^{jk^{loc}}$

From formula (3.5.15) it follows that an element $g \in N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$ belongs to $N_{[-\delta_{jk+1}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$ if and only if $\varphi_{jk}(g) = 0$. Therefore the zero locus of I coincides with the closure of $N_{[-\delta_{jk+1}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$. It remains to show that I is radical.

Indeed, let $c \in \mathbb{C}[G^p]$ be such that $c^n \in I$ for some $n \in \mathbb{N}$, $n > 0$, i.e. $c^n = f + g$, $f = \varphi_{jk}^m \rho$ for some $m \in \mathbb{N}$, $m > 0$ and $\rho \in \mathbb{C}[G^p]$, $g \in J^{jk^{loc}}$. Then the restriction of c^n to $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$ coincides with the restriction of f to $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$. In particular, the restriction of c^n to $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$ vanishes on $N_{[-\delta_{jk+1}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$. This implies that the restriction of c to $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$ must also vanish on $N_{[-\delta_{jk+1}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$.

By Lemma 3.5.3 (i) the composition of the isomorphisms (3.5.6) and (3.5.10) yield an isomorphism of varieties

$$G^j \simeq \mathbb{C}^D \times H \times \mathbb{C}^D, \tag{3.5.17}$$

and under this isomorphism the function φ_{jk} becomes the k-th coordinate function on $\mathbb{C}^D$ on the first factor in (3.5.17).

Now formula (3.5.15) and simple induction over k imply that under isomorphism (3.5.17) $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$ becomes a subset of $\mathbb{C}^D \times H \times \mathbb{C}^D$ of the form

$$\underbrace{0 \times \ldots \times 0}_{k-1 \text{ factors}} \times \mathbb{C} \times X_{j_k},$$

where the factor $\mathbb{C}$ in the last product corresponds to the k-th component in $\mathbb{C}^D$ in the first factor in the right hand side of (3.5.17), X_{j_k} is a subset of all the remaining factors in the right hand side of (3.5.17), and the function φ_{jk} becomes the coordinate function on the factor $\mathbb{C}$.

From formula (3.5.15) it follows that an element $g \in N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$ belongs to $N_{[-\delta_{jk+1}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$ if and only if $\varphi_{jk}(g) = 0$. Hence $X_{j_k} \simeq N_{[-\delta_{jk+1}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$, and under isomorphism (3.5.17) we have $N_{[-\delta_{jk+1}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j \simeq 0 \times \ldots \times 0 \times 0 \times X_{j_k}$.

We conclude that if c vanishes on $N_{[-\delta_{jk+1}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$, its restriction to $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$ coincides with the restriction to this subset of the function $\varphi_{jk}^p \psi$ for some $p \in \mathbb{N}, p > 0$ and $\psi \in \mathbb{C}[G^p]$. Thus $c = \varphi_{jk}^p(\psi + \psi') + \psi''$, where $\psi', \psi'' \in J^{jk^{loc}}$, and hence $c \in I$ by the definition of I.

We deduce that I is radical. This implies $I = J^{jk+1^{loc}}$ and completes the proof of part (ii).

Part (iii) follows from (ii) using induction over k.

$\square$

Now we come back to the description of the projection operator Π.

Proposition 3.5.6. *Let $g = nn_s z s m_s n^{-1}$ be the unique presentation (3.5.1) for an element $g \in N_- Z s M_-$. Then n can be uniquely factorized as $n = X_{-\beta_{11}}(u_{11}) \ldots X_{-\beta_{R-1n_{R-1}}}(u_{R-1n_{R-1}})$ where we assume that the root vectors $X_{-\beta_{jk}}$ used in the definition of the one–parameter subgroups are related to the root vectors $X_{-\delta_{jk}}$ used in*

the definition of the functions φ_{jk} as follows $X_{-\beta_{jk}} = \text{Ad}(w_1 \ldots w_{j-1})X_{-\delta_{jk}}$, and the numbers u_{jk} can be found inductively by the following formula

$$u_{jk} = \varphi_{jk}((w_1 \ldots w_{j-1})^{-1}g_{jk}(w_1 \ldots w_{j-1})), \tag{3.5.18}$$

where $g_{jk} = n_{jk}^{-1}gn_{jk}$, $n_{jk} = X_{-\beta_{11}}(u_{11}) \ldots X_{-\beta_{jk-1}}(u_{jk-1})$, $j = 1, \ldots, R-1$, $k = 1, \ldots, n_j$ and it is assumed that $n_{10} = 1$ and $X_{-\beta_{j0}}(u_{j0}) = X_{-\beta_{j-1n_{j-1}}}(u_{jn_{j-1}})$.

Proof. The numbers u_{jk} can be found by induction starting with u_{11}. We shall establish the induction step. The case of u_{11} corresponding to the base of the induction can be considered in a similar way.

Assume that $u_{11}, \ldots, u_{jk-1}$ have already been found. Then

$$g_{jk} = n_{jk}^{-1}gn_{jk} = X_{-\beta_{jk}}(u_{jk}) \ldots X_{-\beta_{R-1n_{R-1}}}(u_{R-1n_{R-1}})n_s zsm_s X_{-\beta_{R-1n_{R-1}}}(-u_{R-1n_{R-1}}) \ldots X_{-\beta_{jk}}(-u_{jk}).$$

Now $(w_1 \ldots w_{j-1})^{-1}g_{jk}(w_1 \ldots w_{j-1})$ has the form of g from Lemma 3.5.5 with $u = u_{jk}$. Therefore we obtain (3.5.18) by formula (3.5.15).

$\square$

For a representative $w \in G$ of a Weyl group element $w \in W$ we denote the operator on $\mathbb{C}(G)$ induced by the conjugation by w on G by the same letter,

$$(wf)(g) = f(wgw^{-1}), f \in \mathbb{C}(G).$$

Observing that in the notation of Proposition 3.5.6 for $g = nn_s zsm_s n^{-1} \in N_- ZsM_-$ we have $n^{-1}gn = n_s zsm_s$ and recalling the definition of the operator Π in (3.5.2) we infer the following theorem from Proposition 3.5.6.

Theorem 3.5.7. *Let Π_{jk} be the operator on the space of rational functions $\mathbb{C}(G)$ on G induced by conjugation by the element $\exp(-\varphi_{jk}X_{-\delta_{jk}})$,*

$$(\Pi_{jk}f)(g) = f(\exp(-\varphi_{jk}(g)X_{-\delta_{jk}})g\exp(\varphi_{jk}(g)X_{-\delta_{jk}})). \tag{3.5.19}$$

Then the restriction of the composition

$$\Pi_{11} \ldots \Pi_{1n_1} \circ w_1^{-1} \circ \Pi_{21} \ldots \Pi_{2n_2} \circ w_2^{-1} \ldots \circ w_{R-2}^{-1} \circ \Pi_{R-11} \ldots \Pi_{R-1n_{R-1}} \circ w_1 \ldots w_{R-2}$$

to $\mathbb{C}[N_- ZsM_-]$ is equal to the projection operator Π onto the subspace $\mathbb{C}[N_- ZsM_-]^{M_-}$ of M_-–invariant regular functions on $N_- ZsM_-$, $\Pi : \mathbb{C}[N_- ZsM_-] \to \mathbb{C}[N_- ZsM_-]^{M_-}$,

$$\Pi = \Pi_{11} \ldots \Pi_{1n_1} \circ w_1^{-1} \circ \Pi_{21} \ldots \Pi_{2n_2} \circ w_2^{-1} \ldots \circ w_{R-2}^{-1} \circ \Pi_{R-11} \ldots \Pi_{R-1n_{R-1}} \circ w_1 \ldots w_{R-2}. \tag{3.5.20}$$

Corollary 3.5.8. *The operator $\Pi_c : \mathbb{C}[(w_1 \ldots w_{R-2})^{-1}N_- ZsM_-(w_1 \ldots w_{R-2})] \to \mathbb{C}[N_- ZsM_-]^{M_-}$ defined by*

$$\Pi_c = \Pi_{11} \ldots \Pi_{1n_1} \circ w_1^{-1} \circ \Pi_{21} \ldots \Pi_{2n_2} \circ w_2^{-1} \ldots \circ w_{R-2}^{-1} \circ \Pi_{R-11} \ldots \Pi_{R-1n_{R-1}} = \Pi \circ (w_1 \ldots w_{R-2})^{-1}. \tag{3.5.21}$$

is surjective.

This corollary has a quantum counterpart which will be formulated and proved in the next chapter. The operator Π_c has a direct quantum analogue.

3.6 Vanishing ideals

Let $\iota : \mathbb{C}[G] \to \mathbb{C}[G^j] = \mathbb{C}[G][\mathcal{S}_j^{-1}]$ be the canonical ring homomorphism, $J^{jk} := \iota^{-1}J^{jk^{loc}}$.

By this definition

$$J^{jk} = \mathcal{V}_G(N_{[-\delta_{jk}, -\delta_{jD}]}Z_-^j HZ_+^j \widehat{s}_j M_-^j). \tag{3.6.1}$$

Lemma 3.6.1. *(i) $J^{jk^{loc}} = J^{jk}[\mathcal{S}_j^{-1}]$.*

(ii) $J^{jk^{loc}} = J^{jk'}[\mathcal{S}_j^{-1}]$, where $J^{jk'} \subset \mathbb{C}[G]$ is the ideal generated by J^{j1} and by $(v_{jm}, \cdot \widehat{s}_j^{-1}v_{\mu_{jm}})$, $m = 1, \ldots, k-1$, where we assume that $J^{j1'} = J^{j1}$.

Proof. (i) If $f \in J^{jk \, loc}$ then $f = g/g'$, $g \in \mathbb{C}[G]$, $g' \in \mathcal{S}_j$ and f vanishes on $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$. Since by Lemma 3.5.5 (ii)

$$N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j \subset G^j,$$

and by (3.5.11) g' does not vanish on G^j we obtain that g vanishes on $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j$, i.e. $g \in J^{jk}$. We deduce that $J^{jk \, loc} \subset J^{jk}[S_p^{-1}]$. The opposite inclusion is obvious.

(ii) Note that by the definition of φ_{jm} one has

$$(v_{jm}, g\widetilde{s}_j^{-1} v_{\mu_{jk}}) = \varphi_{jm}(g)(v_{\mu_{jm}}, g\widetilde{s}_j^{-1} v_{\mu_{jm}}),$$

and $(v_{\mu_{jm}}, \cdot \widetilde{s}_j^{-1} v_{\mu_{jm}}) \in \mathcal{S}_j$. Therefore the statement in part (ii) follows from Lemma 3.5.5 (iii). $\qquad \square$

Next we give several descriptions of the ideals J^{jk}. Lemma 3.5.5 (ii) and (iii), Corollary 3.5.8 and the statements of this section are the only results of this chapter which will be used in Chapter 4 for the purposes of quantization.

Proposition 3.6.2. *(i) For $j = 1, \ldots, R - 1$, $k = 1, \ldots, n_j$ let $\mathfrak{n}_{jk}, \mathfrak{z}_\pm^j, \mathfrak{m}_-^j \subset \mathfrak{g}$ be the Lie subalgebras of the subgroups $N_{[-\delta_{jk}, -\delta_{jD}]}$, $Z_\pm^j$, and M_-^j of G, respectively. Then the ideal J^{jk} consists of the matrix elements of the form $(w, \cdot v) \in \mathbb{C}[G]$, where $w, v \in V$, V is a finite-dimensional representation of $\mathfrak{g}$, and $(w, y\widehat{s}_j h z_+ x v) = 0$ for any $y \in U(\mathfrak{n}_{jk})$, $h \in U(\mathfrak{h})$, $z_+ \in U(\mathfrak{z}_+^j)$, $x \in U(\mathfrak{m}_-^j)$.*

(ii) The ideal J^{j1} is generated by the matrix elements of the form $(u, \cdot v) \in \mathbb{C}[G]$, where u is a highest weight vector in a finite-dimensional representation V of $\mathfrak{g}$, and $v \in V$ satisfies, and $(v, \widehat{s}_p z_+ x v) = 0$ for any $z_+ \in U(\mathfrak{z}_+^j)$, $x \in U(\mathfrak{m}_-^j)$.

The proof of this proposition follows from the following lemma.

Lemma 3.6.3. *Let $G_1, \ldots, G_k \subset G$ be the Lie subgroups corresponding to Lie subalgebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_k \subset \mathfrak{g}$, respectively, and $g \in G$. The following statements are true.*

(i) The ideal $\mathcal{V}_G(G_1 g G_2 \ldots G_k) \subset \mathbb{C}[G]$ consists of the matrix elements of the form $(w, \cdot v) \in \mathbb{C}[G]$, where $w, v \in V$, V is a finite-dimensional representation V of $\mathfrak{g}$, and $(w, x_1 g x_2 \ldots x_k v) = 0$ for any $x_i \in U(\mathfrak{g}_i)$, $i = 1, \ldots, k$.

(ii) If $G_1 = B_-$ the ideal $\mathcal{V}_G(G_1 g G_2 \ldots G_k) \subset \mathbb{C}[G]$ is generated by the matrix elements of the form $(u, \cdot v) \in \mathbb{C}[G]$, where u is a highest weight vector in a finite-dimensional representation V of $\mathfrak{g}$, and $v \in V$ is such that $(u, g x_2 \ldots x_k v) = 0$ for any $x_i \in U(\mathfrak{g}_i)$, $i = 2, \ldots, k$.

Proof of Proposition 3.6.2. Since s fixes all roots from Δ_0, Z_+ is generated by one-parameter subgroups corresponding to roots from $(\Delta_0)_+$, and any representative of any Weyl group element normalizes H, we have

$$H Z_+^j \widehat{s}_j = H(w_1 \ldots w_{j-1})^{-1} Z_+ s w_1 \ldots w_{j-1} = H(w_1 \ldots w_{j-1})^{-1} s Z_+ w_1 \ldots w_{j-1}) =$$

$$= (w_1 \ldots w_{j-1})^{-1} s H Z_+ w_1 \ldots w_{j-1} = H \widehat{s}_j Z_+^j = \widehat{s}_j H Z_+^j.$$

Therefore we can write

$$J^{jk} = \mathcal{V}_G(N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j H Z_+^j \widehat{s}_j M_-^j) = \mathcal{V}_G(N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j \widehat{s}_j H Z_+^j M_-^j). \qquad (3.6.2)$$

Since for $j = 1, \ldots, R - 1$, $k = 1, \ldots n_j$ $(\Delta_0)_- \subset [-\beta_{jk}, -\beta_D]$ we deduce that $(w_1 \ldots w_{j-1})^{-1}((\Delta_0)_-) \subset (w_1 \ldots w_{j-1})^{-1}([-\beta_{jk}, -\beta_D]) \subset [-\delta_{jk}, -\delta_{jD}]$, and hence $Z_-^j \subset N_{[-\delta_{jk}, -\delta_{jD}]} \subset N_-$. Thus $N_{[-\delta_{jk}, -\delta_{jD}]} Z_-^j = N_{[-\delta_{jk}, -\delta_{jD}]}$, and (3.6.2) takes the form

$$J^{jk} = \mathcal{V}_G(N_{[-\delta_{jk}, -\delta_{jD}]} \widehat{s}_j H Z_+^j M_-^j) = \mathcal{V}_G(N_{[-\delta_{jk}, -\delta_{jD}]} H \widehat{s}_j Z_+^j M_-^j). \qquad (3.6.3)$$

Now part (i) of Proposition 3.6.2 follows from part (i) of Lemma 3.6.3, with $k = 4$, $G_1 = N_{[-\delta_{jk}, -\delta_{jD}]}$, $G_2 = H$, $G_3 = Z_+^j$, $G_4 = M_-^j$, and $g = \widehat{s}_j$.

Part (ii) of Proposition 3.6.2 follows from parts (ii) of Lemma 3.6.3 with $k = 3$, $G_1 = N_{[-\delta_{jk}, -\delta_{jD}]} H = B_-$, $G_2 = Z_+^j$, $G_3 = M_-^j$, and $g = \widehat{s}_p$. $\qquad \square$

Proof of part (i) of Lemma 3.6.3.

Since for any $j = 1, \ldots, k$, $i, i_j = 1, 2, \ldots$, $r_j^i \in \mathbb{R}$, $x_j^i \in \mathfrak{g}_j$ one has

$$e^{r_1^1 x_1^1} \ldots e^{r_1^{i_1} x_1^{i_1}} g e^{r_2^1 x_2^1} \ldots e^{r_2^{i_2} x_2^{i_2}} \ldots e^{r_k^1 x_k^1} \ldots e^{r_k^{i_k} x_k^{i_k}} \in G_1 g G_2 \ldots G_k,$$

we obtain that for every element $(w, \cdot v) \in \mathcal{V}_G(G_1 g G_2 \ldots G_k)$

$$0 = \frac{d^n}{\prod_{j=1}^k \prod_{i=1}^{i_j} dr_j^i} \Big|_{r_i^j = 0} (w, e^{r_1^1 x_1^1} \ldots e^{r_1^{i_1} x_1^{i_1}} g e^{r_2^1 x_2^1} \ldots e^{r_2^{i_2} x_2^{i_2}} \ldots e^{r_k^1 x_k^1} \ldots e^{r_k^{i_k} x_k^{i_k}} v) =$$

$$= (w, x_1^1 \ldots x_1^{i_1} g x_2^1 \ldots x_2^{i_2} \ldots x_k^1 \ldots x_k^{i_k} v),$$

where $n = \sum_{j=1}^k \sum_{i=1}^{i_j} 1 = \sum_{j=1}^k \frac{(i_j+1)i_j}{2}$. Thus $(w, x_1 g x_2 \ldots x_k v) = 0$ for any $x_j \in U(\mathfrak{g}_j)$, $j = 1, \ldots, k$ of the form $x_j = x_j^1 \ldots x_j^{i_j}$ as in the previous formula. By linearity $(w, x_1 g x_2 \ldots x_k v) = 0$ for any $x_j \in U(\mathfrak{g}_j)$, $j = 1, \ldots, k$.

Conversely, let $(w, \cdot v)$ be any element of $\mathbb{C}[G]$ such that $(w, x_1 g x_2 \ldots x_k v) = 0$ for any $x_j \in U(\mathfrak{g}_j)$, $j = 1, \ldots, k$. Observe that the Lie groups G_j, $j = 1, \ldots k$ are connected, so that each G_j is generated by $\exp(\mathfrak{g}_j)$. Thus any element of $G_1 g G_2 \ldots G_k$ can be written in the form $g_1 g g_2 \ldots g_k$, where $g_j = e^{x_j^1} \ldots e^{x_j^{i_j}}$, $j = 1, \ldots, k$, $i_j \in \{1, 2, \ldots\}$, $x_j^i \in \mathfrak{g}_j$, $i = 1, \ldots, i_j$. Now we have

$$(w, g_1 g g_2 \ldots g_k v) = (w, e^{x_1^1} \ldots e^{x_1^{i_1}} g e^{x_2^1} \ldots e^{x_2^{i_2}} \ldots e^{x_k^1} \ldots e^{x_k^{i_k}} v) =$$

$$= \sum_{n_j^i \in \mathbb{N}} \frac{1}{\prod_{j=1}^k \prod_{i=1}^{i_j} n_j^i!} (w, (x_1^1)^{n_1^1} \ldots (x_1^{i_1})^{n_1^{i_1}} g (x_2^1)^{n_2^1} \ldots (x_2^{i_2})^{n_2^{i_2}} \ldots (x_k^1)^{n_k^1} \ldots (x_k^{i_k})^{n_k^{i_k}} v) = 0.$$

Thus $(w, \cdot v) \in \mathcal{V}_G(G_1 g G_2 \ldots G_k)$. This completes the proof.

$\square$

The proof of part (ii) of Lemma 3.6.3 is based on the description of closed subvarieties in $B_- \backslash G$ in terms of the so–called generalized Plücker coordinates.

Recall that matrix elements of the form $(u, \cdot v)$, where u is a highest weight vector in a finite-dimensional representation V of $\mathfrak{g}$, and $v \in V$, can be viewed as sections of line bundles on $B_- \backslash G$ (see [38], Section 3.1). They are also called generalized Plücker coordinates on $B_- \backslash G$. More precisely if V has highest weight λ then $(u, \cdot v)$ is a section of the line bundle on $B_- \backslash G$ associated to the one–dimensional representation of $B_+ = \omega(B_-)$ corresponding to λ.

Lemma 3.6.4. *Any closed subvariety in $B_- \backslash G$ is the zero locus of a finite set of generalized Plücker coordinates.*

Proof. Indeed, the flag variety $B_- \backslash G$ can be realized as the G–orbit $\mathcal{O}$ of the line $[u]$ defined by a non–zero lowest weight vector u in the projectivisation $P(V_\mu^*)$ of a finite-dimensional irreducible representation V_μ^* of G dual to a highest weight irreducible representation V_μ with a regular dominant highest weight μ (see e.g. [9], Section 2 or [46], §4). Note that $B_- \backslash G$ is a projective complete variety (see [124], Section 6.2, in particular, Lemma 6.2.2), and hence by Proposition 6.1.2 (iv) in [124] it is closed in $P(V_\mu^*)$.

If we identify V_μ with V_μ^* using the contravariant form then a highest weight vector u of V_μ becomes a lowest weight vector in V_μ^* and the class $[g] \in B_- \backslash G$ of an element $g \in G$ corresponds to $\omega(g)[u] = [\omega(g)u] \in \mathcal{O} \subset P(V_\mu^*)$.

Now let $v \in V_\mu$. Then

$$(u, gv) = (\omega(g)u, v) = (x, v), x = \omega(g)u \in V_\mu^*, \tag{3.6.4}$$

so that $[x] = [\omega(g)u] \in \mathcal{O} \subset P(V_\mu^*)$.

Any $y \in V_\mu^*$ can be written in the form $y = \sum_{n=1}^S c_n e_n \in V_\mu^*$, where e_n, $n = 1, \ldots, S$ is a weight basis of V_μ^*. The functions $\phi_n(y) = c_n$ are linear coordinates on V_μ^*. Moreover, since $V_\mu \simeq V_\mu^*$ one can find elements $v_n \in V_\mu$ such that $\phi_n(y) = (y, v_n)$. In particular, ϕ_n generate the algebra of polynomial functions on V_μ^*, and any closed subvariety in $P(V_\mu^*)$, and hence in $\mathcal{O}$, which is closed in $P(V_\mu^*)$, is the zero locus of a finite collection of some polynomials homogeneous in ϕ_n (see [50], Ch. 1, §2).

If $f(\phi_1, \ldots \phi_S)$ is such a polynomial of degree d then the equation $f(\phi_1(y), \ldots \phi_S(y)) = 0$ is well–defined in $P(V_\mu^*)$, i.e. if $y \in V_\mu^*$ is its solution then any element of the line $[y] \in P(V_\mu^*)$ is also its solution. Now using (3.6.4), the definition of ϕ_n and the homogeneity of f we deduce that for $[x] \in \mathcal{O}$ $f(\phi_1(x), \ldots \phi_S(x)) = 0$ if and only if $f((u, gv_1), \ldots, (u, gv_S)) = 0$ for any $g \in G$, where $[g] \in B_- \backslash G$ corresponds to $[x]$ under the isomorphism $\mathcal{O} \simeq B_- \backslash G$.

Note also that using algebraic rules for matrix elements we immediately obtain that $f((u, \cdot v_1), \ldots, (u, \cdot v_S))$ is a generalized Plücker coordinate defined using the representation $V_\mu^{\otimes d}$ and the highest weight vector

$$u^{\otimes d} = \underbrace{u \otimes \ldots \otimes u}_{d \text{ times}} \in V_\mu^{\otimes d},$$

i.e. $f((u, \cdot v_1), \ldots, (u, \cdot v_S)) = (u^{\otimes d}, \cdot v)$ for some $v \in V_\mu^{\otimes d}$. The last two facts imply that any closed subvariety in $B_- \backslash G$ is the zero locus of a finite set of generalized Plücker coordinates.

$\square$

Proof of part (ii) of Lemma 3.6.3.

Observe that the projection $p : G \to B_- \backslash G$ is an open map by Theorem 5.5.5 in [124], so that for any $U \subset B_- \backslash G$ $p^{-1}(\overline{U}) = \overline{p^{-1}(U)}$. In particular,

$$p^{-1}(\overline{B_- \backslash B_- g G_2 \ldots G_k}) = \overline{p^{-1}(B_- \backslash B_- g G_2 \ldots G_k)} = \overline{B_- g G_2 \ldots G_k}. \qquad (3.6.5)$$

By Lemma 3.6.4 the closure $\overline{B_- \backslash B_- g G_2 \ldots G_k}$ is the zero locus of a finite set of generalized Plücker coordinates. Therefore by (3.6.5) $h \in \overline{B_- g G_2 \ldots G_k}$ if and only if all generalized Plücker coordinates from this set vanish on the class $[h]$ of h in $B_- \backslash G$. But if $(u, \cdot v)$ is the matrix element corresponding to one of these Plücker coordinates then by the definition this Plücker coordinate vanishes on $[h] \in B_- \backslash G$ if and only if $(u, hv) = 0$. Therefore the matrix elements $(u, \cdot v)$ corresponding to the Plücker coordinates the common zero locus of which is $\overline{B_- \backslash B_- g G_2 \ldots G_k}$ generate $\mathcal{V}_G(\overline{B_- g G_2 \ldots G_k}) = \mathcal{V}_G(B_- g G_2 \ldots G_k)$.

If $(u, \cdot v) \in \mathcal{V}_G(B_- g G_2 \ldots G_k)$, where u is a highest weight vector in a finite-dimensional representation V of $\mathfrak{g}$, and $v \in V$ then by part (i) of Lemma 3.6.3 $(u, x_1 g x_2 \ldots x_k v) = 0$ for any $x_i \in U(\mathfrak{g}_i)$, $i = 1, \ldots, k$, where $\mathfrak{g}_1 = \mathfrak{b}_-$. Since $\omega(U(\mathfrak{b}_-))v = \omega(U(\mathfrak{b}_+))v = \mathbb{C}v$ the last condition is equivalent to $(u, g x_2 \ldots x_k v) = 0$ for any $x_i \in U(\mathfrak{g}_i)$, $i = 2, \ldots, k$. Thus the ideal $\mathcal{V}_G(B_- g G_2 \ldots G_k) \subset \mathbb{C}[G]$ is generated by the matrix elements of the form $(u, \cdot v) \in \mathbb{C}[G]$, where u is a highest weight vector in a finite-dimensional representation V of $\mathfrak{g}$, and $v \in V$ satisfies $(u, g x_2 \ldots x_k v) = 0$ for any $x_i \in U(\mathfrak{g}_i)$, $i = 2, \ldots, k$. This completes the proof.

$\square$

3.7 Bibliographic comments

The results on Poisson–Lie groups used in this book can be found in [20], [32], [100], [106].

Proposition 3.1.1 is stated in [20] as Theorem 1.3.2 and Proposition 3.1.2 and the relevant properties of classical r-matrices can be found in [8], [104].

The result stated in Proposition 3.2.1 can be found in [106], Section 2.

Q-W–algebras for realizations of quantum groups associated to Weyl group elements were introduced in [110, 111] in the case of Coxeter elements and in [114] in the general situation. However, in the definitions given in those papers other forms of the quantum group are used. The definition of q-W–algebras in this book is more close to the one given in [116]; it uses the Ad locally finite part of the quantum group (see [59], Chapter 7) which reduces to the algebra of regular functions on G when $q = 1$. However, in this book we define all algebras over slightly different rings.

The exposition in Sections 3.2 and 3.4 follows [114, 116] with some appropriate modifications.

The presentation of the results on Poisson reduction in Section 3.3 is close to [112], Section 2.3. More details on the notion and the properties of dual pairs and Poisson reduction can be found in [105], and for statements related to the moment map for Poisson–Lie group actions the reader is referred to [79].

The original definition of the Poisson algebras $W^s(G)$ using the classical Poisson reduction only was given in [113].

The definition of the classical Zhelobenko type operator Π in Section 3.5 is a modified version of the definition given in [119].

Chapter 4

Zhelobenko type operators for q-W–algebras

In this chapter we define a quantum analogue Π_c^q of the operator Π_c and apply it to describe q-W–algebras. Observe that the operator Π is defined using the conjugation action and operators of multiplication by the functions φ_{jk}. The conjugation action has a natural quantum group analogue, the adjoint action. But multiplication by functions in $\mathbb{C}[G]$ is quite far from the multiplication in the algebra $\mathbb{C}_{\mathcal{B}}^s[G_*]$ which is used in the definition of q-W–algebras. However, using isomorphism (3.2.14) of Ad–modules $\mathbb{C}_{\mathcal{B}}^s[G_*]$ and $\mathbb{C}_{\mathcal{B}}^s[G]$ we can try to describe q-W–algebras in terms of the space $\mathbb{C}_{\mathcal{B}}^s[G]$ multiplication in which is more closely related to that of $\mathbb{C}[G]$. Therefore it is natural to expect that a quantum analogue of the operator Π_c, if it exists at all, should be defined in terms of the adjoint action and of operators of multiplication in $\mathbb{C}_{\mathcal{B}}^s[G]$ using appropriate quantum analogues of formulas (3.5.19) and (3.5.21). We shall see that this conjecture is almost correct. In fact $\mathbb{C}_{\mathcal{B}}^s[G]$ should be replaced with a certain localization. More precisely, recall that the operator Π_c is defined using the functions φ_{jk} given by (3.5.9). Natural analogues of matrix elements which appear in formula (3.5.9) can be defined. But formula (3.5.9) contains some artificial denominators zeroes of which do not correspond to any singularities of the functions φ_{jk}, which are in fact regular, in the formula for Π_c. It turns out that in the quantum case formulas similar to (3.5.9) make sense but the denominators in them are not canceled in the formula for Π_c^q, and we are forced to use localizations containing all such denominators. This will also force us to replace the algebra $W_{\mathcal{B}}^s(G)$ with a certain localization $W_{\mathcal{B}}^{s,loc}(G)$ of it.

The main difficulty in defining a quantum analogue Π_c^q of the operator Π_c is that the proof of the fact that the operator defined by (3.5.20) is a projection operator onto $W^s(G)$ is based on isomorphism (3.4.32) a quantum counterpart of which does not make sense. Recall that $W_{\mathcal{B}}^s(G)$ is the space of invariants with respect to the adjoint action of $\mathbb{C}_{\mathcal{B}}^s[M_+]$ on $Q_{\mathcal{B}}$. Although quantum analogues of operators (3.5.19) can be defined the proof of the fact that their composition similar to (3.5.21) is an with the image being the localization $W_{\mathcal{B}}^{s,loc}(G)$ of $W_{\mathcal{B}}^s(G)$ should only use the algebra structure of $\mathbb{C}_{\mathcal{B}}^s[G]$, the properties of the adjoint action of $\mathbb{C}_{\mathcal{B}}^s[M_+]$ on $Q_{\mathcal{B}}$, and the structure of $Q_{\mathcal{B}}$. These are the only technical tools in our disposal.

Thus our first task is to describe in terms of $\mathbb{C}_{\mathcal{B}}^s[G]$ the $\mathbb{C}_{\mathcal{B}}^s[M_+]$–module $Q_{\mathcal{B}}$ originally defined using $\mathbb{C}_{\mathcal{B}}^s[G_*]$. In the classical case this would correspond to describing the vanishing ideal of the closed subvariety $N_-ZsM_- \simeq N_-ZsM_-^s \subset G$ as by Lemma 3.4.1 and by Theorem 3.4.5 $Q_{\mathcal{B}}/(q^{\frac{1}{dr^2}}-1)Q_{\mathcal{B}} \simeq \mathbb{C}[N_-ZsM_-^s]$. It turns out that not all elements of $\mathbb{C}[G]$ generating the vanishing ideal of $N_-ZsM_-^s$ have nice quantum counterparts in $\mathbb{C}_{\mathcal{B}}^s[G]$. Recall that $\mathbb{C}[G]$ is $P \times P$–graded via the left and the right regular action of H on G. The subvariety $N_-ZsM_-^s \subset G$ is closed and some generators of its vanishing ideal belong to the graded components and some do not. It turns out that at least some of the generators of the latter type have no nice quantum counterparts. But for our purposes it suffices to replace $N_-ZsM_-^s$ with a larger set $N_-LsM_-^s$ the vanishing ideal of which has a nice quantum counterpart $I_{\mathcal{B}}^{11}$ in $\mathbb{C}_{\mathcal{B}}^s[G]$. This counterpart is described in Proposition 4.1.2 and its image under the natural map $\mathbb{C}_{\mathcal{B}}^s[G] \simeq \mathbb{C}_{\mathcal{B}}^s[G_*] \to Q_{\mathcal{B}}$ is zero.

After recollecting some facts on the algebra $\mathbb{C}_{\mathcal{B}}^s[G]$ and on the adjoint action in Section 4.2 we study properties of $I_{\mathcal{B}}^{11}$ in Section 4.3.

In order to show that Π_c^q is an operator with the image $W_{\mathcal{B}}^{s,loc}(G)$ we shall need some relations which resemble relations in the algebras $\mathbb{C}[G^j]/J^{jk^{loc}}$.

In Section 4.5 we introduce the localizations mentioned above and study their properties and the relevant

properties of the adjoint action. The results obtained in Sections 4.1, 4.2, 4.3, 4.4 and 4.5 are prerequisites for the study of the properties of the quantum analogues P_{jk} of the operators Π_{jk} and of their compositions in Section 4.6, the main properties being summarized in Proposition 4.6.1. In Proposition 4.6.7 we also define quantum analogues of monomials in variables φ_{jk} which play a crucial role in the study of equivariant modules over a quantum group and in the proof of the De Concini–Kac–Procesi conjecture.

Finally in Section 4.7 we prove that the image of the operator Π_c^q almost coincides with the localization $W_{\mathcal{B}}^{s,loc}(G)$ of the algebra $W_{\mathcal{B}}^s(G)$.

4.1 A quantum analogue of the level surface of the moment map for q-W–algebras

In this section we describe a quantum counterpart $I_{\mathcal{B}}^{11} \subset \mathbb{C}_{\mathcal{B}}^s[G]$ of the vanishing ideal J^{11}. As we mentioned in the introduction to this chapter isomorphism (3.2.14) of Ad_s–modules $\mathbb{C}_{\mathcal{B}}^s[G_*]$ and $\mathbb{C}_{\mathcal{B}}^s[G]$ plays a central role in the description of $I_{\mathcal{B}}^{11}$. Note that since both ω_0 and the antipode S_s are algebra antiautomorphisms, the compositions $\omega_0 S_s^{-1}$ and $S_s\omega_0$ are algebra automorphisms. For technical reasons we shall replace the adjoint action of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ on $\mathbb{C}_{\mathcal{B}}^s[G]$ with the twisted adjoint action defined by

$$(\mathrm{Ad}_s^0 x f)(w) = f(\omega_0 S_s^{-1}(\mathrm{Ad}_s' x(S_s\omega_0 w))), \tag{4.1.1}$$

where $f \in \mathbb{C}_{\mathcal{B}}^s[G], x, w \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$. Since ω_0 is an algebra antiautomorphism and a coalgebra automorphism we can also write

$$(\mathrm{Ad}_s^0 x f)(w) = f((\omega_0 S_s^{-1})(x^1) w \omega_0(x^2)) = f(S_s(\omega_0 x^1) w \omega_0(x^2)) = f(\mathrm{Ad}_s \omega_0(x)(w)). \tag{4.1.2}$$

Consider isomorphism (3.2.14) twisted by the automorphism $\omega_0 S_s^{-1}$,

$$\varphi : \mathbb{C}_{\mathcal{B}}^s[G] \to \mathbb{C}_{\mathcal{B}}^s[G_*], f \mapsto (id \otimes f)(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s). \tag{4.1.3}$$

If $\kappa = 1$, by the definition of $Q_{\mathcal{B}} = \rho_{\chi_q^s}(\mathbb{C}_{\mathcal{B}}^s[G_*])$, where $\rho_{\chi_q^s} : \mathbb{C}_{\mathcal{B}}^s[G^*] \to \mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}} := Q_{\mathcal{B}}'$ is the canonical projection, and by Lemma 3.2.11 φ induces a homomorphism of $\mathbb{C}_{\mathcal{B}}^s[M_+]$–modules

$$\phi : \mathbb{C}_{\mathcal{B}}^s[G] \to Q_{\mathcal{B}}, \phi(f) = \varphi(f)1, \tag{4.1.4}$$

where $\mathbb{C}_{\mathcal{B}}^s[G]$ is equipped with the restriction of action (4.1.1) to $\mathbb{C}_{\mathcal{B}}^s[M_+]$, $Q_{\mathcal{B}}$ with the action induced by the adjoint action Ad of $\mathbb{C}_{\mathcal{B}}^s[M_+]$, and 1 is the image of $1 \in \mathbb{C}_{\mathcal{B}}^s[G_*]$ in $Q_{\mathcal{B}}$ under $\rho_{\chi_q^s}$.

Now we make some preparations to state a quantum counterpart of Proposition 3.4.3. The proof of Proposition 3.4.3 was based on the Chevalley commutation relations between one–parameter subgroups in G as described in Lemma 3.4.4 and on formula (3.4.9) for representatives of Weyl group elements in G. In the quantum case instead of the Chevalley commutation relations we have commutation relations between quantum root vectors, and the Weyl group is replaced with the corresponding braid group generators of which are also expressed in terms on generators of the quantum group by formula (2.2.4). However, in the quantum case the generators of the braid group do not square to identity automorphisms of the quantum group and we are only allowed to use the braid group relations. The action of the braid group on quantum root vectors is also very difficult to control. It is much more complicated than the conjugation action of representatives in G of Weyl group elements on root vectors in $\mathfrak{g}$. All this brings additional complications to the proof of Proposition 4.1.2 below which is a quantum counterpart of Proposition 3.4.3.

As before we assume that a Weyl group element $s \in W$ and a normally ordered system of positive roots Δ_+ associated to s are fixed as in Definition 1.6.19, and denote by $\beta_1, \ldots, \beta_D$ the ordered roots in Δ_+. So if $\alpha_1, \ldots, \alpha_l$ are the simple roots in Δ_+ and $\overline{w} = s_{i_1} \ldots s_{i_D}$ the corresponding decomposition of the longest element $\overline{w} \in W$ then

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \ldots, \beta_D = s_{i_1} \ldots s_{i_{D-1}}\alpha_{i_D}.$$

Unless explicitly stated otherwise, we shall assume that all quantum root vectors are defined using this normal ordering of Δ_+.

Let $U_q^{res}(w'(\mathfrak{b}_+)) = U_{U_q^{res}(H)}^{res}([\beta_{k_{l'}+1}, -\beta_{k_{l'}}])$ be the subalgebra in $U_q^{res}(\mathfrak{g})$ generated by the elements

$$(X_{\beta_1}^-)^{(n_1)}, \ldots, (X_{\beta_{k_{l'}}}^-)^{(n_{k_{l'}})}, (X_{\beta_{k_{l'}+1}}^+)^{(n_{k_{l'}+1})}, \ldots, (X_{\beta_D}^+)^{(n_D)}, n_i \in \mathbb{N}, i = 1, \ldots, D,$$

where $\beta_{k_{l'}} = \gamma_{l'}$, and by $U_q^{res}(H)$. The reason for the use of the symbol $U_q^{res}(w'(\mathfrak{b}_+))$ for this subalgebra will be clear later from the definition of $w' \in W$ in formula (4.1.73) and Remark 4.1.10.

Below we denote the multiplication in the algebra $\mathbb{C}_{\mathcal{B}}^s[G]$ by $\otimes$. We recall the notation introduced in Section 3.2 for elements of $\mathbb{C}_{\mathcal{B}}^s[G]$. Let V^{res} be a $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$–lattice in a finite rank $U_h(\mathfrak{g})$–module V. Recall that there is a contravariant non–degenerate form $(\cdot, \cdot)$ on V such that $(u, xv) = (\omega(x)u, v)$ for any $u, v \in V$, $x \in U_h(\mathfrak{g})$. Assume that u is such that $(u, w) \in \mathcal{B}$ for any $w \in V^{res}$. Then $(u, \cdot)$ is an element of the dual module V^{res*}. Since V and V^{res} are of finite ranks and $(\cdot, \cdot)$ is non–degenerate all elements of V^{res*} can be obtained this way. Clearly, for any $v \in V^{res}$ $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}^s[G]$, and by the definition $\mathbb{C}_{\mathcal{B}}^s[G]$ is generated by such elements.

Let $c_\beta \in \mathcal{B}$, $\beta \in \Delta_{\mathfrak{m}_+}$ be elements such that

$$c_\beta = \begin{cases} k_i \in \mathcal{B} & \text{if } \beta = \gamma_i, i = 1, \ldots, l' \\ 0 & \text{otherwise} \end{cases} .$$

As we observed in Proposition 3.2.7 (ii) the elements $\tilde{e}_{\beta_1}^{n_1} \ldots \tilde{e}_{\beta_D}^{n_D} V_i \tilde{f}_{\beta_D}^{m_D} \ldots \tilde{f}_{\beta_1}^{m_1}$ with $n_j, m_j, i \in \mathbb{N}$, $j = 1, \ldots, D$ form a $\mathcal{B}$–basis in $\mathbb{C}_{\mathcal{B}}^s[G^*]$.

Clearly, the elements $\tilde{e}_{\beta_1}^{n_1} \ldots \tilde{e}_{\beta_D}^{n_D} V_i \tilde{f}_{\beta_D}^{m_D} \ldots \tilde{f}_{\beta_{c+1}}^{m_{c+1}} (\tilde{f}_{\beta_c} - c_{\beta_c})^{m_c} \ldots (\tilde{f}_{\beta_1} - c_{\beta_1})^{m_1}$ with $n_j, m_j, i \in \mathbb{N}$, $j = 1, \ldots, D$ also form a $\mathcal{B}$–basis in $\mathbb{C}_{\mathcal{B}}^s[G^*]$. Let $I_{\mathcal{B}}^{\mathbf{k}}$ be the $\mathcal{B}$–submodule in $\mathbb{C}_{\mathcal{B}}^s[G^*]$ generated by the elements

$$\tilde{e}_{\beta_1}^{n_1} \ldots \tilde{e}_{\beta_D}^{n_D} V_i \tilde{f}_{\beta_D}^{m_D} \ldots \tilde{f}_{\beta_{c+1}}^{m_{c+1}} (\tilde{f}_{\beta_c} - c_{\beta_c})^{m_c} \ldots (\tilde{f}_{\beta_1} - c_{\beta_1})^{m_1}$$

with $n_j, m_j, i \in \mathbb{N}$, $j = 1, \ldots, D$, and where at least one $m_j > 0$ for $j < c + 1$. Since these elements are linearly independent they form a $\mathcal{B}$–basis in $I_{\mathcal{B}}^{\mathbf{k}}$.

Proposition 4.1.1. *Let $J_{\mathcal{B}}^{11'}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}^s[G]$ generated by the elements $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}^s[G]$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V^{res}$ is such that $(u, T_s xv) = 0$ for any $x \in U_q^{res}(w'(\mathfrak{b}_+))$. Denote $I_{\mathcal{B}}^{11} = (J_{\mathcal{B}}^{11'} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}_{\mathcal{B}}^s[G]$. Let $Q_{\mathcal{B}}^{\mathbf{k}}$ be the image of $\mathbb{C}_{\mathcal{B}}^s[G_*] \subset \mathbb{C}_{\mathcal{B}}^s[G^*]$ under the canonical projection $\mathbb{C}_{\mathcal{B}}^s[G^*] \to \mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}^{\mathbf{k}}$. Denote by $1 \in Q_{\mathcal{B}}^{\mathbf{k}}$ the image of $1 \in \mathbb{C}_{\mathcal{B}}^s[G^*]$ in $\mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}^{\mathbf{k}}$. Then the following statements are true.*

(i) $\varphi(J_{\mathcal{B}}^{11'}) \subset I_{\mathcal{B}}^{\mathbf{k}} \cap \mathbb{C}_{\mathcal{B}}^s[G_*]$ and $\varphi(I_{\mathcal{B}}^{11}) \subset I_{\mathcal{B}}^{\mathbf{k}} \cap \mathbb{C}_{\mathcal{B}}^s[G_*]$.

(ii) If u is a highest weight vector in a finite rank indecomposable representation V_λ of $U_h(\mathfrak{g})$ of highest weight λ such that $(u, u) = 1$ then for any $f \in \mathbb{C}_{\mathcal{B}}^s[G]$ we have

$$\varphi(f \otimes (u, \cdot T_s^{-1} u))1 = c_\lambda \varphi(\mathrm{Ad}_s^0(q^{-(\kappa \frac{1+s}{1-s} P_{\mathfrak{b}'} + id)\lambda^\vee})(f))q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{b}'})\lambda^\vee}1 = \tag{4.1.5}$$

$$= c_\lambda q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{b}'})\lambda^\vee} \varphi(\mathrm{Ad}_s^0(q^{(-\kappa \frac{1+s}{1-s} s^{-1} P_{\mathfrak{b}'} + s^{-1})\lambda^\vee})(f))1 \in Q_{\mathcal{B}}^{\mathbf{k}},$$

where $c_\lambda = c \prod_{i=1}^{l'} k_i^{n_i}$, $c \in \mathcal{B}^*$ is an invertible element of $\mathcal{B}$ which only depends on λ, $\gamma_1, \ldots \gamma_{l'}$, and $n_i = \lambda^\vee(\gamma_i) \geq 0$ for $i = 1, \ldots, n$, $n_i = \lambda^\vee(s^1 \gamma_i) \geq 0$ for $i = n + 1, \ldots, l'$. The classes in the quotient $\mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}^{\mathbf{k}}$ of the elements of $\mathbb{C}_{\mathcal{B}}^s[G^*]$ in the right hand side of (4.1.5) belong to $Q_{\mathcal{B}}^{\mathbf{k}} \subset \mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}^{\mathbf{k}}$.

In particular,

$$\varphi((u, \cdot T_s^{-1} u))1 = c_\lambda q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{b}'})\lambda^\vee} 1 \in Q_{\mathcal{B}}^{\mathbf{k}}.$$

$q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{b}'})\lambda^\vee} 1$ should be understood as the class of the element $q^{(s^{-1}+id)(id - \kappa P_{\mathfrak{b}'})\lambda^\vee} \in \mathbb{C}_{\mathcal{B}}^s[G^*]$ in the quotient $\mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}^{\mathbf{k}}$. This class belongs to $Q_{\mathcal{B}}^{\mathbf{k}}$.

Proof. The proof of this proposition is based on Lemma 4.1.8 which will be proved in the end of this section.

(i) We start proving this proposition by obtaining a useful expression for $(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s)$. In order to do that we recall some properties of universal R-matrices (see (2.5.4), (2.5.5)),

$$(S_s \otimes id)\mathcal{R}^s = (id \otimes S_s^{-1})\mathcal{R}^s = \mathcal{R}^{s-1}, (S_s \otimes S_s)\mathcal{R}^s = \mathcal{R}^s, \tag{4.1.6}$$

Using the first identity above we can write

$$\mathcal{R}_{21}^s \mathcal{R}^s = \mathcal{R}_{21}^s (id \otimes S_s)(\mathcal{R}^{s-1}) = (id \otimes S_s^{-1})(\mathcal{R}_{21}^{s\ -1})(id \otimes S_s)(\mathcal{R}^{s-1}) = (id \otimes S_s)((id \otimes S_s^{-2})(\mathcal{R}_{21}^{s\ -1}) \circ \mathcal{R}^{s-1}),$$

where

$$(a \otimes b) \circ (c \otimes d) = ac \otimes db.$$

Now since ω_0 an algebra antiautomorphism we have

$$(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s) = (id \otimes \omega_0)((id \otimes S_s^{-2})(\mathcal{R}_{21}^{s\,-1}) \circ \mathcal{R}^{s-1}) = (id \otimes \omega_0 S_s^{-2})(\mathcal{R}_{21}^{s\,-1})(id \otimes \omega_0)(\mathcal{R}^{s-1}). \quad (4.1.7)$$

Recalling the definition (2.6.16) of $\mathcal{R}^s$ and (2.8.14), (2.8.15) we obtain

$$\mathcal{R}^{s-1} = \exp\left[-h(\sum_{i=1}^{l} Y_i \otimes H_i - \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i)\right] \times \qquad (4.1.8)$$
$$\times \prod_{\beta \in \Delta_+} \exp_{q_\beta^{-1}}[(1-q_\beta^2)f_\beta \otimes e_\beta e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee}] =$$

$$= \prod_{\beta \in \Delta_+} \exp_{q_\beta^{-1}}[(1-q_\beta^2)e^{h(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}-id)\beta^\vee} f_\beta \otimes e_\beta q^{\beta^\vee}] \times$$
$$\times \exp\left[-h(\sum_{i=1}^{l} Y_i \otimes H_i - \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i)\right],$$

where the order of the terms in the products over the positive roots is opposite to the normal ordering in Δ_+.

Let ρ be a half of the sum of the positive roots. Using the fact that

$$S_s^{-2} = \mathrm{Ad}_s \ q^{2\rho^\vee},$$

which can be derived straightforwardly from the definition (2.6.14) of S_s, we also deduce

$$(id \otimes S_s^{-2})(\mathcal{R}_{21}^{s\,-1}) = \prod_{\beta \in \Delta_+} \exp_{q_\beta^{-1}}[(1-q_\beta^2)q^{-2\beta(\rho^\vee)}e_\beta q^{\beta^\vee} \otimes e^{h(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}-id)\beta^\vee} f_\beta] \times \qquad (4.1.9)$$
$$\exp\left[-h(\sum_{i=1}^{l}(Y_i \otimes H_i) + \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i)\right].$$

The order of the terms in the products in the formulas above is such that the α–term appears to the left of the β–term if $\alpha > \beta$ with respect to the normal ordering of Δ_+.

Substituting (4.1.8) and (4.1.9) into (4.1.7) we arrive at the following expression for $(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s)$

$$(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s) = \overset{\leftarrow}{\underset{\rightarrow}{\prod}} \exp_{q_\beta^{-1}}[(1-q_\beta^2)q^{-2\beta(\rho^\vee)}e_\beta q^{\beta^\vee} \otimes \omega_0(e^{h(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}-id)\beta^\vee} f_\beta)] \times$$
$$\times \exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i^f) + \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i^f)\right] \times \qquad (4.1.10)$$
$$\times \exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i^r) - \sum_{i=1}^{l} \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i^r)\right] \times$$
$$\times \overset{\leftarrow}{\underset{\rightarrow}{\prod}} \exp_{q_\beta^{-1}}[(1-q_\beta^2)f_\beta \otimes \omega_0(e_\beta e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee})],$$

where in the product

$$\overset{\leftarrow}{\underset{\rightarrow}{\prod}}$$

the upper (the lower) arrow indicates the order of the terms in the first (the second) factor of the tensor product relative to the normal ordering of Δ_+, and superscripts f (resp. r) indicate that the corresponding term appears in the front (resp. in the rear) of all the other terms in the product.

Assume now that $u \in V$ has highest weight λ and $v \in V^{res}$ is any vector of weight μ such that $g(\cdot) := (u, \cdot v) \in \mathbb{C}_\mathcal{B}^s[G]$. Observe that by (2.3.13) and (2.8.11) all elements $z_\beta = \omega_0(e^{h(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}-id)\beta^\vee} f_\beta)$ in the first product of the q-exponentials in (4.1.10) have strictly negative weights. As u has the highest possible weight in V, we deduce by

Lemma 2.8.4 (i) that only constant terms in the expansions (2.2.7) of the q-exponentials in the first product in (4.1.10) will contribute to the formula for $\varphi(g) = (id \otimes g)(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s)$. Using also the definition of the action of the generators H_i^r, H_i^f, Y_i^r, Y_i^f on u and v we obtain from (4.1.10)

$$\varphi((u, \cdot v)) = (id \otimes g)(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s) = q^{\lambda^\vee + \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\lambda^\vee + \mu^\vee - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\mu^\vee} \times$$

$$\times (id \otimes g)(\overset{\leftarrow}{\underset{\rightarrow}{\prod}} \exp_{q_\beta^{-1}}[(1 - q_\beta^2)f_\beta \otimes \omega_0(e_\beta e^{-h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee})]).$$

Using the definition (2.2.7) of the q-exponential and formulas (2.3.13) and (2.8.10) this can be rewritten as follows

$$\varphi((u, \cdot v)) = q^{\lambda^\vee + \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\lambda^\vee + \mu^\vee - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\mu^\vee} \times$$

$$\times \sum_{\substack{m_i \in \mathbb{N} \\ i = 1, \ldots D}} \bar{d}(m_1, \ldots, m_D) \tilde{f}_{\beta_D}^{m_D} \ldots \tilde{f}_{\beta_1}^{m_1}(u, e^{-m_1 h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_1^\vee} e_{\beta_1}^{(m_1)} \ldots e^{-m_D h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_D^\vee} e_{\beta_D}^{(m_D)} v), \qquad (4.1.11)$$

where $\bar{d}(m_1, \ldots, m_D) \in \mathcal{B}^*$, and the sum is finite as the elements $e_\beta^{(n)}$ act by zero on V for large enough n.

Recalling that according to the definition in Proposition 2.6.2 $e_\beta^{(n)} = (X_\beta^+)^{(n)} q^{n K_s \beta^\vee}$, where $q^{n K_s \beta^\vee} = e^{nh \frac{\kappa}{2} \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta^\vee}$, and using commutation relations (2.3.5) we can write

$$e^{-m_1 h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_1^\vee} e_{\beta_1}^{(m_1)} \ldots e^{-m_D h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_D^\vee} e_{\beta_D}^{(m_D)} =$$

$$= e^{-m_1 h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_1^\vee}(X_{\beta_1}^+)^{(m_1)} q^{m_1 K_s \beta_1^\vee} \ldots e^{-m_D h\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\beta_D^\vee}(X_{\beta_D}^+)^{(m_D)} q^{m_D K_s \beta_D^\vee} =$$

$$= b(m_1, \ldots, m_D)(X_{\beta_1}^+)^{(m_1)} \ldots (X_{\beta_D}^+)^{(m_D)} q^{-K_s(m_1 \beta_1^\vee + \ldots + m_D \beta_D^\vee)}, \qquad (4.1.12)$$

where $b(m_1, \ldots, m_D) \in \mathcal{B}^*$.

Next, since v has weight μ, we can rewrite (4.1.11) using (4.1.12) as follows

$$\varphi((u, \cdot v)) = q^{\lambda^\vee + \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\lambda^\vee + \mu^\vee - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\mu^\vee} \times$$

$$\times \sum_{\substack{m_i \in \mathbb{N} \\ i = 1, \ldots D}} \bar{d}'(m_1, \ldots, m_D) \tilde{f}_{\beta_D}^{m_D} \ldots \tilde{f}_{\beta_1}^{m_1}(u, (X_{\beta_1}^+)^{(m_1)} \ldots (X_{\beta_D}^+)^{(m_D)} v), \qquad (4.1.13)$$

where $\bar{d}'(m_1, \ldots, m_D) = \bar{d}(m_1, \ldots, m_D) b(m_1, \ldots, m_D) q^{-\mu(K_s(m_1 \beta_1^\vee + \ldots + m_D \beta_D^\vee))} \in \mathcal{B}^*$.

Now using the definitions of $I_\mathcal{B}^{\mathbf{k}}$ and of $Q_\mathcal{B}^{\mathbf{k}} = \mathrm{Im}(\mathbb{C}_\mathcal{B}^s[G_*] \to \mathbb{C}_\mathcal{B}^s[G^*]/I_\mathcal{B}^{\mathbf{k}})$ we have in $Q_\mathcal{B}^{\mathbf{k}}$

$$\varphi((u, \cdot v))1 = q^{\lambda^\vee + \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\lambda^\vee + \mu^\vee - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\mu^\vee} \times \qquad (4.1.14)$$

$$\times \sum_{\substack{n_i, m_j \in \mathbb{N} \\ i = 1, \ldots, l' \\ j = c+1, \ldots, D}} c(n_1, \ldots, n_{l'}, m_{c+1}, \ldots, m_D) \prod_{i=1}^{l'} k_i^{n_i} \tilde{f}_{\beta_D}^{m_D} \ldots \tilde{f}_{\beta_{c+1}}^{m_{c+1}}(u, (X_{\gamma_1}^+)^{(n_1)} \ldots (X_{\gamma_{l'}}^+)^{(n_{l'})}(X_{\beta_{c+1}}^+)^{(m_{c+1})} \ldots (X_{\beta_D}^+)^{(m_D)} v)1,$$

where $c(n_1, \ldots, n_{l'}, m_{c+1}, \ldots, m_D) \in \mathcal{B}^*$, and the sum is finite.

Lemma 4.1.8 applied to the products $(X_{\gamma_1}^+)^{(n_1)} \ldots (X_{\gamma_{l'}}^+)^{(n_{l'})}$ in the previous formula and the fact that by the definition of the algebra $U_q^{res}(w'(\mathfrak{b}_+))$ one has $(X_{\beta_{c+1}}^+)^{(m_{c+1})} \ldots (X_{\beta_D}^+)^{(m_D)} \in U_q^{res}(w'(\mathfrak{b}_+))$, imply that in $Q_\mathcal{B}^{\mathbf{k}}$

$$\varphi((u, \cdot v))1 = q^{\lambda^\vee + \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\lambda^\vee + \mu^\vee - \kappa \frac{1+s}{1-s} P_{\mathfrak{h}'}\mu^\vee} \times \qquad (4.1.15)$$

$$\times \sum_{\substack{n_i, m_j \in \mathbb{N} \\ i = 1, \ldots, l' \\ j = c+1, \ldots, D}} c(n_1, \ldots, n_{l'}, m_{c+1}, \ldots, m_D) \prod_{i=1}^{l'} k_i^{n_i} \tilde{f}_{\beta_D}^{m_D} \ldots \tilde{f}_{\beta_{c+1}}^{m_{c+1}}(u, T_s X(n_1, \ldots, n_{l'})(X_{\beta_{c+1}}^+)^{(m_{c+1})} \ldots (X_{\beta_D}^+)^{(m_D)} v)1 =$$

$$= \sum_i x_i(u, T_s y_i v)1,$$

where $x_i \in \mathbb{C}_\mathcal{B}^s[G_*]$, $X(n_1, \ldots, n_{l'})$, $y_i \in U_q^{res}(w'(\mathfrak{b}_+))$.

Since every element of V^{res} is the sum of its weight components a formula similar to that which appears in the last line of (4.1.15) holds for $\varphi((u,\cdot v))1$ with arbitrary $(u,\cdot v) \in \mathbb{C}^s_{\mathcal{B}}[G]$, where u is a highest weight vector.

If v is chosen in such a way that $(u, T_s x v) = 0$ for any $x \in U^{res}_q(w'(\mathfrak{b}_+))$ we deduce from (4.1.15) that $(u, T_s y_i v) = 0$. Thus $\varphi((u,\cdot v))1 = 0$ in $Q^{\mathbf{k}}_{\mathcal{B}}$. This implies $\varphi((u,\cdot v)) \in I^{\mathbf{k}}_{\mathcal{B}} \cap \mathbb{C}^s_{\mathcal{B}}[G_*]$, as $\varphi((u,\cdot v)) \in \mathbb{C}^s_{\mathcal{B}}[G_*]$ by the definition of φ.

Now using the properties (see formulas (2.5.2))

$$(\Delta_s \otimes id)\mathcal{R}^s = \mathcal{R}^s_{13}\mathcal{R}^s_{23}, (id \otimes \Delta_s)\mathcal{R}^s = \mathcal{R}^s_{13}\mathcal{R}^s_{12},$$

and the fact that ω_0 is a coalgebra automorphism (see Section 2.8) and S_s is an anti-coautomorphism we get

$$(id \otimes \Delta_s)(id \otimes \omega_0 S_s^{-1})(\mathcal{R}^s_{21}\mathcal{R}^s) = (id \otimes \omega_0 S_s^{-1} \otimes \omega_0 S_s^{-1})(id \otimes \Delta_s^{opp})(\mathcal{R}^s_{21}\mathcal{R}^s) =$$

$$= (id \otimes \omega_0 S_s^{-2} \otimes \omega_0 S_s^{-2})(\mathcal{R}^s_{31}{}^{-1}\mathcal{R}^s_{21}{}^{-1})(id \otimes \omega_0 \otimes \omega_0)(\mathcal{R}^s_{12}{}^{-1}\mathcal{R}^s_{13}{}^{-1}).$$

In the case when v has weight μ, from this identity we obtain, similarly to (4.1.14), that for any $f \in \mathbb{C}^s_{\mathcal{B}}[G]$ in $Q^{\mathbf{k}}_{\mathcal{B}}$

$$\varphi(f \otimes (u,\cdot v))1 = (id \otimes f \otimes (u,\cdot v))((id \otimes \Delta_s)(id \otimes \omega_0 S_s^{-1})(\mathcal{R}^s_{21}\mathcal{R}^s))1 = \qquad (4.1.16)$$

$$= q^{(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee}\varphi(f)q^{(-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\mu^\vee} \times$$

$$\times \sum_{\substack{n_i,\,m_j\,\in\,\mathbb{N} \\ i=1,\dots,l' \\ j=c+1,\dots,D}} c(n_1,\dots,n_{l'},m_{c+1},\dots,m_D)\prod_{i=1}^{l'} k_i^{n_i}\tilde{f}^{m_D}_{\beta_D}\dots\tilde{f}^{m_{c+1}}_{\beta_{c+1}}(u,(X^+_{\gamma_1})^{(n_1)}\dots(X^+_{\gamma_{l'}})^{(n_{l'})}(X^+_{\beta_{c+1}})^{(m_{c+1})}\dots(X^+_{\beta_D})^{(m_D)}v)1 =$$

$$= q^{(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee}\varphi(f)q^{(-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\mu^\vee} \times$$

$$\times \sum_{\substack{n_i,\,m_j\,\in\,\mathbb{N} \\ i=1,\dots,l' \\ j=c+1,\dots,D}} c(n_1,\dots,n_{l'},m_{c+1},\dots,m_D)\prod_{i=1}^{l'} k_i^{n_i}\tilde{f}^{m_D}_{\beta_D}\dots\tilde{f}^{m_{c+1}}_{\beta_{c+1}}(u,T_sX(n_1,\dots,n_{l'})(X^+_{\beta_{c+1}})^{(m_{c+1})}\dots(X^+_{\beta_D})^{(m_D)}v)1,$$

which implies, similarly to (4.1.15), that for arbitrary v

$$\varphi(f \otimes (u,\cdot v))1 = \sum_i x'_i\varphi(f)x''_i(u,T_sy'_iv)1,$$

where $x'_i, x''_i \in \mathbb{C}^s_{\mathcal{B}}[G_*]$, $y'_i \in U^{res}_q(w'(\mathfrak{b}_+))$. Hence $\varphi(f \otimes (u,\cdot v))1 = 0$ in $Q^{\mathbf{k}}_{\mathcal{B}}$ by the choice of v, i.e. $\varphi(J^{11'}_{\mathcal{B}}) \subset I^{\mathbf{k}}_{\mathcal{B}} \cap \mathbb{C}^s_{\mathcal{B}}[G_*]$ as $\varphi(f \otimes (u,\cdot v)) \in \mathbb{C}^s_{\mathcal{B}}[G_*]$ by the definition of φ.

In order to show that $\varphi(I^{11}_{\mathcal{B}}) \subset I^{\mathbf{k}}_{\mathcal{B}} \cap \mathbb{C}^s_{\mathcal{B}}[G_*]$ we naturally extend φ to and Ad–module isomorphism $\varphi : \mathbb{C}_q[G] \to \mathbb{C}_q[G_*]$, where $\mathbb{C}_q[G] = \mathbb{C}^s_{\mathcal{B}}[G] \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $\mathbb{C}_q[G_*] = \mathbb{C}^s_{\mathcal{B}}[G_*] \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$. By the definition of $I^{11}_{\mathcal{B}}$ we have $\varphi(I^{11}_{\mathcal{B}}) \subset (I^{\mathbf{k}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}^s_{\mathcal{B}}[G_*]$ as obviously $\varphi(J^{11'}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \subset I^{\mathbf{k}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$ since we already proved that $\varphi(J^{11'}_{\mathcal{B}}) \subset I^{\mathbf{k}}_{\mathcal{B}} \cap \mathbb{C}^s_{\mathcal{B}}[G_*]$, and $\varphi(\mathbb{C}^s_{\mathcal{B}}[G]) \subset \mathbb{C}^s_{\mathcal{B}}[G_*]$.

We also have $(I^{\mathbf{k}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}^s_{\mathcal{B}}[G_*] \subset (I^{\mathbf{k}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}^s_{\mathcal{B}}[G^*]$ as $\mathbb{C}^s_{\mathcal{B}}[G_*] \subset \mathbb{C}^s_{\mathcal{B}}[G^*]$.

Recall that by the definition the elements $\tilde{e}^{n_1}_{\beta_1}\dots\tilde{e}^{n_D}_{\beta_D}V_i\tilde{f}^{m_D}_{\beta_D}\dots\tilde{f}^{m_{c+1}}_{\beta_{c+1}}(\tilde{f}_{\beta_c}-c_{\beta_c})^{m_c}\dots(\tilde{f}_{\beta_1}-c_{\beta_1})^{m_1}$ with $n_j,m_j,i \in \mathbb{N}$, $j=1,\dots,D$, and where at least one $m_j > 0$ for $j < c+1$ form a $\mathcal{B}$–basis in $I^{\mathbf{k}}_{\mathcal{B}}$, and this basis can be completed to a $\mathcal{B}$–basis of $\mathbb{C}^s_{\mathcal{B}}[G^*]$ which consists of the elements $\tilde{e}^{n_1}_{\beta_1}\dots\tilde{e}^{n_D}_{\beta_D}V_i\tilde{f}^{m_D}_{\beta_D}\dots\tilde{f}^{m_{c+1}}_{\beta_{c+1}}(\tilde{f}_{\beta_c}-c_{\beta_c})^{m_c}\dots(\tilde{f}_{\beta_1}-c_{\beta_1})^{m_1}$ with $n_j,m_j,i \in \mathbb{N}$, $j=1,\dots,D$.

This implies $(I^{\mathbf{k}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}^s_{\mathcal{B}}[G^*] = I^{\mathbf{k}}_{\mathcal{B}}$, and hence $\varphi(I^{11}_{\mathcal{B}}) \subset (I^{\mathbf{k}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}^s_{\mathcal{B}}[G_*] = ((I^{\mathbf{k}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})) \cap \mathbb{C}^s_{\mathcal{B}}[G^*]) \cap \mathbb{C}^s_{\mathcal{B}}[G_*] = I^{\mathbf{k}}_{\mathcal{B}} \cap \mathbb{C}^s_{\mathcal{B}}[G_*]$. This completes the proof of part (i).

(ii) Consider formula (4.1.16) with $v = T_s^{-1}u$,

$$\varphi(f \otimes (u,\cdot T_s^{-1}u))1 = \qquad (4.1.17)$$

$$= q^{(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee}\varphi(f)q^{s^{-1}(-\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee} \times$$

$$\times \sum_{\substack{n_i,\,m_j\,\in\,\mathbb{N} \\ i=1,\dots,l' \\ j=c+1,\dots,D}} c(n_1,\dots,n_{l'},m_{c+1},\dots,m_D)\prod_{i=1}^{l'} k_i^{n_i}\tilde{f}^{m_D}_{\beta_D}\dots\tilde{f}^{m_{c+1}}_{\beta_{c+1}}(u,(X^+_{\gamma_1})^{(n_1)}\dots(X^+_{\gamma_{l'}})^{(n_{l'})}(X^+_{\beta_{c+1}})^{(m_{c+1})}\dots(X^+_{\beta_D})^{(m_D)}T_s^{-1}u)1 =$$

$$= q^{(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} \varphi(f) q^{s^{-1}(-\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} \times$$

$$\sum_{\substack{n_i, m_j \in \mathbb{N} \\ i = 1, \ldots, l' \\ j = c+1, \ldots, D}} c(n_1, \ldots, n_{l'}, m_{c+1}, \ldots, m_D) \prod_{i=1}^{l'} k_i^{n_i} \tilde{f}_{\beta_D}^{m_D} \ldots \tilde{f}_{\beta_{c+1}}^{m_{c+1}} (u, T_s X(n_1, \ldots, n_{l'})(X_{\beta_{c+1}}^+)^{(m_{c+1})} \ldots (X_{\beta_D}^+)^{(m_D)} T_s^{-1} u) 1.$$

Observe that the roots $-\beta_1, \ldots, -\beta_{k_{l'}}, \beta_{k_{l'}+1}, \ldots, \beta_D$ corresponding to the elements

$$(X_{\beta_1}^-)^{(n_1)}, \ldots, (X_{\beta_{k_{l'}}}^-)^{(n_{k_{l'}})}, (X_{\beta_{k_{l'}+1}}^+)^{(n_{k_{l'}+1})}, \ldots, (X_{\beta_D}^+)^{(n_D)},$$

which generate together with $U_q^{res}(H)$ the subalgebra $U_q^{res}(w'(\mathfrak{b}_+))$, form a minimal segment $\{\alpha \in \Delta, -\beta_1 \leq \alpha < \gamma_{l'}\}$ which is, in fact, a system of positive roots (see Figure 5). Therefore if an element of $U_q^{res}(w'(\mathfrak{b}_+))$ has a non-zero weight, the product of this element and of any other element of $U_q^{res}(w'(\mathfrak{b}_+))$ has also a non-zero weight.

On the other hand, from the expression in the last line of (4.1.17) it follows that only terms with

$$X(n_1, \ldots, n_{l'})(X_{\beta_{c+1}}^+)^{(m_{c+1})} \ldots (X_{\beta_D}^+)^{(m_D)} \in U_q^{res}(w'(\mathfrak{b}_+))$$

of zero weight may give non-trivial contributions to the right hand side of (4.1.17). Since the elements $(X_\beta^+)^{(n)}$, $\beta \in \{\alpha \in \Delta_+ : \alpha > \gamma_{l'}\} \subset \{\alpha \in \Delta, -\beta_1 \leq \alpha < \gamma_{l'}\}$ have non-zero weights if $n > 0$, only the terms with $m_{c+1} = \ldots = m_D = 0$ will give non-trivial contributions to the right hand side of (4.1.17).

Now the second expression for $\varphi(f \otimes (u, \cdot T_s^{-1} u)) 1$ in (4.1.17) yields

$$\varphi(f \otimes (u, \cdot T_s^{-1} u)) 1 = \tag{4.1.18}$$

$$= q^{(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} \varphi(f) q^{s^{-1}(-\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} \sum_{\substack{n_i \in \mathbb{N} \\ i = 1, \ldots, l'}} d(n_1, \ldots, n_{l'}) \prod_{i=1}^{l'} k_i^{n_i} (u, (X_{\gamma_1}^+)^{(n_1)} \ldots (X_{\gamma_{l'}}^+)^{(n_{l'})} T_s^{-1} u) 1,$$

where $d(n_1, \ldots, n_{l'}) \in \mathcal{B}^*$.

Now observe that since u has weight λ, and different weight spaces in V are orthogonal with respect to the contravariant form $(\cdot, \cdot)$, only elements

$$(X_{\gamma_1}^+)^{(n_1)} \ldots (X_{\gamma_{l'}}^+)^{(n_{l'})} T_s^{-1} u \in V^{res}$$

of weight λ can contribute to the right hand side of formula (4.1.18).

Because the roots $\gamma_1, \ldots, \gamma_{l'}$ are linearly independent, for $n_1, \ldots, n_{l'} \in \mathbb{N}$ the element

$$(X_{\gamma_1}^+)^{(n_1)} \ldots (X_{\gamma_{l'}}^+)^{(n_{l'})} T_s^{-1} u \in V^{res}$$

has weight λ if and only if $n_1 \gamma_1 + \ldots + \gamma_{l'} n_{l'} = \lambda - s^{-1}\lambda = \sum_{i=1}^n \lambda^\vee(\gamma_i)\gamma_i + \sum_{i=n+1}^{l'} \lambda^\vee(s^1\gamma_i)\gamma_i$, where to obtain the last expression we also used the fact that the roots $\gamma_1, \ldots \gamma_n$ are mutually orthogonal and the roots $\gamma_{n+1}, \ldots \gamma_{l'}$ are also mutually orthogonal. This implies $n_i = \lambda^\vee(\gamma_i)$, $i = 1, \ldots, n$, and $n_i = \lambda^\vee(s^1\gamma_i)$, $i = n+1, \ldots, l'$. Note that for $i = n+1, \ldots, l'$, by Proposition 1.6.6 (i) one has $s^1\gamma_i \in s^1(\Delta_{s^2}^s) \subset \Delta_+^s \setminus \Delta_{s^1}^s \subset \Delta_+$, so $n_i = \lambda^\vee(s^1\gamma_i) \geq 0$ as $\lambda \in P_+$. Also, for the last reason $n_i = \lambda^\vee(\gamma_i) \geq 0$ for $i = 1, \ldots, n$.

By Proposition B.6. from [2], in this case we have $(X_{\gamma_1}^+)^{(n_1)} \ldots (X_{\gamma_{l'}}^+)^{(n_{l'})} T_s^{-1} u = \epsilon c_0 u$, where $\epsilon \in \{\pm 1\}$ and $c_0 \in q^{\mathbb{Z}}$ only depend on $\gamma_1, \ldots, \gamma_{l'}$ and λ. Therefore formula (4.1.18) takes the form

$$\varphi(f \otimes (u, \cdot T_s^{-1} u)) 1 = c \prod_{i=1}^{l'} k_i^{n_i} q^{(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} \varphi(f) q^{s^{-1}(-\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} 1,$$

where $c \in \mathcal{B}^*$ is an invertible element of $\mathcal{B}$ which only depends on λ, $\gamma_1, \ldots \gamma_{l'}$, and $n_i = \lambda^\vee(\gamma_i)$, $i = 1, \ldots, n$, $n_i = \lambda^\vee(s^1\gamma_i)$, $i = n+1, \ldots, l'$.

The last formula can also be rewritten as follows

$$\varphi(f \otimes (u, \cdot T_s^{-1} u)) 1 = c \prod_{i=1}^{l'} k_i^{n_i} q^{\frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} \varphi(f) q^{s^{-1}(-\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} 1 =$$

$$= c\prod_{i=1}^{l'} k_i^{n_i} q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee} \mathrm{Ad}_s(q^{(-\kappa\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}+s^{-1})\lambda^\vee})(\varphi(f))1 =$$

$$= c\prod_{i=1}^{l'} k_i^{n_i} q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee} \varphi(\mathrm{Ad}_s^0(q^{(-\kappa\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}+s^{-1})\lambda^\vee})(f))1 =$$

$$= c\prod_{i=1}^{l'} k_i^{n_i} \mathrm{Ad}_s(q^{-(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(\varphi(f))q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}1 =$$

$$= c\prod_{i=1}^{l'} k_i^{n_i} \varphi(\mathrm{Ad}_s^0(q^{-(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f))q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}1.$$

In particular, in $Q_\mathcal{B}^\mathbf{k}$

$$\varphi((u, \cdot T_s^{-1}u))1 = c\prod_{i=1}^{l'} k_i^{n_i} q^{(s^{-1}+id)(id-\kappa P_{\mathfrak{h}'})\lambda^\vee}1.$$

This completes the proof.

$\square$

Now consider the case when $\kappa = 1$ and $k_i \in \mathcal{B}$, $i = 1, \ldots, l'$ are defined in (3.2.16). Recall that the elements $\tilde{e}_{\beta_1}^{n_1} \ldots \tilde{e}_{\beta_D}^{n_D} V_i \tilde{f}_{\beta_D}^{m_D} \ldots \tilde{f}_{\beta_{c+1}}^{m_{c+1}} (\tilde{f}_{\beta_c} - c_{\beta_c})^{m_c} \ldots (\tilde{f}_{\beta_1} - c_{\beta_1})^{m_1}$ with $n_j, m_j, i \in \mathbb{N}$, $j = 1, \ldots, D$ form a $\mathcal{B}$–basis in $\mathbb{C}_\mathcal{B}^s[G^*]$ and observe that when $\kappa = 1$ the elements $(\tilde{f}_{\beta_c} - c_{\beta_c})^{m_c} \ldots (\tilde{f}_{\beta_1} - c_{\beta_1})^{m_1}$ with $n_j, m_j, i \in \mathbb{N}$, $j = 1, \ldots, D$, and where at least one $m_j > 0$ for $j < c+1$, form a $\mathcal{B}$–basis of $\mathrm{Ker}\chi_q^s$. Therefore, if $\kappa = 1$, we have $I_\mathcal{B}^\mathbf{k} = I_\mathcal{B}$, and hence $Q_\mathcal{B}^\mathbf{k} = Q_\mathcal{B}$, so we can apply the previous proposition to get the following statement.

Proposition 4.1.2. *Assume that $\kappa = 1$ and $k_i \in \mathcal{B}$ are defined in (3.2.16). Then the following statements are true.*

(i) $J_\mathcal{B}^{11'}, I_\mathcal{B}^{11} \subset \mathrm{Ker}\,\phi$.

(ii) If u is a highest weight vector in a finite rank indecomposable representation V_λ of $U_h(\mathfrak{g})$ of highest weight λ such that $(u, u) = 1$ then for any $f \in \mathbb{C}_\mathcal{B}^s[G]$

$$\phi(f \otimes (u, \cdot T_s^{-1}u)) = c_\lambda \varphi(\mathrm{Ad}_s^0(q^{-(\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f))q^{2P_{\mathfrak{h}'\perp}\lambda^\vee}1 = \tag{4.1.19}$$

$$= c_\lambda q^{2P_{\mathfrak{h}'\perp}\lambda^\vee} \phi(\mathrm{Ad}_s^0(q^{(-\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}+s^{-1})\lambda^\vee})(f)) \in Q_\mathcal{B},$$

where $c_\lambda = c\prod_{i=1}^{l'} k_i^{n_i}$, $c \in \mathcal{B}^*$ is an invertible element of $\mathcal{B}$ which only depends on $\lambda, \gamma_1, \ldots \gamma_{l'}$, and $n_i = \lambda^\vee(\gamma_i) \geq 0$, $i = 1, \ldots, n$, $n_i = \lambda^\vee(s^1\gamma_i) \geq 0$, $i = n+1, \ldots, l'$. Here the classes in the quotient $\mathbb{C}_\mathcal{B}^s[G^*]/I_\mathcal{B} = Q_\mathcal{B}'$ of the elements of $\mathbb{C}_\mathcal{B}^s[G^*]$ in the right hand side of (4.1.19) belong to $Q_\mathcal{B} \subset \mathbb{C}_\mathcal{B}^s[G^*]/I_\mathcal{B} = Q_\mathcal{B}'$.

In particular,

$$\phi((u, \cdot T_s^{-1}u)) = c_\lambda q^{2P_{\mathfrak{h}'\perp}\lambda^\vee}1 \in Q_\mathcal{B}.$$

$q^{2P_{\mathfrak{h}'\perp}\lambda^\vee}1$ *should be understood as the class of the element $q^{2P_{\mathfrak{h}'\perp}\lambda^\vee} \in \mathbb{C}_\mathcal{B}^s[G^*]$ in the quotient $\mathbb{C}_\mathcal{B}^s[G^*]/I_\mathcal{B} = Q_\mathcal{B}'$. This class belongs to $Q_\mathcal{B}$.*

The rest of this section will be devoted to the proof of Lemma 4.1.8. This proof is in turn split into several other lemmas.

Lemma 4.1.3. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u, v \in V$ weight vectors. Let $\overline{w} = s_{i_1} \ldots s_{i_D}$ be a reduced decomposition of the longest element of the Weyl group W. Then for any $\beta = s_{i_1} \ldots s_{i_{k-1}}\alpha_{i_k} \in \Delta_+$ and $k \in \mathbb{N}$*

$$(u, (X_\beta^+)^{(k)}v) = \sum_{p,p'} (u, K_{p,p'}(X_\beta^-)^{(p)}(X_\beta^+)^{(p')}T_\beta v), \tag{4.1.20}$$

where the sum in the right hand side is finite, $X_\beta^\pm = T_{i_1} \ldots T_{i_{k-1}}X_{i_k}^\pm$, $K_{p,p'} \in \mathbb{C}[q, q^{-1}]$, and

$$T_\beta = T_{i_1} \ldots T_{i_{k-1}} T_{i_k}^{-1} T_{i_{k-1}}^{-1} \ldots T_{i_1}^{-1}. \tag{4.1.21}$$

If v has weight λ then identity (4.1.20) is not trivial if and only if u has weight $\lambda + k\beta$. In this case the finite sum in (4.1.20) is taken over p and p' subject to the condition $p' - p - \beta^\vee(\lambda) = k$, so all terms $K_{p,p'}(X_\beta^-)^{(p)}(X_\beta^+)^{(p')}T_\beta v$ have also weight $\lambda + k\beta$.

Proof. Assume that v is a weight vector of weight λ. Then the left hand side of (4.1.20) is not zero if u is of weight $\lambda + k\beta$ as different weight subspaces of V are orthogonal with respect to the bilinear form $(\cdot, \cdot)$, so we can assume that u has weight $\lambda + k\beta$.

Conjugating (2.2.6) by $T_{i_1} \ldots T_{i_{k-1}}$ we get

$$\exp'_{q_i}(-X_\beta^+) = \exp'_{q_i^{-1}}(-q_i X_\beta^- K_\beta^{-1}) q_i^{\frac{H_\beta(H_\beta+1)}{2}} \exp'_{q_i^{-1}}(q_i^{-1} X_\beta^+) T_\beta, \tag{4.1.22}$$

where $K_\beta = q^{\beta^\vee} = q_\beta^{H_\beta}$, $H_\beta = T_{i_1} \ldots T_{i_{k-1}} H_{i_k}$.

Evaluating this identity on the matrix element $(u, \cdot v)$ and using (2.2.3) we obtain

$$-q_i^{\frac{1}{2}k(k-1)}(u, (X_\beta^+)^{(k)}v) = (u, \sum_{p=0}^\infty q_i^{-\frac{1}{2}p(p-1)} \frac{(-q_i X_\beta^- K_\beta^{-1})^p}{[p]_{q_i}!} q_i^{\frac{H_\beta(H_\beta+1)}{2}} \sum_{p'=0}^\infty q_i^{-\frac{1}{2}p'(p'-1)} \frac{(q_i^{-1} X_\beta^+)^{p'}}{[p']_{q_i}!} v)$$

Observing that the elements H_β act on weight spaces of V by multiplication by integer numbers, that the elements $(X_\beta^-)^p$ and $(X_\beta^+)^{p'}$ map weight spaces of V to weight spaces and for large enough p and p' they act on V by zero endomorphisms, we obtain (4.1.20) for v of weight λ, where the sum in the right hand side is such that all terms $K_{p,p'}(X_\beta^-)^{(p)}(X_\beta^+)^{(p')}T_\beta v$ have weight $\lambda + k\beta$, i.e. $p' - p - \beta^\vee(\lambda) = k$, and the number of these terms is finite. This completes the proof. $\square$

Next we obtain some useful relations in the Weyl group which lead to important formulas for the action of braid group elements on quantum root vectors. Recall that according to Lemma 1.6.14 (i) s^1 is the longest element in the Weyl group $W(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ of the semisimple part $\mathfrak{m}_{s^1}$ of a Levi subalgebra of $\mathfrak{g}$, the Cartan subalgebra of $\mathfrak{m}_{s^1}$ is denoted by $\mathfrak{h}_{s^1}$.

The system of positive roots $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) := \Delta_+ \cap \Delta(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ of the root system $\Delta(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) = \Delta_{s^1}^{-1} \bigcup (-\Delta_{s^1}^{-1})$ is the set (we use the notation of (1.6.9))

$$\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) = \{\gamma_1, \ldots, \gamma_2, \ldots, \gamma_3, \ldots, \gamma_n, -\beta_{t+1}^1, \ldots, -\beta_{t+\frac{p-n}{2}}^1\} \tag{4.1.23}$$

and s^1 acts on the elements of this set by multiplication by -1. According to (1.6.9) the number of roots in $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ is equal to p. The roots in (4.1.23) are ordered as in the normal ordering of Δ_+ associated to s. With respect to this normal ordering the set (4.1.23) is the disjoint union of an initial segment and a final segment in the normal ordering of Δ_+. Therefore $\mathfrak{m}_{s^1}$ is in fact the semisimple part of a standard Levi subalgebra of $\mathfrak{g}$, for otherwise by Lemma 1.6.14 (iii) there would be some roots preceding those from the set (4.1.23) in the normal ordering of Δ_+. Thus by Lemma 1.6.14 (iii)

$$\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) = \{\alpha \in \Delta_+ : s^1\alpha \in \Delta_-\} = \{\alpha \in \Delta_+ : s^1\alpha = -\alpha\}. \tag{4.1.24}$$

This also implies that one can define the subalgebra $U_h(\mathfrak{m}_{s^1}) \subset U_h(\mathfrak{g})$ generated by the elements $X_i^\pm$ and H_i for $\alpha_i \in \Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ and its restricted specialization $U_q^{res}(\mathfrak{m}_{s^1}) \subset U_q^{res}(\mathfrak{g})$.

Let

$$s^1 = s_{i_1} \ldots s_{i_p} = s_{i_{k_1}} \ldots s_{i_{k_2}} \ldots s_{i_{k_n}} s_{i_{k_n+1}} \ldots s_{i_p}, i_{k_1} = i_1, \tag{4.1.25}$$

be the reduced decomposition of s^1 in $W(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ corresponding to (4.1.23) as described in Lemma 1.6.14 (iii), where $s_i = s_{\alpha_i}$, α_i are simple roots in $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$,

$$\gamma_m = s_{i_{k_1}} \ldots s_{i_{k_2}} \ldots s_{i_{k_m-1}} \alpha_{i_{k_m}}, \ m = 1, \ldots, n. \tag{4.1.26}$$

Since s^1 is an involution we also have the following reduced decomposition

$$s^1 = s_{i_p} \ldots s_{i_1} = s_{i_p} \ldots s_{i_{k_n+1}} s_{i_{k_n}} \ldots s_{i_{k_2}} \ldots s_{i_{k_1}}. \tag{4.1.27}$$

Let $\gamma_1 \le \beta_q \le \gamma_n$, $\beta_q = s_{i_1} \ldots s_{i_{q-1}} \alpha_{i_q}$, $q = 1, \ldots, k_n$. Note that according to (4.1.23) (see also (1.6.11))

$$k_n = p - \frac{p-n}{2} = \frac{p+n}{2}. \tag{4.1.28}$$

Since $s^1 = -1$ in $W(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$

$$s^1 = s_{i_q} \ldots s_{i_1} s^1 s_{i_1} \ldots s_{i_q} = s_{i_q} \ldots s_{i_1} s_{i_1} \ldots s_{i_p} s_{i_1} \ldots s_{i_q} = s_{i_{q+1}} \ldots s_{i_p} s_{i_1} \ldots s_{i_q},$$

and in the right hand side we obtain a reduced decomposition of s^1 as the number of simple reflections in it is equal to p, i.e. to the length of s^1.

Now by the previous formula

$$s^1 \alpha_{i_q} = -\alpha_{i_q} = s_{i_{q+1}} \ldots s_{i_p} s_{i_1} \ldots s_{i_q} \alpha_{i_q} = -s_{i_{q+1}} \ldots s_{i_p} s_{i_1} \ldots s_{i_{q-1}} \alpha_{i_q},$$

and hence

$$s_{i_{q+1}} \ldots s_{i_p} s_{i_1} \ldots s_{i_{q-1}} \alpha_{i_q} = \alpha_{i_q}. \tag{4.1.29}$$

Form the expressions $\gamma_m = s_{i_{k_1}} \ldots s_{i_{k_2}} \ldots s_{i_{k_m-1}} \alpha_{i_{k_m}}$, $m = 1, \ldots, n$ we deduce

$$s_{\gamma_1} \ldots s_{\gamma_{q-1}} = \tag{4.1.30}$$

$$= s_{i_{k_1}} \big(s_{i_{k_1}} \ldots s_{i_{k_2-1}} s_{i_{k_2}} s_{i_{k_2-1}}^{-1} \ldots s_{i_{k_1}}^{-1}\big)\big(s_{i_{k_1}} \ldots s_{i_{k_3-1}} s_{i_{k_3}} s_{i_{k_3-1}}^{-1} \ldots s_{i_{k_1}}^{-1}\big) \ldots \big(s_{i_{k_1}} \ldots s_{i_{k_2}} \ldots s_{i_{k_{q-1}-1}} s_{i_{k_{q-1}}} s_{i_{k_{q-1}-1}}^{-1} \ldots s_{i_{k_1}}^{-1}\big) =$$

$$= s_{i_{k_1+1}} \ldots s_{i_{k_2-1}} s_{i_{k_2+1}} \ldots s_{i_{k_{q-1}-1}} s_{i_{k_{q-1}}} s_{i_{k_{q-1}-1}} \ldots s_{i_{k_1}}.$$

In particular, in the right hand sides of the following formulas we have reduced decompositions

$$s^1 = s_{\gamma_1} \ldots s_{\gamma_n} = s_{i_{k_1+1}} \ldots s_{i_{k_2-1}} s_{i_{k_2+1}} \ldots s_{i_{k_n-1}} s_{i_{k_n}} s_{i_{k_n-1}} \ldots s_{i_{k_1}} \tag{4.1.31}$$

and

$$s^1 = s_{i_{k_n}} \ldots s_{i_1} s^1 s_{i_1} \ldots s_{i_{k_n}} = s_{i_{k_n}} s_{i_{k_n-1}} \ldots s_{i_{k_1}} s_{i_{k_1+1}} \ldots s_{i_{k_2-1}} s_{i_{k_2+1}} \ldots s_{i_{k_n-1}} \tag{4.1.32}$$

as by (4.1.28) the number of simple reflections in them is equal to $k_n + k_n - n = 2k_n - n = p + n - n = p$, i.e. to the length of s^1.

Multiplying (4.1.31) and (4.1.27) by $\big(s_{i_{k_n}} s_{i_{k_n-1}} \ldots s_{i_{k_1}}\big)^{-1}$ on the right we obtain the following identity for reduced decompositions

$$s_{i_{k_1+1}} \ldots s_{i_{k_2-1}} s_{i_{k_2+1}} \ldots s_{i_{k_n-1}} = s_{i_p} \ldots s_{i_{k_n+1}}. \tag{4.1.33}$$

As the roots γ_m, $m = 1, \ldots, n$ are mutually orthogonal we deduce using (4.1.30) and (4.1.26)

$$s_{\gamma_1} \ldots s_{\gamma_{q-1}} \gamma_q = s_{i_{k_1+1}} \ldots s_{i_{k_2-1}} s_{i_{k_2+1}} \ldots s_{i_{k_{q-1}-1}} s_{i_{k_{q-1}}} s_{i_{k_{q-1}-1}} \ldots s_{i_{k_1}} s_{i_{k_1}} \ldots s_{i_{k_{q-1}}} \alpha_{i_{k_q}} =$$

$$= s_{i_{k_1+1}} \ldots s_{i_{k_2-1}} s_{i_{k_2+1}} \ldots s_{i_{k_{q-1}-1}} s_{i_{k_{q-1}+1}} \ldots s_{i_{k_q-1}} \alpha_{i_{k_q}} = \gamma_q = s_{i_{k_1}} \ldots s_{i_{k_q-1}} \alpha_{i_{k_q}}. \tag{4.1.34}$$

Therefore

$$s_{i_{k_q-1}} \ldots s_{i_{k_1}} s_{i_{k_1+1}} \ldots s_{i_{k_2-1}} s_{i_{k_2+1}} \ldots s_{i_{k_{q-1}-1}} s_{i_{k_{q-1}+1}} \ldots s_{i_{k_q-1}} \alpha_{i_{k_q}} = \alpha_{i_{k_q}},$$

where

$$s_{i_{k_q-1}} \ldots s_{i_{k_1}} s_{i_{k_1+1}} \ldots s_{i_{k_2-1}} s_{i_{k_2+1}} \ldots s_{i_{k_{q-1}-1}} s_{i_{k_{q-1}+1}} \ldots s_{i_{k_q-1}}$$

is a reduced decomposition since it is a part of reduced decomposition (4.1.32).

The last two properties and (2.2.13) imply

$$T_{i_{k_q-1}} \ldots T_{i_{k_1}} T_{i_{k_1+1}} \ldots T_{i_{k_2-1}} T_{i_{k_2+1}} \ldots T_{i_{k_{q-1}-1}} T_{i_{k_{q-1}+1}} \ldots T_{i_{k_q-1}} X_{i_{k_q}}^{\pm} = X_{i_{k_q}}^{\pm},$$

or

$$T_{i_{k_1+1}} \ldots T_{i_{k_2-1}} T_{i_{k_2+1}} \ldots T_{i_{k_{q-1}-1}} T_{i_{k_{q-1}+1}} \ldots T_{i_{k_q-1}} X_{i_{k_q}}^{\pm} = T_{i_{k_1}}^{-1} \ldots T_{i_{k_q-1}}^{-1} X_{i_{k_q}}^{\pm}. \tag{4.1.35}$$

Using the definition of T_{γ_i} (see (4.1.21)) we obtain

$$T_{\gamma_1} \ldots T_{\gamma_{q-1}} = \tag{4.1.36}$$

$$T_{i_{k_1}}^{-1}\big(T_{i_{k_1}} \ldots T_{i_{k_2-1}} T_{i_{k_2}}^{-1} T_{i_{k_2-1}}^{-1} \ldots T_{i_{k_1}}^{-1}\big)\big(T_{i_{k_1}} \ldots T_{i_{k_3-1}} T_{i_{k_3}}^{-1} T_{i_{k_3-1}}^{-1} \ldots T_{i_{k_1}}^{-1}\big) \ldots \big(T_{i_{k_1}} \ldots T_{i_{k_{q-1}-1}} T_{i_{k_{q-1}}}^{-1} T_{i_{k_{q-1}-1}}^{-1} \ldots T_{i_{k_1}}^{-1}\big) =$$

$$= T_{i_{k_1+1}} \ldots T_{i_{k_2-1}} T_{i_{k_2+1}} \ldots T_{i_{k_{q-1}-1}} T_{i_{k_{q-1}}}^{-1} T_{i_{k_{q-1}-1}}^{-1} \ldots T_{i_{k_1}}^{-1}.$$

Recalling also that $X_{\gamma_q}^{\pm} = T_{i_{k_1}} \ldots T_{i_{k_q-1}} X_{i_{k_q}}^{\pm}$ and applying formula (4.1.35) we arrive at

$$T_{\gamma_1} \ldots T_{\gamma_{q-1}} X_{\gamma_q}^{\pm} = T_{i_{k_1+1}} \ldots T_{i_{k_2-1}} T_{i_{k_2+1}} \ldots T_{i_{k_{q-1}-1}} T_{i_{k_{q-1}}}^{-1} T_{i_{k_{q-1}-1}}^{-1} \ldots T_{i_{k_1}}^{-1} T_{i_{k_1}} \ldots T_{i_{k_q-1}} X_{i_{k_q}}^{\pm} =$$

$$= T_{i_{k_1+1}} \ldots T_{i_{k_2-1}} T_{i_{k_2+1}} \ldots T_{i_{k_{q-1}-1}} T_{i_{k_{q-1}+1}} \ldots T_{i_{k_q-1}} X_{i_{k_q}}^{\pm} = T_{i_{k_1}}^{-1} \ldots T_{i_{k_q-1}}^{-1} X_{i_{k_q}}^{\pm} = \overline{X}_{\gamma_q}^{\pm}. \tag{4.1.37}$$

Lemma 4.1.4. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u, v \in V$ weight vectors. Then for any $m_1, \ldots, m_n \in \mathbb{N}$ the following statements are true.*

(i)

$$(u, (X^+_{\gamma_1})^{(m_1)} \ldots (X^+_{\gamma_n})^{(m_n)} v) = (u, K(m_1, \ldots, m_n) T_{\gamma_1} \ldots T_{\gamma_n} v), \tag{4.1.38}$$

where $K(m_1, \ldots, m_n) \in U^{res}_q(\mathfrak{m}^-_{s^1}) \overline{U}^{res}_q([\gamma_1, \gamma_n])$, $U^{res}_q(\mathfrak{m}^-_{s^1}) \subset U^{res}_q(\mathfrak{g})$ is the subalgebra generated by $(X^-_\beta)^{(k)}$ for simple roots $\beta \in \Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ and $k \geq 0$. Moreover, $K(m_1, \ldots, m_n)$ belongs to a weight subspace of $U^{res}_q(\mathfrak{g})$.

(ii) If $u \in V$ is a highest weight vector then

$$(u, (X^+_{\gamma_1})^{(m_1)} \ldots (X^+_{\gamma_n})^{(m_n)} v) = (u, K'(m_1, \ldots, m_n) T_{\gamma_1} \ldots T_{\gamma_n} v), \tag{4.1.39}$$

where $K'(m_1, \ldots, m_n) \in \overline{U}^{res}_q([\gamma_1, \gamma_n])$. Moreover, $K'(m_1, \ldots, m_n)$ belongs to a weight subspace of $U^{res}_q(\mathfrak{g})$.

Proof. By Lemma 4.1.3

$$(u, (X^+_{\gamma_1})^{(m_1)} \ldots (X^+_{\gamma_n})^{(m_n)} v) =$$

$$= \sum_{\substack{c_1, \ldots, c_n \\ c'_1, \ldots, c'_n}} K^{c_1, \ldots, c_n}_{c'_1, \ldots, c'_n} (u, (X^-_{\gamma_1})^{(c_1)} (X^+_{\gamma_1})^{(c'_1)} T_{\gamma_1} (X^-_{\gamma_2})^{(c_2)} (X^+_{\gamma_2})^{(c'_2)} T_{\gamma_2} \ldots (X^-_{\gamma_n})^{(c_n)} (X^+_{\gamma_n})^{(c'_n)} T_{\gamma_n} v),$$

where the sum in the right hand side is finite and $K^{c_1, \ldots, c_n}_{c'_1, \ldots, c'_n} \in \mathbb{C}[q, q^{-1}]$.

Using (4.1.37) and observing that $X^\pm_{\gamma_1} = \overline{X}^\pm_{\gamma_1}$, as γ_1 the first simple root in the normal ordering of Δ_+ associated to s, we obtain

$$(u, (X^+_{\gamma_1})^{(k_1)} \ldots (X^+_{\gamma_n})^{(k_n)} v) = \tag{4.1.40}$$

$$= \sum_{\substack{c_1, \ldots, c_n \\ c'_1, \ldots, c'_n}} K^{c_1, \ldots, c_n}_{c'_1, \ldots, c'_n} (u, (\overline{X}^-_{\gamma_1})^{(c_1)} (\overline{X}^+_{\gamma_1})^{(c'_1)} (\overline{X}^-_{\gamma_2})^{(c_2)} (\overline{X}^+_{\gamma_2})^{(c'_2)} \ldots (\overline{X}^-_{\gamma_n})^{(c_n)} (\overline{X}^+_{\gamma_n})^{(c'_n)} T_{\gamma_1} \ldots T_{\gamma_n} v).$$

Now to justify (4.1.38) we show that all monomials

$$(\overline{X}^-_{\gamma_1})^{(c_1)} (\overline{X}^+_{\gamma_1})^{(c'_1)} (\overline{X}^-_{\gamma_2})^{(c_2)} (\overline{X}^+_{\gamma_2})^{(c'_2)} \ldots (\overline{X}^-_{\gamma_n})^{(c_n)} (\overline{X}^+_{\gamma_n})^{(c'_n)} \tag{4.1.41}$$

belong to $U^{res}_q(\mathfrak{m}^-_{s^1}) \overline{U}^{res}_q([\gamma_1, \gamma_n]) U^{res}_q(H)$.

Indeed, denote by β_p the last root in normal ordering (4.1.23). Then $(\overline{X}^+_{\gamma_{n-1}})^{(c'_{n-1})} (\overline{X}^-_{\gamma_n})^{(c_n)} \in \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_n, \gamma_{n-1}])$, where here and in this proof below we consider only minimal segments $[\alpha, \beta]$ of the circular ordering of $\Delta(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ corresponding normal ordering (4.1.23) of $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$, and the corresponding subalgebras of $U^{res}_q(\mathfrak{m}_{s^1})$, so, in particular, $[-\gamma_n, \gamma_{n-1}] \subset \Delta(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$, and $\overline{U}^{res}_{U^{res}_q(H)}([-\gamma_n, \gamma_{n-1}]) \subset U^{res}_q(\mathfrak{m}_{s^1})$.

Then by Corollary 2.4.4 (i) we obtain

$$(\overline{X}^+_{\gamma_{n-1}})^{(c'_{n-1})} (\overline{X}^-_{\gamma_n})^{(c_n)} (\overline{X}^+_{\gamma_n})^{(c'_n)} \in \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_n, \gamma_{n-1}]) (\overline{X}^+_{\gamma_n})^{(c'_n)} = \tag{4.1.42}$$

$$= \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_n, -\beta_p]) \overline{U}^{res}_{U^{res}_q(H)}([\gamma_1, \gamma_{n-1}]) (\overline{X}^+_{\gamma_n})^{(c'_n)} \subset \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_n, -\beta_p]) \overline{U}^{res}_{U^{res}_q(H)}([\gamma_1, \gamma_n]).$$

Next, $(\overline{X}^+_{\gamma_{n-2}})^{(c'_{n-2})} (\overline{X}^-_{\gamma_{n-1}})^{(c_{n-1})} \in \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_{n-1}, \gamma_{n-2}])$, and by (4.1.42) and by Corollary 2.4.4 (i) one has

$$(\overline{X}^+_{\gamma_{n-2}})^{(c'_{n-2})} (\overline{X}^-_{\gamma_{n-1}})^{(c_{n-1})} (\overline{X}^+_{\gamma_{n-1}})^{(c'_{n-1})} (\overline{X}^-_{\gamma_n})^{(c_n)} (\overline{X}^+_{\gamma_n})^{(c'_n)} \in$$

$$\in \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_{n-1}, \gamma_{n-2}]) \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_n, -\beta_p]) \overline{U}^{res}_{U^{res}_q(H)}([\gamma_1, \gamma_n]) =$$

$$= \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_{n-1}, -\beta_p]) \overline{U}^{res}_{U^{res}_q(H)}([\gamma_1, \gamma_{n-2}]) \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_n, -\beta_p]) \overline{U}^{res}_{U^{res}_q(H)}([\gamma_1, \gamma_n]) =$$

$$= \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_{n-1}, -\beta_p]) \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_n, \gamma_{n-2}]) \overline{U}^{res}_{U^{res}_q(H)}([\gamma_1, \gamma_n]) =$$

$$= \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_{n-1}, -\beta_p]) \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_n, -\beta_p]) \overline{U}^{res}_{U^{res}_q(H)}([\gamma_1, \gamma_{n-2}]) \overline{U}^{res}_{U^{res}_q(H)}([\gamma_1, \gamma_n]) \subset$$

$$\subset \overline{U}^{res}_{U^{res}_q(H)}([-\gamma_{n-1}, -\beta_p]) \overline{U}^{res}_{U^{res}_q(H)}([\gamma_1, \gamma_n]).$$

One can proceed by induction in a similar way to get

$$(\overline{X}_{\gamma_1}^-)^{(c_1)}(\overline{X}_{\gamma_1}^+)^{(c_1')}(\overline{X}_{\gamma_2}^-)^{(c_2)}(\overline{X}_{\gamma_2}^+)^{(c_2')}\ldots(\overline{X}_{\gamma_n}^-)^{(c_n)}(\overline{X}_{\gamma_n}^+)^{(c_n')} \in \overline{U}_{U_q^{res}(H)}^{res}([-\gamma_1,-\beta_p])\overline{U}_{U_q^{res}(H)}^{res}([\gamma_1,\gamma_n]). \quad (4.1.43)$$

Next, Corollary 2.4.4 (ii) implies

$$\overline{U}_{U_q^{res}(H)}^{res}([-\gamma_1,-\beta_p])\overline{U}_{U_q^{res}(H)}^{res}([\gamma_1,\gamma_n]) = \overline{U}_q^{res}([-\gamma_1,-\beta_p])U_q^{res}(H)\overline{U}_q^{res}([\gamma_1,\gamma_n])U_q^{res}(H) =$$

$$= \overline{U}_q^{res}([-\gamma_1,-\beta_p])\overline{U}_q^{res}([\gamma_1,\gamma_n])U_q^{res}(H) = U_q^{res}(\mathfrak{m}_{s^1}^-)\overline{U}_q^{res}([\gamma_1,\gamma_n])U_q^{res}(H),$$

where at the last step we also used the fact that by the definition $\overline{U}_q^{res}([-\gamma_1,-\beta_p]) = U_q^{res}(\mathfrak{m}_{s^1}^-)$.

Now (4.1.43) takes the form

$$(\overline{X}_{\gamma_1}^-)^{(c_1)}(\overline{X}_{\gamma_1}^+)^{(c_1')}(\overline{X}_{\gamma_2}^-)^{(c_2)}(\overline{X}_{\gamma_2}^+)^{(c_2')}\ldots(\overline{X}_{\gamma_n}^-)^{(c_n)}(\overline{X}_{\gamma_n}^+)^{(c_n')} \in U_q^{res}(\mathfrak{m}_{s^1}^-)\overline{U}_q^{res}([\gamma_1,\gamma_n])U_q^{res}(H). \quad (4.1.44)$$

Thus, recalling that v is a weight vector we deduce (4.1.38) from (4.1.40) and (4.1.44) using Lemma 2.8.6.

Finally note that u and $T_{\gamma_1}\ldots T_{\gamma_n}v$ are weight vectors, and different weight subspaces of V are orthogonal with respect to the bilinear form $(\cdot,\cdot)$. Therefore we can assume also that $K(m_1,\ldots,m_n)$ in (4.1.38) belongs to a weight subspace of $U_q^{res}(\mathfrak{g})$, so that the weight of u is equal to that of $K(m_1,\ldots,m_n)T_{\gamma_1}\ldots T_{\gamma_n}v$.

(4.1.39) follows from (4.1.38) by Lemma 2.8.4 (i) because the only elements of $U_q^{res}(\mathfrak{m}_{s^1}^-)$, whose weights are not strictly negative, belong to $\mathbb{C}[q,q^{-1}]$. This completes the proof.

$\square$

Lemma 4.1.5. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u,v \in V$. Suppose that u is a highest weight vector, and v is a weight vector. Then for any $m_1,\ldots,m_n \in \mathbb{N}$*

$$(u,(X_{\gamma_1}^+)^{(m_1)}\ldots(X_{\gamma_n}^+)^{(m_n)}v) = (u,T_{\gamma_1}\ldots T_{\gamma_n}U(m_1,\ldots,m_n)v), \quad (4.1.45)$$

where $U(m_1,\ldots,m_n) \in U_q^{res}([-\gamma_1,-\gamma_n])$ belongs to a weight subspace of $U_q^{res}(\mathfrak{g})$.

Proof. Since u is a highest weight vector, Lemma 4.1.4 (ii) implies

$$(u,(X_{\gamma_1}^+)^{(m_1)}\ldots(X_{\gamma_n}^+)^{(m_n)}v) = (u,K'(m_1,\ldots,m_n)T_{\gamma_1}\ldots T_{\gamma_n}v), \quad (4.1.46)$$

where $K'(m_1,\ldots,m_n) \in \overline{U}_q^{res}([\gamma_1,\gamma_n])$.

Denote $T^1 = T_{\gamma_1}\ldots T_{\gamma_n}$. We find the action of $(T^1)^{-1}$ on the generators of the algebra $\overline{U}_q^{res}([\gamma_1,\gamma_n])$.

Consider reduced decomposition (4.1.25) of s^1,

$$s^1 = s_{i_1}\ldots s_{i_p} = s_{i_{k_1}}\ldots s_{i_{k_2}}\ldots s_{i_{k_n}}s_{i_{k_n+1}}\ldots s_{i_p},\, i_{k_1} = i_1, \quad (4.1.47)$$

and the roots $\beta_q = s_{i_1}\ldots s_{i_{q-1}}\alpha_{i_q}$, $q = 1,\ldots,k_n$ forming the segment $[\gamma_1,\gamma_n]$.

From (4.1.36) with $q-1 = n$ and (4.1.33) we obtain

$$T^1 = T_{\gamma_1}\ldots T_{\gamma_n} = T_{i_{k_1}+1}\ldots T_{i_{k_2}-1}T_{i_{k_2}+1}\ldots T_{i_{k_n}-1}T_{i_{k_n}}^{-1}T_{i_{k_n}-1}^{-1}\ldots T_{i_1}^{-1} = \quad (4.1.48)$$

$$= T_{i_p}\ldots T_{i_{k_n+1}}T_{i_{k_n}}^{-1}T_{i_{k_n}-1}^{-1}\ldots T_{i_{k_1}}^{-1}.$$

Therefore for the generators $(\overline{X}_{\beta_q}^+)^{(k)} = T_{i_1}^{-1}\ldots T_{i_{q-1}}^{-1}(X_{i_q}^+)^{(k)}$, $q = 1,\ldots,k_n$, $k \in \mathbb{N}$, of the algebra $\overline{U}_q^{res}([\gamma_1,\gamma_n])$ we obtain

$$(T^1)^{-1}(\overline{X}_{\beta_q}^+)^{(k)} = T_{i_1}\ldots T_{i_{k_n}}T_{i_{k_n+1}}^{-1}\ldots T_{i_p}^{-1}T_{i_1}^{-1}\ldots T_{i_{q-1}}^{-1}(X_{i_q}^+)^{(k)} = \quad (4.1.49)$$

$$= T_{w_1}T_{i_{k_n+1}}^{-1}\ldots T_{i_p}^{-1}T_{i_1}^{-1}\ldots T_{i_{q-1}}^{-1}(X_{i_q}^+)^{(k)},$$

where $T_{w_1} = T_{i_1}\ldots T_{i_{k_n}}$ for the reduced decomposition $w_1 = s_{i_1}\ldots s_{i_{k_n}}$ which is an initial part of reduced decomposition (4.1.47).

By (4.1.29), (2.2.13)

$$T_{i_{q+1}}^{-1}\ldots T_{i_p}^{-1}T_{i_1}^{-1}\ldots T_{i_{q-1}}^{-1}(X_{i_q}^+)^{(k)} = (X_{i_q}^+)^{(k)},$$

and hence

$$T_{i_{k_n+1}}^{-1}\ldots T_{i_p}^{-1}T_{i_1}^{-1}\ldots T_{i_{q-1}}^{-1}(X_{i_q}^+)^{(k)} = T_{i_{k_n}}\ldots T_{i_{q+1}}(X_{i_q}^+)^{(k)}.$$

Substituting this into (4.1.49) we infer

$$(T^1)^{-1}(\overline{X}_{\beta_q}^+)^{(k)} = T_{w_1} T_{i_{k_n}} \ldots T_{i_{q+1}} (X_{i_q}^+)^{(k)} = T_{w_1} (X_{\delta_q}^+)^{(k)}, \qquad (4.1.50)$$

where for $q = 1, \ldots, k_n$ we denote $\delta_q = s_{i_{k_n}} \ldots s_{i_{q+1}} \alpha_{i_q}$, and $X_{\delta_q}^+ = T_{i_{k_n}} \ldots T_{i_{q+1}} X_{i_q}^+$.

Let $U_q^{res}(\mathfrak{m}_{s^1}^+)$ be the subalgebra of $U_q^{res}(\mathfrak{m}_{s^1})$ generated by $(X_\beta^+)^{(k)}$ for simple roots $\beta \in \Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ and $k \geq 0$. We show that for $q = 1, \ldots, k_n$ and $k \geq 0$ one has $(X_{\delta_q}^+)^{(k)} \in U_q^{res}(\mathfrak{m}_{s^1}^+)$.

Indeed, observe that $s_{i_{k_n}} \ldots s_{i_1} s_{i_p} \ldots s_{i_{k_n+1}}$ is the reduced decomposition of s^1, with respect to the system of simple roots in $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$, obtained according to (4.1.32) in the following way

$$s^1 = s_{i_{k_n}} \ldots s_{i_1} s^1 s_{i_1} \ldots s_{i_{k_n}} = s_{i_{k_n}} \ldots s_{i_1} s_{i_p} \ldots s_{i_1} s_{i_1} \ldots s_{i_{k_n}} = s_{i_{k_n}} \ldots s_{i_1} s_{i_p} \ldots s_{i_{k_n+1}}. \qquad (4.1.51)$$

$s_{i_{k_n}} \ldots s_{i_1}$ is an initial part of this reduced decomposition.

Now by Lemma 2.4.2 (i) we deduce that for $q = 1, \ldots, k_n$ $X_{\delta_q}^+ = T_{i_{k_n}} \ldots T_{i_{q+1}} X_{i_q}^+ \in U_q^{res}(\mathfrak{m}_{s^1}^+)$, and also obviously

$$(X_{\delta_q}^+)^{(k)} \in U_q^{res}(\mathfrak{m}_{s^1}^+) \qquad (4.1.52)$$

for any $k \geq 0$. Since for $q = 1, \ldots, k_n$ and $k \geq 0$ the elements $(\overline{X}_{\beta_q}^+)^{(k)}$ generate $\overline{U}_q^{res}([\gamma_1, \gamma_n])$, we deduce from (4.1.52) and (4.1.50) that

$$(T^1)^{-1}(\overline{U}_q^{res}([\gamma_1, \gamma_n])) \subset T_{w_1}(U_q^{res}(\mathfrak{m}_{s^1}^+)).$$

By this inclusion $(T^1)^{-1}(K'(m_1, \ldots, m_n)) \in T_{w_1}(U_q^{res}(\mathfrak{m}_{s^1}^+))$, and (4.1.46) takes the form

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} v) = (u, K'(m_1, \ldots, m_n) T_{\gamma_1} \ldots T_{\gamma_n} v) = \qquad (4.1.53)$$

$$= (u, T_{\gamma_1} \ldots T_{\gamma_n} (T^1)^{-1}(K'(m_1, \ldots, m_n)) v) = (u, T_{\gamma_1} \ldots T_{\gamma_n} K''(m_1, \ldots, m_n) v),$$

where $K''(m_1, \ldots, m_n) = (T^1)^{-1}(K'(m_1, \ldots, m_n)) \in T_{w_1}(U_q^{res}(\mathfrak{m}_{s^1}^+))$.

Now we factorize $U_q^{res}(\mathfrak{m}_{s^1}^+)$ in an appropriate way to bring (4.1.53) to form (4.1.45). Observe that using reduced decomposition (4.1.51) one can define the corresponding normal ordering of $\Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$,

$$\phi_{k_n} = \alpha_{i_{k_n}}, \ldots, \phi_1 = s_{i_{k_n}} \ldots s_{i_2} \alpha_{i_1}, \phi_p = s_{i_{k_n}} \ldots s_{i_1} \alpha_{i_p}, \ldots, \phi_{k_n+1} = s_{i_{k_n}} \ldots s_{i_1} s_{i_p} \ldots s_{i_{k_n+2}} \alpha_{i_{k_n+1}},$$

and the elements

$$Y_{\phi_{k_n}} = X_{i_{k_n}}^+, \ldots,$$

$$Y_{\phi_1} = T_{i_{k_n}}^{-1} \ldots T_{i_2}^{-1} X_{i_1}^+,$$

$$Y_{\phi_p} = T_{i_{k_n}}^{-1} \ldots T_{i_1}^{-1} X_{i_p}^+, \ldots,$$

$$Y_{\phi_{k_n+2}} = T_{i_{k_n}}^{-1} \ldots T_{i_1}^{-1} T_{i_p}^{-1} \ldots T_{i_{k_n+3}}^{-1} X_{i_{k_n+2}}^+,$$

$$Y_{\phi_{k_n+1}} = T_{i_{k_n}}^{-1} \ldots T_{i_1}^{-1} T_{i_p}^{-1} \ldots T_{i_{k_n+2}}^{-1} X_{i_{k_n+1}}^+,$$

which belong to $U_q^{res}(\mathfrak{m}_{s^1}^+)$ by Lemma 2.4.2 (v).

Let $\overline{U}_q^{res}([\phi_{k_n}, \phi_1]) \subset U_q^{res}(\mathfrak{m}_{s^1}^+)$ be the subalgebra generated by $(Y_{\phi_i})^{(k)}, i = 1, \ldots k_n, k \in \mathbb{N}$, and $\overline{U}_q^{res}([\phi_p, \phi_{k_n+1}]) \subset U_q^{res}(\mathfrak{m}_{s^1}^+)$ the subalgebra generated by $(Y_{\phi_i})^{(k)}, i = k_n + 1, \ldots p, k \in \mathbb{N}$.

Now by Corollary 2.4.4 (iii),

$$U_q^{res}(\mathfrak{m}_{s^1}^+) = \overline{U}_q^{res}([\phi_p, \phi_{k_n+1}]) \overline{U}_q^{res}([\phi_{k_n}, \phi_1]). \qquad (4.1.54)$$

We find the action of the automorphism $T_{w_1} = T_{i_1} \ldots T_{i_{k_n}}$ on $\overline{U}_q^{res}([\phi_p, \phi_{k_n+1}])$ and $\overline{U}_q^{res}([\phi_{k_n}, \phi_1])$.

By the definition of the elements Y_{ϕ_r}, for $r = k_n + 1, \ldots p$

$$T_{w_1}(Y_{\phi_r})^{(k)} = (T_{i_1} \ldots T_{i_{k_n}})(Y_{\phi_r})^{(k)} = (T_{i_p}^{-1} \ldots T_{i_{r+1}}^{-1})(X_{i_r}^+)^{(k)}. \qquad (4.1.55)$$

Since u is a highest weight vector, by Lemma 2.8.7 we have $T_{\gamma_n} \ldots T_{\gamma_1} u = \omega(c) T_{i_p} \ldots T_{i_1} u$, where $c \in \mathbb{C}[q, q^{-1}]^*$. Therefore for any element $w \in V$ we can write

$$(u, T_{\gamma_1} \ldots T_{\gamma_n} w) = c(u, T_{i_1} \ldots T_{i_p} w) = c(u, T_{s^1} w), \qquad (4.1.56)$$

where $T_{s^1} = T_{i_1} \ldots T_{i_p}$ is the braid group element corresponding to reduced decomposition (4.1.47). Now by (4.1.55), using the definition (2.2.2) of the braid group action and commutation relations (2.3.6), we obtain for all generators $(Y_{\phi_r})^{(k)}$, $r = k_n + 1, \ldots p$, $k \in \mathbb{N}$ of $\overline{U}_q^{res}([\phi_p, \phi_{k_n+1}])$

$$T_{s^1} T_{w_1} (Y_{\phi_r})^{(k)} = T_{i_1} \ldots T_{i_p} T_{i_1} \ldots T_{i_{k_n}} (Y_{\phi_r})^{(k)} = T_{i_1} \ldots T_{i_p} T_{i_p}^{-1} \ldots T_{i_{r+1}}^{-1} (X_{i_r}^+)^{(k)} =$$

$$= T_{i_1} \ldots T_{i_r} (X_{i_r}^+)^{(k)} = \frac{1}{[k]_{q_{i_r}}!} T_{i_1} \ldots T_{i_{r-1}} (-X_{i_r}^- K_{i_r})^k = \frac{1}{[k]_{q_{i_r}}!} (-X_{\beta_r}^- K_{\beta_r})^k =$$

$$= (-1)^k q_{\beta_r}^{-k(k-1)} (X_{\beta_r}^-)^{(k)} K_{\beta_r}^k \in U_{U_q^{res}(H)}^{res}([-\beta_{k_n+1}, \ldots, -\beta_p]),$$

where $K_{\beta_r} = T_{i_1} \ldots T_{i_{r-1}} K_{i_r}$. Thus

$$T_{s^1} T_{w_1} (\overline{U}_q^{res}([\phi_p, \phi_{k_n+1}])) \subset U_{U_q^{res}(H)}^{res}([-\beta_{k_n+1}, \ldots, -\beta_p]). \tag{4.1.57}$$

On the other hand by the definition of the elements Y_{ϕ_r}, for $r = 1, \ldots k_n$ we have for all generators $(Y_{\phi_r})^{(k)}$, $r = 1, \ldots k_n$, $k \in \mathbb{N}$ of $\overline{U}_q^{res}([\phi_{k_n}, \phi_1])$

$$T_{w_1} (Y_{\phi_r})^{(k)} = T_{i_1} \ldots T_{i_{k_n}} (Y_{\phi_r})^{(k)} = \frac{1}{[k]_{q_{i_r}}!} T_{i_1} \ldots T_{i_r} (X_{i_r}^+)^k = \frac{1}{[k]_{q_{i_r}}!} T_{i_1} \ldots T_{i_{r-1}} (-X_{i_r}^- K_{i_r})^k = \tag{4.1.58}$$

$$= \frac{1}{[k]_{q_{i_r}}!} (-X_{\beta_r}^- K_{\beta_r})^k =$$

$$= (-1)^k q_{\beta_r}^{-k(k-1)} (X_{\beta_r}^-)^{(k)} K_{\beta_r}^k \in U_{U_q^{res}(H)}^{res}([-\gamma_1, -\gamma_n]),$$

where $K_{\beta_r} = T_{i_1} \ldots T_{i_{r-1}} K_{i_r}$, and we also used the definition (2.2.2) of the braid group action and commutation relations (2.3.6).

Thus

$$T_{w_1} (\overline{U}_q^{res}([\phi_{k_n}, \phi_1])) \subset U_{U_q^{res}(H)}^{res}([-\gamma_1, -\gamma_n]). \tag{4.1.59}$$

Now recall that in (4.1.53) $K''(m_1, \ldots, m_n) \in T_{w_1}(U_q^{res}(\mathfrak{m}_{s^1}^+))$, and hence we can write $K''(m_1, \ldots, m_n) = T_{w_1} A$, $A \in U_q^{res}(\mathfrak{m}_{s^1}^+)$.

Factorizing A according to (4.1.54),

$$A = \sum_i a_i b_i, \ a_i \in \overline{U}_q^{res}([\phi_p, \phi_{k_n+1}]), \ b_i \in \overline{U}_q^{res}([\phi_{k_n}, \phi_1]),$$

we obtain

$$K''(m_1, \ldots, m_n) = T_{w_1}(A) = \sum_i (T_{w_1}(a_i))(T_{w_1}(b_i)) = \sum_i (T_{w_1}(a_i)) c_i,$$

where $c_i = T_{w_1}(b_i) \in U_{U_q^{res}(H)}^{res}([-\gamma_1, -\gamma_n])$ by (4.1.59).

Now recalling (4.1.56), (4.1.53) can be rewritten as follows

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} v) = c(u, T_{i_1} \ldots T_{i_p} K''(m_1, \ldots, m_n) v) =$$

$$= c \sum_i (u, T_{s^1}((T_{w_1})(a_i)) c_i v) = c \sum_i (u, ((T_{s^1} T_{w_1})(a_i)) T_{s^1} c_i v) =$$

$$= c \sum_i (u, g_i T_{s^1} c_i v),$$

where $g_i = (T_{s^1} T_{w_1})(a_i) \in U_{U_q^{res}(H)}^{res}([-\beta_{k_n+1}, \ldots, -\beta_p])$ by (4.1.57).

Observe that the only elements of the algebra $U_{U_q^{res}(H)}^{res}([-\beta_{k_n+1}, \ldots, -\beta_p])$ whose weights are not strictly negative belong to $U_q^{res}(H)$, and hence by Lemma 2.8.4 (i) the last formula takes the form

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} v) = c(u, T_{s^1} K''(m_1, \ldots, m_n) v) = c \sum_i (u, g_i' T_{s^1} c_i v) =$$

$$= c(u, T_{s^1} U'v) = (u, T_{\gamma_1} \ldots T_{\gamma_n} U'v), \tag{4.1.60}$$

where $g_i' \in U_q^{res}(H)$, and $U' = \sum_i (T_{s^1})^{-1}(g_i') c_i \in U_{U_q^{res}(H)}^{res}([-\gamma_1, -\gamma_n])$.

Recall finally that by Corollary 2.4.4 (ii) we have $U_{U_q^{res}(H)}^{res}([-\gamma_1, -\gamma_n]) = U_q^{res}([-\gamma_1, -\gamma_n]) U_q^{res}(H)$ and that according to Lemma 2.8.6 elements of $U_q^{res}(H)$ act on weight vectors by multiplication by elements of $\mathbb{C}[q, q^{-1}]$. Therefore (4.1.60) immediately implies (4.1.45).

Finally note that v and $T_{\gamma_n} \ldots T_{\gamma_1} u$ are weight vectors, and different weight subspaces of V are orthogonal with respect to the bilinear form $(\cdot, \cdot)$. Therefore we can assume also that $U(m_1, \ldots, m_n)$ in (4.1.45) belongs to a weight subspace of $U_q^{res}(\mathfrak{g})$, so that the weight of $T_{\gamma_n} \ldots T_{\gamma_1} u$ is equal to that of $U(m_1, \ldots, m_n)v$. This completes the proof.

$\square$

Now we obtain an analogue of Lemma 4.1.4 for s^2. We argue in the way similar to the previous discussion in the case of s^1. According to Lemma 1.6.14 (i) s^2 is the longest element in the Weyl group $W(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ of the semisimple part $\mathfrak{m}_{s^2}$ of a Levi subalgebra of $\mathfrak{g}$, the Cartan subalgebra of $\mathfrak{m}_{s^2}$ is denoted by $\mathfrak{h}_{s^2}$.

The system of positive roots $\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) := \Delta_+ \cap \Delta(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ of the root system $\Delta(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) = \Delta_{s^2}^{-1} \bigcup (-\Delta_{s^2}^{-1})$ is the set (we again use the notation of (1.6.9))

$$\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) = \{\gamma_{n+1}, \ldots, \gamma_{n+2}, \ldots, \gamma_{n+3}, \ldots, \gamma_{l'}, \beta_{t' + \frac{q+l'-n}{2}+1}^2, \ldots, \beta_{t'+q}^2\},$$

which is in fact a segment, and s^2 acts on the elements of this set by multiplication by -1. The roots in (4.1.61) are ordered as in the normal ordering of Δ_+ associated to s. Note that, in fact, $\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) = \Delta_+^s(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) = \Delta(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) \cap \Delta_+^s = \Delta_{s^2}^{-1}$.

As before if $\overline{w} = s_{i_1} \ldots s_{i_D}$ is the reduced decomposition of the longest element $\overline{w} \in W$ corresponding to Δ_+, we write $\gamma_m = s_{i_1} \ldots s_{i_{k_{n+1}}} \ldots s_{i_{k_{m-1}}} \alpha_{i_{k_m}}$ for $m = n+1, \ldots, l'$.

Let $w = s_{i_1} \ldots s_{i_{k_{n+1}-1}}$, $T_w = T_{i_1} \ldots T_{i_{k_{n+1}-1}}$. Then $\widetilde{\Delta}_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) = w^{-1}(\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}))$ is the set of positive roots of the semisimple part $\widetilde{\mathfrak{m}}_{s^2} = \mathrm{Ad} w^{-1}(\mathfrak{m}_{s^2})$ of a standard Levi subalgebra of $\mathfrak{g}$. Indeed, by part (iv) of Lemma 1.6.3 the reduced decomposition $s_{i_{k_{n+1}}} \ldots s_{i_D}$, which is a part of the reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$, can be completed to a reduced decomposition of $\overline{w}$, $\overline{w} = s_{i_{k_{n+1}}} \ldots s_{i_D} s_{j_1} \ldots s_{j_{k_{n+1}-1}}$, and by the definition the roots from the set $\widetilde{\Delta}_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) = w^{-1}(\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}))$ form an initial segment in the corresponding normal ordering of Δ_+. Therefore, if $\widetilde{\mathfrak{m}}_{s^2}$ was not the semisimple part of a standard Levi subalgebra of $\mathfrak{g}$, by Lemma 1.6.14 (iii) there would be some roots preceding those from the set $w^{-1}(\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}))$ in this normal ordering.

Therefore one can define the subalgebra $U_h(\widetilde{\mathfrak{m}}_{s^2}) \subset U_h(\mathfrak{g})$ generated by the elements $X_i^{\pm}$ and H_i for $\alpha_i \in \widetilde{\Delta}_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ and its restricted specialization $U_q^{res}(\widetilde{\mathfrak{m}}_{s^2}) \subset U_q^{res}(\mathfrak{g})$.

Lemma 1.6.14 (i) also implies that $\widetilde{s}^2 = w^{-1} s^2 w$ is the longest element in the Weyl group $\widetilde{W}(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) = w^{-1} W(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}) w$ with respect to the system of simple roots in $w^{-1}(\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2}))$.

For any root $\beta \in \Delta_+$ denote

$$\widetilde{X}_\beta^{\pm} = T_w^{-1}(X_\beta^{\pm}), \quad \widetilde{\overline{X}}_\beta^{\pm} = \overline{T}_w^{-1}(\overline{X}_\beta^{\pm}), \quad \widetilde{T}_\beta = T_w^{-1} T_\beta T_w. \tag{4.1.61}$$

Denote by $\beta_r := \beta_{t'+q}^2$ the greatest root in the segment $\Delta_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ with respect to ordering (4.1.61). Then by the definition $T_w^{-1}(U_q^{res}([-\gamma_{n+1}, -\beta_r])) := U_q^{res}(\widetilde{\mathfrak{m}}_{s^2}^-)$ is the subalgebra in the algebra $U_q^{res}(\widetilde{\mathfrak{m}}_{s^2})$ generated by $(X_\beta^-)^{(k)}$ for simple roots $\beta \in \widetilde{\Delta}_+(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$ and $k \geq 0$.

Similarly to Lemma 4.1.4 we have the following lemma.

Lemma 4.1.6. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u, v \in V$ weight vectors. Then for any $m_{n+1}, \ldots, m_{l'} \in \mathbb{N}$ one has*

$$(u, (\widetilde{X}_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (\widetilde{X}_{\gamma_{l'}}^+)^{(m_{l'})} v) = (u, \widetilde{K}(m_{n+1}, \ldots, m_{l'}) \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} v), \tag{4.1.62}$$

where $\widetilde{K}(m_{n+1}, \ldots, m_{l'}) \in T_w^{-1}(U_q^{res}([-\gamma_{n+1}, -\beta_r])) \overline{T}_w^{-1}(\overline{U}_q^{res}([\gamma_{n+1}, \gamma_{l'}])) = U_q^{res}(\widetilde{\mathfrak{m}}_{s^2}^-) \overline{T}_w^{-1}(\overline{U}_q^{res}([\gamma_{n+1}, \gamma_{l'}]))$, and $\widetilde{K}(m_{n+1}, \ldots, m_{l'})$ belongs to a weight subspace of $U_q^{res}(\mathfrak{g})$.

From Lemmas 4.1.5 and 4.1.6 we obtain the following statement.

Lemma 4.1.7. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u, v \in V$ weight vectors. Suppose that u is a highest weight vector. Then for any $m_1, \ldots, m_{l'} \in \mathbb{N}$*

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} (X_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (X_{\gamma_{l'}}^+)^{(m_{l'})} v) =$$

$$= (u, T_{\gamma_1} \ldots T_{\gamma_n} T_{\gamma_{n+1}} \ldots T_{\gamma_{l'}} T_{w'} X(m_1, \ldots, m_{l'}) T_{w'}^{-1} v), \tag{4.1.63}$$

where $X(m_1, \ldots, m_{l'}) \in U_q^{res}(\mathfrak{n}_+)$, $T_{w'} = T_{i_1} \ldots T_{i_{k_{l'}}}$, for $w' = s_{i_1} \ldots s_{i_{k_{l'}}}$, and $X(m_1, \ldots, m_{l'})$ belongs to a weight subspace of $U_q^{res}(\mathfrak{g})$.

Proof. By Lemmas 4.1.5 and 4.1.6 we have, using the notation introduced in these lemmas,

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} (X_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (X_{\gamma_{l'}}^+)^{(m_{l'})} v) =$$

$$= (u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} T_w (\widetilde{X}_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (\widetilde{X}_{\gamma_{l'}}^+)^{(m_{l'})} T_w^{-1} v) =$$

$$= (u, T_{\gamma_1} \ldots T_{\gamma_n} U(m_1, \ldots, m_n) T_w \widetilde{K}(m_{n+1}, \ldots, m_{l'}) \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v), \tag{4.1.64}$$

where $U(m_1, \ldots, m_n) \in U_q^{res}([-\gamma_1, -\gamma_n])$, $\widetilde{K}(m_{n+1}, \ldots, m_{l'}) \in T_w^{-1}(U_q^{res}([-\gamma_{n+1}, -\beta_r]))\overline{T}_w^{-1}(\overline{U}_q^{res}([\gamma_{n+1}, \gamma_{l'}]))$, so we can write $\widetilde{K}(m_{n+1}, \ldots, m_{l'}) = \sum_j T_w^{-1}(a_j) b_j$, $a_j \in U_q^{res}([-\gamma_{n+1}, -\beta_r])$, $b_j \in \overline{T}_w^{-1}(\overline{U}_q^{res}([\gamma_{n+1}, \gamma_{l'}]))$. Substituting this into (4.1.64) we obtain

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} (X_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (X_{\gamma_{l'}}^+)^{(m_{l'})} v) = \tag{4.1.65}$$

$$= (u, T_{\gamma_1} \ldots T_{\gamma_n} U(m_1, \ldots, m_n) T_w \sum_j T_w^{-1}(a_j) b_j \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v) =$$

$$= \sum_j (u, T_{\gamma_1} \ldots T_{\gamma_n} U(m_1, \ldots, m_n) a_j T_w b_j \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v),$$

where $U(m_1, \ldots, m_n) a_j \in U_q^{res}([-\gamma_1, -\gamma_n]) U_q^{res}([-\gamma_{n+1}, -\beta_r]) \subset U_q^{res}([-\gamma_1, -\beta_r])$.

Observe that by Corollary 2.4.4 (iii) $U_q^{res}([-\gamma_1, -\beta_r]) = U_q^{res}([-\beta_{k_{n+1}}, -\beta_r]) U_q^{res}([-\gamma_1, -\gamma_n])$, so $U(m_1, \ldots, m_n) a_j = \sum_i c_i^j o_i^j$ for some $c_i^j \in U_q^{res}([-\beta_{k_{n+1}}, -\beta_r])$, $o_i^j \in U_q^{res}([-\gamma_1, -\gamma_n])$.

Now (4.1.65) takes the form

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} (X_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (X_{\gamma_{l'}}^+)^{(m_{l'})} v) = \tag{4.1.66}$$

$$= \sum_{i,j} (u, T_{\gamma_1} \ldots T_{\gamma_n} c_i^j o_i^j T_w b_j \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v) = \sum_{i,j} (u, (T_{\gamma_1} \ldots T_{\gamma_n}(c_i^j)) T_{\gamma_1} \ldots T_{\gamma_n} o_i^j T_w b_j \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v).$$

If $\gamma_n < \beta \leq \beta_r$ then by (4.1.24) we have $\beta \notin \Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1}) = \{\alpha \in \Delta_+ : s^1 \alpha \in \Delta_-\}$. This implies $s^1 \beta \in \Delta_+$, and hence by the definition of the braid group action, for $\gamma_n < \beta \leq \beta_r$, $k \in \mathbb{N}$ the element $T_{\gamma_1} \ldots T_{\gamma_n} (X_\beta^-)^{(k)}$ has weight $-k s^1 \beta < 0$. Since the elements $(X_\beta^-)^{(k)}$ for $\gamma_n < \beta \leq \beta_r$, $k \in \mathbb{N}$ generate $U_q^{res}([-\beta_{k_{n+1}}, -\beta_r])$, we deduce that the only elements of $T_{\gamma_1} \ldots T_{\gamma_n}(U_q^{res}([-\beta_{k_n+1}, -\beta_r]))$ which do not have strictly negative weights belong to $\mathbb{C}[q, q^{-1}]$. Thus by Lemma 2.8.4 (i) the right hand side of (4.1.66) takes the form

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} (X_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (X_{\gamma_{l'}}^+)^{(m_{l'})} v) = \tag{4.1.67}$$

$$= \sum_{i,j} (u, (T_{\gamma_1} \ldots T_{\gamma_n}(c_i'^j)) T_{\gamma_1} \ldots T_{\gamma_n} o_i^j T_w b_j \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v) = \sum_j (u, T_{\gamma_1} \ldots T_{\gamma_n} p_j T_w b_j \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_w^{-1} v),$$

where $c_i'^j \in \mathbb{C}[q, q^{-1}]$, $p_j = \sum_i c_i'^j o_i^j \in U_q^{res}([-\gamma_1, -\gamma_n])$.

Now we proceed as in the proof of Lemma 4.1.5. Namely, denote $\widetilde{T}^2 = \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}}$. We find the action of $(\widetilde{T}^2)^{-1}$ on the generators of the algebra $\overline{T}_w^{-1}(\overline{U}_q^{res}([\gamma_{n+1}, \gamma_{l'}]))$.

For $\gamma_{n+1} \leq \beta_q \leq \gamma_{l'}$ we have by the definition $\beta_q = w s_{k_{n+1}} \ldots s_{i_{q-1}} \alpha_{i_q}$, $q = k_{n+1}, \ldots, k_{l'}$, where we use the reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$, and $w = s_{i_1} \ldots s_{i_{k_{n+1}}-1}$.

Similarly to (4.1.50) we infer

$$(\widetilde{T}^2)^{-1} \widetilde{X}_{\beta_q}^+ = T_{i_{n+1}} \ldots T_{i_{k_{l'}}} T_{i_{k_{l'}}} \ldots T_{i_{q+1}} X_{i_q}^+. \tag{4.1.68}$$

Denote for $q = k_{n+1}, \ldots, k_{l'}$, $\delta_q = s_{i_{k_{l'}}} \ldots s_{i_{q+1}} \alpha_{i_q}$, and

$$\widehat{X}_{\delta_q}^+ = T_{i_{k_{l'}}} \ldots T_{i_{q+1}} X_{i_q}^+. \tag{4.1.69}$$

We show that $\widehat{X}_{\delta_q}^+ \in U_q^{res}(\mathfrak{n}_+)$ for $q = k_{n+1}, \ldots, k_{l'}$.

Indeed, the element $w_2^{-1} = s_{i_{k_{l'}}} \ldots s_{k_{n+1}}$ is a part of the reduced decomposition

$$\overline{w} = s_{i_D} \ldots s_{i_{k_{l'}+1}} s_{i_{k_{l'}}} \ldots s_{i_{k_{n+1}}} \ldots s_{i_1} \tag{4.1.70}$$

inverse to that corresponding to the normal ordering in Δ_+ associated to s. Therefore by part (iv) of Lemma 1.6.3 we can consider it as an initial part of another reduced decomposition of the longest element $\overline{w} \in W$. Now by Lemma 2.4.2 (i) one has $\widehat{X}_{\delta_q}^+ \in U_q^{res}(\mathfrak{n}_+)$ for $q = k_{n+1}, \ldots, k_{l'}$ which obviously implies that for the generators $(\widetilde{\overline{X}}_{\beta_q}^+)^{(k)}$, $q = k_{n+1}, \ldots, k_{l'}$, $k \in \mathbb{N}$ of the algebra $\overline{T}_w^{-1}(\overline{U}_q^{res}([\gamma_{n+1}, \gamma_{l'}]))$ we have by (4.1.68)

$$(\widetilde{T}^2)^{-1}(\widetilde{\overline{X}}_{\beta_q}^+)^{(k)} = T_{i_{k_{n+1}}} \ldots T_{i_{k_{l'}}}(\widehat{X}_{\delta_q}^+)^{(k)} = T_{w_2}(\widehat{X}_{\delta_q}^+)^{(k)} \in T_{i_{k_{n+1}}} \ldots T_{i_{k_{l'}}}(U_q^{res}(\mathfrak{n}_+)) = T_{w_2}(U_q^{res}(\mathfrak{n}_+)),$$

where $T_{w_2} = T_{i_{k_{n+1}}} \ldots T_{i_{k_{l'}}}$. Thus

$$(\widetilde{T}^2)^{-1}(\overline{T}_w^{-1}(\overline{U}_q^{res}([\gamma_{n+1}, \gamma_{l'}]))) \subset T_{w_2}(U_q^{res}(\mathfrak{n}_+)). \tag{4.1.71}$$

Now, from (4.1.67) and (4.1.71) we obtain

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} (X_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (X_{\gamma_{l'}}^+)^{(m_{l'})} v) = \tag{4.1.72}$$

$$= \sum_j (u, T_{\gamma_1} \ldots T_{\gamma_n} p_j T_w \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} ((\widetilde{T}^2)^{-1}(b_j)) T_w^{-1} v) = \sum_j (u, T_{\gamma_1} \ldots T_{\gamma_n} p_j T_w \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} (T_{w_2}(b_j')) T_w^{-1} v) =$$

$$= \sum_j (u, T_{\gamma_1} \ldots T_{\gamma_n} p_j T_w \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_{w_2} b_j' T_{w_2}^{-1} T_w^{-1} v) = \sum_j (u, T_{\gamma_1} \ldots T_{\gamma_n} p_j T b_j' T_{w'}^{-1} v),$$

where $(\widetilde{T}^2)^{-1}(b_j) = T_{w_2}(b_j')$, $b_j' \in U_q^{res}(\mathfrak{n}_+)$,

$$T = T_w \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_{w_2},$$

and

$$T_{w'} = T_w T_{w_2} = T_{i_1} \ldots T_{i_{k_{l'}}},$$

for

$$w' = w w_2 = s_{i_1} \ldots s_{i_{k_{l'}}}. \tag{4.1.73}$$

We find the action of T^{-1} on the algebra $U_q^{res}([-\gamma_1, -\gamma_n])$. First we obtain a convenient expression for T. If

$$\overline{w} = s_{i_1} \ldots s_{i_{k_{n+1}}} \ldots s_{i_{k_{l'}}} s_{i_{k_{l'}+1}} \ldots s_{i_r} \ldots s_{i_D} \tag{4.1.74}$$

is the reduced decomposition of the longest element of the Weyl group corresponding to the normal ordering in Δ_+ associated to s, so that $\widetilde{s}^2 = s_{i_{k_{n+1}}} \ldots s_{i_{k_{l'}}} s_{i_{k_{l'}+1}} \ldots s_{i_r}$ is the corresponding reduced decomposition of $\widetilde{s}^2$, then, similarly to (4.1.48), we have

$$\widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} = T_{i_r} \ldots T_{i_{k_{l'}+1}} T_{i_{k_{l'}}}^{-1} T_{i_{k_{l'}-1}}^{-1} \ldots T_{i_{k_{n+1}}}^{-1},$$

and hence

$$\widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_{w_2} = \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_{i_{k_{n+1}}} \ldots T_{i_{k_{l'}}} = T_{i_r} \ldots T_{i_{k_{l'}+1}}. \tag{4.1.75}$$

This identity together with the definition of T_w imply

$$T = T_w \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_{w_2} = T_{i_1} \ldots T_{i_{k_{n+1}-1}} T_{i_r} \ldots T_{i_{k_{l'}+1}}. \tag{4.1.76}$$

Observe that T is the braid group element corresponding to the reduced decomposition

$$s_{i_1} \ldots s_{i_{k_{n+1}-1}} s_{i_r} \ldots s_{i_{k_{l'}+1}} \tag{4.1.77}$$

which is a part of the reduced decomposition

$$\overline{w} = s_{i_1} \ldots s_{i_{k_{n+1}-1}} s_{i_r} \ldots s_{i_{k_{l'}+1}} s_{i_{k_{l'}}} \ldots s_{i_{k_{n+1}}} \ldots s_{i_D}$$

obtained from reduced decomposition (4.1.74),

$$\overline{w} = s_{i_1} \ldots s_{i_{k_{n+1}-1}} s_{i_{k_{n+1}}} \ldots s_{i_{k_{l'}}} s_{i_{k_{l'}+1}} \ldots s_{i_r} \ldots s_{i_D},$$

by inverting the part $\widetilde{s}^2 = s_{i_{k_{n+1}}} \ldots s_{i_{k_{l'}}} s_{i_{k_{l'}+1}} \ldots s_{i_r}$. This inversion gives a reduced decomposition again because $\widetilde{s}^2 = -1$ is the longest element in $\widetilde{W}(\mathfrak{m}_{s^2}, \mathfrak{h}_{s^2})$.

Now for $\beta_1 \leq \beta_t \leq \beta_{k_n}$, $\beta_t = s_{i_1} \ldots s_{i_{t-1}} \alpha_{i_t}$, $X_{\beta_t}^- = T_{i_1} \ldots T_{i_{t-1}} X_{i_t}^-$, $t = 1, \ldots, k_n$, we have by (4.1.76), by commutation relations (2.3.6), and by the definition of the braid group action (see (2.2.1)), for all generators $(X_{\beta_t}^-)^{(k)}$, $t = 1, \ldots, k_n$, $k \in \mathbb{N}$, of the algebra $U_q^{res}([-\gamma_1, -\gamma_n])$

$$T^{-1}(X_{\beta_t}^-)^{(k)} = \frac{1}{[k]_{q_{i_t}}!} T_{i_{k_{l'}+1}}^{-1} \ldots T_{i_r}^{-1} T_{i_{k_{n+1}-1}}^{-1} \ldots T_{i_{t+1}}^{-1} T_{i_t}^{-1} \ldots T_{i_1}^{-1} T_{i_1} \ldots T_{i_{t-1}} (X_{i_t}^-)^k = \tag{4.1.78}$$

$$= \frac{1}{[k]_{q_{i_t}}!} (T_{i_{k_{l'}+1}}^{-1} \ldots T_{i_r}^{-1} T_{i_{k_{n+1}-1}}^{-1} \ldots T_{i_{t+1}}^{-1} T_{i_t}^{-1} X_{i_t}^-)^k = \frac{1}{[k]_{q_{i_t}}!} (-T_{i_{k_{l'}+1}}^{-1} \ldots T_{i_r}^{-1} T_{i_{k_{n+1}-1}}^{-1} \ldots T_{i_{t+1}}^{-1} X_{i_t}^+ K_{i_t})^k =$$

$$= \frac{1}{[k]_{q_{i_t}}!} (-T_{i_{k_{l'}+1}}^{-1} \ldots T_{i_r}^{-1} T_{i_{k_{n+1}-1}}^{-1} \ldots T_{i_{t+1}}^{-1} (X_{i_t}^+) R_t)^k = (-1)^k q_{i_t}^{k(k-1)} T_{i_{k_{l'}+1}}^{-1} \ldots T_{i_r}^{-1} T_{i_{k_{n+1}-1}}^{-1} \ldots T_{i_{t+1}}^{-1} ((X_{i_t}^+)^{(k)}) R_t^k,$$

where

$$R_t = T_{i_{k_{l'}+1}}^{-1} \ldots T_{i_r}^{-1} T_{i_{k_{n+1}-1}}^{-1} \ldots T_{i_{t+1}}^{-1} (K_{i_t}).$$

Since $s_{i_{k_{l'}+1}} \ldots s_{i_r} s_{i_{k_{n+1}-1}} \ldots s_{i_1}$ is a reduced decomposition by (4.1.77), by part (iv) of Lemma 1.6.3 we can consider it as an initial part of a reduced decomposition of the longest element $\overline{w} \in W$, and hence by Lemma 2.4.2 (v)

$$Z_t^{(k)} = T_{i_{k_{l'}+1}}^{-1} \ldots T_{i_r}^{-1} T_{i_{k_{n+1}-1}}^{-1} \ldots T_{i_{t+1}}^{-1} (X_{i_t}^+)^{(k)} \in U_q^{res}(\mathfrak{n}_+). \tag{4.1.79}$$

Therefore for $\beta_1 \leq \beta_t \leq \beta_{k_n}$ using (4.1.79) one has from (4.1.78) for all generators $(X_{\beta_t}^-)^{(k)}$, $t = 1, \ldots, k_n$, $k \in \mathbb{N}$, of the algebra $U_q^{res}([-\gamma_1, -\gamma_n])$

$$T^{-1}(X_{\beta_t}^-)^{(k)} = (-1)^k q_{i_t}^{k(k-1)} Z_t^{(k)} R_t^k \in U_q^{res}(\mathfrak{b}_+). \tag{4.1.80}$$

This implies

$$T^{-1}(U_q^{res}([-\gamma_1, -\gamma_n])) \subset U_q^{res}(\mathfrak{b}_+),$$

and hence (4.1.72) takes the form

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} (X_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (X_{\gamma_{l'}}^+)^{(m_{l'})} v) = \tag{4.1.81}$$

$$= \sum_j (u, T_{\gamma_1} \ldots T_{\gamma_n} T p_j' b_j' T_{w'}^{-1} v) = (u, T_{\gamma_1} \ldots T_{\gamma_n} T X' T_{w'}^{-1} v),$$

where $p_j' = T^{-1}(p_j) \in U_q^{res}(\mathfrak{b}_+)$, so that $X' = \sum_j p_j' b_j' \in U_q^{res}(\mathfrak{b}_+)$.

We also have by the definitions of $\widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} = T_w^{-1}(T_{\gamma_{n+1}} \ldots T_{\gamma_{l'}})$ and of $T_{w'} = T_w T_{w_2}$ that

$$T_{\gamma_1} \ldots T_{\gamma_n} T = T_{\gamma_1} \ldots T_{\gamma_n} T_w \widetilde{T}_{\gamma_{n+1}} \ldots \widetilde{T}_{\gamma_{l'}} T_{w_2} = T_{\gamma_1} \ldots T_{\gamma_n} T_{\gamma_{n+1}} \ldots T_{\gamma_{l'}} T_w T_{w_2} =$$

$$= T_{\gamma_1} \ldots T_{\gamma_n} T_{\gamma_{n+1}} \ldots T_{\gamma_{l'}} T_{w'}.$$

Therefore we can rewrite (4.1.81) as follows

$$(u, (X_{\gamma_1}^+)^{(m_1)} \ldots (X_{\gamma_n}^+)^{(m_n)} (X_{\gamma_{n+1}}^+)^{(m_{n+1})} \ldots (X_{\gamma_{l'}}^+)^{(m_{l'})} v) = (u, T_{\gamma_1} \ldots T_{\gamma_n} T_{\gamma_{n+1}} \ldots T_{\gamma_{l'}} T_{w'} X' T_{w'}^{-1} v). \tag{4.1.82}$$

Now recall that by Corollary 2.4.4 (ii) $X' \in U_q^{res}(\mathfrak{b}_+) = U_q^{res}(\mathfrak{n}_+)U_q^{res}(H)$. Observe also that $T_{w'}^{-1}v \in V$ is a weight vector, and by Lemma 2.8.6 elements of $U_q^{res}(H)$ act on it by multiplication by elements of $\mathbb{C}[q,q^{-1}]$. Therefore (4.1.82) immediately implies (4.1.63).

Finally note that $T_{w'}^{-1}v$ and $T_{w'}^{-1}T_{\gamma_{l'}}\ldots T_{\gamma_1}u$ are weight vectors, and different weight subspaces of V are orthogonal with respect to the bilinear form $(\cdot,\cdot)$. Therefore we can assume also that $X(m_1,\ldots,m_{l'})$ in (4.1.63) belongs to a weight subspace of $U_q^{res}(\mathfrak{g})$, so that the weight of $T_{w'}^{-1}T_{\gamma_{l'}}\ldots T_{\gamma_1}u$ is equal to that of $X(m_1,\ldots,m_{l'})T_{w'}^{-1}v$. This completes the proof.

$\square$

Lemma 4.1.8. *Let V be a finite rank representation of $U_h(\mathfrak{g})$, $u,v \in V$ weight vectors. Suppose that u is a highest weight vector. Then for any $m_1,\ldots,m_{l'} \in \mathbb{N}$*

$$(u,(X_{\gamma_1}^+)^{(m_1)}\ldots(X_{\gamma_n}^+)^{(m_n)}(X_{\gamma_{n+1}}^+)^{(m_{n+1})}\ldots(X_{\gamma_{l'}}^+)^{(m_{l'})}v) =$$

$$= (u,T_{\gamma_1}\ldots T_{\gamma_n}T_{\gamma_{n+1}}\ldots T_{\gamma_{l'}}Yv) = c(u,T_sYv),$$

where

$$Y = \sum_{n_1,\ldots,n_D} F_{n_1,\ldots,n_D}(X_{\beta_{k_{l'}}}^-)^{(n_{k_{l'}})}\ldots(X_{\beta_1}^-)^{(n_1)}(X_{\beta_{k_{l'}+1}}^+)^{(n_{k_{l'}+1})}\ldots(X_{\beta_D}^+)^{(n_D)} \in U_q^{res}(w'(\mathfrak{b}_+)),$$

$F_{n_1,\ldots,n_D} \in \mathbb{C}[q,q^{-1}]$*, the sum is finite, and $c \in \mathbb{C}[q,q^{-1}]^*$ is an integer power of q up to a numeric factor.*

Proof. We bring the right hand side of formula (4.1.63) to the form stated in this lemma. Firstly, using part (iv) of Lemma 1.6.3 we complete the final part $s_{i_{k_{l'}}}\ldots s_{i_1}$ of reduced decomposition (4.1.70) to a reduced decomposition $\overline{w} = s_{i_{k_{l'}}}\ldots s_{i_1}s_{p_{k_{l'}+1}}\ldots s_{p_D}$, and consider the corresponding normal ordering $\beta'_{k_{l'}},\ldots,\beta'_1,\beta'_{p_{k_{l'}+1}},\ldots,\beta'_{p_D}$ of Δ_+. Define the corresponding quantum root vectors

$$\overline{X}_{\beta'_q}^+ = \begin{cases} T_{i_{k_{l'}}}^{-1}\ldots T_{i_{q-1}}^{-1}X_{i_q}^+ & 1 \leq q \leq k_{l'} \\ T_{w'}^{-1}T_{p_{k_{l'}+1}}^{-1}\ldots T_{p_{q-1}}^{-1}X_{p_q}^+ & k_{l'}+1 \leq q \leq D \end{cases},$$

and the basis of $U_q^{res}(\mathfrak{n}_+)$ as in Lemma 2.4.2 (v),

$$(\overline{X}_{\beta'_{k_{l'}}}^+)^{(n_{k_{l'}})}\ldots(\overline{X}_{\beta'_1}^+)^{(n_1)}(\overline{X}_{\beta'_{k_{l'}+1}}^+)^{(n_{k_{l'}+1})}\ldots(\overline{X}_{\beta'_D}^+)^{(n_D)}. \tag{4.1.83}$$

Similarly, we also complete the final segment $s_{i_{k_{l'}+1}}\ldots s_{i_D}$ of reduced decomposition (4.1.74) to another reduced decomposition $\overline{w} = s_{i_{k_{l'}+1}}\ldots s_{i_D}s_{p_1}\ldots s_{p_{k_{l'}}}$, and consider the corresponding normal ordering $\beta''_{k_{l'}},\ldots,\beta''_1,\beta''_{p_{k_{l'}+1}},\ldots,\beta''_{p_D}$ of Δ_+. Define the corresponding quantum root vectors

$$X_{\beta''_q}^+ = \begin{cases} T_{w''}T_{p_1}\ldots T_{p_{q-1}}X_{p_q}^+ & 1 \leq q \leq k_{l'} \\ T_{i_{k_{l'}+1}}\ldots T_{i_{q-1}}X_{i_q}^+ & k_{l'}+1 \leq q \leq D \end{cases},$$

where $T_{w''} = T_{i_{k_{l'}+1}}\ldots T_{i_D}$ for $w'' = s_{i_{k_{l'}+1}}\ldots s_{i_D}$, and the basis of $U_q^{res}(\mathfrak{n}_+)$ as in Lemma 2.4.2 (i),

$$(X_{\beta''_{k_{l'}+1}}^+)^{(n_{k_{l'}+1})}\ldots(X_{\beta''_D}^+)^{(n_D)}(X_{\beta''_1}^+)^{(n_1)}\ldots(X_{\beta''_{k_{l'}}}^+)^{(n_{k_{l'}})}. \tag{4.1.84}$$

We can represent the monomials $(\overline{X}_{\beta'_{k_{l'}+1}}^+)^{(n_{k_{l'}+1})}\ldots(\overline{X}_{\beta'_D}^+)^{(n_D)}$ using basis (4.1.84),

$$(\overline{X}_{\beta'_{k_{l'}+1}}^+)^{(n_{k_{l'}+1})}\ldots(\overline{X}_{\beta'_D}^+)^{(n_D)} = \tag{4.1.85}$$

$$= \sum_{q_1,\ldots,q_D} C_{q_1,\ldots q_D}(X_{\beta''_{k_{l'}+1}}^+)^{(q_{k_{l'}+1})}\ldots(X_{\beta''_D}^+)^{(q_D)}(X_{\beta''_1}^+)^{(q_1)}\ldots(X_{\beta''_{k_{l'}}}^+)^{(q_{k_{l'}})},$$

where $C_{q_1,\ldots q_D} \in \mathbb{C}[q,q^{-1}]$. Applying $T_{w'}$ to this identity we get

$$(\widehat{\overline{X}}_{\beta'_{k_{l'}+1}}^+)^{(n_{k_{l'}+1})}\ldots(\widehat{\overline{X}}_{\beta'_D}^+)^{(n_D)} = \tag{4.1.86}$$

$$= \sum_{q_1,\ldots,q_D} C_{q_1,\ldots q_D}(X^+_{\beta_{k_{l'}+1}})^{(q_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(q_D)}(\check{X}^+_{\beta''_1})^{(q_1)}\ldots(\check{X}^+_{\beta''_{k_{l'}}})^{(q_{k_{l'}})},$$

where by Lemma 2.4.2 (v) for $k_{l'}+1 \leq q \leq D$

$$\widehat{\overline{X}}^+_{\beta'_q} = T_{w'}\overline{X}^+_{\beta'_q} = T^{-1}_{p_{k_{l'}+1}}\ldots T^{-1}_{p_{q-1}}X^+_{p_q} \in U^{res}_q(\mathfrak{n}_+), \tag{4.1.87}$$

and for $1 \leq q \leq k_{l'}$

$$\check{X}^+_{\beta''_q} = T_{w'}X^+_{\beta''_q} = T_{w'}T_{w''}T_{p_1}\ldots T_{p_{q-1}}X^+_{p_q} = T_{\overline{w}}T_{p_1}\ldots T_{p_{q-1}}X^+_{p_q} \in U^{res}_q(\mathfrak{b}_-), \tag{4.1.88}$$

where we used the fact that $T_{p_1}\ldots T_{p_{q-1}}X^+_{p_q} \in U^{res}_q(\mathfrak{n}_+)$ by Lemma 2.4.2 (i) and $T_{\overline{w}}U^{res}_q(\mathfrak{n}_+) \subset U^{res}_q(\mathfrak{b}_-)$ by Lemma 2.4.6.

Observe that for $1 \leq q \leq k_{l'}$, $s_{p_1}\ldots s_{p_{q-1}}\alpha_{p_q} \in \Delta_+$ by Lemma 1.6.3 (iv) as $s_{p_1}\ldots s_{p_{k_{l'}}}$ is a reduced decomposition being a part of the reduced decomposition $\overline{w} = s_{i_{k_{l'}+1}}\ldots s_{i_D}s_{p_1}\ldots s_{p_{k_{l'}}}$. Therefore $\overline{w}s_{p_1}\ldots s_{p_{q-1}}\alpha_{p_q} \in \Delta_-$.

Thus the elements $\check{X}^+_{\beta''_q} \in U^{res}_q(\mathfrak{b}_-)$, $1 \leq q \leq k_{l'}$ have strictly negative weights, $\mathrm{wt}\check{X}^+_{\beta''_q} = \overline{w}s_{p_1}\ldots s_{p_{q-1}}\alpha_{p_q} < 0$. Now by (4.1.87) and (4.1.88) we have that in (4.1.86)

$$(\widehat{\overline{X}}^+_{\beta'_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(\widehat{\overline{X}}^+_{\beta'_D})^{(n_D)} \in U^{res}_q(\mathfrak{n}_+),$$

$$(X^+_{\beta_{k_{l'}+1}})^{(q_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(q_D)} \in U^{res}_q(\mathfrak{n}_+),$$

and the elements

$$(\check{X}^+_{\beta''_1})^{(q_1)}\ldots(\check{X}^+_{\beta''_{k_{l'}}})^{(q_{k_{l'}})} \in U^{res}_q(\mathfrak{b}_-) = U^{res}_q(H)U^{res}_q(\mathfrak{n}_-)$$

have strictly negative weights unless $q_1 = \ldots = q_{k_{l'}} = 0$.

By the uniqueness of the Poincaré–Birkhoff–Witt decomposition (2.4.2) in Lemma 2.4.2 we deduce that (4.1.86) takes the form

$$(\widehat{\overline{X}}^+_{\beta'_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(\widehat{\overline{X}}^+_{\beta'_D})^{(n_D)} =$$

$$= \sum_{q_{k_{l'}+1},\ldots,q_D} C_{q_{k_{l'}+1},\ldots q_D}(X^+_{\beta_{k_{l'}+1}})^{(q_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(q_D)},$$

where $C_{q_{k_{l'}+1},\ldots,q_D} \in \mathbb{C}[q,q^{-1}]$.

Applying $T^{-1}_{w'}$ to this identity we obtain that (4.1.85) takes the form

$$(\overline{X}^+_{\beta'_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(\overline{X}^+_{\beta'_D})^{(n_D)} = \sum_{q_{k_{l'}+1},\ldots,q_D} C_{q_{k_{l'}+1},\ldots q_D}(X^+_{\beta''_{k_{l'}+1}})^{(q_{k_{l'}+1})}\ldots(X^+_{\beta''_D})^{(q_D)}. \tag{4.1.89}$$

Recalling basis (4.1.83) we infer that every element of $U^{res}_q(\mathfrak{n}_+)$ is a $\mathbb{C}[q,q^{-1}]$–linear combination of monomials of the form

$$(\overline{X}^+_{\beta'_{k_{l'}}})^{(n_{k_{l'}})}\ldots(\overline{X}^+_{\beta'_1})^{(n_1)}(X^+_{\beta''_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta''_D})^{(n_D)}. \tag{4.1.90}$$

Kostant's formula shows that they form a $\mathbb{C}[q,q^{-1}]$–basis of $U^{res}_q(\mathfrak{n}_+)$.

Now for $1 \leq q \leq k_{l'}$ we have

$$T_{w'}\overline{X}^+_{\beta'_q} = T_{w'}T^{-1}_{i_{k_{l'}}}\ldots T^{-1}_{i_{q-1}}X^+_{i_q} = T_{i_1}\ldots T_{i_q}X^+_{i_q} = -T_{i_1}\ldots T_{i_{q-1}}X^-_{i_q}K_{i_q} = -X^-_{\beta_q}K_{\beta_q}, \tag{4.1.91}$$

where

$$K_{\beta_q} = T_{i_1}\ldots T_{i_{q-1}}K_{i_q},$$

and for $k_{l'}+1 \leq q \leq D$

$$T_{w'}X^+_{\beta''_q} = T_{w'}T_{i_{k_{l'}+1}}\ldots T_{i_{q-1}}X^+_{i_q} = X^+_{\beta_q}. \tag{4.1.92}$$

The last two identities and commutation relations between elements K_{β_q} and quantum root vectors following from relations (2.3.5) imply

$$T_{w'}\left((\overline{X}^+_{\beta'_{k_{l'}}})^{(n_{k_{l'}})}\ldots(\overline{X}^+_{\beta'_1})^{(n_1)}(X^+_{\beta''_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta''_D})^{(n_D)}\right) = \tag{4.1.93}$$

$$= Q_{n_1,\ldots,n_{k_{l'}}}(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})}\ldots(X^-_{\beta_1})^{(n_1)}(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(n_D)},$$

where $Q_{n_1,\ldots,n_{k_{l'}}}$ is a monomial in $K_1^{\pm 1},\ldots,K_l^{\pm 1}$.

Recalling basis (4.1.90) we can express $X \in U_q^{res}(\mathfrak{n}_+)$ in (4.1.63) as follows

$$X = \sum_{n_1,\ldots,n_D} \overline{F}_{n_1,\ldots,n_D}(\overline{X}^+_{\beta'_{k_{l'}}})^{(n_{k_{l'}})}\ldots(\overline{X}^+_{\beta'_1})^{(n_1)}(X^+_{\beta''_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta''_D})^{(n_D)},$$

where $\overline{F}_{n_1,\ldots,n_D} \in \mathbb{C}[q,q^{-1}]$ and the sum is finite.

Now using (4.1.93) we deduce that in (4.1.63)

$$T_{w'}XT_{w'}^{-1} = \sum_{n_1,\ldots,n_D} F'_{n_1,\ldots,n_D}(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})}\ldots(X^-_{\beta_1})^{(n_1)}(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(n_D)}, \tag{4.1.94}$$

where $F'_{n_1,\ldots,n_D} = \overline{F}_{n_1,\ldots,n_D}Q_{n_1,\ldots,n_{k_{l'}}} \in U_q(H) \subset U_q^{res}(H)$.

Observing that $T_{\gamma_{l'}}\ldots T_{\gamma_{n+1}}T_{\gamma_n}\ldots T_{\gamma_1}u$ is a weight vector and using Lemma 2.8.6, formula (4.1.63) can be rewritten as follows

$$(u,(X^+_{\gamma_1})^{(m_1)}\ldots(X^+_{\gamma_n})^{(m_n)}(X^+_{\gamma_{n+1}})^{(m_{n+1})}\ldots(X^+_{\gamma_{l'}})^{(m_{l'})}v) = \tag{4.1.95}$$

$$= \sum_{n_1,\ldots,n_D} F_{n_1,\ldots,n_D}(u,T_{\gamma_1}\ldots T_{\gamma_n}T_{\gamma_{n+1}}\ldots T_{\gamma_{l'}}(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})}\ldots(X^-_{\beta_1})^{(n_1)}(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(n_D)}v) =$$

$$= (u,T_{\gamma_1}\ldots T_{\gamma_n}T_{\gamma_{n+1}}\ldots T_{\gamma_{l'}}Yv),$$

where

$$Y = \sum_{n_1,\ldots,n_D} F_{n_1,\ldots,n_D}(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})}\ldots(X^-_{\beta_1})^{(n_1)}(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(n_D)} \in U_q^{res}(w'(\mathfrak{b}_+)),$$

$F_{n_1,\ldots,n_D} \in \mathbb{C}[q,q^{-1}]$ and the sum is finite. This proves the first formula in the statement of this lemma.

To justify the last formula in the statement of the lemma we observe that $T_{\gamma_{l'}}\ldots T_{\gamma_{n+1}}T_{\gamma_n}\ldots T_{\gamma_1}$ and T_s^{-1} act as the same transformations of $\mathfrak{h} \subset U_h(\mathfrak{h})$ and apply Lemma 2.8.7. This completes the proof. $\qquad\square$

In the course of the proof of the previous lemma we obtained the following result.

Corollary 4.1.9. *The products*

$$(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})}\ldots(X^-_{\beta_1})^{(n_1)}(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(n_D)} \tag{4.1.96}$$

or

$$(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})}\ldots(X^+_{\beta_D})^{(n_D)}(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})}\ldots(X^-_{\beta_1})^{(n_1)} \tag{4.1.97}$$

form $U_q^{res}(H)$–bases in the subalgebra $U_q^{res}(w'(\mathfrak{b}_+))$ of $U_q^{res}(\mathfrak{g})$ generated over $U_q^{res}(H)$ by the elements

$$(X^-_{\beta_{k_{l'}}})^{(n_{k_{l'}})},\ldots,(X^-_{\beta_1})^{(n_1)},(X^+_{\beta_{k_{l'}+1}})^{(n_{k_{l'}+1})},\ldots,(X^+_{\beta_D})^{(n_D)}, n_i \in \mathbb{N}, i = 1,\ldots,D. \tag{4.1.98}$$

Proof. If $Y \in U_q^{res}(w'(\mathfrak{b}_+))$ then by (4.1.91) and (4.1.92) we have $T_{w'}^{-1}(Y) \in U_q^{res}(\mathfrak{b}_+)$. By (4.1.94) $Y = T_{w'}(T_{w'}^{-1}(Y))$ can be represented as a $U_q^{res}(H)$–linear combination of elements (4.1.96). By Kostant's formula they form a $U_q^{res}(H)$–basis in $U_q^{res}(w'(\mathfrak{b}_+))$.

The case of elements (4.1.97) is considered in a similar way. This completes the proof. $\qquad\square$

Remark 4.1.10. *The use of the symbol $U_q^{res}(w'(\mathfrak{b}_+))$ is motivated by the fact that the algebra $U_q^{res}(w'(\mathfrak{b}_+))$ is generated by elements (4.1.98) which are defined with the help of the quantum root vectors corresponding to the roots $-\beta_{k_{l'}},\ldots,-\beta_1,\beta_{k_{l'}+1},\ldots,\beta_D$. They form a system of positive roots $\Delta^{w'}_+$ in Δ such that $w'^{-1}\Delta^{w'}_+ = \Delta_+$. Indeed, recalling (4.1.73) we obtain that $\Delta_{w'^{-1}} = \{\beta_1,\ldots,\beta_{k_{l'}}\}$ by Lemma 1.6.3 (iv), and hence $w'^{-1}\Delta^{w'}_+ = \Delta_+$ as the number of the roots in $\Delta^{w'}_+$ is the same as in Δ_+. So, we infer that $\Delta^{w'}_+ = w'\Delta_+$. Thus $w'(\mathfrak{b}_+) := \mathrm{Ad}w'(\mathfrak{b}_+)$ is the Borel subalgebra of $\mathfrak{g}$ the nilradical of which is generated by the root vectors corresponding to the roots from $\Delta^{w'}_+$, and $U_q^{res}(w'(\mathfrak{b}_+))$ is the restricted version of the quantum counterpart of the enveloping algebra $U(w'(\mathfrak{b}_+))$.*

4.2   Some auxiliary results on the quantized algebra of regular functions on an algebraic Poisson–Lie group

In this section we give several formulas related to the adjoint action and commutation relations in the algebra $\mathbb{C}^s_{\mathcal{B}}[G]$.

Let $\beta_1, \ldots, \beta_N$ be a normal order on Δ_+, $f_{\beta_1}, \ldots, f_{\beta_D} \in U^{s,res}_{\mathcal{B}}(\mathfrak{g})$ the corresponding root vectors defined with the help of this normal ordering.

Firstly, following [12], Theorem I.8.16 we recall the commutation relations in the algebra $\mathbb{C}^s_{\mathcal{B}}[G]$ which follow from the fact that $U^s_h(\mathfrak{g})$ is quasitriangular. Namely, if V, V' are finite rank representations of $U_h(\mathfrak{g})$, $(V)_\eta, (V')_\rho$, $(V)_\beta, (V')_\gamma$ their weight subspaces of weights η, ρ, β and γ, respectively, and $v \in (V)_\eta, v_1 \in (V')_\rho$, $u \in (V)_\beta, u_1 \in (V')_\gamma$ then evaluating the identity $\Delta^{opp}_s(x)\mathcal{R}^s = \mathcal{R}^s\Delta_s(x)$ on the matrix element $(u, \cdot\, v) \otimes (u_1, \cdot\, v_1)$ and recalling formula (2.6.16) we obtain

$$q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\eta^\vee,\rho^\vee)}\left((u_1, \cdot\, v_1) \otimes (u, \cdot\, v) + \sum_{\nu \in Q_+, \nu \neq 0} \sum_{\substack{n_1, \ldots, n_D \in \mathbb{N}, \\ n_1\beta_1 + \ldots + n_D\beta_D = \nu}} (u_1, \cdot\, u_{n_1,\ldots,n_D}v_1) \otimes (u, \cdot\, u_{-n_1,\ldots,-n_D}v)\right) =$$

$$= q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\beta^\vee,\gamma^\vee)}(u, \cdot\, v) \otimes (u_1, \cdot\, v_1) + \tag{4.2.1}$$

$$+ \sum_{\nu \in Q_+, \nu \neq 0} \sum_{\substack{n_1, \ldots, n_D \in \mathbb{N}, \\ n_1\beta_1 + \ldots + n_D\beta_D = \nu}} q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)(\beta^\vee+\nu^\vee),\gamma^\vee-\nu^\vee)}(\omega(u_{-n_1,\ldots,-n_D})u, \cdot\, v) \otimes (\omega(u_{n_1,\ldots,n_D})u_1, \cdot\, v_1),$$

where

$$u_{-n_1,\ldots,-n_D} = c_{-n_1,\ldots,-n_D}f^{(n_1)}_{\beta_1}\ldots f^{(n_D)}_{\beta_D},$$

$$u_{n_1,\ldots,n_D} = c_{n_1,\ldots,n_D}e^{n_1}_{\beta_1}\ldots e^{n_D}_{\beta_D},$$

$c_{\pm n_1,\ldots,\pm n_D} \in \mathcal{B}$, and similarly evaluating the identity $\Delta^{opp}_s(x)\mathcal{R}^{s\,-1}_{21} = \mathcal{R}^{s\,-1}_{21}\Delta_s(x)$ on the matrix element $(u, \cdot\, v) \otimes (u_1, \cdot\, v_1)$ we get

$$q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\eta^\vee,\rho^\vee)}\left((u_1, \cdot\, v_1) \otimes (u, \cdot\, v) + \sum_{\nu \in Q_+, \nu \neq 0} \sum_{\substack{n_1, \ldots, n_D \in \mathbb{N}, \\ n_1\beta_1 + \ldots + n_D\beta_D = \nu}} (u_1, \cdot\, u'_{-n_1,\ldots,-n_D}v_1) \otimes (u, \cdot\, u'_{n_1,\ldots,n_D}v)\right) =$$

$$= q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta^\vee,\gamma^\vee)}(u, \cdot\, v) \otimes (u_1, \cdot\, v_1) +$$

$$+ \sum_{\nu \in Q_+, \nu \neq 0} \sum_{\substack{n_1, \ldots, n_D \in \mathbb{N}, \\ n_1\beta_1 + \ldots + n_D\beta_D = \nu}} q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)(\beta^\vee+\nu^\vee),\gamma^\vee-\nu^\vee)}(\omega(u'_{n_1,\ldots,n_D})u, \cdot\, v) \otimes (\omega(u'_{-n_1,\ldots,-n_D})u_1, \cdot\, v_1),$$

where

$$u'_{-n_1,\ldots,-n_D} = c'_{-n_1,\ldots,-n_D}f^{(n_D)}_{\beta_D}\ldots f^{(n_1)}_{\beta_1},$$

$$u'_{n_1,\ldots,n_D} = c'_{n_1,\ldots,n_D}e^{n_D}_{\beta_D}\ldots e^{n_1}_{\beta_1},$$

$c'_{\pm n_1,\ldots,\pm n_D} \in \mathcal{B}$.

If $v = T^{-1}_s v_\lambda \in (V)_{s^{-1}\lambda}, v_1 = T^{-1}_s v_\mu \in (V')_{s^{-1}\mu}$, $u \in (V)_\beta, u_1 = v_\mu \in (V')_\mu$, where $v_\lambda \in V$ and $v_\mu \in V'$ are highest weight vectors, then the previous identity yields

$$q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\lambda^\vee,\mu^\vee)}(v_\mu, \cdot\, T^{-1}_s v_\mu) \otimes (u, \cdot\, T^{-1}_s v_\lambda) = \tag{4.2.2}$$

$$= q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta^\vee,\mu^\vee)}(u, \cdot\, T^{-1}_s v_\lambda) \otimes (v_\mu, \cdot\, T^{-1}_s v_\mu).$$

The next lemma shows how the adjoint action behaves with respect to the multiplication in $\mathbb{C}^s_{\mathcal{B}}[G]$.

Lemma 4.2.1. *For any $f, g \in \mathbb{C}^s_{\mathcal{B}}[G], x \in U^{s,res}_{\mathcal{B}}(\mathfrak{g})$ we have*

$$\mathrm{Ad}^0_s x(f \otimes g)(\cdot\,,\cdot\,) = (\mathrm{Ad}^0_s x^2 f)(\cdot\,) \otimes g((\omega_0 S^{-1}_s)(x^1)\cdot\,\omega_0 x^3),$$

where $\Delta^2_s x = (\Delta_s \otimes id)\Delta_s x = x^1 \otimes x^2 \otimes x^3$ in the Sweedler notation.

In particular,

$$\mathrm{Ad}^0_s f_\beta (f \otimes g)(\cdot\,,\cdot\,) = f(\cdot\,)\otimes g((\omega_0 S_s^{-1})(f_\beta)\cdot\,) + (\mathrm{Ad}^0_s f_\beta f)(\cdot\,)\otimes g(G_\beta \cdot\,) + \sum_i (\mathrm{Ad}^0_s x_i f)(\cdot\,)\otimes g((\omega_0 S_s^{-1})(y_i)\cdot\,) + \quad (4.2.3)$$

$$+ f(G_\beta \cdot\ G_\beta^{-1}) \otimes g(G_\beta \cdot\ \omega_0(f_\beta)) + \sum_i (\mathrm{Ad}^0_s y_i^2 f)(\cdot\,) \otimes g((\omega_0 S_s^{-1})(y_i^1)\cdot\ \omega_0 x_i) =$$

$$= f(\cdot\,) \otimes g((\omega_0 S_s^{-1})(f_\beta)\cdot\,) + (\mathrm{Ad}^0_s f_\beta f)(\cdot\,) \otimes g(G_\beta \cdot\,) + f(G_\beta \cdot\ G_\beta^{-1}) \otimes g(G_\beta \cdot\ \omega_0(f_\beta)) +$$

$$+ \sum_i (\mathrm{Ad}^0_s y_i f)(\cdot\,) \otimes g(G_\beta \cdot\ \omega_0 x_i) + \sum_i (\mathrm{Ad}^0_s x_i^1 f)(\cdot\,) \otimes g((\omega_0 S_s^{-1})(y_i)\cdot\ \omega_0 x_i^2)$$

and

$$\mathrm{Ad}^0_s f_\beta^{(n)}(f \otimes g)(\cdot\,,\cdot\,) = \sum_{k=0}^{n}\sum_{p=0}^{n-k} q_\beta^{-k(n-k)-p(n-k-p)}(\mathrm{Ad}^0_s(G_\beta^{-k} f_\beta^{(p)})f)(\cdot\,) \otimes g(\omega_0 S_s^{-1}(G_\beta^{-k-p} f_\beta^{(n-k-p)})\cdot\ \omega_0(f_\beta^{(k)})) +$$

$$+ \sum_{k=0}^{n-1}\sum_i q_\beta^{-k(n-k)}(\mathrm{Ad}^0_s(G_\beta^{-k} x_i^{(n-k)})f)(\cdot\,) \otimes g((\omega_0 S_s^{-1})(G_\beta^{-k} y_i^{(n-k)})\cdot\ \omega_0(f_\beta^{(k)})) + \quad (4.2.4)$$

$$+ \sum_i (\mathrm{Ad}^0_s(y_i^{(n)2})f)(\cdot\,) \otimes g((\omega_0 S_s^{-1})(y_i^{(n)1})\cdot\ \omega_0(x_i^{(n)})),$$

where G_β, x_i, y_i, $x_i^{(p)}$, $y_i^{(p)}$ are defined in (2.8.13) and (2.8.16), and $\Delta_s(x_i) = x_i^1 \otimes x_i^2$, $\Delta_s(y_i) = y_i^1 \otimes y_i^2$, $\Delta_s(y_i^{(p)}) = y_i^{(p)1} \otimes y_i^{(p)2}$ in the Sweedler notation.

Proof. Denote, using the Sweedler notation,

$$\Delta_s^3 x = (\Delta_s \otimes id \otimes id)(\Delta_s \otimes id)\Delta_s x = x^1 \otimes x^2 \otimes x^3 \otimes x^4$$

and observe that the definition of Ad'_s implies that for any $x, z \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$

$$\Delta_s^{opp}(\mathrm{Ad}'_s xz) = (x^2 \otimes x^1)(z^2 \otimes z^1)(S_s x^3 \otimes S_s x^4) = \mathrm{Ad}'_s x^2 z^2 \otimes x^1 z^1 S_s x^3.$$

Let $z = S_s \omega_0 y$, $y \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$. Then, since $\omega_0 S_s^{-1}$ is an algebra homomorphism and a coalgebra anti-homomorphism, we deduce

$$\Delta_s \omega_0 S_s^{-1}(\mathrm{Ad}'_s x S_s \omega_0 y) = (\omega_0 S_s^{-1} \otimes \omega_0 S_s^{-1})\Delta_s^{opp}(\mathrm{Ad}'_s xz) = (\omega_0 S_s^{-1} \otimes \omega_0 S_s^{-1})\mathrm{Ad}'_s x^2 z^2 \otimes x^1 z^1 S_s x^3 =$$

$$= (\omega_0 S_s^{-1} \otimes \omega_0 S_s^{-1})\mathrm{Ad}'_s x^2 (S_s \omega_0)(y^1) \otimes x^1 (S_s \omega_0)(y^2) S_s x^3 = \omega_0 S_s^{-1} \mathrm{Ad}'_s x^2 ((S_s \omega_0)(y^1)) \otimes (\omega_0 S_s^{-1})(x^1)y^2 \omega_0 x^3.$$

Evaluating the last identity on $f \otimes g$ we get the first formula in the statement of the lemma. (4.2.3) and (4.2.4) are obtained from it using (2.8.13) and (2.8.16).

$\square$

It will be convenient to consider a left action Ad^s of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ on $\mathbb{C}_{\mathcal{B}}^s[G]$ defined as follows

$$(\mathrm{Ad}^s xf)(w) = f(\mathrm{Ad}_s x(w)). \quad (4.2.5)$$

It is related to Ad^0_s by the formula

$$(\mathrm{Ad}^s xf)(w) = (\mathrm{Ad}^0_s \omega_0(x)f)(w). \quad (4.2.6)$$

To define quantum analogues of the ideals J^{jk} we need to introduce quantum counterparts of matrix elements (3.5.3). By Proposition 8.3 in [48] there exist integral dominant weights μ_p, $p = 1, \ldots, D$, and elements $v_p \in (V_{\mu_p})_{\mu_p - \beta_p}$ such that $(v_p, \cdot v_{\mu_p}) \in \mathbb{C}_{\mathcal{B}}^s[G]$, and

$$(v_p, \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_1}^{(n_1)})v_{\mu_p}) = \begin{cases} 1 & \text{if } f_{\beta_D}^{(n_D)} \ldots f_{\beta_1}^{(n_1)} = f_{\beta_p} \\ 0 & \text{otherwise} \end{cases}, \quad (4.2.7)$$

where the highest weight vectors $v_{\mu_p} \in V_{\mu_p}$ are normalized by the condition $(v_{\mu_p}, v_{\mu_p}) = 1$.

Since $\omega(\omega_0 S_s^{-1}(f_{\beta_p}))v_p \in (V_{\mu_p})_{\mu_p}$ and the subspace $(V_{\mu_p})_{\mu_p}$ is one–dimensional this definition implies

$$\omega(\omega_0 S_s^{-1}(f_{\beta_p}))v_p = v_{\mu_p}. \tag{4.2.8}$$

In particular, $(v_{\mu_p}, \cdot v_{\mu_p}) \in \mathbb{C}_{\mathcal{B}}^s[G]$, and

$$(v_p, \omega_0 S_s^{-1}(f_{\beta_p}) \cdot T_s^{-1} v_{\mu_p}) = (v_{\mu_p}, \cdot T_s^{-1} v_{\mu_p}). \tag{4.2.9}$$

Let

$$A_p = (v_p, \cdot T_s^{-1} v_{\mu_p}), A_p^0 = (v_{\mu_p}, \cdot T_s^{-1} v_{\mu_p}). \tag{4.2.10}$$

From (4.2.2) we deduce

$$(v_{\mu_p}, \cdot T_s^{-1} v_{\mu_p}) \otimes (v_p, \cdot T_s^{-1} v_{\mu_p}) = q^{((-2K_s+id)\beta_p^\vee, \mu_p^\vee)}(v_p, \cdot T_s^{-1} v_{\mu_p}) \otimes (v_{\mu_p}, \cdot T_s^{-1} v_{\mu_p}),$$

i.e.

$$A_p^0 \otimes A_p = q^{((-2K_s+id)\beta_p^\vee, \mu_p^\vee)} A_p \otimes A_p^0. \tag{4.2.11}$$

More generally, denote $\Delta_\mu^s(\cdot) = (v_\mu, \cdot T_s^{-1} v_\mu) \in \mathbb{C}_{\mathcal{B}}^s[G]$, $\mu \in P_+$, where $v_\mu \in V_\mu$ are highest weight vectors normalized by $(v_\mu, v_\mu) = 1$. Then

$$\Delta_\mu^s \otimes A_p = q^{((-2K_s+id)\alpha_{i_p}^\vee, \mu^\vee)} A_p \otimes \Delta_\mu^s, \tag{4.2.12}$$

The following statement is a quantum counterpart of Lemma 3.5.1. In fact Lemma 3.5.1 can be proved by specializing the statement of Lemma 4.2.2 at $q^{\frac{1}{dr^2}} = 1$.

Lemma 4.2.2. *(i) For any $1 \le q < p \le D$, $y \in \omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}([-\beta_p, -\beta_D]))$ of the form $y = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)})$, $n_i \in \mathbb{N}$, $y \ne 1$, and any $u \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ one has*

$$(v_q, yuv_{\mu_q}) = 0. \tag{4.2.13}$$

In particular, $\omega(y)v_q = 0$ in this case.

(ii) For any $1 \le p \le D$, $y \in \omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}([-\beta_p, -\beta_D]))$ of the form $y = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)})$, $n_i \in \mathbb{N}$, $y \ne 1, \omega_0 S_s^{-1}(f_{\beta_p})$, and any $u \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ one has

$$(v_p, yuv_{\mu_p}) = 0. \tag{4.2.14}$$

In particular, $\omega(y)v_p = 0$ in this case.

Proof. (i) Observe that, by the definitions of ω_0 and of S_s, $\omega_0 S_s^{-1}$ is an algebra automorphism of $U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ such that $\omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}(\mathfrak{n}_\pm)) \subset U_{\mathcal{B}}^{s,res}(\mathfrak{b}_\pm)$ and $\omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}(\mathfrak{h})) = U_{\mathcal{B}}^{s,res}(\mathfrak{h})$. Therefore, since $v_{\mu_p} \in V_{\mu_p}$ is a highest weight vector generating V_{μ_p}, by Lemmas 2.8.2 (iii), 2.8.4 (ii) and 2.8.6 we have

$$V_{\mu_p} = \omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}(\mathfrak{g}))v_{\mu_p} = \omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-))\omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}(\mathfrak{h}))\omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}(\mathfrak{n}_+))v_{\mu_p} = \tag{4.2.15}$$

$$= \omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-))U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_{\mathcal{B}}^{s,res}(\mathfrak{b}_+)v_{\mu_p} = \omega_0 S_s^{-1}(U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-))v_{\mu_p}.$$

Therefore we can write (4.2.13) in the form

$$(v_q, yu)v_{\mu_q} = (v_q, y\omega_0 S_s^{-1}(y')v_{\mu_q}) \tag{4.2.16}$$

for some $y' \in U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$. We claim that that the right hand side of (4.2.16)vanishes.

Indeed, $y\omega_0 S_s^{-1}(y') = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)} y')$ and $f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)} y'$ belongs to the right ideal Y_p in $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ generated by $f_{\beta_u}^{(n_u)}$, $u \ge p > q$ and $n_u > 0$. By Lemma 2.8.2 (vii) the elements $f_{\beta_D}^{(n_D)} \ldots f_{\beta_1}^{(n_1)}$ with at least one $n_u > 0$ for $u \ge p > q$ form a linear basis of this ideal, and this basis does not contain multiples of f_{β_q}. So by (4.2.7) we have

$$(v_q, \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)} y')v_{\mu_q}) = 0.$$

The result that we proved implies that $\omega(y)v_q$ is orthogonal, with respect to the contravariant form, to any element of the form uv_{μ_q}, $u \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$. Since any element of V_{μ_q} is of this form and the contravariant form is non–degenerate we deduce that $\omega(y)v_q = 0$.

(ii) By (4.2.15) we can write, similarly to (4.2.16),

$$(v_p, yuv_{\mu_p}) = (v_p, y\omega_0 S_s^{-1}(y')v_{\mu_p}) = (v_p, \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)} y')v_{\mu_p}) \tag{4.2.17}$$

for some $y' \in U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$.

If at least one $n_i > 0$ for some $i > p$, $f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)} y'$ belongs to the right ideal Y_{p+1} of $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ generated by $f_{\beta_u}^{(m_u)}$, $u > p$ and $m_u > 0$. By Lemma 2.8.2 (vii) the elements $f_{\beta_D}^{(m_D)} \ldots f_{\beta_1}^{(m_1)}$ with at least one $m_u > 0$ for $u > p$ form a linear basis of this ideal, and this basis does not contain multiples of f_{β_p}. Therefore according to (4.2.7)

$$(v_p, \omega_0 S_s^{-1}(y'')v_{\mu_p}) = 0, \text{ for any } y'' \in Y_{p+1}, \tag{4.2.18}$$

and hence the right hand side of (4.2.17) vanishes if at least one $n_i > 0$ for some $i > p$.

If $n_i = 0$ for all $i > p$ we have $f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)} y' = f_{\beta_p}^{(n_p)} y'$, $n_p > 1$. By Lemma 2.8.2 (ix) $y' \in U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-) = U_{\mathcal{B}}^{s,res}([-\beta_{p+1}, -\beta_D])U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_p])$, and hence by Lemma 2.8.2 (viii)

$$f_{\beta_p}^{(n_p)} y' \in f_{\beta_p}^{(n_p)} U_{\mathcal{B}}^{s,res}([-\beta_{p+1}, -\beta_D])U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_p]) \subset \sum_{i=0}^{n_p-1} (U_{\mathcal{B}}^{s,res}([-\beta_{p+1}, -\beta_D]))_0 f_{\beta_p}^{(i)} U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_p]) +$$

$$+ U_{\mathcal{B}}^{s,res}([-\beta_{p+1}, -\beta_D]) f_{\beta_p}^{(n_p)} U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_p]),$$

where $(U_{\mathcal{B}}^{s,res}([-\beta_{p+1}, -\beta_D]))_0 = Y_{p+1} \cap U_{\mathcal{B}}^{s,res}([-\beta_{p+1}, -\beta_D])$, so for $i < n_p$ one has

$$(U_{\mathcal{B}}^{s,res}([-\beta_{p+1}, -\beta_D]))_0 f_{\beta_p}^{(i)} U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_p]) \subset Y_{p+1}.$$

Thus by (4.2.18)

$$(v_p, \omega_0 S_s^{-1}(f_{\beta_p}^{(n_p)} y')v_{\mu_p}) = (v_p, \omega_0 S_s^{-1}(y''')v_{\mu_p}), y''' \in U_{\mathcal{B}}^{s,res}([-\beta_{p+1}, -\beta_D]) f_{\beta_p}^{(n_p)} U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_p]).$$

By Lemma 2.8.2 (vi) and by Remark 2.8.3 the decompositions of elements of $U_{\mathcal{B}}^{s,res}([-\beta_{p+1}, -\beta_D]) f_{\beta_p}^{(n_p)} U_{\mathcal{B}}^{s,res}([-\beta_1, -\beta_p])$ with respect to the basis $f_{\beta_D}^{(m_D)} \ldots f_{\beta_1}^{(m_1)}$ only contain non-zero multiples of elements of the basis with $m_p \geq n_p > 1$. None of these elements is equal to f_{β_p}. Therefore by (4.2.7)

$$(v_p, yuv_{\mu_p}) = (v_p, y\omega_0 S_s^{-1}(y')v_{\mu_p}) = (v_p, \omega_0 S_s^{-1}(f_{\beta_p}^{(n_p)} y')v_{\mu_p}) = (v_p, \omega_0 S_s^{-1}(y''')v_{\mu_p}) = 0.$$

Thus
$$(v_p, yuv_{\mu_p}) = (v_p, f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)} uv_{\mu_p}) = 0$$

if $y = \omega_0 S_s^{-1}(f_{\beta_D}^{(n_D)} \ldots f_{\beta_p}^{(n_p)}) \neq 1, \omega_0 S_s^{-1}(f_{\beta_p})$.

The result that we proved implies that $\omega(y)v_p$ is orthogonal, with respect to the contravariant form, to any element of the form uv_{μ_p}, $u \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$. Since any element of V_{μ_p} is of this form and the contravariant form is non–degenerate we deduce that $\omega(y)v_p = 0$.

This completes the proof of Lemma 4.2.2.

$\square$

In order co define quantum counterparts of functions φ_p we have to introduce a certain localization $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]$ of $\mathbb{C}_{\mathcal{B}}^s[G]$. By Lemma 9.1.10 in [59] it is possible to define a localization of $\mathbb{C}_{\mathcal{B}}^s[G]$ in the sense of localization for non–commutative algebras which contains the quantum counterparts of the denominators in the definition of the functions φ_p given by (4.2.10). We shall not need these results in full generality. In fact we shall only need right denominators and $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]$ will be defined as a "right" localization of $\mathbb{C}_{\mathcal{B}}^s[G]$. The exact meaning of this term will be explained below.

We start by introducing a subalgebra $\mathbb{C}_{\mathcal{B}}^s[G]_0 \subset \mathbb{C}_{\mathcal{B}}^s[G]$ a proper localization of which contains all the required denominators.

Lemma 4.2.3. *The set of elements* $(u, \cdot T_s^{-1} v) \in \mathbb{C}_{\mathcal{B}}^s[G]$, *where v is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$ and $u \in V$ form a subalgebra $\mathbb{C}_{\mathcal{B}}^s[G]_0$ in $\mathbb{C}_{\mathcal{B}}^s[G]$.*

Proof. It suffices to show that the product of two elements from $\mathbb{C}_{\mathcal{B}}^s[G]_0$ belongs to $\mathbb{C}_{\mathcal{B}}^s[G]_0$. Let $s = s_{i_1} \ldots s_{i_k}$ be a reduced decomposition of s. Then by (2.6.17)

$$\Delta_s(T_s) = \prod_{p=1}^{k} \theta_{\beta_p}^s \mathcal{F}_s(T_s \otimes T_s)(\mathcal{F}_s^{-1})T_s \otimes T_s, \tag{4.2.19}$$

where in the products $\theta_{\beta_p}^s$ appears on the left from $\theta_{\beta_q}^s$ if $p < q$, and for $p = 1, \ldots, k$

$$e_{\beta_p} = \psi_{\{n_{ij}\}}^{-1}(X_{\beta_p}^+ e^{hK_s\beta_p^\vee}), f_\beta = \psi_{\{n_{ij}\}}^{-1}(e^{-hK_s\beta_p^\vee}X_{\beta_p}^-), \beta_p = s_{i_1}\ldots s_{i_{p-1}}\alpha_{i_p},$$

$$X_{\beta_p}^\pm = T_{i_1}\ldots T_{i_{p-1}}X_{i_p}^\pm,$$

$$\theta_{\beta_p}^s = \exp_{q_{\beta_p}}[(1 - q_{\beta_p}^{-2})e_{\beta_p}e^{-h\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta_p^\vee} \otimes f_{\beta_p}].$$

Next,

$$\mathcal{F}_s(T_s \otimes T_s)(\mathcal{F}_s^{-1}) = q^{\sum_{i=1}^l (T_sY_i \otimes T_sK_sH_i - Y_i \otimes K_sH_i)} = q^{\sum_{i=1}^l (T_sY_i \otimes K_sT_{s_p}H_i - Y_i \otimes K_sH_i)} = q^{\sum_{i=1}^l (Y_i \otimes K_sH_i - Y_i \otimes K_sH_i)} = 1. \tag{4.2.20}$$

Here we also used the identity $\sum_{i=1}^l T_sY_i \otimes K_sT_sH_i = \sum_{i=1}^l Y_i \otimes K_sH_i$ which holds since $T_sY_i, T_sH_i, i = 1, \ldots, l$ is a pair of dual bases in $\mathfrak{h}$.

Now by (4.2.20) formula (4.2.19) takes the form

$$\Delta_s(T_s) = \prod_{p=1}^{k} \theta_{\beta_p}^s T_s \otimes T_s, \tag{4.2.21}$$

and for two highest weight vectors $v \in V, v' \in V'$ we have by Lemma 2.8.4 (ii)

$$\Delta_s(T_s^{-1})v \otimes v' = T_s^{-1}v \otimes T_s^{-1}v'. \tag{4.2.22}$$

Therefore for any $u \in V, u' \in V'$ we have by (4.2.22)

$$(u, \cdot T_s^{-1}v) \otimes (u', \cdot T_s^{-1}v') = (u \otimes u', \cdot T_s^{-1}v \otimes T_s^{-1}v') = (u \otimes u', \cdot T_s^{-1}(v \otimes v')). \tag{4.2.23}$$

Since $v \otimes v'$ is a highest weight vector in $V \otimes V'$, the last identity implies $(u, \cdot T_s^{-1}v) \otimes (u', \cdot T_s^{-1}v') \in \mathbb{C}_{\mathcal{B}}^s[G]_0$. This completes the proof. $\square$

From formulas (4.2.2) and (4.2.23) it follows that the set $\mathfrak{S}_s = \{cq^{n\frac{1}{dr^2}}\Delta_\mu^s | \mu \in P_+, c \in \mathbb{C}^*, n \in \mathbb{Z}\}$ is a multiplicative set of normal elements in $\mathbb{C}_{\mathcal{B}}^s[G]_0$.

In particular, if $v_\lambda \in V$ and $v_\mu \in V'$ are highest weight vectors, $u \in (V)_\beta$, then by (4.2.2)

$$\Delta_\mu^s \otimes (u, \cdot T_s^{-1}v_\lambda) = q^{((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)(\beta^\vee-\lambda^\vee),\mu^\vee)}(u, \cdot T_s^{-1}v_\lambda) \otimes \Delta_\mu^s. \tag{4.2.24}$$

For $u = v_p$, $\lambda = \lambda_p$ this yields

$$\Delta_\mu^s \otimes A_p = q^{-((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta_p^\vee,\mu^\vee)}A_p \otimes \Delta_\mu^s, \tag{4.2.25}$$

and for $u = v_\lambda$ we obtain

$$\Delta_\mu^s \otimes \Delta_\lambda^s = \Delta_\lambda^s \otimes \Delta_\mu^s. \tag{4.2.26}$$

Let $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0$ be the localization of $\mathbb{C}_{\mathcal{B}}^s[G]_0$ by $\mathfrak{S}_s$. Denote $\mathfrak{S}_s^* = \{f \otimes g^{-1} \in \mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0 | f, g \in \mathfrak{S}_s\}$, $\mathfrak{S}_s^{-1} = \{f^{-1} \in \mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0 | f \in \mathfrak{S}_s\}$.

We shall need more information on the structure of subalgebras U_k in $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0$ generated by the elements

$$B_p = A_p^{0^{-1}} \otimes A_p \in \mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0, \tag{4.2.27}$$

where $p = 1, \ldots, k$, and the multiplication in $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0$ is still denoted by $\otimes$. The elements B_p are quantum analogues of functions φ_p. From (4.2.25) and (4.2.26) we also obtain

$$\Delta_\mu^s \otimes B_p = q^{-((\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}-id)\beta_p^\vee,\mu^\vee)}B_p \otimes \Delta_\mu^s. \tag{4.2.28}$$

The following Lemma is similar to Proposition 8.3 in [48].

Lemma 4.2.4. *Let U_k be the (non-unital) subalgebra in $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0$ generated by the elements $B_p = A_p^{0^{-1}} \otimes A_p \in \mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0$, $p = 1, \ldots, k$, and $U_{\mathcal{B}}^k$ the (non-unital) subalgebra in $\mathbb{C}_{\mathcal{B}}^s[G_*]$ generated by the elements $e_p = (q_{\beta_p}^{-1} - q_{\beta_p})q^{\beta_p^\vee} e_{\beta_p}$, $p = 1, \ldots, k$. Then the map*

$$B_p \mapsto e_p$$

gives rise to an algebra isomorphism $\vartheta : U_k \to U_{\mathcal{B}}^k$.

In particular, the elements B_p satisfy the following commutation relations

$$B_p B_r - q^{(\beta_p, \beta_r) + (\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'*} \beta_p, \beta_r)} B_r B_p = \sum_{k_{p+1}, \ldots, k_{r-1}} C(k_{p+1}, \ldots, k_{r-1}) B_{p+1}^{k_{p+1}} \ldots B_{r-1}^{k_{r-1}}, \ p < r, \tag{4.2.29}$$

where $C(k_{p+1}, \ldots, k_{r-1}) \in \mathcal{B}$.

Proof. First observe that the map

$$\psi : \mathbb{C}_{\mathcal{B}}^s[G] \to \mathbb{C}_{\mathcal{B}}^s[B_-], (u, \cdot v) \mapsto (u \otimes id, (\omega_0 \otimes id)(\mathcal{R})v \otimes id)$$

is an algebra homomorphism. Indeed, since ω_0 is a coautomorphism we have by the first property in (2.5.2) in the case of Δ_s and $\mathcal{R}^s$

$$(\Delta_s \otimes id)((\omega_0 \otimes id)(\mathcal{R}^s)) = (\omega_0 \otimes \omega_0 \otimes id)(\Delta_s \otimes id)\mathcal{R}^s = (\omega_0 \otimes \omega_0 \otimes id)(\mathcal{R}_{13}^s \mathcal{R}_{23}^s) = (\omega_0 \otimes id)(\mathcal{R}_{13}^s)(\omega_0 \otimes id)(\mathcal{R}_{23}^s),$$

and hence

$$\psi((u, \cdot v) \otimes (u', \cdot v')) = (u \otimes id, (\omega_0 \otimes id)(\mathcal{R}_{13}^s)v \otimes id)(u' \otimes id, (\omega_0 \otimes id)(\mathcal{R}_{23}^s)v' \otimes id) = \psi((u, \cdot v))\psi((u', \cdot v')),$$

i.e. ψ is an algebra homomorphism.

Recall that by Lemma 4.2.3 elements of the form $(u, \cdot T_s^{-1}v) \in \mathbb{C}_{\mathcal{B}}^s[G]$, where v is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$ and $u \in V$, form a subalgebra $\mathbb{C}_{\mathcal{B}}^s[G]_0$ in $\mathbb{C}_{\mathcal{B}}^s[G]$.

From (4.2.22) we also deduce that the map

$$\mathbb{C}_{\mathcal{B}}^s[G]_0 \to \mathbb{C}_{\mathcal{B}}^s[G], (u, \cdot T_s^{-1}v) \mapsto (u, \cdot v)$$

is an algebra homomorphism. Composing this map with ψ we obtain another algebra homomorphism ψ^0.

Next using (4.1.6), (4.1.8) and the definition of A_p we obtain

$$\psi^0(A_p) = (v_p \otimes id, (\omega_0 S_s^{-1} S_s \otimes id)(\mathcal{R})v_{\mu_p} \otimes id) = (v_p \otimes id, (\omega_0 S_s^{-1} \otimes id)(\mathcal{R}^{s-1})v_{\mu_p} \otimes id) =$$

$$= (q_{\beta_p}^{-1} - q_{\beta_p})q^{(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\mu_p^\vee} q^{\beta_p^\vee} e_{\beta_p}.$$

Similarly

$$\psi^0(A_p^0) = q^{(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\mu_p^\vee}.$$

From the last two formulas we deduce that ψ^0 gives rise to an algebra homomorphism $\vartheta : U_k \to U_{\mathcal{B}}^k$ such that

$$\vartheta(B_p) = \vartheta(A_p^{0^{-1}} \otimes A_p) = (q_{\beta_p}^{-1} - q_{\beta_p})q^{\beta_p^\vee} e_{\beta_p} = e_p.$$

This homomorphism is surjective by construction. ϑ is also injective as ψ^0 is injective (see Proposition 8.3 in [48] and [130], Theorem 2.6 for the proof).

Commutation relations (4.2.29) follow from (3.2.15) by applying ω and by multiplying by $q^{\beta_p^\vee} q^{\beta_r^\vee}$. $\qquad\square$

Denote $\mathbb{C}_{\mathcal{B}}^{s,loc}[G] = \mathbb{C}_{\mathcal{B}}^s[G] \otimes_{\mathbb{C}_{\mathcal{B}}^s[G]_0} \mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0$. $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]$ is naturally a left $\mathbb{C}_{\mathcal{B}}^s[G]$–module and a right $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0$–module. We denote by $\otimes$ both the left $\mathbb{C}_{\mathcal{B}}^s[G]$–action and the right $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0$–action on $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]$ and call these actions multiplications. We shall often omit the symbol $\otimes$ to shorten the notation if it does not lead to a confusion.

From (4.2.25) it follows that

$$\mathfrak{S}_s^* A_p \subset \mathcal{B} A_p \mathfrak{S}_s^*, \mathfrak{S}_s^* B_p \subset \mathcal{B} A_p \mathfrak{S}_s^*, U_r \Delta_\mu^s = \Delta_\mu^s U_r, \tag{4.2.30}$$

where $\mathcal{B} A_p \mathfrak{S}_s^*$ (resp. $\mathcal{B} A_p \mathfrak{S}_s^*$) is the $\mathcal{B}$–submodule in $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]$ generated by $A_p \mathfrak{S}_s^*$ (resp. by $A_p \mathfrak{S}_s^*$).

By (4.2.29) we also have

$$U_p B_r \subset B_r U_{r-1} + U_{r-1}, r > p,\qquad(4.2.31)$$

and hence if we denote by $\overline{U}_p$ the non-unital subalgebra in $\mathbb{C}_{\mathcal{B}}^s[G]$ generated by $A_1, \ldots, A_p$ then by (4.2.25) and by the relations $A_r = \Delta_{\mu_r} B_r$, $r = 1, \ldots, D$ we have

$$\overline{U}_p A_r \subset A_r \overline{U}_{r-1} \mathfrak{S}_s^* + \overline{U}_{r-1} \mathfrak{S}_s^*, r > p.\qquad(4.2.32)$$

Also by (4.2.25)

$$\overline{U}_p \Delta_\mu^s = \Delta_\mu^s \overline{U}_p.\qquad(4.2.33)$$

Let $J^p = \mathbb{C}_{\mathcal{B}}^s[G]\overline{U}_p \subset \mathbb{C}_{\mathcal{B}}^s[G]$ be the left ideal generated by $\overline{U}_p$, $J^{ploc} \subset \mathbb{C}_{\mathcal{B}}^{s,loc}[G]$ the image of $J^p \otimes \mathfrak{S}_s^* \subset \mathbb{C}_{\mathcal{B}}^s[G] \otimes \mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0$ under the projection $\mathbb{C}_{\mathcal{B}}^s[G] \otimes \mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0 \to \mathbb{C}_{\mathcal{B}}^s[G] \otimes_{\mathbb{C}_{\mathcal{B}}^s[G]_0} \mathbb{C}_{\mathcal{B}}^{s,loc}[G]_0 = \mathbb{C}_{\mathcal{B}}^{s,loc}[G]$.

Lemma 4.2.5. $J^{ploc} A_r = J^{ploc} B_r \subset J^{r-1 loc}$ for $r > p$.

Proof. Firstly, the identity $J^{ploc} A_r = J^{ploc} B_r$ follows from the definitions of A_r, B_r and of J^{ploc}.

Now we establish the inclusion. Since by the definition $J^{ploc} = J^p \mathfrak{S}_s^* = \mathbb{C}_{\mathcal{B}}^s[G]\overline{U}_p \mathfrak{S}_s^*$ for any $r > p$ one has by the first formula in (4.2.30)

$$J^{ploc} A_r = J^p \mathfrak{S}_s^* A_r = J^p A_r \mathfrak{S}_s^*.\qquad(4.2.34)$$

By (4.2.32) we have

$$J^p A_r = \mathbb{C}_{\mathcal{B}}^s[G]\overline{U}_p A_r \subset \mathbb{C}_{\mathcal{B}}^s[G] A_r \overline{U}_{r-1} \mathfrak{S}_s^* + \overline{U}_{r-1} \mathfrak{S}_s^*) \subset J_{\mathcal{B}}^r \mathfrak{S}_s^* = J^{rloc}.\qquad(4.2.35)$$

This completes the proof.

$\square$

4.3   Quantized vanishing ideals

In this section we introduce and study quantum analogues of the ideals J^{j1}, $j = 2, \ldots, R-1$.

Firstly we obtain an alternative description of $J_{\mathcal{B}}^{11'}$ which agrees at the classical level with the description of J^{11} in Proposition 3.6.2 (ii). We start with a technical lemma.

Lemma 4.3.1. Let $\Delta_{\mathfrak{m}_+}^s = \Delta_{\mathfrak{m}_+} \cap \Delta_s^s$ and note that $(\Delta_0)_+ = \Delta_0 \cap \Delta_+^s = \Delta_0 \cap \Delta_+$. Both $\Delta_{\mathfrak{m}_+}^s, (\Delta_0)_+ \subset \Delta_+$ are minimal segments. Denote by $U_q^{res}(-\Delta_{\mathfrak{m}_+}^s)$ and $U_q^{res}((\Delta_0)_+)$ the subalgebras of $U_q^{res}(\mathfrak{g})$ corresponding to $-\Delta_{\mathfrak{m}_+}^s$ and $(\Delta_0)_+$, respectively (see Proposition 2.4.2 (iv) for their definition). Let $b \in U_q^{res}(w'(\mathfrak{b}_+))$, $(u, \cdot u') \in \mathbb{C}_{\mathcal{B}}^s[G]$ an element of $\mathbb{C}_{\mathcal{B}}^s[G]$ such that u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, $u' \in V$. Then

$$(u, T_s b u') = (u, T_s \sum_i z_+^i x_i' u'),\qquad(4.3.1)$$

where $z_+^i \in U_q^{res}((\Delta_0)_+)$, $x_i' \in U_q^{res}(-\Delta_{\mathfrak{m}_+}^s)$.

Proof. First note that $U_q^{res}(w'(\mathfrak{b}_+)) = U_{U_q^{res}(H)}^{res}([\beta_{k_{l'}+1}, -\beta_{k_{l'}}])$, and we have a disjoint union of minimal segments (see Figure 5)

$$[\beta_{k_{l'}+1}, -\beta_{k_{l'}}] = (-\Delta_{\mathfrak{m}_+}^s) \cup (-(\Delta_+^s \setminus (\Delta_s^s \cup (\Delta_0)_+))) \cup (\Delta_0)_+ \cup (\Delta_{s^2}^s \setminus \Delta_{\mathfrak{m}_+}),$$

where the order of the segments in the union agrees with the circular normal ordering of Δ corresponding to normal ordering (1.6.9) of Δ_+^s.

Applying iteratively Corollary 2.4.4 (i) we obtain using this union

$$U_q^{res}(w'(\mathfrak{b}_+)) = U_{U_q^{res}(H)}^{res}([\beta_{k_{l'}+1}, -\beta_{k_{l'}}]) =$$

$$= U_{U_q^{res}(H)}^{res}((-(\Delta_+^s \setminus (\Delta_s^s \cup (\Delta_0)_+))) \cup (\Delta_0)_+ \cup (\Delta_{s^2}^s \setminus \Delta_{\mathfrak{m}_+})) U_{U_q^{res}(H)}^{res}(-\Delta_{\mathfrak{m}_+}^s) =$$

$$= U_{U_q^{res}(H)}^{res}((-(\Delta_+^s \setminus (\Delta_s^s \cup (\Delta_0)_+))) U_{U_q^{res}(H)}^{res}((\Delta_0)_+ \cup (\Delta_{s^2}^s \setminus \Delta_{\mathfrak{m}_+})) U_{U_q^{res}(H)}^{res}(-\Delta_{\mathfrak{m}_+}^s) =$$

$$= U_{U_q^{res}(H)}^{res}(-(\Delta_+^s \setminus (\Delta_s^s \cup (\Delta_0)_+))) U_{U_q^{res}(H)}^{res}(\Delta_{s^2}^s \setminus \Delta_{\mathfrak{m}_+}) U_{U_q^{res}(H)}^{res}((\Delta_0)_+) U_{U_q^{res}(H)}^{res}(-\Delta_{\mathfrak{m}_+}^s).$$

Note that $(\Delta_0)_+ \subset \Delta_+$ and $-\Delta^s_{\mathfrak{m}_+} \subset \Delta_-$. Therefore by Corollary 2.4.4 (ii)

$$U_q^{res}(w'(\mathfrak{b}_+)) = U_{U_q^{res}(H)}^{res}(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+)))U_{U_q^{res}(H)}^{res}(\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+})U_{U_q^{res}(H)}^{res}((\Delta_0)_+)U_q^{res}(H)U_q^{res}(-\Delta^s_{\mathfrak{m}_+}) =$$

$$U_{U_q^{res}(H)}^{res}(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+)))U_{U_q^{res}(H)}^{res}(\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+})U_q^{res}(H)U_q^{res}((\Delta_0)_+)U_q^{res}(-\Delta^s_{\mathfrak{m}_+}) = \qquad (4.3.2)$$

$$= U_{U_q^{res}(H)}^{res}(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+)))U_{U_q^{res}(H)}^{res}(\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+})U_q^{res}((\Delta_0)_+)U_q^{res}(-\Delta^s_{\mathfrak{m}_+}).$$

Now observe that by the definition of Δ^s_s we have $s(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+))) \subset -\Delta^s_+$, and that $s^{-1}(s(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+)))) = -(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+)) \subset \Delta^s_-$, so in fact $s(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+))) \subset -(\Delta^s_+ \setminus \Delta^s_{s^{-1}}) \subset \Delta_-$, where the last inclusion follows from the definition of Δ_- (see also Figure 5). Recalling that for $\mu \in Q$ one has $T_s(U_q^{res}(\mathfrak{g}))_\mu = (U_q^{res}(\mathfrak{g}))_{s\mu}$ we deduce

$$T_s(U_{U_q^{res}(H)}^{res}(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+)))) \subset \bigoplus_{\mu \leq 0}(U_q^{res}(\mathfrak{g}))_\mu. \qquad (4.3.3)$$

Next, $\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+} \subset \Delta^s_{s^2}$, so

$$s(\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+}) \subset s^1 s^2(\Delta^s_{s^2}) = -s^1(\Delta^s_{s^2}),$$

where at the last step we used the fact that s^2 is an involution, and hence $s^2(\Delta^s_{s^2}) = -\Delta^s_{s^2}$. Now by Proposition 1.6.6 (i)

$$s(\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+}) \subset -s^1(\Delta^s_{s^2}) \subset -(\Delta^s_{s^{-1}} \setminus \Delta^s_{s^1}) \subset \Delta_-,$$

where at the last step we used the definition of Δ_- (see also Figure 5). Similarly to (4.3.3) the last inclusion implies

$$T_s(U_{U_q^{res}(H)}^{res}(\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+})) \subset \bigoplus_{\mu \leq 0}(U_q^{res}(\mathfrak{g}))_\mu. \qquad (4.3.4)$$

Now we can express any $b \in U_q^{res}(w'(\mathfrak{b}_+))$ using (4.3.2) as follows

$$b = \sum_i y_i w_i z^i_+ x_i, \qquad (4.3.5)$$

where $y_i \in U_{U_q^{res}(H)}^{res}(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+)))$, $w_i \in U_{U_q^{res}(H)}^{res}(\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+})$, $z^i_+ \in U_q^{res}((\Delta_0)_+)$, $x_i \in U_q^{res}(-\Delta^s_{\mathfrak{m}_+})$.

Let $(u, \cdot\, u') \in \mathbb{C}^s_{\mathcal{B}}[G]$ be such that u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $u' \in V$. Using (4.3.5) we obtain

$$(u, T_s b u') = (u, \sum_i T_s(y_i)T_s(w_i)T_s z^i_+ x_i u').$$

Note that by (4.3.3) and (4.3.4) the weights of the elements $T_s(y_i)$ and $T_s(w_i)$ are non–positive, and hence Lemma 2.8.4 yields

$$(u, T_s b u') = (u, \sum_i T_s(y'_i)T_s(w'_i)T_s z^i_+ x_i u'), \qquad (4.3.6)$$

where $y'_i \in U_{U_q^{res}(H)}^{res}(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+)))$, $w'_i \in U_{U_q^{res}(H)}^{res}(\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+})$ have zero weights. Recalling that $-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+))$ and $\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+}$ are minimal segments we deduce that the only zero weight elements of the algebras $U_{U_q^{res}(H)}^{res}(-(\Delta^s_+ \setminus (\Delta^s_s \cup (\Delta_0)_+)))$ and $U_{U_q^{res}(H)}^{res}(\Delta^s_{s^2} \setminus \Delta_{\mathfrak{m}_+})$ belong to $U_q^{res}(H)$. Therefore $y'_i, w'_i \in U_q^{res}(H)$, and by Lemma 2.8.6 (4.3.6) takes the form

$$(u, T_s b u') = \sum_i k_i(u, T_s z^i_+ x_i u') = (u, \sum_i T_s z^i_+ x'_i u'),$$

where $k_i \in \mathbb{C}[q, q^{-1}]$, $z^i_+ \in U_q^{res}((\Delta_0)_+)$, $x'_i = k_i x_i \in U_q^{res}(-\Delta^s_{\mathfrak{m}_+})$. This completes the proof

$\square$

The following description of $J^{11'}_{\mathcal{B}}$ is analogous to the description of J^{11} in Proposition 3.6.2 (ii).

Lemma 4.3.2. $J^{11'}_{\mathcal{B}}$ *coincides with the left ideal generated by the elements* $(u, \cdot v) \in \mathbb{C}^s_{\mathcal{B}}[G]$ *such that u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ satisfies $(u, t_s z x v) = 0$ for any $z \in U^{s,res}_{\mathcal{B}}((\Delta_0)_+)$, $x \in U^{s,res}_{\mathcal{B}}(-\Delta_{\mathfrak{m}_+})$ and any element t_s of the braid group acting on $\mathfrak{h} \subset U_h(\mathfrak{h})$ in the same way as s.*

Proof. Firstly we prove the statement for $t_s = T_s$ as in the definition of $J_{\mathcal{B}}^{11'}$.

Assume that $(u, T_s bv) = 0$ for any $b \in U_q^{res}(w'(\mathfrak{b}_+))$. Observe that $U_{\mathcal{B}}^{s,res}((\Delta_0)_+) \subset U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_q^{res}((\Delta_0)_+)$ and $U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+}) \subset U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_q^{res}(-\Delta_{\mathfrak{m}_+})$. Thus

$$U_{\mathcal{B}}^{s,res}((\Delta_0)_+)U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+}) \subset U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_q^{res}((\Delta_0)_+)U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_q^{res}(-\Delta_{\mathfrak{m}_+}) = \tag{4.3.7}$$

$$= U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_q^{res}((\Delta_0)_+)U_q^{res}(-\Delta_{\mathfrak{m}_+})$$

as $U_q^{res}((\Delta_0)_+)U_{\mathcal{B}}^{s,res}(\mathfrak{h}) = U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_q^{res}((\Delta_0)_+)$.

Since $(\Delta_0)_+, -\Delta_{\mathfrak{m}_+} \subset [\beta_{k_{l'}+1}, -\beta_{k_{l'}}]$, we have $U_q^{res}((\Delta_0)_+)U_q^{res}(-\Delta_{\mathfrak{m}_+}) \subset U_{U_q^{res}(H)}^{res}([\beta_{k_{l'}+1}, -\beta_{k_{l'}}]) = U_q^{res}(w'(\mathfrak{b}_+))$. Thus by (4.3.7)

$$U_{\mathcal{B}}^{s,res}((\Delta_0)_+)U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+}) \subset U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_q^{res}(w'(\mathfrak{b}_+)),$$

and for any $z \in U_{\mathcal{B}}^{s,res}((\Delta_0)_+)$, $x \in U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+})$ one has $zx = \sum_i h_i b_i$, $h_i \in U_{\mathcal{B}}^{s,res}(\mathfrak{h})$, $b_i \in U_q^{res}(w'(\mathfrak{b}_+))$. So by Lemma 2.8.6

$$(u, T_s z_+ xv) = \sum_i (u, h_i b_i v) = \sum_i c_i (u, b_i v) = 0,$$

where $c_i \in \mathcal{B}$ are defined by $c_i u = h_i u$.

Conversely, if $(u, T_s z_+ xv) = 0$ for any $z \in U_{\mathcal{B}}^{s,res}((\Delta_0)_+)$, $x \in U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+}^s)$ then by Lemma 4.3.1 with $u' = v$

$$(u, T_s bv) = (u, T_s \sum_i z_i x_i v), \tag{4.3.8}$$

where for all i $z_i \in U_q^{res}((\Delta_0)_+)$, $x_i \in U_q^{res}(-\Delta_{\mathfrak{m}_+}^s) \subset U_q^{res}(-\Delta_{\mathfrak{m}_+})$.

Observe that $U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_{\mathcal{B}}^{s,res}((\Delta_0)_+) \supset U_q^{res}((\Delta_0)_+)$ and $U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+}) \supset U_q^{res}(-\Delta_{\mathfrak{m}_+})$. Thus

$$U_q^{res}((\Delta_0)_+)U_q^{res}(-\Delta_{\mathfrak{m}_+}) \subset U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_{\mathcal{B}}^{s,res}((\Delta_0)_+)U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+}) = \tag{4.3.9}$$

$$= U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_{\mathcal{B}}^{s,res}((\Delta_0)_+)U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+})$$

as $U_{\mathcal{B}}^{s,res}((\Delta_0)_+)U_{\mathcal{B}}^{s,res}(\mathfrak{h}) = U_{\mathcal{B}}^{s,res}(\mathfrak{h})U_{\mathcal{B}}^{s,res}((\Delta_0)_+)$.

By (4.3.9) formula (4.3.8) takes the form

$$(u, T_s bv) = (u, T_s \sum_i h_i' z_i' x_i' v),$$

where for any $z_i' \in U_{\mathcal{B}}^{s,res}((\Delta_0)_+)$, $x_i' \in U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+})$, $h_i' \in U_{\mathcal{B}}^{s,res}(\mathfrak{h})$.

Thus by Lemma 2.8.6

$$(u, T_s bv) = \sum_i (u, h_i' z_i' x_i' v) = \sum_i c_i' (u, z_i' x_i' v) = 0,$$

where $c_i' \in \mathcal{B}$ are defined by $c_i' u = h_i' u$.

Hence $J_{\mathcal{B}}^{11'}$ coincides with the left ideal generated by the elements $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}^s[G]$ such that u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ satisfies $(u, T_s zxv) = 0$ for any $z \in U_{\mathcal{B}}^{s,res}((\Delta_0)_+)$, $x \in U_{\mathcal{B}}^{s,res}(-\Delta_{\mathfrak{m}_+})$.

Finally by Lemma 2.8.7 T_s used in the definition of $J_{\mathcal{B}}^{11'}$ can be replaced in the statement of this lemma with any element t_s of the braid group acting on $\mathfrak{h} \subset U_h(\mathfrak{h})$ in the same way as s. This completes the proof. $\square$

We proceed with the definition of the quantum analogues of the ideals J^{j1}, $j = 2, \ldots, R - 1$. Denote $\delta_{jm_j} = (w_1 \ldots w_{j-1})^{-1}\beta_{R-1n_{R-1}} = (w_1 \ldots w_{j-1})^{-1}\beta_c$. Let $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jm_j}]) = T_{w_1 \ldots w_{j-1}}^{-1} U_{\mathcal{B}}^{s,res}([-\beta_{jk}, -\beta_c])$ be the subalgebra in $U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{g})$ generated by the elements $T_{w_1 \ldots w_{j-1}}^{-1} f_\beta^{(n)}$, $\beta \in [-\beta_{jk}, -\beta_{R-1n_{R-1}}]$, $n \in \mathbb{N}$. Note that from the definition of the elements f_β it follows that $T_{w_1 \ldots w_{j-1}}^{-1} f_\beta^{(n)}(s) = f_{(w_1 \ldots w_{j-1})^{-1}\beta}^{(n)}(\widehat{s}_j)$ for $\beta \in [-\beta_{j1}, -\beta_D]$, where for $\alpha \in \Delta_+$ and $j > 1$ the elements $f_\alpha(\widehat{s}_j)$ are defined using the element $\widehat{s}_j$ and the normal ordering on Δ_+ introduced in Remark 3.5.4. To shorten the notation we write from now on $f_\alpha^{(n)}(\widehat{s}_j) = f_\alpha^{(n)}$ for $\beta \in [-\delta_{j1}, -\delta_{jD}]$ if it does not cause a confusion. We shall also need the subalgebra $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jD}]) \subset U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{g})$.

In complete analogy with the definition of the ideals J^{j1} for $j = 1, \ldots, R - 1$ we define $J_{\mathcal{B}}^{j1'}$ as the left ideal generated by the elements $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ such that u is a highest weight vector in a finite rank representation

V of $U_h(\mathfrak{g})$, and $v \in V$ satisfies $(u, t_{\widehat{s}_j} z_+ xv) = 0$ for any $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j) := T_{w_1 \ldots w_{j-1}}^{-1} U_{\mathcal{B}}^{s, res}((\Delta_0)_+)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ and an arbitrary fixed element $t_{\widehat{s}_j}$ of the braid group acting on $\mathfrak{h} \subset U_h(\mathfrak{h})$ in the same way as $\widehat{s}_j$.

Note that by Lemma 2.8.7 this definition does not depend on the choice of $t_{\widehat{s}_j}$. For $j = 1$ this definition agrees with the previously given definition of $J_{\mathcal{B}}^{11'}$ due to Lemma 4.3.2.

Also similarly to $I_{\mathcal{B}}^{11}$ defined in Proposition 4.1.1 we introduce left ideals $I_{\mathcal{B}}^{j1} = J_q^{j1'} \cap \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ in $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$, where $J_q^{j1'} = J_{\mathcal{B}}^{j1'} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $j = 1, \ldots, R - 1$.

Next, for the left ideals $I_{\mathcal{B}}^{j1}$ we obtain a description similar to the one given in Proposition 3.6.2 (i). We start with a technical lemma which will allow us to use properties of the vanishing ideals J^{jk} to prove some properties of their quantum counterparts.

Lemma 4.3.3. *Let X be a free $\mathcal{B}$–module, $V \subset V' \subset X$ two its submodules such that $V = V' \mod (q^{\frac{1}{dr^2}} - 1)V'$. Let $V_q = V \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $V_q' = V' \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$. Then $V_q = V_q'$.*

Proof. First observe that by Theorem 6.5 in [102] $V' \subset X$ is $\mathcal{B}$–free, as $\mathcal{B}$ is a principal ideal domain, and $V \subset V'$ is a free submodule of V'. Let e_a, $a \in A$ be a basis of V'. As noted in the proof of Theorem 6.5 in [102], A is a well ordered set, i.e. it is a totally ordered set in which any nonempty subset has a smallest element. Then as shown in the proof of Theorem 6.5 in [102] V has a basis elements of which have the form $f_a = \sum_{b \leq a} c_a^b e_b$, $a \in B \subset A$, where $c_a^a \neq 0$, $c_a^b \in \mathcal{B}$, and the sum is finite. Since $V = V' \mod (q^{\frac{1}{dr^2}} - 1)V'$ we must have $B = A$, and $c_a^a \neq 0 \mod (q^{\frac{1}{dr^2}} - 1)$ for all $a \in A$.

Now using a simple transfinite induction we can express the elements of the basis e_a regarded as elements of V_q' in terms of f_a. Indeed, if $a_0 \in A$ is the minimal element then $e_{a_0} = c_{a_0}^{a_0}{}^{-1} f_{a_0}$, and assuming that $e_d = \sum_{b \leq d} g_d^b f_b$, $g_d^b \in \mathbb{C}(q^{\frac{1}{dr^2}})$, $g_d^d \neq 0$ holds for all $d < a$ we get $e_a = c_a^{a}{}^{-1}(f_a - \sum_{b < a} c_a^b e_b) = c_a^{a}{}^{-1}(f_a - \sum_{b < a} c_a^b \sum_{h \leq b} g_b^h f_h)$, where all sums are finite. This establishes the induction step and completes the proof. $\square$

Lemma 4.3.4. *(i) For $j = 1, \ldots, R - 1$, let $t_{\widehat{s}_j}$ be an arbitrary element of the braid group acting on $\mathfrak{h} \subset U_h(\mathfrak{h})$ in the same way as $\widehat{s}_j$. Let $J_{\mathcal{B}}^{j1}$ be the left ideal in $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ generated by the elements $(w, \cdot v)$ such that $(w, y t_{\widehat{s}_j} z_+ xv) = 0$ for any $y \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-), z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j), x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.*

Then $J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{j1}$ and $J_{\mathcal{B}}^{j1'} = J_{\mathcal{B}}^{j1} = J^{j1} \mod (q^{\frac{1}{dr^2}} - 1)J_{\mathcal{B}}^{j1}$.

(ii) Denote $J_q^{j1'} = J_{\mathcal{B}}^{j1'} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $J_q^{j1} = J_{\mathcal{B}}^{j1} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$. Then $J_q^{j1'} = J_q^{j1}$. Thus $I_{\mathcal{B}}^{j1} = J_q^{j1} \cap \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G] = J_q^{j1'} \cap \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$.

(iii) $J_{\mathcal{B}}^{j1'}$, $J_{\mathcal{B}}^{j1}$ and $I_{\mathcal{B}}^{j1}$ are stable under the $\mathrm{Ad}_{\widehat{s}_j}^0$-action of $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$. Thus $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]/J_{\mathcal{B}}^{j1'}$ and $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]/I_{\mathcal{B}}^{j1}$ are naturally equipped with the $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$-action induced by the $\mathrm{Ad}_{\widehat{s}_j}^0$-action of $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ on $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$.

Proof. (i) If $(u, \cdot v) \in J_{\mathcal{B}}^{j1'}$ is one of the elements generating $J_{\mathcal{B}}^{j1'}$, i.e. u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ obeys $(u, t_{\widehat{s}_j} z_+ xv) = 0$ for any $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, then for any $y \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-)$, $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ one has by Lemma 2.8.4 (i) and by the definition of $J_{\mathcal{B}}^{j1'}$ that

$$(u, y t_{\widehat{s}_j} z_+ xv) = (u, y_0 t_{\widehat{s}_j} z_+ xv) = c(u, t_{\widehat{s}_j} z_+ xv) = 0,$$

where $y_0 \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$ is the zero weight component of y, and $c \in \mathcal{B}$ is defined by $cu = y_0 u$.

Thus $(u, \cdot v)$ is also one of the elements generating $J_{\mathcal{B}}^{j1}$, so that $J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{j1}$. This proves the first claim of part (i) of the lemma.

Next, observe that the specializations of $U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$ and of $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ at $q^{\frac{1}{dr^2}} = 1$ are isomorphic to $U(\mathfrak{z}_+^j)$ and $U(\mathfrak{m}_-^j)$, respectively, the quotient of the specialization of $U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-)$ at $q^{\frac{1}{dr^2}} = 1$ by the ideal generated by $L_i - 1$, and by $t_i - 1$, $i = 1, \ldots, l$ is isomorphic to $U(\mathfrak{b}_-)$, and the elements L_i and t_i act by identity transformations on the specialization of any module V^{res} at $q^{\frac{1}{dr^2}} = 1$. Therefore recalling Proposition 2.8.5 (ii) we deduce that the specialization of $J_{\mathcal{B}}^{j1}$ at $q^{\frac{1}{dr^2}} = 1$ is generated by the matrix elements of the form $(w, \cdot v) \in \mathbb{C}[G]$, where $w, v \in V$, V is a finite-dimensional representation of $\mathfrak{g}$, and $(w, y\widehat{s}_j z_+ xv) = 0$ for any $y \in U(\mathfrak{b}_-)$, $z_+ \in U(\mathfrak{z}_+^j)$, $x \in U(\mathfrak{m}_-^j)$,

and the specialization of $J_{\mathcal{B}}^{j1'}$ at $q^{\frac{1}{dr^2}} = 1$ is generated by the matrix elements of the form $(u, \cdot v) \in \mathbb{C}[G]$, where u is a highest weight vector in a finite-dimensional representation V of $\mathfrak{g}$, and $v \in V$ satisfies, and $(v, \widehat{s}_j z_+ x v) = 0$ for any $z_+ \in U(\mathfrak{z}_+^j)$, $x \in U(\mathfrak{m}_-^j)$.

Since the enveloping algebra of any Lie subalgebra of $\mathfrak{g}$ is also a Hopf subalgebra of $U(\mathfrak{g})$, the product of any element of $\mathbb{C}[G]$ and of $(w, \cdot v) \in \mathbb{C}[G]$, where $w, v \in V$, V is a finite-dimensional representation of $\mathfrak{g}$, and $(w, y\widehat{s}_j z_+ x v) = 0$ for any $y \in U(\mathfrak{b}_-)$, $z_+ \in U(\mathfrak{z}_+^j)$, $x \in U(\mathfrak{m}_-^j)$ satisfies the same condition. So in fact the specialization of $J_{\mathcal{B}}^{j1}$ at $q^{\frac{1}{dr^2}} = 1$ consists of the matrix elements of the form $(w, \cdot v) \in \mathbb{C}[G]$, where $w, v \in V$, V is a finite-dimensional representation of $\mathfrak{g}$, and $(w, y\widehat{s}_j z_+ x v) = 0$ for any $y \in U(\mathfrak{b}_-)$, $z_+ \in U(\mathfrak{z}_+^j)$, $x \in U(\mathfrak{m}_-^j)$.

Now from parts (i) and (ii) of Proposition 3.6.2 it follows that $J_{\mathcal{B}}^{j1'} = J_{\mathcal{B}}^{j1} = J^{j1} \mod (q^{\frac{1}{dr^2}} - 1)J_{\mathcal{B}}^{j1}$.

(ii) Note that by part (i) $J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{j1}$ are submodules of the $\mathcal{B}$–module $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ which is free by Proposition 3.2.7 (iv). Also, by part (i) $J_{\mathcal{B}}^{j1} = J_{\mathcal{B}}^{j1'} = J^{j1} \mod (q^{\frac{1}{dr^2}} - 1)J_{\mathcal{B}}^{j1}$. Therefore by Lemma 4.3.3 $J_q^{j1} = J_q^{j1'}$.

(iii) Firstly we prove the statement for $J_{\mathcal{B}}^{j1'}$.

From formula (4.2.4) with $s = \widehat{s}_j$ we have for any $\beta \in [-\delta_{j1}, -\delta_{jm_j}]$

$$
\mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(f \otimes g)(\cdot\,,\cdot\,) = \sum_{k=0}^n \sum_{p=0}^{n-k} q_\beta^{-k(n-k)-p(n-k-p)}(\mathrm{Ad}_{\widehat{s}_j}^0 (G_\beta^{-k} f_\beta^{(p)})f)(\cdot\,) \otimes g(\omega_0 S_{\widehat{s}_j}^{-1}(G_\beta^{-k-p} f_\beta^{(n-k-p)})\cdot\ \omega_0(f_\beta^{(k)}))+
$$

$$
+\sum_{k=0}^{n-1}\sum_i q_\beta^{-k(n-k)}(\mathrm{Ad}_{\widehat{s}_j}^0 (G_\beta^{-k} x_i^{(n-k)})f)(\cdot\,) \otimes g((\omega_0 S_{\widehat{s}_j}^{-1})(G_\beta^{-k} y_i^{(n-k)})\cdot\ \omega_0(f_\beta^{(k)}))+ \qquad (4.3.10)
$$

$$
+\sum_i (\mathrm{Ad}_{\widehat{s}_j}^0 (y_i^{(n)^2})f)(\cdot\,) \otimes g((\omega_0 S_{\widehat{s}_j}^{-1})(y_i^{(n)^1})\cdot\ \omega_0(x_i^{(n)})).
$$

Assume that $g(\cdot\,) = (u, \cdot\ v) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ is such that $(u, t_{\widehat{s}_j} z_+ x v) = 0$ for any $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, and $f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ is arbitrary.

We claim now that all terms in the right hand side of the last formula belong to $J_{\mathcal{B}}^{j1'}$. More generally, one can prove the following statement.

Lemma 4.3.5. *Let $g(\cdot\,) = (u, \cdot\ v) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ is such that $(u, t_{\widehat{s}_j} z_+ x v) = 0$ for any $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$. Then for any $y' \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-)$ and $b \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ the element $g(y' \cdot b)$ satisfies the same properties as g.*

Proof. By Lemma 2.8.6

$$
g(y' \cdot b) = (u, y \cdot bv) = (u, y_0' \cdot bv) = c_0(u, \cdot bv),
$$

where $y_0' \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$ is the zero weight component of y' and $c_0 \in \mathcal{B}$ is defined by $y_0' u = c_0 u$.

Also, for any $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ we have $xb \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, and hence by the definition of g

$$
(u, y' t_{\widehat{s}_j} z_+ x bv) = c_0(u, t_{\widehat{s}_j} z_+ x bv) = 0.
$$

$\square$

Since $S_{\widehat{s}_j}(G_\beta^{-k-p} f_\beta^{(n-k-p)}), S_{\widehat{s}_j}(G_\beta^{-k} y_i^{(n-k)}), S_{\widehat{s}_j}(y_i^{(n)^1}) \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-)$ and $f_\beta^{(k)}, x_i^{(n)} \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ the previous lemma implies that all terms in the right hand side of (4.3.10) belong to $J_{\mathcal{B}}^{j1'}$, i.e. $\mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(f \otimes g)(\cdot\,,\cdot\,) \in J_{\mathcal{B}}^{j1'}$. Since the elements $f \otimes g$, with $f, g \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ as in (4.3.10) span $J_{\mathcal{B}}^{j1'}$ we obtain $\mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(J_{\mathcal{B}}^{j1'}) \subset J_{\mathcal{B}}^{j1'}$. The elements $f_\beta^{(n)}$, $\beta \in [\delta_{j1}, \delta_{jm_j}]$ generate $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, and hence $J_{\mathcal{B}}^{j1'}$ is stable under the $\mathrm{Ad}_{\widehat{s}_j}^0$-action of $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.

If we naturally extend the $\mathrm{Ad}_{\widehat{s}_j}^0$-action of $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ to $\mathbb{C}_q[G]$ we immediately deduce that $J_q^{j1} = J_{\mathcal{B}}^{j1'} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$ is stable under this action.

Hence $I_{\mathcal{B}}^{j1} = J_q^{j1} \cap \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ is also stable under this action as clearly $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ is $\mathrm{Ad}_{\widehat{s}_j}^0$–stable.

Assume now that $g(\cdot) = (w, \cdot\, v) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ is such that $(w, yt_{\widehat{s}_j} z_+ xv) = 0$ for any $y \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-), z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j), x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, and $f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ is arbitrary.

We claim again that all terms in the right hand side of the last formula belong to $J_{\mathcal{B}}^{j1}$. More generally, one can prove the following statement.

Lemma 4.3.6. *Let* $g(\cdot) = (w, \cdot\, v) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ *is such that* $(w, yt_{\widehat{s}_j} z_+ xv) = 0$ *for any* $y \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-), z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j), x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$. *Then for any* $y' \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-)$ *and* $b \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ *the element* $g(y' \cdot b)$ *satisfies the same properties as* g.

Proof. The results is obvious as for any $y \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-), x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ one has $y'y \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-), xb \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, and hence

$$g(y'yt_{\widehat{s}_j} z_+ xb) = 0.$$

$\square$

Since $S_{\widehat{s}_j}(G_\beta^{-k-p} f_\beta^{(n-k-p)}), S_{\widehat{s}_j}(G_\beta^{-k} y_i^{(n-k)}), S_{\widehat{s}_j}(y_i^{(n)^1}) \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-)$ and $f_\beta^{(k)}, x_i^{(n)} \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ the previous lemma implies that all terms in the right hand side of (4.3.10) belong to $J_{\mathcal{B}}^{j1}$, i.e. $\mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(f \otimes g)(\cdot, \cdot) \in J_{\mathcal{B}}^{j1}$. Since the elements $f \otimes g$, with $f, g \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ with g as in the previous lemma span $J_{\mathcal{B}}^{j1'}$ we obtain $\mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(J_{\mathcal{B}}^{j1}) \subset J_{\mathcal{B}}^{j1}$. The elements $f_\beta^{(n)}, \beta \in [\delta_{j1}, \delta_{jm_j}]$ generate $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, and hence $J_{\mathcal{B}}^{j1}$ is stable under the $\mathrm{Ad}_{\widehat{s}_j}^0$-action of $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.

This completes the proof.

$\square$

4.4 Higher quantized vanishing ideals

The next step in our approach is the definition and the study of quantum analogues $J_{\mathcal{B}}^{jk}$ and $J_{\mathcal{B}}^{jk'}$ of the ideals J^{jk} and $J^{jk'}$ for $k > 1$.

Firstly, for each $j = 1, \ldots, R-1$, $k = 1, \ldots, n_j$ we define the matrix elements $(v_{jk}, \cdot v_{\mu_{jk}}) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ by condition (4.2.7), where for $j = 1$ the normal ordering on Δ_+ introduced in Definition 1.6.19 is used, and for $j > 1$ the normal ordering on Δ_+ defined in Remark 3.5.4 is used, and $\beta_p = \delta_{jk}$. Let

$$A_{jk}(\cdot) = (v_{jk}, \cdot\, T_{\widehat{s}_j}^{-1} v_{\mu_{jk}}).$$

Lemma 4.4.1. *For* $j = 1, \ldots, R-1$, $k = 1, \ldots, n_j + 1$, *let* $J_{\mathcal{B}}^{jk'}$ *be the left ideal in* $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ *generated by* $J_{\mathcal{B}}^{j1'}$ *and by* A_{jp}, $p = 1, \ldots, k-1$.

Let $t_{\widehat{s}_j}$ *be an arbitrary element of the braid group acting on* $\mathfrak{h} \subset U_h(\mathfrak{h})$ *in the same way as* $\widehat{s}_j$, $J_{\mathcal{B}}^{jk}$ *the left ideal in* $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ *generated by the elements* $(w, \cdot v) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ *such that* $(w, yht_{\widehat{s}_j} z_+ xv) = 0$ *for any* $y \in \omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jD}]))$, $h \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$, $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j), x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.

Then the following statements are true.

(i) If $p < k$ *then for any* $y' \in \omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jD}]))$ *of the form* $y' = \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_D}^{(n_D^j)} \ldots f_{\delta_{jk}}^{(n_k^j)})$, $n_i^j \in \mathbb{N}$, $b \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, $h' \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$ *one has*

$$A_{jp}(y'h' \cdot b) \in J_{\mathcal{B}}^{jk}.$$

If in addition $y' \neq 1, \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})$ *or the zero weight component of* b *is zero then*

$$A_{jk}(y'h' \cdot b) \in J_{\mathcal{B}}^{jk}. \tag{4.4.1}$$

$$A_{jk}(S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})\cdot) = \Delta_{\mu_{jk}}(\cdot) = A_{jk}^0(\cdot). \tag{4.4.2}$$

(ii) $J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{j1} \subset J_{\mathcal{B}}^{jk}$ *and* $J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{jk'} \subset J_{\mathcal{B}}^{jk}$.

(iii) For any $y' \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-)$, $b \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, $h' \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$, if the zero weight component of y' is zero or the zero weight component of b is zero then

$$\Delta_\mu^{\widehat{s}_j}(y' \cdot b) \in J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{jk'} \subset J_{\mathcal{B}}^{jk}.$$

(iv) $\mathrm{Ad}_{\widehat{s}_j}^0(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jm_j}])) J_{\mathcal{B}}^{jk'} \subset J_{\mathcal{B}}^{jk}$.

Proof. (i) If $p < k$ and $y' \neq 1$ then $A_{jp}(y'h' \cdot b) = 0$ by Lemma 4.2.2 (i).

If $y' = 1$ we have to show that $A_{jp}(h'yht_{\widehat{s}_j}z_+xb) = 0$ for any $y \in \omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jD}]))$, $h \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$, $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.

By linearity it suffices to check this condition for y of the form $y = \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_D^j}^{(n_D^j)} \ldots f_{\delta_{jk}}^{(n_k^j)})$, $n_i^j \in \mathbb{N}$ as these elements form a basis of $\omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jD}]))$.

Again if $y \neq 1$ then $A_{jp}(h'yht_{\widehat{s}_j}z_+xb) = 0$ by Lemma 4.2.2 (i).

If $y = 1$

$$A_{jp}(h'yht_{\widehat{s}_j}z_+xb) = (v_{jp}, h'ht_{\widehat{s}_j}z_+xbT_{\widehat{s}_j}^{-1}v_{\mu_{jp}}) = c_{jp}c'_{jp}(v_{jp}, t_{\widehat{s}_j}z_+xbt_{\widehat{s}_j}^{-1}v_{\mu_{jp}}) = c_{jp}c'_{jp}(v_{jp}, t_{\widehat{s}_j}(z_+xb)v_{\mu_{jp}}),$$

where $h'hv_{jp} = c_{jp}v_{jp}$ and $T_{\widehat{s}_j}^{-1}v_{\mu_{jp}} = c'_{jp}t_{\widehat{s}_j}^{-1}v_{\mu_{jp}}$, $c_{jp}, c'_{jp} \in \mathcal{B}$.

Lemma 4.4.2. *(i) For any $v \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ the element $t_{\widehat{s}_j}(v)$ has no weight components of weights which belong to $-(w_1 \ldots w_{j-1})^{-1}\Delta^j$.*

(ii) For any $v \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ and any $b \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ which has no non-trivial zero weight component the element $t_{\widehat{s}_j}(vb)$ has no non-trivial zero weight component.

Proof. (i) The weights of the weight components of $t_{\widehat{s}_j}(v)$ belong to $-\mathbb{N}(w_1 \ldots w_{j-1})^{-1}s(\Delta_+^j \cup \Delta_0)$, and by Lemma 1.6.22 (ii) this set has empty intersection with $-(w_1 \ldots w_{j-1})^{-1}\Delta^j$.

(ii) The weights of the weight components of $t_{\widehat{s}_j}(vb)$ belong to $\mathbb{N}(\widehat{s}_j([-\delta_{j1}, -\delta_{jm_j}]) \cup (w_1 \ldots w_{j-1})^{-1}s(\Delta_0)_+) = \mathbb{N}(w_1 \ldots w_{j-1})^{-1}s(([-\beta_{j1}, -\beta_c]) \cup (\Delta_0)_+)$, and the set $[-\beta_{j1}, -\beta_c] \cup (\Delta_0)_+$ is contained in the minimal segment $[\beta_1^0, -\beta_c]$. Therefore if b has no non-trivial zero weight component the element vb has no non-trivial zero weight component as well. Hence $t_{\widehat{s}_j}(vb)$ has no non-trivial zero weight component. □

Since $z_+xb \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ by part (i) of the previous lemma the element $t_{\widehat{s}_j}(z_+xb)$ has no weight components of weights which belong to $-(w_1 \ldots w_{j-1})^{-1}\Delta^j$, and $-\delta_{jp} \in -(w_1 \ldots w_{j-1})^{-1}\Delta^j$. Since v_{jp} has weight $\mu_{jp} - \delta_{jp}$,
$$A_{jp}(h'ht_{\widehat{s}_j}z_+xb) = c_{jp}c'_{jp}(v_{jp}, t_{\widehat{s}_j}(z_+xb)v_{\mu_{jp}}) = 0$$
by orthogonality of different weight subspaces with respect to the contravariant form. This proves part (i) for $p < k$.

If $p = k$ $y' \neq 1, \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})$ then $A_{jk}(y' \cdot b) = 0$ by Lemma 4.2.2 (ii).

If $y' = 1$ we have to show that $A_{jk}(h'yht_{\widehat{s}_j}z_+xb) = 0$ for any $y = \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_D^j}^{(n_D^j)} \ldots f_{\delta_{jk}}^{(n_k^j)})$, $n_i^j \in \mathbb{N}$, $h \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$, $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.

Again if $y \neq 1, \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})$ then $A_{jk}(h'yht_{\widehat{s}_j}z_+xb) = 0$ by Lemma 4.2.2 (i).

If $y = 1$

$$A_{jk}(h'yht_{\widehat{s}_j}z_+xb) = (v_{jk}, h'ht_{\widehat{s}_j}z_+xbT_{\widehat{s}_j}^{-1}v_{\mu_{jk}}) = c_{jk}c'_{jk}(v_{jk}, t_{\widehat{s}_j}z_+xbt_{\widehat{s}_j}^{-1}v_{\mu_{jk}}) = c_{jk}c'_{jk}(v_{jk}, t_{\widehat{s}_j}(z_+xb)v_{\mu_{jk}}),$$

where $hh'v_{jk} = c_{jk}v_{jk}$ and $T_{\widehat{s}_j}^{-1}v_{\mu_{jk}} = c'_{jk}t_{\widehat{s}_j}^{-1}v_{\mu_{jk}}$, $c_{jk}, c'_{jk} \in \mathcal{B}$.

Since $z_+xb \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ by Lemma 4.4.2 (i) the element $t_{\widehat{s}_j}(z_+xb)$ has no weight components of weights which belong to $-(w_1 \ldots w_{j-1})^{-1}\Delta^j$, and $-\delta_{jk} \in -(w_1 \ldots w_{j-1})^{-1}\Delta^j$. Since v_{jk} has weight $\mu_{jk} - \delta_{jk}$,
$$A_{jk}(h'yht_{\widehat{s}_j}z_+xb) = c_{jk}c'_{jk}(v_{jk}, t_{\widehat{s}_j}(z_+xb)v_{\mu_{jk}}) = 0$$
by orthogonality of different weight subspaces with respect to the contravariant form.

If $y = \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})$ then

$$A_{jk}(h'yht_{\widehat{s}_j}z_+xb) = (v_{jk}, h'\omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})ht_{\widehat{s}_j}z_+xbT_{\widehat{s}_j}^{-1}v_{\mu_{jk}}) =$$

$$= c''_{jk}c'_{jk}(v_{jk}, \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})ht_{\widehat{s}_j}z_+xbt_{\widehat{s}_j}^{-1}v_{\mu_{jp}}) = c'''_{jk}c''_{jk}c'_{jk}(v_{\mu_{jk}}, t_{\widehat{s}_j}(z_+xb)v_{\mu_{jk}}),$$

where $h'v_{jk} = c''_{jk}v_{jk}$, $hv_{\mu_{jk}} = c'''_{jk}v_{\mu_{jk}}$ and $T_{\widehat{s}_j}^{-1}v_{\mu_{jk}} = c'_{jk}t_{\widehat{s}_j}^{-1}v_{\mu_{jk}}$, $c''_{jk}, c'''_{jk} \in \mathcal{B}$.

The element $t_{\widehat{s}_j}(z_+xb)$ has no non–trivial zero weight component by Lemma 4.4.2 (ii) as by the assumption b has no zero weight components in this case. So we deduce that

$$A_{jk}(y'h'yht_{\widehat{s}_j}z_+xb) = c'''_{jk}c''_{jk}c'_{jk}(v_{\mu_{jk}}, t_{\widehat{s}_j}(z_+xb)v_{\mu_{jk}}) = 0$$

by orthogonality of different weight subspaces with respect to the contravariant form.

If $p = k$ $y' = \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})$ we have to show that $A_{jk}(S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})h'yht_{\widehat{s}_j}z_+xb) = 0$ for any $y = \omega_0 S_{\widehat{s}_j}^{-1}(f_{\delta_D^j}^{(n_D^j)} \dots f_{\delta_{jk}}^{(n_k^j)})$, $n_i^j \in \mathbb{N}$, $h \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$, $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.

In this case

$$A_{jk}(S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})h'yht_{\widehat{s}_j}z_+xb) = (v_{\mu_{jk}}, h'yht_{\widehat{s}_j}z_+xbT_{\widehat{s}_j}^{-1}v_{\mu_{jk}}) = d_{jk}c'_{jk}(v_{\mu_{jk}}, yht_{\widehat{s}_j}(z_+xb)v_{\mu_{jk}}),$$

where $h'v_{\mu jk} = d_{jk}v_{\mu_{jk}}$.

Again if $y \neq 1$ then

$$A_{jk}(S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})h'yht_{\widehat{s}_j}z_+xb) = d_{jk}c'_{jk}(v_{\mu_{jk}}, yht_{\widehat{s}_j}(z_+xb)v_{\mu_{jk}}) = 0$$

as $v_{\mu_{jk}}$ is a highest weight vector.

If $y = 1$

$$A_{jk}(S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})h'yht_{\widehat{s}_j}z_+xb) = d_{jk}c'''_{jk}c'_{jk}(v_{\mu_{jk}}, t_{\widehat{s}_j}(z_+xb)v_{\mu_{jk}}).$$

The element $t_{\widehat{s}_j}(z_+xb)$ has no non–trivial zero weight component by Lemma 4.4.2 (ii) as by the assumption b has no zero weight components in this case. We deduce that

$$A_{jk}(S_{\widehat{s}_j}^{-1}(f_{\delta_{jk}})h'yht_{\widehat{s}_j}z_+xb) = d_{jk}c'''_{jk}c'_{jk}(v_{\mu_{jk}}, t_{\widehat{s}_j}(z_+xb)v_{\mu_{jk}}) = 0$$

by orthogonality of different weight subspaces with respect to the contravariant form.

Formula (4.4.2) follows from (4.2.9).

This completes the proof of part (i).

(ii) We show that $J_{\mathcal{B}}^{jk'} \subset J_{\mathcal{B}}^{jk}$. Indeed, by Lemma 4.3.4 (i) any element from $J_{\mathcal{B}}^{j1'}$ belongs to $J_{\mathcal{B}}^{j1}$. We claim that $J_{\mathcal{B}}^{j1} \subset J_{\mathcal{B}}^{jk}$, where the same braid group element $t_{\widehat{s}_j}$ is used in the definitions of both algebras.

Let $(w, \cdot v) \in J_{\mathcal{B}}^{j1}$ be one of the elements generating $J_{\mathcal{B}}^{j1}$, i.e. $(w, yt_{\widehat{s}_j}z_+xv) = 0$ for any for any $y \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-)$, $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$. Then $(w, yht_{\widehat{s}_j}z_+xv) = 0$ for any $y \in \omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jD}]))$, $h \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$, $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ as $\omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jD}]))U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h}) \subset U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{b}_-)$. Thus $(w, \cdot v) \in J_{\mathcal{B}}^{jk}$ is one of the elements generating $J_{\mathcal{B}}^{jk}$.

We deduce using Lemma 4.3.4 (ii) that $J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{j1} \subset J_{\mathcal{B}}^{jk}$.

By part (i) $A_{jp} \in J_{\mathcal{B}}^{jk}$ for $p = 1, \dots, k-1$. Since $J_{\mathcal{B}}^{j1'}$ and A_{jp} with $p < k$ generate $J_{\mathcal{B}}^{jk'}$ as a left ideal, we obtain that $J_{\mathcal{B}}^{jk'} \subset J_{\mathcal{B}}^{jk}$.

(iii) In order to show that $\Delta_\mu^{\widehat{s}_j}(y' \cdot b) \in J_{\mathcal{B}}^{j1'}$ we shall check that it is one of the generating elements of $J_{\mathcal{B}}^{j1'}$. More precisely, we verify that $\Delta_\mu^{\widehat{s}_j}(y' \cdot b) = c(v_\mu, \cdot bT_{\widehat{s}_p}^{-1}v_\mu)$ for some $c \in \mathcal{B}$, and

$$\Delta_\mu^{\widehat{s}_j}(y't_{\widehat{s}_p}z_+xb) = (v_\mu, y't_{\widehat{s}_p}z_+xbT_{\widehat{s}_p}^{-1}v_\mu) = (\omega(y')v_\mu, t_{\widehat{s}_p}z_+xbT_{\widehat{s}_p}^{-1}v_\mu) = 0 \tag{4.4.3}$$

for any $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.

Since ω changes the signs of the weights, the weights of the weight components of $\omega(y')$ belong to $\mathbb{N}(\Delta_+)$. Thus

$$\Delta_\mu^{\widehat{s}_j}(y' \cdot b) = (\omega(y'_0)v_\mu, \cdot bT_{\widehat{s}_p}^{-1}v_\mu) = c(v_\mu, \cdot bT_{\widehat{s}_p}^{-1}v_\mu),$$

where y_0' is the zero weight component of y', $c \in \mathcal{B}$ is defined by $y_0' v_\mu = c v_\mu$.

Next, for any $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ one has using the previous formula

$$\Delta_\mu^{\widehat{s}_j}(y' t_{\widehat{s}_p} z_+ x b) = c(v_\mu, t_{\widehat{s}_j} z_+ x b T_{\widehat{s}_j}^{-1} v_\mu) = cc''(v_\mu, t_{\widehat{s}_j} z_+ x b t_{\widehat{s}_j}^{-1} v_\mu) = cc''(v_\mu, t_{\widehat{s}_p}(z_+ x b) v_\mu), \tag{4.4.4}$$

where $c'' \in \mathbb{C}q^{\mathbb{Z}}$, $c'' \neq 0$ is given by Lemma 2.8.7 using the condition $T_{\widehat{s}_j}^{-1} v_\mu = c'' t_{\widehat{s}_j}^{-1} v_\mu$.

If the zero weight component of y' is zero then $c = 0$. Thus (4.4.3) holds. In fact $\Delta_\mu^{\widehat{s}_j}(y' \cdot b) = 0$ in this case.

By Lemma 4.4.2 (ii), if the zero weight component of b is zero then the zero weight component of $t_{\widehat{s}_p}(z_+ x b)$ is zero as well, and hence the right hand side of (4.4.4) vanishes as different weight subspaces are orthogonal with respect to the contravariant form. Thus (4.4.3) holds.

This completes the proof of part (iii).

(iv) Firstly, the ideal $J_{\mathcal{B}}^{j1'}$ is stable under the $\mathrm{Ad}_{\widehat{s}_j}^0$–action of $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, and hence under the $\mathrm{Ad}_{\widehat{s}_j}$–action of $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jm_j}]) \subset U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$. Therefore by part (ii) and by the definition of $J_{\mathcal{B}}^{jk'}$ we infer

$$\mathrm{Ad}_{\widehat{s}_j}^0(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jm_j}]))J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{jk'} \subset J_{\mathcal{B}}^{jk}.$$

Since the elements $f_\beta^{(n)}$, $\beta \in [-\delta_{jk}, -\delta_{jm_j}]$, $n \in \mathbb{N}$ generate $U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jm_j}])$, and $J_{\mathcal{B}}^{j1'}$ and A_{jp} with $p < k$ generate $J_{\mathcal{B}}^{jk'}$, it remains to show that for arbitrary $f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$, $\beta \in [-\delta_{jk}, -\delta_{jm_j}]$, and $n \in \mathbb{N}$ one has $\mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(f \otimes A_{jp}) \in J_{\mathcal{B}}^{jk}$ for $p < k$.

Indeed, consider formula (4.3.10) with arbitrary $f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$, $g = A_{jp}$, $p < k$, $n \in \mathbb{N}$, and $\beta \in [-\delta_{jk}, -\delta_{jm_j}]$. By part (i) the second factors in all $\otimes$–products in $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ in the terms in the right hand side of (4.3.10) belong to $J_{\mathcal{B}}^{jk}$. Thus $\mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(f \otimes A_{jp}) \in J_{\mathcal{B}}^{jk}$. This completes the proof of Lemma 4.4.1.

$\qquad\quad$ $\square$

Lemma 4.4.3. *For any $j \in \{1, \ldots, R-1\}$, $1 \leq k \leq n_j + 1$ the following statements are true.*

(i) $J_{\mathcal{B}}^{j1'} A_{jk} \subset J_{\mathcal{B}}^{jk}$.

(ii) *For any $\mu \in P_+$ $J_{\mathcal{B}}^{j1'} \Delta_\mu^{\widehat{s}_j} \subset J_{\mathcal{B}}^{j1} \subset J_{\mathcal{B}}^{jk}$.*

(iii) *For any $\mu \in P_+$ $J_{\mathcal{B}}^{jk'} \Delta_\mu^{\widehat{s}_j} \subset J_{\mathcal{B}}^{jk}$.*

Proof. (i) To show that $J_{\mathcal{B}}^{j1'} A_{jk} \subset J_{\mathcal{B}}^{jk}$ it suffices to verify that for any $(u, \cdot v) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$, where u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ is such that $(u, t_{\widehat{s}_j} z_+ x v) = 0$ for any $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$ one has

$$((u, \cdot v) \otimes A_{jk})(yht_{\widehat{s}_p} z_+ x) = 0 \tag{4.4.5}$$

for any $y \in \omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jk}, -\delta_{jD}]))$, $h \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h})$, $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j)$, $x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.

We shall use this condition with $t_{\widehat{s}_j} = \overline{T}_{\widehat{s}_j}$.

From (2.8.13) and the fact that $\omega_0 S_{\widehat{s}_j}^{-1}$ is an anti-coautomorphism preserving weights we obtain

$$\Delta_{\widehat{s}_j}(y) = 1 \otimes y + \sum_i v_i \otimes w_i, \tag{4.4.6}$$

where the weights of the elements v_i are strictly negative.

From formula (2.6.18) with k equal to the length of $\widehat{s}_j$ we have

$$\Delta_{\widehat{s}_j}(\overline{T}_{\widehat{s}_j}) = \prod_{p=1}^k \overline{\theta}_{\beta_p}^{\widehat{s}_j}{}' q^{\sum_{i=1}^l (-Y_i \otimes K_{\widehat{s}_j} H_i + T_{\widehat{s}_j} Y_i \otimes T_{\widehat{s}_j} K_{\widehat{s}_j} H_i)} \overline{T}_{\widehat{s}_j} \otimes \overline{T}_{\widehat{s}_j} = \prod_{p=1}^k \overline{\theta}_{\beta_p}^{\widehat{s}_j}{}' \overline{T}_{\widehat{s}_j} \otimes \overline{T}_{\widehat{s}_j} \tag{4.4.7}$$

where we used the fact that $K_{\widehat{s}_j} h = \frac{\kappa}{2} \frac{1+\widehat{s}_j}{1-\widehat{s}_j} w_{j-1}^{-1} P_{\mathfrak{h}'} w_{j-1} h$ for any $h \in \mathfrak{h}$, so have $K_{\widehat{s}_j} s_{\widehat{s}_j} = s_{\widehat{s}_j} K_{\widehat{s}_j}$, and hence by (4.2.20)

$$q^{\sum_{i=1}^l (T_{\widehat{s}_j} Y_i \otimes T_{\widehat{s}_j} K_{\widehat{s}_j} H_i - Y_i \otimes K_{\widehat{s}_j} H_i)} = 1. \tag{4.4.8}$$

Since u is a highest weight vector we obtain using Lemma 2.8.4 (i), (4.4.7), the definition of $\overline{\theta}^{s}_{\beta_p}{}'$ after (2.6.18), and (4.4.6) that the left hand side of (4.4.5) takes the form

$$((u, \cdot v) \otimes A_{jk})(yh\overline{T}_{\widehat{s}_j} z_+ x) = (u, h^1 \overline{T}_{\widehat{s}_j} z^1_+ x^1 v)(v_{jk}, yh^2 \overline{T}_{\widehat{s}_j} z^2_+ x^2 T^{-1}_{\widehat{s}_j} v_{\mu_{jk}}) =$$

$$= t(u, h^1 \overline{T}_{\widehat{s}_j} z^1_+ x^1 v)(v_{jk}, yh^2 \overline{T}_{\widehat{s}_j} z^2_+ x^2 \overline{T}^{-1}_{\widehat{s}_j} v_{\mu_{jk}}) = t(u, h^1 \overline{T}_{\widehat{s}_j} z^1_+ x^1 v)(v_{jk}, yh^2 \overline{T}_{\widehat{s}_j} (z^2_+ x^2) v_{\mu_{jk}}), \tag{4.4.9}$$

where t is a non–zero multiple of a power of q defined in Lemma 2.8.7 by the condition $T^{-1}_{\widehat{s}_j} v_{\mu_{jk}} = t\overline{T}^{-1}_{\widehat{s}_j} v_{\mu_{jk}}$ and we use the Sweedler notation for the coproducts.

By linearity it suffices to check condition (4.4.5) for $y \in y \in \omega_0 S^{-1}_{\widehat{s}_j}(U^{\widehat{s}_j, res}_{\mathcal{B}}([-\delta_{jk}, -\delta_{jD}]))$ of the form

$$y = \omega_0 S^{-1}_s (f^{(n^j_D)}_{\delta^j_D} \dots f^{(n^j_k)}_{\delta_{jk}}), n^j_i \in \mathbb{N}.$$

In this case by Lemma 4.2.2 (i) the second factor in the right hand side of (4.4.9) vanishes if $y \neq 1, \omega_0 S^{-1}_s(f_{\delta_{jk}})$, and hence (4.4.5) holds.

If $y = 1$ (4.4.9) takes the form

$$((u, \cdot v) \otimes A_{jk})(yh\overline{T}_{\widehat{s}_j} z_+ x) = tt'(u, \overline{T}_{\widehat{s}_j} z^1_+ x^1 v)(v_{jk}, \overline{T}_{\widehat{s}_j} (z^2_+ x^2) v_{\mu_{jk}}), \tag{4.4.10}$$

where $t' \in \mathbb{B}$ is defined by $t^1 u \otimes t^2 v_{jk} = t' u \otimes v_{jk}$.

From (2.8.13) or (2.8.16) it follows that $x^2 \in U^{\widehat{s}_j, res}_{\mathcal{B}}([-\delta_{1k}, -\delta_{jm_j}])$, so the weights of x^2 belong to $\mathbb{N}[-\delta_{j1}, -\delta_{jm_j}] \subset \mathbb{N}(w_1 \dots w_{j-1})^{-1}(-\Delta^j_+ \cup (\Delta_0)_+)$. The first line in (2.8.17) implies that the weights of the weight components of z^2_+ belong to $\mathbb{N}(w_1 \dots w_{j-1})^{-1}((\Delta_0)_+ \cup [-\beta_{jk}, -\beta^0_{D_0}]) \subset \mathbb{N}(w_1 \dots w_{j-1})^{-1}(-\Delta^j_+ \cup (\Delta_0)_+)$. We conclude that the weights of $\overline{T}_{\widehat{s}_j}(z^2_+ x^2)$ belong to $\mathbb{N}(w_1 \dots w_{j-1})^{-1} s(-\Delta^j_+ \cup (\Delta_0)_+)$, and by Lemma 1.6.22 (ii) this set has empty intersection with the set $-(w_1 \dots w_{j-1})^{-1} \Delta^j$ which contains $-\delta_{jk}$. Since the weight of v_{jk} is $\mu_{jk} - \delta_{jk}$ the right hand side of (4.4.10) vanishes as different weight spaces of $V_{\mu_{jk}}$ are orthogonal with respect to the contravariant form.

If $y = \omega_0 S^{-1}_{\widehat{s}_j}(f_{\delta_{jk}})$ then by (4.4.2) formula (4.4.9) takes the form

$$((u, \cdot v) \otimes A_{jk})(yh\overline{T}_{\widehat{s}_j} z_+ x) = t(u, h^1 \overline{T}_{\widehat{s}_j} z^1_+ x^1 v)(v_{\mu_{jk}}, h^2 \overline{T}_{\widehat{s}_j}(z^2_+ x^2) v_{\mu_{jk}}) = tt''(u, \overline{T}_{\widehat{s}_j} z^1_+ x^1 v)(v_{\mu_{jk}}, \overline{T}_{\widehat{s}_j}(z^2_+ x^2) v_{\mu_{jk}}), \tag{4.4.11}$$

where $t'' \in \mathbb{B}$ is defined by $t^1 u \otimes t^2 v_{\mu_{jk}} = t'' u \otimes v_{\mu_{jk}}$.

Recall that from (2.8.13) or (2.8.16) it follows that $x^2 \in U^{\widehat{s}_j, res}_{\mathcal{B}}([-\delta_{j1}, -\delta_{jm_j}]) = U^{\widehat{s}_j, res}_{\mathcal{B}}((w_1 \dots w_{j-1})^{-1}[-\beta_{j1}, -\beta_c])$, and the second line in (2.8.17) implies that $z^2_+ \in U^{\widehat{s}_j, res}_{\mathcal{B}}((w_1 \dots w_{j-1})^{-1}[\beta^0_1, -\beta_{j-1n_{j-1}}]) U^{\widehat{s}_j, res}_{\mathcal{B}}(\mathfrak{h})$. Since $[-\beta_{j1}, -\beta_c]$, $[\beta^0_1, -\beta_{j-1n_{j-1}}] \subset [\beta^0_1, -\beta_c]$, and $[\beta^0_1, -\beta_c]$ is a minimal segment, the zero weight component $(z^2_+ x^2)_0$ of $z^2_+ x^2$ belongs to $U^{\widehat{s}_j, res}_{\mathcal{B}}(\mathfrak{h})$, and by (2.8.13) or (2.8.16), and by the second line in (2.8.17) we have $z^1_+ x^1 \otimes (z^2_+ x^2)_0 = \sum_n z_n x_n \otimes h_n$, $z_n \in U^{\widehat{s}_p, res}_{\mathcal{B}}((w_1 \dots w_{j-1})^{-1}(\Delta_0)_+)$, $x_n \in U^{\widehat{s}_p, res}_{\mathcal{B}}((w_1 \dots w_{j-1})^{-1}[-\beta_{j1}, -\beta_c])$, $h_n \in U^{\widehat{s}_j, res}_{\mathcal{B}}(\mathfrak{h})$. Thus (4.4.11) takes the form

$$((u, \cdot v) \otimes A_{jk})(yh\overline{T}_{\widehat{s}_j} z_+ x) = tt''(u, \overline{T}_{\widehat{s}_j} z^1_+ x^1 v)(v_{\mu_{jk}}, \overline{T}_{\widehat{s}_j}(z^2_+ x^2)_0 v_{\mu_{jk}}) =$$

$$= tt'' \sum_n (u, \overline{T}_{\widehat{s}_j} z_n x_n v)(v_{\mu_{jk}}, \overline{T}_{\widehat{s}_j} h_n v_{\mu_{jk}}) = 0$$

since $(u, \overline{T}_{\widehat{s}_j} z_n x_n v) = 0$ for all n by the choice of u and v. Thus (4.4.5) holds, and the proof of part (i) is completed.

The proof of part (ii) is similar to that of part one. The same arguments are applied with A_{jk} replaced by $\Delta^{\widehat{s}_j}_\mu$. Formula (4.4.11) will be replaced with

$$((u, \cdot v) \otimes \Delta^{\widehat{s}_j}_\mu)(yh\overline{T}_{\widehat{s}_j} z_+ x) = t\hat{t}(u, \overline{T}_{\widehat{s}_j} z^1_+ x^1 v)(v_\mu, \overline{T}_{\widehat{s}_j}(z^2_+ x^2) v_\mu),$$

where $\hat{t}$ is defined by the condition $\Delta_{\widehat{s}_j}(yh)u \otimes v_\mu = \hat{t} u \otimes v_\mu$. The rest of the proof is repeated verbatim.

For part (iii) we recall that $J^{jk}_{\mathcal{B}}{}'$ is the left ideal in $\mathbb{C}^{\widehat{s}_j}_{\mathcal{B}}[G]$ generated by $J^{j1}_{\mathcal{B}}{}'$ and by A_{jp}, $p < k$. By part (ii) $J^{j1}_{\mathcal{B}}{}' \Delta^{\widehat{s}_j}_\mu \subset J^{j1}_{\mathcal{B}} \subset J^{jk}_{\mathcal{B}}$.

Using commutation relations (4.2.12) we deduce that the left ideal in $\mathbb{C}^{\widehat{s}_j}_{\mathcal{B}}[G]$ generated by A_{jp}, $p < k$ is invariant with respect to multiplication by $\Delta^{\widehat{s}_j}_\mu$ from the right. This observation together with the inclusion $J^{j1}_{\mathcal{B}}{}' \Delta^{\widehat{s}_j}_\mu \subset J^{jk}_{\mathcal{B}}$ imply the inclusion in part (iii).

$\square$

4.5   Localizations

Let $J_{\mathcal{B}}^{jk\,loc}$ and $J_{\mathcal{B}}^{jk'\,loc}$ the images of $J_{\mathcal{B}}^{jk} \otimes \mathfrak{S}_{\widehat{s}_j}^* \subset \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G] \otimes \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]_0$ and of $J_{\mathcal{B}}^{jk'} \otimes \mathfrak{S}_{\widehat{s}_j}^* \subset \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G] \otimes \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]_0$, respectively, in $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]$ under the projection $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G] \otimes \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]_0 \to \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G] \otimes_{\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]_0} \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]_0 = \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]$.

Lemma 4.5.1. *Let* $J_q^{jk\,loc} = J_{\mathcal{B}}^{jk\,loc} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr2}})$, $J_q^{jk'\,loc} = J_{\mathcal{B}}^{jk'\,loc} \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr2}})$.
Then $J_q^{jk\,loc} = J_q^{jk'\,loc}$. *Thus* $I_{\mathcal{B}}^{jk\,loc} := J_q^{jk'\,loc} \cap \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G] = J_q^{jk\,loc} \cap \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]$.

Proof. Let $\iota_j : \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G] \to \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]$ be the canonical $\mathcal{B}$–module homomorphism. Since $J_{\mathcal{B}}^{jk} \subset J_{\mathcal{B}}^{jk'}$ by Lemma 4.4.1 (ii), $J_{\mathcal{B}}^{jk\,loc} \subset J_{\mathcal{B}}^{jk'\,loc}$, and hence

$$\iota_j^{-1}(J_{\mathcal{B}}^{jk\,loc}) \subset \iota_j^{-1}(J_{\mathcal{B}}^{jk'\,loc}). \tag{4.5.1}$$

When specializing $q^{\frac{1}{dr2}}$ to 1 the matrix elements A_{jk} become $(v_{jk}, \cdot\,\widehat{s}_j^{-1} v_{\mu_{jk}})$, and the set $\mathfrak{S}_{\widehat{s}_j}$ becomes the set $\mathcal{S}_j$ by their definitions. By Lemma 4.3.4 (i) $J_{\mathcal{B}}^{j1'} = J_{\mathcal{B}}^{j1} = J^{j1}$ mod $(q^{\frac{1}{dr2}} - 1)J_{\mathcal{B}}^{j1}$. Therefore $J_{\mathcal{B}}^{jk'} = J^{jk'}$ mod $(q^{\frac{1}{dr2}} - 1)$ and $J_{\mathcal{B}}^{jk'\,loc} = J^{jk\,loc}$ mod $(q^{\frac{1}{dr2}} - 1)$ by Lemma 3.6.1 (ii).

On the other hand $J_{\mathcal{B}}^{jk} = J^{jk}$ mod $(q^{\frac{1}{dr2}} - 1)$ by Lemma 3.6.2 (i), and hence $J_{\mathcal{B}}^{jk\,loc} = J^{jk\,loc}$ mod $(q^{\frac{1}{dr2}} - 1)$ by Lemma 3.6.1 (i).

Thus $J_{\mathcal{B}}^{jk'\,loc} = J_{\mathcal{B}}^{jk\,loc} = J^{jk\,loc}$ mod $(q^{\frac{1}{dr2}} - 1)$, and hence $\iota_j^{-1}(J_{\mathcal{B}}^{jk\,loc}) = \iota_j^{-1}(J_{\mathcal{B}}^{jk'\,loc})$ mod $(q^{\frac{1}{dr2}} - 1)$. But $\iota_j^{-1}(J_{\mathcal{B}}^{jk\,loc})$, $\iota_j^{-1}(J_{\mathcal{B}}^{jk'\,loc}) \subset \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ which is $\mathcal{B}$–free. Thus by Lemma 4.3.3 $\iota_j^{-1}(J_{\mathcal{B}}^{jk\,loc}) \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr2}}) = \iota_j^{-1}(J_{\mathcal{B}}^{jk'\,loc}) \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr2}})$, or

$$\iota_j^{-1}(J_q^{jk\,loc}) = \iota_j^{-1}(J_q^{jk'\,loc}), \tag{4.5.2}$$

where we denote by the same symbol the natural extension

$$\iota_j : \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G] \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr2}}) \to \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G] \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr2}}).$$

Formula (4.5.2) implies that $J_q^{jk\,loc} = J_q^{jk'\,loc}$. This completes the proof. $\qquad\square$

Corollary 4.5.2. *For any* $j \in \{1, \ldots, R-1\}$, $1 \le k \le n_j + 1$ *the following statements are true.*
(i) For any $j \in \{1, \ldots, R-1\}$, $1 \le k \le n_j + 1$ *one has* $I_{\mathcal{B}}^{jm\,loc} A_{jk} = I_{\mathcal{B}}^{jm\,loc} B_{jk} \subset I_{\mathcal{B}}^{jk\,loc}$, $m \le k$, *where* $B_{jk} = {A_{jk}^0}^{-1} \otimes A_{jk}$, $A_{jk}^0 = \Delta_{\mu_{jk}}$.
(ii) For any $j \in \{1, \ldots, R-1\}$, $1 \le k \le n_j + 1$, $\mu \in P_+$ *one has* $I_{\mathcal{B}}^{jk\,loc} \Delta_\mu^{\widehat{s}_j} \subset I_{\mathcal{B}}^{jk\,loc}$.
(iii) For any $j \in \{1, \ldots, R-1\}$, $1 \le k \le n_j$ *one has* $I_{\mathcal{B}}^{jk\,loc} \subset I_{\mathcal{B}}^{jk+1\,loc}$.

Proof. Part (i) follows from Lemma 4.4.3 (i), Lemma 4.2.5 with $s = \widehat{s}_j$, $\overline{U}_p$ being the algebra generated by A_{jn}, $n < m$, and from Lemma 4.5.1.

Part (ii) follows the definition of $I_{\mathcal{B}}^{jk\,loc}$.

Pert (iii) follows from the obvious inclusion $J_{\mathcal{B}}^{jk'} \subset J_{\mathcal{B}}^{jk+1'}$ and from Lemma 4.5.1 $\qquad\square$

For $j \in \{1, \ldots, R-1\}$, $1 \le k \le n_j + 1$ let

$$\mathbb{C}_{jk}^{loc}[G] = \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]/I_{\mathcal{B}}^{jk\,loc}.$$

$\mathbb{C}_{jk}^{loc}[G]$ is naturally a left $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$–module.

From Corollary 4.5.2 (i) and (ii) we deduce the following statement.

Lemma 4.5.3. *For* $j \in \{1, \ldots, R-1\}$, $1 \le k \le n_j$ *multiplication from the right on* $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]$ *induces a natural action of* $\Delta_\mu^{\widehat{s}_j} \in \mathfrak{S}_{\widehat{s}_j}$, $\mu \in P_+$ *on* $\mathbb{C}_{jk}^{loc}[G]$, *and for* $k \ge m$, *right multiplication by* A_{jk} *and by* B_{jk} *gives rise to well–defined homomorphisms of left* $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$–modules $\mathbb{C}_{jm}^{loc}[G] \to \mathbb{C}_{jk}^{loc}[G]$.

In the next two lemmas we study the properties of the adjoint action. These properties will be needed to study properties of quantum analogues of the operators Π_{jk} defined in (3.5.19).

Lemma 4.5.4. *(i) For any $j = 1, \ldots, R-1$, $n \in \mathbb{N}$, $\beta \in [\delta_{j1}, \delta_{jm_j}]$ and $f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ we have*

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(f \otimes \Delta_\mu^{\widehat{s}_j}) = q^{n((2K_{\widehat{s}_j} - id)\beta^\vee, \mu^\vee)} \mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(f) \otimes \Delta_\mu^{\widehat{s}_j} \tag{4.5.3}$$

in $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]/J_{\mathcal{B}}^{j1'}$.

(ii) The adjoint action of $U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{j1}, -\delta_{jm_j}])$ on $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]/J_{\mathcal{B}}^{j1'}$ defined in Lemma 4.3.4 (iii) induces an action on $\mathbb{C}_{j1}^{loc}[G]$ satisfying

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(f \otimes \Delta_\mu^{\widehat{s}_j}{}^{-1}) = q^{-n((2K_{\widehat{s}_j} - id)\beta^\vee, \mu^\vee)} \mathrm{Ad}_{\widehat{s}_j}^0 f_\beta^{(n)}(f) \otimes \Delta_\mu^{\widehat{s}_j}{}^{-1}, f \in \mathbb{C}_{j1}^{loc}[G]. \tag{4.5.4}$$

This action is locally finite.

Proof. The proof of part (i) follows from formula (4.2.4) applied to $g = \Delta_\mu^{\widehat{s}_j}$. By Lemma 4.4.1 (iii) all terms in the right hand side of (4.2.4) belong to $J_{\mathcal{B}}^{j1'}$, except for term in the first sum which corresponds to $p = n$, $k = 0$. It gives the right hand side of (4.5.3).

Part (ii) follows from Lemma 4.5.1 and from part (i).

Local finiteness of the action of $U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{j1}, -\delta_{jm_j}])$ on $\mathbb{C}_{j1}^{loc}[G]$ follows from the local finiteness of the action of $U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{j1}, -\delta_{jm_j}])$ on $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ and from formula (4.5.4). $\square$

Lemma 4.5.5. *The $\mathrm{Ad}_{\widehat{s}_j}^0$–action of $U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jk}, -\delta_{jm_j}])$ on $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ induces a locally finite action on $\mathbb{C}_{jk}^{loc}[G] = \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]/I_{\mathcal{B}}^{jk\,loc}$ satisfying (4.5.4) for any $f \in \mathbb{C}_{jk}^{loc}[G]$.*

Proof. By Lemma 4.4.1 (iv) $\mathrm{Ad}_{\widehat{s}_j}^0(U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jk}, -\delta_{jm_j}])J_{\mathcal{B}}^{jk'} \subset J_{\mathcal{B}}^{jk}$. Since $J_{\mathcal{B}}^{j1'} \subset J_{\mathcal{B}}^{jk'}$ the adjoint action on $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]/J_{\mathcal{B}}^{j1'}$ defined in Lemma 4.5.4 (i) satisfies the property $\mathrm{Ad}_{\widehat{s}_j}^0(U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jk}, -\delta_{jm_j}])(J_{\mathcal{B}}^{jk'}/J_{\mathcal{B}}^{j1'}) \subset J_{\mathcal{B}}^{jk}/J_{\mathcal{B}}^{j1'}$. Hence by Lemma 4.5.1 the action of $U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jk}, -\delta_{jm_j}])$ on $\mathbb{C}_{j1}^{loc}[G]$ defined in part (ii) of Lemma 4.5.4 satisfies

$$\mathrm{Ad}_{\widehat{s}_j}^0(U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jk}, -\delta_{jm_j}])(I_{\mathcal{B}}^{jk\,loc}/I_{\mathcal{B}}^{j1\,loc}) \subset I_{\mathcal{B}}^{jk\,loc}/I_{\mathcal{B}}^{j1\,loc}.$$

Thus this action induces an action on $\mathbb{C}_{jk}^{loc}[G] = \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j,loc}[G]/I_{\mathcal{B}}^{jk\,loc}$. This action is locally finite as $\mathrm{Ad}_{\widehat{s}_j}^0$–action of $U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jk}, -\delta_{jm_j}]$ on $\mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ is locally finite. $\square$

Proposition 4.5.6. *For $j = 1, \ldots, R-2$ the morphism of $\mathbb{C}[[h]]$–modules $T_{w_j}^{-1}\psi^{\theta_{w_j}^{\widehat{s}_j}{}^{-1}} : U_h^{\widehat{s}_j}(\mathfrak{g}) \to U_h^{\widehat{s}_{j+1}}(\mathfrak{g})$ induces an invertible morphism of $\mathcal{B}$–modules $\phi_j : \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G] \to \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$ which is defined by*

$$(\phi_j f)(x) = f(T_{w_j}^{-1}\psi^{\theta_{w_j}^{\widehat{s}_j}{}^{-1}}(x)), x \in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{g}), f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G] \tag{4.5.5}$$

and satisfies

$$\phi_j \mathrm{Ad}^{s_{j+1}}(T_{w_j}^{-1}x) = \mathrm{Ad}^{s_j}(x)\phi_j, x \in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{g}), \tag{4.5.6}$$

$$\phi_j(f \otimes g) = \sum_{m,n}(\mathrm{Ad}^{s_j}(d_m^1 c_n)\phi_j f)(\cdot) \otimes g(T_{w_j}^{-1}(v S_{\widehat{s}_j}(c_m) \cdot d_m^2 d_n)), f, g \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G], y \in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{g}), \tag{4.5.7}$$

where $\theta_{w_j}^{\widehat{s}_j}{}^{-1} = \sum_m a_m \otimes b_m$, $v_{\widehat{s}_j} = \sum_m a_m S_{\widehat{s}_j}(b_m)$, $\theta_{w_j}^{\widehat{s}_j} = \sum_n c_n \otimes d_n$, $\Delta_{\widehat{s}_j} d_m = d_m^1 \otimes d_m^2$.

The inverse to ϕ_j is given by

$$(\phi_j^{-1}f)(x) = f((\psi^{\theta_{w_j}^{\widehat{s}_j}{}^{-1}})^{-1}T_{w_j}(x)), x \in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{g}), f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]. \tag{4.5.8}$$

Proof. Recall that by (2.6.17) for the reduced decomposition $w_j = s_{n_{j1}} \ldots s_{n_{jn_j}}$ and $T_{w_j} = T_{n_{j1}} \ldots T_{n_{jn_j}}$ we defined

$$\theta_{w_j}^{\widehat{s}_j} = \prod_{k=1}^{n_j} \theta_{\delta_{jk}}^{\widehat{s}_j}, \tag{4.5.9}$$

where in the products $\theta_{\delta_{jk}}^{\widehat{s}_j}$ appears on the left from $\theta_{\delta_{jm}}^{\widehat{s}_j}$ if $k < m$, and

$$\theta_{\delta_{jk}}^{\widehat{s}_j} = \exp_{q_{\delta_{jk}}}[(1 - q_{\delta_{jk}}^{-2})e_{\delta_{jk}}e^{-2K_{\widehat{s}_j}\delta_{jk}^{\vee}} \otimes f_{\delta_{jk}}], {\theta_{\delta_{jk}}^{\widehat{s}_j}}^{-1} = \exp_{q_{\delta_{jk}}^{-1}}[(1 - q_{\delta_{jk}}^{2})e_{\delta_{jk}}e^{-2K_{\widehat{s}_j}\delta_{jk}^{\vee}} \otimes f_{\delta_{jk}}],$$

$$e_{\delta_{jk}} = (X_{\delta_{jk}}^{+}e^{hK_{\widehat{s}_j}\delta_{jk}^{\vee}}), f_{\delta_{jk}} = (e^{-hK_{\widehat{s}_j}\delta_{jk}^{\vee}}X_{\delta_{jk}}^{-}), \delta_{jk} = s_{n_{j1}} \ldots s_{n_{jk-1}}\alpha_{i_{jk}},$$

$$X_{\delta_{jk}}^{\pm} = T_{n_{j1}} \ldots T_{n_{jk-1}}X_{i_{jk}}^{\pm}.$$

Observe that for $f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G]$ one has $f(T_{w_j}^{-1} \cdot T_{w_j}) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G]$ as the braid group acts on finite rank $U_{\mathcal{B}}^{\widehat{s}_{j+1},res}(\mathfrak{g})$–modules.

Formula (2.7.5) for $\psi^{{\theta_{w_j}^{\widehat{s}_j}}^{-1}}$ is expressed in terms of $\theta_{w_j}^{\widehat{s}_j}$ and of $v_{\widehat{s}_j}$ the definitions of which contain infinite series in divided powers of quantum root vectors (see (4.5.9)). Only finitely many such divided powers act on a given finite rank $U_{\mathcal{B}}^{\widehat{s}_{j+1},res}(\mathfrak{g})$–module in a non–trivial way. Therefore only finitely many terms in the infinite series will contribute to formula (4.5.5).

The elements $e^{-2K_{\widehat{s}_j}\delta_{jk}^{\vee}}$ which appear in the formula for $\theta_{w_j}^{\widehat{s}_j}$ also act on finite rank $U_{\mathcal{B}}^{\widehat{s}_{j+1},res}(\mathfrak{g})$–modules as such modules are direct sums of their weight subspaces, and $e^{-2K_{\widehat{s}_j}\delta_{jk}^{\vee}}$ acts on a subspace of weight λ by multiplication by $e^{\lambda(-2K_{\widehat{s}_j}\delta_{jk}^{\vee})} \in \mathcal{B}$.

We conclude that (4.5.5) defines a morphism of $\mathcal{B}$–modules $\phi_j : \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G] \to \mathbb{C}_{\mathcal{B}}^{\widehat{s}_j}[G]$.

Formulas (4.5.6) and (4.5.7) follow from (2.7.26) and (2.7.27), respectively.

$\square$

Lemma 4.5.7. *(i) For $j = 1, \ldots, R - 2$ one has $\phi_j(J_{\mathcal{B}}^{j+11'}) \subset J_{\mathcal{B}}^{jn_j+1}$.*

(ii) For $j = 1, \ldots, R - 2$, $f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G]$ one has $\phi_j(f \otimes \Delta_{\mu}^{\widehat{s}_{j+1}}) = (\phi_j f) \otimes T_{w_j}^{-1}(\Delta_{\mu}^{\widehat{s}_{j+1}})$ mod $J_{\mathcal{B}}^{jn_j+1}$, where $T_{w_j}^{-1}(\Delta_{\mu}^{\widehat{s}_{j+1}})(\cdot) := \Delta_{\mu}^{\widehat{s}_{j+1}}(T_{w_j}^{-1} \cdot T_{w_j})$.

(iii) If $\mu, \nu \in P_+$ are such that $w_j\mu + \nu \in P_+$ and $j = 1, \ldots, R - 2$ then $T_{w_j}^{-1}(\Delta_{\mu}^{\widehat{s}_{j+1}}) \otimes \Delta_{\nu}^{\widehat{s}_j} = \Delta_{w_j\mu+\nu}^{\widehat{s}_j}$ mod $J_{\mathcal{B}}^{jn_j+1}$.

Proof. (i) By the definition of $J_{\mathcal{B}}^{jn_j}$ and of $J_{\mathcal{B}}^{j+11'}$ it suffices to show that for any $f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G]$ and $g(\cdot) = (u, \cdot v) \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G]$ such that u is a highest weight vector in a finite rank representation V of $U_h(\mathfrak{g})$, and $v \in V$ satisfies $(u, t_{\widehat{s}_{j+1}}z_+xv) = 0$ for any $z_+ \in U_{\mathcal{B}}^{\widehat{s}_{j+1},res}(\mathfrak{z}_+^{j+1})$, $x \in U_{\mathcal{B}}^{\widehat{s}_{j+1},res}([-\delta_{j+11}, -\delta_{j+1}])$ and some element $t_{\widehat{s}_{j+1}}$ of the braid group acting on $\mathfrak{h} \subset U_h(\mathfrak{h})$ in the same way as $\widehat{s}_j$ one has

$$\phi_j(f \otimes g)(w \otimes yht_{\widehat{s}_j}z_+x) = 0$$

for any $w \in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{g})$, $y \in \omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jn_j+1}, -\delta_{jD}]))$, $h \in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{h})$, $z_+ \in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{z}_+^{j})$, $x \in U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{j1}, -\delta_{jm_j}])$.

Indeed, from the explicit formula for $v_{\widehat{s}_j}$ and for $\theta_{w_j}^{\widehat{s}_j}$ and from the fact that $S_{\widehat{s}_j}$ preserves weights, and for $k = 1, \ldots n_j$ $T_{w_j}^{-1}(e_{\delta_{jk}})$ has strictly negative weights it follows by Lemma 2.8.4 (i) and by formula (4.5.7) that

$$\phi_j(f \otimes g)(w \otimes yht_{\widehat{s}_j}z_+x) = \sum_{m,n}(\text{Ad}^{s_j}(d_m^1 c_n)\phi_j f)(w)g(T_{w_j}^{-1}(vS_{\widehat{s}_j}(c_m)yht_{\widehat{s}_j}z_+xd_m^2 d_n)) =$$

$$= \sum_{m,n}(\text{Ad}^{s_j}(d_m^1 c_n)\phi_j f)(w)(u, T_{w_j}^{-1}(vS_{\widehat{s}_j}(c_m)yht_{\widehat{s}_j}z_+xd_m^2 d_n)v) = \sum_{n}(\text{Ad}^{s_j}(c_n)\phi_j f)(w)(u, T_{w_j}^{-1}(yht_{\widehat{s}_j}z_+xd_n)v).$$

Since $w_j^{-1}[-\delta_{jn_j+1}, -\delta_{jD}] \subset \Delta_-$ Lemma 2.8.4 (i) implies that the right hand side of the last formula takes the form

$$\phi_j(f \otimes g)(w \otimes yht_{\widehat{s}_j}z_+x) = y_0 \sum_{n}(\text{Ad}^{s_j}(c_n)\phi_j f)(w)(u, T_{w_j}^{-1}(ht_{\widehat{s}_j}z_+xd_n)v), \tag{4.5.10}$$

where $y_0 \in \mathcal{B}$ is the zero weight component of y.

Next, $xd_n \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, therefore by Lemma 2.8.2 (ix) we can write $xd_n = \sum_p x_{np} x'_{np} + y_n$, where $x_{np} \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jn_j}])$ have strictly negative weights, $y_n, x'_{np} \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jn_j+1}, -\delta_{jm_j}])$, and (4.5.10) can be rewritten as follows

$$\phi_j(f \otimes g)(w \otimes yht_{\widehat{s}_j} z_+ x) = y_0 \sum_{p,n} (\mathrm{Ad}^{s_j}(c_n)\phi_j f)(w)(u, T_{w_j}^{-1}(ht_{\widehat{s}_j} z_+ x_{np} x'_{np})v) + \qquad (4.5.11)$$

$$+ y_0 \sum_n (\mathrm{Ad}^{s_j}(c_n)\phi_j f)(w)(u, T_{w_j}^{-1}(ht_{\widehat{s}_j} z_+ y_n)v) =$$

$$= y_0 \sum_{p,n} (\mathrm{Ad}^{s_j}(c_n)\phi_j f)(w)(u, T_{w_j}^{-1}(ht_{\widehat{s}_j}(z_+ x_{np})) T_{w_j}^{-1}(t_{\widehat{s}_j} x'_{np})v) + y_0 c \sum_n (\mathrm{Ad}^{s_j}(c_n)\phi_j f)(w)(u, t_{\widehat{s}_{j+1}} T_{w_j}^{-1}(z_+ y_n)v),$$

where $c \in \mathcal{B}$ is defined by the condition $T_{w_j}^{-1}(h)u = cu$ and $t_{\widehat{s}_{j+1}} = T_{w_j}^{-1} t_{\widehat{s}_j} T_{w_j}^{-1}$.

The weights of the weight components of $t_{\widehat{s}_j}(z_+ x_{np})$ belong to $\mathbb{N}\widehat{s}_j((w_1 \ldots w_{j-1})^{-1}(\Delta_0)_+ \cup [-\delta_{j1}, -\delta_{jn_j}]) = \mathbb{N}(w_1 \ldots w_{j-1})^{-1}(s(-\Delta^j \cup (\Delta_0)_+))$ and by the choice of x_{np} they do not belong to $\mathbb{N}(w_1 \ldots w_{j-1})^{-1}(\Delta_0)_+$. Since by Lemma 1.6.22 (i) $s(\Delta^j) \subset \Delta_+^{j+1}$, and $(w_1 \ldots w_j)^{-1}\Delta_+^{j+1} = \Delta_+$ we deduce that the weights of the weight components of $T_{w_j}^{-1}(ht_{\widehat{s}_j}(z_+ x_{np}))$ are non–zero; they belong to $\mathbb{N}(\Delta_- \cup (w_1 \ldots w_j)^{-1}(\Delta_0)_+)$ and do not belong to $\mathbb{N}(w_1 \ldots w_j)^{-1}(\Delta_0)_+ \subset \Delta_+$. In particular, these weights are not non–negative, and hence by Lemma 2.8.4 (i) the first term in the left hand side of (4.5.11) vanishes as u is a highest weight vector.

For the second term we note that by the definition

$$T_{w_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_{j+1}, res}(\mathfrak{z}_+^j)) = U_{\mathcal{B}}^{\widehat{s}_{j+1}, res}(\mathfrak{z}_+^{j+1}), T_{w_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jn_j+1}, -\delta_{jm_j}])) = U_{\mathcal{B}}^{\widehat{s}_{j+1}, res}([-\delta_{j+11}, -\delta_{j+1}]),$$

so that $T_{w_j}^{-1}(z_+ y_n) \in U_{\mathcal{B}}^{\widehat{s}_{j+1}, res}(\mathfrak{z}_+^{j+1}) U_{\mathcal{B}}^{\widehat{s}_{j+1}, res}([-\delta_{j+11}, -\delta_{j+1}])$. Therefore the second term in the right hand side of (4.5.11) vanishes by the choice of u and v. This completes the proof of part (i)

(ii) By the definition of $J_{\mathcal{B}}^{jn_j}$ it suffices to show that for any $f \in \mathbb{C}_{\mathcal{B}}^{\widehat{s}_{j+1}}[G]$ one has

$$\phi_j(f \otimes \Delta_\mu^{\widehat{s}_{j+1}})(w \otimes yht_{\widehat{s}_j} z_+ x) = (\phi_j f)(w) \Delta_\mu^{\widehat{s}_{j+1}}(T_{w_j}^{-1} yht_{\widehat{s}_j} z_+ x T_{w_j})$$

for any $w \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{g}), y \in \omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jn_j+1}, -\delta_{jD}])), h \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{h}), z_+ \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j), x \in U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$.

Similarly to (4.5.10) we obtain

$$\phi_j(f \otimes \Delta_\mu^{\widehat{s}_{j+1}})(w \otimes yht_{\widehat{s}_j} z_+ x) = y_0 \sum_n (\mathrm{Ad}^{s_j}(c_n)\phi_j f)(w)(v_\mu, T_{w_j}^{-1}(ht_{\widehat{s}_j} z_+ xd_n) t_{\widehat{s}_{j+1}} v_\mu) = \qquad (4.5.12)$$

$$= y_0 \sum_n (\mathrm{Ad}^{s_j}(c_n)\phi_j f)(w)(v_\mu, T_{w_j}^{-1}(ht_{\widehat{s}_j}(z_+ xd_n))v_\mu),$$

where $y_0 \in \mathcal{B}$ is the zero weight component of y, and we assume without loss of generality that $t_{\widehat{s}_{j+1}} = T_{w_j}^{-1} t_{\widehat{s}_j} T_{w_j}^{-1}$.

Note that only the zero degree component of $z_+ xd_n$ can contribute to the right hand side of (4.5.12). Since $z_+ xd_n \in U_{\mathcal{B}}^{\widehat{s}_j, res}(\mathfrak{z}_+^j) U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{j1}, -\delta_{jm_j}])$, the weights of the weight components of $z_+ xd_n$ belong to

$$\mathbb{N}((w_1 \ldots w_{j-1})^{-1}(\Delta_0)_+ \cup [-\delta_{j1}, -\delta_{jm_j}] = \mathbb{N}(w_1 \ldots w_{j-1})^{-1}((\Delta_0)_+ \cup [-\beta_{j1}, -\beta_c]).$$

Observe that $(\Delta_0)_+ \cup [-\beta_{j1}, -\beta_c] \subset [\beta_1^0, -\beta_c]$ which is a minimal segment. Therefore only the product of the zero degree components of z_+, x and d_n contribute to the right hand side of (4.5.12). From the formula (4.5.9) for $\theta_{w_j}^{\widehat{s}_j} = \sum_n c_n \otimes d_n$ it follows that only one term with $d_n = 1$ has this property, and the corresponding $c_n = 1$ as well. We conclude that (4.5.12) takes the form

$$\phi_j(f \otimes \Delta_\mu^{\widehat{s}_{j+1}})(w \otimes yht_{\widehat{s}_j} z_+ x) = y_0(\phi_j f)(w)(v_\mu, T_{w_j}^{-1}(ht_{\widehat{s}_j}(z_+ x))v_\mu).$$

Finally since $w_j^{-1}[-\delta_{jn_j+1}, -\delta_{jD}] \subset \Delta_-$ and $y \in \omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j, res}([-\delta_{jn_j+1}, -\delta_{jD}]))$, Lemma 2.8.4 (i) implies that the right hand side of the last formula takes the form

$$\phi_j(f \otimes \Delta_\mu^{\widehat{s}_{j+1}})(w \otimes yht_{\widehat{s}_j} z_+ x) = (\phi_j f)(w)(v_\mu, T_{w_j}^{-1} yht_{\widehat{s}_j} z_+ x T_{w_j} t_{\widehat{s}_{j+1}} v_\mu) = (\phi_j f)(w) \Delta_\mu^{\widehat{s}_{j+1}}(T_{w_j}^{-1} yht_{\widehat{s}_j} z_+ x T_{w_j}).$$

This completes the proof of part (ii).

(iii) By the definition of $J_{\mathcal{B}}^{jn_j}$ it suffices to show that

$$(\Delta_\mu^{\widehat{s}_{j+1}}(T_{w_j}^{-1}\cdot T_{w_j})\otimes\Delta_\nu^{\widehat{s}_j})(yht_{\widehat{s}_j}z_+x) = \Delta_{w_j\mu+\nu}^{\widehat{s}_j}(yht_{\widehat{s}_j}z_+x) \qquad (4.5.13)$$

for any $w\in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{g})$, $y\in\omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jn_j+1},-\delta_{jD}]))$, $h\in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{h})$, $z_+\in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{z}_+^j)$, $x\in U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{j1},-\delta_{jm_j}])$.
We shall use this condition with $t_{\widehat{s}_j}=T_{\widehat{s}_j}$.

From (2.8.13) and the fact that $\omega_0 S_{\widehat{s}_j}^{-1}$ is an anti-coautomorphism preserving weights we obtain

$$\Delta_{\widehat{s}_j}(y) = y_0 + \sum_i y_i\otimes h_i' + \sum_k v_k\otimes w_k, \qquad (4.5.14)$$

where the weights of the elements w_k are strictly negative, $y_i\in\omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jn_j+1},-\delta_{jD}]))$ have non–zero weights, $h_i'\in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{h})$, and $y_0\in\mathcal{B}$ is the zero weight component of y.

By (4.2.21)

$$\Delta_{\widehat{s}_j}(T_{\widehat{s}_j}) = \theta_{\widehat{s}_j}^{\widehat{s}_j}T_{\widehat{s}_j}\otimes T_{\widehat{s}_j}$$

Recalling the definition of $\theta_{\widehat{s}_j}^{\widehat{s}_j}$ and observing that v_μ and v_ν are highest weight vectors and that the weights of the elements of $T_{w_j}^{-1}(\omega_0 S_{\widehat{s}_j}^{-1}(U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jn_j+1},-\delta_{jD}])))$ are non–positive as $w_j^{-1}[-\delta_{jn_j+1},-\delta_{jD}]\subset\Delta_-$ we deduce using also (4.5.14) that the left hand side of (4.5.13) takes the form

$$(\Delta_\mu^{\widehat{s}_{j+1}}(T_{w_j}^{-1}\cdot T_{w_j})\otimes\Delta_\nu^{\widehat{s}_j})(yhT_{\widehat{s}_j}z_+x) = y_0(v_\mu,T_{w_j}^{-1}h^1T_{\widehat{s}_j}z_+^1x^1T_{w_j}T_{\widehat{s}_{j+1}}v_\mu)(v_\nu,h^2T_{\widehat{s}_j}z_+^2x^2T_{\widehat{s}_j}v_\nu) = \qquad (4.5.15)$$

$$= y_0(v_\mu,T_{w_j}^{-1}(h^1T_{\widehat{s}_j}(z_+^1x^1))v_\mu)(v_\nu,h^2T_{\widehat{s}_j}(z_+^2x^2)v_\nu).$$

We deduce that only the zero weight components of $z_+^1x^1$ and of $z_+^2x^2$ can contribute to the right hand side of the last formula. Since the comultiplication preserves weights and $z_+x\in U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{z}_+^j)U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{j1},-\delta_{jm_j}])$, the weights of the weight components of z_+x belong to

$$\mathbb{N}((w_1\ldots w_{j-1})^{-1}(\Delta_0)_+\cup[-\delta_{j1},-\delta_{jm_j}] = \mathbb{N}(w_1\ldots w_{j-1})^{-1}((\Delta_0)_+\cup[-\beta_{j1},-\beta_c]).$$

Observe that $(\Delta_0)_+\cup[-\beta_{j1},-\beta_c]\subset[\beta_1^0,-\beta_c]$ which is a minimal segment. Therefore only the product of the zero degree components $(z_+)_0$ and x_0 of z_+ and x, respectively, can contribute to the right hand side of (4.5.15). By the definition of the algebras $U_{\mathcal{B}}^{\widehat{s}_j,res}(\mathfrak{z}_+^j)$ and $U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{j1},-\delta_{jm_j}])$ we have $(z_+)_0,x_0\in\mathcal{B}$. Therefore we can rewrite (4.5.15) as follows

$$(\Delta_\mu^{\widehat{s}_{j+1}}(T_{w_j}^{-1}\cdot T_{w_j})\otimes\Delta_\nu^{\widehat{s}_j})(yhT_{\widehat{s}_j}z_+x) = y_0(z_+)_0x_0(v_\mu,T_{w_j}^{-1}(h^1)v_\mu)(v_\nu,h^2v_\nu) = \qquad (4.5.16)$$

$$= y_0(z_+)_0x_0(T_{w_j}v_\mu\otimes v_\nu,hT_{w_j}v_\mu\otimes v_\nu) = y_0(z_+)_0x_0(v_{w_j\mu+\nu},hv_{w_j\mu+\nu}) = y_0(v_{w_j\mu+\nu},hT_{\widehat{s}_j}(z_+x)v_{w_j\mu+\nu}) =$$

$$= (v_{w_j\mu+\nu},yhT_{\widehat{s}_j}z_+xT_{\widehat{s}_j}^{-1}v_{w_j\mu+\nu}) = \Delta_{w_j\mu+\nu}^{\widehat{s}_j}(yhT_{\widehat{s}_j}z_+x).$$

This establishes (4.5.13) and completes the proof.

$\square$

Proposition 4.5.8. *(i) If $\mu,\nu\in P_+$ are such that $w_j\mu+\nu\in P_+$ then for $j=1,\ldots,R-2$ the following relation holds in $\mathbb{C}_{jn_j+1}^{loc}[G]$:*

$$T_{w_j}^{-1}(\Delta_\mu^{\widehat{s}_{j+1}}) = \Delta_{w_j\mu+\nu}^{\widehat{s}_j}\otimes\Delta_\nu^{\widehat{s}_j\,-1}.$$

(ii) For $j=1,\ldots,R-2$ ϕ_j induces a morphism of $\mathcal{B}$–modules

$$\phi_j:\mathbb{C}_{j+11}^{loc}[G]\to\mathbb{C}_{jn_j+1}^{loc}[G] \qquad (4.5.17)$$

which satisfies

$$\phi_j\mathrm{Ad}^{\widehat{s}_{j+1}}(T_{w_j}^{-1}x) = \mathrm{Ad}^{\widehat{s}_j}(x)\phi_j,\quad x\in U_{\mathcal{B}}^{\widehat{s}_j,res}([-\delta_{jn_j+1},-\delta_{jm_j}]), \qquad (4.5.18)$$

and

$$\phi_j(f\otimes\Delta_\mu^{\widehat{s}_{j+1}}) = (\phi_j f)\otimes\Delta_{w_j\mu+\nu}^{\widehat{s}_j}\otimes\Delta_\nu^{\widehat{s}_j\,-1},f\in\mathbb{C}_{j+11}^{loc}[G],$$

$$\phi_j(f\otimes\Delta_\mu^{\widehat{s}_{j+1}\,-1}) = (\phi_j f)\otimes\Delta_\nu^{\widehat{s}_j}\otimes\Delta_{w_j\mu+\nu}^{\widehat{s}_j\,-1},f\in\mathbb{C}_{j+11}^{loc}[G],$$

where $\mu,\nu\in P_+$ are such that $w_j\mu+\nu\in P_+$.

Proof. (i) follows from Lemma 4.5.7 (iii) and from the definition of $\mathbb{C}^{loc}_{jn_j+1}[G]$.

(ii) follows from part (i) of this proposition, from Lemma 4.5.7 (i) and (ii), formula (4.5.6) and from the definitions of $\mathbb{C}^{loc}_{j+11}[G]$ and $\mathbb{C}^{loc}_{jn_j+1}[G]$.

$\square$

4.6   Zhelobenko type operators for q-W–algebras

This section is central in this part and in the whole book. We are going to introduce and study some quantum analogues of the operators Π_{jk} and Π_c defined in (3.5.19) and (3.5.21). It turns out that the analogues P_{jk} of Π_{jk} can be obtained by proper extrapolation of the expansion of the conjugation operator in (3.5.19) in terms of the adjoint action operator and by replacing the coefficients φ_{jk} with their quantum counterparts B_{jk} introduced in Corollary 4.5.2 (i). However, the proof of Proposition 4.6.1 which asserts that the image of their composition P_c consists of invariant elements with respect to the adjoint action of $U^{s,res}_{\mathcal{B}}(\mathfrak{m}_-)$ is rather complicated. It entirely relies on the properties of $I^{jk\,loc}_{\mathcal{B}}$, of the quotients $\mathbb{C}^{loc}_{jk}[G]$ and of the adjoint action, which were obtained in the previous sections of this chapter.

The point is that in the quantum case we do not have in our disposal the isomorphism (3.4.32) which plays a crucial role in the proof of a similar property for the operator Π_c, given by (3.5.19), (3.5.21), as one can see from the proof of Proposition 3.5.6.

We start with the definition of quantum analogues P_{jp} of the operators Π_{jp}. For technical reasons we shall also need more general operators P^k_{jp}. More precisely, by Lemmas 4.5.3 and 4.5.5 for $f \in \mathbb{C}^{loc}_{jp}[G]$, $j = 1, \ldots, R - 1$, $p = 1, \ldots, n_j$, $n, k \in \mathbb{N}$, $n \geq k$, we have a well–defined element $\mathrm{Ad}^0_{\widehat{s}_j} f^{(n-k)}_{\delta_{jp}}(f) \otimes B^n_{jp} \in \mathbb{C}^{loc}_{jp}[G]$. Note that by Lemma 4.5.5 for each $f \in \mathbb{C}^{loc}_{jp}[G]$ $\mathrm{Ad}^0_{\widehat{s}_j} f^{(n-k)}_{\delta_{jp}}(f) = 0$ for $n - k > N(f)$, where $N(f) \in \mathbb{N}$ depends on f. Therefore we can define define an element $P^k_{jp}(f) \in \mathbb{C}^{loc}_{jp}[G]$ by

$$P^k_{jp}(f) = \sum_{n=k}^{\infty} (-1)^n q_{\delta_{jp}}^{\frac{(n-1)(n-2k)}{2}} \mathrm{Ad}^0_{\widehat{s}_j} f^{(n-k)}_{\delta_{jp}}(f) \otimes B^n_{jp}. \tag{4.6.1}$$

In particular, formula (4.6.1) defines an operator

$$P^k_{jp} : \mathbb{C}^{loc}_{jp}[G] \to \mathbb{C}^{loc}_{jp}[G].$$

Denote $P^0_{jp} = P_{jp}$.

The following proposition summarizes the main properties of the operators P_{jp}.

Proposition 4.6.1. *(i) For any $j = 1, \ldots, R - 1$, $p = 1, \ldots, n_j$ the composition*

$$P_{\leq jp} = P_{11} \ldots P_{1n_1} \circ \phi_1 \circ P_{21} \ldots P_{2n_2} \circ \phi_2 \ldots \circ \phi_{j-1} \circ P_{j1} \ldots P_{jp} : \mathbb{C}^{loc}_{jp}[G] \to \mathbb{C}^{loc}_{11}[G] \tag{4.6.2}$$

is well–defined. In fact $P_{\leq jp}$ is well–defined as an operator with the domain $\mathbb{C}^{loc}_{jp+1}[G]$.

(ii) For any $\beta \in [\beta_{11}, \beta_{jp}]$, $n > 0$ and for any $f \in \mathbb{C}^{loc}_{jp+1}[G]$ we have

$$\mathrm{Ad}^0_s f^{(n)}_\beta (P_{\leq jp}(f)) = 0. \tag{4.6.3}$$

In particular, for any $x \in U^{s,res}_{\mathcal{B}}([-\beta_{11}, -\beta_{jp}])$, $f \in \mathbb{C}^{loc}_{jp+1}[G]$

$$\mathrm{Ad}^0_s x(P_{\leq jp}(f)) = \varepsilon_s(x) P_{\leq jp}(f),$$

and if we denote $P_c = P_{\leq R-1 n_{R-1}}$ then for any $x \in U^{s,res}_{\mathcal{B}}(\mathfrak{m}_-)$, $f \in \mathbb{C}^{loc}_{R-1 n_{R-1}+1}[G]$

$$\mathrm{Ad}^0_s x(P_c(f)) = \varepsilon_s(x) P_c(f).$$

Moreover, for $j = 1, \ldots, R - 1$, $p = 1, \ldots, n_j$, $(j, p) \neq (R - 1, n_{R-1})$ one has

$$\mathrm{Ad}^0_s f^{(n)}_{\beta_{jp+1}} (P_{\leq jp}(f)) = c_{jp+1} P_{\leq jp}(\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp+1}} f), \tag{4.6.4}$$

where we assume that $\beta_{jn_j+1} = \beta_{j+11}$ and $c_{jp+1} \in \{\pm q^{\mathbb{Z}}\}$.

The proof of Proposition 4.6.1 is quite long. It will be split into several lemmas. Firstly we shall show that $P_{\leq jp}$ is well–defined as an operator with the domain of definition $\mathbb{C}^{loc}_{jp+1}[G]$, i.e. $I^{jp+1\,loc}_{\mathcal{B}}/I^{jp\,loc}_{\mathcal{B}} \subset \mathbb{C}^{loc}_{jp}[G]$ belongs to the kernel of this operator.

Then we prove that for $n > 0$ $\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}} P_{jp}$ is the zero operator. This is done using an explicit calculation and observing that $\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}}$ has some properties analogues to properties of derivations of order n.

Finally to show that for $\beta \in [\beta_{11}, \beta_{jp}]$ and $n > 0$ $\mathrm{Ad}^0_s f^{(n)}_\beta(P_{\leq jp}(f)) = 0$ we shall use the property of the images of operators P_{jp} mentioned in the previous paragraph and commutation relations between P_{jp} and $\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_\beta$ for $\beta > \delta_{jp}$ which, in particular, lead to commutation relations (4.6.4).

Firstly, we are going to show by an explicit calculation that for $n > 0$ $\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}} P_{jp}$ is the zero operator. Since the right hand side of formula (4.6.1) contains products of elements from $\mathbb{C}^{\widehat{s}_j, loc}_{\mathcal{B}}[G]$ and of B^n_{jp}, we have to study the adjoint action on such products.

Lemma 4.6.2. *(i) Let $B_{jp} = A^{0\,-1}_{jp} \otimes A_{jp} \in \mathbb{C}^{\widehat{s}_j, loc}_{\mathcal{B}}[G]_0$, $f \in \mathbb{C}^{loc}_{jp}[G]$, $j = 1, \ldots, R-1$, $p = 1, \ldots, n_j$. Then for any $k, n \in \mathbb{N}$ the identity*

$$\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}}(f \otimes B^k_{jp}) = \sum_{r=0}^{\min(n,k)} q^{(2n-r)k + \frac{r(r-1)}{2} - rn}_{\delta_{jp}} \begin{bmatrix} k \\ r \end{bmatrix}_{q_{\delta_{jp}}} \mathrm{Ad}^0_{\widehat{s}_j} f^{(n-r)}_{\delta_{jp}}(f) \otimes B^{k-r}_{jp} \tag{4.6.5}$$

holds in $\mathbb{C}^{loc}_{jp}[G]$.

(ii) For any two roots $\alpha, \beta \in \Delta$ we denote

$$c^s_{\alpha\beta} = q^{((2K_s + id)\alpha, \beta)}.$$

Then for any $j = 1, \ldots, R-1$, $p = 1, \ldots, n_j$, $m, n \in \mathbb{N}$, $f \in \mathbb{C}^{loc}_{jp}[G]$, $\beta \in [\delta_{jp+1}, \delta_{jm_j}]$ the identity

$$\mathrm{Ad}^0_{\widehat{s}_j} f^{(m)}_\beta(f \otimes B^n_{jp}) = (c^{\widehat{s}_j}_{\delta_{jp}\beta})^{mn} \mathrm{Ad}^0_{\widehat{s}_j} f^{(m)}_\beta(f) \otimes B^n_{jp} \tag{4.6.6}$$

holds in $\mathbb{C}^{loc}_{jp}[G]$.

Proof. (i) We prove (4.6.5) by induction over k. We start with the case when f is the image of an element of $\mathbb{C}^{\widehat{s}_j}_{\mathcal{B}}[G]$ in $\mathbb{C}^{loc}_{jp}[G]$ under the canonical map $\mathbb{C}^{\widehat{s}_j}_{\mathcal{B}}[G] \to \mathbb{C}^{\widehat{s}_j, loc}_{\mathcal{B}}[G] \to \mathbb{C}^{loc}_{jp}[G]$. Firstly by (4.2.4), we have for any $n \in \mathbb{N}$

$$\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}}(f \otimes A_{jp}) = q^{n(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \delta_{jp} - \delta_{jp}, \mu_{jp} - \delta_{jp})} \mathrm{Ad}^0_{\widehat{s}_j}(f^{(n)}_{\delta_{jp}})(f) \otimes A_{jp} +$$

$$+ q^{(n-1)(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \delta_{jp} - \delta_{jp}, \mu_{jp} - \delta_{jp})} q^{-(n-1)}_{\delta_{jp}} \mathrm{Ad}^0_{\widehat{s}_j}(f^{(n-1)}_{\delta_{jp}})(f) \otimes A_{jp}(\omega_0 S^{-1}_{\widehat{s}_j}(f_{\delta_{jp}}) \cdot) +$$

$$+ \sum_{k=1}^n q^{-k(n-k)}_{\delta_{jp}} q^{n(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \delta_{jp} - \delta_{jp}, \mu_{jp} - \delta_{jp})} \mathrm{Ad}^0_{\widehat{s}_j}(G^{-k}_{\delta_{jp}} f^{(n-k)}_{\delta_{jp}})(f) \otimes A_{jp}(\cdot \omega_0(f^{(k)}_{\delta_{jp}})) +$$

$$+ \sum_{k=1}^n q^{-k(n-k)-(n-k-1)}_{\delta_{jp}} \mathrm{Ad}^0_{\widehat{s}_j}(G^{-k}_{\delta_{jp}} f^{(n-k-1)}_{\delta_{jp}})(f) \otimes A_{jp}(\omega_0 S^{-1}_{\widehat{s}_j}(G^{-n+1}_{\delta_{jp}} f_{\delta_{jp}}) \cdot \omega_0(f^{(k)}_{\delta_{jp}})) + \tag{4.6.7}$$

$$+ \sum_{k=0}^n \sum_{p=0}^{n-k-2} q^{-k(n-k)-p(n-k-p)}_{\delta_{jp}} \mathrm{Ad}^0_{\widehat{s}_j}(G^{-k}_{\delta_{jp}} f^{(p)}_{\delta_{jp}})(f) \otimes A_{jp}(\omega_0 S^{-1}_{\widehat{s}_j}(G^{-k-p}_{\delta_{jp}} f^{(n-k-p)}_{\delta_{jp}}) \cdot \omega_0(f^{(k)}_{\delta_{jp}})) +$$

$$+ \sum_{k=0}^{n-1} \sum_i q^{-k(n-k)}_{\delta_{jp}} \mathrm{Ad}^0_{\widehat{s}_j}(G^{-k}_{\delta_{jp}} x^{(n-k)}_i)(f) \otimes A_{jp}((\omega_0 S^{-1}_{\widehat{s}_j})(G^{-k}_{\delta_{jp}} y^{(n-k)}_i) \cdot \omega_0(f^{(k)}_{\delta_{jp}})) +$$

$$+ \sum_i \mathrm{Ad}^0_{\widehat{s}_j}(y^{(n)^2}_i)(f) \otimes A_{jp}((\omega_0 S^{-1}_{\widehat{s}_j})(y^{(n)^1}_i) \cdot \omega_0(x^{(n)}_i)).$$

By (4.4.1) all terms in the right hand side of (4.6.7), except for the first two, vanish. Also by (4.4.2) we have $A_{jp}((\omega_0 S^{-1}_{\widehat{s}_j})(f_{\delta_{jp}}) \cdot) = A^0_{jp}$, and hence in both cases we have

$$\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}}(f \otimes A_{jp}) = q^{n(\kappa \frac{1+s}{1-s} P_{\mathfrak{h}'} \delta_{jp} - \delta_{jp}, \mu_{jp} - \delta_{jp})} \mathrm{Ad}^0_{\widehat{s}_j}(f^{(n)}_{\delta_{jp}})(f) \otimes A_{jp} +$$

$$+q^{(n-1)(\kappa\frac{1+s}{1-s}P_{\mathfrak{h}'}\delta_{jp}-\delta_{jp},\mu_{jp}-\delta_{jp})}q_{\delta_{jp}}^{-(n-1)}\mathrm{Ad}_{\widehat{s}_j}^0(f_{\delta_{jp}}^{(n-1)})(f)\otimes A_{jp}^0.$$

Since by definition elements of $\mathfrak{S}_{\widehat{s}_j}^{-1}$ naturally act on $\mathbb{C}_{jp}^{loc}[G]$ by multiplication from the right one can multiply the last identity by $(A_{jp}^0)^{-1}$ from the right. Using commutation relations (4.2.2), formula (4.5.4) and recalling the definition of B_{jp} we obtain

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n)}(f\otimes B_{jp})=q_{\delta_{jp}}^{n-1}\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-1)}(f)+q_{\delta_{jp}}^{2n}\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n)}(f)\otimes B_{jp}. \tag{4.6.8}$$

Multiplying the last identity by Δ_μ^{-1} from the right and using the same arguments we get

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n)}(f\otimes\Delta_\mu^{-1}\otimes B_{jp})=q_{\delta_{jp}}^{n-1}\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-1)}(f\otimes\Delta_\mu^{-1})+q_{\delta_{jp}}^{2n}\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n)}(f\otimes\Delta_\mu^{-1})\otimes B_{jp},$$

Since elements $f\otimes\Delta_\mu^{-1}$ span $\mathbb{C}_{jp}^{loc}[G]$ formula (4.6.8) holds for any $n\in\mathbb{N}$, $f\in\mathbb{C}_{jp}^{loc}[G]$, i.e. (4.6.5) holds for any $n\in\mathbb{N}$, $k=1$. This establishes the base of induction.

Now we assume that (4.6.5) holds for some natural k and for all natural n and prove that it holds for $k+1$ and all natural n. The arguments given below are the same for both cases, and we consider them simultaneously. Since by Lemma 4.5.3 right multiplication gives rise to a well–defined action of B_{jp} on $\mathbb{C}_{jp}^{loc}[G]$, we have for any $f\in\mathbb{C}_{jp}^{loc}[G]$ using the base of induction and the induction assumption

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n)}(f\otimes B_{jp}^{k+1})=q_{\delta_{jp}}^{n-1}\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-1)}(f\otimes B_{jp}^k)+q_{\delta_{jp}}^{2n}\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n)}(f\otimes B_{jp}^k)\otimes B_{jp}=$$

$$=q_{\delta_{jp}}^{n-1}\sum_{r=0}^{\min(n-1,k)}q_{\delta_{jp}}^{(2(n-1)-r)k+\frac{r(r-1)}{2}-r(n-1)}\left[\begin{array}{c}k\\r\end{array}\right]_{q_{\delta_{jp}}}\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-r-1)}(f)\otimes B_{jp}^{k-r}+$$

$$+q_{\delta_{jp}}^{2n}\sum_{r=0}^{\min(n,k)}q_{\delta_{jp}}^{(2n-r)k+\frac{r(r-1)}{2}-rn}\left[\begin{array}{c}k\\r\end{array}\right]_{q_{\delta_{jp}}}\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-r)}(f)\otimes B_{jp}^{k-r+1}=$$

$$=\sum_{r=0}^{\min(n,k+1)}q_{\delta_{jp}}^{(2n-r)(k+1)+\frac{r(r-1)}{2}-rn}\left(q_{\delta_{jp}}^r\left[\begin{array}{c}k\\r\end{array}\right]_{q_{\delta_{jp}}}+q_{\delta_{jp}}^{r-(k+1)}\left[\begin{array}{c}k\\r-1\end{array}\right]_{q_{\delta_{jp}}}\right)\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-r)}(f)\otimes B_{jp}^{k-r+1}=$$

$$=\sum_{r=0}^{\min(n,k+1)}q_{\delta_{jp}}^{(2n-r)(k+1)+\frac{r(r-1)}{2}-rn}\left[\begin{array}{c}k+1\\r\end{array}\right]_{q_{\delta_{jp}}}\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-r)}(f)\otimes B_{jp}^{k+1-r},$$

where we used the identity

$$q_{\delta_{jp}}^r\left[\begin{array}{c}k\\r\end{array}\right]_{q_{\delta_{jp}}}+q_{\delta_{jp}}^{r-(k+1)}\left[\begin{array}{c}k\\r-1\end{array}\right]_{q_{\delta_{jp}}}=\left[\begin{array}{c}k+1\\r\end{array}\right]_{q_{\delta_{jp}}}$$

which can be found e.g. in [62], Proposition 6.1. This establishes the induction step and completes the proof of (4.6.5).

Formula (4.6.6) is proved in a similar way by induction using Lemma 4.4.1 (i), formulas (4.2.4), (4.4.2), (4.5.4), and the definition of B_{jp}.

$\square$

The previous lemma shows that the operator $\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n)}$ acts on $f\otimes B_{jp}^k$ as a quantum analogue of a derivation of order n such that the derivative of B_{jp} by $\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}$ is equal to one. Recalling the definition of the classical counterpart φ_{jp} of B_{jp} one can observe that a similar formula can be obtained in the classical case as well. It will be given by specializing (4.6.5) at $q^{\frac{1}{dr^2}}=1$.

In the next lemma we show that for $n>0$ the composition $\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n)}P_{jp}$ is the zero operator. At the same time this lemma lays a basis for the proof of the fact that the kernel of $P_{\leq jp}$ contains $I_{\mathcal{B}}^{jp+1^{loc}}/I_{\mathcal{B}}^{jp^{loc}}$.

Lemma 4.6.3. *For any $f\in\mathbb{C}_{jp}^{loc}[G]$, $j=1,\ldots,R-1$, $p=1,\ldots,n_j$, $k=0,1,\ldots$ we have*

$$P_{jp}^k(f\otimes B_{jp})=0, \tag{4.6.9}$$

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n)}(P_{jp}(f))=0,\ n>0. \tag{4.6.10}$$

Proof. Since by Lemma 4.5.3 right multiplication by B_{jp} gives rise to an action of B_{jp} on $\mathbb{C}^{loc}_{jp}[G]$ we have by (4.6.1) and (4.6.5) with $k = 1$

$$P^k_{jp}(f \otimes B_{jp}) = \sum_{n=k+1}^{\infty} (-1)^n q_{\delta_{jp}}^{\frac{(n-1)(n-2k)}{2}} q_{\delta_{jp}}^{n-k-1} \mathrm{Ad}^0_{\widehat{s}_j} f^{(n-k-1)}_{\delta_{jp}}(f) \otimes B^n_{jp} +$$

$$+ \sum_{n=k}^{\infty} (-1)^n q_{\delta_{jp}}^{\frac{(n-1)(n-2k)}{2}} q_{\delta_{jp}}^{2(n-k)} \mathrm{Ad}^0_{\widehat{s}_j} f^{(n-k)}_{\delta_{jp}}(f) \otimes B^{n+1}_{jp} =$$

$$= - \sum_{n=k}^{\infty} (-1)^n q_{\delta_{jp}}^{\frac{n(n+1-2k)}{2}+n-k} \mathrm{Ad}^0_{\widehat{s}_j} f^{(n-k)}_{\delta_{jp}}(f) \otimes B^{n+1}_{jp} +$$

$$+ \sum_{n=k}^{\infty} (-1)^n q_{\delta_{jp}}^{\frac{(n-1)(n-2k)}{2}+2n-2k} \mathrm{Ad}^0_{\widehat{s}_j} f^{(n-k)}_{\delta_{jp}}(f) \otimes B^{n+1}_{jp} = 0$$

which proves (4.6.9).

Similarly by (4.6.1) and (4.6.5) we obtain

$$\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}}(P_{jp}(f)) = \mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}} \left(\sum_{k=0}^{\infty} (-1)^k q_{\delta_{jp}}^{\frac{(k-1)k}{2}} \mathrm{Ad}^0_{\widehat{s}_j} f^{(k)}_{\delta_{jp}}(f) \otimes B^k_{jp} \right) =$$

$$= \sum_{k=0}^{\infty} \sum_{t=0}^{\min(n,k)} (-1)^k q_{\delta_{jp}}^{\frac{(k-1)k}{2}+(2n-t)k+\frac{t(t-1)}{2}-tn} \begin{bmatrix} k \\ t \end{bmatrix}_{q_{\delta_{jp}}} \mathrm{Ad}^0_{\widehat{s}_j} (f^{(k)}_{\delta_{jp}} f^{(n-t)}_{\delta_{jp}})(f) \otimes B^{k-t}_{jp}.$$

Introducing a new variable of summation $k - t = r$ and using the identity

$$f^{(k)}_{\delta_{jp}} f^{(n-t)}_{\delta_{jp}} = \begin{bmatrix} n+k-t \\ k \end{bmatrix}_{q_{\delta_{jp}}} f^{(n+k-t)}_{\delta_{jp}}$$

we get

$$\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}}(P_{jp}(f)) =$$

$$= \sum_{r=0}^{\infty} \sum_{t=0}^{n} (-1)^{r+t} q_{\delta_{jp}}^{\frac{(r+t-1)(r+t)}{2}+(2n-t)(r+t)+\frac{t(t-1)}{2}-tn} \begin{bmatrix} r+t \\ t \end{bmatrix}_{q_{\delta_{jp}}} \begin{bmatrix} r+n \\ r+t \end{bmatrix}_{q_{\delta_{jp}}} \mathrm{Ad}^0_{\widehat{s}_j} (f^{(n+r)}_{\delta_{jp}})(f) \otimes B^r_{jp}.$$

Now recalling that

$$\begin{bmatrix} r+t \\ t \end{bmatrix}_{q_{\delta_{jp}}} \begin{bmatrix} r+n \\ r+t \end{bmatrix}_{q_{\delta_{jp}}} = \begin{bmatrix} n \\ t \end{bmatrix}_{q_{\delta_{jp}}} \begin{bmatrix} r+n \\ r \end{bmatrix}_{q_{\delta_{jp}}}$$

we obtain

$$\mathrm{Ad}^0_{\widehat{s}_j} f^{(n)}_{\delta_{jp}}(P_{jp}(f)) =$$

$$= \sum_{r=0}^{\infty} (-1)^r q_{\delta_{jp}}^{\frac{(r-1)r}{2}+2nr} \begin{bmatrix} r+n \\ r \end{bmatrix}_{q_{\delta_{jp}}} \left(\sum_{t=0}^{n} (-1)^t q_{\delta_{jp}}^{-t+tn} \begin{bmatrix} n \\ t \end{bmatrix}_{q_{\delta_{jp}}} \right) \mathrm{Ad}^0_{\widehat{s}_j} (f^{(n+r)}_{\delta_{jp}})(f) \otimes B^r_{jp} = 0,$$

where we used the identity

$$\sum_{t=0}^{n} (-1)^t q_{\delta_{jp}}^{-t+tn} \begin{bmatrix} n \\ t \end{bmatrix}_{q_{\delta_{jp}}} = 0$$

which follows from the q–binomial theorem (see e.g. [43], Ch. 1).

This proves (4.6.10).

$\square$

Lemma 4.6.4. *(i) Let $k \in \mathbb{N}$. Then for any $j = 1, \ldots, R-1$, $p-1, \ldots, n_j$ one has $P_{jp}^k(I_{\mathcal{B}}^{jp+1\,loc}/I_{\mathcal{B}}^{jp\,loc}) = 0$. Thus P_{jp}^k is well–defined as an operator with the domain $\mathbb{C}_{jp+1}^{loc}[G]$.*

(ii) Moreover, for any $1 \leq i \leq j \leq R-1$ $1 \leq q \leq n_i$, $1 \leq p \leq n_j$ such that $q \leq p$ if $i = j$, and any $k_{ij} \in \mathbb{N}$ the composition

$$P_{iq}^{k_{iq}} \ldots P_{in_i}^{k_{in_i}} \circ \phi_i \ldots \circ \phi_{j-1} \circ P_{j1}^{k_{j1}} \ldots P_{jp}^{k_{jp}}$$

is well–defined and gives rise to an operator with domain $\mathbb{C}_{jp+1}^{loc}[G]$, and the target space being $\mathbb{C}_{iq}^{loc}[G]$.

In particular, the composition $P_{\leq jp} = P_{11} \ldots P_{jp}$ is well–defined and gives rise to an operator with domain $\mathbb{C}_{jp+1}^{loc}[G]$ and the target space being $\mathbb{C}_{11}^{loc}[G]$.

Proof. Let us show that $P_{jp}^k(I_{\mathcal{B}}^{jp+1\,loc}/I_{\mathcal{B}}^{jp\,loc}) = 0$. Indeed, by the definition of $I_{\mathcal{B}}^{jp+1\,loc}$ and by commutation relations (4.2.2) for any element $f \in I_{\mathcal{B}}^{jp+1\,loc}$ there exists an element $u \in \mathcal{B}$, $u \neq 0$ and $g, h \in I_{\mathcal{B}}^{jp\,loc}$ such that $uf = g + h \otimes B_{jp}$. Let $\bar{f}, \bar{g}, \overline{h \otimes B_{jp}}$ and $\bar{h}$ be the classes of the corresponding elements in $\mathbb{C}_{jp}^{loc}[G]$. Note that $\overline{h \otimes B_{jp}} = \bar{h} \otimes B_{jp}$ and $\bar{g} = 0$, and by the previous Lemma $P_{jp}^{k_{jp}}(\overline{h \otimes B_{jp}}) = P_{jp}^{k_{jp}}(\bar{h} \otimes B_{jp}) = 0$. Thus $P_{jp}^{k_{jp}}(\overline{uf}) = uP_{jp}^{k_{jp}}(\bar{f}) = 0$. Since $u \neq 0$ this implies $P_{jp}^{k_{jp}}(\bar{f}) = 0$. This completes the proof of the first statement.

The remaining statements of the lemma are simple corollaries of the first assertion and of (4.5.17). $\qquad \square$

Next, we are going to study how the adjoint action of quantum root vectors commutes with the operators P_{jp}. For this purpose we shall need some commutation relations between quantum root vectors stated in the following lemma.

Lemma 4.6.5. *Let $f_\beta \in U_{\mathcal{B}}^{s,res}(\mathfrak{g})$ be quantum root vectors defined with the help of an arbitrary normal ordering on Δ_+. Then for any $\alpha < \beta$, $\alpha, \beta \in \Delta_+$ and any $m, n \in \mathbb{N}$ we have*

$$f_\alpha^{(m)} f_\beta^{(n)} = (c_{\alpha\beta}^s)^{mn} \sum_{p=0}^{m} q_\alpha^{p(m-1)} \sum_{\substack{\alpha \,<\, \delta_1 \,<\, \ldots \,<\, \delta_n \,\leq\, \beta \\ p_1, \ldots, p_n \,\in\, \mathbb{N}}} d_{p_1, \ldots, p_n}^p f_{\delta_1}^{(p_1)} \ldots f_{\delta_n}^{(p_n)} f_\alpha^{(m-p)}, \qquad (4.6.11)$$

where the coefficients $d_{p_1, \ldots, p_n}^p \in \mathcal{B}$ do not depend on m.

Proof. To prove this lemma it suffices to show that

$$f_\alpha^m f_\beta^n = (c_{\alpha\beta}^s)^{mn} \sum_{p=0}^{m} \sum_{\substack{\alpha \,<\, \delta_1 \,<\, \ldots \,<\, \delta_n \,\leq\, \beta \\ p_1, \ldots, p_n \,\in\, \mathbb{N}}} c_{p_1, \ldots, p_n}^p S_m^p f_{\delta_1}^{p_1} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-p}, \qquad (4.6.12)$$

where

$$S_m^p = q_\alpha^{p(m-1)} \begin{bmatrix} m \\ p \end{bmatrix}_{q_\alpha},$$

and the coefficients $c_{p_1, \ldots, p_n}^p \in \mathcal{A}$ do not depend on m.

Indeed, dividing (4.6.12) by $[n]_{q_\alpha}![m]_{q_\beta}!$ we arrive at an identity of the form (4.6.11) where the coefficients $d_{p_1, \ldots, p_n}^p$ a priori belong to $\mathbb{C}(q^{\frac{1}{dr^2}})$. But by the uniqueness of the Poincaré-Birkhoff-Witt decomposition in $U_{\mathcal{B}}^{s,res}(\mathfrak{n}_-)$ (see Lemma 2.8.2) we have $d_{p_1, \ldots, p_n}^p \in \mathcal{B}$.

Now we establish (4.6.12). Firstly we consider the case $n = 1$. By commutation relations (2.8.12) we have

$$f_\alpha^m f_\beta = (c_{\alpha\beta}^s)^m f_\beta f_\alpha^m + \sum_{k_1=0}^{m-1} (c_{\alpha\beta}^s)^{m-k_1-1} f_\alpha^{k_1} \sum_{\substack{\alpha \,<\, \delta_1 \,<\, \ldots \,<\, \delta_n \,<\, \beta \\ p_1, \ldots, p_n \,\in\, \mathbb{N}}} C(p_1, \ldots, p_n) f_{\delta_1}^{p_1} f_{\delta_2}^{p_2} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-k_1-1}, \qquad (4.6.13)$$

where $C(p_1, \ldots, p_n) \in \mathcal{A}$. The first term in the right hand side of this formula agrees with the term in the right hand side of (4.6.12) corresponding to $p = 0$. The other terms in the right hand side of (4.6.13) will contribute to the terms in the right hand side of (4.6.12) with $p > 0$.

Denote

$$D_\beta f_\alpha = \sum_{\substack{\alpha \,<\, \delta_1 \,<\, \ldots \,<\, \delta_n \,<\, \beta \\ p_1, \ldots, p_n \,\in\, \mathbb{N}}} C(p_1, \ldots, p_n) f_{\delta_1}^{p_1} f_{\delta_2}^{p_2} \ldots f_{\delta_n}^{p_n}.$$

Then (4.6.13) takes the form

$$f_\alpha^m f_\beta = (c_{\alpha\beta}^s)^m f_\beta f_\alpha^m + \sum_{k_1=0}^{m-1} (c_{\alpha\beta}^s)^{m-k_1-1} f_\alpha^{k_1} D_\beta f_\alpha f_\alpha^{m-k_1-1}. \tag{4.6.14}$$

To get the term in the right hand side of (4.6.12) corresponding to $p = 1$ we have to move $f_\alpha^{k_1}$ to the right in (4.6.13) using commutation relations (2.8.12),

$$f_\alpha f_\delta = c_{\alpha\delta}^s f_\delta f_\alpha + \sum_{\substack{\alpha < \delta_1 < \ldots < \delta_n < \delta \\ p_1, \ldots, p_n \in \mathbb{N}}} C(p_1, \ldots, p_n) f_{\delta_1}^{p_1} f_{\delta_2}^{p_2} \ldots f_{\delta_n}^{p_n} = c_{\alpha\delta}^s f_\delta f_\alpha + D_\delta f_\alpha, \; \alpha < \delta, \tag{4.6.15}$$

and keep the leading term $c_{\alpha\delta}^s f_\delta f_\alpha$ for each $\delta = \delta_1, \ldots, \delta_n$. Since for weight reasons $\delta_1 p_1 + \ldots + \delta_n p_n = \alpha + \beta$, so the weight of $D_\beta f_\alpha$ is $\alpha + \beta$, this yields

$$f_\alpha^m f_\beta = (c_{\alpha\beta}^s)^m f_\beta f_\alpha^m + \sum_{k_1=0}^{m-1} (c_{\alpha\beta}^s)^{m-k_1-1} c_{\alpha\alpha+\beta}^{k_1} D_\beta f_\alpha f_\alpha^{m-1} + R, \tag{4.6.16}$$

where R stands for the terms contributing to the terms in the right hand side of (4.6.12) with $p > 1$. Now

$$\sum_{k_1=0}^{m-1} (c_{\alpha\beta}^s)^{m-k_1-1} c_{\alpha\alpha+\beta}^{k_1} = (c_{\alpha\beta}^s)^{m-1} \sum_{k_1=0}^{m-1} (c_{\alpha\beta}^s)^{-k_1} (c_{\alpha\beta}^s)^{k_1} c_{\alpha\alpha}^{k_1} = (c_{\alpha\beta}^s)^{m-1} \sum_{k_1=0}^{m-1} q_\alpha^{2k_1} = (c_{\alpha\beta}^s)^{m-1} \frac{q_\alpha^{2m}-1}{q_\alpha^2-1} =$$

$$= (c_{\alpha\beta}^s)^{m-1} q_\alpha^{m-1} [m]_{q_\alpha} = (c_{\alpha\beta}^s)^{m-1} S_m^1,$$

and (4.6.16) takes the form

$$f_\alpha^m f_\beta = (c_{\alpha\beta}^s)^m f_\beta f_\alpha^m + (c_{\alpha\beta}^s)^{m-1} S_m^1 \sum_{\substack{\alpha < \delta_1 < \ldots < \delta_n < \beta \\ p_1, \ldots, p_n \in \mathbb{N}}} C(p_1, \ldots, p_n) f_{\delta_1}^{p_1} f_{\delta_2}^{p_2} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-1} + R. \tag{4.6.17}$$

The second term in the right hand side of this formula agrees with the term in the right hand side of (4.6.12) corresponding to $p = 1$.

To get the term in the right hand side of (4.6.12) corresponding to $p = 2$ we have to move $f_\alpha^{k_1}$ to the right in (4.6.13) using commutation relations (2.8.12),

$$f_\alpha^{k_1} f_{\delta_l} = (c_{\alpha\delta_l}^s)^k f_{\delta_l} f_\alpha^{k_1} + \sum_{k_2=0}^{k_1-1} (c_{\alpha\delta_l}^s)^{k_1-k_2-1} f_\alpha^{k_2} D_{\delta_l} f_\alpha f_\alpha^{k_1-k_2-1}, \tag{4.6.18}$$

and keep the terms containing one "differentiation" $D_{\delta_l} f_\alpha$ for some $l = 1, \ldots, n$. Since the weight of $D_{\delta_l} f_\alpha$ is $\alpha + \delta_l$, this yields

$$R = \sum_{k_1=0}^{m-1} \sum_{k_2=0}^{k_1-1} \sum_{l=1}^{n} \sum_{q=1}^{p_l} (c_{\alpha\beta}^s)^{m-k_1-1} (c_{\alpha\delta_1}^s)^{k_1 p_1} \ldots (c_{\alpha\delta_{l-1}}^s)^{k_1 p_{l-1}} (c_{\alpha\delta_l}^s)^{k_1(p_l-q)} (c_{\alpha\delta_l}^s)^{k_1-k_2-1} \times \tag{4.6.19}$$

$$\times (c_{\alpha\alpha+\delta_l}^s)^{k_2} (c_{\alpha\delta_l}^s)^{(k_1-1)(q-1)} (c_{\alpha\delta_{l+1}}^s)^{(k_1-1)p_{l+1}} \ldots (c_{\alpha\delta_n}^s)^{(k_1-1)p_n} \times$$

$$\times C(p_1, \ldots, p_n) f_{\delta_1}^{p_1} \ldots f_{\delta_{l-1}}^{p_{l-1}} f_{\delta_l}^{p_l-q} D_{\delta_l} f_\alpha f_{\delta_l}^{q-1} f_{\delta_{l+1}}^{p_{l+1}} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-2} + R_1,$$

where R_1 contains only terms with more than one "differentiation" $D_{\delta_l} f_\alpha$ for some $l = 1, \ldots, n$.

Now by the definition of the coefficients $(c_{\alpha\beta}^s)$ and using the identity $\delta_1 p_1 + \ldots + \delta_n p_n = \alpha + \beta$ we have

$$(c_{\alpha\beta}^s)^{m-k_1-1} (c_{\alpha\delta_1}^s)^{k_1 p_1} \ldots (c_{\alpha\delta_{l-1}}^s)^{k_1 p_{l-1}} (c_{\alpha\delta_l}^s)^{k_1(p_l-q)} (c_{\alpha\delta_l}^s)^{k_1-k_2-1} (c_{\alpha\alpha+\delta_l}^s)^{k_2} (c_{\alpha\delta_l}^s)^{(k_1-1)(q-1)} (c_{\alpha\delta_{l+1}}^s)^{(k_1-1)p_{l+1}} \ldots (c_{\alpha\delta_n}^s)^{(k_1-1)p_n} =$$

$$= (c_{\alpha\beta}^s)^{m-1} (c_{\alpha\alpha}^s)^{k_1+k_2} (c_{\alpha\delta_l}^s)^{-q} (c_{\alpha\delta_{l+1}}^s)^{-p_{l+1}} \ldots (c_{\alpha\delta_n}^s)^{-p_n},$$

so

$$\sum_{k_1=0}^{m-1} \sum_{k_2=0}^{k_1-1} (c_{\alpha\beta}^s)^{m-k_1-1} (c_{\alpha\delta_1}^s)^{k_1 p_1} \ldots (c_{\alpha\delta_{l-1}}^s)^{k_1 p_{l-1}} (c_{\alpha\delta_l}^s)^{k_1(p_l-q)} (c_{\alpha\delta_l}^s)^{k_1-k_2-1} (c_{\alpha\alpha+\delta_l}^s)^{k_2} (c_{\alpha\delta_l}^s)^{(k_1-1)(q-1)} (c_{\alpha\delta_{l+1}}^s)^{(k_1-1)p_{l+1}} \ldots$$

$$\ldots (c_{\alpha\delta_n}^s)^{(k_1-1)p_n} =$$

$$= (c_{\alpha\beta}^s)^{m-1}(c_{\alpha\delta_l}^s)^{-q}(c_{\alpha\delta_{l+1}}^s)^{-p_{l+1}} \ldots (c_{\alpha\delta_n}^s)^{-p_n} \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1-1} (c_{\alpha\alpha}^s)^{k_1+k_2} =$$

$$= (c_{\alpha\beta}^s)^{m-1}(c_{\alpha\delta_l}^s)^{-q}(c_{\alpha\delta_{l+1}}^s)^{-p_{l+1}} \ldots (c_{\alpha\delta_n}^s)^{-p_n} \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1-1} q_\alpha^{2(k_1+k_2)},$$

and, after combining terms containing the same monomials $f_{\delta_1}^{p_1} f_{\delta_2}^{p_2} \ldots f_{\delta_n}^{p_n}$, equation (4.6.19) takes the form

$$R = (c_{\alpha\beta}^s)^m \sum_{\substack{\alpha\,<\,\delta_1\,<\,\ldots\,<\,\delta_n\,\leq\,\beta \\ p_1,\,\ldots,\,p_n\,\in\,\mathbb{N}}} c_{p_1,\ldots,p_n}^2 S_m^2 f_{\delta_1}^{p_1} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-2} + R_1, \tag{4.6.20}$$

where

$$S_m^2 = \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1-1} q_\alpha^{2(k_1+k_2)},$$

and the coefficients $c_{p_1,\ldots,p_n}^2 \in \mathcal{A}$ do not depend on m. The first term in (4.6.20) will agree with with the term in the right hand side of (4.6.12) corresponding to $p = 2$ if we show that $S_m^2 = q_\alpha^{2(m-1)} \begin{bmatrix} m \\ 2 \end{bmatrix}_{q_\alpha}$.

Now we can continue in the same way taking into account more "derivatives" $D_{\delta_l} f_\alpha$ to obtain formula (4.6.12) with $n = 1$, where

$$S_m^p = \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_p=0}^{k_{p-1}-1} q_\alpha^{2(k_1+k_2+\ldots+k_p)}.$$

From this formula it follows that the coefficients S_m^p satisfy the following relations

$$S_m^p = S_{m-1}^p + q_\alpha^{2(m-1)} S_{m-1}^{p-1}. \tag{4.6.21}$$

On the other hand for the q-binomial coefficients we have

$$\begin{bmatrix} m \\ p \end{bmatrix}_{q_\alpha} = q_\alpha^{-p} \begin{bmatrix} m-1 \\ p \end{bmatrix}_{q_\alpha} + q_\alpha^{m-p} \begin{bmatrix} m-1 \\ p-1 \end{bmatrix}_{q_\alpha}.$$

If we denote

$$C_m^p = q_\alpha^{p(m-1)} \begin{bmatrix} m \\ p \end{bmatrix}_{q_\alpha}$$

this implies

$$C_m^p = C_{m-1}^p + q_\alpha^{2(m-1)} C_{m-1}^{p-1},$$

i.e. the coefficients C_m^p satisfy the same relations (4.6.21) as the coefficients S_m^p. Since these relations completely determine them once S_2^p are known and $C_2^0 = S_2^0 = 1$, $C_2^2 = S_2^2 = q_\alpha^2$, $C_2^1 = S_2^1 = q_\alpha[2]_{q_\alpha}$, we deduce that $C_m^p = S_m^p$ for all m and p (compare with Theorem 6.1 in [62]). Thus formula (4.6.12) is established for $n = 1$.

For $n > 1$ we argue by induction over n. The base of the induction is already established. Now assuming that the statement is true for $n - 1$ we show that it holds for n. Indeed,

$$f_\alpha^m f_\beta^n = (c_{\alpha\beta}^s)^{m(n-1)} \sum_{q=0}^m \sum_{\substack{\alpha\,<\,\delta_1\,<\,\ldots\,<\,\delta_n\,\leq\,\beta \\ p_1,\,\ldots,\,p_n\,\in\,\mathbb{N}}} c_{p_1,\ldots,p_n}^q S_m^q f_{\delta_1}^{p_1} \ldots f_{\delta_n}^{p_n} f_\alpha^{m-q} f_\beta = \tag{4.6.22}$$

$$= (c_{\alpha\beta}^s)^{m(n-1)} \sum_{q=0}^m \sum_{k=0}^{m-q} \sum_{\substack{\alpha\,<\,\delta_1\,<\,\ldots\,<\,\delta_n\,\leq\,\beta \\ p_1',\,\ldots,\,p_n'\,\in\,\mathbb{N}}} (c_{\alpha\beta}^s)^{m-p} c_{p_1',\ldots,p_n'}' S_m^q S_{m-q}^k f_{\delta_1}^{p_1'} \ldots f_{\delta_n}^{p_n'} f_\alpha^{m-q-k},$$

where $c_{p_1',\ldots,p_n'}' \in \mathcal{A}$ do not depend on m.

Now

$$S_m^q S_{m-q}^k = \frac{q_\alpha^{-kq}}{[k]_{q_\alpha}!} S_m^{k+q}.$$

Introducing new summation variable $p = k + q$ in (4.6.22) we can rewrite (4.6.22) in the form

$$f_\alpha^m f_\beta^n = (c_{\alpha\beta}^s)^{mn} \sum_{p=0}^{m} \sum_{q=0}^{p} \sum_{\substack{\alpha \,<\, \delta_1 \,<\, \ldots \,<\, \delta_n \,\le\, \beta \\ p_1', \ldots, p_n' \,\in\, \mathbb{N}}} (c_{\alpha\beta}^s)^{-p} \frac{q_\alpha^{-(p-q)q}}{[p-q]_{q_\alpha}!} S_m^p c_{p_1', \ldots, p_n'}' f_{\delta_1}^{p_1'} \ldots f_{\delta_n}^{p_n'} f_\alpha^{m-p}.$$

Combining terms containing the same monomials $f_{\delta_1}^{p_1'} f_{\delta_2}^{p_2'} \ldots f_{\delta_n}^{p_n'}$ we arrive at an identity of the form (4.6.12) where the coefficients $c_{p_1, \ldots, p_n}^p$ a priori belong to $\mathbb{C}(q^{\frac{1}{dr^2}})$. But by the uniqueness of the Poincaré-Birkhoff-Witt decomposition in $U_{\mathcal{A}}^s(\mathfrak{n}_-)$ (see Lemma 2.8.2) we have $c_{p_1, \ldots, p_n}^p \in \mathcal{A}$. This completes the proof of (4.6.12). $\qquad\square$

The next lemma shows how the adjoint action of the quantum root vectors commutes with the operators P_{jp}.

Lemma 4.6.6. *For any $j = 1, \ldots, R - 1$, $p = 1, \ldots, n_j$, $p < q \le m_j$, and any $f \in \mathbb{C}_{jp}^{loc}[G]$ we have*

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jq}}^{(m)}(P_{jp}(f)) = \sum_{k=0}^{\infty} P_{jp}^k(\mathrm{Ad}_{\widehat{s}_j}^0(\sum_{p_{jp+1}, \ldots, p_{jq}} d_{p_{jp+1}, \ldots, p_{jq}}^k f_{\delta_{jp+1}}^{(p_{jp+1})} \ldots f_{\delta_{jq}}^{(p_{jq})})(f)), \qquad (4.6.23)$$

where only a finite number of terms in the sum are non-zero, $d_{p_{jp+1}, \ldots, p_{jq}}^k \in \mathcal{B}$,

$$d_{p_{jp+1}, \ldots, p_{jq-1}, m}^k = \begin{cases} 1 & \text{if} \quad k = p_{jp+1} = \ldots = p_{jq-1} = 0 \\ 0 & \text{otherwise} \end{cases},$$

and for $0 \le n < m$ and any k

$$d_{0, \ldots, 0, n}^k = 0.$$

In particular,

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp+1}}^{(m)}(P_{jp}(f)) = P_{jp}(\mathrm{Ad}_{\widehat{s}_j}^0(f_{\delta_{jp+1}}^{(m)})(f)). \qquad (4.6.24)$$

Proof. By formula (4.6.6) and by the definition of P_{jp} we have

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jq}}^{(m)}(P_{jp}(f)) = \sum_{n=0}^{\infty} (-1)^n q_{\delta_{jp}}^{\frac{(n-1)n}{2}} d_{\delta_{jp}\delta_{jq}}^{mn} \mathrm{Ad}_{\widehat{s}_j}^0(f_{\delta_{jp}}^{(n)} f_{\delta_{jq}}^{(m)})(f) \otimes B_{jp}^n.$$

Using (4.6.11) we can rewrite this formula as follows

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jq}}^{(m)}(P_{jp}(f)) =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n q_{\delta_{jp}}^{\frac{(n-1)n}{2}} q_{\delta_{jp}}^{-k(n-1)} (c_{\delta_{jp}\delta_{jq}}^{\widehat{s}_j})^{mn} \omega_0((c_{\delta_{jp}\delta_{jq}}^{\widehat{s}_j})^{mn}) \mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-k)}(\mathrm{Ad}_{\widehat{s}_j}^0(\sum_{p_{jp}, \ldots, p_{jq}} d_{p_{jp}, \ldots, p_{jq}}^k f_{\delta_{jp+1}}^{(p_{jp+1})} \ldots f_{\delta_{jq}}^{(p_{jq})})(f)).$$

Now recalling that $\omega_0(q) = q^{-1}$ and swapping the order of summation in the last formula we get

$$\mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jq}}^{(m)}(P_{jp}(f)) =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n q_{\delta_{jp}}^{\frac{(n-1)n}{2}} q_{\delta_{jp}}^{-k(n-1)} \mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-k)}(\mathrm{Ad}_{\widehat{s}_j}^0(\sum_{p_{jp+1}, \ldots, p_{jq}} d_{p_{jp+1}, \ldots, p_{jq}}^k f_{\delta_{jp+1}}^{(p_{jp+1})} \ldots f_{\delta_{jq}}^{(p_{jq})})(f)) =$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} (-1)^n q_{\delta_{jp}}^{\frac{(n-1)(n-2k)}{2}} \mathrm{Ad}_{\widehat{s}_j}^0 f_{\delta_{jp}}^{(n-k)}(\mathrm{Ad}_{\widehat{s}_j}^0(\sum_{p_{jp+1}, \ldots, p_{jq}} d_{p_{jp+1}, \ldots, p_{jq}}^k f_{\delta_{jp+1}}^{(p_{jp+1})} \ldots f_{\delta_{jq}}^{(p_{jq})})(f)) =$$

$$= \sum_{k=0}^{\infty} P_{jp}^k(\mathrm{Ad}_{\widehat{s}_j}^0(\sum_{p_{jp+1}, \ldots, p_{jq}} d_{p_{jp+1}, \ldots, p_{jq}}^k f_{\delta_{jp+1}}^{(p_{jp+1})} \ldots f_{\delta_{jq}}^{(p_{jq})})(f)).$$

(4.6.24) is obtained in a similar way using the relation $f_{\delta_{jp}}^{(n)} f_{\delta_{jp+1}}^{(m)} = (c_{\delta_{jp}\delta_{jp+1}}^{\widehat{s}_j})^{mn} f_{\delta_{jp+1}}^{(m)} f_{\delta_{jp}}^{(n)}.$

$\qquad\square$

Now we have all prerequisites to prove Proposition 4.6.1.

Proof of Proposition 4.6.1 Firstly, the composition (4.6.2) is well defined by Lemma 4.6.4 (ii).

By Lemmas 4.6.4, 4.6.6, and formula (4.6.10) we have for any f from the domain of P_{jp} and $\delta_{1q} \leq \beta_{jp}$

$$\mathrm{Ad}_s^0 f_{\delta_{1q}}^{(n)}(P_{\leq jp}(f)) = \sum_{k_1=0}^{\infty} P_{11}^{k_1}\left(\mathrm{Ad}_s^0(\sum_{p_{12},\ldots,p_{1q}} d_{p_{12},\ldots,p_{1q}}^{k_1} f_{\delta_{12}}^{(p_{12})}\ldots f_{\delta_{1q}}^{(p_q)})(P_{12}\ldots P_{jp}(f))\right) =$$

$$= \sum_{k_1=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{12},\ldots,p_{1q}} \mathrm{Ad}_s^0(d_{p_{12},\ldots,p_{1q}}^{k_1} f_{\delta_{13}}^{(p_{13})}\ldots f_{\delta_{1q}}^{(p_{1q})})\mathrm{Ad}_s^0 f_{\delta_{12}}^{(p_{12})}(P_{12}\ldots P_{jp}(f))\right) =$$

$$\sum_{k_1=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{13},\ldots,p_{1q}} \mathrm{Ad}_s^0(d_{0,p_{13},\ldots,p_{1q}}^{k_1} f_{\delta_{13}}^{(p_{13})}\ldots f_{\delta_{1q}}^{(p_{1q})})(P_{12}\ldots P_{jp}(f))\right),$$

where, by weight counting in the left hand side and in the right hand side, for each k_1 $d_{0,0\ldots,0}^{k_1} = 0$.

Similarly,

$$\mathrm{Ad}_s^0 f_{\delta_{1q}}^{(n)}(P_{\leq jp}(f)) = \sum_{k_1=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{13},\ldots,p_{1q}} \mathrm{Ad}_s^0(d_{0,p_{13},\ldots,p_{1q}}^{k_1} f_{\delta_{13}}^{(p_{13})}\ldots f_{\delta_{1q}}^{(p_{1q})})(P_{12}\ldots P_{jp}(f))\right) =$$

$$= \sum_{k_1=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{13},\ldots,p_{1q}} \mathrm{Ad}_s^0(d_{0,p_{13},\ldots,p_{1q}}^{k_1} f_{\delta_{14}}^{(p_{14})}\ldots f_{\delta_{1q}}^{(p_{1q})})(P_{12}\mathrm{Ad}_s^0(f_{\delta_{13}}^{(p_{13})})P_{13}\ldots P_{jp}(f))\right) =$$

$$= \sum_{k_1=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{14},\ldots,p_{1q}} \mathrm{Ad}_s^0(d_{0,0,p_{14},\ldots,p_{1q}}^{k_1} f_{\delta_{14}}^{(p_{14})}\ldots f_{\delta_{1q}}^{(p_{1q})})(P_{12}P_{13}\ldots P_{jp}(f))\right).$$

Now we proceed along the same line,

$$\mathrm{Ad}_s^0 f_{\delta_{1q}}^{(n)}(P_{\leq jp}(f)) =$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{14},\ldots,p_{1q}} \mathrm{Ad}_s^0(d_{0,0,p_{14},\ldots,p_{1q}}^{k_1} f_{\delta_{15}}^{(p_{15})}\ldots f_{\delta_{1q}}^{(p_{1q})})(P_{12}^{k_2}\sum_{p'_{13},p'_{14}} \mathrm{Ad}_s^0(d_{p'_{13},p'_{14}}^{k_2} f_{\delta_{13}}^{(p'_{13})} f_{\delta_{14}}^{(p'_{14})})P_{13}\ldots P_{jp}(f))\right) =$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{14},\ldots,p_{1q}} \mathrm{Ad}_s^0(d_{0,0,p_{14},\ldots,p_{1q}}^{k_1} f_{\delta_{15}}^{(p_{15})}\ldots f_{\delta_{1q}}^{(p_{1q})})(P_{12}^{k_2}\sum_{p'_{14}} \mathrm{Ad}_s^0(d_{0,p'_{14}}^{k_2} f_{\beta_{14}}^{(p'_{14})})P_{13}\ldots P_{jp}(f))\right) =$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{14},\ldots,p_{1q}} \mathrm{Ad}_s^0(d_{0,0,p_{14},\ldots,p_{1q}}^{k_1} f_{\delta_{15}}^{(p_{15})}\ldots f_{\delta_{1q}}^{(p_{1q})})(P_{12}^{k_2}\sum_{p'_{14}} P_{13}\mathrm{Ad}_s^0(d_{0,p'_{14}}^{k_2} f_{\delta_{14}}^{(p'_{14})})P_{14}\ldots P_{jp}(f))\right) =$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{14},\ldots,p_{1q}} \omega_0(d_{0,0}^{k_2})\mathrm{Ad}_s^0(d_{0,0,p_{14},\ldots,p_{1q}}^{k_1} f_{\delta_{15}}^{(p_{15})}\ldots f_{\delta_{1q}}^{(p_{1q})})(P_{12}^{k_2}P_{13}P_{14}\ldots P_{jp}(f))\right).$$

But $d_{0,0}^{k_2}$ is zero if $p_{14} > 0$ and we infer

$$\mathrm{Ad}_s^0 f_{\delta_{1q}}^{(n)}(P_{\leq jp}(f)) =$$

$$= \sum_{k_1=0}^{\infty} P_{11}^{k_1}\left(\sum_{p_{15},\ldots,p_{1q}} \mathrm{Ad}_s^0(d_{0,0,0,p_{15},\ldots,p_{1q}}^{k_1} f_{\delta_{15}}^{(p_{15})}\ldots f_{\delta_{1q}}^{(p_{1q})})(P_{12}P_{13}\ldots P_{jp}(f))\right).$$

Now we repeat the same steps until we reach ϕ_1 in the composition (4.6.2). When $\mathrm{Ad}_s^0(f_{\delta_{1q}}^{(a)})$ with $q > n_1$ needs to be moved on the right from ϕ_1 in (4.6.2) we use formulas (4.5.18) and (4.2.6) and the fact that for $q > n_1$ $T_{w_1}^{-1}(f_{\delta_{1q}}(s)) = f_{\delta_{2q-n_1}}(\widehat{s}_2)$.

Similar arguments work for the compositions $P_{i1} \ldots P_{in_i}$, $i > 1$ and for $f_{\delta_{iq}}(\widehat{s}_i)$ with $q > n_i$ and $T_{w_i}^{-1}$. Repeating these arguments we arrive at

$$\mathrm{Ad}_s^0 f_{\delta_{1q}}^{(n)}(P_{\leq jp}(f)) = \sum_{k_1=0}^{\infty} \omega_0(d_{0,0,\ldots,0}^{k_1}) P_{11}^{k_1} \ldots P_{1n_1} \circ \phi_1 \circ P_{21} \ldots P_{2n_2} \circ \phi_2 \ldots \circ \phi_{j-1} \circ P_{j1} \ldots P_{jp}(f) = 0$$

as $d_{0,0,\ldots,0}^{k_1} = 0$ for all k_1. This proves (4.6.3).

(4.6.4) is established in a similar way using (4.6.23), Lemma 4.6.4, (4.5.18), (4.2.6), and formula (4.6.10).

$\square$

In conclusion, using the operators P_{jp} and the elements B_{jp} we define proper quantum analogues of monomials in functions φ_{jp}. They will play a crucial rule in establishing a quantum group version of the Skryabin equivalence in Section 5.2 and in the proof of the De–Concini–Kac–Procesi conjecture.

Proposition 4.6.7. *For $m_{ij}, k_{ij} \in \mathbb{N}$ the elements*

$$B_{\{m_{ij}\}} = P_{11}(\ldots P_{1n_1-1}(P_{1n_1}(\phi_1(P_{21}(\ldots P_{2n_2-1}(P_{2n_2}(\phi_2(\ldots \phi_{R-2}(P_{R-11}(\ldots$$

$$\ldots P_{R-1n_{R-1}-2}(P_{R-1n_{R-1}-1}(B_{R-1n_{R-1}}^{m_{R-1n_{R-1}}})B_{R-1n_{R-1}-1}^{m_{R-1n_{R-1}-1}})\ldots B_{R-11}^{m_{R-11}}))\ldots))B_{2n_2}^{m_{2n_2}})\ldots)B_{21}^{m_{21}}))B_{1n_1}^{m_{1n_1}})\ldots)B_{11}^{m_{11}} \in \mathbb{C}_{11}^{loc}[G]$$

satisfy

$$\mathrm{Ad}_s^0(f_{\beta_{11}}^{(k_{11})} \ldots f_{\beta_{R-1n_{R-1}}}^{(k_{R-1n_{R-1}})})(B_{\{m_{ij}\}}) = \begin{cases} c_{\{m_{ij}\}} \prod_{i=1}^{R-1} \prod_{j=1}^{n_i} q_{\delta_{ij}}^{\frac{m_{ij}(m_{ij}-1)}{2}} & \text{if } m_{ij} = k_{ij} \text{ for } i = 1, \ldots, R-1, j = 1, \ldots n_i \\ 0 \text{ if } k_{ij} = m_{ij} \text{ if } (ij) < (pq) \text{ and } k_{pq} > m_{pq} \text{ for some } (pq) \end{cases},$$

where the pairs (ij) are ordered according to the lexicographic order and $c_{\{m_{ij}\}} \in \{\pm q^{\mathbb{Z}}\}$. Thus

$$\mathrm{Ad}_s^0(f_{\beta_{11}}^{k_{11}} \ldots f_{\beta_{R-1n_{R-1}}}^{k_{R-1n_{R-1}}})(B_{\{m_{ij}\}}) = \begin{cases} c_{\{m_{ij}\}} \prod_{i=1}^{R-1} \prod_{j=1}^{n_i} q_{\delta_{ij}}^{\frac{m_{ij}(m_{ij}-1)}{2}} [m_{ij}]_{q_{\delta_{ij}}}! & \text{if } m_{ij} = k_{ij} \text{ for } i = 1, \ldots, R-1, j = 1, \ldots n_i \\ 0 \text{ if } k_{ij} = m_{ij} \text{ if } (ij) < (pq) \text{ and } k_{pq} > n_{pq} \text{ for some } (pq) \end{cases}.$$

In particular, the elements $B_{\{m_{ij}\}}$ are linearly independent for different $\{m_{ij}\}$, $i = 1, \ldots, R-1$, $j = 1, \ldots n_i$.

Proof. The proof follows from Lemmas 4.4.3, 4.5.5, 4.6.4, 4.6.6, formula (4.6.10) and Proposition 4.6.1. We shall prove the statement by induction.

First observe that by Proposition 4.6.1

$$\mathrm{Ad}_s^0 f_{\beta_{11}}^{(k)} P_{11}(\ldots P_{1n_1-1}(P_{1n_1}(\phi_1(P_{21}(\ldots P_{2n_2-1}(P_{2n_2}(\phi_2(\ldots \phi_{R-2}(P_{R-11}(\ldots$$

$$\ldots P_{R-1n_{R-1}-2}(P_{R-1n_{R-1}-1}(B_{R-1n_{R-1}}^{m_{R-1n_{R-1}}})B_{R-1n_{R-1}-1}^{m_{R-1n_{R-1}-1}})\ldots B_{R-11}^{m_{R-11}}))\ldots))B_{2n_2}^{m_{2n_2}})\ldots)B_{21}^{m_{21}}))B_{1n_1}^{m_{1n_1}})\ldots B_{12}^{m_{12}}) = 0$$

for any $k > 0$ and recall that right multiplication by B_{11} gives rise to an operator on $\mathbb{C}_{11}^{loc}[G]$ by Lemma 4.5.3. Therefore from (4.6.5) for $(jp) = (11)$ we have

$$\mathrm{Ad}_s^0 f_{\beta_{11}}^{(k_{11})}(B_{\{m_{ij}\}}) =$$

$$= q_{\delta_{11}}^{\frac{k_{11}(k_{11}-1)}{2}} P_{11}(\ldots P_{1n_1-1}(P_{1n_1}(\phi_1(P_{21}(\ldots P_{2n_2-1}(P_{2n_2}(\phi_2(\ldots \phi_{R-2}(P_{R-11}(\ldots$$

$$\ldots P_{R-1n_{R-1}-2}(P_{R-1n_{R-1}-1}(B_{R-1n_{R-1}}^{m_{R-1n_{R-1}}})B_{R-1n_{R-1}-1}^{m_{R-1n_{R-1}-1}})\ldots B_{R-11}^{m_{R-11}}))\ldots))B_{2n_2}^{m_{2n_2}})\ldots)B_{21}^{m_{21}}))B_{1n_1}^{m_{1n_1}})\ldots B_{12}^{m_{12}})$$

if $k_{11} = m_{11}$ and

$$\mathrm{Ad}_s^0 f_{\beta_{11}}^{(k_{11})}(B_{\{m_{ij}\}}) = 0$$

if $k_{11} > m_{11}$.

Now assume that for some $1 \leq p \leq R-1$, $1 \leq q \leq n_i$

$$\mathrm{Ad}_s^0(f_{\beta_{11}}^{(k_{11})} \ldots f_{\beta_{pq}}^{(m_{pq})})(B_{\{k_{11},\ldots,k_{pq},m_{pq+1},\ldots,m_{R-1n_{R-1}}\}}) = \tag{4.6.25}$$

$$= d_{pq} \prod_{(ij) \leq (pq)} q_{\delta_{ij}}^{\frac{k_{ij}(k_{ij}-1)}{2}} P_{\leq pq}(P_{pq+1} \ldots P_{pn_p-1}(P_{pn_p}(\phi_p(P_{p+11} \ldots P_{p+1n_{p+1}-1}(P_{p+1n_{p+1}}(\phi_{p+1}(\ldots \phi_{R-2}(P_{R-11} \ldots$$

$$\ldots P_{R-1n_{R-1}-2}(P_{R-1n_{R-1}-1}(B_{R-1n_{R-1}}^{m_{R-1n_{R-1}}})B_{R-1n_{R-1}-1}^{m_{R-1n_{R-1}-1}})\ldots B_{R-11}^{m_{R-11}}))\ldots)B_{p+2n_{p+2}}^{m_{p+2n_{p+2}}})\ldots B_{p+11}^{m_{p+11}}))B_{pn_p}^{m_{pn_p}})\ldots B_{pq+1}^{m_{pq+1}}),$$

where we assume that $m_{pn_p+1} = m_{p+11}$, and $d_{pq} \in \{\pm q^{\mathbb{Z}}\}$.

Then by (4.6.4) and by the induction assumption

$$\mathrm{Ad}_s^0(f_{\beta_{11}}^{(k_{11})}\ldots f_{\beta_{pq+1}}^{(k_{pq+1})})(B_{\{k_{11},\ldots,k_{pq},m_{pq+1},\ldots,m_{R-1n_{R-1}}\}}) =$$

$$= d_{pq}c_{pq+1}\prod_{(ij)\leq(pq)}q_{\delta_{ij}}^{\frac{k_{ij}(k_{ij}-1)}{2}}P_{\leq pq}\mathrm{Ad}_{\hat{s}_p}^0(f_{\delta_{pq+1}}^{(k_{pq+1})})(P_{pq+1}\ldots P_{pn_p-1}(P_{pn_p}(\phi_p(P_{p+11}\ldots P_{p+1n_{p+1}-1}(P_{p+1n_{p+1}}(\phi_{p+1}(\ldots\phi_{R-2}(P_{R-11}\ldots$$

$$\ldots P_{R-1n_{R-1}-2}(P_{R-1n_{R-1}-1}(B_{R-1n_{R-1}}^{m_{R-1n_{R-1}}})B_{R-1n_{R-1}-1}^{m_{R-1n_{R-1}-1}})\ldots B_{R-11}^{m_{R-11}}))\ldots)B_{p+2n_{p+2}}^{m_{p+2n_{p+2}}})\ldots B_{p+11}^{m_{p+11}}))B_{pn_p}^{m_{pn_p}})\ldots B_{pq+1}^{m_{pq+1}}),$$

where $c_{pq+1} \in \{\pm q^{\mathbb{Z}}\}$ is defined in (4.6.4). Denote $d_{pq+1} = d_{pq}c_{pq+1} \in \{\pm q^{\mathbb{Z}}\}$.

The last formula can be simplified using (4.6.5), the fact that

$$\mathrm{Ad}_{\hat{s}_p}^0(f_{\delta_{pq+1}}^{(k)})(P_{pq+1}(\ldots P_{pn_p-1}(P_{pn_p}(\phi_p(P_{p+11}\ldots P_{p+1n_{p+1}-1}(P_{p+1n_{p+1}}(\phi_{p+1}(\ldots\phi_{R-2}(P_{R-11}\ldots$$

$$\ldots P_{R-1n_{R-1}-2}(P_{R-1n_{R-1}-1}(B_{R-1n_{R-1}}^{m_{R-1n_{R-1}}})B_{R-1n_{R-1}-1}^{m_{R-1n_{R-1}-1}})\ldots B_{R-11}^{m_{R-11}}))\ldots)B_{p+2n_{p+2}}^{m_{p+2n_{p+2}}})\ldots B_{p+11}^{m_{p+11}}))B_{pn_p}^{m_{pn_p}})\ldots B_{pq+2}^{m_{pq+2}})) = 0$$

for any $k > 0$, Lemma 4.6.4 and recalling that right multiplication by B_{pq+1} gives rise to an operator on $\mathbb{C}pq+1^{loc}G$ by Lemma 4.5.3. This yields

$$\mathrm{Ad}_s^0(f_{\beta_{11}}^{(k_{11})}\ldots f_{\beta_{pq+1}}^{(k_{pq+1})})(B_{\{k_{11},\ldots,k_{pq},m_{pq+1},\ldots,m_{R-1n_{R-1}}\}}) =$$

$$= d_{pq+1}\prod_{(ij)\leq(pq+1)}q_{\delta_{ij}}^{\frac{k_{ij}(k_{ij}-1)}{2}}(P_{\leq pq+1}(\ldots P_{pn_p-1}(P_{pn_p}(\phi_p(P_{p+11}\ldots P_{p+1n_{p+1}-1}(P_{p+1n_{p+1}}(\phi_{p+1}(\ldots$$

$$\ldots\phi_{R-2}(P_{R-11}\ldots\ldots P_{R-1n_{R-1}-2}(P_{R-1n_{R-1}-1}(B_{R-1n_{R-1}}^{m_{R-1n_{R-1}}})B_{R-1n_{R-1}-1}^{m_{R-1n_{R-1}-1}})\ldots B_{R-11}^{m_{R-11}}))\ldots$$

$$\ldots)B_{p+2n_{p+2}}^{m_{p+2n_{p+2}}})\ldots B_{p+11}^{m_{p+11}}))B_{pn_p}^{m_{pn_p}})\ldots B_{pq+2}^{m_{pq+2}}))$$

if $k_{pq+1} = m_{pq+1}$ and

$$\mathrm{Ad}_s^0(f_{\beta_{11}}^{(k_{11})}\ldots f_{\beta_{pq+1}}^{(k_{pq+1})})(B_{\{k_{11},\ldots,k_{pq},m_{pq+1},\ldots,m_{R-1n_{R-1}}\}}) = 0$$

if $k_{pq+1} > m_{pq+1}$. This establishes the induction step and completes the proof of the proposition.

$\square$

Since the ordering on the set of roots β_{ij}, $i = 1,\ldots,R-1$, $j = 1,\ldots,n_i$ induced by the lexicographic ordering of the pairs (ij) coincides with the normal ordering on the segment $\Delta_{\mathfrak{m}_+}$, we can rewrite the properties of the elements $B_{\{m_{ij}\}}$ as follows to simplify the notation for later use.

Corollary 4.6.8. *For $n_1,\ldots,n_c,k_1,\ldots,k_c \in \mathbb{N}$ there are elements*

$$B_{n_1\ldots n_c} \in \mathbb{C}_{11}^{loc}[G]$$

which satisfy

$$\mathrm{Ad}_s^0(f_{\beta_1}^{(k_1)}\ldots f_{\beta_c}^{(k_c)})(B_{n_1\ldots n_c}) = \begin{cases} c_{n_1\ldots n_c}(q) & \text{if } n_p = k_p \text{ for } p = 1,\ldots,c \\ 0 & \text{if } k_i = n_i, i = 1,\ldots,p-1 \text{ and } k_p > n_p \text{ for some } p \in \{1,\ldots,c\} \end{cases},$$

where $c_{n_1\ldots n_c}(q) \in \{\pm q^{\mathbb{Z}}\}$, and hence

$$\mathrm{Ad}_s^0(f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c})(B_{n_1\ldots n_c}) = \begin{cases} c'_{n_1\ldots n_c} & \text{if } n_p = k_p \text{ for } p = 1,\ldots,d \\ 0 & \text{if } k_i = n_i, i = 1,\ldots,p-1 \text{ and } k_p > n_p \text{ for some } p \in \{1,\ldots,c\} \end{cases},$$

where $c'_{n_1\ldots n_c}(q) = c_{n_1\ldots n_c}(q)\prod_{p=1}^c [n_p]_{q_{\beta_p}}!$.

4.7 A description of q-W–algebras in terms of Zhelobenko type operators

Now we are in a position to describe q-W–algebras in terms of the operator P introduced in the previous section. Recall that q-W–algebras are only defined when the value of the parameter κ is equal to one. Therefore in this section we always assume that $\kappa = 1$. As a $\mathcal{B}$–module the q-W–algebra $W_{\mathcal{B}}^s(G)$ is the space of $\mathbb{C}_{\mathcal{B}}^s[M_+]$–invariants in $Q_{\mathcal{B}}$ with respect to the adjoint action. In order to use the operator P_c for the description of this space we shall transfer the results of Proposition 4.6.1 from $\mathbb{C}_{11}^{loc}[G]$ to a localization $Q_{\mathcal{B}}^{loc}$ of $Q_{\mathcal{B}}$ using a natural extension of the $\mathbb{C}_{\mathcal{B}}^s[M_+]$–module homomorphism $\phi : \mathbb{C}_{\mathcal{B}}^s[G] \to Q_{\mathcal{B}}$ to a homomorphism $\mathbb{C}_{\mathcal{B}}^{s,loc}[G] \to Q_{\mathcal{B}}^{loc}$. Recall that according to Proposition 4.1.2 $I_{\mathcal{B}}^{11}$ belongs to the kernel of the homomorphism ϕ, and as we shall see $I_{\mathcal{B}}^{11\,loc}$ belongs to the kernel of the extension of ϕ to $\mathbb{C}_{\mathcal{B}}^{s,loc}[G]$. Therefore one can compose this extension with the operator P_c, and by Proposition 4.6.1 the image of this composition is invariant with respect to the natural extension of the $\mathbb{C}_{\mathcal{B}}^s[M_+]$–adjoint action to $Q_{\mathcal{B}}^{loc}$. The operator Π_c is a classical counterpart of this composition and using the description of Π_c given in Corollary 3.5.8 we shall show that the image of the composition is a localization of the algebra $W_{\mathcal{B}}^s(G)$.

More precisely, formula (4.1.19) and the surjectivity of the map ϕ imply that one can define a natural action of the algebra generated by the elements $q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee} \in \mathbb{C}_{\mathcal{B}}^s[G^*]$, $\lambda \in P_+$ on $Q_{\mathcal{B}}$ as follows

$$q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}\phi(f) = \varphi(\mathrm{Ad}_s^0(q^{-2P_{\mathfrak{h}'^\perp}\lambda^\vee})(f))q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1 = \tag{4.7.1}$$

$$= c_\lambda^{-1}\mathrm{Ad}_s(q^{-2P_{\mathfrak{h}'^\perp}\lambda^\vee})(\varphi(f))q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1,$$

where the last identity follows from part (i) of Proposition 3.2.6. Let $Q_{\mathcal{B}}^{loc}$ be the localization of $Q_{\mathcal{B}}$ by the elements $q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}$, $\lambda \in P_+$.

Now consider the subalgebra $\mathbb{C}_{\mathcal{B}}^{s,loc}[G_*] \subset \mathbb{C}_{\mathcal{B}}^s[G^*]$ generated by $\mathbb{C}_{\mathcal{B}}^s[G_*]$ and by the elements $q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}$, $\lambda \in P$. Note that the adjoint action of these elements normalizes $\mathbb{C}_{\mathcal{B}}^s[G_*]$ in $\mathbb{C}_{\mathcal{B}}^s[G^*]$ as $\mathbb{C}_{\mathcal{B}}^s[G_*]$ is the direct sum of its weight components. Therefore $Q_{\mathcal{B}}^{loc}$ is the image of $\mathbb{C}_{\mathcal{B}}^{s,loc}[G_*]$ under the natural projection $\rho_{\chi_q^s} : \mathbb{C}_{\mathcal{B}}^s[G^*] \to \mathbb{C}_{\mathcal{B}}^s[G^*]/I_{\mathcal{B}}$, and hence the adjoint action of $\mathbb{C}_{\mathcal{B}}^s[M_+]$ on $Q_{\mathcal{B}}$ naturally extends to $Q_{\mathcal{B}}^{loc}$.

Lemma 4.7.1. *Let $\kappa = 1$. Assume that for $i = 1,\ldots l'$ $k_i \in \mathcal{B}^*$, where k_i are defined in (3.2.16). Then $c_\lambda \in \mathcal{B}^*$ for any $\lambda \in P_+$, and ϕ extends to a $\mathbb{C}_{\mathcal{B}}^s[M_+]$–module homomorphism $\phi : \mathbb{C}_{\mathcal{B}}^{s,loc}[G] \to Q_{\mathcal{B}}^{loc}$,*

$$\phi(f \otimes \Delta_\lambda^{s^{-1}}) = c_\lambda^{-1}q^{2(P_{\mathfrak{h}'}\lambda^\vee,\lambda^\vee)-2P_{\mathfrak{h}'^\perp}\lambda^\vee}\phi(\mathrm{Ad}_s^0(q^{(\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}-s^{-1})\lambda^\vee})(f)), f \in \mathbb{C}_{\mathcal{B}}^s[G], \tag{4.7.2}$$

and $I_{\mathcal{B}}^{11\,loc}$ belongs to the kernel of this homomorphism, so

$$\phi : \mathbb{C}_{11}^{loc}[G] \to Q_{\mathcal{B}}^{loc}.$$

Proof. From formula (4.1.19) it follows that ϕ extends to a $\mathbb{C}_{\mathcal{B}}^s[M_+]$–module homomorphism $\phi : \mathbb{C}_{\mathcal{B}}^{s,loc}[G] \to Q_{\mathcal{B}}^{loc}$ which is defined by (4.7.2), and by Proposition 4.1.2 $I_{\mathcal{B}}^{11\,loc}$ belongs to the kernel of this homomorphism. $\square$

By (4.5.3) for $\lambda \in P_+$ Δ_λ^s is $\mathbb{C}_{\mathcal{B}}^s[M_+]$–invariant with respect to the Ad_s^0–action on $\mathbb{C}_{11}^{loc}[G]$, and hence $\phi(\Delta_\lambda^s) = q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1$ is $\mathbb{C}_{\mathcal{B}}^s[M_+]$–invariant with respect to the Ad_s–action on $Q_{\mathcal{B}}^{loc}$. Thus by (4.1.19) and (3.2.24) we have for $\beta \in \Delta_{\mathfrak{m}_+}$

$$\phi(\mathrm{Ad}_s^0(\tilde{f}_\beta)(f \otimes \Delta_\lambda^s)) = c_\lambda\mathrm{Ad}_s(\tilde{f}_\beta)(\varphi(\mathrm{Ad}_s^0(q^{-(\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f)))q^{2P_{\mathfrak{h}'^\perp}\lambda^\vee}1,$$

and

$$\phi(\mathrm{Ad}_s^0(\tilde{f}_\beta)(f \otimes \Delta_\lambda^{s^{-1}})) = c_\lambda^{-1}q^{2(P_{\mathfrak{h}'}\lambda^\vee,\lambda^\vee)}\mathrm{Ad}_s(\tilde{f}_\beta)(\varphi(\mathrm{Ad}_s^0(q^{(\frac{1+s}{1-s}P_{\mathfrak{h}'}+id)\lambda^\vee})(f)))q^{-2P_{\mathfrak{h}'^\perp}\lambda^\vee}1.$$

The last formula completely determines the adjoint action of $\mathbb{C}_{\mathcal{B}}^s[M_+]$ on $Q_{\mathcal{B}}^{loc}$.

Now we can describe q-W–algebras in terms of the operator P.

Theorem 4.7.2. *Suppose that $\kappa = 1$ and $k_i \in \mathcal{B}^*$ for $i = 1,\ldots l'$, where k_i are defined in (3.2.16). Then the composition ϕP_c gives rise to a well–defined operator*

$$\Pi_c^q = \phi P_c : \mathbb{C}_{R^{-1}n_{R-1}+1}^{loc}[G] \to W_{\mathcal{B}}^{s,loc}(G), \tag{4.7.3}$$

where $W_{\mathcal{B}}^{s,loc}(G) = \mathrm{Hom}_{\mathbb{C}_{\mathcal{B}}^{s}[M_+]}(\mathbb{C}_{\mathcal{B}}, Q_{\mathcal{B}}^{loc})$.

Moreover, we have an imbedding of $\mathbb{C}_{\mathcal{B}}^{s}[M_+]$–modules, $Q_{\mathcal{B}}^{loc} \subset \mathbb{C}_{\mathcal{B}}^{s}[G^*]/I_{\mathcal{B}}$, multiplication in $\mathbb{C}_{\mathcal{B}}^{s}[G^*]$ induces a multiplication in $W_{\mathcal{B}}^{s,loc}(G)$ and we have an imbedding of algebras $W_{\mathcal{B}}^{s}(G) \subset W_{\mathcal{B}}^{s,loc}(G)$.

Let $W_q^{s,loc}(G) = W_{\mathcal{B}}^{s,loc}(G) \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$, $\mathrm{Im}\Pi_c^q$ the image of Π_c^q and denote $\mathrm{Im}_q\Pi_c^q = \mathrm{Im}\Pi_c^q \otimes_{\mathcal{B}} \mathbb{C}(q^{\frac{1}{dr^2}})$.

Then $W_q^{s,loc}(G) = \mathrm{Im}_q\Pi_c^q$ and $W_{\mathcal{B}}^{s,loc}(G) = W_{\mathcal{B}}^{s,loc}(G) \cap \mathrm{Im}_q\Pi_c^q$.

Proof. By Lemma 4.7.1 and by Proposition 4.6.1 the composition $\Pi_c^q = \phi P : \mathbb{C}_{R-1n_{R-1}+1}^{loc}[G] \to W_{\mathcal{B}}^{s,loc}(G)$ is well–defined. All other claims of this theorem, except for the isomorphisms $W_q^{s,loc}(G) = \mathrm{Im}_q\Pi_c^q$ and $W_{\mathcal{B}}^{s,loc}(G) = W_{\mathcal{B}}^{s,loc}(G) \cap \mathrm{Im}_q\Pi_c^q$ follow from the definitions and are established similarly to Proposition 3.2.12.

Using Lemma 4.3.3 one sees that in order to establish these isomorphisms it suffices to verify that the specialization of Π_c^q at $q^{\frac{1}{dr^2}} = 1$ is surjective.

In order to do that we observe that by the definition and by Theorem 3.4.5 the specialization of $Q_{\mathcal{B}}^{loc}$ at $q^{\frac{1}{dr^2}} = 1$ is isomorphic to the localization $\mathbb{C}^{loc}[N_-ZsM_-]$ of the algebra $\mathbb{C}[N_-ZsM_-]$ by the classical counterparts of the elements Δ_λ^s which we denote by the same symbols, $\Delta_\lambda^s = (v_\lambda, \cdot s^{-1}v_\lambda)$. Denote by $\mathbb{C}^{loc}[(w_1\dots w_{R-2})^{-1}N_-ZsM_-(w_1\dots w_{R-2})]$ the image of $\mathbb{C}^{loc}[N_-ZsM_-]$ under the map $f \mapsto (w_1\dots w_{R-2})(f)$. This map sends Δ_λ^s to a function which we denote $\Delta_{(w_1\dots w_{R-2})^{-1}\lambda}^{\widehat{s}_{R-1}}$.

By (4.5.3) for $\lambda \in P_+$ Δ_λ^s regarded as elements of $\mathbb{C}_{11}^{loc}[G]$ are $\mathbb{C}_{\mathcal{B}}^{s}[M_+]$–invariant with respect to the Ad_s^0–action on $\mathbb{C}_{11}^{loc}[G]$, and hence $\phi(\Delta_\lambda^s) = q^{2P_{\mathfrak{b}'\perp}\lambda^\vee}1$ is $\mathbb{C}_{\mathcal{B}}^{s}[M_+]$–invariant with respect to the Ad_s–action on $Q_{\mathcal{B}}^{loc}$. Therefore their classical counterparts $\Delta_\lambda^s = (v_\lambda, \cdot s^{-1}v_\lambda) \in \mathbb{C}[N_-ZsM_-]$ are M_-–invariant, and hence $\mathbb{C}^{loc}[N_-ZsM_-]^{M_-}$ is the localization of $\mathbb{C}[N_-ZsM_-]^{M_-}$ by the elements Δ_λ^s, $\lambda \in P_+$.

This result and explicit formulas (3.5.19), (3.5.20), (3.5.18), (3.5.21) for the operator Π_c, formulas (4.6.1) with $k = 0$ and the definition of the operator P_c imply that the specialization of the operator Π_c^q at $q^{\frac{1}{dr^2}} = 1$ gives rise to a natural extension of the operator $\Pi_c : \mathbb{C}[(w_1\dots w_{R-2})N_-ZsM_-(w_1\dots w_{R-2})^{-1}] \to \mathbb{C}[N_-ZsM_-]^{M_-}$ to an operator $\Pi_c^{loc} : \mathbb{C}^{loc}[(w_1\dots w_{R-2})^{-1}N_-ZsM_-(w_1\dots w_{R-2})] \to \mathbb{C}^{loc}[N_-ZsM_-]^{M_-}$ given by

$$\Pi_c^{loc}(f\Delta_{(w_1\dots w_{R-2})^{-1}\lambda}^{\widehat{s}_{R-1}}{}^{-1}) = \Pi_c(f)\Delta_\lambda^s{}^{-1}, f \in \mathbb{C}[(w_1\dots w_{R-2})N_-ZsM_-(w_1\dots w_{R-2})^{-1}].$$

Since the operator Π_c is surjective, Π_c^{loc} is also surjective.

From the surjectivity of the specialization of the operator Π_c^q at $q^{\frac{1}{dr^2}} = 1$ it follows that $W_{\mathcal{B}}^{s,loc}(G) = \mathrm{Im}\Pi_c^q$ mod $(q^{\frac{1}{dr^2}} - 1)W_{\mathcal{B}}^{s,loc}(G)$. Note also that $\mathrm{Im}\Pi_c^q \subset W_{\mathcal{B}}^{s,loc}(G)$ are submodules of the $\mathcal{B}$–module $Q_{\mathcal{B}}^{loc}$, and $Q_{\mathcal{B}}^{loc}$ is a $\mathcal{B}$–submodule of the $\mathcal{B}$–module $\mathbb{C}_{\mathcal{B}}^{s}[G^*]/I_{\mathcal{B}}$ which is free over $\mathcal{B}$ by Proposition 3.2.10. Since $\mathcal{B}$ is a principal ideal domain $Q_{\mathcal{B}}^{loc}$ is $\mathcal{B}$–free by Theorem 6.5 in [102].

The properties mentioned in the previous paragraph and Lemma 4.3.3 imply that $W_q^{s,loc}(G) = \mathrm{Im}_q\Pi_c^q$ and $W_{\mathcal{B}}^{s,loc}(G) = W_{\mathcal{B}}^{s,loc}(G) \cap \mathrm{Im}_q\Pi_c^q$. This completes the proof.

$\square$

4.8   Bibliographic comments

The results presented in this chapter are entirely new.

Commutation relations in the algebra $\mathbb{C}_{\mathcal{B}}^{s}[G]$ which appear in Section 4.2 can be found, for instance, in [12], Theorem I.8.16.

The definition of the Zhelobenko type operators for q-W–algebras was inspired by the construction of extremal projection operators and of the Zhelobenko operators due to Zhelobenko. The definitions and the statements in this chapter are conceptually close to the definition and the properties of the Zhelobenko operators introduced and studied in [133]–[141] (see also [66]). Below, for the convenience of the reader who is familiar with these papers, we give references to similar statements from them. However, the results of [133]–[141] and [66] are not used in this book and not directly related to it.

For $k = 0$ the operators P_{jp}^k are counterparts of the Zhelobenko operators q_α introduced in [133], §2 and §5, in [141], Definition 5.2.1, and for $k > 0$ the operators P_{jp}^k are counterparts of the operators $q_{\alpha,\mathfrak{m}}^{(k)}$ defined in [66], formula (4.9).

Properties of the Zhelobenko operators similar to those of the operators P_{jp} mentioned in Proposition 4.6.1 can be found in [133], §5, Proposition 1 (iii) and (iv), [141], Proposition 5.2.4 (b) and (c), [66], Lemma 4.5 (ii), (iv) and (v).

Zhelobenko operators q_w conceptually analogous to $P_{\leq jp}$ were defined in [133], §5, Definition 1, [141], Definition 5.2.4.

Properties (4.6.9) and (4.6.10) are counterparts of Proposition 3, parts (i) and (iii) in [133], §2, and of properties (α) and (β) in Section 5.2.4 in [141].

Lemma 4.6.4 is analogous to a similar property for the Zhelobenko operators stated in [141], Proposition and Corollary 5.2.3, and in [66], Lemmas 4.3 and 4.5 (iii).

Formulas similar to (4.6.23) are used in the proof of Proposition 5.2.4 in [141] and in Proposition 4.4 in [66].

Theorem 4.7.2 is analogous to Theorem 2 in [133], §6 and to Theorem 5.5.1 in [141] for the Zhelobenko operators.

Chapter 5

Application of q-W–algebras to the description of the category of equivariant modules over a quantum group

In this rather short chapter we apply Corollary 4.6.8 to establish an equivalence between the category of finitely generated representations of a q-W–algebra and a category $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ of equivariant modules over a quantum group. Categories of this kind were denoted $A - \mathrm{mod}_B^\chi$ in the introduction. The structure of modules from the category $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ is similar to that of $\mathfrak{g} - K$–modules or of principal series representations over complex semisimple Lie algebras.

The proof of the main theorem of this chapter, Theorem 5.2.1, is based on Corollary 4.6.8. In this framework one can give precise values of ε for which the categorical equivalence holds. Remarkably, with slight modifications this method is also applicable to the study of the structure of finite–dimensional representations over quantum groups at roots of unity. This will be done in the next chapter.

5.1 A category of equivariant modules over a quantum group

In this section we define a category of equivariant representations over a quantum group.

Suppose that $\kappa = 1$ and let $\varepsilon \in \mathbb{C}$. Fix a root $\varepsilon^{\frac{1}{dr^2}}$ of ε of order $\frac{1}{dr^2}$. Let $U_\varepsilon^s(\mathfrak{g})$, $U_\varepsilon^s(\mathfrak{m}_-)$, $\mathbb{C}_\varepsilon^{s,loc}[G_*]$, $\mathbb{C}_\varepsilon^s[M_+]$, $\mathbb{C}_\varepsilon^s[B_+]$, $\mathbb{C}_\varepsilon^s[G^*]$, Q_ε^{loc}, $W_\varepsilon^{s,loc}(G)$, χ_ε^s, $\mathbb{C}_{\varepsilon_s}$, I_ε, ϕ_ε, $B_{n_1\ldots n_c}^\varepsilon$, $\mathbb{C}_{11}^{loc}[G]_\varepsilon$ be the natural specializations at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ of $U_\mathcal{A}^s(\mathfrak{g})$, $U_\mathcal{A}^s(\mathfrak{m}_-)$, $\mathbb{C}_\mathcal{B}^{s,loc}[G_*]$, $\mathbb{C}_\mathcal{B}^s[M_+]$, $\mathbb{C}_\mathcal{B}^s[B_+]$, $\mathbb{C}_\mathcal{B}^s[G^*]$, $Q_\mathcal{B}^{loc}$, $W_\mathcal{B}^{s,loc}(G)$, χ_q^s, $\mathcal{B}_{\varepsilon_s}$, $I_\mathcal{B}$, ϕ, $B_{n_1\ldots n_c}$, $\mathbb{C}_{11}^{loc}[G]$, respectively.

We shall always assume that $[n]_{\varepsilon_{\alpha_i}} \neq 0$ and $\varepsilon^{2d_i} \neq 1$ for $i = 1, \ldots, l$, $n \in \mathbb{N}$. Then $\mathbb{C}_\varepsilon^s[M_+] = U_\varepsilon^s(\mathfrak{m}_-) = U_\varepsilon^{s,res}(\mathfrak{m}_-)$, $U_\varepsilon^s(\mathfrak{g}) = U_\varepsilon^{s,res}(\mathfrak{g})$ and $\mathbb{C}_\varepsilon^s[G^*]$ is a subalgebra in $U_\varepsilon^{s,res}(\mathfrak{g})$ as $\mathbb{C}_\mathcal{B}^s[G^*]$ is a subalgebra in $U_\mathcal{B}^{s,res}(\mathfrak{g})$.

Let $J = \mathrm{Ker}\ \varepsilon_s|_{\mathbb{C}_\varepsilon^s[M_+]}$ be the augmentation ideal of $\mathbb{C}_\varepsilon^s[M_+]$ related to the counit ε_s, and $\mathbb{C}_{\varepsilon_s}$ the trivial representation of $\mathbb{C}_\varepsilon^s[M_+]$ given by the counit. Let V be a finitely generated $\mathbb{C}_\varepsilon^{s,loc}[G_*]$–module which satisfies the following conditions:

1. V is a right $\mathbb{C}_\varepsilon^s[M_+]$–module with respect to an action Ad_s such that the action of the augmentation ideal J on V is locally nilpotent.

2. The following compatibility condition holds for the two actions

$$\mathrm{Ad}_s x(yv) = \mathrm{Ad}_s x^1(y)\mathrm{Ad}_s x^2(v),\ x \in \mathbb{C}_\varepsilon^s[M_+],\ y \in \mathbb{C}_\varepsilon^{s,loc}[G_*],\ v \in V, \tag{5.1.1}$$

where $\Delta_s(x) = x^1 \otimes x^2$, $\mathrm{Ad}_s x^1(y)$ is the adjoint action of $x^1 \in \mathbb{C}_\varepsilon^s[B_+]$ on $y \in \mathbb{C}_\varepsilon^{s,loc}[G_*]$.

An element $v \in V$ is called a Whittaker vector if $\mathrm{Ad}_s xv = \varepsilon_s(x)v$ for any $x \in \mathbb{C}_\varepsilon^s[M_+]$. The space

$$\mathrm{Hom}_{\mathbb{C}_\varepsilon^s[M_+]}(\mathbb{C}_{\varepsilon_s}, V) = \mathrm{Wh}(V). \tag{5.1.2}$$

is called the space of Whittaker vectors of V.

Consider the induced $\mathbb{C}_\varepsilon^s[G^*]$–module $W = \mathbb{C}_\varepsilon^s[G^*] \otimes_{\mathbb{C}_\varepsilon^{s,loc}[G_*]} V$. Using the adjoint action of $\mathbb{C}_\varepsilon^s[G^*]$ on itself one can naturally extend the adjoint action of $\mathbb{C}_\varepsilon^s[M_+]$ from V to W in such a way that compatibility condition (5.1.1) is satisfied for the natural action of $\mathbb{C}_\varepsilon^s[G^*]$ and the adjoint action Ad of $\mathbb{C}_\varepsilon^s[M_+]$ on W. As we observed in Section 3.2 $\Delta_s(\mathbb{C}_\varepsilon^s[M_+]) \subset \mathbb{C}_\varepsilon^s[B_+] \otimes \mathbb{C}_\varepsilon^s[M_+]$.

We shall require that

3. For any $x \in \mathbb{C}_\varepsilon^s[M_+]$ the natural action of the element $(S_s \otimes \chi_\varepsilon^s)\Delta_s(x) \in \mathbb{C}_\varepsilon^s[G^*]$ on W coincides with the adjoint action $\mathrm{Ad}_s x$ of x on W.

As in the second part of the proof of Proposition 3.2.12 one can check that the last condition implies that for any $z \in \mathbb{C}_\varepsilon^{s,loc}[G_*] \cap I_\varepsilon$ and $v \in \mathrm{Wh}(V)$ $zv = 0$.

Denote by $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ the category of finitely generated $\mathbb{C}_\varepsilon^{s,loc}[G_*]$–modules which satisfy conditions (1)–(3) above. Morphisms in the category $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ are $\mathbb{C}_\varepsilon^{s,loc}[G_*]$- and $\mathbb{C}_\varepsilon^s[M_+]$–module homomorphisms. We call $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ the category of $(\mathbb{C}_\varepsilon^s[M_+], \chi_\varepsilon^s)$–equivariant modules over $\mathbb{C}_\varepsilon^{s,loc}[G_*]$.

Note that the algebra $W_\mathcal{B}^{s,loc}(G)$ naturally acts in the space of Whittaker vectors for any object V of the category $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$. Indeed, if $w, w' \in \mathbb{C}_\varepsilon^{s,loc}[G_*]$ are two representatives of an element from $W_\mathcal{B}^{s,loc}(G)$ then $w - w' \in \mathbb{C}_\varepsilon^{s,loc}[G_*] \cap I_\varepsilon$, and hence for any $v \in \mathrm{Wh}(V)$ $wv = w'v$. Moreover, by the definition of the algebra $W_\mathcal{B}^{s,loc}(G)$ and by condition (5.1.1) we have

$$\mathrm{Ad}_s x(wv) = \mathrm{Ad}_s x^1(w)\mathrm{Ad}_s x^2(v) = \mathrm{Ad}_s x^1(w)\varepsilon_s(x^2)v = \mathrm{Ad}_s x(w)v = \varepsilon_s(x)wv.$$

Therefore wv is a Whittaker vector independent of the choice of the representative w.

For any $\mathbb{C}$–module R we denote by $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+]), R)$ the subspace in $\mathrm{Hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R)$ which consists of the linear maps vanishing on some power of the augmentation ideal $J = \mathrm{Ker}\ \varepsilon_s$ of $\mathbb{C}_\varepsilon^s[M_+]$, $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R) = \{f \in \mathrm{Hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R) : f(J^n) = 0$ for some $n > 0\}$. Note that for every element f of $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R)$ one has $f(x) = 0$ if x does not belong to a finite–dimensional subspace of $\mathbb{C}_\varepsilon^s[M_+]$, and hence $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R) = \mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C}) \otimes R$.

Equip the space $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R)$ with the right action of $\mathbb{C}_\varepsilon^s[M_+]$ induced by the multiplication in $\mathbb{C}_\varepsilon^s[M_+]$ from the left. To study the properties of this module we shall need a special filtration on the algebra $\mathbb{C}_\varepsilon^s[M_+] = U_\varepsilon^s(\mathfrak{m}_-)$.

Recall that the algebra $U_\varepsilon^s(\mathfrak{g})$ can be equipped with the DeConcini–Kac filtration such that the associated graded algebra is almost commutative. For $\mathbf{r}, \mathbf{t} \in \mathbb{N}^D$ define the height of the element $u_{\mathbf{r}, \mathbf{t}, t} = e^{\mathbf{r}} t f^{\mathbf{t}}$, $t \in U_\varepsilon^s(\mathfrak{h})$ as follows ht $(u_{\mathbf{r}, \mathbf{t}, t}) = \sum_{i=1}^D (t_i + r_i)\mathrm{ht}\ \beta_i \in \mathbb{N}$, where ht β_i is the height of the root β_i. Introduce also the degree of $u_{\mathbf{r}, \mathbf{t}, t}$ by

$$d(u_{\mathbf{r}, \mathbf{t}, t}) = (r_1, \ldots, r_D, t_D, \ldots, t_1, \mathrm{ht}\ (u_{\mathbf{r}, \mathbf{t}, t})) \in \mathbb{N}^{2D+1}.$$

By Lemma 2.8.2 the elements $e^{\mathbf{r}} t f^{\mathbf{t}}$ span $U_\varepsilon^s(\mathfrak{g})$ as a linear space.

Equip $\mathbb{N}^{2D+1}$ with the total lexicographic order and denote by $(U_q^s(\mathfrak{g}))_k$ the span of elements $u_{\mathbf{r}, \mathbf{t}, t}$ with $d(u_{\mathbf{r}, \mathbf{t}, t}) \le k$ in $U_q^s(\mathfrak{g})$. Then Proposition 1.7 in [23] implies that $(U_\varepsilon^s(\mathfrak{g}))_k$ is a filtration of $U_\varepsilon^s(\mathfrak{g})$ such that the associated graded algebra is the complex associative algebra with generators e_α, f_α, $\alpha \in \Delta_+$, $t_i^{\pm 1}$, $i = 1, \ldots l$ subject to the relations

$$t_i t_j = t_j t_i, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i e_\alpha t_i^{-1} = \varepsilon^{\frac{H_i(\alpha)}{d}} e_\alpha, \quad t_i f_\alpha t_i^{-1} = \varepsilon^{-\frac{H_i(\alpha)}{d}} f_j,$$

$$e_\alpha f_\beta = \varepsilon^{(\frac{1+s}{1-s} P_{\mathfrak{h}'^*}\alpha, \beta)} f_\beta e_\alpha,$$

$$e_\alpha e_\beta = \varepsilon^{(\alpha,\beta)+(\frac{1+s}{1-s} P_{\mathfrak{h}'^*}\alpha, \beta)} e_\beta e_\alpha, \ \alpha < \beta, \tag{5.1.3}$$

$$f_\alpha f_\beta = \varepsilon^{(\alpha,\beta)+(\frac{1+s}{1-s} P_{\mathfrak{h}'^*}\alpha, \beta)} f_\beta f_\alpha, \ \alpha < \beta.$$

Such algebras are called semi–commutative.

Lemma 5.1.1. *Let $J = \mathrm{Ker}\ \varepsilon_s$ be the augmentation ideal of $\mathbb{C}_\varepsilon^s[M_+]$, R a $\mathbb{C}$–module, $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R) = \{f \in \mathrm{Hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R) : f(J^p) = 0$ for some $p > 0\}$. Equip $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R)$ with the right action of $\mathbb{C}_\varepsilon^s[M_+]$ induced by multiplication on $\mathbb{C}_\varepsilon^s[M_+]$ from the left. Then the $\mathbb{C}_\varepsilon^s[M_+]$–module $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R)$ is injective.*

Proof. First observe that the algebra $\mathbb{C}_\varepsilon^s[M_+] \simeq U_\varepsilon^s(\mathfrak{m}_-)$ is Notherian and ideal J satisfies the so-called weak Artin–Rees property, i.e. for every finitely generated left $U_\varepsilon^s(\mathfrak{m}_-)$–module M and its submodule N there exists an integer $n > 0$ such that $J^n M \cap N \subset JN$.

Indeed, observe that the algebra $U_\varepsilon^s(\mathfrak{m}_-)$ can be equipped with a filtration induced by the De Concini–Kac filtration on the algebra $U_\varepsilon^s(\mathfrak{g})$ in such a way that the associated graded algebra is finitely generated and semi–commutative (see (5.1.3)). The fact that $U_\varepsilon^s(\mathfrak{m}_-)$ is Notherian follows from the existence of the filtration on it for which the associated graded algebra is semi–commutative and from Theorem 4 in Ch. 5, §3 in [57] (compare also with Theorem 4.8 in [92]). The ideal J satisfies the weak Artin–Rees property because the subring $U_\varepsilon^s(\mathfrak{m}_-) + Jt + J^2t^2 + \ldots \subset U_\varepsilon^s(\mathfrak{m}_-)[t]$, where t is a central indeterminate, is Notherian (see [94], Ch. 11, §2, Lemma 2.1). The last fact follows from the existence of a filtration on $U_\varepsilon^s(\mathfrak{m}_-) + Jt + J^2t^2 + \ldots$ induced by the filtration on $U_\varepsilon^s(\mathfrak{m}_-)$ for which the associated graded algebra is semi–commutative and again from Theorem 4 in Ch. 5, §3 in [57].

Finally, the module $\mathrm{Hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R)$ is obviously injective. By Lemma 3.2 in Ch. 3, [50] the module $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R) = \{f \in \mathrm{Hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], R) : f(J^p) = 0 \text{ for some } p > 0\}$ is also injective since the ideal J satisfies the weak Artin–Rees property.

$\square$

5.2 Skryabin equivalence for equivariant modules over a quantum group

Now we can formulate the main theorem on the structure of the category $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ and on the properties of its objects.

Theorem 5.2.1. *If $[n]_{\varepsilon_{\alpha_i}} \neq 0$ and $\varepsilon^{2d_i} \neq 1$ for $i = 1, \ldots, l$, $n \in \mathbb{N}$ then the functor $E \mapsto Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} E$, is an equivalence of the category of finitely generated left $W_\varepsilon^{s,loc}(G)$–modules and of the category $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$. The inverse equivalence is given by the functor $V \mapsto \mathrm{Wh}(V)$. In particular, the latter functor is exact.*

Every module $V \in \mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ is isomorphic to $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C}) \otimes \mathrm{Wh}(V)$ as a right $\mathbb{C}_\varepsilon^s[M_+]$–module, where $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C})$ is equipped with the right action of $\mathbb{C}_\varepsilon^s[M_+]$ induced by the multiplication in $\mathbb{C}_\varepsilon^s[M_+]$ from the left. Q_ε^{loc} is isomorphic to $\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C}) \otimes W_\varepsilon^{s,loc}(G)$ as a $\mathbb{C}_\varepsilon^s[M_+]$–$W_\varepsilon^{s,loc}(G)$–bimodule, where the right $W_\varepsilon^{s,loc}(G)$–action is induced by the multiplication in $W_\varepsilon^{s,loc}(G)$ from the right. In particular, V is $\mathbb{C}_\varepsilon^s[M_+]$–injective, and $\mathrm{Ext}_{\mathbb{C}_\varepsilon^s[M_+]}^\bullet(\mathbb{C}_{\varepsilon_s}, V) = \mathrm{Wh}(V)$.

Proof. First we prove that Q_ε^{loc} is an object in $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$. We shall prove that the adjoint action of the augmentation ideal J of $\mathbb{C}_\varepsilon^s[M_+]$ on Q_ε^{loc} is locally nilpotent. All the other properties of objects of the category $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ for Q_ε^{loc} were already established in Proposition 3.2.13.

Indeed, recalling the $\mathbb{C}_\varepsilon^s[M_+]$–module homomorphism of $\phi_\varepsilon : \mathbb{C}_{11}^{loc}[G]_\varepsilon \to Q_\varepsilon^{loc}$ and the definition of the adjoint action on $\mathbb{C}_{11}^{loc}[G]_\varepsilon$ in formula (4.5.4) we deduce that in order to show that the adjoint action of the augmentation ideal J of $\mathbb{C}_\varepsilon^s[M_+]$ on Q_ε^{loc} is locally nilpotent it suffices to show that the Ad_s^0–action of the augmentation ideal J of $\mathbb{C}_\varepsilon^s[M_+]$ on $\mathbb{C}_\varepsilon[G]$ is locally nilpotent. But the last fact is true as $\mathbb{C}_\varepsilon[G] = \bigoplus_{\lambda \in P_+} V_\lambda^* \otimes V_\lambda$, where V_λ is the finite–dimensional irreducible representation of $U_\varepsilon^s(\mathfrak{g})$ of highest weight λ, and the action of $\mathbb{C}_\varepsilon^s[M_+]$ on $V_\lambda^* \otimes V_\lambda$ induced by the adjoint action is locally nilpotent since the action of $\mathbb{C}_\varepsilon^s[M_+]$ on finite–dimensional irreducible representations is locally nilpotent.

Now let V be an object in the category $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$. Fix any linear map $\rho : V \to \mathrm{Wh}(V)$ the restriction of which to $\mathrm{Wh}(V)$ is the identity map, and let for any $v \in V$ $\sigma_\varepsilon(v) : \mathbb{C}_\varepsilon^s[M_+] \to \mathrm{Wh}(V)$ be the $\mathbb{C}$–linear homomorphism given by $\sigma_\varepsilon(v)(x) = \rho(\mathrm{Ad}_s x(v))$. Since the adjoint action of J on V is locally nilpotent $\sigma_\varepsilon(v) \in \mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$, and we have a map $\sigma_\varepsilon : V \to \mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V)) \simeq \mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C}) \otimes \mathrm{Wh}(V)$.

By definition σ_ε is a homomorphism of right $\mathbb{C}_\varepsilon^s[M_+]$–modules, where the right action of $\mathbb{C}_\varepsilon^s[M_+]$ on

$$\mathrm{hom}_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$$

is induced by multiplication in $\mathbb{C}_\varepsilon^s[M_+]$ from the left. We claim that σ_ε is an isomorphism.

First we prove that σ_ε is injective. The proof will be based on the following lemma that will be also used later.

Lemma 5.2.2. *Let $\phi : X \to Y$ be a homomorphism of $U_\varepsilon^s(\mathfrak{m}_-)$–modules. Denote by $\mathrm{Wh}(X)$ the subspace of Whittaker vectors of X, i.e. the subspace of X which consists of elements v such that $xv = \varepsilon_s(x)v$, $x \in U_\varepsilon^s(\mathfrak{m}_-)$.*

Assume that the action of the augmentation ideal of $U_\varepsilon^s(\mathfrak{m}_-)$ on X is locally nilpotent and that the restriction of ϕ to the subspace of Whittaker vectors of X is injective. Then ϕ is injective.

Proof. Let $Z \subset X$ be the kernel of ϕ. Assume that Z is not trivial. Observe that Z is invariant with respect to the action induced by the action of $U_\varepsilon^s(\mathfrak{m}_-)$ on X, and that the augmentation ideal of $U_\varepsilon^s(\mathfrak{m}_-)$ acts on X by locally nilpotent transformations. Therefore by Engel theorem Z must contain a nonzero $U_\varepsilon^s(\mathfrak{m}_-)$–invariant vector which is a Whittaker vector $v \in X$. But since the restriction of ϕ to the subspace of Whittaker vectors of X is injective $\phi(v) \neq 0$. Thus we arrive at a contradiction, and hence ϕ is injective. $\square$

Now recall that the action of J on V is locally nilpotent. All non–zero Whittaker vectors in V belong to $\mathrm{Wh}(V)$ and by the definition of σ_ε their images in $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$ are non–zero homomorphisms non-vanishing at 1. Therefore by Lemma 5.2.2 σ_ε is injective.

Next we show that σ_ε is also surjective. Denote $x_{n_1...n_c} = c'_{n_1...n_c}(\varepsilon)^{-1}\phi_\varepsilon(B^\varepsilon_{n_1...n_c})$, where $c'_{n_1...n_c}(\varepsilon)$ are the values of $c'_{n_1...n_c}(q)$ at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ which are all non–zero by the choice of $\varepsilon^{\frac{1}{dr^2}}$. By Corollary 4.6.8

$$\mathrm{Ad}_s(f_{\beta_1}^{k_1}\ldots f_{\beta_d}^{k_c})(x_{n_1...n_c}) = \begin{cases} 1 & \text{if } n_p = k_p \text{ for } p = 1,\ldots,c \\ 0 & \text{if } k_i = n_i, i = 1,\ldots,p-1 \text{ and } k_p > n_p \text{ for some } p \in \{1,\ldots,c\} \end{cases}. \quad (5.2.1)$$

Since for any $v \in \mathrm{Wh}(V)$ and $z \in \mathbb{C}_\varepsilon^{s,loc}[G_*] \cap I_\varepsilon$ we have $zv = 0$, the elements $x_{n_1...n_c}v$ are well–defined and satisfy $\mathrm{Ad}_s x(x_{n_1...n_c}v) = \mathrm{Ad}_s x(x_{n_1...n_c})v$, $x \in \mathbb{C}_\varepsilon^s[M_+]$. For the same reason formula (5.2.1) implies

$$\mathrm{Ad}_s(f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c})(x_{n_1...n_c}v) = \begin{cases} v & \text{if } n_p = k_p \text{ for } p = 1,\ldots,c \\ 0 & \text{if } k_i = n_i, i = 1,\ldots,p-1 \text{ and } k_p > n_p \text{ for some } p \in \{1,\ldots,c\} \end{cases},$$

and hence

$$\sigma_\varepsilon(x_{n_1...n_c}v)(f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c}) = \begin{cases} v & \text{if } n_p = k_p \text{ for } p = 1,\ldots,c \\ 0 & \text{if } k_i = n_i, i = 1,\ldots,p-1 \text{ and } k_p > n_p \text{ for some } p \in \{1,\ldots,c\} \end{cases}. \quad (5.2.2)$$

Observe that the elements $f_{\beta_1}^{k_1}\ldots f_{\beta_c}^{k_c}$ form a linear basis of $\mathbb{C}_\varepsilon^s[M_+]$. Elements of this basis are labeled by elements of the set $\mathbb{N}^c$. Introduce the lexicographic order on this set, so that $(k_1,\ldots,k_c) > (n_1,\ldots,n_c)$ if $k_i = n_i$ for $i = 1,\ldots,p-1$ and $k_p > n_p$ for some $p \in \{1,\ldots,c\}$.

Now let $\mathbf{k} = (k_1,\ldots,k_c) \in \mathbb{N}^c$, $v \in \mathrm{Wh}(V)$ and denote

$$\sigma_\varepsilon(x_{k_1...k_c}v) = f_v^{\mathbf{k}}.$$

Since $f_v^{\mathbf{k}} \in \hom_{\mathbb{C}}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$ it does not vanish only on a finite number of the elements $f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}$ with $\mathbf{k} = (k_1,\ldots,k_c) > (n_1,\ldots,n_c) = \mathbf{n}$. Also by (6.3.8) $f_v^{\mathbf{k}}(f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}) = 0$ for $\mathbf{k} < \mathbf{n} = (n_1,\ldots,n_c)$.

Let $\mathbf{n}^1 = (n_1^1,\ldots,n_c^1) \in \mathbb{N}^c$ be the largest element such that $\mathbf{k} > \mathbf{n}^1$ and

$$f_v^{\mathbf{k}}(f_{\beta_1}^{n_1^1}\ldots f_{\beta_c}^{n_c^1}) = v_1 \neq 0. \quad (5.2.3)$$

Denote

$$f_{v_1}^{\mathbf{n}^1} = -\sigma_\varepsilon(x_{n_1^1...n_c^1}v_1).$$

Then (5.2.2) and (5.2.3) imply that for $g_1 = f_v^{\mathbf{k}} + f_{v_1}^{\mathbf{n}^1} \in \hom_{\mathbb{C}}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$ one has

$$g_1(f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}) = \begin{cases} 0 \text{ if } \mathbf{n} = (n_1,\ldots,n_c) \geq \mathbf{n}^1, \mathbf{n} \neq \mathbf{k} \\ v \text{ if } \mathbf{n} = \mathbf{k} \end{cases}. \quad (5.2.4)$$

Since $g_1 \in \hom_{\mathbb{C}}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$ it does not vanish on a finite number of the elements $f_{\beta_1}^{n_1}\ldots f_{\beta_c}^{n_c}$ with $\mathbf{n}^1 > \mathbf{n} = (n_1,\ldots,n_c)$. Let $\mathbf{n}^2 = (n_1^2,\ldots,n_c^2) \in \mathbb{N}^c$ be the largest element such that $\mathbf{n}^1 > \mathbf{n}^2$ and

$$g_1(f_{\beta_1}^{n_1^2}\ldots f_{\beta_c}^{n_c^2}) = v_2 \neq 0. \quad (5.2.5)$$

Denote

$$f_{v_2}^{\mathbf{n}^2} = -\sigma_\varepsilon(x_{n_1^2...n_c^2}v_2).$$

Then (5.2.2) and (5.2.5) imply that for $g_2 = f_v^{\mathbf{k}} + f_{v_1}^{\mathbf{n}^1} + f_{v_2}^{\mathbf{n}^2} = g_1 + f_{v_2}^{\mathbf{n}^2} \in \hom_{\mathbb{C}}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$ one has

$$g_2(f_{\beta_1}^{n_1} \cdots f_{\beta_c}^{n_c}) = \begin{cases} 0 \text{ if } \mathbf{n} = (n_1, \ldots, n_c) \geq \mathbf{n}^2, \mathbf{n} \neq \mathbf{k} \\ v \text{ if } \mathbf{n} = \mathbf{k} \end{cases}. \tag{5.2.6}$$

Iterating this procedure we obtain a sequence of elements $\mathbf{k} > \mathbf{n}^1 > \mathbf{n}^2 > \ldots > \mathbf{n}^i > \ldots$, $\mathbf{n}^i \in \mathbb{N}^c$ and a sequence of elements $g_i \in \hom_{\mathbb{C}}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$ such that

$$g_i(f_{\beta_1}^{n_1} \cdots f_{\beta_c}^{n_c}) = \begin{cases} 0 \text{ if } \mathbf{n} = (n_1, \ldots, n_c) \geq \mathbf{n}^i, \mathbf{n} \neq \mathbf{k} \\ v \text{ if } \mathbf{n} = \mathbf{k} \end{cases}. \tag{5.2.7}$$

Since by Theorem 1.13 in [51] (see also Theorem 2.4.2 in [6]) the lexicographic order on $\mathbb{N}^c$ is a well–order the sequence $\mathbf{k} > \mathbf{n}^1 > \mathbf{n}^2 > \ldots > \mathbf{n}^i > \ldots$ must be finite, i.e. for some $i \in \mathbb{N}$

$$g_i(f_{\beta_1}^{n_1} \cdots f_{\beta_c}^{n_c}) = \begin{cases} 0 \text{ if } \mathbf{n} = (n_1, \ldots, n_c) \geq \mathbf{n}^i, \mathbf{n} \neq \mathbf{k} \\ v \text{ if } \mathbf{n} = \mathbf{k} \end{cases}, \tag{5.2.8}$$

and there is no element $\mathbf{n} = (n_1, \ldots, n_c) < \mathbf{n}^i$ such that

$$g_i(f_{\beta_1}^{n_1^2} \cdots f_{\beta_c}^{n_c^2}) \neq 0. \tag{5.2.9}$$

Therefore $h_v^{\mathbf{k}} = g_i$ satisfies

$$h_v^{\mathbf{k}}(f_{\beta_1}^{n_1} \cdots f_{\beta_c}^{n_c}) = \begin{cases} v \text{ if } (n_1, \ldots, n_c) = (k_1, \ldots, k_c), \\ 0 \text{ if } (n_1, \ldots, n_c) \neq (k_1, \ldots, k_c) \end{cases}.$$

This implies that the elements $h_v^{\mathbf{k}}$ with $(k_1, \ldots, k_c) \in \mathbb{N}^c$, $v \in \mathrm{Wh}(V)$ generate $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$. We deduce that the elements $\sigma_\varepsilon(x_{n_1 \ldots n_c} v)$ with $(n_1, \ldots, n_c) \in \mathbb{N}^c$, $v \in \mathrm{Wh}(V)$ generate $\hom_{\mathbb{C}}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V))$ as well. Therefore σ_ε is surjective.

Next, we shall need another lemma.

Lemma 5.2.3. *Let $\phi : X \to Y$ be an injective homomorphism of $U_\varepsilon^s(\mathfrak{m}_-)$–modules. As above, denote by $\mathrm{Wh}(X)$ the subspace of Whittaker vectors of X, i.e. the subspace of X which consists of elements v such that $xv = \varepsilon_s(x)v$, $x \in U_\varepsilon^s(\mathfrak{m}_-)$. Assume that ϕ induces an isomorphism of the spaces of Whittaker vectors of X and of Y, and that $\mathrm{Ext}^1_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, X) = 0$, where $\mathbb{C}_{\varepsilon_s}$ is the trivial representation of $U_\varepsilon^s(\mathfrak{m}_-)$. Suppose also that the action of the augmentation ideal J of $U_\varepsilon^s(\mathfrak{m}_-)$ on the cokernel of ϕ is locally nilpotent. Then ϕ is surjective.*

Proof. Consider the exact sequence

$$0 \to X \to Y \to W' \to 0,$$

where W' is the cokernel of ϕ, and the corresponding long exact sequence of cohomology,

$$0 \to \mathrm{Ext}^0_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, X) \to \mathrm{Ext}^0_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, Y) \to \mathrm{Ext}^0_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, W') \to$$

$$\to \mathrm{Ext}^1_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, X) \to \ldots.$$

Since ϕ induces an isomorphism of the spaces of Whittaker vectors of X and of Y, and $\mathrm{Ext}^1_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_{\varepsilon_s}, X) = 0$, the initial part of the long exact cohomology sequence takes the form

$$0 \to \mathrm{Wh}(X) \to \mathrm{Wh}(Y) \to \mathrm{Wh}(W') \to 0,$$

where the second map in the last sequence is an isomorphism. Using the last exact sequence we deduce that $\mathrm{Wh}(W') = 0$. But the augmentation ideal J acts on W' by locally nilpotent transformations. Therefore, by Engel theorem, if W' is not trivial there should exists a nonzero $U_\varepsilon^s(\mathfrak{m}_-)$–invariant vector in it. Thus we arrive at a contradiction, and $W' = 0$. Therefore ϕ is surjective.

$\square$

Now we show that Q_ε^{loc} is isomorphic to $\hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C})) \otimes W_\varepsilon^{s,loc}(G)$ as a $\mathbb{C}_\varepsilon^s[M_+]$–$W_\varepsilon^{s,loc}(G)$–bimodule. Indeed, by the definitions of the spaces $\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), W_\varepsilon^s(G))$ and $\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C})$ we have an obvious right $U_\varepsilon^s(\mathfrak{m}_-)$–module isomorphism $\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), W_\varepsilon^{s,loc}(G)) = \hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C}) \otimes W_\varepsilon^{s,loc}(G)$.

Now consider the $U_\varepsilon^s(\mathfrak{m}_-)$–submodule $\sigma_\varepsilon^{-1}(\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C}))$ of Q_ε^{loc}, where

$$\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C}) \subset \hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), W_\varepsilon^{s,loc}(G)).$$

Obviously $\sigma_\varepsilon^{-1}(\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C})) \simeq \hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C})$ as a right $U_\varepsilon^s(\mathfrak{m}_-)$–module.

Let $\phi_\varepsilon : \sigma_\varepsilon^{-1}(\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C})) \otimes W_\varepsilon^{s,loc}(G) \to Q_\varepsilon^{loc}$ be the map induced by the action of $W_\varepsilon^{s,loc}(G)$ on Q_ε^{loc}. Since this action commutes with the adjoint action of $U_\varepsilon^s(\mathfrak{m}_-)$ on Q_ε^{loc} we infer that ϕ_ε is a homomorphism of $U_\varepsilon^s(\mathfrak{m}_-)$–$W_\varepsilon^{s,loc}(G)$–bimodules.

We claim that ϕ_ε is injective. This follows straightforwardly from Lemma 5.2.2 because all Whittaker vectors of $\sigma_\varepsilon^{-1}(\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C})) \otimes W_\varepsilon^{s,loc}(G)$ belong to the subspace

$$1 \otimes W_\varepsilon^{s,loc}(G) \subset \sigma_\varepsilon^{-1}(\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C})) \otimes W_\varepsilon^{s,loc}(G),$$

and the restriction of ϕ_ε to this subspace is injective.

Now we show that ϕ_ε is surjective. By Lemma 5.1.1 one can immediately deduce that the right $U_\varepsilon^s(\mathfrak{m}_-)$–module $\sigma_\varepsilon^{-1}(\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C})) \otimes W_\varepsilon^{s,loc}(G) \simeq \hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), W_\varepsilon^{s,loc}(G))$ is injective. In particular,

$$\mathrm{Ext}^1_{U_\varepsilon^s(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, \sigma_\varepsilon^{-1}(\hom_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_-), \mathbb{C})) \otimes W_\varepsilon^{s,loc}(G)) = 0.$$

One checks straightforwardly, similarly the case of the map σ_ε, that the other conditions of Lemma 5.2.3 for the map ϕ_ε are satisfied as well. Therefore ϕ_ε is surjective.

Thus Q_ε^{loc} is isomorphic to $\hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C})) \otimes W_\varepsilon^{s,loc}(G)$ as a $\mathbb{C}_\varepsilon^s[M_+]$–$W_\varepsilon^{s,loc}(G)$–bimodule.

Now we prove the main claim of this theorem. Let E be a finitely generated $W_\varepsilon^{s,loc}(G)$–module. Since $Q_\varepsilon^{loc} \simeq \hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C})) \otimes W_\varepsilon^{s,loc}(G)$ as a $\mathbb{C}_\varepsilon^s[M_+]$–$W_\varepsilon^{s,loc}(G)$–bimodule, we have

$$Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} E \simeq \hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C})) \otimes W_\varepsilon^{s,loc}(G) \otimes_{W_\varepsilon^{s,loc}(G)} E \simeq \hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C})) \otimes E$$

as a $\mathbb{C}_\varepsilon^s[M_+]$–module. This implies

$$\mathrm{Wh}(Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} E) \simeq \mathrm{Hom}_{\mathbb{C}_\varepsilon^s[M_+]}(\mathbb{C}_{\varepsilon_s}, Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} E) \simeq \mathrm{Hom}_{\mathbb{C}_\varepsilon^s[M_+]}(\mathbb{C}_{\varepsilon_s}, \hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C}) \otimes E) \simeq$$

$$\simeq \mathrm{Hom}_{\mathbb{C}_\varepsilon^s[M_+]}(\mathbb{C}_{\varepsilon_s}, \hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], E)) \simeq \hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+] \otimes_{\mathbb{C}_\varepsilon^s[M_+]} \mathbb{C}_{\varepsilon_s}, E) \simeq E. \tag{5.2.10}$$

Therefore to prove the theorem it suffices to check that for any $V \in \mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ the canonical map $f : Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V) \to V$ is an isomorphism.

In order to do this we observe that $Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$ is an object of the category $\mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$ since Q_ε^{loc} is an object of this category. The action of the augmentation ideal J on $Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$ is locally nilpotent. By (5.2.10) the space of Whittaker vectors of $Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$ is $1 \otimes \mathrm{Wh}(V)$, and the restriction of f to $1 \otimes \mathrm{Wh}(V)$ induces an isomorphism of the spaces of Whittaker vectors of $Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$ and of V. Therefore f is injective by Lemma 5.2.2.

As we proved above $\sigma_\varepsilon : V \to \hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathrm{Wh}(V)) \simeq \hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C}) \otimes \mathrm{Wh}(V)$ is an isomorphism of $\mathbb{C}_\varepsilon^s[M_+]$–modules for any module $V \in \mathbb{C}_\varepsilon^{s,loc}[G_*] - \mathrm{mod}_{\mathbb{C}_\varepsilon^s[M_+]}^{\chi_\varepsilon^s}$. By Lemma 5.1.1 $\hom_\mathbb{C}(\mathbb{C}_\varepsilon^s[M_+], \mathbb{C}) \otimes \mathrm{Wh}(V)$ is injective over $\mathbb{C}_\varepsilon^s[M_+]$. Therefore V is injective as an $\mathbb{C}_\varepsilon^s[M_+]$–module with respect to the adjoint action. In particular, $Q_\varepsilon^{loc} \otimes_{W_{\varepsilon,\xi}^{s,loc}(G)} \mathrm{Wh}(V)$ is injective over $\mathbb{C}_\varepsilon^s[M_+]$, and hence $\mathrm{Ext}^1_{\mathbb{C}_\varepsilon^s[M_+]}(\mathbb{C}_{\varepsilon_s}, Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)) = 0$.

Recall also that f induces an isomorphism of the spaces of Whittaker vectors of $Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$ and of V and that the adjoint action of J on V is locally nilpotent. Therefore f is surjective by Lemma 5.2.3 with $X = Q_\varepsilon^{loc} \otimes_{W_\varepsilon^{s,loc}(G)} \mathrm{Wh}(V)$, $Y = V$, $\phi = f$. This completes the proof of the theorem.

$\square$

5.3   Bibliographic comments

A categorical equivalence for Lie algebras, called the Skryabin equivalence, similar to that considered in this chapter was established in the Appendix to [97].

The main theorem of this chapter is an improvement of Theorem 7.7 in [116] where a similar equivalence was established in a quantum group case for q specialized to generic values $q = \varepsilon \in \mathbb{C}$. The proof of Theorem 7.7 in [116] relies on homological methods and arguments related to the properties of the quasiclassical limits $W^s(G)$ of q-W–algebras. In this book we use the approach similar to the original Skryabin's idea.

The definition of the category of equivariant representations over a quantum group given in Section 5.1 is a slight modification of a similar definition given in Section 7 in [116], minor changes being related to the fact that in Corollary 4.6.8, and more generally in the previous chapter, we dealt not with quantum groups themselves but with their localizations.

Chapter 6

Application of q-W–algebras to quantum groups at roots of unity and the proof of De Concini–Kac–Procesi conjecture

In this chapter we are going to use the elements $B_{n_1 \ldots n_c}$ introduced in Corollary 4.6.8 to study the structure of representations of quantum groups at roots of unity. De Concini and Kac observed that every irreducible representation of a quantum group $U_\varepsilon(\mathfrak{g})$ at a root of unity ε is in fact a representation of a finite–dimensional quotient $U_\eta(\mathfrak{g})$ of the quantum group, and hence every such representation is finite–dimensional itself. The quotient $U_\eta(\mathfrak{g})$ here depends on the representation. Later De Concini, Kac and Procesi also conjectured that the dimension of every such representation is divisible by a number b which depends on (an isomorphism class of) $U_\eta(\mathfrak{g})$; a precise definition of b will be given in Theorem 6.3.2.

Our main goal in this chapter is to prove this conjecture. We shall also obtain other related results on the structure of finite–dimensional representations of $U_\eta(\mathfrak{g})$. Firstly we are going to use an observation that every finite–dimensional representation of $U_\eta(\mathfrak{g})$ can be equipped with a second right action of a finite–dimensional subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ of the so–called small quantum group, and the dimension of this subalgebra is equal to b. The choice of the subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ depends on $U_\eta(\mathfrak{g})$ and the action of $U_{\eta_1}(\mathfrak{m}_-)$ satisfies a compatibility condition similar to condition (5.1.1) for equivariant modules over quantum groups at generic ε. Thus every finite–dimensional representation of $U_\eta(\mathfrak{g})$ is in fact an equivariant $U_\eta(\mathfrak{g}) - U_{\eta_1}(\mathfrak{m}_-)$–bimodule. Next we prove that every finite–dimensional representation of $U_\eta(\mathfrak{g})$ is cofree over the corresponding subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ which confirms, in particular, the De Concini–Kac–Procesi conjecture. Remarkably, to prove this statement one can apply almost verbatim the arguments from the proof of Theorem 5.2.1 on the quantum group version of the Skryabin equivalence for generic ε which overemphasizes again a striking similarity between the categories of finite–dimensional representations of algebras $U_\eta(\mathfrak{g})$ and the categories of equivariant modules over quantum groups introduced in Section 5.1.

The peculiarity of the quantum group case is that one can explicitly construct cofree bases of finite–dimensional $U_\eta(\mathfrak{g})$–modules using the elements $B_{n_1 \ldots n_c}$ from Corollary 4.6.8.

6.1 Quantum groups at roots of unity

In this section we recall some results on representation theory of quantum groups at roots of unity.

Let m be an odd positive integer number such that $m > d_i$ is coprime to all d_i for all i, ε a primitive m-th root of unity. An appropriate number d, which appears in the definiton of the algebras $U_\varepsilon(\mathfrak{g})$ and $U_\varepsilon^s(\mathfrak{g})$, can be found from the following proposition.

Proposition 6.1.1. *Let Δ be an irreducible root system, Δ_+^s the system of positive roots associated to the conjugacy class of a Weyl group element $s \in W$ in Theorem 1.5.2, $s = s_{\gamma_1} \ldots s_{\gamma_{l'}}$, representation (1.2.1) for s, $\alpha_1, \ldots, \alpha_l$ the system of simple roots in Δ_+^s. Then*

(i) if Δ is of exceptional type the lowest common multiple d' of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$, where $i, j = 1, \ldots, l$ is given in the tables in Appendix 2;

(ii) if Δ is of classical type then the conjugacy class of s corresponds to the sum of a number of blocks as in (1.4.7), (1.4.8), (1.4.12) or (1.4.15). To each block of type X we associate an integer $d_{ij}(X)$, $i, j = 1, \ldots, l$ as follows:

if Δ is not of type A_l, D_l, an orbit with the smallest number of elements for the action of the group $\langle s \rangle$ on E corresponds to a block of type A_n and s does not fix any root from Δ then

for $\Delta = B_l$

$$d_{ij}(A_n) = \begin{cases} 2p+1 & \text{if } n = 2p \text{ is even;} \\ p+1 & \text{if } n = 2p+1, \, n \neq 4p-1 \text{ is odd;} \\ p & \text{if } n = 4p-1 \text{ is odd and } i < j; \\ 2p & \text{if } n = 4p-1 \text{ is odd and } i > j; \end{cases} \tag{6.1.1}$$

for $\Delta = C_l$

$$d_{ij}(A_n) = \begin{cases} 2p+1 & \text{if } n = 2p \text{ is even;} \\ p+1 & \text{if } n = 2p+1, \, n \neq 4p-1 \text{ is odd;} \\ 2p & \text{if } n = 4p-1 \text{ is odd and } i < j; \\ p & \text{if } n = 4p-1 \text{ is odd and } i > j; \end{cases} \tag{6.1.2}$$

for $\Delta = D_l$ if $A_{l-1} \subset D_l$ is the only nontrivial block of the conjugacy class of s then

$$d_{ij}(A_{l-1}) = \begin{cases} 2p+1 & \text{if } l = 2p+1 \text{ is odd;} \\ p+1 & \text{if } l = 2p+2, \, l \neq 4p \text{ is even;} \\ p & \text{if } l = 4p \text{ is even;} \end{cases} \tag{6.1.3}$$

for $\Delta = A_l$ if s is a representative in the Coxeter conjugacy class, i.e. the conjugacy class of s corresponds to the block of type A_l, then

$$d_{ij}(A_l) = 1; \tag{6.1.4}$$

in all other cases

$$d_{ij}(A_k) = \begin{cases} k+1 & \text{if } k \text{ is even;} \\ \frac{k-1}{2} + 1 & \text{if } k \text{ is odd;} \end{cases} \tag{6.1.5}$$

in all cases

$$d_{ij}(C_n) = d_{ij}(B_n) = d_{ij}(D_{v+w}(a_{w-1})) = 1,$$

where, as before, we use the notation of [18], Section 7 for (blocks of) Weyl group conjugacy classes.

Then a common multiple d' of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$, where $i, j = 1, \ldots, l$ is the lowest common multiple of the numbers $d_{ij}(X)$, $i, j = 1, \ldots, l$ for all blocks X of the conjugacy class of s.

If $\alpha'_1, \ldots \alpha'_l$ is another system of simple roots then a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$ will be also a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'_j\right)$ and vice versa.

Proof. First observe that if Δ'_+ is another system of positive roots with the simple roots $\alpha'_1, \ldots, \alpha'_l$ then $\alpha_i = \sum_{k=1}^l c_i^k \alpha'_k$, $\alpha_j^\vee = \sum_{k=1}^l b_i^k \alpha_k'^\vee$, where c_i^k, b_i^k are integer coefficients. Hence

$$\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right) = \left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j^\vee\right) = \sum_{k,p=1}^l c_i^k b_j^p \left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_k, \alpha_p'^\vee\right) = \sum_{k,p=1}^l c_i^k b_j^p \frac{1}{d_p}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_k, \alpha'_p\right),$$

and a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'_j\right)$ will be also a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$ and vice versa.

In case of classical irreducible root systems we shall compute a common multiple d' of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'_j\right)$, where Δ'_+ is chosen in such a way that s is elliptic in a parabolic Weyl subgroup $W' \subset W$ generated by the simple reflections corresponding to roots from a subset of $\alpha'_1, \ldots, \alpha'_l$ (for instance, one can take $\Delta'_+ = \Delta_+(\mathfrak{m}_{s^1}, \mathfrak{h}_{s^1})$ from the proof of Theorem 1.5.2).

Since different blocks of the conjugacy class of s correspond to different disjoint mutually orthogonal subsets of simple roots in $\alpha'_1, \ldots, \alpha'_l$ it suffices to consider the case when the conjugacy class of s corresponds to a diagram with a single nontrivial block. We shall compute d' in case when this block is of type A_k, $k > 1$. Other cases

can be considered in a similar way. Assume that the root system Δ is realized as in Section 1.5, where V is a real Euclidean n–dimensional vector space equipped with the standard scalar product, with an orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$. In that case simple roots are

$\mathbf{A_n}$

$\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \le i \le n$;

$\mathbf{B_n}$

$\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \le i < n$, $\alpha'_n = \varepsilon_n$;

$\mathbf{C_n}$

$\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \le i < n$, $\alpha'_n = 2\varepsilon_n$;

$\mathbf{D_n}$

$\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \le i < n$, $\alpha'_n = \varepsilon_{n-1} + \varepsilon_n$;

Then s is of the form

$$s = s^1 s^2, \ s^1 = s_{\alpha'_{p+1}} s_{\alpha'_{p+3}} \ldots, \ s^2 = s_{\alpha'_{p+2}} s_{\alpha'_{p+4}} \ldots,$$

where in the formulas for s^i, $i = 1, 2$ the products are taken over mutually orthogonal simple roots labeled by indexes of the same parity; the last simple root which appears in those products is $\alpha'_{p+k} = \varepsilon_{p+k} - \varepsilon_{p+k+1}$, so $\gamma_1, \ldots, \gamma_k = \alpha'_{p+1}, \alpha'_{p+3}, \ldots, \alpha'_{p+2}, \alpha'_{p+4}, \ldots$.

We have to compute the numbers $\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_i, \alpha'^\vee_j \right)$. We consider the case when $i < j$. The case when $i > j$ can be obtained from it by observing that

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_i, \alpha'^\vee_j \right) = -\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_j, \alpha'^\vee_i \right) \frac{(\alpha'_i, \alpha'_i)}{(\alpha'_j, \alpha'_j)}. \tag{6.1.6}$$

First recall that by Lemma 2.8.1

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \gamma_i, \gamma_j \right) = \varepsilon_{ij} (\gamma_i, \gamma_j), \tag{6.1.7}$$

where

$$\varepsilon_{ij} = \begin{cases} -1 & i < j \\ 0 & i = j \\ 1 & i > j \end{cases}.$$

Let ω'_t be the fundamental weights of the root subsystem $A_k \subset \Delta$ with respect to the basis of simple roots α'_i, $i = p+1, \ldots, p+k$,

$$\omega'_t = \varepsilon_{p+1} + \ldots + \varepsilon_{p+t} - \frac{t}{k+1} \sum_{j=1}^{k+1} \varepsilon_{p+j}, t = 1, \ldots, k.$$

Since α'_{p+t}, $t = 1, \ldots, k$ form a linear basis of $\mathfrak{h}'^*$, and ω'_t, $t = 1, \ldots, k$ form the dual basis we have

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_i, \alpha'^\vee_j \right) = \sum_{t,u=1}^{k} (\omega'^\vee_t, \alpha'_i) \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_{p+t}, \alpha'^\vee_{p+u} \right) (\omega'_u, \alpha'^\vee_j).$$

Since the scalar product in V is normalized in such a way that $\alpha'^\vee_{p+u} = \alpha'_{p+u}$, $u = 1, \ldots, k$ we obtain using (6.1.7)

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_i, \alpha'^\vee_j \right) = \sum_{t,u=1}^{k} (\omega'^\vee_t, \alpha'_i) \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha'_{p+t}, \alpha'_{p+u} \right) (\omega'_u, \alpha'^\vee_j) = \tag{6.1.8}$$

$$= \sum_{t=1}^{k} (-1)^t (\omega'^\vee_t, \alpha'_i) (\omega'_{t-1} + \omega'_{t+1}, \alpha'^\vee_j),$$

where we assume that $\omega_0' = \omega_{k+1}' = 0$.

Now one has to consider several cases.

If one of the roots α_i', α_j' is orthogonal to $\mathfrak{h}'^*$ then the left hand side of the last equality is zero.

If $\alpha_i', \alpha_j' \in \{\gamma_1, \ldots, \gamma_k\}$ then by (6.1.7) the left hand side of (6.1.8) is equal to ± 1.

If $\alpha_i' = \alpha_{p+t}'$, $1 < t < k$, $\alpha_j' = \alpha_{p+k+1}'$ then

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i', \alpha_j'^\vee \right) = (-1)^t (\omega_{t-1}' + \omega_{t+1}', \alpha_{p+k+1}'^\vee) = (-1)^t (\vartheta - \delta \frac{2t}{k+1}), \qquad (6.1.9)$$

where $\delta = 2$ if $\alpha_j'^\vee = 2\varepsilon_{p+k+1}$ or $\alpha_j'^\vee = \varepsilon_{p+k} + \varepsilon_{p+k+1}$, $\vartheta = 0$ in the former case, and $\vartheta = 1$ in the latter case. In all other cases $\vartheta = 0$ and $\delta = 1$. Note that $\delta \neq 1$ only in case when Δ is of type B_n or D_n; for arbitrary s this situation can only be realized if an orbit with the smallest number of elements for the action of the group $\langle s \rangle$ on E corresponds to a block of type A_k and s does not fix any root from Δ. The denominator r of the number in the right hand side of (6.1.9) is given by

$$r = \begin{cases} 2p+1 & \text{if } k = 2p \text{ is even;} \\ p+1 & \text{if } k = 2p+1, \, k \neq 4p-1 \text{ is odd;} \\ \frac{2p}{\delta} & \text{if } k = 4p-1 \text{ is odd.} \end{cases} \qquad (6.1.10)$$

If $\alpha_i' = \alpha_{p+1}'$, $\alpha_j' = \alpha_{p+k+1}'$ then

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i', \alpha_j'^\vee \right) = -(\omega_2', \alpha_{p+k+1}'^\vee) = \delta \frac{2}{k+1} - \vartheta,$$

where $\delta = 2$ if $\alpha_j'^\vee = 2\varepsilon_{p+k+1}$ or $\alpha_j'^\vee = \varepsilon_{p+k} + \varepsilon_{p+k+1}$, $\vartheta = 0$ in the former case, and $\vartheta = 1$ in the latter case if $k = 2$. In all other cases $\vartheta = 0$ and $\delta = 1$. We again obtain (6.1.10).

If $\alpha_i' = \alpha_{p+k}'$, $\alpha_j' = \alpha_{p+k+1}'$ then

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i', \alpha_j'^\vee \right) = (-1)^k (\omega_{k-1}', \alpha_{p+k+1}'^\vee) = -(-1)^k \delta \frac{k-1}{k+1},$$

and we obtain (6.1.10).

If $\alpha_i' = \alpha_p'$, $\alpha_j' = \alpha_{p+k+1}'$ then

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i', \alpha_j'^\vee \right) = \sum_{t=1}^{k} (-1)^t (\omega_t', \alpha_p'^\vee)(\omega_{t-1}' + \omega_{t+1}', \alpha_{p+k+1}'^\vee) =$$

$$= -\sum_{t=1}^{k-1} (-1)^t \left(-1 + \frac{t}{k+1} \right) \frac{2t\delta}{k+1} - (-1)^k \left(-1 + \frac{k}{k+1} \right) \frac{k-1}{k+1} \delta + (-1)^{k-1} \vartheta \left(-1 + \frac{k-1}{k+1} \right).$$

Using the fact that

$$\sum_{r=1}^{n} (-1)^{r+1} r^2 = (-1)^{n+1} \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{r=1}^{n} (-1)^{r+1} r = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd;} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

we obtain

$$\left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i', \alpha_j'^\vee \right) = \begin{cases} -\frac{\delta}{k+1} + \vartheta \frac{2}{k+1} & \text{if } k \text{ is even;} \\ -\vartheta \frac{2}{k+1} & \text{if } k \text{ is odd.} \end{cases}$$

The denominator r of the number in the right hand side of the last equality is given by

$$r = \begin{cases} 2p+1 & \text{if } k = 2p \text{ is even;} \\ 1 & \text{if } k = 2p+1, \, n \text{ is odd and } \vartheta = 0; \\ p+1 & \text{if } k = 2p+1 \text{ is odd and } \vartheta = 1. \end{cases}$$

Summarizing all cases considered above and adding the case $i > j$ (see (6.1.6)) we arrive at (6.1.1), (6.1.2), (6.1.3), (6.1.4) and (6.1.5).

Other cases can be treated in a similar way.

$\square$

Now one can choose $d = 2d'$, where d' is defined in the previous proiposition. However, we shall not always assume that d is chosen in this way and that the system of positive roots Δ_+^s will not be chosed as in Proposition 6.1.1 unless it is explicitly specified.

We shall always assume that d and m are coprime. This condition is equivalent to the existence of an integer n such that $\varepsilon^{nd-1} = 1$. From now on we shall also assume that $\kappa = nd$. With this choice of κ we have the following relation between the generators t_i and L_i of the quantum group $U_{\mathcal{A}}(\mathfrak{g})$, $t_i = L_i^n$. In particular, the specialization $U_\varepsilon(\mathfrak{g})$ of $U_{\mathcal{A}}(\mathfrak{g})$ coincides with the specialization of the simply connected form of the standard Drinfeld–Jimbo quantum group without generators t_i at $q = \varepsilon$.

Let Z_ε be the center of $U_\varepsilon(\mathfrak{g})$. In the following proposition we summarize the results on the structure of Z_ε. In particular, we recall that in case when ε is a root of unity Z_ε is much larger than in case of a generic ε. In fact in the former case Z_ε contains a remarkable subalgebra Z_0 the properties of which impose very strong restrictions on the structure of irreducible representations of $U_\varepsilon(\mathfrak{g})$.

Proposition 6.1.2. *Fix the normal ordering in the positive root system Δ_+ corresponding a reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$ of the longest element $\overline{w}$ of the Weyl group W of $\mathfrak{g}$ and let $X_\alpha^\pm$ be the corresponding quantum root vectors in $U_\varepsilon(\mathfrak{g})$, and X_α the corresponding root vectors in $\mathfrak{g}$. Let $x_\alpha^- = (\varepsilon_\alpha - \varepsilon_\alpha^{-1})^m (X_\alpha^-)^m$, $x_\alpha^+ = (\varepsilon_\alpha - \varepsilon_\alpha^{-1})^m T_0 (X_\alpha^-)^m$, where $T_0 = T_{i_1} \ldots T_{i_D}$, $\alpha \in \Delta_+$, and $l_i = L_i^m$, $i = 1, \ldots, l$.*
Then the following statements are true.
(i) The elements $x_\alpha^\pm$, $\alpha \in \Delta_+$, l_i, $i = 1, \ldots, l$ lie in Z_ε.
(ii) Let Z_0 ($Z_0^\pm$ and Z_0^0) be the subalgebras of Z_ε generated by the $x_\alpha^\pm$ and the $l_i^{\pm 1}$ (respectively by the $x_\alpha^\pm$ and by the $l_i^{\pm 1}$). Then $Z_0^\pm \subset U_\varepsilon(\mathfrak{n}_\pm)$, $Z_0^0 \subset U_\varepsilon(\mathfrak{h})$, $Z_0^\pm$ is the polynomial algebra with generators $x_\alpha^\pm$, Z_0^0 is the algebra of Laurent polynomials in the l_i, $Z_0^\pm = U_\varepsilon(\mathfrak{n}_\pm) \cap Z_0$, and multiplication defines an isomorphism of algebras

$$Z_0^- \otimes Z_0^0 \otimes Z_0^+ \to Z_0.$$

The subalgebra Z_0 is independent of the choice of the reduced decomposition $\overline{w} = s_{i_1} \ldots s_{i_D}$.
(iii) $U_\varepsilon(\mathfrak{g})$ is a free Z_0–module with basis the set of monomials $(X^+)^{\mathbf{r}} L^{\mathbf{s}} (X^-)^{\mathbf{t}}$ for which $0 \le r_k, t_k, s_i < m$ for $i = 1, \ldots, l$, $k = 1, \ldots, D$, where for $\mathbf{s} = (s_1, \ldots s_l) \in \mathbb{Z}^l$,

$$L^{\mathbf{s}} = L_1^{s_1} \ldots L_l^{s_l}.$$

(iv) $\mathrm{Spec}(Z_0) = \mathbb{C}^{2D} \times (\mathbb{C}^*)^l$ is a complex affine space of dimension equal to $\dim \mathfrak{g}$.
(v) The subalgebra Z_0 is preserved by the action of the braid group automorphisms T_i.
(vi) Let G be the connected simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}$ and G_0^* the solvable algebraic subgroup in $G \times G$ which consists of elements of the form $(L_+, L_-) \in G \times G$,

$$(L_+, L_-) = (t, t^{-1})(n_+, n_-), \ n_\pm \in N_\pm, \ t \in H.$$

Then $\mathrm{Spec}(Z_0^0)$ can be naturally identified with the maximal torus H in G, and the map

$$\widetilde{\pi} : \mathrm{Spec}(Z_0) = \mathrm{Spec}(Z_0^+) \times \mathrm{Spec}(Z_0^0) \times \mathrm{Spec}(Z_0^-) \to G_0^*,$$

$$\widetilde{\pi}(u_+, t, u_-) = (t\mathbf{X}^+(u_+), t^{-1}\mathbf{X}^-(u_-)^{-1}), \ u_\pm \in \mathrm{Spec}(Z_0^\pm), \ t \in \mathrm{Spec}(Z_0^0),$$

$$\mathbf{X}^\pm : \mathrm{Spec}(Z_0^\pm) \to N_\pm,$$

$$\mathbf{X}^- = \exp(x_{\beta_D}^- X_{-\beta_D}) \exp(x_{\beta_{D-1}}^- X_{-\beta_{D-1}}) \ldots \exp(x_{\beta_1}^- X_{-\beta_1}),$$

$$\mathbf{X}^+ = \exp(x_{\beta_D}^+ T_0(X_{-\beta_D})) \exp(x_{\beta_{D-1}}^+ T_0(X_{-\beta_{D-1}})) \ldots \exp(x_{\beta_1}^+ T_0(X_{-\beta_1})),$$

where $x_{\beta_i}^\pm$ should be regarded as complex-valued functions on $\mathrm{Spec}(Z_0)$, is an isomorphism of varieties independent of the choice of reduced decomposition of $\overline{w}$.

Parts (ii) and (iii) of Proposition 6.1.2 can also be reformulated in terms of the quantum root vectors e_α and f_α.

Proposition 6.1.3. *Let $s \in W$ be a Weyl group element, and e_α, f_α the quantum root vectors defined in Proposition 2.6.2. Then the following statements are true.*
(i) The subalgebra Z_0 is the tensor product of the polynomial algebra with generators e_α^m, f_α^m, $\alpha \in \Delta_+$ and of the algebra of Laurent polynomials in l_i, $i = 1, \ldots, l$.
(ii) $U_\varepsilon(\mathfrak{g})$ is a free Z_0–module with basis the set of monomials $f^{\mathbf{r}} L^{\mathbf{s}} e^{\mathbf{t}}$ for which $0 \le r_k, t_k, s_i < m$ for $i = 1, \ldots, l$, $k = 1, \ldots, D$.

Let $\mathbf{K} : \mathrm{Spec}(Z_0^0) \to H$ be the map defined by $\mathbf{K}(h) = h^2$, $h \in H$.

Proposition 6.1.4. *Let $G^0 = N_- H N_+$ be the big cell in G. Then the map*

$$\pi = \mathbf{X}^- \mathbf{K} \mathbf{X}^+ : \mathrm{Spec}(Z_0) \to G^0$$

is independent of the choice of reduced decomposition of $\overline{w}$, and is an unramified covering of degree 2^l.

Denote by $\lambda_0 : G_0^* \to G^0$ the map defined by $\lambda_0(L_+, L_-) = L_-^{-1} L_+$. Then obviously $\pi = \lambda_0 \circ \widetilde{\pi}$.

Another important property of quantum groups at roots of unity, which distinguishes the root of unity case, is the existence of the so–called quantum coadjoint action which is an automorphism group action on an extension of $U_\varepsilon(\mathfrak{g})$. It is defined with the help of derivations $\underline{x}_i^\pm$ of $U_{\mathcal{A}}(\mathfrak{g})$ given by

$$\underline{x}_i^+(u) = \left[\frac{(X_i^+)^m}{[m]_{q_i}!}, u \right], \quad \underline{x}_i^-(u) = T_0 \underline{x}_i^+ T_0^{-1}(u), \quad i = 1, \ldots, l, \quad u \in U_{\mathcal{A}}(\mathfrak{g}). \tag{6.1.11}$$

Let $\widehat{Z}_0$ be the algebra of formal power series in the $x_\alpha^{\pm 1}$, $\alpha \in \Delta_+$, and the $l_i^{\pm 1}$, $i = 1, \ldots, l$, which define holomorphic functions on $\mathrm{Spec}(Z_0) = \mathbb{C}^{2D} \times (\mathbb{C}^*)^l$. Let

$$\widehat{U}_\varepsilon(\mathfrak{g}) = U_\varepsilon(\mathfrak{g}) \otimes_{Z_0} \widehat{Z}_0, \quad \widehat{Z}_\varepsilon = Z_\varepsilon \otimes_{Z_0} \widehat{Z}_0.$$

Proposition 6.1.5. *(i)On specializing to $q = \varepsilon$, (6.1.11) induces a well–defined derivation $\underline{x}_i^\pm$ of $U_\varepsilon(\mathfrak{g})$.*
(ii)The series

$$\exp(t \underline{x}_i^\pm) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\underline{x}_i^\pm)^k$$

converges for all $t \in \mathbb{C}$ to a well–defined automorphism of the algebra $\widehat{U}_\varepsilon(\mathfrak{g})$.
(iii)Let $\mathcal{G}$ be the group of automorphisms generated by the one–parameter groups $\exp(t \underline{x}_i^\pm)$, $i = 1, \ldots, l$. The action of $\mathcal{G}$ on $\widehat{U}_\varepsilon(\mathfrak{g})$ preserves the subalgebras $\widehat{Z}_\varepsilon$ and $\widehat{Z}_0$, and hence $\mathcal{G}$ acts by holomorphic automorphisms on the complex algebraic varieties $\mathrm{Spec}(Z_\varepsilon)$ and $\mathrm{Spec}(Z_0)$.
(iv)Let $\mathcal{O}$ be a conjugacy class in G. The intersection $\mathcal{O}^0 = \mathcal{O} \cap G^0$ is a smooth connected variety, and the variety $\pi^{-1}(\mathcal{O}^0)$ is a $\mathcal{G}$–orbit in $\mathrm{Spec}(Z_0)$.

Given a homomorphism $\eta : Z_0 \to \mathbb{C}$, let
$$U_\eta(\mathfrak{g}) = U_\varepsilon(\mathfrak{g}) / I_\eta,$$

where I_η is the ideal in $U_\varepsilon(\mathfrak{g})$ generated by elements $z - \eta(z)$, $z \in Z_0$. By part (iii) of Proposition 6.1.2 $U_\eta(\mathfrak{g})$ is an algebra of dimension $m^{\dim \mathfrak{g}}$ with linear basis the set of monomials $(X^+)^{\mathbf{r}} L^{\mathbf{s}} (X^-)^{\mathbf{t}}$ for which $0 \leq r_k, t_k, s_i < m$ for $i = 1, \ldots, l$, $k = 1, \ldots, D$.

If $\widetilde{g} \in \mathcal{G}$ then for any $\eta \in \mathrm{Spec}(Z_0)$ we have $\widetilde{g}\eta \in \mathrm{Spec}(Z_0)$ by part (iii) of Proposition 6.1.5, and by part (ii) of the same proposition $\widetilde{g}$ induces an isomorphism of algebras,

$$\widetilde{g} : U_\eta(\mathfrak{g}) \to U_{\widetilde{g}\eta}(\mathfrak{g}). \tag{6.1.12}$$

Since on every irreducible representation of $U_\varepsilon(\mathfrak{g})$ the subalgebra Z_0 of the center Z_ε acts by a character $\eta : Z_0 \to \mathbb{C}$, every irreducible representation of $U_\varepsilon(\mathfrak{g})$ is a representation of some algebra $U_\eta(\mathfrak{g})$ for a unique η. This reduces the study of irreducible representations of $U_\varepsilon(\mathfrak{g})$ to the study of representations of finite–dimensional algebras $U_\eta(\mathfrak{g})$. Moreover, taking into account isomorphisms (6.1.12) it suffices to consider a representative in each isomorphism class of these algebras under the isomorphisms induced by the action of the elements of the group $\mathcal{G}$ on $\widehat{U}_\varepsilon(\mathfrak{g})$.

6.2 Whittaker vectors in modules over quantum groups at roots of unity

It turns out that any finite–dimensional representation V of $U_\eta(\mathfrak{g})$ can be equipped with another action of a subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ of a small quantum group which is a root of unity "truncated" version of the algebra $U_{\mathcal{A}}^s(\mathfrak{m}_-)$ for an appropriate s depending on η. The new action is compatible with the original action of $U_\eta(\mathfrak{g})$ in a certain

equivariant way, and the dimension of $U_{\eta_1}(\mathfrak{m}_-)$ is equal to $b = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}} = m^{\dim \mathfrak{m}_-}$, where $\mathcal{O}_{\pi\eta}$ is the conjugacy class of $\pi\eta \in G$. The existence of the second action is crucial for the proof of the De Concini–Kac–Procesi conjecture and for the study of other properties of finite–dimensional representations of $U_\eta(\mathfrak{g})$.

In this section we define the algebras $U_{\eta_1}(\mathfrak{m}_-)$ and their actions on finite–dimensional representations of $U_\eta(\mathfrak{g})$. These definitions are related to the notion of Whittaker vectors for finite–dimensional $U_\eta(\mathfrak{g})$–modules which are defined using root of unity versions of characters χ_q^s. We start by reminding the definitions of these characters.

Firstly let us observe that $U_\varepsilon^s(\mathfrak{m}_-)$ can be regarded as a subalgebra in $U_\varepsilon(\mathfrak{g})$. Therefore for every character $\eta : Z_0 \to \mathbb{C}$ one can define the corresponding subalgebra in $U_\eta(\mathfrak{g})$ generated by f_α, $\alpha \in \Delta_{\mathfrak{m}_+}$. We denote this subalgebra by $U_\eta(\mathfrak{m}_-)$. By part (ii) of Proposition 6.1.3 we have $\dim U_\eta(\mathfrak{m}_-) = m^{\dim \mathfrak{m}_-}$.

In order to define analogues of characters χ_q^s for quantum groups at roots of unity we shall need some properties of the finite dimensional algebras $U_\eta(\mathfrak{g})$ and $U_\eta(\mathfrak{m}_-)$ and auxiliary results on non–zero irreducible representations of the algebra $U_\eta(\mathfrak{m}_-)$.

Observe that by Proposition 1.6.6 for any two roots $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ can not be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \ldots < \gamma_{i_k} < \beta$, and hence from commutation relations (2.8.12) one can deduce that

$$f_\alpha f_\beta - \varepsilon^{(\alpha,\beta) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'*}\alpha, \beta)} f_\beta f_\alpha = \sum_{\alpha < \delta_1 < \ldots < \delta_n < \beta} C'(k_1, \ldots, k_n) f_{\delta_n}^{k_n} f_{\delta_{n-1}}^{k_{n-1}} \ldots f_{\delta_1}^{k_1} \in \mathcal{J}, \qquad (6.2.1)$$

where at least one of the roots δ_i in the right hand side of the last formula belongs to $\Theta = \{\alpha \in \Delta_{\mathfrak{m}_+} : \alpha \notin \{\gamma_1, \ldots, \gamma_{l'}\}\}$, $\mathcal{J}$ is the ideal in $U_\eta(\mathfrak{m}_-)$ generated by the elements $f_\beta \in U_\eta(\mathfrak{m}_-)$, $\beta \in \Theta$. Thus from part (ii) of Proposition 6.1.3 and commutation relations (6.2.1) it follows that if $\beta_1 < \beta_2 < \ldots < \beta_c$ are the roots in the segment $\Delta_{\mathfrak{m}_+}$, the elements

$$x_{k_1, \ldots, k_b} = f_{\beta_c}^{k_c} f_{\beta_{c-1}}^{k_{c-1}} \ldots f_{\beta_1}^{k_1} \qquad (6.2.2)$$

for $k_i \in \mathbb{N}$, $k_i < m$ form a linear basis of $U_\eta(\mathfrak{m}_-)$, and elements (6.2.2) for $k_i \in \mathbb{N}$, $k_i < m$ and $k_i > 0$ for at least one $\beta_i \in \Theta$ form a linear basis of $\mathcal{J}$.

Lemma 6.2.1. *Let η be an element of $\mathrm{Spec}(Z_0)$. Assume that $\eta(f_{\gamma_i}^m) = a_i \neq 0$ for $i = 1, \ldots, l'$ and that and $\eta(f_\beta^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$, and hence $f_{\gamma_i}^m = \eta(f_{\gamma_i}^m) = a_i \neq 0$ in $U_\eta(\mathfrak{m}_-)$ for $i = 1, \ldots, l'$ and $f_\beta^m = 0$ in $U_\eta(\mathfrak{m}_-)$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$. Then the ideal $\mathcal{J}$ is the Jacobson radical of $U_\eta(\mathfrak{m}_-)$ and $U_\eta(\mathfrak{m}_-)/\mathcal{J}$ is isomorphic to the truncated polynomial algebra*

$$\mathbb{C}[f_{\gamma_1}, \ldots, f_{\gamma_{l'}}]/\{f_{\gamma_i}^m = a_i\}_{i=1, \ldots, l'}$$

.

Proof. First we show that $\mathcal{J}$ is nilpotent.

Let i be the largest number such that $k_j = 0$ for $j > i$ in (6.2.2) and $k_i \neq 0$. Then we define the degree of $x_{k_1, \ldots, k_c}$ by

$$deg(x_{k_1, \ldots, k_c}) = (k_i, i) \in \{1, \ldots, m-1\} \times \{1, \ldots, c\}.$$

Equip $\{1, \ldots, m-1\} \times \{1, \ldots, c\}$ with the order such that $(k, i) < (k', j)$ if $j > i$ or $j = i$ and $k' > k$.

For any given $(k, i) \in \{1, \ldots, m-1\} \times \{1, \ldots, c\}$ denote by $(U_\eta(\mathfrak{m}_-))_{(k,i)}$ the linear span of the elements $x_{k_1, \ldots, k_c}$ with $deg(x_{k_1, \ldots, k_c}) \leq (k, i)$ and define $\mathcal{J}_{(k,i)} = \mathcal{J} \cap (U_\eta(\mathfrak{m}_-))_{(k,i)}$. We also have $(U_\eta(\mathfrak{m}_-))_{(k,i)} \subset (U_\eta(\mathfrak{m}_-))_{(k',j)}$ and $\mathcal{J}_{(k,i)} \subset \mathcal{J}_{(k',j)}$ if $(k, i) < (k', j)$, and $\mathcal{J}_{(m-1,c)} = \mathcal{J}$. Note that for the first few i linear spaces $\mathcal{J}_{(k,i)}$ may be trivial, and these are all possibilities when those spaces can be trivial.

We shall prove that $\mathcal{J}$ is nilpotent by induction over the order in $\{1, \ldots, m-1\} \times \{1, \ldots, c\}$. Let (k, i) be minimal possible such that $\mathcal{J}_{(k,i)}$ is not trivial. Then we must have $k = 1$. If $y \in \mathcal{J}_{(1,i)}$ then y must be of the form

$$y = f_\beta v, \qquad (6.2.3)$$

where v is a linear combination of elements of the form $f_{\beta_{i_1}}^{k_1} \ldots f_{\beta_{i_r}}^{k_r}$ for $\beta_{i_1}, \ldots, \beta_{i_r} \in \{\gamma_1, \ldots, \gamma_n\}$, $\beta > \beta_{i_1}$, and β is the first root from the set Θ greater than γ_1 in the normal ordering of Δ_+ associated to s. Here it is assumed that $f_{\beta_{i_1}}^{k_1} \ldots f_{\beta_{i_r}}^{k_r} = 1$ if the set $\{\gamma_1, \ldots, \gamma_n\}$ is empty.

Now equation (6.2.1) implies that for any $f_{\beta_{i_j}}$ which appears in the expression for v one has

$$f_\beta f_{\beta_{i_j}} - \varepsilon^{(\beta, \beta_{i_j}) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'*}\beta, \beta_{i_j})} f_{\beta_{i_j}} f_\beta \in \mathcal{J}_{(m-1, i-1)} = 0 \qquad (6.2.4)$$

as by our choice of i $\mathcal{J}_{(m-1,i-1)} = 0$.

Formula (6.2.4) implies that the product of m elements of type (6.2.3) can be represented in the form

$$f_\beta^m v',$$

where v' is of the same form as v. Since $f_\beta^m = 0$ we deduce that $\mathcal{J}_{(1,i)}^m = 0$.

Now assume that $\mathcal{J}_{(k,i)}^K = 0$ for some $K > 0$. Let (k', i') be the smallest element of $\{1, \dots, m-1\} \times \{1, \dots, c\}$ which satisfies $(k, i) < (k', i')$. Then by Propositions 2.6.2 and 6.1.3, and by (6.2.1), any element of $\mathcal{J}_{(k',i')}$ is of the form $f_{\beta_{i'}} u + u'$, where $u' \in \mathcal{J}_{(k,i)}$ and if $\beta_{i'} \in \Theta$ then $u \in (U_\eta(\mathfrak{m}_-))_{(k,i)}$; if $\beta_{i'} \notin \Theta$ then $u \in \mathcal{J}_{(k,i)}$.

Now equation (6.2.1) together with (2.8.12) imply that for any $u \in (U_\eta(\mathfrak{m}_-))_{(k,i)}$ one has

$$u f_{\beta_{i'}} = c f_{\beta_{i'}} u + w, \tag{6.2.5}$$

where c is a non–zero constant depending on u, and $w \in \mathcal{J}_{(k,i)}$. By formula (6.2.5) the product of m elements $f_{\beta_{i'}} u_p + u'_p$, $p = 1, \dots, m$ of the type described above can be represented in the form

$$\sum_{j=0}^{m} f_{\beta_{i'}}^j c_j, \tag{6.2.6}$$

where $c_j \in \mathcal{J}_{(k,i)}$ for $j = 0, \dots, m-1$ and if $\beta_{i'} \in \Theta$ then $c_m \in (U_\eta(\mathfrak{m}_-))_{(k,i)}$; if $\beta_{i'} \notin \Theta$ then $c_m \in \mathcal{J}_{(k,i)}$. In the former case $f_{\beta_{i'}}^m = 0$, and the last term in sum (6.2.6) is zero; in the latter case $f_{\beta_{i'}}^m = \eta(f_{\beta_{i'}}^m) \neq 0$, and the last term in sum (6.2.6) is from $\mathcal{J}_{(k,i)}$. So we can combine it with the term corresponding to $j = 0$. In both cases sum (6.2.6) takes the form

$$\sum_{j=0}^{m-1} f_{\beta_{i'}}^j c'_j, \tag{6.2.7}$$

where $c'_j \in \mathcal{J}_{(k,i)}$. By (6.2.5) the product of K sums of type (6.2.7) is of the form

$$\sum_{j=0}^{(m-1)K} f_{\beta_{i'}}^j c''_j,$$

where each c''_j is a linear combination of elements from $\mathcal{J}_{(k,i)}^K$. By our assumption $\mathcal{J}_{(k,i)}^K = 0$, and hence the product of any mK elements of $\mathcal{J}_{(k',i')}$ is zero. This justifies the induction step and proves that $\mathcal{J}_{(m-1,c)} = \mathcal{J}$ is nilpotent. Hence $\mathcal{J}$ is contained in the Jacobson radical of $U_\eta(\mathfrak{m}_-)$.

Using commutation relations (2.8.12) we also have (see the proof of Proposition 3.2.9)

$$f_{\gamma_i} f_{\gamma_j} - f_{\gamma_j} f_{\gamma_i} \in \mathcal{J}.$$

Therefore the quotient algebra $U_\eta(\mathfrak{m}_-)/\mathcal{J}$ is isomorphic to the truncated polynomial algebra

$$\mathbb{C}[f_{\gamma_1}, \dots, f_{\gamma_{l'}}]/\{f_{\gamma_i}^m = a_i\}_{i=1,\dots,l'}$$

which is semisimple. Therefore $\mathcal{J}$ coincides with the Jacobson radical of $U_\eta(\mathfrak{m}_-)$. $\square$

In Proposition 3.2.9 we constructed some characters of the algebra $U_\varepsilon^s(\mathfrak{m}_-)$. Now we show that the algebra $U_\eta(\mathfrak{m}_-)$ has a finite number of irreducible representations which are one–dimensional, and all those representations can be obtained from each other by twisting with the help of automorphisms of $U_\eta(\mathfrak{m}_-)$.

Proposition 6.2.2. *Let η be an element of $\mathrm{Spec}(Z_0)$. Assume that $\eta(f_\beta^m) = a_i \neq 0$ for $i = 1, \dots, l'$ and that and $\eta(f_\beta^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \dots, \gamma_{l'}\}$, and hence $f_{\gamma_i}^m = \eta(f_{\gamma_i}^m) = a_i \neq 0$ in $U_\eta(\mathfrak{m}_-)$ for $i = 1, \dots, l'$ and $f_\beta^m = 0$ in $U_\eta(\mathfrak{m}_-)$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \dots, \gamma_{l'}\}$. Then all non–zero irreducible representations of the algebra $U_\eta(\mathfrak{m}_-)$ are one–dimensional and have the form*

$$\chi(f_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \dots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i, \ i = 1, \dots, l' \end{cases}, \tag{6.2.8}$$

where complex numbers c_i satisfy the conditions $c_i^m = a_i$, $i = 1, \dots, l'$. Moreover, all non–zero irreducible representations of $U_\eta(\mathfrak{m}_-)$ can be obtained from each other by twisting with the help of automorphisms of $U_\eta(\mathfrak{m}_-)$.

Proof. Let V be a non–zero finite–dimensional irreducible $U_\eta(\mathfrak{m}_-)$–module. By Corollary 54.13 in [31] elements of the ideal $\mathcal{J} \subset U_\eta(\mathfrak{m}_-)$ act by zero transformations on V. Hence V is in fact an irreducible representation of the algebra $U_\eta(\mathfrak{m}_-)/\mathcal{J}$ which is isomorphic to the truncated polynomial algebra

$$\mathbb{C}[f_{\gamma_1}, \ldots, f_{\gamma_{l'}}]/\{f_{\gamma_i}^m = a_i\}_{i=1,\ldots,l'}.$$

The last algebra is commutative and all its complex irreducible representations are one–dimensional. Therefore V is one–dimensional, and if v is a nonzero element of V then $f_{\gamma_i}v = c_i v$, for some $c_i \in \mathbb{C}$, $i = 1, \ldots, l'$. Note that $\eta(f_{\gamma_i}^m) = a_i \neq 0$, $i = 1, \ldots, l'$ and hence $c_i^m = a_i \neq 0$, $i = 1, \ldots, l'$. In particular, the elements f_{γ_i} act on V by semisimple automorphisms.

If we denote by $\chi : U_\eta(\mathfrak{m}_-) \to \mathbb{C}$ the character of $U_\eta(\mathfrak{m}_-)$ such that

$$\chi(f_\beta) = \left\{ \begin{array}{ll} 0 & \beta \notin \{\gamma_1, \ldots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i, \ i = 1, \ldots, l' \end{array} \right.$$

and by $\mathbb{C}_\chi$ the corresponding one–dimensional representation of $U_\eta(\mathfrak{m}_-)$ then we have $V = \mathbb{C}_\chi$.

Now we have to prove that the representations $\mathbb{C}_\chi$ for different characters χ are obtained from each other by twisting with the help of automorphisms of $U_\eta(\mathfrak{m}_-)$.

Since $c_i^m = a_i$, $i = 1, \ldots, l'$ there are only finitely many possible characters χ corresponding to the given η in the statement of this proposition. If χ and χ' are two such characters, $\chi(f_{\gamma_i}) = c_i$, $i = 1, \ldots, l'$ and $\chi'(f_{\gamma_i}) = c_i'$, $i = 1, \ldots, l'$ then the relations $c_i^m = c_i'^m = a_i$, $i = 1, \ldots, l'$ imply that $c_i' = \varepsilon^{m_i} c_i$, $0 \leq m_i \leq m - 1$, $m_i \in \mathbb{Z}$, $i = 1, \ldots, l'$.

Now observe that for any $h \in \mathfrak{h}$ the map defined by $f_\alpha \mapsto \varepsilon^{\alpha(h)} f_\alpha$, $\alpha \in \Delta_{\mathfrak{m}_+}$ is an automorphism of the algebra $U_\varepsilon^s(\mathfrak{m}_-)$ generated by elements f_α, $\alpha \in \Delta_{\mathfrak{m}_+}$ with defining relations (2.8.12). Here the principal branch of the analytic function ε^z is used to define $\varepsilon^{\alpha(h)}$, so that $\varepsilon^{\alpha(h)}\varepsilon^{\beta(h)} = \varepsilon^{(\alpha+\beta)(h)}$ for any $\alpha, \beta \in \Delta_{\mathfrak{m}_+}$. If in addition $\varepsilon^{m\gamma_i(h)} = 1$, $i = 1, \ldots, l'$ the above defined map gives rise to an automorphism ς of $U_\eta(\mathfrak{m}_-)$. Indeed in that case $(\varepsilon^{\gamma_i(h)} f_{\gamma_i})^m = f_{\gamma_i}^m$, $i = 1, \ldots, l'$ and all the remaining defining relations $f_{\gamma_i}^m = \eta(f_{\gamma_i}^m) = a_i \neq 0$, $i = 1, \ldots, l'$, $f_\beta^m = \eta(f_\beta^m) = 0$, $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$ of the algebra $U_\eta(\mathfrak{m}_-)$ are preserved by the action of the above defined map ς.

Now fix $h \in \mathfrak{h}$ such that $\gamma_i(h) = m_i$, $i = 1, \ldots, l'$. Obviously we have $\varepsilon^{mm_i} = 1$, $i = 1, \ldots, l'$. We claim that the representation $\mathbb{C}_\chi$ twisted by the corresponding automorphism ς coincides with $\mathbb{C}_{\chi'}$. Indeed, we obtain

$$\chi(\varsigma f_{\gamma_i}) = \chi(\varepsilon^{m_i} f_{\gamma_i}) = \varepsilon^{m_i} c_i = c_i', \ i = 1, \ldots, l'.$$

This completes the proof of the proposition.

$\square$

Now we can define the notion of Whittaker vectors. Let V be a $U_\eta(\mathfrak{g})$–module, where η is an element of $\mathrm{Spec}(Z_0)$ such that $\eta(f_{\gamma_i}^m) = a_i \neq 0$ for $i = 1, \ldots, l'$ and that and $\eta(f_\beta^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$. Let $\chi : U_\eta(\mathfrak{m}_-) \to \mathbb{C}$ be a character defined in the Proposition 6.2.2, $\mathbb{C}_\chi$ the corresponding one–dimensional $U_\eta(\mathfrak{m}_-)$–module. Then the space $V_\chi = \mathrm{Hom}_{U_\eta(\mathfrak{m}_-)}(\mathbb{C}_\chi, V)$ is called the space of Whittaker vectors of V. Elements of V_χ are called Whittaker vectors.

Now we describe the space of Whittaker vectors in terms of a nilpotent action of the unital subalgebra $U_{\eta_1}(\mathfrak{m}_-)$ generated by f_α, $\alpha \in \Delta_{\mathfrak{m}_+}$ in the small quantum group $U_{\eta_1}(\mathfrak{g}) = U_\varepsilon^s(\mathfrak{g})/I_{\eta_1}$ corresponding to the trivial central character η_1 such that $\widetilde{\pi}(\eta_1) = 1 \in G_0^*$ and $\eta_1(x_\alpha^\pm) = 0$, $\alpha \in \Delta_+$, $\eta_1(l_i) = 1$, $i = 1, \ldots, l$.

Recall that $U_\varepsilon^s(\mathfrak{m}_-)$ is a right coideal in $U_\varepsilon^s(\mathfrak{g})$. One can also equip the algebra $U_\varepsilon^s(\mathfrak{m}_-)$ with a character given by formula (3.2.17), where the numbers c_i are the same as in the definition of the character χ. We denote this character by the same letter, $\chi : U_\varepsilon^s(\mathfrak{m}_-) \to \mathbb{C}$.

Note that V can be regarded as a $U_\varepsilon(\mathfrak{g})$–module and a $U_\varepsilon^s(\mathfrak{g})$–module assuming that the ideal I_η acts on V in the trivial way. Now observe that $\Delta_s : U_\varepsilon^s(\mathfrak{m}_-) \to U_\varepsilon^s(\mathfrak{g}) \otimes U_\varepsilon^s(\mathfrak{m}_-)$ is a homomorphism of algebras. Composing it with the tensor product $S_s \otimes \chi$ of the anti–homomorphism S_s and of the character χ, which can be regarded as an anti–homomorphism as well, one can define an anti–homomorphism, $U_\varepsilon^s(\mathfrak{m}_-) \to U_\varepsilon^s(\mathfrak{g})$, $x \mapsto S_s(x_1)\chi(x_2)$, $\Delta_s(x) = x_1 \otimes x_2$, $x \in U_\varepsilon^s(\mathfrak{m}_-)$.

Using this anti–homomorphism one can introduce a right $U_\varepsilon^s(\mathfrak{m}_-)$–action on V which we call the adjoint action and denote it by Ad. It is given by the formula

$$\mathrm{Ad}_s\, xv = S_s(x_1)\chi(x_2)v, x \in U_\varepsilon^s(\mathfrak{m}_-), v \in V, \tag{6.2.9}$$

where $\Delta_s(x) = x_1 \otimes x_2$.

Note that using the Swedler notation for the comultiplication, $(\Delta_s \otimes id \otimes id)(\Delta_s \otimes id)\Delta_s(x) = x_1 \otimes x_2 \otimes x_3 \otimes x_4$, the coassiciativity of the comultiplication and the definition of the antipode we have for any $x \in U_\varepsilon^s(\mathfrak{m}_-)$, $y \in U_\varepsilon^s(\mathfrak{g})$, $v \in V$ (compare with the proof of Lemma 2.2 in [60])

$$\mathrm{Ad}_s x(yv) = S_s(x_1)\chi(x_2)yv = S_s(x_1)yx_2 S_s(x_3)\chi(x_4)v = \mathrm{Ad}_s x_1(y)\mathrm{Ad}_s x_2(v). \tag{6.2.10}$$

We shall need the following formula for the action of the comultiplication on the quantum root elements f_β, $\beta = \sum_{i=1}^l c_i \alpha_i \in \Delta_{\mathfrak{m}_+}$, $c_i \in \mathbb{N}$,

$$\Delta_s(f_\beta) = \prod_{i=1}^l K_i^{c_i} \prod_{i,j=1}^l L_j^{-\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'*}\alpha_i,\alpha_j)c_i} \otimes f_\beta + f_\beta \otimes 1 + \tag{6.2.11}$$
$$+ \sum_i y_i \otimes x_i, \ \ x_i \in U_{<\beta}, y_i \in U_{>\beta}U_\varepsilon^{s^{-1}}(\mathfrak{h}),$$

where $U_{<\beta}$ is the subalgebra (without unit) in $U_\varepsilon^s(\mathfrak{m}_-)$ generated by f_α, $\alpha < \beta$ and $U_{>\beta}$ is the subalgebra (without unit) in $U_\varepsilon^s(\mathfrak{n}_-)$ generated by f_α, $\alpha > \beta$. Formula (6.2.11) is a straightforward consequence of (2.8.13).

Similarly to the Proposition in Section 5.6 in [25] we infer that Z_0 is a Hopf subalgebra in $U_\varepsilon^s(\mathfrak{g})$. Namely,

$$\Delta_s(f_i^m) = K_i^m \prod_{j=1}^l L_j^{-m\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'*}\alpha_i,\alpha_j)} \otimes f_i^m + f_i^m \otimes 1,$$

$$\Delta_s(e_i^m) = e_i^m \otimes K_i^{-m} + \prod_{j=1}^l L_j^{m\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'*}\alpha_i,\alpha_j)} \otimes e_i^m,$$

$$\Delta_s(L_i^m) = L_i^m \otimes L_i^m.$$

Therefore recalling that by the definition of χ for $x \in U_\varepsilon^s(\mathfrak{m}_-) \cap Z_0$ one has $\chi(x) = \eta(x)$ we deduce

$$\mathrm{Ad}_s \, xv = S_s(x_1)\chi(x_2)v = \eta(S_s(x_1)x_2)v = \varepsilon_s(x)v, v \in V,$$

where ε_s is the counit of $U_\varepsilon^s(\mathfrak{g})$. Note that by the definition of the ideal I_{η_1} the ideal $U_\varepsilon^s(\mathfrak{m}_-) \cap I_{\eta_1} \subset U_\varepsilon^s(\mathfrak{m}_-)$ is generated by the elements f_α^m, $\alpha \in \Delta_{\mathfrak{m}_+}$ and $\varepsilon_s(f_\alpha^m) = 0$ for $\alpha \in \Delta_{\mathfrak{m}_+}$ by the definition of ε_s. Hence the adjoint action of $U_\varepsilon^s(\mathfrak{m}_-)$ on V induces an action of the subalgebra $U_{\eta_0}(\mathfrak{m}_-)$ of the small quantum group $U_{\eta_0}(\mathfrak{g})$. We call this action the adjoint action as well.

Note that the small quantum group $U_{\eta_0}(\mathfrak{g})$ is a Hopf algebra with the comultiplication inherited from $U_\varepsilon^s(\mathfrak{g})$.

The space of Whittaker vectors V_χ can be characterized in terms of the adjoint action as follows.

Lemma 6.2.3. *The space of Whittaker vectors V_χ coincides with the space of $U_{\eta_1}(\mathfrak{m}_-)$–invariants for the adjoint action on V,*

$$V_\chi = \{v \in V : \mathrm{Ad}_s \, x(v) = \varepsilon_s(x)v \quad \forall x \in U_{\eta_1}(\mathfrak{m}_-)\}. \tag{6.2.12}$$

Proof. Indeed, denote by T_β the factor $\prod_{i=1}^l K_i^{c_i} \prod_{i,j=1}^l L_j^{-\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'*}\alpha_i,\alpha_j)c_i}$ which appears in (6.2.11), $T_\beta = \prod_{i=1}^l K_i^{c_i} \prod_{i,j=1}^l L_j^{-\frac{nd}{d_j}(\frac{1+s}{1-s}P_{\mathfrak{h}'*}\alpha_i,\alpha_j)c_i}$. Then simiularly to (2.8.18) we obtain

$$S_s(f_\beta) = -S_s(T_\beta)f_\beta - \sum_i S_s(y_i)x_i. \tag{6.2.13}$$

Now for $\beta \in \Delta_{\mathfrak{m}_+}$, (6.2.11), (6.2.13) and definition (6.2.9) of the adjoint action imply

$$\mathrm{Ad}_s \, f_\beta v = T_\beta^{-1}\chi(f_\beta)v - T_\beta^{-1}f_\beta v - \sum_i S_s(y_i)x_i v + \sum_i S_s(y_i)\chi(x_i)v =$$

$$= T_\beta^{-1}(\chi(f_\beta) - f_\beta)v + \sum_i S_s(y_i)(\chi(x_i) - x_i)v, \ x_i \in U_{<\beta}, y_i \in U_{>\beta}U_\varepsilon^s(\mathfrak{h}). \tag{6.2.14}$$

If $v \in V_\chi$ we immediately obtain from (6.2.14) that $\mathrm{Ad}_s \, f_\beta v = 0$ for any $\beta \in \Delta_{\mathfrak{m}_+}$, i.e. v belongs to the right hand side of (6.2.12).

Conversely, suppose that v belongs to the right hand side of (6.2.12). We shall show that $xv = \chi(x)v$ for any $x \in U_\varepsilon^s(\mathfrak{m}_-)$. Let $\overline{U}_{<\beta}$ be the subalgebra with unit generated by $U_{<\beta}$. We proceed by induction over the subalgebras $\overline{U}_{<\delta_k}$, $k = 1, \ldots b+1$, where as before $\delta_1 < \ldots < \delta_b$ is the normally ordered segment $\Delta_{\mathfrak{m}_+}$ and we define $\overline{U}_{<\delta_{b+1}}$ to be the subalgebra $U_\varepsilon^s(\mathfrak{m}_-)$.

Observe that δ_1 is a simple root and hence $U_{<\delta_1} = 0$. Therefore we deduce from (6.2.14) for $\beta = \delta_1$

$$\mathrm{Ad}_s\ f_{\delta_1}v = T_{\delta_1}^{-1}(\chi(f_{\delta_1}) - f_{\delta_1})v = 0.$$

Since $T_{\delta_1}^{-1}$ acts on V by an invertible transformation this implies $(\chi(f_{\delta_1}) - f_{\delta_1})v = 0$, and hence $xv = \chi(x)v$ for any $x \in \overline{U}_{<\delta_2}$ as $\overline{U}_{<\delta_2}$ is generated by f_{δ_1}.

Now assume that for some $k \leq b$ $xv = \chi(x)v$ for any $x \in \overline{U}_{<\delta_k}$. Then by (6.2.14)

$$\mathrm{Ad}_s\ f_{\delta_k}v = T_{\delta_k}^{-1}(\chi(f_{\delta_k}) - f_{\delta_k})v = 0.$$

As above this implies

$$(\chi(f_{\delta_k}) - f_{\delta_k})v = 0. \tag{6.2.15}$$

By Proposition 2.6.2 any element $y \in \overline{U}_{<\delta_{k+1}}$ can be uniquely represented in the form $y = f_{\delta_k}y' + y''$, where $y', y'' \in \overline{U}_{<\delta_k}$. Now by (6.2.15) and by the induction assumption

$$yv = (f_{\delta_k}y' + y'')v = \chi(f_{\delta_k})\chi(y')v + \chi(y'')v = \chi(y)v,$$

i.e. $yv = \chi(y)v$ for any $x \in \overline{U}_{<\delta_{k+1}}$. This establishes the induction step and completes the proof.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following proposition is an analogue of the Engel theorem for quantum groups at roots of unity.

Proposition 6.2.4. *Let η be an element of $\mathrm{Spec}(Z_0)$. Assume that $\eta(f_\beta^m) = a_i \neq 0$ for $i = 1, \ldots, l'$ and that $\eta(f_\beta^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$, and hence $f_{\gamma_i}^m = \eta(f_{\gamma_i}^m) = a_i \neq 0$ in $U_\eta(\mathfrak{m}_-)$ for $i = 1, \ldots, l'$ and $f_\beta^m = 0$ in $U_\eta(\mathfrak{m}_-)$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$. Let $\chi : U_\eta(\mathfrak{m}_-) \to \mathbb{C}$ be any character defined in Proposition 6.2.2. Then any non–zero finite–dimensional $U_\eta(\mathfrak{g})$–module contains a non–zero Whittaker vector.*

Proof. We begin the proof with the following lemma.

Lemma 6.2.5. *The augmentation ideal $\mathcal{J}^1$ of $U_{\eta_1}(\mathfrak{m}_-)$ coincides with its Jacobson radical which is nilpotent.*

Proof. The proof of this fact is similar to that of Lemma 6.2.1, and we shall keep the notation used in that proof.

We define $\mathcal{J}_{(k,i)}^1 = \mathcal{J}^1 \cap (U_{\eta_1}(\mathfrak{m}_-))_{(k,i)}$, so that $\mathcal{J}_{(k,i)}^1 \subset \mathcal{J}_{(k',j)}^1$ if $(k,i) < (k',j)$, and $\mathcal{J}_{(m-1,c)}^1 = \mathcal{J}^1$.

We shall prove that $\mathcal{J}^1$ is nilpotent by induction over the order in $\{1, \ldots, m-1\} \times \{1, \ldots, c\}$. Note that $(k,i) = (1,1)$ is minimal possible such that $\mathcal{J}_{(k,i)}$ is not trivial. If $y \in \mathcal{J}_{(1,1)}$ then y must be of the form

$$y = af_{\beta_1}, a \in \mathbb{C}. \tag{6.2.16}$$

The product of m elements of type (6.2.16) is equal to zero,

$$f_{\beta_1}^m a^m = 0,$$

as $f_{\beta_1}^m = 0$ in $U_{\eta_1}(\mathfrak{m}_-)$. We deduce that $(\mathcal{J}_{(1,1)}^1)^m = 0$.

Now assume that $(\mathcal{J}_{(k,i)}^1)^K = 0$ for some $K > 0$. Let (k', i') be the smallest element of $\{1, \ldots, m-1\} \times \{1, \ldots, c\}$ which satisfies $(k,i) < (k', i')$. Then by Propositions 2.6.2 and 6.1.3 any element of $\mathcal{J}_{(k',i')}^1$ is of the form $f_{\beta_{i'}}u + u'$, where $u' \in \mathcal{J}_{(k,i)}^1$ and $u \in (U_{\eta_1}(\mathfrak{m}_-))_{(k,i)}$.

Now equation (2.8.12) implies that for any $u \in (U_{\eta_1}(\mathfrak{m}_-))_{(k,i)}$ one has

$$uf_{\beta_{i'}} = bf_{\beta_{i'}}u + w, \tag{6.2.17}$$

where b is a non–zero constant depending on u, and $w \in \mathcal{J}_{(k,i)}^1$. By the formula (6.2.17) the product of m elements $f_{\beta_{i'}}u_p + u'_p$, $p = 1, \ldots, m$ of the type described above can be represented in the form

$$\sum_{j=0}^{m} f_{\beta_{i'}}^j c_j, \tag{6.2.18}$$

where $c_j \in \mathcal{J}^1_{(k,i)}$ for $j = 0, \ldots, m-1$ and $c_m \in (U_{\eta_1}(\mathfrak{m}_-))_{(k,i)}$. Since $f^m_{\beta_{i'}} = 0$ the last term in sum (6.2.18) is zero. So sum (6.2.18) takes the form

$$\sum_{j=0}^{m-1} f^j_{\beta_{i'}} c'_j, \tag{6.2.19}$$

where $c'_j \in \mathcal{J}^1_{(k,i)}$. By (6.2.17) the product of K sums of type (6.2.19) is of the form

$$\sum_{j=0}^{(m-1)K} f^j_{\beta_{i'}} c''_j,$$

where each c''_j is a linear combination of elements from $(\mathcal{J}^1_{(k,i)})^K$. By our assumption $(\mathcal{J}^1_{(k,i)})^K = 0$, and hence the product of any mK elements of $\mathcal{J}^1_{(k',i')}$ is zero. This justifies the induction step and proves that $\mathcal{J}^1_{(m-1,c)} = \mathcal{J}^1$ is nilpotent. Hence $\mathcal{J}^1$ is contained in the Jacobson radical of $U_{\eta_1}(\mathfrak{m}_-)$.

The quotient algebra $U_{\eta_1}(\mathfrak{m}_-)/\mathcal{J}^1$ is isomorphic to $\mathbb{C}$. Therefore $\mathcal{J}^1$ coincides with the Jacobson radical of $U_{\eta_1}(\mathfrak{m}_-)$.

$\square$

Now let V be a finite–dimensional $U_\eta(\mathfrak{g})$–module. Then V is also a finite–dimensional $U_{\eta_1}(\mathfrak{m}_-)$–module with respect to the adjoint action. Thus V must contain a non–trivial irreducible $U_{\eta_1}(\mathfrak{m}_-)$–submodule with respect to the adjoint action on which the Jacobson radical $\mathcal{J}^1$ must act trivially. From (6.2.12) it follows that this non–trivial irreducible submodule consists of Whittaker vectors. This completes the proof of the proposition.

$\square$

Now we show that for any $\eta \in \operatorname{Spec}(Z_0)$ subalgebras and characters which appear in Propositions 6.2.2, 6.2.4 and in Lemma 6.2.3 indeed exist. Moreover, we shall see that to each $\eta \in \operatorname{Spec}(Z_0)$ one can associate a subalgebra of this type the dimension of which is equal to $m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}}$, where $\mathcal{O}_{\pi\eta}$ is the conjugacy class of $\pi\eta \in G_\mathbb{C}$.

Proposition 6.2.6. *Let*

$$G = \bigcup_{\mathcal{C} \in C(W)} G_\mathcal{C}.$$

be the Lusztig partition of G, $\eta \in \operatorname{Spec}(Z_0)$ be an element such that $\pi\eta \in G_\mathcal{C}$, $\mathcal{C} \in C(W)$ and $s^{-1} \in \mathcal{C}$. Let Δ^s_+ be the system of positive roots defined for s in Theorem 1.5.2, Δ_+ the corresponding system of positive roots associated to s, $d = 2d'$, where d' is defined in Proposition 6.1.1. Assume that m and d are coprime.

Then there exists a quantum coadjoint transformation $\widetilde{g}$ such that $\xi = \widetilde{g}\eta$ satisfies $\xi(f^m_{\gamma_i}) = a_i \neq 0$ for $i = 1, \ldots, l'$ and $\xi(f^m_\beta) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$, where $f_\alpha \in U_\xi(\mathfrak{m}_-)$ are generators of the corresponding algebra $U_\xi(\mathfrak{m}_-) \subset U_\xi(\mathfrak{g})$. Let $\chi : U_\xi(\mathfrak{m}_-) \to \mathbb{C}$ be any character defined in Proposition 6.2.2. Then any finite–dimensional $U_\xi(\mathfrak{g})$–module contains a non–zero Whittaker vector with respect to the subalgebra $U_\xi(\mathfrak{m}_-)$ and the character χ, and any $U_\eta(\mathfrak{g})$–module contains a non–zero Whittaker vector with respect to the subalgebra $U^{\widetilde{g}}_\eta(\mathfrak{m}_-) = \widetilde{g}^{-1}U_\xi(\mathfrak{m}_-)$ and the character $\chi^{\widetilde{g}}$ given by the composition of χ and $\widetilde{g}$, $\chi^{\widetilde{g}} = \chi \circ \widetilde{g} : U^{\widetilde{g}}_\eta(\mathfrak{m}_-) \to \mathbb{C}$.

Moreover, $\dim U_\xi(\mathfrak{m}_-) = \dim U^{\widetilde{g}}_\eta(\mathfrak{m}_-) = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}} = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\xi}} = m^{\dim \mathfrak{m}_-}$, where $\mathcal{O}_{\pi\eta}$ is the conjugacy class of $\pi\eta \in G_\mathcal{C}$, and $\mathcal{O}_{\pi\xi}$ is the conjugacy class of $\pi\xi \in G_\mathcal{C}$.

Proof. First observe that the system of positive roots Δ^s_+ satisfies the conditions of Theorem 1.5.2 when s is replaced with s^{-1}. Indeed, in the case of classical root systems its definition only depends on the spectral decomposition of $\mathfrak{h}$ under the action of s which is the same as the spectral decomposition of $\mathfrak{h}$ under the action of s^{-1}. In the case of exceptional root systems one has to note in addition that obviously $\dim \Sigma_s = \dim \Sigma_{s^{-1}}$, and hence all properties of Δ^s_+ used in the proof of Theorem 1.5.2 are satisfied if s is replaced with s^{-1} in the proof.

Let $\overline{N}_{s^{-1}} = \{\bar{n} \in \overline{N} : s^{-1}\bar{n}s \in N\}$. Applying Theorem 1.5.2 to s^{-1} and to the system of positive rootrs Δ^s_+ and swapping the roles of N and of $\overline{N}$ we deduce that all conjugacy classes in the stratum $G_\mathcal{C}$ intersect the variety $s^{-1}H^0\overline{N}_{s^{-1}}$ which is a subvariety of the transversal slice $\overline{\Sigma}_{s^{-1}} = s^{-1}Z\overline{N}_{s^{-1}}$ to the set of conjugacy classes in G. Note that $\overline{N}_{s^{-1}}$ is not a subgroup in N_+. But every element of $\bar{n}_{s^{-1}} \in \overline{N}_{s^{-1}}$ can be uniquely factorized as follows $\bar{n}_{s^{-1}} = \bar{n}^+_{s^{-1}}\bar{n}^-_{s^{-1}}$, $\bar{n}^\pm_{s^{-1}} \in \overline{N}_{s^{-1}} \cap N_\pm$, $\overline{N}_{s^{-1}} \cap N_- = \overline{N} \cap N_-$, $\overline{N}_{s^{-1}} \cap N_+ \subset M_+$, where $M_+ \subset N_+$ is the subgroup corresponding to the Lie subalgebra $\mathfrak{m}_+$. Therefore every element $s^{-1}h_0\bar{n}_{s^{-1}} \in s^{-1}H^0\overline{N}_{s^{-1}}$ can be represented as follows $s^{-1}h_0\bar{n}_{s^{-1}} = s^{-1}h_0\bar{n}^+_{s^{-1}}\bar{n}^-_{s^{-1}}$, and conjugating by $\bar{n}^-_{s^{-1}}$ we obtain that $s^{-1}h_0\bar{n}_{s^{-1}}$ is conjugate to $\bar{n}^-_{s^{-1}}s^{-1}h_0\bar{n}^+_{s^{-1}}$.

Since the decomposition $s = s^1 s^2$ is reduced

$$s(\overline{N}_{s^{-1}} \cap N_-)s^{-1} = s(\overline{N} \cap N_-)s^{-1} \subset M_+.$$

Taking into account that H^0 normalizes M_+ we have $\bar{n}^-_{s^{-1}} s^{-1} h_0 \bar{n}^+_{s^{-1}} = s^{-1} h_0 h_0^{-1} s \bar{n}^-_{s^{-1}} s^{-1} h_0 \bar{n}^+_{s^{-1}} = s^{-1} h_0 m_s$, $m_s = h_0^{-1} s \bar{n}^-_{s^{-1}} s^{-1} h_0 \bar{n}^+_{s^{-1}} \in M_+$. We deduce that all conjugacy classes in the stratum $G_{\mathcal{C}}$ intersect the variety $s^{-1} H^0 M_+$.

Recall that by (3.4.25) with the roles of N_+ and N_- swapped we have $v = n'sm$ for some $n' \in N, m \in M_+$, where $v \in G$ is an element of the form

$$v = \prod_{i=1}^{l'} \exp[t_i X_{-\gamma_i}],$$

$t_i \in \mathbb{C}$ are non–zero constants depending on the choice of the representative $s \in G$ and the product over roots is taken in the normal order (1.6.9) associated to s. We deduce that $s^{-1} = mv^{-1}n'$.

Now let $s^{-1} h_0 m_s, h_0 \in H^0, m_s \in M_+$ be an element of $s^{-1} H^0 M_+$. Using the previous expression for s^{-1} we can write $s^{-1} h_0 m_s = mu^{-1} n' h_0 m_s$. Conjugating this element by m^{-1} and recalling that H^0 normalizes N_+ we infer that $s^{-1} h_0 m_s$ is conjugate to

$$v^{-1} n' h_0 m_s m = v^{-1} h_0 n = \lambda_0(h_0^{\frac{1}{2}} n, h_0^{-\frac{1}{2}} v),$$

where $n = h_0^{-1} n' m_s m \in N_+$, $h_0^{\frac{1}{2}} \in H^0$ is any element such that $h_0^{\frac{1}{2}} h_0^{\frac{1}{2}} = h_0$, λ_0 is defined immediately after Proposition 6.1.4. We deduce that all conjugacy classes in the stratum $G_{\mathcal{C}}$ intersect the variety $v^{-1} H^0 N_+$.

By part (iv) of Proposition 6.1.5 we conclude that if $\eta \in \mathrm{Spec}(Z_0)$ satisfies $\pi\eta \in G_{\mathcal{C}}$ then there is a quantum coadjoint transformation $\widetilde{g}$ such that $\widetilde{\pi}(\widetilde{g}\eta) = (h_0^{\frac{1}{2}} n, h_0^{-\frac{1}{2}} v)$ for some $n \in N_+$, $h_0^{\frac{1}{2}} \in H^0$.

Denote $\xi = \widetilde{g}\eta$. From the definition of the map $\widetilde{\pi}$ and of the element v it follows that

$$\exp(\xi(x^-_{\beta_D})X_{-\beta_D})\exp(\xi(x^-_{\beta_{D-1}})X_{-\beta_{D-1}})\ldots\exp(\xi(x^-_{\beta_1})X_{-\beta_1}) = v^{-1}$$

which implies $\xi((X^-_{\gamma_i})^m) = -\frac{t_i}{(\varepsilon_{\gamma_i} - \varepsilon_{\gamma_i}^{-1})^m} \neq 0$ for $i = 1, \ldots, l'$ and that $\xi((X^-_\beta)^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$.

By the definition of the elements f_β with $\beta = \sum_{i=1}^l m_i \alpha_i$ we have $f_\beta = \prod_{i,j=1}^l L_j^{m_i n_{ij}} X_\beta^-$. Therefore the commutation relations between elements L_j and X_β^- imply that $f_\beta^m = c_\beta \prod_{i,j=1}^l L_j^{m m_i n_{ij}} (X_\beta^-)^m$, where c_β are non–zero constants, and hence $\xi(f_\beta^m) = c_\beta \prod_{i,j=1}^l \xi(L_j^m)^{m_i n_{ij}} \xi((X_\beta^-)^m)$. Since $\xi(L_j) \neq 0$ for $j = 1, \ldots, l$, $\xi((X^-_{\gamma_i})^m) = -\frac{t_i}{(\varepsilon_{\gamma_i} - \varepsilon_{\gamma_i}^{-1})^m} \neq 0$ for $i = 1, \ldots, l'$ and $\xi((X^-_\beta)^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$ we deduce $\xi(f^m_{\gamma_i}) = a_i \neq 0$ for $i = 1, \ldots, l'$ and $\xi(f_\beta^m) = 0$ for $\beta \in \Delta_{\mathfrak{m}_+}$, $\beta \notin \{\gamma_1, \ldots, \gamma_{l'}\}$. Thus ξ satisfies the condition of Propositions 6.2.2 and 6.2.4. Let $U_{\widetilde{g}\eta}(\mathfrak{m}_-) = U_\xi(\mathfrak{m}_-)$ be the corresponding subalgebra in $U_\xi(\mathfrak{g})$.

Note that by Theorem 1.5.2 for any $g \in G_{\mathcal{C}}$ we have

$$\dim Z_G(g) = \dim \overline{\Sigma}_{s^{-1}},$$

where $Z_G(g)$ is the centralizer of g in G.

By the definition of $\overline{\Sigma}_{s^{-1}}$ we also have $\dim \overline{\Sigma}_{s^{-1}} = l(s) + 2D_0 + \dim \mathfrak{h}'^\perp$. Observe also that $\dim G = 2D + \dim \mathfrak{h}$ and $\dim \mathfrak{h} - \dim \mathfrak{h}'^\perp = \dim \mathfrak{h}' = l'$, and hence from (1.6.15) we deduce that $\dim \mathfrak{m}_- = D - D_0 - \frac{1}{2}(l(s) - l') = \frac{1}{2}(\dim G - \dim \overline{\Sigma}_{s^{-1}}) = \frac{1}{2}\dim \mathcal{O}_g$ and $\dim U_\eta^{\widetilde{g}}(\mathfrak{m}_-) = \dim U_{\widetilde{g}\eta}(\mathfrak{m}_-) = m^{\dim \mathfrak{m}_-} = m^{\frac{1}{2}\dim \mathcal{O}_g}$, where $\mathcal{O}_g$ is the conjugacy class of any $g \in G_{\mathcal{C}}$.

In particular, $\dim U_\eta^{\widetilde{g}}(\mathfrak{m}_-) = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}}$, where $\mathcal{O}_{\pi\eta}$ is the conjugacy class of $\pi\eta \in G_{\mathcal{C}}$.

The remaining statements of this proposition are consequences of Proposition 6.2.4.

$\square$

6.3 Skryabin equivalence for quantum groups at roots of unity and the proof of De Concini–Kac–Procesi conjecture

In this section we shall study the $U_{\eta_1}(\mathfrak{m}_-)$–action on finite–dimensional $U_\eta(\mathfrak{g})$–modules introduced in the previous section. We shall show that each such module is $U_{\eta_1}(\mathfrak{m}_-)$–cofree. Taking into account that $\dim U_{\eta_1}(\mathfrak{m}_-) = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}}$ this will imply the De Concini–Kac–Procesi conjecture.

The main observation with the help of which we shall prove these statements is that the structure of $U_\eta(\mathfrak{g})$–modules is similar to that of a root of unity analogue Q_χ of the module $Q_\mathcal{B}^{loc}$, and the results of Propositions 4.6.1 and 4.6.8 can be specialized to $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ and transferred to Q_χ and, more generally, to any finite–dimensional $U_\eta(\mathfrak{g})$–module using root of unity analogues of the homomorphism ϕ. In particular, using images of the specializations $B_{n_1...n_c}^\varepsilon$ of the elements $B_{n_1...n_c}$ in finite–dimensional $U_\eta(\mathfrak{g})$–modules one can construct their $U_{\eta_1}(\mathfrak{m}_-)$–cofree bases.

To realize this program for any given $\eta \in \mathrm{Spec}(Z_0)$ we assume that Δ_+^s is the system of positive roots defined for s in Theorem 1.5.2, Δ_+ the corresponding system of positive roots associated to s, $d = 2d'$, where d' is defined in Proposition 6.1.1. Assume that m and d are coprime. Fix a quantum coadjoint transformation $\widetilde{g} \in \mathcal{G}$ as in Theorem 6.2.6 and denote $\xi = \widetilde{g}\eta \in \mathrm{Spec}(Z_0)$. Since according to (6.1.12) $\widetilde{g}$ gives rise to an isomorphism of the algebras $U_\eta(\mathfrak{g})$ and $U_\xi(\mathfrak{g})$ it suffices to consider the case of the algebra $U_\xi(\mathfrak{g})$.

Our first objective is to obtain root of unity analogues of Proposition 4.1.2 and Lemma 4.7.1. We start by introducing the notions required for the formulations of these statements. For indeterminate q these notions were introduced in Section 4.1.

Let χ be a character of $U_\xi(\mathfrak{m}_-)$ defined in Proposition 6.2.2, $\mathbb{C}_\chi$ the corresponding representation of $U_\xi(\mathfrak{m}_-)$. Denote by Q_χ the induced left $U_\xi(\mathfrak{g})$–module, $Q_\chi = U_\xi(\mathfrak{g}) \otimes_{U_\xi(\mathfrak{m}_-)} \mathbb{C}_\chi$. Q_χ can also be naturally regarded as a $U_\varepsilon^s(\mathfrak{g})$–module via the natural projection $U_\varepsilon^s(\mathfrak{g}) = U_\varepsilon(\mathfrak{g}) \to U_\xi(\mathfrak{g})$.

Let $\mathbb{C}_\varepsilon^{loc}[G]$, $\mathbb{C}_\varepsilon^{s,loc}[G_*]$, $\mathbb{C}_\varepsilon^s[G^*]$, $B_{n_1...n_c}^\varepsilon$, $\mathbb{C}_{11}^{loc}[G]_\varepsilon$ be the natural specializations at $q^{\frac{1}{rd^2}} = \varepsilon^{\frac{1}{rd^2}}$ of $\mathbb{C}_\mathcal{B}^{loc}[G]$, $\mathbb{C}_\mathcal{B}^{s,loc}[G_*]$, $\mathbb{C}_\mathcal{B}^s[G^*]$, $B_{n_1...n_c}$, $\mathbb{C}_{11}^{loc}[G]$, respectively.

Define the twisted adjoint action of $U_\varepsilon^{s,res}(\mathfrak{g})$ on $\mathbb{C}_\varepsilon^s[G]$ by

$$(\mathrm{Ad}_s^0 x f)(w) = f(\omega_0 S_s^{-1}(\mathrm{Ad}_s' x(S_s \omega_0 w))) = f((\omega_0 S_s^{-1})(x^1) w \omega_0(x^2)), f \in \mathbb{C}_\varepsilon^s[G], x, w \in U_\varepsilon^{s,res}(\mathfrak{g}). \qquad (6.3.1)$$

Specializing isomorphism (3.2.14) at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ and twisting it by $\omega_0 S_s^{-1}$ we obtain a $U_\varepsilon^{s,res}(\mathfrak{g})$–module homomorphism

$$\varphi_\varepsilon : \mathbb{C}_\varepsilon^s[G] \to \mathbb{C}_\varepsilon^s[G_*], f \mapsto (id \otimes f)(id \otimes \omega_0 S_s^{-1})(\mathcal{R}_{21}^s \mathcal{R}^s). \qquad (6.3.2)$$

Observe that the subalgebra in $U_\varepsilon^{s,res}(\mathfrak{g})$ generated by $\tilde{f}_\beta$, $\beta \in \Delta_{\mathfrak{m}_+}$ is isomorphic to $U_{\eta_1}(\mathfrak{m}_-)$. Therefore composing homomorphism (6.3.2) with the natural projection $\mathbb{C}_\varepsilon^s[G_*] \subset U_\varepsilon^s(\mathfrak{g}) \to U_\xi(\mathfrak{g}) \to Q_\chi$ we obtain a homomorphism of $U_{\eta_1}(\mathfrak{m}_-)$–modules

$$\phi_\xi : \mathbb{C}_\varepsilon^s[G] \to Q_\chi, \phi_\xi(f) = \varphi_\varepsilon(f)1, \qquad (6.3.3)$$

where $\mathbb{C}_\varepsilon^s[G]$ is equipped with the restriction of action (6.3.1) to $U_{\eta_1}(\mathfrak{m}_-)$ and Q_χ with the action induced by the adjoint action Ad of $U_{\eta_1}(\mathfrak{m}_-)$ and 1 is the image of $1 \in \mathbb{C}_\varepsilon^s[G_*]$ in Q_ξ under the map $\mathbb{C}_\varepsilon^s[G_*] \to Q_\xi$.

Similarly, for any finite–dimensional $U_\xi(\mathfrak{g})$–module V and any $w \in V_\chi$ one can define a $U_{\eta_1}(\mathfrak{m}_-)$–module homomorphism $\phi_\xi^w : \mathbb{C}_\varepsilon^s[G] \to V$ by

$$\phi_\xi^w : \mathbb{C}_\varepsilon^s[G] \to V, \phi_\xi(f) = \varphi_\varepsilon(f)w.$$

Proposition 6.3.1. *The specialization $I_\varepsilon^{11} \subset \mathbb{C}_\varepsilon^s[G]$ of the left ideal $I_\mathcal{B}^{11} \subset \mathbb{C}_\mathcal{B}^s[G]$ at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ lies in the kernel of ϕ_ξ^w.*

Moreover, if u is a highest weight vector in the specialization at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ of a finite rank indecomposable representation V_λ of $U_\mathcal{B}^{res}(\mathfrak{g})$ of highest weight λ and such that $(u,u) = 1$ then for any $f \in \mathbb{C}_\varepsilon^s[G]$

$$\phi_\xi^w(f \otimes (u, \cdot T_s^{-1} u)) = \varphi_\varepsilon(\mathrm{Ad}_s^0(\varepsilon^{-(nd\frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee})(f))\phi_\xi^w((u, \cdot T_s^{-1} u)), \qquad (6.3.4)$$

where $\mathrm{Ad}_s^0(\varepsilon^{-(nd\frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee})(f)$ is the adjoint action of the element

$$\varepsilon^{-(nd\frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} = q^{-(nd\frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} \pmod{(q^{\frac{1}{r^2 d}} - \varepsilon^{\frac{1}{r^2 d}})}, \varepsilon^{-(nd\frac{1+s}{1-s} P_{\mathfrak{h}'} + id)\lambda^\vee} \in \mathbb{C}_\varepsilon^s[G^*]$$

on $f \in \mathbb{C}_\varepsilon^s[G]$.

One can define an action of an operator $\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}$ on the image of $\mathbb{C}_\varepsilon^s[G]$ in V by the formula

$$\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \phi_\xi^w(f) = \varphi_\varepsilon(\mathrm{Ad}_s^0(\varepsilon^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})(f))\phi_\xi^w((u, \cdot T_s^{-1} u)), \qquad (6.3.5)$$

where $\mathrm{Ad}_s^0(\varepsilon^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})(f)$ is the adjoint action of the element

$$\varepsilon^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} = q^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \pmod{(q^{\frac{1}{r^2 d}} - \varepsilon^{\frac{1}{r^2 d}})}, \varepsilon^{-(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \in \mathbb{C}_\varepsilon^s[G^*]$$

on $f \in \mathbb{C}_\varepsilon^s[G]$.

Using this operator formula (6.3.4) can be rewritten as follows

$$\phi_\xi^w(f \otimes (u, \cdot T_s^{-1} u)) = \varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \phi_\xi^w(\mathrm{Ad}_s^0(\varepsilon^{(-nd\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}+s^{-1})\lambda^\vee})(f)). \tag{6.3.6}$$

The operator $\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}$ is invertible. More precisely, for some $n_0 \in \mathbb{N}$ the action of $(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})^{n_0}$ coincides with the action of the element

$$\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} = q^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \pmod{(q^{\frac{1}{r^2d}} - \varepsilon^{\frac{1}{r^2d}})}, \varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \in U_\varepsilon^s(\mathfrak{h}),$$

and hence $(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})^{n_0 m} = \xi(\varepsilon^{n_0 m(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}) = c_\lambda \in \mathbb{C}^*$.

ϕ_ξ^w extends to a $U_{\eta_1}(\mathfrak{m}_-)$–module homomorphism $\phi_\xi^w : \mathbb{C}_\varepsilon^{s,loc}[G] \to V$,

$$\phi_\xi^w(f \otimes \Delta_\lambda^{s-1}) = \varepsilon^{(\frac{(1-nd)s^{-1}+(1+nd)s-2}{1-s}P_{\mathfrak{h}'}\lambda^\vee, \lambda^\vee)} \left(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}\right)^{-1} \phi_\xi^w(\mathrm{Ad}_s^0(\varepsilon^{(nd\frac{1+s}{1-s}s^{-1}P_{\mathfrak{h}'}-s^{-1})\lambda^\vee})(f)), \tag{6.3.7}$$

and the specialization $I_\varepsilon^{11\,loc} \subset \mathbb{C}_\varepsilon^{s,loc}[G]$ of $I_{\mathcal{B}}^{11\,loc} \subset \mathbb{C}_{\mathcal{B}}^{s,loc}[G]$ at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ belongs to the kernel of this homomorphism, so

$$\phi_\xi^w : \mathbb{C}_{11}^{loc}[G]_\varepsilon \to V.$$

Proof. In order to prove this proposition one can apply Proposition 4.1.1 for $\kappa = nd$ and $q^{\frac{1}{dr^2}}$ specialized to $\varepsilon^{\frac{1}{dr^2}}$, and the appropriately modified arguments before Proposition 4.1.2 and from the proof of Lemma 4.7.1.

Indeed, by the definition of $I_{\mathcal{B}}^{\mathbf{k}}$ with $\kappa = nd$, $k_i = c_i$, $i = 1, \ldots, l'$ for c_i, $i = 1, \ldots, l'$ used in the definition of χ, the specialization of $I_{\mathcal{B}}^{\mathbf{k}}$ at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ belongs to the annihilator of w, and hence by Proposition 4.1.1 the specialization $I_\varepsilon^{11} \subset \mathbb{C}_\varepsilon^s[G]$ of the left ideal $I_{\mathcal{B}}^{11} \subset \mathbb{C}_{\mathcal{B}}^s[G]$ at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ lies in the kernel of ϕ_ξ^w.

Formulas (6.3.4), (6.3.5) and (6.3.6) are obtained by specializing the corresponding formulas in Proposition 4.1.1 at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ with $\kappa = nd$.

The only essential difference is that the operator $\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}$ is invertible and for some $n_0 \in \mathbb{N}$ its action coincides with the action of the element $\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \in U_\varepsilon^s(\mathfrak{h})$. This can be justified as follows.

Recall that elements $\gamma_i^\vee$, $i = 1, \ldots, l'$ form a linear basis of $\mathfrak{h}'$. Let γ_i^*, $i = 1, \ldots, l'$ be the basis of $\mathfrak{h}'$ dual to $\gamma_i^\vee$, $i = 1, \ldots, l'$ with respect to the restriction of the bilinear form $(\cdot, \cdot)$ to $\mathfrak{h}'$. Since the numbers $(\gamma_i^\vee, \gamma_j^\vee)$ are integer each element γ_i^* has the form $\gamma_i^* = \sum_{j=1}^{l'} m_{ij} \gamma_j^\vee$, where $m_{ij} \in \mathbb{Q}$. Therefore $P_{\mathfrak{h}'}\lambda^\vee = \sum_{p=1}^{l'}(\lambda^\vee, \gamma_p^\vee)\gamma_p^* = \sum_{p,q=1}^{l'}(\lambda^\vee, \gamma_p^\vee) m_{pq} \gamma_q^\vee$ belongs to the rational span of the set of simple coweights Y_i, $i = 1, \ldots, l$, and hence $(id + s^{-1})(id - ndP_{\mathfrak{h}'})\lambda^\vee$ belongs to the rational span of the set of simple coweights Y_i, $i = 1, \ldots, l$ as well. We conclude that there exists an integer $n_0 \in \mathbb{N}$ such that $n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee$ belongs to the integer span of the set of simple coweights Y_i, $i = 1, \ldots, l$, and $q^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \in U_{\mathcal{B}}^s(\mathfrak{h})$. So if we define $\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} = q^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \bmod (q^{\frac{1}{r^2d}} - \varepsilon^{\frac{1}{r^2d}})$ then $\varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \in U_\varepsilon^s(\mathfrak{h})$.

Now using (6.3.4) and (6.3.5) one immediately verifies that

$$(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})^{n_0} \phi_\xi^w(f) = \varphi_\varepsilon(\mathrm{Ad}_s^0(\varepsilon^{-n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})(f))\phi_\xi^w((u, \cdot T_s^{-1} u)^{n_0})$$

and that

$$\phi_\xi^w((u, \cdot T_s^{-1} u)^{n_0}) = \varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} w.$$

Recalling the equivariance of φ_ε with respect to the action of $U_\varepsilon^{s,res}(\mathfrak{g}) \supset U_\varepsilon^s(\mathfrak{h})$ we obtain from the last two identities that

$$(\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee})^{n_0} \phi_\xi^w(f) =$$

$$= \varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \varphi_\varepsilon(f) \varepsilon^{-n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} w =$$

$$= \varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \varphi_\varepsilon(f) w = \varepsilon^{n_0(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee} \phi_\xi^w(f).$$

Finally applying verbatim the arguments from the proof of Lemma 4.7.1 we deduce that ϕ_ξ^w extends to a $U_{\eta_1}(\mathfrak{m}_-)$–module homomorphism $\phi_\xi^w : \mathbb{C}_\varepsilon^{s,loc}[G] \to V$ in such a way that (6.3.7) holds, and the specialization $I_\varepsilon^{11\,loc} \subset \mathbb{C}_\varepsilon^{s,loc}[G]$ of $I_{\mathcal{B}}^{11\,loc} \subset \mathbb{C}_{\mathcal{B}}^{s,loc}[G]$ at $q^{\frac{1}{dr^2}} = \varepsilon^{\frac{1}{dr^2}}$ belongs to the kernel of this homomorphism, so

$$\phi_\xi^w : \mathbb{C}_{11}^{loc}[G]_\varepsilon \to V.$$

Note that due to invertibility of the operator $\varepsilon^{(id+s^{-1})(id-ndP_{\mathfrak{h}'})\lambda^\vee}$ no localization of V, which appears in Lemma 4.7.1 for $Q_\mathcal{B}$ in the case of indeterminate q, is required.

$\square$

Now we define root of unity counterparts of q–W–algebras. Let $W^s_{\varepsilon,\xi}(G) = \operatorname{End}_{U_\xi(\mathfrak{g})}(Q_\chi)^{opp}$ be the algebra of $U_\xi(\mathfrak{g})$–endomorphisms of Q_χ with the opposite multiplication. The algebra $W^s_{\varepsilon,\xi}(G)$ is also called a q–W–algebra associated to $s \in W$ and to $\xi \in \operatorname{Spec}(Z_0)$. Denote by $U_\xi(\mathfrak{g}) - \operatorname{mod}$ the category of finite–dimensional left $U_\xi(\mathfrak{g})$–modules and by $W^s_{\varepsilon,\xi}(G) - \operatorname{mod}$ the category of finite–dimensional left $W^s_{\varepsilon,\xi}(G)$–modules. Observe that if $V \in U_\xi(\mathfrak{g}) - \operatorname{mod}$ then the algebra $W^s_{\varepsilon,\xi}(G)$ naturally acts on the finite–dimensional space $V_\chi = \operatorname{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, V) = \operatorname{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, V)$ by compositions of homomorphisms.

The following theorem is a root of unity analogue of the Skryabin equivalence for equivariant modules over quantum groups. This theorem uncovers some striking similarity between the structure of the category of finite–dimensional representations of $U_\xi(\mathfrak{g})$ and of the category of equivariant modules over a quantum group for generic ε.

Theorem 6.3.2. *Every module $V \in U_\xi(\mathfrak{g}) - \operatorname{mod}$ is isomorphic to $\operatorname{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$ as a right $U_{\eta_1}(\mathfrak{m}_-)$–module, where the right action of $U_{\eta_1}(\mathfrak{m}_-)$ on $\operatorname{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$ is induced by the multiplication in $U_{\eta_1}(\mathfrak{m}_-)$ from the left. Q_χ is isomorphic to $\operatorname{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-)) \otimes W^s_{\varepsilon,\xi}(G)$ as a $U_{\eta_1}(\mathfrak{m}_-)$–$W^s_{\varepsilon,\xi}(G)$–bimodule. In particular, V is $U_{\eta_1}(\mathfrak{m}_-)$–injective, $\operatorname{Ext}^\bullet_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, V) = V_\chi$ and the dimension of V is divisible by $\dim U_\xi(\mathfrak{m}_-) = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\xi}}$.*

The functor $E \mapsto Q_\chi \otimes_{W^s_{\varepsilon,\xi}(G)} E$ establishes an equivalence of the category of finite–dimensional left $W^s_{\varepsilon,\xi}(G)$–modules and the category $U_\xi(\mathfrak{g}) - \operatorname{mod}$. The inverse equivalence is given by the functor $V \mapsto V_\chi$. In particular, the latter functor is exact, and every finite–dimensional $U_\xi(\mathfrak{g})$–module is generated by Whittaker vectors.

Proof. Let V be an object in the category $U_\xi(\mathfrak{g}) - \operatorname{mod}$. Fix any linear map $\rho : V \to V_\chi$ the restriction of which to V_χ is the identity map, and let for any $v \in V$ $\sigma_\varepsilon(v) : U_{\eta_1}(\mathfrak{m}_-) \to V_\chi$ be the $\mathbb{C}$–linear homomorphism given by $\sigma_\varepsilon(v)(x) = \rho(\operatorname{Ad}_s x(v))$, and we have a map $\sigma_\varepsilon : V \to \operatorname{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$.

By definition σ_ε is a homomorphism of right $U_{\eta_1}(\mathfrak{m}_-)$–modules, where the right action of $U_{\eta_1}(\mathfrak{m}_-)$ on

$$\operatorname{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$$

is induced by multiplication in $U_{\eta_1}(\mathfrak{m}_-)$ from the left.

We claim that σ_ε is an isomorphism. Firstly, σ_ε is injective for otherwise its kernel would contain a non–zero Whittaker vector. Indeed by Lemma 6.2.5 the augmentation ideal of $U_{\eta_1}(\mathfrak{m}_-)$ coincides with its Jacobson radical which is nilpotent. Therefore its action on the kernel of σ_ε is nilpotent, and hence the kernel, if it is non–trivial, must contain a non–zero Whittaker vector annihilated by the augmentation ideal of $U_{\eta_1}(\mathfrak{m}_-)$. But all non–zero Whittaker vectors in V belong to V_χ and by the definition of σ_ε their images in $\operatorname{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$ are non–zero homomorphisms non-vanishing at 1.

Next we show that σ_ε is also surjective. Indeed, for $n_1, \ldots, n_c = 1, \ldots m - 1$ and any $v \in V_\chi$ the elements $v^{n_1 \ldots n_c} = c'_{n_1 \ldots n_c}(\varepsilon)^{-1} \phi^v_\xi(B^\varepsilon_{n_1 \ldots n_d})$ are well–defined, all $c'_{n_1 \ldots n_c}(\varepsilon)$ being non–zero by the choice of ε, and Corollary 4.6.8 for $k_1, \ldots, k_c = 1, \ldots m - 1$ implies

$$\operatorname{Ad}_s(f^{k_1}_{\beta_1} \ldots f^{k_c}_{\beta_c})v^{n_1 \ldots n_c} = \begin{cases} v & \text{if } n_p = k_p \text{ for } p = 1, \ldots, c \\ 0 & \text{if } k_i = n_i, i = 1, \ldots, p - 1 \text{ and } k_p > n_p \text{ for some } p \in \{1, \ldots, c\} \end{cases},$$

and hence

$$\sigma_\varepsilon(v^{n_1 \ldots n_c})(f^{k_1}_{\beta_1} \ldots f^{k_c}_{\beta_c}) = \begin{cases} v & \text{if } n_p = k_p \text{ for } p = 1, \ldots, c \\ 0 & \text{if } k_i = n_i, i = 1, \ldots, p - 1 \text{ and } k_p > n_p \text{ for some } p \in \{1, \ldots, c\} \end{cases}. \quad (6.3.8)$$

Observe that the elements $f^{k_1}_{\beta_1} \ldots f^{k_c}_{\beta_c}$, $k_1, \ldots, k_c = 1, \ldots m - 1$ form a linear basis of $U_{\eta_1}(\mathfrak{m}_-)$. Elements of this basis are labeled by elements of the set $\mathbb{N}^c_m$, where $\mathbb{N}_m = \{0, 1, \ldots, m - 1\}$. Introduce the lexicographic order on this set, so that $(k_1, \ldots, k_c) > (n_1, \ldots, n_c)$ if $k_i = n_i$ for $i = 1, \ldots, p - 1$ and $k_p > n_p$ for some $p \in \{1, \ldots, c\}$.

Note that for any $(k_1, \ldots, k_c) \in \mathbb{N}^c_m$ the number of elements $(n_1, \ldots, n_c) \in \mathbb{N}^c$ such that $(k_1, \ldots, k_c) > (n_1, \ldots, n_c)$ is finite.

Now let $(k_1, \ldots, k_c) \in \mathbb{N}^c_m$, $v \in V_\chi$. If for $(n_1, \ldots, n_c) \in \mathbb{N}^c_m$ such that $(k_1, \ldots, k_c) \geq (n_1, \ldots, n_c)$ we denote

$$\sigma_\varepsilon(v^{n_1 \ldots n_c}_{n_1 \ldots n_c}) = f_{n_1 \ldots n_c},$$

where for $(k_1, \ldots, k_c) \geq (n_1, \ldots, n_c)$ $v_{n_1 \ldots n_c} \in V_\chi$ are defined by induction starting from $v_{k_1 \ldots k_c} = v$ as follows

$$v_{n_1 \ldots n_c} = - \sum_{(k_1, \ldots, k_c) \geq (n'_1, \ldots, n'_c) > (n_1, \ldots, n_c)} f_{n'_1 \ldots n'_c}(f_{\beta_1}^{n_1} \ldots f_{\beta_c}^{n_c}), \tag{6.3.9}$$

then using (6.3.8) one obtains

$$f_{n_1 \ldots n_c}(f_{\beta_1}^{n'_1} \ldots f_{\beta_c}^{n'_c}) = \begin{cases} v_{n_1 \ldots n_c} & \text{if } (n'_1, \ldots, n'_c) = (n_1, \ldots, n_c), \\ 0 & \text{if } (n'_1, \ldots, n'_c) > (n_1, \ldots, n_c) \end{cases}.$$

From this property and from (6.3.9) one immediately checks that if we define

$$f_v^{k_1 \ldots k_c} = \sum_{(k_1, \ldots, k_c) \geq (n_1, \ldots, n_c)} f_{n_1 \ldots n_c}$$

then for any $(n_1, \ldots, n_c) \in \mathbb{N}_m^c$

$$f_v^{k_1 \ldots k_c}(f_{\beta_1}^{n_1} \ldots f_{\beta_c}^{n_c}) = \begin{cases} v & \text{if } (n_1, \ldots, n_c) = (k_1, \ldots, k_c), \\ 0 & \text{if } (n_1, \ldots, n_c) \neq (k_1, \ldots, k_c) \end{cases}. \tag{6.3.10}$$

Since the elements $f_{\beta_1}^{k_1} \ldots f_{\beta_c}^{k_c}$, $k_1, \ldots, k_c = 1, \ldots m-1$ form a linear basis of $U_{\eta_1}(\mathfrak{m}_-)$ (6.3.10) implies that the elements $f_v^{k_1 \ldots k_c}$ with $(k_1, \ldots, k_c) \in \mathbb{N}_m^c$, $v \in V_\chi$ generate $\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$, and hence the elements $\sigma_\varepsilon(v^{n_1 \ldots n_c})$ with $(n_1, \ldots, n_c) \in \mathbb{N}_m^c$, $v \in V_\chi$ generate $\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$ as well. Therefore σ_ε is surjective, and the first part of the theorem is proved.

Similarly to the case of generic ε (see last part of the proof of Theorem 5.2.1) one shows that Q_χ is isomorphic to $\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-)) \otimes W_{\varepsilon,\xi}^s(G)$ as a $U_{\eta_1}(\mathfrak{m}_-)$–$W_{\varepsilon,\xi}^s(G)$–bimodule.

Let E be a finite–dimensional $W_{\varepsilon,\xi}^s(G)$–module. Using the isomorphism $Q_\chi \simeq \mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-)) \otimes W_{\varepsilon,\xi}^s(G)$ of $U_{\eta_1}(\mathfrak{m}_-)$–$W_{\varepsilon,\xi}^s(G)$–bimodules and the linear space isomorphism $W_{\varepsilon,\xi}^s(G) = \mathrm{End}_{U_\xi(\mathfrak{g})}(Q_\chi)^{opp} = \mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, Q_\chi) = (Q_\chi)_\chi$, one immediately deduces similarly to the case of generic ε in the proof of Theorem 5.2.1 that $(Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} E)_\chi = E$. Therefore to prove the second statement of the theorem it suffices to check that for any $V \in U_\xi(\mathfrak{g}) - \mathrm{mod}$ the canonical map $f : Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi \to V$ is an isomorphism.

Indeed, f is injective because otherwise by Proposition 6.2.4 its kernel would contain a non–zero Whittaker vector with respect to χ. But all Whittaker vectors of $Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi$ belong to the subspace $1 \otimes V_\chi$, and the restriction of f to $1 \otimes V_\chi$ induces an isomorphism of the spaces of Whittaker vectors of $Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi$ and of V.

In order to prove that f is surjective we consider the exact sequence

$$0 \to Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi \to V \to W \to 0,$$

where W is the cokernel of f, and the corresponding long exact sequence of cohomology,

$$0 \to \mathrm{Ext}_{U_{\eta_1}(\mathfrak{m}_-)}^0(\mathbb{C}_\varepsilon, Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi) \to \mathrm{Ext}_{U_{\eta_1}(\mathfrak{m}_-)}^0(\mathbb{C}_\varepsilon, V) \to \mathrm{Ext}_{U_{\eta_1}(\mathfrak{m}_-)}^0(\mathbb{C}_\varepsilon, W) \to$$

$$\to \mathrm{Ext}_{U_{\eta_1}(\mathfrak{m}_-)}^1(\mathbb{C}_\varepsilon, Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi) \to \ldots.$$

Now recall that f induces an isomorphism of the spaces of Whittaker vectors of $Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi$ and of V. As we proved above the finite–dimensional $U_\xi(\mathfrak{g})$–module $Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi$ is injective over $U_{\eta_1}(\mathfrak{m}_-)$, and hence $\mathrm{Ext}_{U_{\eta_1}(\mathfrak{m}_-)}^1(\mathbb{C}_\varepsilon, Q_\chi \otimes_{W_{\varepsilon,\xi}^s(G)} V_\chi) = 0$. Therefore the initial part of the long exact cohomology sequence takes the form

$$0 \to V_\chi \to V_\chi \to W_\chi \to 0,$$

where the second map in the last sequence is an isomorphism. Using the last exact sequence we deduce that $W_\chi = 0$. But if W were non–trivial it would contain a non–zero Whittaker vector by Proposition 6.2.4. Thus $W = 0$, and f is surjective. This completes the proof of the theorem.

$\square$

By the previous theorem every module $V \in U_\xi(\mathfrak{g}) - \mathrm{mod}$ is isomorphic to

$$\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi) = \mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), \mathbb{C}) \otimes V_\chi$$

as a right $U_{\eta_1}(\mathfrak{m}_-)$–module. In fact, one can show that the algebra $U_{\eta_1}(\mathfrak{m}_-)$ is Frobenius, i.e. its left regular representation is isomorphic to the dual of the right regular representation and its right regular representation is isomorphic to the dual of the left regular representation. Thus as a right $U_{\eta_1}(\mathfrak{m}_-)$–module V is isomorphic to $U_{\eta_1}(\mathfrak{m}_-) \otimes V_\chi$, where the right action of $U_{\eta_1}(\mathfrak{m}_-)$ on $U_{\eta_1}(\mathfrak{m}_-) \otimes V_\chi$ is induced by the multiplication in $U_{\eta_1}(\mathfrak{m}_-)$ from the right. In particular, V is $U_{\eta_1}(\mathfrak{m}_-)$–free.

More generally, we have the following proposition.

Proposition 6.3.3. *For any character $\eta : Z_0 \to \mathbb{C}$ the algebra $U_\eta(\mathfrak{g})$ and its subalgebra $U_\eta(\mathfrak{m}_-)$ are Frobenius algebras.*

Proof. The proof of this proposition is parallel to the proof of a similar statement for Lie algebras over fields of prime characteristic (see Proposition 1.2 in [41]) and for the restricted form of the quantum group in [74]. We shall only briefly outline the main steps of the proof for $U_\eta(\mathfrak{g})$. The proof for $U_\eta(\mathfrak{m}_-)$ is similar.

The key ingredient of the proof is the De Concini-Kac filtration on $U_\varepsilon(\mathfrak{g}) \simeq U_\varepsilon^s(\mathfrak{g})$ defined as in Section 5.1. For $\mathbf{r}, \mathbf{t} \in \mathbb{N}^D$ introduce the element $u_{\mathbf{r},\mathbf{t},t} = e^\mathbf{r} t f^\mathbf{t}$, $t \in U_\varepsilon(\mathfrak{h})$, where we use the notation of Lemma 2.8.2. Here the generators $f_\alpha, e_\alpha, \alpha \in \Delta_+$ and the ordered products of them are defined with the help of the normal ordering of Δ_+ associated to s. Define also the height of the element $u_{\mathbf{r},\mathbf{t},t}$ as follows $\mathrm{ht}\,(u_{\mathbf{r},\mathbf{t},t}) = \sum_{i=1}^D (t_i + r_i) \mathrm{ht}\,\beta_i \in \mathbb{N}$, where $\mathrm{ht}\,\beta_i$ is the height of the root β_i. Introduce also the degree of $u_{\mathbf{r},\mathbf{t},t}$ by

$$d(u_{\mathbf{r},\mathbf{t},t}) = (r_1, \ldots, r_D, t_D, \ldots, t_1, \mathrm{ht}\,(u_{\mathbf{r},\mathbf{t},t})) \in \mathbb{N}^{2D+1}.$$

Equip $\mathbb{N}^{2D+1}$ with the total lexicographic order and for $k \in \mathbb{N}^{2D+1}$ denote by $(U_\varepsilon(\mathfrak{g}))_k$ the span of elements $u_{\mathbf{r},\mathbf{t},t}$ with $d(u_{\mathbf{r},\mathbf{t},t}) \le k$ in $U_\varepsilon(\mathfrak{g})$. Then Proposition 1.7 in [23] implies that $(U_\varepsilon(\mathfrak{g}))_k$ is a filtration of $U_\varepsilon(\mathfrak{g})$ such that the associated graded algebra is the associative algebra over $\mathbb{C}$ with generators e_α, f_α, $\alpha \in \Delta_+$, $L_i^{\pm 1}$, $i = 1, \ldots l$ subject to the relations

$$L_i L_j = L_j L_i, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad L_i e_\alpha L_i^{-1} = \varepsilon^{\alpha(Y_i)} e_\alpha, \quad L_i f_\alpha L_i^{-1} = \varepsilon^{-\alpha(Y_i)} f_\alpha,$$

$$e_\alpha f_\beta = \varepsilon^{nd(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha, \beta)} f_\beta e_\alpha,$$

$$e_\alpha e_\beta = \varepsilon^{(\alpha,\beta) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha, \beta)} e_\beta e_\alpha, \quad \alpha < \beta, \tag{6.3.11}$$

$$f_\alpha f_\beta = \varepsilon^{(\alpha,\beta) + nd(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha, \beta)} f_\beta f_\alpha, \quad \alpha < \beta.$$

Such algebras are called semi–commutative.

By Theorem 61.3 in [31] it suffices to show that there is a non–degenerate bilinear form $B_\eta : U_\eta(\mathfrak{g}) \times U_\eta(\mathfrak{g}) \to \mathbb{C}$ which is associative in the sense that

$$B_\eta(ab, c) = B_\eta(a, bc), \quad a, b, c \in U_\eta(\mathfrak{g}).$$

Consider the free Z_0–basis of $U_\varepsilon(\mathfrak{g})$ introduced in part (ii) of Proposition 6.1.3. This basis consists of the monomials $x_I = f^\mathbf{r} L^\mathbf{s} e^\mathbf{t}$, $I = (r_1, \ldots, r_D, s_1, \ldots, s_l, t_1, \ldots, t_D)$ for which $0 \le r_k, t_k, s_i < m$ for $i = 1, \ldots, l$, $k = 1, \ldots, D$. Set $I' = (m - 1 - r_1, \ldots, m - 1 - r_D, m - 1 - s_1, \ldots, m - 1 - s_l, m - 1 - t_1, \ldots, m - 1 - t_D)$ and $P = (m - 1, \ldots, m - 1)$.

Let $\Phi : U_\varepsilon(\mathfrak{g}) \to Z_0$ be the Z_0–linear map defined on the basis x_I of monomials by

$$\Phi(x_I) = \begin{cases} 1 & I = P \\ 0 & \text{otherwise} \end{cases}.$$

Let $x = \sum_I c_I x_I, c_I \in Z_0$ be an element of $U_\varepsilon(\mathfrak{g})$, and $c_K \ne 0$ a coefficient such that $d(x_K)$ is maximal possible with $c_K \ne 0$ in the sum defining x.

Using the definition of the De Concini–Kac filtration and commutation relations (6.3.11) one can check that $\Phi(x x_{K'}) = a_x c_K$, where a_x is a nonzero complex number (see [74], proof of Theorem 2.2, Assertion I for details).

Therefore the bilinear form $B_\eta : U_\eta(\mathfrak{g}) \times U_\eta(\mathfrak{g}) \to \mathbb{C}$ associated to the associative Z_0–bilinear pairing $B : U_\varepsilon(\mathfrak{g}) \otimes_{Z_0} U_\varepsilon(\mathfrak{g}) \to Z_0$, $B(x, y) = \Phi(xy)$ is non–degenerate and associative. This completes the proof. $\qquad \square$

We restate the results of the discussion before the previous proposition as its corollary.

Corollary 6.3.4. *As a right $U_{\eta_1}(\mathfrak{m}_-)$–module, every module $V \in U_\xi(\mathfrak{g}) - \mathrm{mod}$ is isomorphic to $U_{\eta_1}(\mathfrak{m}_-) \otimes V_\chi$, where the right action of $U_{\eta_1}(\mathfrak{m}_-)$ on $U_{\eta_1}(\mathfrak{m}_-) \otimes V_\chi$ is induced by the multiplication in $U_{\eta_1}(\mathfrak{m}_-)$ from the right. In particular, V is $U_{\eta_1}(\mathfrak{m}_-)$–free.*

6.4 Properties of q-W–algebras associated to quantum groups at roots of unity

In conclusion we study some further properties of q-W–algebras at roots of unity and of the module Q_χ. We keep the notation introduced in the previous section. First we prove the following lemma.

Lemma 6.4.1. *The left $U_\xi(\mathfrak{g})$–module Q_χ is projective in the category $U_\xi(\mathfrak{g}) - \mathrm{mod}$.*

Proof. We have to show that the functor $\mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, \cdot)$ is exact. Let $V^\bullet$ be an exact complex of finite–dimensional $U_\xi(\mathfrak{g})$–modules. Since by the previous theorem any object V of $U_\xi(\mathfrak{g}) - \mathrm{mod}$ is isomorphic to $\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), V_\chi)$ as a right $U_{\eta_1}(\mathfrak{m}_-)$–module we have

$$V^\bullet \simeq \mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), \overline{V}^\bullet),$$

where $\overline{V}^\bullet$ is an exact complex of vector spaces and the action of $U_{\eta_1}(\mathfrak{m}_-)$ on $\mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), \overline{V}^\bullet)$ is induced by multiplication from the left on $U_{\eta_1}(\mathfrak{m}_-)$.

Now by the Frobenius reciprocity we have obvious isomorphisms of complexes,

$$\mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, V^\bullet) \simeq \mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, V^\bullet) = \mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, V^\bullet) \simeq$$

$$\simeq \mathrm{Hom}_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, \mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-), \overline{V}^\bullet)) \simeq \mathrm{Hom}_\mathbb{C}(U_{\eta_1}(\mathfrak{m}_-) \otimes_{U_{\eta_1}(\mathfrak{m}_-)} \mathbb{C}_\varepsilon, \overline{V}^\bullet) = \overline{V}^\bullet,$$

where the last complex is exact, and we used the fact that by Lemma 6.2.3 for any finite–dimensional $U_\xi(\mathfrak{g})$–module V one has $\mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, V) \simeq \mathrm{Hom}_{U_{\eta_1}(\mathfrak{m}_-)}(\mathbb{C}_\varepsilon, V)$. We conclude that the functor $\mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(Q_\chi, \cdot)$ is exact. $\square$

The properties of q-W–algebras at roots of unity are summarized in the following proposition.

Proposition 6.4.2. *Denote $b = m^{\dim \mathfrak{m}_-} = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\xi}}$. Then $Q_\chi^b \simeq U_\xi(\mathfrak{g})$ as left $U_\xi(\mathfrak{g})$–modules, $U_\xi(\mathfrak{g}) \simeq \mathrm{Mat}_b(W_{\varepsilon,\xi}^s(G))$ as algebras and $Q_\chi \simeq (W_{\varepsilon,\xi}^s(G)^{opp})^b$ as right $W_{\varepsilon,\xi}^s(G)$–modules.*

Proof. Let E_i, $i = 1, \ldots, C$ be the simple finite–dimensional modules over the finite–dimensional algebra $U_\xi(\mathfrak{g})$. Denote by P_i the projective cover of E_i. Since by Theorem 6.3.2 the dimension of E_i is divisible by b we have $\dim E_i = br_i$, $r_i \in \mathbb{N}$, where r_i is the rank of E_i over $U_{\eta_1}(\mathfrak{m}_-)$ equal to the dimension of the space of Whittaker vectors in E_i. By Proposition 2.1 in [97]

$$U_\xi(\mathfrak{g}) = \mathrm{Mat}_b(\mathrm{End}_{U_\xi(\mathfrak{g})}(P)^{opp}),$$

where $P = \bigoplus_{i=1}^C P_i^{r_i}$. Therefore to prove the second statement of the proposition it suffices to show that $P \simeq Q_\chi$. Since by the previous lemma Q_χ is projective we only need to verify that

$$r_i = \dim \mathrm{Hom}_{U_\xi(\mathfrak{g})}(P, E_i) = \dim \mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, E_i).$$

Indeed, by the Frobenius reciprocity we have

$$\dim \mathrm{Hom}_{U_\xi(\mathfrak{g})}(Q_\chi, E_i) = \dim \mathrm{Hom}_{U_\xi(\mathfrak{m}_-)}(\mathbb{C}_\chi, E_i) = r_i.$$

This proves the second statement of the proposition. From Proposition 2.1 in [97] we also deduce that $P^b \simeq U_\xi(\mathfrak{g})$ as left $U_\xi(\mathfrak{g})$–modules. Together with the isomorphism $P \simeq Q_\chi$ this gives the first statement of the proposition.

Using results of Section 6.4 in [95] and the fact that Q_χ is projective one can find an idempotent $e \in U_\xi(\mathfrak{g})$ such that $Q_\chi \simeq U_\xi(\mathfrak{g})e$ as modules and $(W_{\varepsilon,\xi}^s(G))^{opp} \simeq eU_\xi(\mathfrak{g})e$ as algebras.

By the first two statements of this proposition one can also find idempotents $e = e_1, e_2, \ldots, e_b \in U_\xi(\mathfrak{g})$ such that $e_1 + \ldots + e_b = 1$, $e_i e_j = 0$ if $i \neq j$ and $e_i U_\xi(\mathfrak{g}) \simeq e U_\xi(\mathfrak{g})$ as right $U_\xi(\mathfrak{g})$–modules. Therefore $e_i U_\xi(\mathfrak{g})e \simeq e U_\xi(\mathfrak{g})e$ as right $eU_\xi(\mathfrak{g})e$–modules, and

$$Q_\chi \simeq U_\xi(\mathfrak{g})e = \bigoplus_{i=1}^b e_i U_\xi(\mathfrak{g})e \simeq (eU_\xi(\mathfrak{g})e)^b \simeq (W_{\varepsilon,\xi}^s(G)^{opp})^b$$

as right $W_{\varepsilon,\xi}^s(G)$–modules. This completes the proof of the proposition

$\square$

Corollary 6.4.3. *The algebra $W_{\varepsilon,\xi}^s(G)$ is finite–dimensional, and* $\dim W_{\varepsilon,\xi}^s(G) = m^{\dim \Sigma_s}$.

Proof. By Proposition 3.4.5 $2\dim \mathfrak{m}_- + \dim \Sigma_s = \dim G$. Therefore by the definition of Q_χ we have $\dim Q_\chi = m^{\dim G - \dim \mathfrak{m}_-} = m^{\dim \mathfrak{m}_- + \dim \Sigma_s}$. Finally from the last statement of the previous theorem one obtains that $\dim W_{\varepsilon,\xi}^s(G) = \dim Q_\chi / m^{\dim \mathfrak{m}_-} = m^{\dim \Sigma_s}$.

$\square$

Using Proposition 6.1.1 we deduce from Proposition 6.4.2 the following statement on the structure of the algebra $U_\eta(\mathfrak{g})$.

Corollary 6.4.4. *Let $\eta \in \mathrm{Spec}(Z_0)$ be an element such that $\pi\eta \in G_{\mathcal{C}}$, $\mathcal{C} \in C(W)$ and $s^{-1} \in \mathcal{C}$, $d = 2d'$, where d' is defined in Proposition 6.1.1. Assume that m and d are coprime.*

Then $U_\eta(\mathfrak{g}) \simeq \mathrm{Mat}_b(W_{\varepsilon,\xi}^s(G))$, where $\xi \in \mathrm{Spec}Z_0$ is chosen as in Proposition 6.4.2, and $b = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}}$.

Let $\mathcal{L}$ be a sheaf of algebras over $\mathrm{Spec}Z_0$ the stalk of which over $\eta \in \mathrm{Spec}Z_0$ is $U_\eta(\mathfrak{g})$. Assume that the conditions imposed on m are satisfied for all Weyl group conjugacy classes in $C(W)$. Then the sheaf $\mathcal{L}$ is isomorphic to a sheaf the stalk of which over any $\eta \in \mathrm{Spec}Z_0$ with $\pi\eta \in G^0 \cap G_{\mathcal{C}}$, $\mathcal{C} \in C(W)$ is $\mathrm{Mat}_b(W_{\varepsilon,\xi}^s(G))$, where $\xi \in \mathrm{Spec}Z_0$ is chosen as in Proposition 6.4.2, $s^{-1} \in \mathcal{C}$, $b = m^{\frac{1}{2}\dim \mathcal{O}_{\pi\eta}}$.

6.5 Bibliographic comments

The study of representations of quantum groups at roots of unity was initiated in [23], where the quantum coadjoint action was defined as well. This action was studied in detail in [25] where the De Concini–Kac–Procesi conjecture on the dimensions of irreducible representations of quantum groups at roots of unity was formulated.

The results on quantum groups at roots of unity stated in Section 6.1 can be found in [23] and [25]. Proposition 6.1.1 first appeared in Appendix A to [118]. The statements of Proposition 6.1.2 can be found in [23], Corollary 3.3, [25], Theorems 3.5, 7.6 and Proposition 4.5. Proposition 6.1.4 is Corollary 4.7 in [25], and the statements of Proposition 6.1.5 appear in Propositions 3.4, 3.5, [23], and in Proposition 6.1 and Theorem 6.6 in [25]. Finite–dimensional quotients $U_\eta(\mathfrak{g})$ were introduced in [24].

The notions of Whittaker vectors for representations of quantum groups at roots of unity, of the algebras $U_{\eta_1}(\mathfrak{m}_-)$, and of their actions on finite–dimensional representations of $U_\eta(\mathfrak{g})$ were introduced in [117], and the exposition in Section 6.2 follows [117] as well.

In the representation theory of Lie algebras in prime characteristic there is a conjecture similar to the De Concini–Kac–Procesi conjecture. It is called the Kac–Weisfeiler conjecture. Our proof of the De Concini–Kac–Procesi conjecture is conceptually similar to the proof of the Kac–Weisfeiler conjecture given in [120] which is in turn a straightforward prime characteristic generalization of the proof of the Skryabin equivalence for reductive Lie algebras over algebraically closed fields of zero characteristic suggested in the Appendix to [96]. All these proofs go back to the original Kostant's idea on the proof of the classification theorem for Whittaker representations of complex semisimple Lie algebras in [72], the proof of the Skryabin equivalence in [96] being a significantly refined and simplified version of the proof of the main Theorem 3.3 in [72].

The properties of q-W–algeras at roots of unity are similar to those of W–algebras associated to semisimple Lie algebras in prime characteristic proved in [97], Proposition 6.4.2 being an analogue of Theorem 2.3 in [97].

Appendix

Appendix 1. Normal orderings of root systems compatible with involutions in Weyl groups

By Theorem A in [101] every involution w in the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ is the longest element of the Weyl group of a Levi subalgebra in $\mathfrak{g}$ with respect to some system of positive roots, and w acts by multiplication by -1 in the Cartan subalgebra $\mathfrak{h}_w \subset \mathfrak{h}$ of the semisimple part $\mathfrak{m}_w$ of that Levi subalgebra. By Lemma 5 in [18] the involution w can also be expressed as a product of $\dim \mathfrak{h}_w$ reflections from the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$, with respect to mutually orthogonal roots, $w = s_{\gamma_1} \dots s_{\gamma_n}$, and the roots $\gamma_1, \dots, \gamma_n$ span the subalgebra $\mathfrak{h}_w$.

If w is the longest element in the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$ with respect to some system of positive roots, where $\mathfrak{m}_w$ is a simple Lie algebra and $\mathfrak{h}_w$ is a Cartan subalgebra of $\mathfrak{m}_w$, then w is an involution acting by multiplication by -1 in $\mathfrak{h}_w$ if and only if $\mathfrak{m}_w$ is of one of the following types: $A_1, B_l, C_l, D_{2n}, E_7, E_8, F_4, G_2$.

Fix a system of positive roots $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$. Let $w = s_{\gamma_1} \dots s_{\gamma_n}$ be a representation of w as a product of $\dim \mathfrak{h}_w$ reflections from the Weyl group of the pair $(\mathfrak{m}_w, \mathfrak{h}_w)$, with respect to mutually orthogonal positive roots. A normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ is called compatible with the decomposition $w = s_{\gamma_1} \dots s_{\gamma_n}$ if it is of the following form

$$\beta_1, \dots, \beta_{\frac{p-n}{2}}, \gamma_1, \dots, \gamma_2, \dots, \gamma_3, \dots, \gamma_n,$$

where p is the number of positive roots, and for any two positive roots $\alpha, \beta \in \Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ such that $\gamma_1 \leq \alpha < \beta$ the sum $\alpha + \beta$ cannot be represented as a linear combination $\sum_{k=1}^{q} c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \dots < \gamma_{i_k} < \beta$. Note that from the definition it also follows that

$$|[\beta_1, \beta_{\frac{p-n}{2}}]| = \frac{p-n}{2}, \quad |[\gamma_1, \dots, \gamma_n]| = \frac{p+n}{2}. \tag{A1.1}$$

Existence of such compatible normal orderings is checked straightforwardly for all simple Lie algebras of types $A_1, B_l, C_l, D_{2n}, E_7, E_8, F_4$ and G_2. In case A_1 this is obvious since there is only one positive root. In the other cases normal orderings defined by the properties described below for each of the types $B_l, C_l, D_{2n}, E_7, E_8, F_4, G_2$ exist and are compatible with decompositions of nontrivial involutions in Weyl group. We use the Bourbaki notation for the systems of positive and simple roots (see [10]).

- B_l

  Dynkin diagram:

  Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = \varepsilon_l$.

  Positive roots: $\varepsilon_i \ (1 \leq i \leq l)$, $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \ (1 \leq i < j \leq l)$.

  The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{\varepsilon_1} \dots s_{\varepsilon_l}$.

  Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{\varepsilon_1} \dots s_{\varepsilon_l}$:

  $$\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_1, \dots, \varepsilon_2, \dots, \varepsilon_l,$$

where the roots $\varepsilon_i - \varepsilon_j$ $(1 \leq i < j \leq l)$ forming the subsystem $\Delta_+(A_{l-1}) \subset \Delta_+(B_l)$ are situated to the left from ε_1, and the roots $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$ are situated to the right from ε_1.

- C_l

  Dynkin diagram:

$$\alpha_1 \quad\quad \alpha_2 \quad\quad\quad\quad\quad \alpha_{l-2} \quad\quad \alpha_{l-1} \quad\quad \alpha_l$$
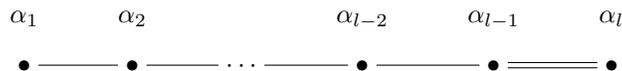
  Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l, \alpha_l = 2\varepsilon_l$.

  Positive roots: $2\varepsilon_i$ $(1 \leq i \leq l)$, $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$.

  The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{2\varepsilon_1} \ldots s_{2\varepsilon_l}$.

  Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{2\varepsilon_1} \ldots s_{2\varepsilon_l}$:

$$\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, 2\varepsilon_1, \ldots, 2\varepsilon_2, \ldots, 2\varepsilon_l,$$

  where the roots $\varepsilon_i - \varepsilon_j$ $(1 \leq i < j \leq l)$ forming the subsystem $\Delta_+(A_{l-1}) \subset \Delta_+(C_l)$ are situated to the left from $2\varepsilon_1$, and the roots $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$ are situated to the right from $2\varepsilon_1$.

- D_{2n}

  Dynkin diagram:

$$\alpha_{2n-1}$$
$$\alpha_1 \quad\quad \alpha_2 \quad\quad\quad\quad \alpha_{2n-3} \quad\quad \alpha_{2n-2}$$

$$\alpha_{2n}$$
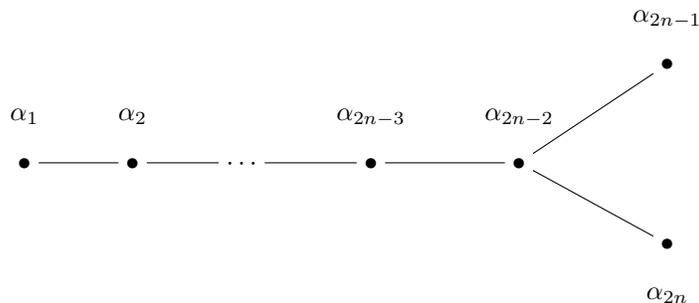
  Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{2n-1} = \varepsilon_{2n-1} - \varepsilon_{2n}, \alpha_{2n} = \varepsilon_{2n-1} + \varepsilon_{2n}$.

  Positive roots: $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 2n)$.

  The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots:

$$w = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2} \ldots s_{\varepsilon_{2n-1} - \varepsilon_{2n}} s_{\varepsilon_{2n-1} + \varepsilon_{2n}}.$$

  Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression

$$w = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2} \ldots s_{\varepsilon_{2n-1} - \varepsilon_{2n}} s_{\varepsilon_{2n-1} + \varepsilon_{2n}} :$$

$$\varepsilon_2 - \varepsilon_3, \varepsilon_4 - \varepsilon_5, \ldots, \varepsilon_{2n-2} - \varepsilon_{2n-1}, \ldots, \varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4, \ldots, \varepsilon_{2n-1} - \varepsilon_{2n-2},$$
$$\varepsilon_1 + \varepsilon_2, \ldots, \varepsilon_3 + \varepsilon_4, \ldots, \varepsilon_{2n-1} + \varepsilon_{2n},$$

  where the roots $\varepsilon_i - \varepsilon_j$ $(1 \leq i < j \leq l)$ forming the subsystem $\Delta_+(A_{l-1}) \subset \Delta_+(C_l)$ are situated to the left from $\varepsilon_1 + \varepsilon_2$, and the roots $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq l)$ are situated to the right from $\varepsilon_1 + \varepsilon_2$.

- E_7

  Dynkin diagram:

Simple roots: $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)$, $\alpha_2 = \varepsilon_1 + \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_1, \alpha_4 = \varepsilon_3 - \varepsilon_2, \alpha_5 = \varepsilon_4 - \varepsilon_3, \alpha_6 = \varepsilon_5 - \varepsilon_4, \alpha_7 = \varepsilon_6 - \varepsilon_5$.

Positive roots: $\pm\varepsilon_i + \varepsilon_j$ $(1 \le i < j \le 6)$, $\varepsilon_8 - \varepsilon_7$, $\frac{1}{2}(\varepsilon_8 - \varepsilon_7 + \sum_{i=1}^{6}(-1)^{\nu(i)}\varepsilon_i)$ with $\sum_{i=1}^{6}\nu(i)$ odd.

The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots:

$$w = s_{\varepsilon_2 - \varepsilon_1} s_{\varepsilon_2 + \varepsilon_1} s_{\varepsilon_4 - \varepsilon_3} s_{\varepsilon_4 + \varepsilon_3} s_{\varepsilon_6 - \varepsilon_5} s_{\varepsilon_6 + \varepsilon_5} s_{\varepsilon_8 - \varepsilon_7}.$$

Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression

$$w = s_{\varepsilon_2 - \varepsilon_1} s_{\varepsilon_2 + \varepsilon_1} s_{\varepsilon_4 - \varepsilon_3} s_{\varepsilon_4 + \varepsilon_3} s_{\varepsilon_6 - \varepsilon_5} s_{\varepsilon_6 + \varepsilon_5} s_{\varepsilon_8 - \varepsilon_7} :$$

$$\alpha_1, \varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \dots, \varepsilon_8 - \varepsilon_7, \dots, \varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \dots,$$
$$\varepsilon_6 + \varepsilon_5, \dots, \varepsilon_4 + \varepsilon_3, \dots, \varepsilon_2 + \varepsilon_1,$$

where the roots $\pm\varepsilon_i + \varepsilon_j$ $(1 \le i < j \le 6)$ forming the subsystem $\Delta_+(D_6) \subset \Delta_+(E_7)$ are placed as in case of the compatible normal ordering of the system $\Delta_+(D_6)$, the only roots from the subsystem $\Delta_+(A_5) \subset \Delta_+(D_6)$ situated to the right from the maximal root $\varepsilon_8 - \varepsilon_7$ are $\varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5$, the roots $\varepsilon_i + \varepsilon_j$ $(1 \le i < j \le 6)$ are situated to the right from $\varepsilon_6 + \varepsilon_5$, and a half of the positive roots which do not belong to the subsystem $\Delta_+(D_6) \subset \Delta_+(E_7)$ are situated to the left from $\varepsilon_8 - \varepsilon_7$ and the other half of those roots are situated to the right from $\varepsilon_8 - \varepsilon_7$.

- E_8

  Dynkin diagram:

Simple roots: $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)$, $\alpha_2 = \varepsilon_1 + \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_1, \alpha_4 = \varepsilon_3 - \varepsilon_2, \alpha_5 = \varepsilon_4 - \varepsilon_3, \alpha_6 = \varepsilon_5 - \varepsilon_4, \alpha_7 = \varepsilon_6 - \varepsilon_5, \alpha_8 = \varepsilon_7 - \varepsilon_6$.

Positive roots: $\pm\varepsilon_i + \varepsilon_j$ $(1 \le i < j \le 8)$, $\frac{1}{2}(\varepsilon_8 + \sum_{i=1}^{7}(-1)^{\nu(i)}\varepsilon_i)$ with $\sum_{i=1}^{7}\nu(i)$ even.

The longest element of the Weyl group expressed as a product of dim $\mathfrak{h}_w$ reflections with respect to mutually orthogonal roots:

$$w = s_{\varepsilon_2 - \varepsilon_1} s_{\varepsilon_2 + \varepsilon_1} s_{\varepsilon_4 - \varepsilon_3} s_{\varepsilon_4 + \varepsilon_3} s_{\varepsilon_6 - \varepsilon_5} s_{\varepsilon_6 + \varepsilon_5} s_{\varepsilon_8 - \varepsilon_7} s_{\varepsilon_8 + \varepsilon_7}.$$

Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression

$$w = s_{\varepsilon_2 - \varepsilon_1} s_{\varepsilon_2 + \varepsilon_1} s_{\varepsilon_4 - \varepsilon_3} s_{\varepsilon_4 + \varepsilon_3} s_{\varepsilon_6 - \varepsilon_5} s_{\varepsilon_6 + \varepsilon_5} s_{\varepsilon_8 - \varepsilon_7} s_{\varepsilon_8 - \varepsilon_7} :$$

$$\alpha_1, \varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \varepsilon_7 - \varepsilon_6, \ldots, \varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5,$$
$$\varepsilon_8 - \varepsilon_7, \ldots, \varepsilon_8 + \varepsilon_7, \ldots, \varepsilon_6 + \varepsilon_5, \ldots, \varepsilon_4 + \varepsilon_3, \ldots, \varepsilon_2 + \varepsilon_1,$$

where the roots $\pm \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 8)$ forming the subsystem $\Delta_+(D_8) \subset \Delta_+(E_8)$ are placed as in case of the compatible normal ordering of the system $\Delta_+(D_8)$, the roots $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 8)$ are situated to the right from $\varepsilon_8 + \varepsilon_7$; the positive roots which do not belong to the subsystem $\Delta_+(D_8) \subset \Delta_+(E_8)$ can be split into two groups: the roots from the first group contain $\frac{1}{2}(\varepsilon_8 + \varepsilon_7)$ in their decompositions with respect to the basis $\varepsilon_i, i = 1, \ldots, 8$, and the roots from the second group contain $\frac{1}{2}(\varepsilon_8 - \varepsilon_7)$ in their decompositions with respect to the basis $\varepsilon_i, i = 1, \ldots, 8$; a half of the roots from the first group are situated to the left from $\varepsilon_2 - \varepsilon_1$ and the other half of those roots are situated to the right from $\varepsilon_8 + \varepsilon_7$; a half of the roots from the second group are situated to the left from $\varepsilon_2 - \varepsilon_1$ and the other half of those roots are situated to the right from $\varepsilon_8 - \varepsilon_7$.

- F_4

  Dynkin diagram:

$$\alpha_1 \qquad \alpha_2 \qquad \alpha_3 \qquad \alpha_4$$
$$\bullet \!\!-\!\!-\!\! \bullet \!\!=\!\!=\!\! \bullet \!\!-\!\!-\!\! \bullet$$

  Simple roots: $\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$.

  Positive roots: ε_i $(1 \leq i \leq 4)$, $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq 4)$, $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$.

  The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{\varepsilon_1} s_{\varepsilon_2} s_{\varepsilon_3} s_{\varepsilon_4}$.
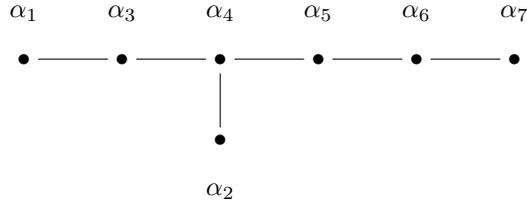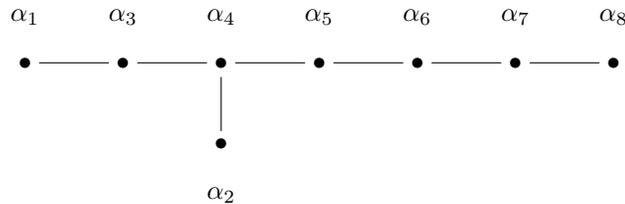
  Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{\varepsilon_1} s_{\varepsilon_2} s_{\varepsilon_3} s_{\varepsilon_4}$:

$$\alpha_4, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_3 - \varepsilon_4, \ldots, \varepsilon_1, \ldots, \varepsilon_2, \ldots, \varepsilon_4,$$

  where the roots $\varepsilon_i \pm \varepsilon_j$ $(1 \leq i < j \leq l)$ forming the subsystem $\Delta_+(B_4) \subset \Delta_+(F_4)$ are situated as in case of B_4, and a half of the positive roots which do not belong to the subsystem $\Delta_+(B_4) \subset \Delta_+(F_4)$ are situated to the left from ε_1 and the other half of those roots are situated to the right from ε_1.

- G_2

  Dynkin diagram:

$$\alpha_1 \qquad \alpha_2$$
$$\bullet \!\!\equiv\!\!\equiv\!\! \bullet$$

  Simple roots: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

  Positive roots: $\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_2$.

  The longest element of the Weyl group expressed as a product of $\dim \mathfrak{h}_w$ reflections with respect to mutually orthogonal roots: $w = s_{\alpha_1} s_{3\alpha_1 + 2\alpha_2}$.

  Normal ordering of $\Delta_+(\mathfrak{m}_w, \mathfrak{h}_w)$ compatible with expression $w = s_{\alpha_1} s_{3\alpha_1 + 2\alpha_2}$:

$$\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, \alpha_1.$$

Appendix 2. Transversal slices for simple exceptional algebraic groups.

In this appendix, for simple exceptional algebraic groups we present the data related to the varieties $\Sigma_{\mathbf{k},s}$ defined in Theorem 1.5.2. Let $G_{\mathbf{k}}$ be a connected simple algebraic group of an exceptional type over an algebraically closed field $\mathbf{k}$, and $\mathcal{O} \in \widehat{\underline{N}}(W)$. Let $H_{\mathbf{k}}$ be a maximal torus of $G_{\mathbf{k}}$, W the Weyl group of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$, and $s \in W$ an element from the conjugacy class $\Psi^W(\mathcal{O})$.

Let Δ be the root system of the pair $(G_{\mathbf{k}}, H_{\mathbf{k}})$ and Δ_+^s the system of positive roots in Δ associated to s and defined in Section 1.2 with the help of decomposition (1.2.14) where the subspaces $\mathfrak{h}_i$ are ordered in such a way that in sum (1.2.14) $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s, the one–dimensional subspaces $\mathfrak{h}_i$ on which s^1 acts by multiplication by -1 are immediately preceding $\mathfrak{h}_0$ in (1.2.14), and if $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\mathfrak{h}_j = \mathfrak{h}_\mu^l$ and $0 \le \lambda < \mu < 1$ then $i < j$. We also use a decomposition $s = s^1 s^2$ for which the direct sum $\bigoplus_{k=0, i_k > 0}^r \mathfrak{h}_{i_k}$ of the one–dimensional subspaces $\mathfrak{h}_{i_k}$ on which s^1 acts by multiplication by -1 is trivial. Such decomposition always exists. As a consequence condition (1.6.8) is satisfied. Let $\Sigma_{\mathbf{k},s}$ be the corresponding variety defined in Proposition 1.3.4 (ii).

Then straightforward calculation shows that

$$\dim Z_{G_p}(n) = \dim \Sigma_{\mathbf{k},s}$$

for any $n \in \mathcal{O} \in \underline{N}(G_p) \subset \widehat{\underline{N}}(W)$. The numbers $\dim Z_{G_p}(n)$ can be found in [75], Chapter 22 (note, however, that the notation in [75] for some classes is different from ours; we follow [85, 123]). The numbers $\dim \Sigma_{\mathbf{k},s}$ are contained in the tables below. These two numbers coincide in all cases. The tables below contain also the following information for each $\mathcal{O} \in \widehat{\underline{N}}(W)$:

– The Weyl group conjugacy class $\Psi^W(\mathcal{O})$ which can be found in [85];

– The two involutions s^1 and s^2 in the decomposition $s = s^1 s^2 \in \Psi^W(\mathcal{O})$; they are represented by sets of natural numbers which are the numbers of roots appearing in decompositions $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \dots s_{\gamma_{l'}}$, where the system of positive roots Δ_+^s is chosen as in Theorem 1.5.2, and the numeration of positive roots is given in Appendix 3;

– The dimension of the fixed point space $\mathfrak{h}_0$ for the action of s on $\mathfrak{h}$;

– The number $|\Delta_0|$ of roots fixed by s;

– The type of the root system Δ_0 fixed by s;

– The Dynkin diagram Γ_0^s of Δ_0, where the numbers at the vertices of Γ_0^s are the numbers of simple roots in Δ_+^s which appear in Γ_0^s; the numeration of simple roots is given in Appendix 3;

– The length $l(s)$ of s with respect to the system of simple roots in Δ_+^s;

– $\dim \Sigma_{\mathbf{k},s} = \dim \mathfrak{h}_0 + |\Delta_0| + l(s)$;

– The lowest common multiple d' of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$, where $i, j = 1, \dots, l$;

G₂.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	$\dim \mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	$\dim \Sigma_{\mathbf{k},s}$	d'
A_1	A_1	$-$	6	1	2	A_1	$\underset{\bullet}{1}$	5	8	1
$(\tilde{A}_1)_3$	$\tilde{A}_1$	$-$	4	1	2	A_1	$\underset{\bullet}{2}$	5	8	1
$\tilde{A}_1$	$A_1 + \tilde{A}_1$	$-$	$\begin{matrix}1\\6\end{matrix}$	0	0	$-$	$-$	6	6	1
$G_2(a_1)$	A_2	5	2	0	0	$-$	$-$	4	4	3
G_2	G_2	1	2	0	0	$-$	$-$	2	2	1

F$_4$.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
A_1	A_1	–	24	3	18	C_3	$\overset{4}{\bullet}\!-\!\overset{3}{\bullet}\!=\!\overset{2}{\bullet}$	15	36	1
$(\tilde{A}_1)_2$	$\tilde{A}_1$	–	21	3	18	B_3	$\overset{1}{\bullet}\!-\!\overset{2}{\bullet}\!=\!\overset{3}{\bullet}$	15	36	1
$\tilde{A}_1$	$2A_1$	–	16 24	2	8	B_2	$\overset{2}{\bullet}\!=\!\overset{3}{\bullet}$	20	30	1
$A_1+\tilde{A}_1$	$4A_1$	–	5 11 18 23	0	0	–	–	24	24	1
A_2	A_2	23	1	2	6	A_2	$\overset{3}{\bullet}\!-\!\overset{4}{\bullet}$	14	22	3
$\tilde{A}_2$	$\tilde{A}_2$	19	4	2	6	A_2	$\overset{1}{\bullet}\!-\!\overset{2}{\bullet}$	14	22	3
$(B_2)_2$	B_2	16	8	2	8	B_2	$\overset{2}{\bullet}\!=\!\overset{3}{\bullet}$	10	20	2
$A_2+\tilde{A}_1$	$A_2+\tilde{A}_1$	23	1 7	1	0	–	–	17	18	3
$(\tilde{A}_2+A_1)_2$	$\tilde{A}_2+A_1$	19	4 5	1	0	–	–	17	18	3
$\tilde{A}_2+A_1$	$A_2+\tilde{A}_2$	23 3	1 4	0	0	–	–	16	16	3
B_2	A_3	16	1 14	1	2	A_1	$\overset{3}{\bullet}$	13	16	2
$(C_3(a_1))_2$	B_2+A_1	16	8 9	1	2	A_1	$\overset{2}{\bullet}$	13	16	2
$C_3(a_1)$	$A_3+\tilde{A}_1$	16	5 6 11	0	0	–	–	14	14	2
$F_4(a_3)$	$D_4(a_1)$	16 2	5 11	0	0	–	–	12	12	1
B_3	D_4	1	16 9 2	0	0	–	–	10	10	1
C_3	C_3+A_1	4	1 3 14	0	0	–	–	10	10	1
$F_4(a_2)$	$F_4(a_1)$	1 3	9 10	0	0	–	–	8	8	1
$F_4(a_1)$	B_4	9 2	1 4	0	0	–	–	6	6	1
F_4	F_4	1 3	2 4	0	0	–	–	4	4	1

E$_6$.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
A_1	A_1	–	36	5	30	A_5	$\overset{1}{\bullet}\!-\!\overset{3}{\bullet}\!-\!\overset{4}{\bullet}\!-\!\overset{5}{\bullet}\!-\!\overset{6}{\bullet}$	21	56	1
$2A_1$	$2A_1$	–	23 36	4	12	A_3	$\overset{3}{\bullet}\!-\!\overset{4}{\bullet}\!-\!\overset{5}{\bullet}$	30	46	1
$3A_1$	$4A_1$	–	8 19 27 35	2	0	–	–	36	38	1
A_2	A_2	35	2	4	12	$2A_2$	$\overset{1}{\bullet}\!-\!\overset{3}{\bullet}\;\;\overset{5}{\bullet}\!-\!\overset{6}{\bullet}$	20	36	3
A_2+A_1	A_2+A_1	35	2 7	3	6	A_2	$\overset{5}{\bullet}\!-\!\overset{6}{\bullet}$	23	32	3
$2A_2$	$2A_2$	1 35	2 3	2	6	A_2	$\overset{5}{\bullet}\!-\!\overset{6}{\bullet}$	22	30	3
A_2+2A_1	A_2+2A_1	35	2 7 11	2	0	–	–	26	28	3
A_3	A_3	23	2 24	3	4	$2A_1$	$\overset{3}{\bullet}\;\;\;\overset{5}{\bullet}$	19	26	2
$2A_2+A_1$	$3A_2$	1 6 35	2 3 5	0	0	–	–	24	24	3
A_3+A_1	A_3+2A_1	23	2 3 5 24	1	0	–	–	21	22	2
$D_4(a_1)$	$D_4(a_1)$	4 23	8 19	2	0	–	–	18	20	2
A_4	A_4	21 24	1 2	2	2	A_1	$\overset{5}{\bullet}$	14	18	5
D_4	D_4	2	23 4 15	2	0	–	–	16	18	1
A_4+A_1	A_4+A_1	21 24	1 2 5	1	0	–	–	15	16	5
A_5	A_5+A_1	1 6	8 9 10 19	0	0	–	–	14	14	1
$D_5(a_1)$	$D_5(a_1)$	2 7	15 12 16	1	0	–	–	13	14	2
A_5+A_1	$E_6(a_2)$	1 2 6	9 10 19	0	0	–	–	12	12	1

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
D_5	D_5	4 15	1 2 6	1	0	–	–	9	10	2
$E_6(a_1)$	$E_6(a_1)$	6 8 9	1 2 5	0	0	–	–	8	8	1
E_6	E_6	1 4 6	2 3 5	0	0	–	–	6	6	1

E_7.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
A_1	A_1	–	63	6	60	D_6	nodes 7–6–5–4, branches 3 and 2	33	99	1
$2A_1$	$2A_1$	–	49 63	5	26	$A_1 + D_4$	node 7; nodes 2–4, branches 3 and 5	50	81	1
$(3A_1)''$	$(3A_1)'$	–	7 49 63	4	24	D_4	nodes 2–4, branches 3 and 5	51	79	1
$(3A_1)'$	$(4A_1)''$	–	19 40 41 63	3	6	$3A_1$	nodes 2, 3, 6	60	69	1
A_2	A_2	62	1	5	30	A_5	nodes 2–4–5–6–7	32	67	3
$4A_1$	$7A_1$	–	21 62 33 44 18 19 16	0	0	–	–	63	63	1
$A_2 + A_1$	$A_2 + A_1$	62	1 30	4	12	A_3	nodes 4–5–6	41	57	3
$A_2 + 2A_1$	$A_2 + 2A_1$	62	1 18 30	3	2	A_1	node 5	46	51	3

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
$A_2 + 3A_1$	$A_2 + 3A_1$	62	1, 5, 18, 30	2	0	–	–	47	49	3
$2A_2$	$2A_2$	23, 62	1, 7	3	6	A_2	$\overset{4}{\bullet}\!-\!\overset{5}{\bullet}$	40	49	3
A_3	A_3	49	1, 37	4	14	$A_1 + A_3$	$\overset{7}{\bullet}\quad\overset{2}{\bullet}\!-\!\overset{4}{\bullet}\!-\!\overset{5}{\bullet}$	31	49	2
$(A_3 + A_1)''$	$(A_3 + A_1)'$	49	1, 7, 37	3	12	A_3	$\overset{2}{\bullet}\!-\!\overset{4}{\bullet}\!-\!\overset{5}{\bullet}$	32	47	2
$2A_2 + A_1$	$3A_2$	5, 25, 62	1, 2, 6	1	0	–	–	42	43	3
$(A_3 + A_1)'$	$(A_3 + 2A_1)''$	49	7, 14, 26, 28	2	2	A_1	$\overset{3}{\bullet}$	37	41	2
$A_3 + 2A_1$	$A_3 + 3A_1$	49	1, 4, 7, 16, 37	1	0	–	–	38	39	2
$D_4(a_1)$	$D_4(a_1)$	3, 49	8, 32	3	6	$3A_1$	$\overset{2}{\bullet}\quad\overset{5}{\bullet}\quad\overset{7}{\bullet}$	30	39	2
$D_4(a_1) + A_1$	$D_4(a_1) + A_1$	3, 49	7, 8, 32	2	4	$2A_1$	$\overset{2}{\bullet}\quad\overset{5}{\bullet}$	31	37	2
D_4	D_4	1	3, 28, 49	3	6	$3A_1$	$\overset{2}{\bullet}\quad\overset{5}{\bullet}\quad\overset{7}{\bullet}$	28	37	1
$(A_3 + A_2)_2$	$A_3 + A_2$	22, 49	4, 20, 21	2	2	A_1	$\overset{7}{\bullet}$	33	37	6
$A_3 + A_2$	$D_4(a_1) + 2A_1$	3, 49	2, 7, 8, 32	1	2	A_1	$\overset{5}{\bullet}$	32	35	2
$A_3 + A_2 + A_1$	$2A_3 + A_1$	3, 49	7, 9, 11, 14, 26	0	0	–	–	33	33	2
A_4	A_4	37, 45	1, 6	3	6	A_2	$\overset{2}{\bullet}\!-\!\overset{4}{\bullet}$	24	33	5
A_5''	A_5'	6, 40	7, 20, 21	2	6	A_2	$\overset{3}{\bullet}\!-\!\overset{4}{\bullet}$	23	31	3

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
D_4+A_1	D_4+3A_1	1	5 18 28 29 30 31	0	0	–	–	31	31	1
A_4+A_1	A_4+A_1	37 45	1 6 9	2	0	–	–	27	29	5
A_4+A_2	A_4+A_2	4 37 45	1 2 6	1	0	–	–	26	27	15
$D_5(a_1)$	$D_5(a_1)$	1 18	24 28 36	2	2	A_1	5 •	23	27	2
$(A_5+A_1)''$	A_5+A_2	22 23 24	4 7 20 21	0	0	–	–	25	25	3
A_5'	$(A_5+A_1)''$	6 19	8 15 17 32	1	0	–	–	24	25	3
$D_5(a_1)+A_1$	$D_5(a_1)+A_1$	1 18	5 24 28 36	1	0	–	–	24	25	2
$D_6(a_2)$	$D_6(a_2)+A_1$	1 2	4 6 15 31 40	0	0	–	–	23	23	1
$(A_5+A_1)'$	$E_6(a_2)$	1 4 16	28 29 31	1	0	–	–	22	23	3
D_5	D_5	3 28	1 6 19	2	2	A_1	2 •	17	21	2
$D_6(a_2)+A_1$	$E_7(a_4)$	1 4 7	12 22 31 35	0	0	–	–	21	21	1
D_5+A_1	D_5+A_1	3 28	1 2 6 19	1	0	–	–	18	19	2
A_6	A_6	11 19 26	6 9 10	1	0	–	–	18	19	7
$D_6(a_1)$	$D_6(a_1)$	3 5 28	1 12 13	1	2	A_1	2 •	16	19	1

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
$D_6(a_1)+A_1$	A_7	1 12 13	9 10 11 22	0	0	–	–	17	17	1
D_6	D_6+A_1	1 6	2 3 5 7 28	0	0	–	–	15	15	1
$E_6(a_1)$	$E_6(a_1)$	8 19 22	1 4 6	1	0	–	–	14	15	3
E_6	E_6	3 6 19	1 9 11	1	0	–	–	12	13	3
D_6+A_1	$E_7(a_3)$	1 2 6	7 10 11 22	0	0	–	–	13	13	1
$E_7(a_2)$	$E_7(a_2)$	1 4 16	2 3 12 13	0	0	–	–	11	11	1
$E_7(a_1)$	$E_7(a_1)$	6 9 10	1 2 5 7	0	0	–	–	9	9	1
E_7	E_7	1 4 6	2 3 5 7	0	0	–	–	7	7	1

E$_8$.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
A_1	A_1	–	120	7	126	E_7	nodes 1—3—4—5—6—7 with 2 below 4	57	190	1
$2A_1$	$2A_1$	–	97 120	6	60	D_6	nodes 7—6—5—4 with branches 2 and 3 off 4	90	156	1

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
$3A_1$	$(4A_1)'$	–	7 61 97 120	4	24	D_4	D_4 diagram: nodes 2—4, 4—3, 4—5	108	136	1
A_2	A_2	119	8	6	72	E_6	E_6 diagram: 1—3—4—5—6, 4—2	56	134	3
$4A_1$	$8A_1$	–	9 13 19 50 67 69 83 119	0	0	–	–	120	120	1
A_2+A_1	A_2+A_1	119	8 69	5	30	A_5	A_5 diagram: 1—3—4—5—6	77	112	3
A_2+2A_1	A_2+2A_1	119	8 31 69	4	12	A_3	A_3 diagram: 3—4—5	86	102	3
A_3	A_3	97	8 74	5	40	D_5	D_5 diagram: 6—5—4, 4—2, 4—3	55	100	2
A_2+3A_1	A_2+4A_1	119	2 8 32 45 57	2	0	–	–	92	94	3
$2A_2$	$2A_2$	63 119	2 8	4	12	$2A_2$	$2A_2$ diagram: 1—3, 5—6	76	92	3
$2A_2+A_1$	$3A_2$	6 63 119	2 5 8	2	6	A_2	A_2 diagram: 1—3	78	86	3
A_3+A_1	$(A_3+2A_1)'$	97	7 22 61 62	3	12	A_3	A_3 diagram: 2—4—3	69	84	2
$D_4(a_1)$	$D_4(a_1)$	7 97	15 68	4	24	D_4	D_4 diagram: nodes 2—4, 4—3, 4—5	54	82	2

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
D_4	D_4	8	7 61 97	4	24	D_4	D_4 diagram (nodes 2 — 4, with 3 and 5)	52	80	1
$2A_2+2A_1$	$4A_2$	1 6 63 119	2 3 5 8	0	0	–	–	80	80	3
A_3+2A_1	A_3+4A_1	97	5 7 32 36 50 61	1	0	–	–	75	76	2
$D_4(a_1)+A_1$	$D_4(a_1)+A_1$	7 97	15 32 68	3	6	$3A_1$	$3A_1$ (nodes 2, 3, 5)	63	72	2
$(A_3+A_2)_2$	A_3+A_2	55 97	6 29 56	3	4	$2A_1$	$2A_1$ (nodes 2, 3)	65	72	6
A_3+A_2	$(2A_3)'$	7 97	13 22 40 62	2	4	$2A_1$	$2A_1$ (nodes 2, 3)	64	70	2
A_4	A_4	74 93	1 8	4	20	A_4	A_4 (nodes 2 — 4 — 5 — 6)	44	68	5
$A_3+A_2+A_1$	$2A_3+2A_1$	7 97	5 26 27 32 36 50	0	0	–	–	66	66	2
$D_4(a_1)+A_2$	$D_4(a_1)+A_2$	7 25 97	4 15 68	2	0	–	–	62	64	6
D_4+A_1	D_4+4A_1	8	9 13 25 35 59 63 80	0	0	–	–	64	64	1
$2A_3$	$2D_4(a_1)$	2 3 7 97	11 12 15 68	0	0	–	–	60	60	1
A_4+A_1	A_4+A_1	74 93	1 8 26	3	6	A_2	A_2 (nodes 4 — 5)	51	60	5

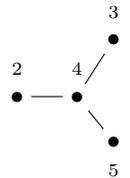| $\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
$D_5(a_1)$	$D_5(a_1)$	8 31	39 61 75	3	12	A_3	$\overset{3}{\bullet}-\overset{4}{\bullet}-\overset{5}{\bullet}$	43	58	2
$(D_4+A_2)_2$	D_4+A_2	8 63	2 49 59 71	2	0	–	–	54	56	3
A_4+2A_1	A_4+2A_1	74 93	1 8 12 26	2	0	–	–	54	56	5
A_4+A_2	A_4+A_2	18 74 93	1 6 8	2	2	A_1	$\overset{4}{\bullet}$	50	54	15
$A_4+A_2+A_1$	$A_4+A_2+A_1$	18 74 93	1 4 6 8	1	0	–	–	51	52	15
A_5	$(A_5+A_1)'$	1 44	15 34 35 68	2	6	A_2	$\overset{4}{\bullet}-\overset{5}{\bullet}$	44	52	3
$D_5(a_1)+A_1$	$D_5(a_1)+A_1$	8 31	19 39 61 75	2	2	A_1	$\overset{4}{\bullet}$	48	52	2
D_4+A_2	D_4+A_3	8 31	2 32 53 61 64	1	0	–	–	49	50	2
$(A_5+A_1)''$	$E_6(a_2)$	1 8 44	34 35 68	2	6	A_2	$\overset{4}{\bullet}-\overset{5}{\bullet}$	42	50	3
D_5	D_5	7 61	1 8 44	3	12	A_3	$\overset{2}{\bullet}-\overset{4}{\bullet}-\overset{5}{\bullet}$	33	48	2
A_4+A_3	$2A_4$	2 5 74 93	1 4 6 8	0	0	–	–	48	48	5
$D_5(a_1)+A_2$	$D_5(a_1)+A_3$	4 8 31	3 5 39 61 75	0	0	–	–	46	46	2
$(A_5+A_1)'$	$A_5+A_2+A_1$	23 24 25	1 15 34 35 68	0	0	–	–	46	46	3

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
$D_6(a_2)$	$2D_4$	2 8	11 12 31 53 61 64	0	0	–	–	44	44	1
A_5+2A_1	$E_6(a_2)+A_2$	1 5 8 44	4 34 35 68	0	0	–	–	44	44	3
A_5+A_2	$E_7(a_4)+A_1$	2 5 8	9 19 41 59 76	0	0	–	–	42	42	1
D_5+A_1	D_5+2A_1	7 61	3 8 16 30 32	1	0	–	–	39	40	2
$2A_4$	$E_8(a_8)$	1 2 6 8	19 41 59 76	0	0	–	–	40	40	1
$D_6(a_1)$	$D_6(a_1)$	3 7 61	8 9 37	2	4	$2A_1$	$\overset{2}{\bullet}\quad\overset{5}{\bullet}$	32	38	2
A_6	A_6	40 44 62	1 13 14	2	2	A_1	$\overset{2}{\bullet}$	34	38	7
A_6+A_1	A_6+A_1	40 44 62	1 2 13 14	1	0	–	–	35	36	7
$D_6(a_1)+A_1$	A_7'	8 9 37	20 33 34 35	1	2	A_1	$\overset{5}{\bullet}$	33	36	2
$(D_5+A_2)_2$	D_5+A_2	7 25 61	4 8 23 24	1	0	–	–	35	36	6
D_5+A_2	A_7+A_1	8 9 37	5 20 33 34 35	0	0	–	–	34	34	2
$E_6(a_1)$	$E_6(a_1)$	15 44 55	1 6 8	2	6	A_2	$\overset{2}{\bullet}\!-\!\overset{4}{\bullet}$	26	34	3

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
D_6	D_6+2A_1	1 8	4 14 17 26 27 55	0	0	–	–	32	32	1
$D_7(a_2)$	$D_7(a_2)$	4 7 61	3 8 16 30	1	0	–	–	31	32	2
E_6	E_6	1 7 44	8 26 27	2	6	A_2	$\underset{\bullet}{4}\!\!-\!\!\underset{\bullet}{5}$	24	32	3
$(A_7)_3$	A_7''	24 38 48	10 11 14 15	1	0	–	–	31	32	4
A_7	$D_8(a_3)$	3 5 7 61	4 8 23 24	0	0	–	–	30	30	1
$E_6(a_1)+A_1$	$E_6(a_1)+A_1$	15 44 55	1 6 8 10	1	0	–	–	29	30	3
$D_8(a_3)$	A_8	22 23 26 31	7 11 12 25	0	0	–	–	28	28	3
D_6+A_1	$E_7(a_3)$	1 2 8	27 28 32 41	1	2	A_1	$\underset{\bullet}{5}$	25	28	1
$(D_7(a_1))_2$	$D_7(a_1)$	1 8 12	19 21 33 49	1	0	–	–	27	28	2
$D_7(a_1)$	$D_8(a_2)$	1 6 8	12 21 25 27 49	0	0	–	–	26	26	1
E_6+A_1	E_6+A_2	1 4 7 44	5 8 26 27	0	0	–	–	26	26	3
$E_7(a_2)$	$E_7(a_2)+A_1$	6 8 48	7 10 12 16 30	0	0	–	–	24	24	1

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\Delta_0|$	Δ_0	Γ_0^s	$l(s)$	dim $\Sigma_{\mathbf{k},s}$	d'
A_8	$E_8(a_6)$	1 2 5 8	21 25 27 49	0	0	–	–	24	24	1
D_7	$D_8(a_1)$	10 11 15 25	1 20 21 22	0	0	–	–	22	22	1
$E_7(a_2)+A_1$	$E_8(a_7)$	4 7 23 24	5 8 20 33	0	0	–	–	22	22	1
$E_7(a_1)$	$E_7(a_1)$	1 13 14	3 5 8 32	1	2	A_1	${}_2$ $\bullet$	17	20	1
$D_8(a_1)$	$E_8(a_3)$	3 7 23 24	4 8 26 27	0	0	–	–	20	20	1
$E_7(a_1)+A_1$	D_8	1 13 14	4 8 17 18 19	0	0	–	–	18	18	1
D_8	$E_8(a_5)$	10 16 20 22	2 3 5 7	0	0	–	–	16	16	1
E_7	E_7+A_1	1 6 8	2 3 5 7 32	0	0	–	–	16	16	1
E_7+A_1	$E_8(a_4)$	7 11 12 25	1 2 6 8	0	0	–	–	14	14	1
$E_8(a_2)$	$E_8(a_2)$	2 3 5 7	1 8 10 20	0	0	–	–	12	12	1
$E_8(a_1)$	$E_8(a_1)$	6 8 10 11	1 2 5 7	0	0	–	–	10	10	1
E_8	E_8	1 4 6 8	2 3 5 7	0	0	–	–	8	8	1

Appendix 3. Irreducible root systems of exceptional types.

In this Appendix we give the lists of positive roots in irreducible root systems of exceptional types. All simple roots are numbered as shown at the Dynkin diagrams. The other roots in each list are given in terms of their coordinates with respect to the basis of simple roots. The coordinates are indicated in the brackets ( ). Each set of coordinates is preceded by the number of the corresponding root. These numbers are used to indicate roots which appear in the columns s^1, s^2 and Γ_0^s in the tables in Appendix 2.

G$_2$.

```
  1       2
  •═══════•
```

1   (1 0)

2   (0 1)

3   (1 1)

4   (2 1)

5   (3 1)

6   (3 2)

F4.

```
  1       2       3       4
  •───────•═══════•───────•
```

1	(1 0 0 0)	9	(0 1 2 0)	17	(1 2 2 1)
2	(0 1 0 0)	10	(0 1 1 1)	18	(1 1 2 2)
3	(0 0 1 0)	11	(1 1 2 0)	19	(1 2 3 1)
4	(0 0 0 1)	12	(1 1 1 1)	20	(1 2 2 2)
5	(1 1 0 0)	13	(0 1 2 1)	21	(1 2 3 2)
6	(0 1 1 0)	14	(1 2 2 0)	22	(1 2 4 2)
7	(0 0 1 1)	15	(1 1 2 1)	23	(1 3 4 2)
8	(1 1 1 0)	16	(0 1 2 2)	24	(2 3 4 2)

E$_6$.

```
  1       3       4       5       6
  •───────•───────•───────•───────•
                  │
                  •
                  2
```
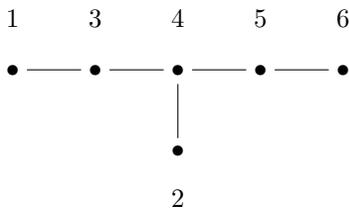
1	(1 0 0 0 0 0)	13	(0 1 1 1 0 0)	25	(0 1 1 1 1 1)
2	(0 1 0 0 0 0)	14	(0 1 0 1 1 0)	26	(1 1 1 2 1 0)
3	(0 0 1 0 0 0)	15	(0 0 1 1 1 0)	27	(1 1 1 1 1 1)
4	(0 0 0 1 0 0)	16	(0 0 0 1 1 1)	28	(0 1 1 2 1 1)
5	(0 0 0 0 1 0)	17	(1 1 1 1 0 0)	29	(1 1 2 2 1 0)
6	(0 0 0 0 0 1)	18	(1 0 1 1 1 0)	30	(1 1 1 2 1 1)
7	(1 0 1 0 0 0)	19	(0 1 1 1 1 0)	31	(0 1 1 2 2 1)
8	(0 1 0 1 0 0)	20	(0 1 0 1 1 1)	32	(1 1 2 2 1 1)
9	(0 0 1 1 0 0)	21	(0 0 1 1 1 1)	33	(1 1 1 2 2 1)
10	(0 0 0 1 1 0)	22	(1 1 1 1 1 0)	34	(1 1 2 2 2 1)
11	(0 0 0 0 1 1)	23	(1 0 1 1 1 1)	35	(1 1 2 3 2 1)
12	(1 0 1 1 0 0)	24	(0 1 1 2 1 0)	36	(1 2 2 3 2 1)

E₇.
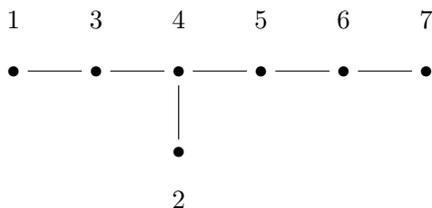
1	(1 0 0 0 0 0 0)	16	(0 1 0 1 1 0 0)	31	(0 0 1 1 1 1 1)
2	(0 1 0 0 0 0 0)	17	(0 0 1 1 1 0 0)	32	(1 1 1 2 1 0 0)
3	(0 0 1 0 0 0 0)	18	(0 0 0 1 1 1 0)	33	(1 1 1 1 1 1 0)
4	(0 0 0 1 0 0 0)	19	(0 0 0 0 1 1 1)	34	(1 0 1 1 1 1 1)
5	(0 0 0 0 1 0 0)	20	(1 1 1 1 0 0 0)	35	(0 1 1 2 1 1 0)
6	(0 0 0 0 0 1 0)	21	(1 0 1 1 1 0 0)	36	(0 1 1 1 1 1 1)
7	(0 0 0 0 0 0 1)	22	(0 1 1 1 1 0 0)	37	(1 1 2 2 1 0 0)
8	(1 0 1 0 0 0 0)	23	(0 1 0 1 1 1 0)	38	(1 1 1 2 1 1 0)
9	(0 1 0 1 0 0 0)	24	(0 0 1 1 1 1 0)	39	(1 1 1 1 1 1 1)
10	(0 0 1 1 0 0 0)	25	(0 0 0 1 1 1 1)	40	(0 1 1 2 2 1 0)
11	(0 0 0 1 1 0 0)	26	(1 1 1 1 1 0 0)	41	(0 1 1 2 1 1 1)
12	(0 0 0 0 1 1 0)	27	(1 0 1 1 1 1 0)	42	(1 1 2 2 1 1 0)
13	(0 0 0 0 0 1 1)	28	(0 1 1 2 1 0 0)	43	(1 1 1 2 2 1 0)
14	(1 0 1 1 0 0 0)	29	(0 1 1 1 1 1 0)	44	(1 1 1 2 1 1 1)
15	(0 1 1 1 0 0 0)	30	(0 1 0 1 1 1 1)	45	(0 1 1 2 2 1 1)

46	(1 1 2 2 2 1 0)	52	(1 1 1 2 2 2 1)	58	(1 2 2 3 2 2 1)
47	(1 1 2 2 1 1 1)	53	(1 2 2 3 2 1 0)	59	(1 1 2 3 3 2 1)
48	(1 1 1 2 2 1 1)	54	(1 1 2 3 2 1 1)	60	(1 2 2 3 3 2 1)
49	(0 1 1 2 2 2 1)	55	(1 1 2 2 2 2 1)	61	(1 2 2 4 3 2 1)
50	(1 1 2 3 2 1 0)	56	(1 2 2 3 2 1 1)	62	(1 2 3 4 3 2 1)
51	(1 1 2 2 2 1 1)	57	(1 1 2 3 2 2 1)	63	(2 2 3 4 3 2 1)

E$_8$.


```
      1     3     4     5     6     7     8
      •————•————•————•————•————•————•
                  |
                  •
                  2
```

1	(1 0 0 0 0 0 0 0)	22	(0 0 0 0 0 1 1 1)	43	(0 0 1 1 1 1 1 1)
2	(0 1 0 0 0 0 0 0)	23	(1 1 1 1 0 0 0 0)	44	(1 1 2 2 1 0 0 0)
3	(0 0 1 0 0 0 0 0)	24	(1 0 1 1 1 0 0 0)	45	(1 1 1 2 1 1 0 0)
4	(0 0 0 1 0 0 0 0)	25	(0 1 1 1 1 0 0 0)	46	(1 1 1 1 1 1 1 0)
5	(0 0 0 0 1 0 0 0)	26	(0 1 0 1 1 1 0 0)	47	(1 0 1 1 1 1 1 1)
6	(0 0 0 0 0 1 0 0)	27	(0 0 1 1 1 1 0 0)	48	(0 1 1 2 2 1 0 0)
7	(0 0 0 0 0 0 1 0)	28	(0 0 0 1 1 1 1 0)	49	(0 1 1 2 1 1 1 0)
8	(0 0 0 0 0 0 0 1)	29	(0 0 0 0 1 1 1 1)	50	(0 1 1 1 1 1 1 1)
9	(1 0 1 0 0 0 0 0)	30	(1 1 1 1 1 0 0 0)	51	(1 1 2 2 1 1 0 0)
10	(0 1 0 1 0 0 0 0)	31	(1 0 1 1 1 1 0 0)	52	(1 1 1 2 2 1 0 0)
11	(0 0 1 1 0 0 0 0)	32	(0 1 1 2 1 0 0 0)	53	(1 1 1 2 1 1 1 0)
12	(0 0 0 1 1 0 0 0)	33	(0 1 1 1 1 1 0 0)	54	(1 1 1 1 1 1 1 1)
13	(0 0 0 0 1 1 0 0)	34	(0 1 0 1 1 1 1 0)	55	(0 1 1 2 2 1 1 0)
14	(0 0 0 0 0 1 1 0)	35	(0 0 1 1 1 1 1 0)	56	(0 1 1 2 1 1 1 1)
15	(0 0 0 0 0 0 1 1)	36	(0 0 0 1 1 1 1 1)	57	(1 1 2 2 2 1 0 0)
16	(1 0 1 1 0 0 0 0)	37	(1 1 1 2 1 0 0 0)	58	(1 1 2 2 1 1 1 0)
17	(0 1 1 1 0 0 0 0)	38	(1 1 1 1 1 1 0 0)	59	(1 1 1 2 2 1 1 0)
18	(0 1 0 1 1 0 0 0)	39	(1 0 1 1 1 1 1 0)	60	(1 1 1 2 1 1 1 1)
19	(0 0 1 1 1 0 0 0)	40	(0 1 1 2 1 1 0 0)	61	(0 1 1 2 2 2 1 0)
20	(0 0 0 1 1 1 0 0)	41	(0 1 1 1 1 1 1 0)	62	(0 1 1 2 2 1 1 1)
21	(0 0 0 0 1 1 1 0)	42	(0 1 0 1 1 1 1 1)	63	(1 1 2 3 2 1 0 0)
				64	(1 1 2 2 2 1 1 0)
				65	(1 1 2 2 1 1 1 1)

66	(1 1 1 2 2 2 1 0)	85	(1 2 2 3 3 2 1 0)	104	(2 2 3 4 3 2 2 1)
67	(1 1 1 2 2 1 1 1)	86	(1 2 2 3 2 2 1 1)	105	(1 2 3 4 3 3 2 1)
68	(0 1 1 2 2 2 1 1)	87	(1 1 2 3 3 2 1 1)	106	(1 2 2 4 4 3 2 1)
69	(1 2 2 3 2 1 0 0)	88	(1 1 2 3 2 2 2 1)	107	(2 2 3 4 3 3 2 1)
70	(1 1 2 3 2 1 1 0)	89	(1 2 2 4 3 2 1 0)	108	(1 2 3 4 4 3 2 1)
71	(1 1 2 2 2 2 1 0)	90	(1 2 2 3 3 2 1 1)	109	(2 2 3 4 4 3 2 1)
72	(1 1 2 2 2 1 1 1)	91	(1 2 2 3 2 2 2 1)	110	(1 2 3 5 4 3 2 1)
73	(1 1 1 2 2 2 1 1)	92	(1 1 2 3 3 2 2 1)	111	(2 2 3 5 4 3 2 1)
74	(0 1 1 2 2 2 2 1)	93	(1 2 3 4 3 2 1 0)	112	(1 3 3 5 4 3 2 1)
75	(1 2 2 3 2 1 1 0)	94	(1 2 2 4 3 2 1 1)	113	(2 3 3 5 4 3 2 1)
76	(1 1 2 3 2 2 1 0)	95	(1 2 2 3 3 2 2 1)	114	(2 2 4 5 4 3 2 1)
77	(1 1 2 3 2 1 1 1)	96	(1 1 2 3 3 3 2 1)	115	(2 3 4 5 4 3 2 1)
78	(1 1 2 2 2 2 1 1)	97	(2 2 3 4 3 2 1 0)	116	(2 3 4 6 4 3 2 1)
79	(1 1 1 2 2 2 2 1)	98	(1 2 3 4 3 2 1 1)	117	(2 3 4 6 5 3 2 1)
80	(1 2 2 3 2 2 1 0)	99	(1 2 2 4 3 2 2 1)	118	(2 3 4 6 5 4 2 1)
81	(1 2 2 3 2 1 1 1)	100	(1 2 2 3 3 3 2 1)	119	(2 3 4 6 5 4 3 1)
82	(1 1 2 3 3 2 1 0)	101	(2 2 3 4 3 2 1 1)	120	(2 3 4 6 5 4 3 2)
83	(1 1 2 3 2 2 1 1)	102	(1 2 3 4 3 2 2 1)		
84	(1 1 2 2 2 2 2 1)	103	(1 2 2 4 3 3 2 1)		

Bibliography

[1] Andersen, H. H., Polo, P., Kexin, W., Representations of quantum algebras, *Invent. Math.* **104** (1991), 1–60.

[2] Andersen, H. H., Jantzen, J. C., Soergel, W., Representations of quantum groups at a p–th root of unity and of semisimple groups in characteristic p: independence of p, *Astérisque* **220** (1994).

[3] Asherova, R. M., Smirnov, Yu. F., Tolstoy, V. N., Projection operators for the simple Lie groups, *Theor. Math. Phys.* **8**(1971), 813–825.

[4] Asherova, R. M., Smirnov, Yu. F., Tolstoy, V. N., Projection operators for the simple Lie groups. II. General scheme for construction of lowering operators. The case of the group SU(n), *Theor. Math. Phys.* **15** (1973), 392–393.

[5] Asherova, R. M., Smirnov, Yu. F., Tolstoy, V. N., A description of some class of projection operators for semisimple complex Lie algebras, *Matem. Zametki* **26** (1979), 499–504.

[6] Baader, F., Nipkow, T., Term Rewriting and All That, Cambridge University Press, Cambridge (1999).

[7] Bezrukavnikov, R., Mirkovic, I., Rumynin, D., Localization of modules for a semisimple Lie algebra in prime characteristic. With an appendix by Bezrukavnikov and Simon Riche, *Ann. of Math.* **167** (2008), 945–991.

[8] Belavin, A. A., Drinfeld, V. G., Solutions of the classical Yang-Baxter equation for simple Lie algebras, *Funct. Anal. Appl.* **16** (1981), 159–180.

[9] Bernstein, J. H., Gelfand, I. M., Gelfand, S. I., Schubert cells and cohomology of the spaces G/P, *Uspekhi Mat. Nauk* **28**, no. 3 (171) (1973), 3–26; *Russian Math. Surveys* **28**, no. 3 (1973), 1–26.

[10] Bourbaki, N., Groupes et algebras de Lie, Chap. 4,5,6, Paris, Hermann (1968).

[11] Brieskorn, E., Singular elements of semisimple algebraic groups, *Actes Congrès Intern. Math.* **2** (1970), 279–284.

[12] Brown, K. A., Goodearl, K. R., Lectures on Algebraic Quantum Groups, Birkhäuser (2002).

[13] Cantarini, N., Carnovale, G., Costantini, M., Spherical orbits and representations of $U_\varepsilon(\mathfrak{g})$, *Transf. Groups* **10** (2005), 29—62.

[14] Cantarini, N., The quantized enveloping algebra $U_q(sl(n))$ at the roots of unity, *Comm. Math. Phys.* **211** (2000), 207—230.

[15] Cantarini, N., Spherical orbits and quantized enveloping algebras, *Comm. Algebra* **27** (1999), 3439–3458.

[16] Cantarini, N., Mod-p reduction for quantum groups, *J. of Alg.* **202** (1998), 357—366.

[17] Carter, R. W., Simple groups of Lie type, John Wiley & Sons, Inc., New York (1989).

[18] Carter, R. W., Conjugacy classes in the Weyl group, *Compositio Math.* **25** (1972), 1–59.

[19] Carter, R. W., Finite groups of Lie type. Conjugacy classes and complex characters, John Wiley & Sons, Ltd., Chichester (1993).

[20] Chari, V., Pressley, A., A guide to quantum groups, Cambridge Univ. Press (1994).

[21] Coxeter, H. S. M., Regular Polytopes, Methuen & Co., London (1948).

[22] De Boer, J., Tjin, T., Quantization and representation theory of finite W–algebras, *Comm. math. Phys.*, **158** (1993), 485–516.

[23] De Concini, C., Kac, V. G., Representations of quantum groups at roots of 1. Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), *Progr. Math.* **92**, Birkhäuser Boston, Boston, MA (1990), 471–506.

[24] De Concini, C., Kac, V. G., Representations of quantum groups at roots of 1: reduction to the exceptional case, *Int. J. Mod. Phys. A* **7** *Suppl. 1A* (1992), 141–149.

[25] De Concini, C., Kac, V. G., Procesi, C., Quantum coadjoint action, *J. Amer. Math. Soc.* **5** (1992), 151–189.

[26] De Concini, C., Kac, V. G., Procesi, C., Some remarkable degenerations of quantum groups, *Comm. Math. Phys.* **157** (1993), 405–427.

[27] De Concini, C., Kac, V. G., Procesi, C., Some quantum analogues of solvable Lie groups. In: Geometry and analysis, Proceedings of the International Colloquium on Geometry and Analysis, Bombay, 1992, Oxford University Press (1995), 41–66.

[28] De Concini, C., Lyubashenko, V., Quantum function algebra at roots of 1, *Adv. in Math.* **108** (1994), 205–262.

[29] De Concini, C., Procesi, C., Quantum Schubert cells and representations at roots of 1. In: Algebraic groups and Lie groups (G. I. Lehrer ed.), *Austral. Math. Soc. Lect. Ser.* **9**, Cambridge Univ. Press, Cambridge (1997), 127–160.

[30] De Concini, C., Procesi, C., Quantum Groups. In: D-modules, Representation Theory, and Quantum Groups (L. Boutet de Monvel et al.), Proc. Venezia 1992, *Lecture Notes in Mathematics* **1565**, Springer, Berlin (1993), 31–140.

[31] Curtis, C. W., Reiner, I., Representation theory of finite groups and associative algebras, Interscience Publishers (1962).

[32] Drinfeld, V.G., Quantum groups, Proc. Int. Congr. Math. Berkley, California, 1986, Amer. Math. Soc., Providence (1987), 718–820.

[33] Drinfeld, V.G., Quasi–Hopf algebras, *Leningrad Math. Journal* **1** (1990), 1419–1457.

[34] Drupieski, C. M., On injective modules and support varieties for the small quantum group, *Int. Math. Res. Not.* **2011** (2011), 2263–2294.

[35] Ellers, E. W., Gordeev, N., Intersection of conjugacy classes with Bruhat cells in Chevalley groups, *Pacific J. Math.* **214** (2004), 245–260.

[36] Faddeev, L. D., Reshetikhin, N. Yu., Takhtajan, L. A., Quantization of Lie groups and Lie algebras, *Leningrad Math. J.* **1** (1989), 178–206.

[37] Fomin, S., Zelevinsky, A., Double Bruhat cells and total positivity, *J. Amer. Math. Soc.* **12** (1999), 335–380.

[38] Fomin, S., Zelevinsky, A., Recognizing Schubert cells, *Journal of Algebraic Combinatorics* **12** (2000), 37–57.

[39] Friedlander, E. M., Parshall, B. J., Support varieties for restricted Lie algebras *Invent. Math.* **86** (1986), 553–562.

[40] Friedlander, E. M., Parshall, B. J., Geometry of p-unipotent Lie algebras, *J. of Alg.* **109** (1987), 25–45.

[41] Friedlander, E. M., Parshall, B. J., Modular representation theory of Lie algebras, *Amer. J. Math.* **110** (1988), 1055–1093.

[42] Gan, W. L., Ginzburg, V., Quantization of Slodowy slices, *Int. Math. Res. Not.* **5** (2002), 243–255.

[43] Gasper, G., Rahman, M., Basic hypergeometric series, Cambridge Univ. Press (1990).

[44] Geck, M., Pfeiffer, G., Characters of finite Coxeter groups and Iwahori-Hecke algebras, *London Mathematical Society Monographs. New Series* **21**, The Clarendon Press, Oxford University Press, New York (2000).

[45] Geck, M., Malle, G., On the existence of a unipotent support for the irreducible characters of a finite group of Lie type, *Trans. Amer. Math. Soc.* **352** (2000), 429—456.

[46] Gelfand, I. M., Serganova, V. V., Combinatorial geometries and the strata of a torus on homogeneous compact manifolds, *Russian Math. Surveys* **42** (1987), 133–168.

[47] Ginzburg, V., Kumar, S., Cohomology of quantum groups at roots of unity, *Duke Math. J.* **69** (1993), 179–198.

[48] Goodearl, K. R., Yakimov, M. T., The Berenstein-Zelevinsky quantum cluster algebra conjecture, *J. Eur. Math. Soc.*, **22** (2020), 2453–2509.

[49] Goto, M., and Grosshans, F. D., Semisimple Lie algebras, Marcel Dekker, Inc., New York and Basel (1978).

[50] Hartshorne, R., Algebraic geometry, Springer, New York (2006).

[51] Harzheim, E., Ordered Sets, Springer, New York (2006).

[52] He, X., Lusztig, G., A generalization of Steinberg's cross-section, *J. Amer. Math. Soc.* **25** (2012), 739–757.

[53] He, X., Nie, S., Minimal length elements of finite Coxeter groups, *Duke Math. J.* **161** (2012), 2945–2967.

[54] Hesselink, W. H., Nilpotency in classical groups over a field of characteristic 2, *Math. Z.* **166** (1979), 165—181.

[55] Humphreys, J., E., Linear algebraic groups, Springer, New York (1981).

[56] Iwahori, N., On the structure of a Hecke ring of a Chevalley group over a finite field, *J. Fac, Sci. Uni. Tokyo*, Sect. **IA 10** (1964), 215–236.

[57] Jacobson, N., Lie algebras, Interscience Publishers, New York (1962).

[58] Jantzen, J. C., Lectures on Quantum Groups, *Graduate Studies in Mathematics* **6**, AMS (1996).

[59] Joseph, A., Quantum groups and their primitive ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **29**, Springer-Verlag, Berlin (1995).

[60] Joseph, A., Letzter, G., Local finiteness of the adjoint action for quantized enveloping algebras, *J. of Alg.* **153** (1992), 289–318.

[61] Kac, V. G., Weisfeiler, B., The irreducible representations of Lie p-algebras, *Funk. Anal. i ego Pril.* **5** (1971), no. 2, 28—36.

[62] Kac, V., Cheung, P., Quantum Calculus, Springer, New York (2002).

[63] Kawanaka, N., Generalized Gelfand–Graev representations and Ennola duality, in: Algebraic Groups and Related Topics, *Advanced Studies in Pure Mathematics* **6**, North-Holland, Amsterdam/New York/Oxford (1985), 175–206.

[64] Kawanaka, N., Generalized Gelfand–Graev representations of exceptional simple groups over a finite field I, *Invent. Math.* **84** (1986), 575–616.

[65] Kazhdan, D., Lusztig, G., Fixed point varieties on affine flag manifolds, *Israel J. Math.* **62** (1988), 129–168.

[66] Khoroshkin, S. M., Ogievetsky O., Mickelsson algebras and Zhelobenko operators, *J. of Alg.* **319** (2008), 2113–2165.

[67] Khoroshkin, S. M., Tolstoy, V. N., Twisting of quantized Lie (super)algebras. Quantum groups (Karpacz, 1994), 63–84, Wydawnictwo Naukowe PWN, Warsaw (1995).

[68] Khoroshkin, S. M., Tolstoy, V. N., The Cartan-Weyl basis and the universal R-matrix for quantum Kac-Moody algebras and superalgebras. Quantum symmetries (Clausthal, 1991), 336–351, World Sci. Publ., River Edge, NJ (1993).

[69] Khoroshkin, S. M., Tolstoy, V. N., Universal R–matrix for quantized (super)algebras, *Comm. Math. Phys.* **141** (1991), 599–617.

[70] Klimyk, A., Schmüdgen, K., Quantum groups and their representations, Springer-Verlag, Heidelberg (1997).

[71] Kolb, S., Stokman, J. V., Reflection equation algebras, coideal subalgebras, and their centres, *Selecta Math. (New Ser.)* **15** (2009), 621–664.

[72] Kostant, B., On Whittaker vectors and representation theory, *Invent. Math.* **48** (1978), 101–184.

[73] Kremnizer, K., Proof of the De Concini-Kac-Procesi conjecture, arXiv:math/0611236.

[74] Kumar, S., Representations of quantum groups at roots of unity, *In: Quantum topology* (ed. by D.N. Yetter), World Scientific, Singapore (1994) 187–224.

[75] Liebeck, M. W., Seitz, G. M., Unipotent and nilpotent classes in simple algebraic groups and Lie algebras, *Mathematical Surveys and Monographs* **180**, American Mathematical Society, Providence, RI (2012).

[76] Losev, I., Finite dimensional representations of W-algebras, *Duke Math. J.* **159** (2011), 99–143.

[77] Losev, I., Ostrik, V., Classification of finite dimensional irreducible modules over W-algebras, *Compositio Math.* **150** (2014), 1024–1076.

[78] Löwdin, P.-O., Angular momentum wave functions constructed by projector operators, *Rev. Mod. Phys.* **36** (1964), 966–976.

[79] Lu J. H., Momentum mapping and reduction of Poisson actions. *In*: Symplectic geometry, groupoids and integrable systems, Berkeley, 1989. P.Dazord and A.Weinstein (eds), pp.209-226. Springer-Verlag.

[80] Lu, J. H., Weinstein, A., Poisson Lie groups, dressing transformations, and Bruhat decompositions, *J. Diff. Geom* **31** (1990), 501–526.

[81] Lusztig, G., Introduction to quantum groups, Birkhäuser (1994).

[82] Lusztig, G., Quantum groups at roots of 1, *Geom. Dedicata* **35** (1990), no. 1–3, 89–113.

[83] Lusztig, G., On conjugacy classes in a reductive group. In: Representations of Reductive Groups (M. Nevins, P. E. Trapa eds.), *Progress in Mathematics* **312**, Birkhäuser (2015), 333–363.

[84] Lusztig, G., Spaltenstein, N., On the generalized Springer correspondence for classical groups. Algebraic groups and related topics (Kyoto/Nagoya, 1983), 289–316, *Adv. Stud. Pure Math.* **6**, North-Holland, Amsterdam, (1985).

[85] Lusztig, G., From conjugacy classes in the Weyl group to unipotent classes II. *Represent. Theory* **16** (2012), 189–211.

[86] Lusztig, G., From conjugacy classes in the Weyl group to unipotent classes, *Represent. Theory* **15** (2011), 494—530.

[87] Lusztig, G., Unipotent elements in small characteristic, *Transform. Groups* **10** (2005), 449–487.

[88] Lusztig, G., Quantum Deformations of Certain Simple Modules over Enveloping Algebras, *Adv. Math.* **70** (1988), 237–249.

[89] Lynch, T. E., Generalized Whittaker vectors and representation theory, Thesis, M.I.T. (1979).

[90] Malten, W., Minimally dominant elements of finite Coxeter groups, arXiv:2110.09266.

[91] Malten, W., From braids to transverse slices in reductive groups, arXiv:2111.01313.

[92] McConnell, J.C., Quantum groups, filtered rings and Gelfand–KIrillov dimension, In *Lecture Notes in Math* **1448**, Springer, Berlin (1991), 139–149.

[93] Ostrik, V. V., Cohomological supports for quantum groups, *Funct. Anal. Appl.* **32** (1999), 237–246.

[94] Passman, D. S., The algebraic structure of group rings, Jonh Wiley & Sons, New York (1977).

[95] Pierce, R.S., Associative algebras. *Graduate Texts in Mathematics* **88**, Springer-Verlag, New York-Berlin (1982).

[96] Premet, A., Skryabin, S., Representations of restricted Lie algebras and families of associative $\mathcal{L}$–algebras, *J. Reine Angew. Math.* **507** (1999), 189–218.

[97] Premet, A., Special transverse slices and their enveloping algebras. With an appendix by Serge Skryabin, *Adv. Math.* **170** (2002), 1–55.

[98] Premet, A., Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture, *Invent. Math.* **121** (1995), 79–117.

[99] Premet, A., Support varieties of non-restricted modules over Lie algebras of reductive groups, *J. London Math. Soc.* (2) **55** (1997), 236–250.

[100] Reshetikhin, N.Yu., Semenov-Tian-Shansky, M.A., Factorization problems for quantum groups, in: *Geometry and Physics, essays in honour of I. M. Gelfand (S. Gindikin and I. M. Singer eds.)*, North Holland, Amsterdam - London - New York (1991), 533–550.

[101] Richardson, R.W., Conjugacy classes of involutions in Coxeter groups, *Bull. Austral. Math. Soc.* **26** (1982), 1–15.

[102] Rotman, S., Advanced Linear Algebra, Springer, New York (2005).

[103] Saito, Y., PBW Basis of Quantized Universal Enveloping Algebras, *Publ. Math. RIMS* **30** (1994), 209–232.

[104] Semenov-Tian-Shansky, M.A., What is a classical r-matrix, *Funct. Anal. Appl.* **17** (1983), 17–33.

[105] Semenov-Tian-Shansky, M.A., Dressing transformations and Poisson–Lie group actions, *Publ. Math. RIMS* **21** (1985), 1237–1260.

[106] Semenov-Tian-Shansky, M. A., Poisson Lie groups, quantum duality principle and the quantum double, *Contemp. Math.* **175** (1994) , 219–248.

[107] Semenov-Tian-Shansky, M. A., Sevostyanov, A., Drinfeld-Sokolov reduction for difference operators and deformations of W-algebras. II. General Semisimple Case, *Comm. Math. Phys.* **192** (1998), 631–647.

[108] Sevostyanov, A., Regular nilpotent elements and quantum groups, *Comm. Math. Phys.* **204** (1999) 1–16.

[109] Sevostyanov, A., Reduction of quantum systems with arbitrary first-class constraints and Hecke algebras, *Comm. Math. Phys.* **204** (1999), 137–146.

[110] Sevostyanov, A., The Whittaker model of the center of the quantum group and Hecke algebras, Ph.D. thesis, Uppsala (1999).

[111] Sevostyanov, A., Quantum deformation of Whittaker modules and the Toda lattice, *Duke Math. J.* **105** (2000), 211–238.

[112] Sevostyanov, A., Towards Drinfeld-Sokolov reduction for quantum groups, *J. Geom. Phys.* **33** (2000), 235–256.

[113] Sevostyanov, A., Algebraic group analogues of Slodowy slices and deformations of Poisson W–algebras, *Int. Math. Res. Not.* **2011** (2011), 1880–1925.

[114] Sevostyanov, A., Conjugacy classes in Weyl groups and q-W algebras, *Adv. Math.* **228** (2011), 1315–1376.

[115] Sevostyanov, A., The geometric meaning of Zhelobenko operators, *Transf. Groups.* **18** (2013), 865—875.

[116] Sevostyanov, A., Localization of quantum biequivariant $\mathcal{D}$-modules and q–W algebras, *Proc. London Math. Soc.* **113** (2016), 583—626.

[117] Sevostyanov A., Representations of quantum groups at roots of unity, Whittaker vectors and q-W algebras, *J. of Alg.* **511** (2018), 63–94.

[118] Sevostyanov A., Strictly transversal slices to conjugacy classes in algebraic groups, *J. of Alg.* **520** (2019), 309–366.

[119] Sevostyanov A., The structure of q-W algebras, *Transf. Groups* **25**, no. 1 (2020), 279–304.

[120] Skryabin, S., Representations of the Poisson algebra in prime characteristic, *Math. Z.* **243** (2003), 563–597.

[121] Slodowy, P., *Simple Singularities and Simple Algebraic Groups*, *Lecture Notes in Mathematics* **815**, Springer-Verlag, New York-Berlin (1980).

[122] Spaltenstein, N., Classes unipotentes et sous-groupes de Borel, *Lecture Notes in Mathematics* **946**, Springer-Verlag, Berlin-New York (1982).

[123] Spaltenstein, N., On the generalized Springer correspondence for exceptional groups. Algebraic groups and related topics (Kyoto/Nagoya, 1983), *Adv. Stud. Pure Math.* **6**, North-Holland, Amsterdam (1985), 317–338 .

[124] Springer, T. A., Linear Algebraic Groups, Birkhäuser, Boston (2009).

[125] Springer, T. A., Some Results on Algebraic Groups with Involutions. Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983), *Adv. Stud. Pure Math.* **6**, North-Holland, Amsterdam (1985), 525–543.

[126] Steinberg, R., Finite Reflection Groups, *Trans. Amer. Math. Soc.* **91** (1959), 493–504.

[127] Steinberg, R., Regular elements of semisimple algebraic groups, *Publ. Math. I.H.E.S.* **25** (1965), 49–80.

[128] Tolstoy, V. N., Fortieth anniversary of extremal projector method for Lie symmetries. Noncommutative geometry and representation theory in mathematical physics, in: *Contemp. Math.* **391**, Amer. Math. Soc., Providence, RI (2005), 371–384.

[129] Weinstein, A., Local structure of Poisson manifolds, *J. Different. Geom.* **18** (1983), 523–558.

[130] Yakimov, M., On the spectra of quantum groups, *Mem. Amer. Math. Soc.* **229** (2014).

[131] Yun, Z., Minimal reduction type and the Kazhdan-Lusztig map, *Indag. Math.* **32**, (2021), 1240–1274.

[132] Zelobenko, D. P., Compact Lie groups and their representations, Amer. Math. Soc., Providence (1973).

[133] Zelobenko, D. P., Extremal cocycles on Weyl groups, *Funk. Anal. i ego Pril.* **21**, no. 3 (1987), 11–21.

[134] Zelobenko, D. P., Constructive Modules and Extremal Projectors over Chevalley Algebras, *Funk. Anal. i ego Pril.* **27**, no. 3 (1993), 5–14; *Funct. Anal. Appl.* **27**, no. 3 (1993), 158–165.

[135] Zelobenko, D. P., S–algebras and Harish–Chandra modules over symmetric Lie algebras, *Izv. Akad. Nauk SSSR Ser. Mat.* **54**, no. 4 (1990), 659–675; *Math. USSR-Izv.* **37**, no 1 (1991), 1–17.

[136] Zelobenko, D. P., An introduction to the theory of S–algebras over reductive Lie algebras. In: Representation of Lie groups and related topics, *Adv. Stud. Contemp. Math.* **7**, Gordon and Breach, New York (1990), 155–221.

[137] Zelobenko, D. P., Extremal projectors and generalized Mickelsson algebras over reductive Lie algebras, *Izv. Akad. Nauk SSSR Ser. Mat.* **52**, no. 4 (1988), 758–773; *Math. USSR-Izv.* **33**, no. 1 (1989), 85–100.

[138] Zelobenko, D. P., S–algebras and Harish–Chandra modules over reductive Lie algebras, *Dokl. Akad. Nauk SSSR* **283**, no 6 (1985), 1306–1308.

[139] Zelobenko, D. P., Z-algebras over reductive Lie algebras, *Dokl. Akad. Nauk SSSR* **273**, no. 6 (1983), 1301–1304.

[140] Zelobenko, D. P., S-algebras and Verma modules over reductive Lie algebras, *Dokl. Akad. Nauk SSSR* **273**, no. 4 (1983), 785–788.

[141] Zelobenko, D. P., Representation Theory of Reductive Lie Algebras (in Russian), Nauka, Moscow (1994).