Local class field theory: perfect residue field case

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LOCAL CLASS FIELD THEORY: PERFECT RESIDUE FIELD CASE

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Let F be a complete (or Henselian) discrete valuation field with a perfect residue field $\overline{F}$ of characteristic $p > 0$. Let $\wp(X)$ denote as usually the polynomial $X^p - X$. It induces the additive homomorphism $\wp\colon \overline{F} \to \overline{F}$. Let

$$\kappa = \dim_{\mathbf{F}_p} \overline{F}/\wp(\overline{F}).$$

Further it will be assumed that $\kappa \neq 0$, the case $\kappa = 0$ when the field $\overline{F}$ is algebraically p-closed may be treated similarly to Serre's geometric class field theory [Sr].

Let F^{ur} be the maximal unramified extension of F in the fixed separable closure F^{sep} of F, F^{abur}/F the maximal abelian subextension in F^{ur}/F, F^{ab}/F the maximal abelian extension in F^{sep}/F. Recall that for totally ramified abelian extensions L_1/F, L_2/F there is a totally ramified abelian extension L_3/F such that $L_3^{\mathrm{ur}} = (L_1L_2)^{\mathrm{ur}}$ (see [Hz1, (2.8.G)]). Thus, one may introduce the group

$$G_F^{\mathrm{abr}} = \varprojlim \operatorname{Gal}(L/F)$$

where the projective limit is taken over the directed system of abelian totally ramified extensions L/F. Then G_F^{abr} is isomorphic to $\operatorname{Gal}(T/F)$ where T/F is any maximal totally ramified subextension in F^{ab}/F (see [Hz2, Subsection 2.4]) and

$$\operatorname{Gal}(F^{\mathrm{ab}}/F) \simeq \operatorname{Gal}(F^{\mathrm{abur}}/F) \times G_F^{\mathrm{abr}},$$

the group $\operatorname{Gal}(F^{\mathrm{abur}}/F)$ is canonically isomorphic to $\operatorname{Gal}(\overline{F}^{\mathrm{ab}}/\overline{F})$.

To describe the maximal abelian extension F^{ab}/F one must study abelian non-p-extensions and abelian p-extensions. Totally tamely ramified abelian extensions over F are easily described by the Kummer theory, since any such extension L/F is generated by adjoining a root $\sqrt[l]{\pi}$ for a suitable prime element π in F and a primitive lth root of unity belongs to F.

Treating abelian p-extensions one deduces at once the description of the maximal unramified abelian p-extension using the Witt theory. Thus, one is reduced to the study of abelian totally ramified p-extensions of F. A variant of description of the group G_F^{abr} in terms of constant pro-quasi-algebraic groups was furnished by M. Hazewinkel ([Hz1, Hz2]).

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Another description of abelian totally ramified p-extensions which is cohomology-free and of more explicit nature will be established below.

Let $\widetilde{F}$ denote the maximal abelian unramified p-extension of F and let L/F be a Galois totally ramified p-extension. For a character

$$\chi \in \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Gal}(\widetilde{L}/L), \mathrm{Gal}(L/F))$$

let Σ_χ be the fixed field of all $\chi(\varphi)\varphi \in \mathrm{Gal}(\widetilde{L}/F)$, where φ runs a topological $\mathbf{Z}_p$-basis of $\mathrm{Gal}(\widetilde{L}/L)$. Let π_χ be a prime element in Σ_χ, and π_L be a prime element in L. Let U_L be the group of units of L. Put

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F}\pi_\chi N_{L/F}\pi_L^{-1} \mod N_{L/F}U_L.$$

We show in Theorem (1.7) below that $\Upsilon_{L/F}$ induces an isomorphism of

$$\mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Gal}(\widetilde{L}/L), \mathrm{Gal}(L/F)^{\mathrm{ab}})$$

onto $U_F/N_{L/F}U_L$ where $\mathrm{Gal}(L/F)^{\mathrm{ab}}$ is the maximal abelian quotient of $\mathrm{Gal}(L/F)$. This construction of $\Upsilon_{L/F}$ can be regarded as a generalization of the Neukirch construction in the classical cases ([N1],[N2]). We describe the inverse isomorphism to $\Upsilon_{L/F}$ as well. Passing to the projective limit one obtains the reciprocity map

$$\Psi_F \colon U_{1,F} \to \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(F^{\mathrm{abp}}/\widetilde{F}))$$

where $U_{1,F}$ is the group of principal units, F^{abp}/F is the maximal p-subextension in F^{ab}/F. The existence theorem deduced in Section 3 describes norm subgroups in $U_{1,F}$ and clarifies the properties of Ψ_F. For its proof theory of decomposable additive polynomials over $\overline{F}$ derived in Section 2 will be used.

The local class field theory exposed has a lot of applications. Among them in ramification theory it justifies the metatheorem that a statement about ramification groups of normal totally ramified extension of a local field which holds in classical cases when the residue field is finite or quasi-finite is true in general ([Sn],[L1], [Ma]), in theory of fields of norms it connects the its constructions by class field theory ([FW],[Wn], [L2], see also [D]).

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§1. Reciprocity map

1.1. The Witt theory immediately shows that if $\kappa = \dim_{\mathbf{F}_p} \overline{F}/\wp(\overline{F})$, then

$$\mathrm{Gal}(\widetilde{F}/F) \simeq \prod_{\kappa} \mathbf{Z}_p.$$

Let L/F be a finite Galois totally ramified p-extension. Then $\mathrm{Gal}(L/F)$ can be identified with $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$, and $\mathrm{Gal}(\widetilde{L}/F)$ is isomorphic with $\mathrm{Gal}(\widetilde{L}/\widetilde{F}) \times \mathrm{Gal}(\widetilde{L}/L)$. Let

$\mathrm{Gal}(L/F)^* = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Gal}(\widetilde{L}/L), \mathrm{Gal}(L/F))$ denote the group of continuous homomorphisms of $\mathbf{Z}_p$-module $\mathrm{Gal}(\widetilde{L}/L)$ ($a \cdot \sigma = \sigma^a$, $a \in \mathbf{Z}_p$) to the discrete $\mathbf{Z}_p$-module $\mathrm{Gal}(L/F)$. This group is isomorphic (non-canonically) with $\oplus_\kappa \mathrm{Gal}(L/F)$. Let $\chi \in \mathrm{Gal}(L/F)^*$ and Σ_χ be the fixed field of $\{\chi(\varphi)\varphi\}$ where φ runs through $\mathrm{Gal}(\widetilde{L}/L)$ and the element $\chi(\varphi)$ of $\mathrm{Gal}(L/F)$ is identified with the corresponding element in $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$. Then, obviously, $\Sigma_\chi \cap \widetilde{F} = F$, i.e., Σ_χ/F is a totally ramified p-extension. Let U_F and U_L be the groups of units of F and L respectively. Let π_χ be a prime element of Σ_χ. Put

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F}\pi_\chi N_{L/F}\pi_L^{-1} \mod N_{L/F}U_L$$

where π_L is a prime element in L;

1.2. Lemma. *The map* $\Upsilon_{L/F}\colon \mathrm{Gal}(L/F)^* \to U_F/N_{L/F}U_L$ *is well defined.*

Proof. $\Upsilon_{L/F}$ does not depend on the choice of π_L. Let M be the compositum of Σ_χ and L. Then M/Σ_χ is unramified and any prime element in Σ_χ can be written as $\pi_\chi N_{M/\Sigma_\chi}\varepsilon$ for a suitable $\varepsilon \in U_M$. As $N_{M/F}\varepsilon = N_{L/F}(N_{M/L}\varepsilon) \in N_{L/F}U_L$ we complete the proof. □

Let $U_{i,F}$ denote the subgroup of principal units $\equiv 1 \mod \pi_F^i$. Then $\Upsilon_{L/F}$ acts in fact from $(\mathrm{Gal}(L/F)^{\mathrm{ab}})^*$ to $U_{1,F}/N_{L/F}U_{1,L}$.

1.3. In order to go further we consider the behavior of the norm map. Let L/F be a cyclic totally ramified extension of degree p. Let π_L be a prime element in L. Then $\pi_F = N_{L/F}\pi_L$ is prime in F. Let σ be a generator of $\mathrm{Gal}(L/F)$,

$$\frac{\sigma\pi_L}{\pi_L} = 1 + \theta_0\pi_L^s + \dots$$

with $\theta_0 \in U_L$, $s = s(L|F) > 0$. Then it is well known that

$$\begin{aligned} N_{L/F}(1+\theta\pi_L^i) &= 1 + \theta^p\pi_F^i + \dots && \text{for } i < s,\ \theta \in U_F \\ N_{L/F}(1+\theta\pi_L^s) &= 1 + (\theta^p - \theta_0^{p-1}\theta)\pi_F^s + \dots && \text{for } \theta \in U_F \\ N_{L/F}(1+\theta\pi_L^{s+pi}) &= 1 - \theta_0^{p-1}\theta\pi_F^{s+i} + \dots && \text{for } i > 0,\ \theta \in U_F. \end{aligned}$$

Then $U_{1,F}/N_{L/F}U_{1,L}$ is generated by $1 + \theta\pi_F^s$ when θ runs element of U_F the residues of which are linearly independent over $\overline{\theta}_0^p\wp(\overline{F})$. Hence $U_{1,F}/N_{L/F}U_{1,L}$ is isomorphic to $\overline{F}/\overline{\theta}_0^p\wp(\overline{F}) \simeq \oplus_\kappa \mathbf{F}_p$.

1.4. Let $\widehat{F}$ denote the completion of $\widetilde{F}$. If L/F is a Galois totally ramified p-extension, then $\mathrm{Gal}(L/F)$ is solvable. We will assume always when it is necessary that $\widehat{F} \subset \widehat{L}$. It follows from (1.3) that $N_{\widehat{L}/\widehat{F}}U_{1,\widehat{L}} = U_{1,\widehat{F}}$. For $\sigma \in \mathrm{Gal}(L/F)$ put

$$i(\sigma) = \frac{\sigma\pi_L}{\pi_L} \mod V(L|F),$$

where π_L is a prime element in L, and

$$V(L|F) = \left\{ \frac{\sigma(\varepsilon)}{\varepsilon} : \varepsilon \in U_{1,\widehat{L}}; \sigma \in \mathrm{Gal}(L/F) \right\}.$$

Then the sequence

$$1 \to \mathrm{Gal}(L/F)^{\mathrm{ab}} \to U_{1,\widehat{L}}/V(L|F) \xrightarrow{N_{\widehat{L}/\widehat{F}}} U_{1,\widehat{F}} \to 1$$

is exact (see [Hz1, (2.7)] or [I, Subsection 2.2]).

If M/F is a Galois subextension in L/F, then $V(M|F) = N_{L/M}V(L|F)$ because $U_{1,\widehat{M}} = N_{\widehat{L}/\widehat{M}}U_{1,\widehat{L}}$.

Lemma. *Assume that L/F is a Galois totally ramified p-extension. Let φ_ν, $\nu \in I$, be elements in $\mathrm{Gal}(\widetilde{F}/F)$ which are $\mathbf{Z}_p$-linearly independent, and ψ_ν, $\nu \in I$, be their extensions on $\widehat{L}$. Let $\psi \in \mathrm{Gal}(\widetilde{L}/F)$ be such that its restriction on $\widetilde{F}$ is $\mathbf{Z}_p$-linearly independent with $\{\varphi_\nu\}_{\nu\in I}$. We will use the same notation for the continuous extension of ψ on $\widehat{L}$. Let E be the fixed subfield of $\{\psi_\nu\}_{\nu\in I}$ and $\varepsilon \in U_{i,E}$. Then there exists $\eta \in U_{i,E}$ such that $\varepsilon = \eta^{\psi-1}$.*

Proof. Note that E is a complete field and one can construct the desired element η step by step modulo higher principal units. For instance, let $\varepsilon \equiv 1 + \theta\pi_L^i \mod \pi_L^{i+1}$ where $\theta \in U_E$, $\bar\psi_\nu(\bar\theta) = \bar\theta$ for $\nu \in I$. Then there is $\xi \in U_E$ such that $\overline{\psi}(\overline{\xi}) - \overline{\xi} = \overline{\theta}$. Then for $\eta \equiv 1 + \xi\pi_L^i \mod \pi_L^{i+1}$ we deduce that $\varepsilon \equiv \eta^{\psi-1} \mod \pi_L^{i+1}$. □

1.5. Now we introduce the map inverse to $\Upsilon_{L/F}$. Let L/F be a Galois totally ramified p-extension. Let $\varepsilon \in U_{1,F}$. According to (1.4) there exists an element $\eta \in U_{1,\widehat{L}}$ such that $N_{\widehat{L}/\widehat{F}}\eta = \varepsilon$. Let φ be a continuous extension of $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$ on $\widehat{L}$. Since $N_{\widehat{L}/\widehat{F}}\big(\eta^{-1}\varphi(\eta)\big) = 1$, we deduce from the exact sequence of (1.4) that $\eta^{-1}\varphi(\eta) \equiv \pi_L\sigma(\pi_L^{-1})$ mod $V(L|F)$ for a suitable $\sigma \in \mathrm{Gal}(\widehat{L}/\widehat{F})$ where π_L is a prime element in L. Set $\chi(\varphi) = \sigma$. Then it is easy to verify that $\chi(\varphi_1\varphi_2) = \sigma_1\sigma_2$. This means $\chi \in \mathrm{Gal}(L/F)^*$. Put $\Psi_{L/F}(\varepsilon) = \chi$.

Lemma. *The map $\Psi_{L/F}\colon U_{1,F}/N_{L/F}U_{1,L} \to \mathrm{Gal}(L/F)^*$ is well defined and a homomorphism.*

Proof. If $N_{\widehat{L}/\widehat{F}}\rho = \varepsilon$, then for $\mu = \eta^{-1}\rho$ the element $\mu^{-1}\varphi(\mu)$ belongs to $V(L|F)$. If $\varepsilon = \varepsilon_1\varepsilon_2$, then one may assume $\eta = \eta_1\eta_2$, consequently $\sigma = \sigma_1\sigma_2$ in $\mathrm{Gal}(L/F)^{\mathrm{ab}}$. Thus, $\Psi_{L/F}(\varepsilon_1\varepsilon_2) = \Psi_{L/F}(\varepsilon_1)\Psi_{L/F}(\varepsilon_2)$. □

In fact $\Psi_{L/F}$ acts from $U_{1,F}/N_{L/F}U_{1,L}$ to $\big(\mathrm{Gal}(L/F)^{\mathrm{ab}}\big)^*$.

1.6. For Theorem to follow we need to consider the following

Proposition. *Assume that L/F is a Galois totally ramified p-extension, M/F is a Galois subextension in L/F, $M \neq L$. Let φ_ν, $\nu \in I$, be $\mathbf{Z}_p$-linearly independent in $\mathrm{Gal}(\widetilde{F}/F)$, and ψ_ν be their continuous extensions on $\widehat{L}$. Let $\psi \in \mathrm{Gal}(\widetilde{L}/F)$ be such that $\psi|_{\widetilde{F}}$ is $\mathbf{Z}_p$-linearly* *independent with $\{\psi_\nu\}_{\nu\in I}$. For a set $S \subset \widehat{L}$ let $S^{\langle\psi_\nu\rangle}$ denote the set of the fixed elements under the action of all ψ_ν. Let $\alpha \in \widehat{L}$. Then*

1) $V(L|M)^{\langle\psi_\nu\rangle} = \left(V(L|M)^{\langle\psi_\nu\rangle}\right)^{\psi-1}$;

2) *if* $\alpha^{\psi_\nu - 1} \in V(L|M)$ *for all* $\nu \in I$, *then* $\alpha \in V(L|M)\widehat{L}^{\langle\psi_\nu\rangle}$;

3) $V(M|F)^{\langle\psi_\nu\rangle} = N_{\widehat{L}/\widehat{M}}(V(L|F)^{\langle\psi_\nu\rangle})$.

Proof. First assume that $|L{:}M| = p$.

1) Let $E = \widehat{L}^{\langle\psi_\nu\rangle}$ be the fixed field of ψ_ν, $\nu \in I$. Let σ be a generator of $\mathrm{Gal}(\widehat{L}/\widehat{M})$. If $\varepsilon \in V(L|M)^{\langle\psi_\nu\rangle}$, then $\varepsilon = \varepsilon_1^{\sigma-1}$ for some $\varepsilon_1 \in U_{1,E}$, and it follows from (1.3) that $\varepsilon \in U_{s+1,E}$ where $s = s(\widehat{L}|\widehat{M})$. By Lemma of (1.4) there exists an element $\eta \in U_{s+1,E}$ such that $\varepsilon = \eta^{\psi-1}$. Then for $\rho = N_{\widehat{L}/\widehat{M}}\eta$ we obtain $\rho \in U_{s+1,\widehat{M}} \cap E^{\langle\psi\rangle}$. Applying (1.3) once again we deduce that $\rho = N_{\widehat{L}/\widehat{M}}\xi$ for some $\xi \in E^{\langle\psi\rangle} \cap U_{s+1,E}$. Then $N_{\widehat{L}/\widehat{M}}(\eta\xi^{-1}) = 1$ and $\eta\xi^{-1} \in U_{s+1,E}$. Therefore, $\eta = \xi\mu^{\sigma-1}$ for some $\mu \in U_{1,E}$. Thus, $\varepsilon = \mu^{(\sigma-1)(\psi-1)} \in \left(V(L|M)^{\langle\psi_\nu\rangle}\right)^{\psi-1}$.

2) Proceed by induction on the cardinality of I. Let $\alpha = \alpha_1\alpha_2$ with $\alpha_1 \in V(L|M)$, $\alpha_2 \in \widehat{L}_J^{\langle\psi_\nu\rangle}$, $\nu \in J$, $J = I - \{i\}$. Then $\alpha_2^{\psi_i - 1} \in V(L|M)^{\langle\psi_\nu\rangle}$, $\nu \in J$, and by 1) we deduce $\alpha_2^{\psi_i-1} = \alpha_3^{\psi_i-1}$ for a suitable $\alpha_3 \in V(L|M)^{\langle\psi_\nu\rangle}$, $\nu \in J$. Then $\alpha_2\alpha_3^{-1} \in \widehat{L}^{\langle\psi_\nu\rangle}$, $\nu \in I$, and $\alpha = (\alpha_1\alpha_3)(\alpha_2\alpha_3^{-1}) \in V(L|M)\widehat{L}^{\langle\psi_\nu\rangle}$.

3) Let $\alpha \in V(M|F)^{\langle\psi_\nu\rangle}$. By (1.4) we get $\alpha = N_{\widehat{L}/\widehat{M}}\beta$ with $\beta \in V(L|F)$. As $\beta^{\psi_\nu-1} \in V(L|M)$, we deduce using 2) that $\beta \in V(L|M)V(L|F)^{\langle\psi_\nu\rangle}$. Therefore, $\alpha \in N_{\widehat{L}/\widehat{M}}\left(V(L|F)^{\langle\psi_\nu\rangle}\right)$, as desired.

In the general case we proceed by induction on $|L{:}M|$ using the solvability of totally ramified extensions.

1) If $\alpha \in V(L|M)^{\langle\psi_\nu\rangle}$, then by the inductional assumption $N_{\widehat{L}/\widehat{K}}\alpha = \beta^{\psi-1}$ for some $\beta \in V(K|M)^{\langle\psi_\nu\rangle}$ where K/M is a non-trivial subextension in L/M. Applying 3) for the extension L/K, we obtain that $\beta = N_{\widehat{L}/\widehat{K}}\gamma$ for some $\gamma \in V(L|K)^{\langle\psi_\nu\rangle}$. Then $N_{\widehat{L}/\widehat{K}}\left(\alpha\gamma^{1-\psi}\right) = 1$ and $\alpha\gamma^{1-\psi} \in V(L|K)$. Applying 1) for the extension L/K, we conclude $\alpha\gamma^{1-\psi} = \delta^{1-\psi}$ with some $\delta \in V(L|K)^{\langle\psi_\nu\rangle}$ and $\alpha \in \left(V(L|M)^{\langle\psi_\nu\rangle}\right)^{\psi-1}$.

Now 2) formally follows from 1) and 3) follows from 2) as just above. □

1.7. Theorem. *Assume that L/F is a Galois totally ramified p-extension. The map $\Upsilon_{L/F}\colon \left(\mathrm{Gal}(L/F)^{\mathrm{ab}}\right)^* \to U_{1,F}/N_{L/F}U_{1,L}$ is an isomorphism, and the map $\Psi_{L/F}$ is the inverse one.*

Proof. First we verify that $\Psi_{L/F} \circ \Upsilon_{L/F} = \mathrm{id}$. Indeed, let $\pi_\chi = \pi_L\eta$ with $\eta \in U_{\widehat{L}}$. Let $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$, $\sigma = \chi(\varphi) \in \mathrm{Gal}(\widetilde{L}/\widetilde{F})$. Then

$$\pi_L^{1-\sigma} = \eta^{\varphi\sigma-1} = \eta^{\varphi-1}\eta^{\varphi(\sigma-1)} \equiv \eta^{\varphi-1} \mod V(L|F)$$

and $N_{\widehat{L}/\widehat{F}}\eta = N_{\Sigma_\chi/F}\pi_\chi N_{L/F}\pi_L^{-1}$. Therefore, $\chi = \Psi_{L/F}(\Upsilon_{L/F}\chi)$.

Next we show that $\Upsilon_{L/F} \circ \Psi_{L/F} = \mathrm{id}$. Let $\varepsilon \in U_{1,F}$ and $\varepsilon = N_{\widehat{L}/\widehat{F}}\eta$ for some $\eta \in U_{1,\widehat{L}}$. Assume that $\eta^{\varphi_\nu-1} \equiv \pi_L^{1-\sigma_\nu} \mod V(L|F)$ for $\varphi_\nu \in \mathrm{Gal}(\widetilde{L}/L)$, $\sigma_\nu \in \mathrm{Gal}(\widetilde{L}/\widetilde{F})$. Put

$\psi_\nu = \varphi_\nu \sigma_\nu$ and apply 2) of the previous Proposition. Then, as $(\pi_L \eta)^{\psi_\nu - 1} \in V(L|F)$, we obtain that $\pi_L \eta = \eta_1 \eta_2$ with $\eta_1 \in V(L|F)$, $\eta_2 \in \widehat{L}^{\langle \psi_\nu \rangle}$. This means that $\pi_L \eta \eta_1^{-1} \in \Sigma_\chi$ where $\chi(\varphi_\nu) = \sigma_\nu$, and

$$\varepsilon \equiv N_{\Sigma_\chi/F}(\pi_L \eta \eta_1^{-1}) N_{L/F} \pi_L^{-1} \mod N_{L/F} U_{1,L}.$$

Thus, $\Upsilon_{L/F} \circ \Psi_{L/F} = \mathrm{id}$. □

Corollary. *Let M/F be the maximal abelian subextension in a Galois totally ramified p-extension L/F. Then $N_{M/F} U_{1,M} = N_{L/F} U_{1,L}$.*

1.8. Now we establish functorial properties of $\Upsilon_{L/F}$ and $\Psi_{L/F}$.

Proposition.

1) *Let L/F, L'/F' be Galois totally ramified p-extensions, F'/F, L'/L be totally ramified. Then the diagram*

$$\begin{array}{ccc} \mathrm{Gal}(L'/F')^* & \longrightarrow & U_{1,F'}/N_{L'/F'}U_{1,L'} \\ \downarrow & & \downarrow{\scriptstyle N_{F'/F}} \\ \mathrm{Gal}(L/F)^* & \longrightarrow & U_{1,F}/N_{L/F}U_{1,L} \end{array}$$

is commutative where the left vertical homomorphism is induced by the natural restriction $\mathrm{Gal}(L'/F') \to \mathrm{Gal}(L/F)$ and the canonical isomorphism $\mathrm{Gal}(\widetilde{L}'/L') \xrightarrow{\sim} \mathrm{Gal}(\widetilde{L}/L)$.

2) *Let L/F be a Galois totally ramified p-extension, and let σ be an automorphism. Then the diagram*

$$\begin{array}{ccc} \mathrm{Gal}(L/F)^* & \longrightarrow & U_{1,F}/N_{L/F}U_{1,L} \\ {\scriptstyle \sigma^*}\downarrow & & \downarrow \\ \mathrm{Gal}(\sigma L/\sigma F)^* & \longrightarrow & U_{1,\sigma F}/N_{\sigma L/\sigma F}U_{1,\sigma L} \end{array}$$

is commutative, where $(\sigma^*\chi)(\sigma\varphi\sigma^{-1}) = \sigma\chi(\varphi)\sigma^{-1}$.

3) *Let L/F be a Galois totally ramified p-extension and M/F be its subextension. Then the diagram*

$$\begin{array}{ccc} (\mathrm{Gal}(L/F)^{\mathrm{ab}})^* & \longrightarrow & U_{1,F}/N_{L/F}U_{1,L} \\ {\scriptstyle \mathrm{Ver}^*}\downarrow & & \downarrow \\ (\mathrm{Gal}(L/M)^{\mathrm{ab}})^* & \longrightarrow & U_{1,M}/N_{L/M}U_{1,L} \end{array}$$

is commutative, where Ver^* is induced by $\mathrm{Ver}\colon \mathrm{Gal}(L/F)^{\mathrm{ab}} \to \mathrm{Gal}(L/M)^{\mathrm{ab}}$.

Proof.

1) Let $\chi' \in \mathrm{Gal}(L'/F')^*$ and $\chi \in \mathrm{Gal}(L/F)^*$ be the corresponding character. Put $\Sigma' = \Sigma_{\chi'}$. Then $\Sigma_\chi = \Sigma' \cap \widetilde{L}$ and Σ/Σ_χ is totally ramified. Therefore, $\pi = N_{\Sigma'/\Sigma_\chi} \pi_{\Sigma'}$ is a prime element in Σ_χ and $N_{\Sigma_\chi/F}\pi = N_{F'/F} N_{\Sigma'/F'} \pi_{\Sigma'}$.

2) It follows from $\Sigma_{\sigma^*\chi} = \sigma \Sigma_\chi$.

3) Let $\varepsilon = N_{\widehat{L}/\widehat{F}}\eta$ and $\eta^{\varphi-1} = \pi_L^{1-\sigma}\gamma$ for a prime element π_L in L, $\sigma \in \mathrm{Gal}(\widehat{L}/\widehat{F})$, $\gamma \in V(L|F)$. Then $\sigma = \chi(\varphi)$, $\chi = \Psi_{L/F}(\varepsilon)$. Let $\tau_i \in \mathrm{Gal}(\widehat{L}/\widehat{F})$ be a set of representatives of $\mathrm{Gal}(\widehat{L}/\widehat{F})$ over $\mathrm{Gal}(\widehat{L}/\widehat{M})$. Then $\varepsilon = N_{\widehat{L}/\widehat{M}}\eta_1$ with $\eta_1 = \prod \eta^{\tau_i}$ and $\eta_1^{\varphi-1} = \prod \pi_L^{(1-\sigma)\tau_i} \prod \eta^{\tau_i}$. Let $\sigma\tau_i = \tau_{i'}h_i(\sigma)$ with $h_i(\sigma) \in \mathrm{Gal}(\widehat{L}/\widehat{M})$. Now we deduce

$$\prod \pi_L^{(1-\sigma)\tau_i} = \prod \pi_L^{\tau_{i'}(1-h_i(\sigma))} \equiv \pi_L^{\prod 1-h_i(\sigma)} = \pi_L^{1-\mathrm{Ver}(\sigma)} \mod V(L|M).$$

Since $\prod \eta^{\tau_i} \in V(L|M)$ we deduce that $\eta_1^{\varphi-1} \equiv \pi_L^{1-\mathrm{Ver}(\sigma)} \mod V(L|M)$, as desired. □

Corollary. *Let L_1/F, L_2/F, L_1L_2/F be abelian totally ramified p-extensions. Put $L_3 = L_1L_2$, $L_4 = L_1 \cap L_2$. Then*

$$N_{L_3/F}U_{1,L_3} = N_{L_1/F}U_{1,L_1} \cap N_{L_2/F}U_{1,L_2},$$
$$N_{L_4/F}U_{1,L_4} = N_{L_1/F}U_{1,L_1}N_{L_2/F}U_{1,L_2}.$$

Moreover, $N_{L_1/F}U_{1,L_1} \subset N_{L_2/F}U_{1,L_2}$ if and only if $L_1 \supset L_2$.

Proof. Put $H_i = \mathrm{Gal}(L_3/L_i)$, $i = 1,2$. Then

$$\begin{aligned} N_{L_3/F}U_{1,L_3} &= \Psi_{L_3/F}^{-1}(1) = \Psi_{L_3/F}^{-1}(H_1 \cap H_2) = \Psi_{L_3/F}^{-1}(H_1) \cap \Psi_{L_3/F}^{-1}(H_2) \\ &= N_{L_1/F}U_{1,L_1} \cap N_{L_2/F}U_{1,L_2}, \\ N_{L_4/F}U_{1,L_4} &= \Psi_{L_3/F}^{-1}(H_1H_2) = N_{L_1/F}U_{1,L_1}N_{L_2/F}U_{1,L_2}. \end{aligned}$$

If $N_{L_1/F}U_{1,L_1} \subset N_{L_2/F}U_{1,L_2}$, then $N_{L_1/F}U_{1,L_1} = N_{L_3/F}U_{1,L_3}$ and $|L_1{:}F| = |L_3{:}F|$, i.e., $L_2 \subset L_1$. □

Remark. Let F^{abp}/F be the maximal p-subextension in F^{ab}/F. Let $\{\psi_\nu\}$ be a set of automorphisms in $\mathrm{Gal}(F^{\mathrm{abp}}/F)$ such that $\psi_\nu|_{\widetilde{F}}$ are linearly independent and generate $\mathrm{Gal}(\widetilde{F}/F)$. Then the group $\mathrm{Gal}(\Sigma/F)$ for the fixed field Σ of ψ_ν is isomorphic to the group $\mathrm{Gal}(F^{\mathrm{abp}}/\widetilde{F})$.

In the definition of $\Psi_{L/F}$ one can replace the group $\mathrm{Hom}_{\mathbf{Z}_p}\big(\mathrm{Gal}(\widetilde{L}/L), \mathrm{Gal}(L/F)\big)$ by the group $\mathrm{Hom}_{\mathbf{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)\big)$. Indeed, let $\psi_1, \psi_2 \in \mathrm{Gal}(\widetilde{L}/F)$ be such that $\psi_1|_{\widetilde{F}} = \psi_2|_{\widetilde{F}}$. Then $\psi_2^{-1}\psi_1 = \tau \in \mathrm{Gal}(\widetilde{L}/\widetilde{F})$, and

$$\eta^{\psi_1-1} = \eta^{\psi_2\tau-1} = \eta^{\psi_2-1}\eta^{\psi_2(\tau-1)} \equiv \eta^{\psi_2-1} \mod V(L|F).$$

Thus, we get an isomorphism

$$\Psi_{L/F}\colon U_{1,F}/N_{L/F}U_{1,L} \to \mathrm{Hom}_{\mathbf{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)^{\mathrm{ab}}\big).$$

Passing to the projective limit we obtain the reciprocity map

$$\Psi_F\colon U_{1,F} \to \mathrm{Hom}_{\mathbf{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(F^{\mathrm{abp}}/\widetilde{F})\big).$$

This map possesses functional properties analogous to stated in Proposition. The kernel of Ψ_F coincides with the intersection of all norm groups $N_{L/F}U_{1,L}$ for abelian totally ramified p-extensions L/F, $L \subset \Sigma$.

1.9. The following assertion can be applied to the study of ramification groups.

Proposition. *Assume that L/F is an abelian totally ramified p-extension and $G = \mathrm{Gal}(L/F)$. Let $h = \psi_{L/F}$ be the Hasse-Herbrand function. Then $\Psi_{L/F}$ maps the quotient group $U_{i,F}/N_{L/F}U_{h(i),L}$ isomorphically onto the ramification group $G_{h(i)}$.*

Proof. Let $\sigma \in G_{h(i)}$. Then $\pi^{-1}\sigma(\pi) \in U_{h(i),L}$ for a prime element π in L. According to Lemma of (1.4) there exists an element $\beta \in U_{h(i),\widehat{L}}$ such that $\beta^{\varphi-1} = \pi^{1-\sigma}$ for a continuous extension φ on $\widehat{L}$ of $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$. Then $N_{\widehat{L}/\widehat{F}}\beta \in U_{i,\widehat{F}}$ and $\Upsilon_{L/F}(\chi) \in U_{i,F}N_{L/F}U_{h(i),L}$ for $\chi \in \mathrm{Gal}(L/F)^*$, $\chi(\varphi) = \sigma$. Thus, $\Upsilon_{L/F}$ induces the homomorphisms

$$G_{h(i)}/G_{h(i)+1} \longrightarrow U_{i,F}/U_{i+1,F}N_{L/F}U_{h(i),L}.$$

Since $\Upsilon_{L/F}$ is an isomorphism, we obtain the required assertion. □

Remark. One can deduce from the proof of the previous Proposition that $G_{h(i)+1} = G_{h(i+1)}$ (so-called Hasse-Arf theorem) .

1.10. We finally consider pairings of $U_{1,F}$.

The first pairing is the Hilbert norm residue symbol. Assume that $\mathrm{char}(F) = 0$ and a primitive p^mth root of unity belongs to F. Let μ_{p^m} be the group of p^mth roots. For $\alpha \in U_{1,F}$, $\beta \in F^*$ put

$$(\alpha,\beta)_m(\varphi) = \gamma^{\Psi_F(\alpha)(\varphi)-1}$$

where $\gamma^{p^m} = \beta$, $\varphi \in \mathrm{Gal}(\widetilde{F}/F)$, $\Psi_F(\alpha)(\varphi) \in \mathrm{Gal}(\widetilde{F}(\gamma)/\widetilde{F})$. Thus, we obtain the pairing

$$(\cdot,\cdot)_m : U_{1,F} \times F^* \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\widetilde{F}/F), \mu_{p^m})$$

(note that the last group is non-canonically isomorphic to $\oplus_\kappa \mu_{p^m}$).

Proposition.

1) *Let $F(\sqrt[p^m]{\beta})/F$ be totally ramified. Then $(\alpha,\beta)_m = 1$ if and only if*

$$\alpha \in N_{F(\sqrt[p^m]{\beta})/F}U_{1,F(\sqrt[p^m]{\beta})}.$$

2) $(\alpha,\beta)_m = 1$ *for all* $\alpha \in U_{1,F}$ *if and only if* $F(\sqrt[p^m]{\beta})/F$ *is unramified.*
3) $(1-\beta,\beta)_m = 1$ *for* $1-\beta \in U_{1,F}$.
4) $(-\beta,\beta)_m = 1$ *for* $-\beta \in U_{1,F}$.
5) $(\alpha,\beta)_m = (\beta,\alpha)_m^{-1}$ *for* $\alpha,\beta \in U_{1,F}$.
6) $(\alpha,\beta)_m = 1$ *for all* $\beta \in F^*$ *if and only if* $\alpha \in U_{1,F}^{p^m}$.

Proof. 1) immediately follows. If $F(\sqrt[p^m]{\beta})/F$ is not unramified, then $\widetilde{F}(\gamma) \neq \widetilde{F}$ for $\gamma^{p^m} = \beta$ and one can take $\alpha \notin N_{\Sigma/F}U_{1,\Sigma}$, where Σ/F is a totally ramified extension such that $\widetilde{\Sigma} = \widetilde{F}(\gamma)$. Then $(\alpha,\beta)_m \neq 1$, and we get 2). 3) and 4) follow from 1). If $\alpha,\beta \in U_{1,F}$, then

$$1 = (\alpha\beta,-\alpha\beta)_m = (\alpha,-\alpha)_m(\beta,-\beta)_m(\alpha,\beta)_m(\beta,\alpha)_m = (\alpha,\beta)_m(\beta,\alpha)_m.$$

If $(\alpha,\beta)_m = 1$ for all $\beta \in F^*$, then $(\beta,\alpha)_m = 1$ for all $\beta \in U_{1,F}$ and $F(\sqrt[p]{\alpha})/F$ is unramified. If $\alpha \notin F^{*p}$, then in the case under consideration $\alpha \equiv 1+\theta\pi_F^{pe/(p-1)} \mod \pi_F^{pe/(p-1)+1}$ where e is the absolute index of ramification of F. Then $\alpha \notin N_{F(\sqrt[p]{\pi_F})/F}F(\sqrt[p]{\pi_F})^*$ as it follows from (1.3). Therefore, $\alpha = \alpha_1^p$ for some $\alpha_1 \in U_{1,F}$. Now $(\alpha_1^p,\beta)_m = (\alpha_1,\beta)_{m-1} = 1$. Proceeding by induction on m, we conclude that $\alpha \in U_{1,F}^{p^m}$. □

Remark. One can extend the Hilbert symbol on $F^* \times F^*$: for $\alpha = \pi^a\theta\varepsilon$, $\beta = \pi^b\theta'\eta$ with $\varepsilon,\eta \in U_{1,F}$ and $\theta,\eta \in \mathcal{R}^*$, where $\mathcal{R}^*$ is the set of multiplicative representatives of $\overline{F}^*$ in F, put

$$(\alpha,\beta)_m = \begin{cases} (\varepsilon^b\eta^a,\pi)_m(\varepsilon,\eta)_m & \text{for } p>2,\\ (-1,\pi^{ab})_m(\varepsilon^b\eta^a,\pi)_m(\varepsilon,\eta)_m & \text{for } p=2.\end{cases}$$

Proposition implies that this pairing is well defined. It induces a non-degenerate pairing

$$F^*/F^{*p^m} \times F^*/F^{*p^m} \to \operatorname{Hom}_{\mathbb{Z}_p}(\operatorname{Gal}(\widetilde{F}/F),\mu_{p^m}).$$

There is another way to determine this pairing as

$$F^*/F^{*p^m} \times F^*/F^{*p^m} \to H^2(F,\mu_{p^m}) \xrightarrow{\sim}_{p^m} \operatorname{Br}(F)\otimes\mu_{p^m}$$

via the natural isomorphism between the last group in the preceding line and the group

$$\operatorname{Hom}_{\mathbb{Z}_p}(\operatorname{Gal}(\widetilde{F}/F),\mu_{p^m}),$$

see [Wt]. Employing the description of p^m-primary elements ([Hs, Sh]) one can deduce explicit formula for the Hilbert symbol ([Sh],[V]).

1.11. The second pairing is the Artin-Schreier pairing. Let F be of characteristic p. For $\alpha \in U_{1,F}$, $\beta \in F$ put

$$(\alpha,\beta](\varphi) = \Psi_F(\alpha)(\varphi)(\gamma) - \gamma,$$

where $\varphi \in \operatorname{Gal}(\widetilde{F}/F)$, γ is a root of the polynomial $\wp(X) - \beta$. We get the pairing

$$(\cdot,\cdot]\colon U_{1,F} \times F \to \operatorname{Hom}_{\mathbb{Z}_p}(\operatorname{Gal}(\widetilde{F}/F),\mathbb{F}_p).$$

In the same way as in the previous Proposition one can verify that:

1) Let $F(\gamma)/F$ be a totally ramified extension. Then $(\alpha,\beta] = 0$ if and only if $\alpha \in N_{F(\gamma)/F}F(\gamma)^*$.

2) $(\alpha,-\alpha] = 0$ for $\alpha \in U_{1,F}$.

3) $(\alpha,\beta] = 0$ for all $\alpha \in U_{1,F}$ if and only if $F(\gamma)/F$ is unramified.

Moreover, it is easy to deduce an explicit formula for $(\cdot,\cdot]$:

Proposition. $(\alpha,\beta](\varphi) = \varphi(\lambda) - \lambda$ *where* λ *is a root of the polynomial* $\wp(X) - \delta$ *and* $\delta = \operatorname{res}_\pi(\alpha^{-1}\dfrac{d\alpha}{d\pi}\beta)$.

Proof. Let $\varphi(\lambda) - \lambda$ be denoted as $d_\pi(\alpha,\beta)(\varphi)$. It suffices to verify the assertion for $\beta = \eta\pi^{-i}$, $\eta \in \overline{F}, p \nmid i$, $i > 0$. Let $L = F(\gamma)$, where $\gamma^p - \gamma = \eta\pi^{-i}$. Let π_L be a prime element in L. Then $\gamma \equiv \eta_1 \pi_L^{-i} \mod \pi_L^{-i+1}\mathcal{O}_L$ with $\eta_1 \in \overline{F}$ such that $\eta_1^p = \eta$. Let σ be a generator of $\mathrm{Gal}(L/F)$ and

$$\frac{\sigma\pi_L}{\pi_L} = 1 + \theta_0\pi_L^i + \dots \quad , \qquad \theta_0 \in \overline{F}.$$

It follows from (1.3) that $U_{1,F}/N_{L/F}U_{1,L}$ is generated by units $1 + \theta\pi^i$ with $\theta \in \overline{F}$, $\notin \theta_0^p\wp(\overline{F})$. Then $(\alpha, \eta\pi^{-i}]$ is determined by its values on $\alpha = 1 + \theta\pi^i$ where $\theta \in \overline{F}$ because of the first property of $(\cdot,\cdot]$. We also obtain that for $\alpha = N_{L/F}\alpha'$

$$\begin{aligned} d_\pi(\alpha,\beta) = d_\pi(N_{L/F}\alpha',\beta) &= d_\pi(N_{L/F}\alpha', \mathrm{Tr}_{L/F}\beta') \\ &= d_{\pi_L}(N_{L/F}\alpha',\beta') = d_{\pi_L}(\alpha', \mathrm{Tr}_{L/F}\beta') = d_{\pi_L}(\alpha',\beta) = 0 \end{aligned}$$

where β' is an element in L with $\mathrm{Tr}_{L/F}\beta' = \beta$. These equalities follow from the properties of residues and from the relation $\beta \in \wp(L)$.

Thus, it remains to verify the assertion for $\alpha = 1 + \theta\pi^i$, $\beta = \eta\pi^{-i}$. In this case

$$d_\pi(\alpha,\beta)(\varphi) = (\varphi - 1)\lambda \qquad \text{where } \lambda^p - \lambda = i\theta\eta.$$

Let $\alpha = N_{\widehat{L}/\widehat{F}}\widehat{\alpha}$, $\widehat{\alpha} = 1 + \xi\pi_L^i + \dots$, $\xi \in \overline{F}^{\mathrm{abp}}$. Then $\xi^p - \xi\theta_0^{p-1} = \theta$ by (1.3) and $\wp(i\xi\eta_1) = i\theta\eta = \wp(\lambda)$. Therefore, $\lambda - i\xi\eta_1 \in \mathbb{F}_p$ and $(\varphi-1)\lambda = i\eta_1(\varphi-1)(\xi) = -i\eta_1\theta_0$ because

$$\pi_L^{-1}\sigma(\pi_L) = 1 + \theta_0\pi_L^i + \dots \equiv (1 + \xi\pi_L^i + \dots)^{1-\varphi} \mod V(L|F).$$

On another hand, $\sigma(\pi_L^{-i}) = \pi_L^{-i} - i\theta_0 \mod \pi_L$, hence $\sigma(\gamma) - \gamma = -i\eta_1\theta_0$ and $d_\pi(\alpha,\beta) = (\alpha,\beta]$. □

Corollary. *If $(\alpha,\beta] = 0$ for all $\beta \in F$, then $\alpha \in U_{1,F}^p$. The pairing $(\cdot,\cdot]$ induces the non-degenerate pairing*

$$U_{1,F}/U_{1,F}^p \times F/(\wp(F) + \overline{F}) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\widetilde{F}/F), \mathbb{F}_p).$$

Remark. One can generalize $(\cdot,\cdot]$ using Witt vectors to obtain the non-degenerate pairing

$$U_{1,F}/U_{1,F}^{p^m} \times W_m(F)/(\wp W_m(F) + W_m(\overline{F})) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Gal}(\widetilde{F}/F), W_m(\mathbb{F}_p)).$$

§2. Additive polynomials

In this section we extend the properties of additive polynomials over quasi-finite fields ([Wh2, CW]) on perfect fields.

2.1. Let K be a perfect field of characteristic $p > 0$. A polynomial $f(X)$ over K is called ***additive*** if $f(a+b) = f(a) + f(b)$ for any $a, b \in K$. It is easy to show that if $\deg f(X) \leqslant \mathrm{card}(K)$, then $f(X)$ is additive if and only if $f(X+Y) = f(X) + f(Y)$ in the ring $K[X][Y]$, i.e., $f(X) = \sum_{m=o}^{m=n} a_m X^{p^m}$, $a_m \in K$.

Furhter we will assume that K is infinite. The ring of additive polynomials with respect to addition and composition is isomorphic to the ring $K[\Lambda]$ of non-commutative polynomials: $\sum a_m X^{p^m} \to \sum a_m \Lambda^m$, $(a\Lambda)(b\Lambda) = ab^p\Lambda^2$ for $a, b \in K$.

In the decomposition $f = g \circ h$ the polynomial g (resp. h) is called an outer (resp. inner) component of f, and f is called an outer (resp. inner) multiple of g (resp. h). For any two additive polynomials $f(X)$, $g(X)$ there exist and uniquely determined additive polynomials $h_1(X)$, $q_1(X)$ (resp. $h_2(X)$, $q_2(X)$) such that $f = h_1 \circ g + q_1$ (resp. $f = g \circ h_2 + q_2$), $\deg q_i < \deg g$. The ring of additive polynomials is a left and a right Euclidean principal ideal ring. If $f_3(X)$ is a least common outer multiple of additive polynomials $f_1(X)$, $f_2(X)$, then $f_3(K) \subset f_1(K) \cap f_2(K)$. If $f_4(X)$ is a greatest common outer component of f_1, f_2, then $f_4 = f_1 \circ g_1 + f_2 \circ g_2$ for a suitable additive polynomials g_1, g_2 and $f_4(K) = f_1(K) + f_2(K)$.

One can also introduce the notion of a generalized additive polynomial over K as a finite sum $\sum a_m X^{p^m}$ with $a_m \in K$, $m \in \mathbb{Z}$. There is an involution $f \to f^*$ in the ring of generalized additive polynomials, for $f(X) = \sum a_m X^{p^m}$ put $f^*(X) = \sum a_m^{p^{-m}} X^{p^{-m}}$.

2.2. For a non-zero additive polynomial $f(X)$ over K its set of roots is an additive finite subgroup in K^{sep}. Conversely, for an additive finite subgroup H in K^{sep} the polynomial $f_H(X) = \prod_{\alpha_i \in H}(X - \alpha_i)$ is an additive polynomial with $\ker f_H = H$. If $f(X)$, $g(X)$ are non-zero additive polynomials and $f'(0) \neq 0$, $\ker f \subset K$, $\ker g \subset K$, then $\ker f \subset \ker g$ if and only if $f(X)$ is an inner component of $g(X)$.

We call an additive polynomial $f(X)$ over K with $\ker f \subset K$ *K-decomposable*. We denote the set of K-decomposable polynomials by DP_K.

Lemma. *If $f(X) \in DP_K$ and $f'(0) \neq 0$, then*

$$f(X) = d_1 X \circ \wp(X) \circ d_2 X \circ \cdots \circ \wp(X) \circ d_{n+1} X$$

where $d_i^{-1} \in \big(\wp(X) \circ d_{i+1} X \circ \cdots \circ d_{n+1} X\big)(K)$. Conversely, any such polynomial is K-decomposable.

Proof. Let $\alpha \in \ker f$. Then $\wp(\alpha^{-1}X)$ is an inner component of $f(X)$ and one can put $d_{n+1} = \alpha^{-1}$. If $f(X) = g(X) \circ \wp(\alpha^{-1}X)$, then $g \in DP_K$ and by inductional arguements we deduce a decomposition of $f(X)$. The conditions on d_i follow from the condition $\ker f \subset K$. □

2.3. Let G_K^{abp} denote the group $\mathrm{Gal}(K^{\mathrm{abp}}/K)$.

Proposition. *Let $f(X) \in DP_K$. Then there is a homomorphism*

$$\lambda\colon K/f(K) \to \mathrm{Hom}_{\mathbb{Z}_p}\big(G_K^{\mathrm{abp}}, \ker f\big),$$
$$\lambda(a)(\varphi) = \varphi b - b \qquad \text{where } f(b) = a.$$

The homomorphism λ is an isomorphism.

Proof. First note that $b \in K^{\mathrm{abp}}$. Indeed, if $\sigma\tau \in \mathrm{Gal}(K(b)/K)$, then $\sigma b = b + c_1$, $\tau b = b + c_2$ with $c_1, c_2 \in \ker f$, and $\sigma\tau b = \tau\sigma b$. The homomorphism λ is evidently injective. If $\chi \in \mathrm{Hom}_{\mathbb{Z}_p}\big(G_K^{\mathrm{abp}}, \ker f\big)$, then let a_φ be an element of K^{abp} such that $(\varphi - 1)a_\varphi = \chi(\varphi)$

and $(\psi - 1)a_\varphi = 0$ for any $\psi \in G_K^{\mathrm{abp}}$ with $\psi \notin \langle\varphi\rangle$. It exists by Lemma (1.4). Then for $b = \sum a_\varphi$ where φ runs topological generators of G_K^{abp} (the sum contains in fact a finite number of non-zero addends) we obtain that $f(b) \in K$ and $\chi(\varphi) = (\varphi - 1)b$ for any $\varphi \in G_K^{\mathrm{abp}}$. □

Corollary. *Let $g \in DP_K$, $g'(0) \neq 0$. Let $f(X)$ be an additive polynomial over K. Then g is an outer component of f if and only if $f(K) \subset g(K)$.*

Proof. Let $d(X)$ be a greatest common outer component of $f(X)$ and $g(X)$. If $f(K) \subset g(K)$, then $d(K) = f(K) + g(K) = g(K)$. As $\ker g \subset K$ we obtain $\ker d \subset K$. Then, by Proposition g is an outer component of $d(X)$ and of $f(X)$. □

2.4. A generalized additive polynomial over K is called K-decomposable if its kernel belongs to K.

Proposition. *An additive polynomial $f(X)$ is K-decomposable if and only if $f^*(X)$ is K-decomposable.*

Proof. One may assume without loss of generality that $f'(0) \neq 0$. By (2.2) $\alpha \in \ker f^*$ if and only if $\wp(\alpha^{-1}X)$ is an inner component of $f^*(X)$, i.e., $\alpha^{-1}\wp(X)$ is an outer component of $f(X)$, i.e., $\alpha^{-1}\wp(K) \supset f(K)$ by Corollary of (2.3). Therefore, the cardinality of $\ker f^* \cap K$ coincides with the cardinality of the set $\{\alpha \in K : \alpha^{-1}\wp(K) \supset f(K)\}$. Let $\deg f = p^n$. Since there are $(p^n - 1)(p-1)^{-1}$ subgroups of order p in $\ker f$, we deduce applying the previous Proposition that there are $(p^n - 1)(p-1)^{-1}$ elements α in K such that all $\alpha^{-1}\wp(K)$ are distinct and $\alpha^{-1}\wp(K) \supset f(K)$. Thus, the cardinality of $\ker f^* \cap K$ is p^n, i.e., $\ker f^* \subset K$. □

Corollary. *Let $f(X) \in DP_K$. Then*

$$f(K) = \cap \alpha^{-1}\wp(K)$$

where α runs a set of the cardinality equal to the cardinality of $\ker f$, such that $\alpha^{-1}\wp(K) \supset f(K)$.

2.5. Proposition. *Let $f_1, f_2 \in DP_K$.*

1) *Let f_3 (resp. f_4) be a least common outer (resp. inner) multiple of f_1, f_2; f_5 (resp. f_6) be a greatest common outer (resp. inner) component of f_1, f_2. Then $f_i \in DP_K$ and $f_3(K) = f_1(K) \cap f_2(K)$.*

2) *$\{a \in K : f_1(a) \in f_2(K)\} = h(K)$ for some $h \in DP_K$.*

Proof.

1) Let $f_3 = f_1 \circ g_1 = f_2 \circ g_2$ with additive polynomials g_1, g_2. First assume that $f_5 = X$. As $\ker f_1^*$, $\ker f_2^*$ are contained in $\ker f_3^*$, we deduce that $\ker f_3^* \subset K$ and by Proposition (2.4) $f_3 \in DP_K$. According to Proposition (2.3) we get the surjective homomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}_p}(G_K^{\mathrm{abp}}, \ker f_3) \to K/f_3(K) \to K/f_1(K) \oplus K/f_2(K)\\ \to \mathrm{Hom}_{\mathbb{Z}_p}(G_K^{\mathrm{abp}}, \ker f_1 \oplus \ker f_2),\end{aligned}$$

which is injective as well. Therefore, $f_3(K) = f_1(K) \cap f_2(K)$.

Now let $f_1 = f_5 \circ h_1$, $f_2 = f_5 \circ h_2$ and $f_3 = f_5 \circ h_3$ with $h_1, h_2 \in DP_K$. If $a \in f_1(K) \cap f_2(K)$, then $a = f_5(h_1(c)) = f_5(h_2(d))$ and $h_2(d) - h_1(c) \in \ker f_5$. As $\ker f_5 \subset h_1(K)$, we obtain $a = f_5(b)$ for some $b \in h_1(K) \cap h_2(K) = h_3(K)$ and $a \in f_3(K)$. We deduce also that $f_3 \in DP_K$.

The polynomials f_4, f_5, f_6 are K-decomposable by Proposition (2.4).

2) One may assume by 1) that $\deg f_1 = \deg f_2 = p$. Then $f_1^{-1}(f_2(K)) \cap K = f_1^{-1}(f_3(K)) \cap K = g_1(K)$, where $f_3 = f_1 \circ g_1$, $g_1 \in DP_K$. □

2.6. Finally we consider an analog of some remarkable property of additive polynomials.

Lemma. *Let $f(X)$ be a polynomial over K, $f(0) = 0$, $f(X) \neq 0$. Let $g(X)$ be a non-zero K-decomposable polynomial. Then there exist finite sequences $q_i(X)$, $h_i(X)$ of polynomials over K such that $g(X)$ is an outer component of $\sum f(q_i(X))$ and of $\sum f(h_i(X))$, where $\sum f(q_i(X)) \neq 0$ and $\sum h_i(X)$ is a non-zero K-decomposable polynomial.*

Proof. According to Corollary 1.1 of [CW] one can find linear polynomials $g_i(X)$, $h_i(X)$ such that $\sum f \circ g_i$ is a non-zero additive polynomial, $\sum f \circ h_i$ is an additive polynomial, and $\sum h_i(X) = X$. Hence it suffices to show that for a non-zero additive polynomial $p(X)$ and $g(X) \in DP_K$ there exists a non-zero K-decomposable polynomial $r(X)$ such that $p \circ r = g \circ s$ for some additive polynomial $s(X)$. Let $g = g_1 \circ g_2$, $g_i \in DP_K$ and $p \circ r_1 = g_1 \circ s_1$, $s_1 \circ r_2 = g_2 \circ s$. Then $p \circ r_1 \circ r_2 = g \circ s$. Therefore, it remains to consider the case of $\deg g(X) = p$. Let H be a finite additive subgroup in K which contains $p^*(\ker g^*)$. Let $r(X)$ be an additive polynomial with $\ker r^* = H$. Then $r \in DP_K$ and $\ker g^* \subset \ker(r^* \circ p^*)$. By (2.2) we obtain $r^* \circ p^* = s^* \circ g^*$ for some additive polynomial $s(X)$. Then $p \circ r = g \circ s$, as desired. □

§3. Existence theorem

In this section we describe the norm groups of totally ramified p-extensions.

3.1. A subgroup H in $\overline{F}$ is called *polynomial* if

$$H = f(\overline{F})$$

for some non-zero $\overline{F}$-decomposable polynomial $f(X)$. Let π be a prime element in F. A subgroup $\mathcal{N}$ in $U_{1,F}$ is called *normic* if

1) $\mathcal{N}$ is open;

2) for any $i > 0$ there exists a polynomial $f_i(X) \in \mathcal{O}_F[X]$ such that $\overline{f}_i$ is non-zero $\overline{F}$-decomposable and $1 + f_i(\mathcal{O}_F)\pi^i \subset \mathcal{N}$;

3) for any $i > 0$ the image of $(U_{i,F} \cap \mathcal{N})U_{i+1,F}$ under the projection

$$U_{i,F} \to U_{i,F}/U_{i+1,F} \xrightarrow{\sim} \overline{F},$$

where $1 + \theta\pi^i \to \overline{\theta}$, is polynomial, and for almost all i this image coincides with $\overline{F}$. It immediately follows that the notion of a normic subgroup does not depend on the choice of a prime element π in F. Our aim is to show that the class of normic subgroup coincides with the class of norm groups of abelian totally ramified p-extensions.

Proposition. *Let L/F be an abelian totally ramified p-extension. Then $N_{L/F}U_{1,L}$ is a normic subgroup in $U_{1,F}$.*

Proof. The first and second properties of normic subgroups for $N_{L/F}U_{1,L}$ are verified in the same way as in the proof of Proposition 15 in [Wh1, II]. The third property for an extension L/F of degree p follows from (1.3). Now we proceed by induction on degree of L/F. Let M/F be a subextension in L/F of degree p. The proof of Proposition (1.9) shows that $N_{L/F}U_{1,L} \cap U_{i,F} = N_{L/F}U_{h(i),L}$ where $h = \psi_{L/F}$ is the Hasse-Herbrand function of L/F. Using inductional arguements it suffices to consider the case of $i = s$ where $s = s(M|F)$ (see (1.3)). Let σ be an element of $\mathrm{Gal}(L/F)$ such that its restriction $\sigma|_M$ is a generator of $\mathrm{Gal}(M/F)$. Let π_L be a prime element in L. Then $\pi_M = N_{L/M}\pi_L$ is prime in M and $\pi_M^{-1}\sigma(\pi_M) = N_{L/M}(\pi_L^{-1}\sigma(\pi_L))$. Let $N_{L/M}$ map $U_{h(s),L}/U_{h(s)+1,L}$ to $U_{s,M}/U_{s+1,M}$ by the polynomial $f_1(X)$ where the residue $\overline{f}_1(X)$ is $\overline{F}$-decomposable, and let $N_{M/F}$ map $U_{s,M}/U_{s+1,M}$ to $U_{s,F}/U_{s+1,F}$ by the polynomial $f_2(X) = \theta_0^p\wp(\theta_0^{-1}X)$, where $\pi_M^{-1}\sigma(\pi_M) \equiv 1 + \theta_0\pi_M^s \mod \pi_M^{s+1}$. Then $\bar{\theta}_0 \in \bar{f}_1(\overline{F})$ and the residue polynomial $\overline{f_2 \circ f_1}$ is $\overline{F}$-decomposable by Lemma of (2.1). □

3.2. Proposition. *Let L/F be an abelian totally ramified p-extension. Let $\mathcal{N}$ be a normic subgroup in $U_{1,F}$. Then $N_{L/F}^{-1}(\mathcal{N})$ is a normic subgroup in $U_{1,L}$.*

Proof. It suffices to verify the assertion for a cyclic totally ramified extension L/F of degree p. Then the first and second properties of $N_{L/F}^{-1}(\mathcal{N})$ can be established similarly with the proof of Lemma 5 in [Wh1, II] by Lemma (2.6). The third property of $N_{L/F}^{-1}(\mathcal{N})$ follows immediately from (1.3) and Proposition (2.5),2). □

3.3. Let π be a prime element in F. Let $\mathcal{E}_\pi$ denote the set of abelian totally ramified p-extensions L/F with $\pi \in N_{L/F}L^*$. If L_1/F, $L_2/F \in \mathcal{E}_\pi$, then $L_1 \cap L_2/F \in \mathcal{E}_\pi$. Moreover, $L_1L_2/F \in \mathcal{E}_\pi$. Indeed, let $M = L_1 \cap L_2$. Assume that $N_{L_1/F}\pi_1 = N_{L_2/F}\pi_2 = \pi$ for prime elements π_1, π_2 in L_1, L_2. Then $N_{M/F}\varepsilon = 1$ for $\varepsilon = N_{L_1/M}\pi_1 N_{L_2/M}\pi_2^{-1}$. Using the first diagram of Proposition (1.8) we deduce that $\varepsilon \in N_{L/M}U_{1,L}$, consequently there is a prime element π_M in M such that $N_{M/F}\pi_M = \pi$ and $\pi_M \in N_{L_1/M}L_1^* \cap N_{L_2/M}L_2^*$. Thus, it suffices to treat the case of $L_1 \cap L_2 = F$ where L_1/F, L_2/F are cyclic of degree p. Assume that L_1L_2/F is not totally ramified. Then there is an unramified cyclic extension E/F of degree p, $E \in L_1L_2$. As $\pi \in N_{L_1/F}L_1^* \cap N_{L_2/F}L_2^*$ one can deduce $\pi \in N_{E/F}E^*$ using Chevalley lemma [C, p. 449], that is impossible. Therefore, L_1L_2/F is totally ramified. By Corollary of (1.8) we obtain $N_{L_1/F}U_{1,L_1} \cap N_{L_2/F}U_{1,L_2} = N_{L_1L_2/F}U_{1,L_1L_2}$. Let $\pi' \in N_{L_1L_2/F}(L_1L_2)^*$ for some prime element π' in F. Then $\pi' \in N_{L_1/F}L_1^* \cap N_{L_2/F}L_2^*$, hence $\varepsilon = \pi'\pi^{-1} \in N_{L_1L_2/F}U_{1,L_1L_2}$. This means that $L_1L_2/F \in \mathcal{E}_\pi$.

3.4. Proposition. *Let π be a prime element in F, and let $\mathcal{N}$ be a normic subgroup in $U_{1,F}$. Then there is precisely one abelian totally ramified p-extension L/F such that $\mathcal{N} = N_{L/F}U_{1,L}$ and $\pi \in N_{L/F}L^*$.*

Proof. First let $U_{1,F}/\mathcal{N}$ be isomorphic with $\oplus_\kappa \mathbb{F}_p$. In this case $U_{s+1,F} \subset \mathcal{N}$ for some $s > 0$ and

$$\mathcal{N} \cap U_{s,F}/U_{s+1,F} \simeq a\wp(\overline{F}), \qquad \mathcal{N}U_{i+1,F} \cap U_{i,F}/U_{i+1,F} \simeq \overline{F}$$

(isomorphisms are given by $1+\theta\pi^i \to \overline{\theta}$), $a \in \overline{F}$.

It is known that there is an Artin-Schreier extension $M = F(\lambda)$ with $\wp(\lambda) \in F$ or a Kummer extension M/F such that $U_{s+1,F} \subset N_{M/F}U_{1,M}$, $\pi \in N_{M/F}M^*$, $N_{M/F}U_{1,M} \cap U_{s,F}/U_{s+1,F} \simeq a\wp(\overline{F})$, $N_{M/F}U_{1,M} \cdot U_{i+1,F} \cap U_{i,F}/U_{i+1,F} \simeq \overline{F}$ for $i < s$ (see Corollary 10.5 in [Wh2] and Lemma 6 in [Wh1, II]). If $s = 1$, then $N_{M/F}U_{1,M} = \mathcal{N}$. If $s > 1$ we proceed by induction on s. Assume that $N_{M/F}U_{1,M} \neq \mathcal{N}$. By Proposition (3.2) the group $N_{M/F}^{-1}(\mathcal{N})$ is normic in $U_{1,M}$ and it is easy to verify that $U_{1,M}/N_{M/F}^{-1}(\mathcal{N})$ is isomorphic with $\oplus_\kappa \mathbb{F}_p$ and $U_{s',M} \subset N_{M/F}^{-1}(\mathcal{N})$ for some $s' < s$. Then by the inductional arguements $N_{M/F}^{-1}(\mathcal{N}) = N_{E/M}U_{1,E}$ for some cyclic extension E/M of degree p and $\mathcal{N} \supset N_{E/F}U_{1,E}$, $\pi \in N_{E/F}E^*$. As for any $\alpha \in U_{1,F}$ the element $\alpha^{1-\sigma}$ for $\sigma \in \mathrm{Gal}(M/F)$ belongs to $N_{E/M}U_{1,E}$, we deduce from Proposition (1.8),2) and Corollary of (1.8) that E/F is abelian. Now $\mathcal{N} = N_{L/F}U_{1,L}$, $\pi \in N_{L/F}L^*$ for the fixed field L of the subgroup H of $\mathrm{Gal}(E/F)$ such that $H^* = \Psi_{E/F}(\mathcal{N}/N_{E/F}U_{1,E})$.

Now let $U_{1,F}/\mathcal{N}$ be isomorphic with $\oplus_\kappa G$ for an abelian p-group G. We argue by induction on the order of G. Let $\mathcal{N}_1$ be a normic subgroup which contains $\mathcal{N}$ and such that $U_{1,F}/\mathcal{N}_1 \simeq \oplus_\kappa \mathbb{F}_p$. Then $\mathcal{N} = N_{M/F}U_{1,M}$ for a suitable cyclic extension M/F of degree p and $\pi = N_{M/F}\pi_M$ for some prime element π_M in M. By the inductional arguements there is an abelian extension L/M with $N_{M/F}^{-1}(\mathcal{N}) = N_{L/M}U_{1,L}$ and such that $\pi_M \in N_{L/M}L^*$. By the same reasons as above L/F is abelian and $N_{L/F}U_{1,L} = \mathcal{N}$, $L/F \in \mathcal{E}_\pi$. The uniqueness follows from (3.3) and Corollary of (1.8). □

Corollary. *Let F_π be the compositum of all fields L with $L/F \in \mathcal{E}_\pi$. Then $F_\pi \cap \widetilde{F} = F$ and $F_\pi\widetilde{F} = F^{\mathrm{abp}}$.*

Proof. Let $\alpha \in F^{\mathrm{abp}}$. There exists an unramified extension $M/F(\alpha)$ such that $\mathrm{Gal}(M/F)$ is isomorphic to $\mathrm{Gal}(M/M_0) \times \mathrm{Gal}(M/E)$, where $M_0 = M \cap \widetilde{F}$, E/F is a suitable abelian totally ramified p-extension, $E \subset M$. Let $N_{E/F}\pi_E = \pi\varepsilon$ for a prime element π_E in E and $\varepsilon \in U_F$. It follows from (1.3) that there is a finite abelian unramified p-extension F_1/F such that $\varepsilon \in N_{E_1/F_1}U_{E_1}$ where $E_1 = EF_1$. Then $\pi \in N_{E_1/F_1}E_1^*$. The group $N_{E_1/F}U_{1,E_1} = N_{E/F}U_{1,E}$ is normic in $U_{1,F}$. Hence there exists an extension $L/F \in \mathcal{E}_\pi$ such that $N_{L/F}U_{1,L} = N_{E_1/F}U_{1,E_1}$. Then $N_{L_1/F}U_{1,L_1} = N_{E_1/F}U_{1,E_1}$ for $L_1 = LF_1$. Since $\ker N_{F_1/F}$ is generated by $\eta^{\varphi-1}$ with $\eta \in U_{1,F_1}$, $\varphi \in \mathrm{Gal}(F_1/F)$, the second commutative diagram of Proposition (1.8) implies that $\ker N_{F_1/F} \subset N_{L_1/F_1}U_{1,L_1}$. Therefore, $N_{L_1/F_1}U_{1,L_1} = N_{E_1/F_1}U_{1,E_1}$ because $|L{:}F| = |E{:}F|$. We get $L_1/F_1, E_1/F_1 \in \mathcal{E}_\pi$. Now by Proposition $L_1 = E_1$. Thus, $E \subset L_1 \subset F_\pi\widetilde{F}$. This means that $F^{\mathrm{abp}} = F_\pi\widetilde{F}$. □

3.5. Existence Theorem. *Let π be a prime element in F. There is an order reversing bijection between the lattice of normic subgroups in $U_{1,F}$ with respect to the intersection and product and $L/F \in \mathcal{E}_\pi$ with respect to the intersection and compositum: $\mathcal{N} \leftrightarrow N_{L/F}U_{1,L}$.*

Proof. It follows from Proposition (3.4) and Corollary of (1.8). □

Corollary. *The reciprocity map*

$$\Psi_F \colon U_{1,F} \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(F_\pi/F)\big)$$

is injective.

Proof. The description of normic subgroups in (3.1) or standard arguements using the Hilbert norm residue symbol and the Artin-Schreier pairing imply the injectivity of Ψ_F. □

Remark. Ψ_F is not surjective when $\overline{F}$ is infinite.

3.6. Another description of normic subgroups can be developed by applying the method of K. Sekiguchi [Sk, Subsection 3.2]. Let $E(\cdot, X)\colon W(\overline{F}) \to 1+X\mathcal{O}_F[[X]]$ be the Artin-Hasse map, c.f. [Wh1, III]. Then, if $\operatorname{char}(F) = p$, one can take as the normic subgroups the finite intersection of the sets

$$E\big(p^n W(\overline{F}) + a\wp W(\overline{F}), \pi^m\big) \prod_{\substack{(i,p)=1\\ i\neq m, i\geqslant 1}} E\big(W(\overline{F}), \pi^i\big)$$

for $n \geqslant 0$, $m \geqslant 1$, $(m,p) = 1$, $a \in W(\overline{F})$ and a prime element π in F. If $\operatorname{char}(F) = 0$ and a group of primitive p^nth roots of unity belongs to F (and n is the maximal number with this property) , then one can take as the normic subgroups the finite intersections of the sets

$$E\big(p^m W(\overline{F}) + a\wp W(\overline{F}), \pi^{pe_1}\big) E\big(p^l W(\overline{F}) + b\wp W(\overline{F}), \pi^k\big) \prod_{\substack{(i,p)=1\\ 1\leqslant i<pe_1\\ i\neq k}} E\big(W(\overline{F}), \pi^i\big)$$

for $m, l \geqslant 0$, $a, b \in W(\overline{F})$, $1 \leqslant k < pe_1$, $e_1 = e/(p-1)$ and a prime element π in F, where e is the absolute index of ramification of F.

3.7. Now we indicate the connections of the established theory with the Hazewinkel local class field theory [Hz1–Hz2]. Let L/F be a Galois totally ramified extension. Then there is an exact sequence

$$1 \to \operatorname{Gal}(L/F)^{\mathrm{ab}} \to U_{\widehat{L^{\mathrm{ur}}}}/V(L|F) \to U_{\widehat{F^{\mathrm{ur}}}} \to 1$$

(similarly with the exact sequence in (1.4)). Involving the pro-quasi-algebraic structure of the group $U_{\widehat{F^{\mathrm{ur}}}}$ and observing that $V(L|F)$ is the maximal reduced subscheme of the connected component of $\ker N_{\widehat{L^{\mathrm{ur}}}/\widehat{F^{\mathrm{ur}}}}$, one deduces the exact sequence

$$\pi_1(U_L) \xrightarrow{N_{L/F}} \pi_1(U_F) \to \operatorname{Gal}(L/F)^{\mathrm{ab}} \to 1.$$

As the quasi-algebraic group $\operatorname{Gal}(L/F)^{\mathrm{ab}}$ is constant, we obtain the exact sequence

$$\tilde{\pi}_1(U_L) \xrightarrow{N_{L/F}} \tilde{\pi}_1(U_F) \to \operatorname{Gal}(L/F)^{\mathrm{ab}} \to 1,$$

where $\tilde{\pi}_1$ is the maximal constant quotient of π_1. Then $\tilde{\pi}_1(U_F)/N_{L/F}\tilde{\pi}_1(U_L)$ is isomorphic with $\operatorname{Gal}(L/F)^{\mathrm{ab}}$. Passing to the projective limit we obtain a homomorphism

$$\Psi\colon \tilde{\pi}_1(U_F) \to G_F^{\mathrm{abr}},$$

which is an isomorphism as it was proved by Hazewinkel.

The group $\tilde{\pi}_1(U_F)$ has no an explicit description with except of the case of finite $\overline{F}$. On the other hand, it is clear that $\mathrm{Gal}(F^{\mathrm{abp}}/\widetilde{F})^*$ is isomorphic with the projective limit $\varprojlim U_{1,F}/N_{L/F}U_{1,L}$ for $L/F \in \mathcal{E}_\pi$. The constant pro-quasi-algebraic group $\tilde{\pi}_1(U_F)$ is the projective limit of the constant kernels of isogenies $X \to U_{\widehat{F^{\mathrm{ur}}}} \to 1$. If we consider a similar isogeny with $U_{1,F}$ instead of U_F, then one has the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), A\big) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), X\big) & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & U_{1,F} & \longrightarrow & U_{1,\widehat{F^{\mathrm{ur}}}} & \xrightarrow{\ \theta\ } & \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), U_{1,\widehat{F^{\mathrm{ur}}}}\big) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

where $\theta(\varepsilon)(\varphi) = \varepsilon^{\varphi - 1}$. Then we obtain a homomorphism

$$U_{1,F} \to \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), \tilde{\pi}_1(U_F)\big)$$

and its composition with Ψ gives the reciprocity map Ψ_F.

3.8. Finally we note that an expansion of the method exposed above and methods employed to furnish class field theory of multidimensional local fields with a finite residue field [F1–F3] will provide a description of abelian totally ramified p-extensions of multidimensional local fields with a perfect residue field of characteristic p.

References

[C] C. Chevalley, *Sur la théorie du corps de classes dans les corps finis et les corps locaux*, J. Fac. Sci. Tokyo Imp. Univ. **2** (1933), 363–476.

[CW] T. H. M. Crampton and G. Whaples, *Additive polinomials.* II, Trans. Amer. Math. Soc. **78** (1955), 239–252.

[D] P. Deligne, *Les corps locaux des caracteristique p, limites de corps locaux caracteristique* 0, J.-N. Bernstein, P. Deligne, D. Kazhdan, M.-F. Vigneras, Representations des groupes réductifs sur un corps local, Hermann, Paris, 1984.

[F1] I. B. Fesenko, *Class field theory of multidimensional local fields of characteristic* 0 *with the residue field of positive characteristic*, Algebra i Analiz **3** (1991), no. 3, 165–196; English transl. in Sankt Petersburg Math. J. **3** (1992).

[F2] ———, *On class field theory of multidimensional local fields of positive characteristic*, Adv. Soviet Math. **4** (1991), 103–127.

[F3] ______, *Multidimensional local class field theory*. II, Algebra i Analiz **3** (1991), no. 5, 168–190; English transl. in Sankt Petersburg Math. J. **3** (1992).

[FW] J.-M. Fontaine and J.-P. Wintenberger, *Le "corps des normes" de certaines extensions algebriques de corps locaux*, C. R. Acad. Sci. Paris Sér. I Math. **288** (1979), 367–370.

[Hs] H. Hasse, *Gruppe der p^n-primären Zahlen für einen Primteiler $\mathfrak{p}$ von p*, J. Reine Angew. Math. **176** (1936), 174–183.

[Hz1] M. Hazewinkel, *Abelian extensions of local fields*, Thesis, Amsterdam Univ., 1969.

[Hz2] ______, *Corps de classes local*, H. Demasure, P. Gabriel, Groupes Algebriques. T. 1, North Holland, Amsterdam, 1970.

[Hz3] ______, *Local class field theory is easy*, Adv. Math. **18** (1975), 148–181.

[I] K. Iwasawa, *Local Class Field Theory*, Iwanami-Shoten, Tokyo, 1980 (Japan); Russian transl., Mir, Moscow, 1983.

[L1] F. Laubie, *Sur la ramification des extensions de Lie*, Comp. Math. **55** (1985), 253–262.

[L2] ______, *Extensions de Lie et groups d'automorphismes de corps locaux*, Comp. Math. **67** (1988), 165–190.

[Ma] M. A. Marshall, *Ramification groups of abelian local field extensions*, Canad. J. Math. **23** (1971), 184–203.

[N1] J. Neukirch, *Neubegründung der Klassenkörpertheorie*, Math. Z. **186** (1984), 557–574.

[N2] ______, *Class Field Theory*, Springer, Berlin etc., 1986.

[Sh] I. R. Shafarevich, *General reciprocity law*, Mat. Sbornik **26** (1950), 113–146; English transl. in Amer. Math. Soc. Transl. ser. **4** (1956), 59–72.

[Sk] K. Sekiguchi, *Class field theory of p-extensions over a formal power series field with a p-quasi-finite coefficient field*, Tokyo J. Math. **6** (1983), 167–190.

[Sn] S. Sen, *Ramification in p-adic Lie extensions*, Invent. Math. **17** (1972), 44–50.

[Sr] J.-P. Serre, *Sur les corps locaux á corps residuel algébriquement clos*, Bull. Soc. Math. France **89** (1961), 105–154.

[V] S.V. Vostokov, *Explicit form of the law of reciprocity*, Iz. AN SSSR. Ser. math. **42** (1978), 1288–1231; English transl. in Math. USSR Izv. **13** (1979).

[Wh1] G. Whaples, *Generalised local class field theory*. I, Duke Math. J. **19** (1952), 505–517; II, Duke Math. J. **21** (1954), 247–256; III, Duke Math. J. **21** (1954), 575–581; IV, Duke Math. J. **21** (1954), 583–586; V, Proc. Amer. Math. Soc. **8** (1957), 137–140.

[Wh2] ______, *Additive polynomials*, Duke Math. J. **21** (1954), 55–66.

[Wn] J.-P. Wintenberger, *Le corps des normes de certaines extensions infinies des corps locaux, applications*, Ann. Sci. École Norm. Sup. **16** (1983), 59–89.

[Wt] E. Witt, *Schiefkörper über diskret bewerteten Körpern*, J. Reine Angew. Math. **176** (1936), 153–156.