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RANDOM WALKS IN HYPERBOLIC SPACES: MARTIN BOUNDARIES AND BISECTORS

WERNER BALLMANN AND PANAGIOTIS POLYMERAKIS

Abstract. We discuss certain random walks on discrete groups of isometries of hyperbolic spaces and their Martin boundaries.

INTRODUCTION

In this note, we discuss an error in the article [3], which is about a relation between the Martin boundary of a connected and complete Riemannian manifold H and random walks on discrete subsets of the manifold, obtained by the Lyons-Sullivan discretization procedure of Brownian motion [8]. The error concerns the nature of elements of the Martin boundary of random walks, namely whether they are harmonic or only super-harmonic. Here we correct the error in the case where H is a Riemannian symmetric space with negative sectional curvature, that is, where H is a hyperbolic space, $H = H^k_{\mathbb{F}}$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, endowed with its standard metric. The most important application in [3] is covered by our discussion, namely the corollary on [3, page 80], which is Corollary B here.

From now on, let $H = H^k_{\mathbb{F}}$ and normalize the metric of H so that the maximum of its sectional curvature equals -1, and choose an origin $x_0 \in H$.

The geometric compactification $\bar{H}$ of H is homeomorphic to the closed Euclidean unit ball such that the unit sphere corresponds to the *geometric boundary* $\partial_{\mathrm{geo}} H$ of H.

Brownian motion $\mathcal{B}$ on H is the diffusion process associated to the *Laplace operator* Δ of H. Recall that $\mathcal{B}$ is transient. Therefore it has a positive *Green's function* $G = G(x, y)$ and *Martin kernel*

$$K = K(x, y) = G(x, y)/G(x_0, y).$$

Equating points $x \in H$ with Martin kernels $K(., x)$ induces an identification of the Martin compactification of H with $\bar{H}$ such that the *Martin boundary* $\partial_\Delta H$ is identified with the geometric boundary of H; see [1, 2, 7]. In this sense, we write $\partial_\Delta H = \partial_{\mathrm{geo}} H$.

Let Γ be a discrete subgroup of isometries of H with volume $|\Gamma\backslash H| < \infty$. Let $X = \Gamma x_0$ and $\mu = (\mu_x)_{x \in X}$ be a Γ-equivariant family of probability

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measures on X. Assume that the associated *random walk* $\mathcal{R}$ on X is non-degenerate and transient with Green's function $g = g(x,y) > 0$. Axiomatizing the corresponding notion from [4], we say that $\mathcal{R}$ is *adapted* if the Martin kernel

$$k = k(x,y) = g(x,y)/g(x_0,y)$$

of $\mathcal{R}$ satisfy $k(x,y) = K(x,y)$ for all $y \neq x, x_0$ in X. The Martin boundary of X with respect to μ is denoted by $\partial_\mu X$.

Remark 1.1. In the case at hand, Lyons and Sullivan construct a family μ, the LS-discretization of $\mathcal{B}$, depending on the choice of specific data, LS-data [8]. Since $\mathcal{B}$ is transient, they can be chosen so that the corresponding μ is adapted, symmetric, has finite first moment (w.r.t. the geometric norm on Γ) and finite entropy; see [3, Theorem 3.1 and Theorem 3.2(b)]. To avoid technicalities, we choose the above more axiomatic approach. Nevertheless, in Section 2, we discuss some aspects of LS-discretization.

The following result corrects [3, Theorem 3.2(a)] in the case of hyperbolic spaces.

Theorem A. *If $\mathcal{R}$ is adapted, then restriction to X induces a homeomorphism $\partial_{\mathrm{geo}} H \to \partial_\mu X$.*

Recall that the free group F_k with $k \geq 2$ generators admits actions on the real hyperbolic plane with cofinite volume (and corresponding geometric norm). As in [3], we obtain therefore the following application of Theorem A.

Corollary B. *There exists a symmetric random walk $\mathcal{R}$ on the free group F_k with $k \geq 2$ generators which has finite first moment (w.r.t. a geometric norm on F_k as above) and finite entropy such that the Martin boundary of $\mathcal{R}$ is equal to a circle.*

The error underlying the discussions of Martin boundaries of random walks in [3, 4] is the implicit assumption that Martin kernels $k(.,\xi)$ are μ-harmonic for all $\xi \in \partial_\mu X$. In fact, rethinking the matter, we have the following result in that direction.

Theorem C. *If $\mathcal{R}$ is adapted and a horoball with center $\xi \in \partial_{\mathrm{geo}} H$ is* precisely invariant *under Γ, then $k(.,\xi)$ is not μ-harmonic.*

Recall that a horoball B is said to be precisely invariant under Γ if $gB \cap B \neq \emptyset$ implies that $gB = B$, for any $g \in \Gamma$. Notice that ξ has precisely Γ-invariant horoballs centered at ξ if ξ corresponds to a cusp of $\Gamma\backslash H$. The set of such points is non-empty if Γ is not cocompact, and then it is a countable dense subset of $\partial_{\mathrm{geo}} H$.

Question 1.2. Are there natural assumptions which allow for a geometric characterization of those points $\xi \in \partial_{\mathrm{geo}} H$ such that $k(.,\xi)$ is μ-harmonic?

In our proofs of Theorem A and Theorem C, we use that the Martin kernels $K(.,\xi)$ are constant along horospheres centered at ξ, for all ξ in $\partial_{\mathrm{geo}} H$. More precisely, we have, with $m = \dim H$ and $d = \dim_{\mathbb{R}} \mathbb{F}$,

$$K(x,\xi) = e^{-hb(x)}, \tag{1.3}$$

where

$$h = m - d + 2(d-1) = m + d - 2 > 0 \tag{1.4}$$

and b is the Busemann function associated to ξ such that $b(x_0) = 0$.

For two points x, y in a metric space, their *bisector* is the set of points in the space of equal distance to x and y. Here we need a geometric description of bisectors of pairs of different points in hyperbolic spaces. In real hyperbolic geometry, the bisector is the hyperbolic hyperplane through the midpoint between the two points and perpendicular to the geodesic segment connecting them. In complex hyperbolic geometry, the situation is more complex, and bisectors are described in Goldman's [6, Section 5]. There is a corresponding description for quaternionic hyperbolic spaces and the hyperbolic octonionic plane, and this might be known to experts. However, we were not able to locate a reference for the latter two cases and discuss bisectors in an appendix. The analog of Goldman's description is as follows.

Proposition D. *Let $H = H^k_{\mathbb{F}}$ with $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $x \neq y$ be points in H. Let L be the unique totally geodesic real hyperbolic space in H of dimension $\dim_{\mathbb{R}} \mathbb{F}$ containing x and y of sectional curvature -4, and let z be a point in $H \setminus L$. Then there is a unique totally geodesic real hyperbolic plane in H of curvature -1 containing x, z, and πz. In particular,*

$$\cosh(d(x,z)) = \cosh(d(x,\pi z)) \cosh(d(z,\pi z)),$$

where $\pi \colon H \to L$ is the nearest point projection.

The above L is also called an $\mathbb{F}$-*line*. It is *spanned* by $x \neq y$. If, in the real hyperbolic case, the geodesic through x and y is understood to be the $\mathbb{R}$-line spanned by x and y, the following consequence holds in all four hyperbolic geometries.

Corollary E. *The bisector between $x \neq y$ in H is equal to the preimage, under the nearest point projection onto the $\mathbb{F}$-line L spanned by x and y, of the bisector between x and y in L.*

Recall that totally geodesic submanifolds of $H = H^k_{\mathbb{F}}$ are of codimension at least d so that bisectors are *not* totally geodesic unless $\mathbb{F} = \mathbb{R}$. This is one of the reasons that the geometry of H is more difficult for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

2. On Lyons-Sullivan discretization

In what follows, we use mostly the notation and terminology from [3, 4]: We let M be a connected Riemannian manifold, Δ its Laplacian, $\mathcal{B}$ its Brownian motion, and X a discrete subset of M. Given M and X, we denote by $(F, V) \sim (F_x, V_x)_{x \in X}$ (regular Lyons-Sullivan) *LS-data* for X. That is, we have

(D1) $x \in \mathring{F}_x$ and $F_x \subseteq V_x$ for all $x \in X$;
(D2) $F_x \cap V_y = \emptyset$ for all $x \neq y$ in X;
(D3) $F = \cup_{x \in X} F_x$ is closed and recurrent with respect to $\mathcal{B}$;
(D4) for all $x \in X$ and $y \in F_x$, the exit measures from V_x satisfy

$$1/C < d\varepsilon_y / d\varepsilon_x < C$$

for some constant $C > 1$, which does not depend on x and y.

Suppose from now on that Brownian motion $\mathcal{B}$ is transient. For that case, the notion of *balanced LS-data* was introduced in [3]; cf. [4, (D5) in Section 3]. The additional requirement is that there is a constant B such that the Green's functions of the V_x satisfy

(D5) $G_{V_x}(z,x) = B$ for all $x \in X$ and $z \in \partial F_x$.

For a properly discontinuous group Γ of isometries of M and an orbit X of Γ in M we say that LS-data (F,V) for X are Γ-*adapted* if X is an orbit of Γ and the data are Γ-equivariant.

Proposition 2.1. *For a properly discontinuous group Γ of isometries of M, an orbit $X \subseteq M$ admits Γ-adapted LS-data if $\mathcal{B}$ is recurrent modulo Γ. If, in addition, $\mathcal{B}$ is transient on M, then the data can be chosen balanced and Γ-adapted.*

Proof. Let $X = \Gamma x_0$ and Γ_0 be the stabilizer of x_0. Let $F_0 \subseteq V_0$ be compact respectively relatively compact open neighborhoods of x_0 which are invariant under the stabilizer Γ_0 of x_0 in Γ and such that $gF_0 \cap V_0 = \emptyset$ unless $g \in \Gamma_0$. For $g \in \Gamma$ and $x = gx_0$, let $F_x = gF_0$ and $V_x = \gamma V_0$. Since $\mathcal{B}$ is recurrent modulo Γ, $F = \cup_{x \in X} F_x$ is $\mathcal{B}$-recurrent. Hence $(F_x, V_x)_{x \in X}$ are Γ-adapted LS-data. The assertion that the data can be chosen balanced follows as in the (short) proof of [3, Theorem 3.2]. □

Since $\mathcal{B}$ is transient, M admits a Green's function, Martin kernels, and Martin boundary $\partial_\Delta M$. Suppose that (F,V) are Γ-adapted and balanced LS-data. Denote by $\mu = (\mu_y)_{y \in M}$ the associated family of LS-measures and by $\mathcal{R}$ the associated random walk on X. It is shown in [4, Theorem 3.29] that $\mathcal{R}$ is transient, and hence X has an associated Martin boundary $\partial_\mu X$.

Denote by $\mathcal{H}^+_F(M,\Delta)$ the space of positive harmonic functions on M, by $\mathcal{H}^+_F(M,\Delta)$ those, which are swept by F, by $\mathcal{H}^+(X,\mu)$ the space of positive μ-harmonic functions on X, and by $\partial_\Delta X$ the accumulation points of X in the Martin boundary $\partial_\Delta M$. By [5, Theorem 2.2] we have

Theorem 2.2. *Restriction $R\colon \partial_\Delta X \to \partial_\mu X$ of Martin kernels from M to X is Γ-equivariant, continuous, and surjective. Furthermore,*

$$R\colon \partial_\Delta X \cap \mathcal{H}^+_F(M,\Delta) \to \partial_\mu X \cap \mathcal{H}^+(X,\mu)$$

is a Γ-equivariant homeomorphism.

3. Negative curvature

In this section, we let H be a complete and simply connected Riemannian manifold with pinched negative sectional curvature, $-b^2 \le K \le -a^2 < 0$. Recall that the geometric compactification $\bar{H}$ of H is homeomorphic to the closed Euclidean unit ball such that the unit sphere corresponds to the geometric boundary $\partial_{\mathrm{geo}} H$ of H. We choose an origin $x_0 \in H$.

Brownian motion $\mathcal{B}$ on H is transient, and therefore it has a positive Green's function $G = G(x,y)$ and Martin kernels $K = K(x,y)$. Identifying points $x \in H$ with the corresponding Martin kernels $K(.,x)$ induces an identification of the geometric compactification $\bar{H}$ with the Martin compactification of H, such that the Martin boundary $\partial_\Delta H$ is identified $\partial_{\mathrm{geo}} H$; see [1, 2, 7]. Moreover, $\partial_\Delta H$ is equal to the space of minimal positive harmonic functions on H, normalized to be equal to one at x_0. We identify $\xi \in \partial_{\rm geo} H$ with the corresponding positive harmonic function $K(.,\xi) \in \partial_\Delta H$.

For a subset $A \subseteq H$, we denote by $\partial_{\rm geo} A$ the closed set of accumulation points of A in $\partial_{\rm geo} H$.

Lemma 3.1. *If $X \subseteq H$ is a discrete subset which admits LS-data (F,V), then $\partial_{\rm geo} F = \partial_{\rm geo} H$.*

Proof. Suppose that X admits LS-data $(F,V) = (F_x, V_x)_{x\in X}$ and that $\partial_{\rm geo} F \neq \partial_{\rm geo} H$. Then there is a cone

$$C = C(x,\xi,\alpha) = \{y \in H \setminus \{x\} \mid \angle_x(y,\xi) < \alpha\},$$

where $x \in H$, $\xi \in \partial_{\rm geo} H$, and $\alpha > 0$, such that $F \cap C = \emptyset$. By the geometry of H, the exit measure of Brownian motion from C starting at any point $z \in C$ is positive on the boundary $\partial_{\rm geo} C$ of C at infinity; cf. [10, (2) and (3) on page 730]. Hence there is a positive probability that a Brownian path starting at z does not hit F before leaving H. Therefore F is not $\mathcal{B}$-recurrent, which is a contradiction. □

Remark 3.2. If there is a uniform bound on the diameters of the F_x, as in the equivariant case below, then $\partial_{\rm geo} X = \partial_{\rm geo} F$.

We now consider a properly discontinuous group Γ of isometries of M and let X be an orbit of Γ in H.

Corollary 3.3. *If $\mathcal{B}$ is recurrent modulo Γ, then $\partial_{\rm geo} X = \partial_{\rm geo} H$.*

Remark 3.4. By the Poincaré recurrence theorem, if $|\Gamma\backslash H| < \infty$, then Brownian motion on H is recurrent modulo Γ.

Proof of Corollary 3.3. Since $\mathcal{B}$ is transient on H, Proposition 2.1 implies that X admits Γ-adapted LS-data. Hence the assertion, by Lemma 3.1. □

Our corrected version of [3, Theorem 3.2(a)] follows now immediately from Theorem 2.2.

Corollary 3.5. *Suppose that (F,V) are balanced Γ-adapted and balanced LS-data for X. Then restriction of Martin kernels from H to X induces a Γ-equivariant continuous surjection $\partial_{\rm geo} H \to \partial_\mu X$.*

4. Hyperbolic spaces

In this section, let $H = H^k_{\mathbb{F}}$ be a hyperbolic space, Γ a discrete group of isometries of H, $X \subseteq H$ a Γ-orbit, and $\mu = (\mu_x)_{x\in X}$ a Γ-equivariant family of probability measures on X such that the associated random walk $\mathcal{R}$ on X is non-degenerate and transient.

The *limit set* Λ_Γ is the closed and Γ-invariant subset of points $\xi \in \partial_{\rm geo} H$, such that $\xi = \lim g_n x$ for some sequence (g_n) in Γ and some or, equivalently, any $x \in H$. In other words, $\Lambda_\Gamma = \partial_{\rm geo} X$, and $\partial_{\rm geo} X$ depends on Γ, but not on the choice of Γ-orbit X.

Recall that Λ_Γ is either finite with at most two points or else infinite. Recall also that $\Lambda_\Gamma = \partial_{\rm geo} H$ if $|\Gamma\backslash H| < \infty$ or, more generally, if the geodesic flow of H is non-wandering modulo Γ.

Theorem 4.1. *If $\mathcal{R}$ is adapted and $\Lambda_\Gamma = \partial_{\rm geo} H$, then restriction of Martin kernels from H to X induces a homeomorphism $\partial_{\rm geo} H \to \partial_\mu X$.*

Proof. Since restriction already a continuous surjection and the underlying spaces are compact, it suffices to show injectivity. That is, given $\xi \neq \eta$ in $\partial_{\rm geo} H$, we need to show that $K(.,\xi) \neq K(.,\eta)$ on X. Let c be the unit speed geodesic from $\xi = c(-\infty)$ to $\eta = c(\infty)$ and choose $c(0)$ as the origin of M. By (1.3) it suffices to show that $b_\xi \neq b_\eta$.

For any $t \in \mathbb{R}$, the bisectors N_t between the points $c(t-s)$ and $c(t+s)$ do not depend on $s > 0$, and the family of N_t foliates H. This is obvious in the real hyperbolic case and follows immediately from Proposition D in the other cases. Note also that the nearest point projection onto the $\mathbb{F}$-line L containing c extends continuously to $\partial_{\rm geo} H$, so that the complement of the closure of N_0 at $\partial_{\rm geo} H$ is non-empty and open.

By the construction of Busemann functions, the difference $b_\xi - b_\eta = 2t$ along each of the N_t. Hence N_0 is equal to the set of point where $b_\xi = b_\eta$. If the restrictions of b_ξ and b_η would coincide on X, then X would be contained in N_0 and hence $\partial_{\rm geo} X$ in the closure of N_0 at $\partial_{\rm geo} H$. However, since the limit set of Γ is equal to $\partial_{\rm geo} H$, we have $\partial_{\rm geo} X = \partial_{\rm geo} H$, a contradiction. □

We say that a point $\xi \in \partial_{\rm geo} H$ is *cuspidal* (with respect to Γ) if horoballs with center ξ, which are sufficiently close to ξ, are precisely invariant under Γ, and if the stabilizer Γ_ξ of ξ in Γ acts uniformly on horospheres centered at Γ. Cuspidal points correspond to cusps of $\Gamma \backslash H$. Note also that cuspidal points belong to Λ_Γ.

Theorem 4.2. *If $\mathcal{R}$ is adapted, Λ_Γ is infinite, and $\xi \in \partial_{\rm geo} H$ is cuspidal, then $k(.,\xi)$ is not μ-harmonic.*

Proof. Let B be a precisely invariant horoball centered at ξ and b the Busemann function associated to ξ with $b(x_0) = 0$. Then either X does not intersect B or $X \cap B$ is non-empty and contained in a horosphere centered at ξ. In the latter case, it is evident that there exists $x \in X$ such that $b(x,\xi)$ is minimal. In the former case, the existence of such $x \in X$ follows from the fact that Γ_ξ acts uniformly on horospheres centered at ξ and that X is discrete.

In any case, it is clear that $b(.,\xi)$ attains its minimum over X at points of $\Gamma_\xi x$. By virtue of (1.3), this means that $k(y,\xi) \leq k(x,\xi)$ for all $y \in X$ and the equality holds if and only if $y \in \Gamma_\xi x$. Since the limit set of Γ is infinite and the random walk $\mathcal{R}$ is non-degenerate, the support of μ_x cannot be contained in $\Gamma_\xi x$. Hence there is at least one $y \in X \setminus \Gamma_\xi x$ for which $\mu_x(y) > 0$. Hence $\mu_x(k(.,\xi)) < k(x,\xi)$. □

Appendix A. Hyperbolic planes in symmetric spaces

In this appendix, our presentation follows the standard one in the theory of symmetric spaces.

Let S be a symmetric space of noncompact type, represented by a symmetric pair (G,K), where G is a semisimple Lie group which acts almost effectively on $S = G/K$. Let s be a corresponding involution of G, $\theta = s_{*e}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition of the Lie algebra of G. Endow $\mathfrak{g}$ with an Ad_G-invariant bilinear form $\langle .,. \rangle$, which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, such that the identification $T_xS = \mathfrak{p}$ is an orthogonal transformation, where $x = K/K \in S$. Then ad_X is a symmetric transformation of $\mathfrak{g}$ with respect to the positive definite inner product $-\langle \theta .,. \rangle$, for all $X \in \mathfrak{p}$.

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. For a linear map $\alpha\colon \mathfrak{a} \to \mathbb{R}$, let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Then $\mathfrak{g}_0 = \mathfrak{z}_\mathfrak{k}\mathfrak{a} \oplus \mathfrak{a}$ and, since all ad_H, $H \in \mathfrak{a}$, are symmetric with respect to the above inner product and pairwise commuting,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where the set Δ of *roots* consists of those $\alpha \neq 0$ for which $\mathfrak{g}_\alpha \neq \{0\}$. For any root α, we have the corresponding *root vector* $H_\alpha \in \mathfrak{a}$ such that $\alpha = \langle H_\alpha .,. \rangle$. Straightforward computations show that

$$\theta\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha} \quad \text{and} \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}.$$

In particular, $\Delta = -\Delta$. Choosing a partition $\Delta = \Delta^+ \cup \Delta^-$, we get

$$\mathfrak{g} = \mathfrak{z}_\mathfrak{k}\mathfrak{a} \oplus \mathfrak{a} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha,$$

with

$$\mathfrak{k}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k} \quad \text{and} \quad \mathfrak{p}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}.$$

For any $\alpha \in \Delta^+$, $\dim \mathfrak{k}_\alpha = \dim \mathfrak{p}_\alpha$. Furthermore, for any $X \in \mathfrak{k}_\alpha$, there is a unique $Y \in \mathfrak{p}_\alpha$ such that $X + Y \in \mathfrak{g}_\alpha$. We say that X and Y are *related.* Related vectors satisfy $|X| = |Y|$ and

$$[H, X] = \alpha(H)Y \quad \text{and} \quad [H, Y] = \alpha(H)X,$$

for all $H \in \mathfrak{a}$. Moreover, $[X, Y] = |X||Y|H_\alpha$.

Proposition A.1. *For any $\alpha \in \Delta^+$ and unit vector $Y \in \mathfrak{p}_\alpha$, H_α and Y span a totally geodesic hyperbolic plane in S through x of curvature $-|\alpha|^2$. In particular, for $y = \exp(sY)$ and $z = \exp(tH_\alpha)$,*

$$\cosh(|\alpha| d(y, z)) = \cosh(|\alpha| s) \cosh(|\alpha|^2 t).$$

Proof. By what we said above, the linear hull $\mathfrak{q} \subseteq \mathfrak{p}$ of H_α and Y satisfies

$$[\mathfrak{q}, [\mathfrak{q}, \mathfrak{q}]] \subseteq \mathfrak{q}.$$

Hence $\mathfrak{q}$ spans a totally geodesic plane through x. Since

$$|\alpha|^2 = |H_\alpha|^2 = \alpha(H_\alpha) \quad \text{and} \quad -R(Y, H_\alpha)H_\alpha = [H_\alpha, [H_\alpha, Y]] = |\alpha|^4 Y,$$

we obtain the assertion about the curvature. The assertion about the distance follows from hyperbolic trigonometry, where we note that H_α and Y are perpendicular. □

Remark A.2. Since $R(Y, H)H = -[H, [H, X]]$, the spaces $\mathfrak{p}_\alpha$ correspond to the eigenspaces of the curvature endomorphisms $R(., H)H$, for $H \in \mathfrak{a}$.

Appendix B. Bisectors in hyperbolic geometry

In a metric space H, the *bisector* of points $y, z \in H$ is the set

$$\{x \in H \mid d(x,y) = d(x,z)\}.$$

In this section, we give a description of bisectors in hyperbolic geometry. The description is certainly known to experts and is trivial in the case of real hyperbolic spaces. However, we were not able to locate a presentation for quaternionic hyperbolic spaces and the octonionic hyperbolic plane. We follow the nice account in [6, Section 5], where the case of complex hyperbolic spaces is treated. Our main contribution is Proposition A.1 respectively its consequence, Proposition B.1.

Let $H = H^k_{\mathbb{F}}$ with $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and normalize the standard metric on H such that its maximal sectional curvature is -1. Set $d = \dim_{\mathbb{R}} \mathbb{F}$ so that the (real) dimension of $H^k_{\mathbb{F}}$ is equal to $m = kd$.

Let v be a unit tangent vector of H at a point $x \in H$. Then there is a unique d-dimensional totally geodesic submanifold $L \subseteq H$ passing through x with $T_xL = V$, where $V \subseteq T_xH$ is the linear hull of v and the subspace U_2 consisting of vectors $u \in T_xH$ perpendicular to v such that $R(u,v)v = -4u$. Metrically, $L \cong H^d_{\mathbb{R}}$, but with constant sectional curvature -4. The orthogonal complement U_1 of V in T_xH consists of vectors $u \in T_xH$ such that $R(u,v)v = -u$.

Proposition B.1. *Any $u \in U_1$ determines a totally geodesic real hyperbolic plane P through x of curvature -1 such that T_xP is spanned by u, v.*

Proof. With respect to the setup in Appendix A, we choose $\mathfrak{a} = \mathbb{R}v$. Then U_1 corresponds to a restricted root space $\mathfrak{p}_1$ with $R(u,v)v = -u$ for all $u \in U_1$. Now the assertion follows from the first part of Proposition A.1. □

Denote by $\pi \colon H \to L$ be the orthogonal, that is, nearest point projection.

Proposition B.2. *Let $y \in L$ and $z \in H$. Then*

$$\cosh d(y,z) = \cosh d(y,\pi z) \cosh d(\pi z, z)$$

Proof. We can assume that $\pi z \neq y$ and that $z \notin L$. Then we let $v \in T_{\pi z}L$ be the unit tangent vector pointing at y and $u \in T_{\pi z}H$ the one pointing at z. Then u is perpendicular to $T_{\pi z}L$, and hence $u \in U_1$ in the above notation. The formula for the distances is now just the formula in Proposition A.1, since the real hyperbolic plane P in Proposition B.1 has curvature -1. □

For all $y_1 \neq y_2$ in H, there is a unique totally geodesic subspace $L = L_{y_1,y_2}$ as above containing both of them.

Corollary B.3. *The bisector of $y_1 \neq y_2$ in H is the preimage under the nearest point projection onto L as above of the bisector of y_1, y_2 in L.*

Proof. This follows readily from Proposition B.2. □

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