ON RELATIONS BETWEEN CONNECTED AND DISCONNECTED JULIA SETS

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0. Introduction and notations

We collect here some notations to be used in the paper.

1. $f_c(z) = x^2 + c$ is the quadratic family

J_c is the Julia set of f_c

$M = \{c \in \mathbb{C} : J_c \text{ is connected}\}$ is the Mandelbrot set

2. Denote $c \mapsto w_c$ a Riemann map of $\mathbb{C} \setminus M$ onto $\{w : |w| > 1\}$

$w \mapsto c_w$ denotes the inverse map

3. $D_c = \{z : f_c^n(z) \to \infty, n \to \infty\}$ is the basin of infinity for f_c

u_c is Green's function of the basin of infinity with the pole at infinity:
$u_c(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |f_c^n(z)|$

Denote $\Omega_c = \{z : u_c(z) < 1\}$

4. B_c is the Bottcher function at infinity, i.e. B_c is holomorphic in a neighborhood of ∞, $B_c(z) \sim z$ as $z \to \infty$, and $B_c \circ f_c(z) = [B_c(z)]^2$.

5. $\sigma(t) = 2t(mod 1), t \in [0, 1)$, $\sigma^n(t) = 2^n t(mod 1)$.

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In [DH1] the connectedness of M is proved by giving a formula for the Riemann map $c \to w_c$:

$$w_c = B_c(c). \tag{1}$$

MLC Conjecture [DH2]. *M is locally connected.*

Assume for a moment that M is locally connected. Then the set of hyperbolic dynamical systems in the space of all complex quadratic polynomials is dense [DH2].

Then also there is a topological model of ∂M as $S^1/\sim$, where $w_1 \sim w_2$ on the unit circle S^1 iff $c_{w_1} = c_{w_2}$, and the relation $\sim$ is known explicitly [DH1],[Dalg],[T].

In fact, the MLC Conjecture tells us about similarity between the dynamical plane and the parameter plane (It is already expressed in the equality (1), see also Sect. 2).

Due to Caratheodory theorem, one can restate the MLC conjecture as follows:

Every external ray $R_t^M = \{c = c_w : w = r\exp(2\pi it), r > 1\}$ of the Mandelbrot set converges to a unique point on ∂M according to the topological model of M.

The local connectivity of M has been proven at the following points $c \in \partial M$: for preperiodic $f = f_c$ [DH2], for f with an indifferent periodic point, and for finitely renormalizable f (Yoccoz, see [Y],[H]). There are examples due to Douady and Hubbard of infinitely renormalizable polynomials $f = f_c$ such that M is locally connected at c but the Julia set of f is not locally connected. Lyubich [Ly] gives combinatorial conditions on the renormalizations of $f = f_c$ which yield the local connectivity of M at c and the local connectivity of the Julia set. In [McM] it is proved that all infinitely renormalizable f which are robust belong to the boundary of M. The paper [S] in which the generalized Feigenbaum universality was proved is also very important in this circle of problems.

An aim of the present paper is to give an approach to the MLC conjecture for the infinitely renormalizable f's with locally connected Julia set.

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1. Content of the paper

We start with a construction which produces quadratic-like mappings $\widetilde{Q}$ with disconnected Julia sets from a quadratic-like mapping Q with connected one (cf.

[DH1], [DHpol-l], [G], [LP]): see Sect. 2. It appears in many parts of the paper (sometimes implicitly). We call $\widetilde{Q}$ by the M-deformation of Q.

Finitely renormalizable case is considered in Sect. 3.

In Sect. 4 we introduce M-stable polynomials. This notion plays a crucial role in the rest of the paper. Let us stress that this is a condition on the polynomial itself.

In sections 4-5 we prove that *for a polynomial with locally connected Julia set the M-stability is equivalent to the continuity of the map $w \to c_w$ at the points which correspond to the accesses to the critical value in the dynamical plane of the polynomial.* This reduces the problem about the parameter space to some extremal length problem in the dynamical plane.

Sections 6-8 are devoted to a proof of the following below Theorem 1.1 (see also the remarks after the theorem).

In section 9 we prove Theorem 1.2 (we state it in the present section).

In the Appendix we extract a path through a disconnected Julia set, on which the dynamics is "real".

Let us formulate the Theorems 1.1 and 1.2.

Let $f = f_{c_*}$, where

c_* is real, and f is infinitely renormalizable.

Given a renormalization $Q = f^m$ of f, let us introduce an annulus $A = A(Q)$ and a domain $U_* = U_*(Q)$ as follows (it is a standard construction: [S], [deM-vanSt]). As usual, $I_* = [Q(0), Q^2(0)]$ (up to the order), and $F = f^{-(m-1)}$ is a branch homeomorphic on I_* and passing through all intervals $f^i(I_*), i = 1, 2, ..., m-1$, of the renormalization, $\widetilde{I} \supset I(0)$ is the maximal interval such that the branch $F : \widetilde{I} \to \mathbb{R}$ is still injective. Denote $\Omega = \{z : u(z) < 1\}$ and $\widetilde{\Omega} = ((\mathbb{C} \setminus \mathbb{R}) \bigcup \widetilde{I}) \bigcap \Omega$. Then F extends to a univalent function in $\widetilde{\Omega}$. We let

$$V_* = f^{-1} \circ F(\widetilde{\Omega}).$$

Note that $Q : V_* \to \widetilde{\Omega}$ is a quadratic-like mapping in the sense of [DHpol-l]. We let the domain

$$U_* = Q^{-1}(V_*)$$

and the annulus

$$A = V_* \setminus U_*.$$

Let $\Lambda(U_*)$ be the set of the domains U such that, for some $i = i_U > 0$, $f^i : U \to U_*$ is an isomorphism. Define

$$\rho(A, U_*) = \frac{area[\bigcup_{U \in \Lambda(U_*)} U \bigcap A]}{area[A]},$$

where *area* denotes the Lebesgue measure on the plane. Denote $m(A)$ the modulus of the annulus A.

Theorem 1.1. *There exists a positive increasing function $\rho_*(m)$, $m > 0$, such that $\rho_*(\infty) = 1$, as follows. Assume there exists a sequence of renormalizations Q_i of f, such that for the corresponding annuli $A^i = A(Q_i)$ and the domains $U^i = U_*(Q_i)$ we have:*

$$\rho(A^i, U^i) < \rho_*(m(A^i)), \tag{1.1}$$

for all i. Then the point $c_* \in \partial M$ (remind $f(z) = z^2 + c_*$) is accessible from $\mathbb{C} \setminus M$. Moreover, to every external argument t_0 of the critical value c_* in the dynamical plane of f_{c_*} there corresponds a ray in the Mandelbrot plane of the same argument t_0 that lands at c_*.

Remarks.

1. Note that if a priori the point c_* is accessible by an external ray of M of argument t_0, then t_0 *is* an external argument of the critical value of f_{c_*} in its dynamical plane (see Theorem 5.2).

2. In fact, we prove a stronger statement because we replace the domain U_* by a domain $W \subset U_*$. The domain W is constructed in section 7. It satisfies a Markov property (like Yoccoz's pieces): the components of $f^{-i}(W)$ and $f^{-j}(W)$ are either disjoint, or one coveres other. Moreover, W contains the ("small") Julia set

$$E = \bigcap_{n=0}^{\infty} Q^{-n}(V_*)$$

of the renormalization Q, and if some preimage W' of W intersects the compact E, then $W' = Q^{-n}(W)$, for some $n > 0$. The boundary of W consists of arcs of the Julia set of f, arcs of equipotentials, and arcs of preimages of the external rays of arguments 0 and 1/2. Since we use here the external rays orthogonal to equipotentials, we speak only about accessibility (along an external ray of M). If would replace above the orthogonal external rays to the β-fixed point of f and to its preimage by rays crossing equipotentials at other angles, one would expect locall connectivity at c_*. The problem is that the modulus m of the annulus A is changed if we change the angle in the definition of the external rays.

3. The main property we need from the Markov piece W is that one can control (in terms of the modulus of A) how the boundary ∂W changes as we M-deformate the quadratic-like Q. For example, if the renormalization is of disjoint type [McM], it is enough to require that ∂W belongs to a fundamental annulus of a given modulus, say to $Q^{-1}(A)$. In the present paper we construct the Markov piece with this property for real polynomials, that's why the Theorem 1.1 and the Theorem 1.2 (below) are about this case only.

4. The condition (1.1) means that the areas of a neighborhood of the Julia set of f with respect to some special metrics, which we construct to check M-stability of f, are finite and, moreover, are uniformly bounded for the sequence of renormalizations Q_i.

5. It was shown in [JH], [J] based on [S] that the Julia set of an infinite renormalizable real f is locally connected if f is of a bounded combinatorics. In [LSvS] we prove a much stronger result, namely, the local connectivity of the Julia set for a broad class of the polynomials $z \mapsto z^\ell + c$ with $\ell \geq 2$ even and c real, including the *all* infinitely renormalizable polynomials of this form.

In order to state the Theorem 1.2, let us fix some extra notations. Again $f = f_{c_*}$ is a real infinitely renormalizable quadratic polynomial. Let us fix an external argument t_0 (access) of the critical value c_* from the basin of infinity of f_{c_*}. Consider again some renormalization $Q = f^m$ of f. By Douady-Hubbard theory [DHpol-l], there is a unique $c = c(Q) \in \partial M$, such that Q and f_c are quasi-conformally conjugated. Moreover, the accesses $t_0/2$ and $t_0/2 + 1/2$ to the critical point of f define a unique access $t = t(Q)$ to the critical value $c(Q)$ from the basin of infinity of $f_{c(Q)}$. Given $f_{c(Q)}$ and $t(Q)$, we assign a non-negative number

$$d(Q) = \max |c - c(Q)|,$$

where the maximum is taken *over the all c from the limit set of the ray $R^M_{t(Q)}$ of the argument $t(Q)$ in the Mandelbrot plane.* Then $d(Q) = 0$ if and only if this ray lands at $c(Q)$, and, moreover, if and only if the ray $R^M_{t_0}$ lands at c_*. On the other hand, we define another quantity $\rho(W)$ similar to the $\rho(A, U_*)$ as follows. Here W is the Markov piece for the renormalization Q (see Remarks above). Denote $\Lambda(W)$ the set of domains U s.t. $f^i : U \to W$ is one-to-one, for some $i = i_U$, and s.t. *U is contained in W*. Then

$$\rho(W) = \frac{area[\bigcup_{U \in \Lambda(W)} U] + area[E]}{area[W]},$$

where E is the Julia set of the renormalization Q. Then $\rho(W)$ is some number between 0 and 1. In the following statement $A(Q)$ denotes the annulus for Q constructed befor Theorem 1.1.

Theorem 1.2. *There exists a positive function $D(\rho, m)$ as follows. Assume that for some sequence of the renormalizations Q_i of f and for the corresponding sequences of the annuli $A^i = A(Q_i)$ and the Markov pieces W_i we have: $\inf_i m(A^i) = m_0 > 0$ and*

$$d(Q_i) < D(\rho(W_i), m_0).$$

Then $d(Q_i) = 0$, that is the ray $R^M_{t_0}$ lands at the polynomial f.

Remark. In what follows we don't really use that the polynomial is quadratic except for the section 3, where the Yoccoz theorem on local connectivity of the Mandelbrot set is discussed. All other results of the present paper hold for the maps of the form $z \mapsto z^\ell + c$ with locally connected Julia set.

2. Construction of disconnected Julia sets from a connected one

Let us fix a quadratic-like mapping $P : U \to V$ with *connected* Julia set $J(P)$. We assume that $U \setminus J(P)$ is also connected (i.e. the *Fatou set* is empty). Assume also that the critical value v of P is accessible from the domain $D = U \setminus J(P)$ (it is not a principal assumption: one can avoid it). Fix a point $z_0 \in D$ and let L be a locally rectifible arc in D, which starts at z_0 and converges to v and such that all preimages of L under the iterates of P are pairwise disjoint (see Remark 2.1). We consider L as a semi-open arc: it includes z_0 but does not include v. Denote Σ the collection of the preimages of L under all iterates of P. Every arc l in Σ consists of two semi-open ones which converge to a common preimage z of the critical value v (we will say that l crosses $J(P)$ at z). Let now E be a closed connected subset of $J(P)$ such that the preimages of v are dense in E (example: $E = J(P)$). Denote Σ_E a subset of Σ consisting of arcs that cross E. Given Riemann surface $V \setminus E$ with the standard complex structure, we construct a new Riemann surface S_E as follows. Let us cut $V \setminus E$ along an arc $l \in \Sigma_E$, and identify pairs of points x, y on different sides of the cut according to the rule: first, a=$P^i(x) = P^i(y) \in L$, for some $i > 0$; second, if X is a neighborhood of the point a devided by L on two semi-neighborhoods X_1, X_2, then $x \in \partial P^{-i}(X_1)$ and $y \in \partial P^{-i}(X_2)$. Making this procedure with all arcs $l \in \Sigma$, we obtain the Riemann surface S_E, which inherits a complex structure from $V \setminus E$. S_E is a *planar Riemann surface* since every closed loop on it separates it. By Uniformization Theorem, we can consider S_E as a domain in $\mathbb{C}$ with the standard complex structure. Let $\Pi : V \setminus (E \bigcup \Sigma_E) \to S_E$ be the projection. It is a conformal isomorphism on its image S'_E. The complement $\widetilde{\Sigma}_E = S_E \setminus S'_E$ consists of open arcs corresponding to the cuts from Σ_E. Denote $\widetilde{E}$ the union of the bounded components of the complement $\mathbb{C} \setminus S_E$. Apriory, it is a union of points and discs. Note also that P induces a holomorphic map $\widetilde{P}$ on the part S'_E of S_E. *In the sequel the compact E is a Julia set of a quadratic-like map (renormalization)* $Q = P^n : U' \to V', V' \subset V$. Then a proof from [McM] shows that $\widetilde{E}$ has *absolute measure zero* (in particular, each component of $\widetilde{E}$ is a point), that is $\widetilde{E}$ is removable for the holomorphic maps outside this compact. Consider now the map $\widetilde{Q} = \widetilde{P}^n$. It is holomorphic in $\Pi(U' \setminus (E \bigcup \Sigma_E))$. Since E is invariant under Q, the map $\widetilde{Q}$ extends to a continuous map on $\Pi(U' \setminus E)$, and, hence, holomorphic there. Then, by removability of $\widetilde{E}$, the map $\widetilde{Q}$ is in fact holomorphic in the domain $\widetilde{U}'$ which is $\Pi(U' \setminus E)$ united with $\widetilde{E}$. Thus $\widetilde{Q}$ is a quadratic-like mapping on $\widetilde{U}'$ with the *disconnected Julia set* $\widetilde{E}$.

Lemma 2.1. *Let R_1, R_2 be curves in $V \setminus (E \bigcup \Sigma_E)$ such that they converge to the same point x of E. Then the curves $\Pi(R_1), \Pi(R_2)$ converge to the same point $\widetilde{x}$ of $\widetilde{E}$.*

Proof. The limit set of each curve $\Pi(R_i)$ is a point $\widetilde{x}_i$ of $\widetilde{E}$ since it is connected and it belongs to the disconnected set $\widetilde{E}$. It is well known that each point $z \in \widetilde{E}$ is defined by a sequence ϵ_j of $0, 1$ as follows. Let γ be a loop around $\widetilde{E}$ containing the critical value of $\widetilde{Q}$, and γ_0 be its preimage. It is "eight"-curve which surrounds two discs. Then 0 is the sign of one disc, and 1 is the sign of other one. Now ϵ_j is the sign of the disc containing $\widetilde{Q}^j(z)$. Finally, the either point $\widetilde{x}_i, i = 1, 2$ has the same sequence because Π conjugates Q and $\widetilde{Q}$ outside $E \bigcup \Sigma_E$ (note that the curve $Q^j(R_i)$ again does not intersect $E \bigcup \Sigma_E$ and it lands at $Q^j(x)$).

The map $\widetilde{Q}$ (more exactly, a class of conformally conjugated maps) depends on the point z_0 but does not depend on the initial arc L. Indeed, let L_1 be other arc in $V \setminus J(P)$ joining z_0 and v. Using the fact that Π conjugates Q and $\widetilde{Q}$ far from E, it is easy to see (extending the conjugacy Π deeper and deeper from a fundamental annulus) that then the map $\widetilde{Q}_1$ corresponding to L_1 is holomorphically conjugate to $\widetilde{Q}$ in its domain of definition. Thus the map $\widetilde{Q}$ depends only on the point z_0.

We will call $\widetilde{Q}$ by M**-deformation of the renormalization** Q **of** P. If we choose $Q = P$, then $\widetilde{P}$ is called M**-deformation of the quadratic-like mapping** P

At the same time, the set of arcs Σ_E (in the plane of Q) as well as the set of their projections $\Pi(\Sigma_E)$ (in the plane of $\widetilde{Q}$) play an important role since they relate the maps Q and $\widetilde{Q}$.

If P is just a quadratic polynomial, then Π extends to a holomorphic map at infinity. In this case we always normalize Π by the condition $\Pi'(\infty) = 1$.

Here is an analoge of MLC conjecture in this framework. Let $E = J(P)$, and $Q = P$.

Is it true that $\widetilde{P} \to P$ as $z_0 \to v$ inside of $V \setminus J(P)$?

(Observe that the set of quadratic-like maps $\widetilde{P}$ (with an appropriate normalization) corresponding to $z_0 \to v$ is compact, and a priori every limit map is a quadratic-like map with a connected Julia set.)

It is the analoge of the MLC conjecture for the Mandelbrot set M. Really, let $P = f_{c_0}$ be a quadratic polynomial, $U = \{z : u_{c_0}(z) < 1\}, V = \{z : u_{c_0}(z) < 2\}$, $E = J_{c_0}$. Then $\widetilde{P}$ is a quadratic polynomial f_c with disconnected Julia set such that

$$B_{c_0}(z_0) = B_c(c).$$

In this case $\Pi = B_c^{-1} \circ B_{c_0}$, and MLC conjecture says that $\Pi \to$id as $z_0 \to c_0$.

Moreover, the question above is *equivalent* to the MLC conjecture itself: if f is a unique quadratic polynomial quasi-conformally conjugated to P (by the Streightening Theorem [DHpol-l]), then $\widetilde{P} \to P$ iff the map $w \to c_w$ is continuous at the corresponding point. The proof is as follows. Let h be the quasi-conformal conjugacy between f and P. If $\widetilde{f}$ is a quadratic polynomial with disconnected Julia set constructed from the f and the curve $h(L)$, then the quasi-conformal distance between $\widetilde{P}$ and $\widetilde{f}$ stays bounded from above as $z_0 \to v$. Hence, any limit map for $\widetilde{P}$ and the corresponding for $\widetilde{f}$ are quasi-conformally conjugate. Since they are with connected Julia sets, this proves the statement (we use the fact from [DHpol-l], that if f_{c_1} and f_{c_2} are quasi-conformally conjugate, and $c_1 \in \partial M$, then $c_1 = c_2$).

Remark 2.1. Given a quadratic-like map $P : U \to V$ with connected and locally connected Julia set $J(P)$, and given an access to the critical value v from the domain $D = V \setminus J(P)$, let us find an arc L, which converges to v from D, homotopic in D to the access, and such that the preimages of L under the iterates are pairwise disjoint. Take a fundamental annulus A_0 for P such that its boundary consists of two Jordan curves. Let A_0' be a standard (round) annulus which is conformally equivalent to A_0. Let $\{R\}$ be the set of curves in A_0 corresponding to the set of intervals in A_0' orthogonal to the family of concentric circles in A_0'. Then the union of all preimages of the set $\{R\}$ under the iterates of P is a set of rays (call it again $\{R\}$) filled-in the domain D. Every ray from $\{R\}$ *lands.* This is because of the expanding property of the external map of P [DHpol-l] and the local connectivity of $J(P)$. Moreover, to each access to v there corresponds a curve from $\{R\}$ as needed.

In the case of quadratic polynomial the role of the curves $\{R\}$ plays usually the curves crossing every equipotential $\{z : u(z) = const\}$ at an angle $\tau \in (0, \pi)$ (so-called τ-rays). The use of τ-external rays instead of usual $\pi/2$-rays is mostly the matter of a taste. An equivalent (maybe less geometrical) way is given by Branner-Hubbard's twists [BH], which are extremal maps in a component of the quasiconformal deformations of the polynomial f_c. In fact, they transform orthogonal rays to τ-rays. In the sequel, whenever we speak about accessibility, usually we use orthogonal, or $\pi/2$-rays. But if we want to prove continuity of the map $w \to c_w$, we need the τ-rays (or something like them).

3. On finitely renormalizable quadratic polynomials

Let f_a be a finitely renormalizable quadratic polynomial without attracting and indifferent periodic points. Yoccoz has proved the following two fundamental results studying his puzzle [H],[M],[Y].

Theorem Y1. *J_a is locally connected.*

Theorem Y2. *M is locally connected at the point $a \in M$.*

To demonstrate some of ideas of the paper we prove here Theorem 3.1 which is a part of the Yoccoz's result (see [H]). We indicate a comparably easy proof of Theorem 3.1 (cf. [H]) making use the theory of the Yoccoz's puzzle and the following

Theorem. *(J. Kahn) The Julia set J_a of the finitely renormalizable polynomial f_a is removable for the homeomorphisms holomorphic outside of J_a.*

For simplicity we assume that f_a is not renormalizable at all (or not simple renormalizable [McM]).

Theorem 3.1. *(see [H] for proof) Let t_0 be an external argument of $a \in J_a$. Then t_0 is an external argument of $a \in M$ and $c_w \to a$ as $w \to w_a = \exp(2\pi i t_0)$.*

Proof. Let $w_k \to w_a$ and $c_k = c_{w_k} \to b$. We have to prove $b = a$.

There is a unique angle τ_k, such that the arc l_k starting at w_k, crossing circles $|w| = const$ at the angle τ_k and containing in the disc $\{w : |w - w_a| \leq |w_k - w_a\}$, converges to w_a. Let $z_k = B_a^{-1}(w_k)$, $L_k = B_a^{-1}(l_k)$, and Σ_k be the set of all preimages of L_k of all orders. We form the Yoccoz's pazzle as usual using the orthogonal external rays to α-fixed point of f_a. Let $\{P\}_d$ be the set of all (closed) pieces of a fixed depth d. Then, for all k big enough, the arcs in Σ_k and the boundaries of the pieces of depth d are disjoint. For given piece P of depth d and for such k we define P^{c_k} as a closed domain with the boundary $B_{c_k}^{-1} \circ B_a(\partial P)$. As $k \to \infty$, then P^{c_k} tends to P^b, a piece of f_b. Indeed, define also

$$\Pi_k = B_{c_k}^{-1} \circ B_a$$

and

$$\Pi = B_b^{-1} \circ B_a$$

on $D_a = \mathbb{C} \setminus J_a$. From the condition, $\Pi_k \to \Pi$ on every compact in D_a. Moreover, it is easy to see that the convergence extends to the set $\bigcup_{i=0}^{\infty} f_a^{-i}(\alpha)$. It follows,

$P^{c_k} \to P^b$. Given nested sequence of pieces P_i of f_a, there is a sequence $i(k)$ of indexes such that the pieces $P_i^{c_k}$, $i = 1, ..., i(k)$, of f_{c_k} are defined, and $i(k) \to \infty$ together with k.

Our aim is to prove that the conjugacy Π extends to a homeomorphism J_a onto J_b. It is well defined also on the preimages of the fixed point α of f_a. On the rest of J_a we want to define Π on J_a as follows. Given $z_0 \in J_a$ and a sequence of pieces $P_i(z_0)$ shrinking to z_0, we want to show that the compact $P_{i(k)}^{c_k} = \bigcap_{i=1}^{i(k)} P_i^{c_k}$ tends to a point, if $k \to \infty$. It will be $\Pi(z_0)$.

Let $A = P' \setminus P''$ be an annulus given by the puzzle of f_a, and A^{c_k}, A^b the corresponding annuli of f_{c_k}, f_a. Note that $f_a : A \to f(A)$ is an unbranged covering if and only if the same is true for $f_{c_k} : A^{c_k} \to f_{c_k}(A^{c_k})$, for given A and all big k. The reason that the pieces $P_{i(k)}^{c_k}$ of f_{c_k} (corresponding to $z_0 \in J_a$) shrink to a point is the same as for the usual pieces for f_a: starting from a non-generated annulus we find many "good" descendents of it. Details follow(cf [M]):

I. The orbit of z_0 is not accumulated by zero. Then it does not hit a thickened puzzle piece $\hat{P}_N(0)$. We construct the thickened puzzle pieces of depth $N-1$ for the maps f_{c_k} (k big) and repeating arguments from [M], we get that diam$P_{N+h}^{c_k} < \lambda^h C$, for some fixed $\lambda < 1$ and $C > 0$ and $k > k(h)$. But $P_{N+h}^{c_k} \to P_{N+h}^b, k \to \infty$.

II. The critical point is recurrent for f_a. Then there is a non-generate annulus $A_m(0)$ and a sequence of critical annuli A_{i_l} such that $f_a^{r_l} : A_{i_l} \to A_m$ is unbranged of degree d_l so that

$$\sum modA_{i_l} = \sum 2^{-d_l} modA_m = \infty.$$

If $k \to \infty$, then $A_m^{c_k}$ tends to a non-generate annulus A_m^b of f_b. Consider a summand $modA_{i_l}$ in the above sum. Then for $k \to \infty$ the $modA_{i_l}^{c_k}$ tends to $2^{-d_l} modA_m^b$. It implies $P_{i(k)}^{c_k}(0) \to \{0\}$ in this case. Let now z_0 be a point of J_a with the orbit accumulated at zero. Then there are infinitely many different annuli $A_{n+d}(z_0)$ which are conformally isomorphic to $A_d(0)$. Passing to f_{c_k} we obtain the same for given n, d and all big k: $modA_{n+d}^{c_k}(z_0) = modA_d^{c_k}(0)$. Again $P_{i(k)}^{c_k}(z_0)$ tends to a point. If the orbit of z_0 stays away from 0, this is the previous case.

III. The critical orbit is not recurrent and orbit of z_0 accumulates at zero. Then again we can repeat argument from [M] and obtain arbitrary many (as $k \to \infty$) dynamically defined annuli $A_d^{c_k}(z_0)$ with moduli bounded away from zero.

Thus we have defined $\Pi : J_a \to J_b$. The map Π is "onto" because the union of the pieces of a depth d coveres the Julia set. It is injective because given $x \neq y$ from J_a we separate them by two pieces P and P and then P^{c_k} and P^{c_k} will separate $\Pi(x)$ and $\Pi(y)$ for big k. Let us show the continuity of Π. It is convenient to pass to an open covering by (new) pieces P^* uniting some of P (see [M]). Let $x_i \to x$, where $x \in J_a$. If $x_i \in P_d^*(x)$, then $\Pi(x_i) \in P_d^{*c_k}(x)$ k big.

Thus we have constructed a homeomorphism Π which conjugates f_a and f_b and holomorphic outside the Julia sets. To complete the proof we apply the Theorem by Kahn stated above.

4. M-STABLE POLYNOMIALS

In this section we consider a quadratic polynomial $f = f_{c_*}$, such that:

(4.1) the Julia set J of f is connected and locally connected,
(4.2) f has no attracting and indifferent periodic orbits.

Let $B = B_{c_*}, u = u_{c_*}$. Let t_0 be an external argument of the critical value $c_* = f(0)$, that is $B^{-1}(\exp(2\pi i t_0)) = f(0)$. Let V_0 be a small semi-neighborhood of $\exp(2\pi i t_0)$, and U_0 be the image under B^{-1} of V_0. A given point z_0 from U_0,*there exists a unique angle $\tau(z_0)$ such that an arc $L = L(z_0)$ that crosses equipotentials at the constant angle $\tau(z_0)$ and is contained in the domain U_0 converges to $f(0)$.* Let $\Sigma(z_0)$, or just Σ, be a collection of the preimages of L under f of all orders $n > 0$. The arcs in Σ are pairwise disjoint. Fix a domain $\Omega = \{z : u(z) < 1\}$ and define a new Riemann surface $S = S(z_0)$ as in section 2. We cut Ω along every arc from Σ, and, on either side of the cut, we identify pairs of points with the same value of the Green function u. If we exclude the Julia set J from the Ω, cut along the arcs in Σ and identify points as above, we obtain the Riemann surface S, which inherits a complex structure σ_0 from Ω.

Let $\Pi : \Omega \setminus (J \bigcup \Sigma) \to S$ be the projection. It is a bijection on its image S'. The complement $\Sigma' = S \setminus S'$ consists of open arcs corresponding to the cuts from Σ.

Definition 4.1. *Given round annulus $C = \{z : r_1 < |z - a| < r_2\}$ in Ω, define a quantity $m(C, z_0)$ as follows. The number $m(C, z_0)$ is the extremal length of a family of curves $\Gamma(C, z_0) = \{\gamma\}$ on the surface S such that: (a) γ and the curves from Σ' have no common arcs (but, of cause, can cross each other), and (b) if x is an end of γ, then either $x \in \Pi(\partial C)$, or x belongs to a curve from Σ', such that this curve meets $\Pi(\partial C)$. We call the polynomial f by M-***stable on the access*** t_0, if there exists a positive δ such that, for each round annulus C of the modulus not big and not small (say, between 1/2 and 2),*

$$\liminf m(C, z_0) > \delta,$$

as $z_0 \to f(0)$ inside U_0.

*We say that f is M-***stable on its external ray of argument*** t_0, if* $\liminf m(C, z_0) > \delta$, *as $z_0 \to f(0)$ along the external ray of f of the argument t_0.*

Remark 4.1. One can define also the M-stability of f on a sequence of points $\{z_n\}_{n=1}^{\infty}$, which tends to z_0 inside the access t_0 (that is the sequence $B(z_n)$ tends to the point $\exp(2\pi i t_0)$ of the unit circle). Then all statements on the M-stability can be obviously adapted to this notion.

Definition 4.2. *To every point $z_0 \in U_0$ we correspond a unique $c = c(z_0)$ according to the rule:*

$$w_c = B(z_0).$$

(Cf. the end of Sect. 2.)

Set $\Omega_c = \{z : u_c(z) < 1\}$, and $S_c = \Omega_c \setminus J_c$.

Proposition 4.1. *The Riemann surface S_c is conformally equivalent to the $S(z_0)$.*

Proof. (In fact, we have proved it already in Sect. 2.) With the slope $\tau(z_0)$ chosen above, we cut the basin of infinity of f_c along the curves that start at 0 or at any its preimage and cross the equipotentials $u_c = const$ at the angle $\tau(z_0)$. We get a domain $A_{c,\tau(z_0)}$. Then we extend the Bottcher function B_c of f_c from infinity to a univalent function B_c^τ in $A_{c,\tau(z_0)}$. By the construction, the map

$$(B_c^\tau)^{-1} \circ B : \Omega \setminus (J \bigcup \Sigma) \to S_c$$

is well-defined and agrees with the dynamics. Therefore, it extends to an isomorphism between $S(z_0)$ and S_c.

Remark 4.2. We have seen that the Riemann surface $S(z_0)$ is isomorphic to (a part of) the basin of infinity for a polynomial outside the set M. So the M-stability actually tells us about moduli of annuli for polynomials outside M and, on the other hand, allows us to use the dynamics of the only map f to check the condition (see sections 5-6).

The next important statement explains why the Julia set J is not included in the definition of $S(z_0)$.

Proposition 4.2. *Let C_c be any annulus in Ω_c. Then the modulus of C_c equals to the extremal length of the family of curves, which join the boundary components of C_c and avoid the Julia set J_c.*

Proof. The Julia set J_c has absolute mesaure zero [McM], and such sets are removable for extremal lengths [AB].

The first application of the M-stability gives the

Proposition 4.3. *Let $R(c(z_0))$ be the $\tau(z_0)$-ray of $f_{c(z_0)}$, which passes via its critical value $c(z_0)$, and let $l(z_0)$ be an arc of the $R(c(z_0))$ between $c(z_0)$ and the Julia set. If f is M-stable, then the Euclidean diameter of $l(z_0)$ tends to zero as z_0 tends to $f(0)$ inside U_0.*

Proof. Let $C(z_0)$ be the annulus $\Omega_{c(z_0)} \setminus l(z_0)$. The domains Ω_c have geometry uniformaly bounded on c in a neighborhood of M. Hence, it is enough to prove that the modulus of $C(z_0)$ tends to infinity as z_0 tends to $f(0)$. If z_0 is close to $f(0)$, we find a large integer n as follows. There exist n pairwise disjoint closed round annuli $C_1, ..., C_n$, each of modulus 1, such that: C_1 encloses $L(z_0)$ and $\partial\Omega$, and each next C_i encloses the preceeding one and $\partial\Omega$. If z_0 is close enough to $f(0)$, then $m(C_i, z_0) > \delta, i = 1, ..., n$. Now, let us consider any curve γ, which joins $l(z_0)$ and $\partial\Omega_{c(z_0)}$. It contains n pairwise disjoint arcs $\gamma_1, ..., \gamma_n$, where γ_i is a curve of the family $\Gamma(C_i, z_0)$ coming from the definition of M-stability. If we take into account Proposition 4.2, we get modulus of $C(z_0)$ is greater than $n\delta$, i.e. tends to ∞ as $z_0 \to f(0)$.

We will see that M-stability is equivalent to continuity of the map $w \mapsto c_w$ on the unit circle provided the Julia set of polynomial is locally connected. We start with the following

Proposition 4.4. *Assume that the map $w \mapsto c_w$ is continuous at a point w_0, and $c_{w_0} = a \in \partial M$. Assume the polynomial f_a has no indifferent periodic points and obey the property:*

for every $\varepsilon > 0$, there exists a finite number of domains $P_1, ..., P_n$, such that (a)Euclidean diameters of all P_i, $diam(P_i) < \varepsilon$, (b)∂P_i consists of arcs of equipotentials, external rays, and finitely many periodic points of f_a or their preimages, (c)among the points $P_i \bigcap J_a$ there are no zero and its preimages.

Then f_a is M-stable.

Proof. Fix a round annulus C of modulus $m \in [\frac{1}{2}, 2]$. Choose ε so small that the modulus m' of more narrow annulus C', which is $2 \times \varepsilon$-neighborhood of C, is almost the same: $m'/m > 1/2$. Given ε, find the domains P_i from the condition of the proposition. Observe that the condition (c) yields that for all z_0 so that $B_a(z_0)$ is close enough to w_0, the set of arcs $\Sigma(z_0)$ is disjoint with the boundaries of all P_i. Denote $P_1, ..., P_m$ those of P_i that intersect ∂C, and forget about others. Let now w be close to w_0, $z_0 = B_a^{-1}(w)$, and $c = c_w$. By the condition, $c \to a$ as $w \to w_0$. Then (1)$\Pi^c = B_c^{-1} \circ B_a$ tends to the identity uniformely on every compact subset of $D_a = \mathbb{C} \setminus J_a$, and (2)domain P_i^c with the boundary $\Pi^c(\partial P_i)$ is well defined and tends to P_i as $w \to w_0$. Choose w so close to w_0 that (1)every P_i^c is in ε-neighborhood of P_i, (2)$\Pi^c(\partial C \setminus \bigcup_{i=1}^m P_i)$ stays in ε-neighborhood of ∂C. After this, if γ is a curve of the family $\Gamma(C, z_0)$, then it joins at least the opposite sides of the annulus C', and we are done.

It follows from the above proposition, Theorem 3.1, and Theorem Y1 (Sect. 3)

Corollary 4.1. *The finitely renormalizable quadratic polynomials without attracting and indifferent periodic points are M-stable.*

To prove the main result of this section, we need

Lemma 4.1. *Let f be as above, i.e. its Julia set J is connected and locally connected, and f has no attracting and indifferent periodic orbits. Then for every $\epsilon > 0$ there is a neighborhood P^ϵ of the zero such that $diam(P^\epsilon) < \epsilon$, and ∂P^ϵ consists of finitely many arcs of equipotentials, arcs of external rays, and periodic points of f and their preimages.*

Proof. Assume f is infinitely renormalizable, otherwise this follows from the Yoccoz's puzzle (see [M]). By [M], one can assume also that the critical point zero is recurrent. By [McM], there exists a sequence of renormalizations $f^{n_i} : U_i \to V_i$, where $U_i \subset V_i$ are neighborhoods of zero, such that the small Julia sets E_i of these renormalizations are ordered by inclusion: $E_{i+1} \subset E_i$. Note that $0 \in E_i$ and the sequence (E_i) of the sets contains a sequence of iterates of zero that tends to zero (otherwise zero would not be a recurrent point). Since all E_i are connected and J is locally connected, $diam(E_i) \to 0$ as $i \to \infty$. Now it is easy to end the proof using the Yoccoz's puzzles.

Theorem 4.1. *Let $f = f_{c_*}$ be M-stable on its external ray of argument t_0. Then c_* is accessible from $\mathbb{C} \setminus M$ and has external argument t_0, i.e. $c_w \to c_*$ as $r \to 1$, where $w = r\exp(2\pi t_0)$. If, moreover, f is M-stable on the access t_0, then c_w is continuous at $\exp(2\pi t_0)$.*

Proof. We will prove only the first statement. Let $w_k \to \exp(2\pi t_0)$ along the ray and $c_k = c_{w_k} \to b$. We have to prove $b = c_*$. We will construct a quasi-conformal conjugacy H between the given map f and the f_b. Let c be c_k, or c_*. If $c = c_k$, we denote J_c^* the Julia set J_c completed by the arcs of $\pi/2$-external rays via the critical point and its preimages (see Remark 2.1). For two points x, y in the dynamical plane of f_c, define $d_c(x, y)$ as Euclidean diameter of the following arc $[x, y]_c$ joining x and y. If x, y are in the Julia set, or in the completed Julia set, then $[x, y]$ is the unique arc joining x, y inside this set. If x and (or) y are outside of the set, then we add to the above arc the arcs of $\pi/2$-external rays from the points to the Julia set (external rays are defined well since t_0 is not periodic under the doubling σ). Given a point x *outside the Julia set* J of f, we define $H_k(x) = B_{c_k}^{-1} \circ B(x)$ whenever this expression makes sense, i.e. for all x not in the set of arcs $\Sigma(z)$, with $z = B^{-1} \circ B_{c_k}(c_k)$. Particularly, $H_k(x)$ is well defined for fixed x and all big k. As $c_k \to b$, we have: $H_k(x) \to H(x) = B_b^{-1} \circ B(x)$. Now, given $x \in J$ *preimage of zero* under an iterate of f, and given c_k, we define $H_k(x)$ as the arc in $J_{c_k}^*$ in the construction of M-deformation f_{c_k} of f. Using our assumption $f_{c_k} \to f_b$ and the Proposition 4.3, we conclude that for every x preimage of 0, the arc $H_k(x)$ tends to a point (as $k \to \infty$), which will be $H(x)$. Given c_k, define $H_k(x)$ for *any other point* $x \in J$, i.e. for x not a preimage of zero. By Lemma 2.1, all $\pi/2$-external rays converging to x, still land at one point after the M-deformation of f to f_{c_k}. This point is said to be $H_k(x)$.

Let x, y be two points on which H_k have been already defined.

Claim 1. There exist constants $C > 0, \nu > 0$ such that, for any two points x, y as above and for any k big enough,

$$d_{c_k}(H_k(x), H_k(y)) \leq C.d_{c_*}(x, y)^{\nu}.$$

Proof of the Claim 1. Certainly, we use M-stability of f_{c_*}. Let A_k be the annulus $\Omega_{c_k} \setminus [H_k(x), H_k(y)]_{c_k}$, and A be the annulus $\Omega_{c_*} \setminus [x, y]_{c_*}$. Like in the proof of Proposition 4.3, we derive that the modulus of A_k is greater than $\delta_1 \times$ the modulus of A, where δ_1 depends only on f_{c_*} and on the δ from the definition of M-stability. This relation between moduli implies the statement.

Using Claim 1, it is easy to check that the sequence $H_k(x)$ is a Cauchy sequence. Its limit is said to be $H(x)$. Given x, y, we get:

$$|H(x) - H(y)| = \lim_{k\to\infty} |H_k(x) - H_k(y)| \leq \lim_{k\to\infty} d_{c_k}(H_k(x), H_k(y)) \leq C.d_{c_*}(x, y)^{\nu}.$$

So, H is continuous. Since $H(J) = J_b$, the Julia set J_b is locally connected too. The maps f and f_b have the same M-deformations f_{c_k}. It allows us to define $H^{-1} : J_b \to J$. Thus, $H : \mathbb{C} \to \mathbb{C}$ is a homeomorphism conjugated f and f_b. It remains to show that H is quasi-conformal. For this, it is enough if H does not change the moduli of the all round annuli too much. To prove it, let us again use M-stability of f (cf. proof of Proposition 4.4). Fix a round annulus C in the dynamical plane of f of modulus $m \in [1/2, 2]$. Choose $\varepsilon > 0$ so small that the modulus m' of more narrow annulus C', which is $2 \times \varepsilon$-neighborhood of C, is almost the same: $m'/m > 1/2$. Given ε, one can find $\epsilon > 0$ with the property that for the domain P^{ϵ} from Lemma 4.1 we will have: $\sup_{i \geq 0} diam(f^{-i}(P^{\epsilon})) < \varepsilon$ (we use local connectivity of J and Caratheodory theorem). Observe that the condition on the boundary of P^{ϵ} yields that for all z_0 so that $B(z_0)$ is close enough to $\exp(2\pi t_0)$, the set of arcs $\Sigma(z_0)$ is disjoint with the boundaries of all $f^{-i}(P^\epsilon)$. Especially, H_k are defined on $f^{-i}(P^\epsilon)$ for all $i \geq 0$ and all k.

Let C^ε be a new annulus obtained by joining with ∂C those preimages of P^ε that intersect ∂C. Then $H(C^\varepsilon)$ is an annulus in the dynamical plane of f_b, and its modulus tends to the modulus of the annulus $H(C)$ as $\varepsilon \to 0$. Now let us look at the annulus $H_k(C^\varepsilon)$, when k is big. On the one hand, its modulus is not less than a definite part of m (the modulus of C), just by M-stability of f. On the other hand, its modulus tends to the modulus of $H(C^\varepsilon)$ as $k \to \infty$. Let us prove it. If it is not the case, then there is a sequence x_j of points on boundaries of some $f^{-i}(P^\epsilon)$, and a sequence of indexes k_j, such that x_j tends to y but $H_{k_j}(x_j) - H_{k_j}(y)$ does not tend to zero as $j \to \infty$. It is impossible by the following reason. If x_j is close to y, we find many round annuli around $[x_j, y]_{c_*}$ like in the proof of Proposition 4.3, and then, by M-stability, the modulus of the annulus $\Omega_{c_{k_j}} \setminus [H_{k_j}(x_j), H_{k_j}(y)]_{c_{k_j}}$ is as big as we wish, when $j \to \infty$.

5. A SUFFICIENT CONDITION FOR M-STABILITY

We fix a quadratic polynomial f without attracting and indifferent periodic orbit, and with connected locally connected Julia set. Let W be a neighborhood of zero of f with Markov property: if i, j are positive integers, then the components of $f^{-i}(W)$ and $f^{-j}(W)$ are either disjoint, or one coveres other. We call W critical piece for f. An example of W is a critical Yoccoz's piece.

We assume that the following conditions are fulfilled:

(5.1) Let z_0 belong to an access to $f(0)$ (see Definition of M-stability). If z_0 is close to $f(0)$, then the set of arcs $\Sigma(z_0)$ and the boundary ∂W are disjoint.

According to this condition, we can consider the part $W(z_0) = \Pi(W)$ of the Riemann surface $S(z_0)$.

(5.2) There exists a K-quasi-conformal mapping φ_W of $W(z_0)$ onto W such that $\varphi_W = \Pi^{-1}$ on $\partial W(z_0)$.

Accordind to this condition, the map $H_W = \varphi_W \circ \Pi$ is K-quasi-conformal where it is defined, and $H_W|_{\partial W} =$id.

Theorem 5.1. *Assume there exists a sequence W_n of the critical pieces for f such that every W_n satisfies (5.1)-(5.2) and, moreover,*

(A) $\sup_n K_n < \infty$,

where K_n is the maximal dilatation of φ_{W_n}, and

(B) $diam(W_n) \to 0$ as $n \to \infty$.

Them f is M-stable.

Proof. For each W_n and each z_0 close enough to $f(0)$ (so that (5.1)-(5.2) hold) we define a conformal metric $\sigma_n(z)|dz|$ on the set $S(z_0) \setminus (J \bigcup \Sigma(z_0))$ as follows.

1. If $f^j(z) \notin W_n$ for all $j = 0, 1, 2, ...$ we let $\sigma_n(z) = 1$ (Euclidean metric).
2. If $z \in W_n$, then $\sigma_n(z) = |(H_{W_n})'_z(z) + (H_{W_n})'_{\bar{z}}(z)|$ a.e.
3. Let $f^j(z) \in W_n$ and $f^i(z) \notin W_n$ for $0 \leq i \leq j-1$. Then there is a domain U such that $f^j : U \to W_n$ is one-to-one. We define $H_U = f^{-j} \circ H_{W_n} \circ f^j$ and $\sigma_n(z) = |(H_U)'_z(z) + (H_U)'_{\bar{z}}(z)|$.[1]

[1]Here and in the next section we apply an idea of spreading a metric to a domain according to scales. I learned it from J.-C. Yoccos, who used this idea in other situation.

Given a round annulus C of radii r_1, r_2 with the modulus $\frac{1}{2\pi}\log\frac{r_2}{r_1} \in (1/2, 2)$ and using condition (B) of the theorem, we fix ϵ small enough and n large enough so that if $\Lambda(W_n)$ is the set of all preimages $f^{-j}(W_n)$, then

$$diam(U) < \epsilon,$$

for every $U \in \Lambda(W_n)$. (We use that the Julia set is locally connected.) It is easy to check that then for all z_0 close enough to $f(0)$ and for the metric σ_n the σ_n-length of any S-curve in C is bounded from below by $r_2 - r_1 - 2\epsilon$ while σ_n-area is bounded from above by $K_*[(r_2+\epsilon)^2 - (r_1-\epsilon)^2]$, where $K_* = \sup K_n < \infty$.

We derive from this statement and from Lemma 4.1

Theorem 5.2. *Let $f = f_{c_*}$ don't have an attracting and indifferent periodic point, and its Julia set be locally connected. If the map $w \to c_w$ is continuous at a point $w_* = \exp(2\pi i t_0)$ and $c_{w_*} = c_*$, then t_0 is an external argument of the critical value of f, and f is M-stable on the access t_0. If c_w has only radial limit c_*, then f is M-stable on the external ray of argument t_0.*

Proof. First, we show that t_0 is also an external argument of c_* in the dynamical plane of f_*. Take a positive sequence $\epsilon_n \to 0$, and choose the sequence of neighborhoods $P_n = P^{\epsilon_n}$ of zero from the Lemma 4.1. Fix n. If w is close to w_*, then by the condition, $c = c_w$ is close to c_*, and P_n is transformed to a neighborhood P_n^c of c. Hence, the external ray of f_* at the argument t_0 crosses ∂P_n. Since P_n shrink to c_*, we are done. Second, we set $W_n = P_n$ and use the Theorem 5.1.

6. On M-stability of infinitely renormalizable real quadratic polynomials

Fix a real infinitely renormalizable quadratic polynomial $f = f_{c_0}$. It has the locally connected Julia set [LSvS]. Consider a renormalization $Q = f^m : U_0 \to V_0$, where the domain U_0 contains the critical point zero. Let $E = \bigcap_{n=0}^{\infty} Q^{-n}(U)$ the Julia set of the renormalization, and denote $J_i = f^i(E), i = 1, 2, ..., m$, where $J_m = E$. Let $R(c_0)$ be an orthogonal external ray of argument t_0 to the critical value $c_0 = f(0)$. Let $z_0 \in R(c_0)$ and L be a part of $R(c_0)$ between z_0 and c_0.

Let us consider the M-deformation $\widetilde{Q}$ of the renormalization Q of f (see Sect. 2). As we know, $\widetilde{Q}$ is a quadratic-like mapping with a disconnected Julia set $\widetilde{E}$, and the natural holomorphic projection $\Pi : \mathbb{C}\setminus(E \bigcup \Sigma_E) \to \mathbb{C}\setminus(\widetilde{E} \bigcup \widetilde{\Sigma}_E)$ is defined (remind Σ_E is the set of those preimages of L under the iterates of f that intersect the small Julia set E, otherwords, Σ_E coincides with $\bigcup_{n=1}^{\infty} Q^{-n}(f^{-1}(L))$). Moreover,

$$\Pi \circ Q = \widetilde{Q} \circ \Pi$$

whenever both hand-sides are defined.

We want to find a simply-connected neighborhood W of zero which obey the following properties:

(W1) W is a critical piece for f: if i, j are positive integers, then components of $f^{-i}(W)$ and $f^{-j}(W)$ are either disjoint, or one coveres other.

(W2) $\overline{W}$ contains the small Julia set E. There is a bigger domain W' such that the annulus $W' \setminus W$ has a positive modulus $m(W' \setminus W) > 0$ and $W' \bigcap \bigcup_{i \geq 0} f^i(0) = W' \bigcap \bigcup_{i \geq 0} f^{mi}(0)$.

(W3) If a $\pi/2$-ray $R(f^i(0))$ to the iteration of the critical point $f^i(0)$ intersects W then $f^i(0) \in W$ and $R(f^i(0)) \bigcap W$ is connected.

(W4) The boundary ∂W does not intersect the arcs $l \in \Sigma_E \bigcup f(\Sigma_E) ... \bigcup f^{m-1}(\Sigma_E)$.

(W5) ∂W and E have at most finitely many common points. By Lemma 2.1, $\Pi(\partial W)$ is a simple closed curve. We denote by $\widetilde{W}$ a bounded domain with $\partial \widetilde{W} = \Pi(\partial W)$. We assume there exists a K-quasiconformal smooth map $\varphi : \widetilde{W} \to W$ suth that $\varphi = \Pi^{-1}$ on the boundary. Let us extend φ to $\mathbb{C} \setminus \widetilde{W}$ by Π^{-1}.

Definition 6.1. $\Lambda(W)$ is the set of components of $f^{-j}(W), j = 1, 2, ...,$, which lie in W and for which there is an index $i = i_U$ such that $f^i : U \to W$ is an isomorphism. By the condition (W1), given $U_0 \in \Lambda(W)$ there exists a maximal finite chain of domains $U_j \in \Lambda(W), j = 1, ..., l$ such that $U_0 \subset U_1 \subset U_2 ... \subset U_l$. We call $l + 1$ the *level* of U_0 and the chain $\{U_j\}_{j=0}^l$ is *associated* to U_0.

Lemma 6.1.

I. Let $U \in \Lambda(W)$. Then $\partial U \bigcap \Sigma_E = \emptyset$. Moreover, $U \bigcap \Sigma_E = \emptyset$.

II. Let $U \in \Lambda(W)$ be of level k. If $V \neq U$ is a domain from the chain associated to U, then $\partial U \bigcap f^{-i_V}(\Sigma_E) = \emptyset$, where the branch f^{-i_V} corresponds to $f^{i_V} : V \to W$. Moreover, $U \bigcap f^{-i_V}(\Sigma_E) = \emptyset$.

Proof. I. Assume some $l \in \Sigma_E$ intersects ∂U. We have two indexes: $i = i_U$ and j such that $f^{j+1}(l) = L_0$.

1. $i > j$. Then W intersects the ray $R(f^{i-j}(0))$ and, by condition (W3), $0 \in f^j(U)$. A contradiction.

2. $i < j$. Then ∂W intersects $f^i(l)$. This is a contradiction with the condition (W4) since the latter arc belongs to $\Sigma_E \bigcup f(\Sigma_E) ... \bigcup f^{m-1}(\Sigma_E)$.

3. $i = j$. This again contradicts to (W4).

Then assume that l belongs to U. Since l intersects the small Julia set E, we obtain that in fact $U = Q^{-n}(W)$ for some $n > 0$. Now by conditions (W1)-(W2) the map $f^{mn} : U \to W$ is of degree two. It contradicts to the definition of U.

Part II is proved by induction on the level k.

Definition *of the metric σ.*

We set $h = \varphi \circ \Pi$. It is a continuous map of $W \setminus (E \bigcup \Sigma_E)$ into W so that $h|_{\partial W} = \mathrm{id}$.

We define a map H from $W \setminus (E \bigcup \Sigma_E)$ into W as follows. If $x \in W \setminus (E \bigcup \Sigma_E)$ does not lie in any $U \in \Lambda(W)$, then $H(x) = h(x)$. Otherwise x belongs to a finite *maximal* chain of domains $\{U_j\}_{j=0}^k$ of $\Lambda(W)$, $U_0 \subset U_1 ... \subset U_k$. Let $i_j = i_{U_j}$ denote the corresponding indexes, $i_k < i_{k-1} < ... < i_0$, that is for some branch $f^{-i_j} : W \to U_j$ is an isomorphism. We define

$$H = h \circ \prod_{j=0}^{k} (f^{-i_j} \circ h \circ f^{i_j}) = h \circ (f^{-i_k} \circ h \circ f^{i_k}) \circ ... \circ (f^{-i_0} \circ h \circ f^{i_0}).$$

Now we will define the metric $\sigma(z)$, where z is a point outside of the Julia set J of the polynomial f and outside of the set of arcs Σ (Remind that Σ is the collection of the preimages of L_0 under all iterates of f.)

If $f^j(z) \notin W$ for all $j = 0, 1, 2, \ldots$ we let $\sigma(z) = 1$ (Euclidean metric).

If $z \in W$, then $\sigma(z)|dz| = |dH(z)|$, so that $\sigma(z) = |(H)'_z(z) + (H)'_{\bar{z}}(z)|$.

Let $f^j(z) \in W$ and $f^i(z) \notin W$ for $0 \leq i \leq j-1$. Then there is a domain U such that $f^j : U \to W$ is one-to-one. We define $H_U = f^{-j} \circ H \circ f^j$ and $\sigma(z) = |(H_U)'_z(z) + (H_U)'_{\bar{z}}(z)|$.

Properties of the map H and the metric σ.

1. $H(z) = z$ if $z \in \partial W$.
2. $H(z)$ is continuous if $z \in W \setminus (\Sigma \bigcup J)$.
3. If γ is a S-curve in W which has only two common points x_1, x_2 with the boundary of W, then $H(\gamma)$ is a usual curve in W joining x_1 and x_2.
4. Denote $W^{(l)}$ the union of the points of domains $U \in \Delta(W)$ of the level $l = 1, 2, \ldots$, $W^{(0)} = W \setminus W^{(1)}$. Then the σ-area of W,

$$A(W) = \int_W (\sigma(z))^2 dxdy \leq \sum_{l=0}^{\infty} K^{l+1} area[H(W^{(l)})].$$

Indeed, if $z \in W^{(l)}$ then H is K^{l+1}-quasiconformal, and

$$A(W) \leq \int \frac{|H_z| + |H_{\bar{z}}|}{|H_z| - |H_{\bar{z}}|}(|H_z|^2 - |H_{\bar{z}}|^2)dxdy \leq \sum_{l=0}^{\infty} K^{l+1} \int_{W^{(l)}} (|H_z|^2 - |H_{\bar{z}}|^2)dxdy.$$

Notation 6.1:

$$A'(W) = \sum_{l=0}^{\infty} K^{l+1} area[H(W^{(l)})].$$

5. There exists a positive decreasing function $C = C(m)$, $C \to 1$ as $m \to \infty$, as follows. If $f^i : U \to W$ is one-to-one, then

$$A(U) = \int_U (\sigma(z))^2 dxdy \leq C \frac{A'(W)}{area(W)} area(U),$$

where $C = C(m(W' \setminus W))$ (see property (W2) of the definition of the domain W).

Proof. Set $G = f^i$. Then G^{-1} extends to a univalent map in W'. By Koebe distortion theorem, there is a function C as above such that $|(G^{-1})'(x)/(G^{-1})'(y)|^2 \leq C(m(W' \setminus W))$, for all $x, y \in W$. Note that that quasiconformal dilatations of $H_U = G^{-1} \circ H \circ G$ at z and H at $G(z)$ are equal. Then

$$A(U) \leq \sum_{l=0}^{\infty} K^{l+1} area[G^{-1}(H(W^{(l)}))],$$

where

$$area[G^{-1}(H(W^{(l)}))]/area[U] \leq C area[H(W^{(l)})]/area[W].$$

Theorem 6.1. *Given the infinitely renormalizable polynomial f with locally connected Julia set, assume the existence of a sequence of domains W_n with the properties (W1)-(W5) such that $diam W_n \to 0$ as $n \to \infty$ and*

$$\sup C_n \frac{A'(W_n)}{area[W_n]} < \infty,$$

where $C_n = C(m(W_n' \setminus W_n))$. Then f is M-stable on the external ray of the argument t_0.

Proof follows the one of Theorem 5.1.

7. Construction of the Markov piece W for real polynomials

Let us remind notations from the end of Sect. 1.

$Q = f^m$ is a fixed renormalization of a real infinitely renormalizable f, E is its Julia set, $J_i = f^i(E), i = 1, 2, ..., m-1$ are the other "small" Julia sets of the renormalization. The interval $I_* = [Q(0), Q^2(0)]$ (up to the order), and $F = f^{-(m-1)}$ is a branch homeomorphic on I_* and passing through all intervals $f^i(I_*), i = 1, 2, ..., m-1$ of the renormalization, $\widetilde{I} \supset I_*$ is the maximal interval such that the branch $F : \widetilde{I} \to \mathbb{R}$ is still injective. Denote $\Omega = \{z : u(z) < 1\}$ and $\widetilde{\Omega} = ((\mathbb{C} \setminus \mathbb{R}) \bigcup \widetilde{I}) \bigcap \Omega$. Then F extends to a univalent function in $\widetilde{\Omega}$. We let

$$V_* = f^{-1} \circ F(\widetilde{\Omega}).$$

Note that $Q : V_* \to \widetilde{\Omega}$ is a quadratic-like mapping. We let

$$U_* = Q^{-1}(V_*)$$

and

$$A = V_* \setminus U_*.$$

Denote $[-a', a']$ the trace of V_* on $\mathbb{R}$. The points $\pm a'$ are the critical points of f^m closest to zero. Let k_* be such that $f^{k_*}(a') = 0$.

Let $V(i) = f^i(U_*), i = 1, 2, ..., m-1$. Then $V(i)$ is a component of $f^{-(m-i)}(V_*)$ that contains $f^i(I_*)$.

Define $\Lambda(V_*)$ as the set of domains U such that $f^i : U \to V_*$ is one-to-one, for some $i = i_U > 0$. For example, $V(i) \in \Lambda(V_*)$ while U_* is not.

Note that the intersection of the domain V_* with the Julia set J of f consists of two "horizontal" arcs (crossing the imaginary axis) and two "vertical" arcs (crossing the real axes). These notions are still in force for an arbitrary $U \in \Lambda(V_*)$ as it is homeomorphic to V_*.

First, we want to study relations between domains of $\Lambda(V_*)$.

Lemma 7.1. *The point a' (or $-a'$) hits a "small" Julia set J_l if and only if E and J_l have a common point (the renormalization Q is of β-type [McM]). In this case $m = 2.l$ (i.e. an even number), and, moreover, f^l is again a renormalization. Besides, in this case a' separates the points $f^{2.l}(0)$ and $f^l(0)$.*

Proof. If Q is of β-type, and a is the common end point of E and J_l, which is a periodic point of f of period k, then E and J_l are interchanging by f^k, i.e. $k = l$

and $m = 2.l$. Moreover, by [McM,Theorem ???], f^l is again a renormalization, and its Julia set contains $E \bigcup J_l$. The rest of the statement "if" is clear from the picture. Let us prove "only if". Let $a' \in J_l$. Then $b = f^m(a') = f^l(0)$ since b is the iterate of zero in J_l of order $< m$. Assume the sets E and J_l don't touch. It follows the closest points of E and J_l are two fixed points of f^m. Then a' must be in between of them because otherwise a point of J_l would belong to the E. A contradiction.

Lemma 7.2. *1. There exists a unique $U \in \Lambda(V_*)$ which contains a'. Then $i_U = k_*$, and $U \bigcap V_*$ is a subarc of a "vertical" arc of V_*.*

2. The above U coincides with one of the domains $V(j)$ if and only if the renormalization Q is of β-type.

Proof. 1. Let some $U \in \Lambda(V_*)$ contain a'. If $i_U < k_*$, then a' is not a closest to zero critical point of f^m. If $i_U > k_*$, then $0 \in f^{k_*}(U)$, i.e. $f^{i_U} : U \to V_*$ is not one-to-one. Thus $i_U = k_*$. To prove the existence, it is enough to show that there are no iterates $f^i(0), i = 1, 2, ..., m-1$ on $[-a', a']$. If there is, then it contradicts to Lemma 9.1. The rest is obvious.

2. If $U = V(j)$, then $a' \in J_j$, and we apply Lemma 7.1.

Notation 7.1: U' is the unique domain in $\Lambda(V_*)$ containing a'. So that $f^{k_*} : U' \to V_*$ is one-to-one.

Lemma 7.3. *U' intersects U_* if and only if Q is of β-type and U' is one of $V(j)$.*

Proof. If U' intersects U_*, then V_* intersects $V(k_*)$, and we apply Lemma 7.2. If, conversly, Q is of β-type, then U' contains even a point of E.

Lemma 7.4. *If $U \in \Lambda(V_*)$, $\overline{U} \bigcap \partial V_* \neq \emptyset$, but $U \neq \pm U'$, then the following two possibilities can occur:*

(a) $U \bigcap \partial V_*$ is a smooth arc, $i_U > m$, i.e. $f^m(U) \in \Lambda(V_*)$, and the domain $f^m(U)$ is symmetric with respect to the real axes;

(b) $U \bigcap \partial V_*$ consists of two smooth arcs meeting at a point $x \in U$ at the angle $\pi/2$, $i_U < m$, and $f^{i_U}(x) = 0$. Moreover, for each $x \in \partial V_*$ such that $f^i(x) = 0$, $i < m$, there exists a unique $U \in \Lambda(V_*)$ with $i_U = i$ and $x \in U$.

Proof. Let $T = \overline{U} \bigcap \partial V_*$. Let T be a smooth arc. Assume $i_U \leq m$. There are no preimages of zero of order $< i_U$ on T (since there are no iterates of zero of order $< m$ on $\overline{V}_*$). There exists a minimal $j < i_U$ such that $T' = f^j(T)$ is an arc of the imaginary axes. If $f^j(U)$ would not be symmetric about T', then we would complete U by a bigger domain from $\Lambda(V_*)$ with the same i_U. Thus $f^{j+1}(U)$ is symmetric about $\mathbb{R}$ and then $f^{i_U - j}(T') = V_* \bigcap \mathbb{R}$. It is a contradiction with a fact that f^m is a homeomorphism on its image on the part of ∂V_* in the upper (lower) halfplains. The rest follows. Now let T consist of a finitely many smooth arcs (l_i) such that l_i meets l_{i+1} at a point x_i at the angle $\pi/2$. Then $f^{k_i}(x_i) = 0$. Let $k = min(k_i)$. Let us show that $k = i_U$. The case $k < i_U$ is impossible because otherwise J would contain a closed curve. The case $k > i_U$ is ruled out since $f^k : U \to V_*$ is one-to-one. Thus $k = i_U$. It follows $U \bigcap \partial V_*$ consists of two smooth arcs.

The Lemma 7.4 implies the following

Lemma 7.5. *If $U_1, U_2 \in \Lambda(V_*)$, U_1 intersects U_2 and $i_{U_1} \le i_{U_2}$, then $U_1 \bigcap U_2 = f^{-i_{U_2}}(I)$, where I is either one of the intervals $I_* = V_* \bigcap \mathbb{R}$ or $V_* \bigcap i\mathbb{R}$,*

or I consists of two orthogonal intervals meeting at zero: one is a "half" of $V_* \bigcap \mathbb{R}$ and the other one is a "half" of $V_* \bigcap i\mathbb{R}$.

It yields, in its turn,

Lemma 7.6. *(An order in $\Lambda(V_*)$.) Let $U_i \in \Lambda(V_*)$, $i = 0, 1, 2$. If U_0 intersects U_1, and U_1 intersects U_2, but U_0 does not intersect U_2, then the possibility $i_{U_1} \ge i_{U_0}$ and $i_{U_1} \ge i_{U_2}$ is not realized.*

Definition 7.1 (of the Markov piece) W.

We distinct two cases.

The disjoint type renormalization, i.e. the small Julia set E of the renormalization $Q = f^m$ is disjoint from the all other small Julia sets J_i. Then $W = Q^{-1}(W_1)$, where the domain W_1 is defined by the condition:

$W_1 \subset V_*$ such that the domain $V_* \setminus \overline{W}_1$ consists of the points x with the property: there exist a finitely many $U_i \in \Lambda(V_*), i = 1, 2, ..., i(x)$ such that $x \in U_{i(x)}$, $U_1 \bigcap \partial V_* \neq \emptyset$, and for each i the domain U_i intersects U_{i+1} and does not intersect U_{i+2}.

The β-type renormalization, i.e. E meets other Julia set J_l at a point a, $f^l(a) = a$, $(f^l)'(a) < -1$. Let a-half plane be a half plane (right or left) containing a. Denote U'' the intersection of $V_* \setminus U'$ with the a-half plane. Let F be a branch of f^{-m} contracting to a. It is well defined at least in the intersection of V_* with the a-half plane. Then we define a domain $V'_* = U''_* \bigcup (-U''_*)$, where $U''_* = U'' \bigcup F(U'') \bigcup F^2(U'') \bigcup F^3(U'') \bigcup ....$ Now we define the domain W_2 exactly like we defined the domain W in the first case (the disjoint type renormalization) by replacing V_* by the new domain V'_*. Finally, we let $W = Q^{-1}(W_2)$.

Remark 7.1. We need a preimage of W_1 by Q in the first case because we need an annulus $A = V_* \setminus U_*$ around W such that it does not intersect $\pi/2$-external rays to the iterates of 0 not belonging to the "central" Julia set E. By exactly the same reason, we need the extra preimage by Q in the second case (β-renormalization).

Theorem 7.1. *The domain W constructed above is connected and simply-connected and it satisfies the properties (W1)-(W5) (Sect. 6). For the disjoint type renormalization ∂W belongs to $\overline{U}_* \setminus Q^{-1}(U_*)$. In the both cases one can choose the K-quasiconformal homeomorphism $\varphi : \widetilde{W} \to W$ (see (W5)) in such a way that its maximal dilatation K depends only on the modulus m_* of the annulus $V_* \setminus U_*$, and, moreover, φ is smooth in the domain $\widetilde{W}$ with the distortion $\sup|\varphi_z|/inf|\varphi_z|$ bounded from above by a function which depends only on m_*.*

Proof. (Sketch.) Using the Lemmas 7.4-7.6 we show that W is connected and simply-connected. The properties (W1)-(W4) follow from the construction of W (we use the orthogonal rays only). Prove (W5). In the case of disjoint renormalization ∂W_1 belongs to $A = \overline{V}_* \setminus U_*$ because of Lemma 7.6. Then in the case of disjoint renormalization we have a Koebe space for the projection Π as Π is univalent in $Q^{-2} \bigcup Q^{-1}(A) \bigcup A$. In the case of β-renormalization the domain W_2 has exactly 4 joint points with the Julia set E: the fixed point a (jointly with other small Julia set), its preimage $-a$, and two preimages of $-a$ under Q. Then in neighborhoods of these points we use local dynamics. It can be done in terms of the modulus m_*. Out of these neighborhoods we use the Koebe space for Π as above (maybe we use more preimages of A under Q, but their number is defined by m_*).

8. End of the proof of the Theorem 1.1

We are going to apply the Theorem 6.1. Remind that a domain $U \in \Lambda(W)$ is of level $l > 0$ if the maximal chain of domains $U_0 = U, U_1, ..., U_l$ from $\Lambda(W)$, $U_i \subset U_{i+1}$, contains exactly l elements. $W^{(l)}$ is the set of points of the domains of l-level. Let $U^j, j = 1, 2, ...$ be all domains of level one. They are pairwise disjoint and the *induced map* T is defined on $W^{(1)}$: $T|_{U^j} = f^i$, where $i = i_{U^j}$, so that $T : U^j \to W$ is an isomorphism. Let $h = \varphi \circ \Pi$ (see Sect. 6) and $\widetilde{T} = T \circ h^{-1}$. In these notations we have (see definitions in Sect. 6):

$$area[H(W^{(l)}] < area[\widetilde{T}^{-l}(W)].$$

If $T|_U = f^i$, for some 1-level domain $U \in \Lambda(W)$, then, for any $x \in W$,

$$T^{-1}(x) = Q^{-r(x)} \circ f^{-L(x)}(x),$$

where $Q = f^m$ is the considered renormalization, $i = r(x)m + L(x)$, and $r(x)$ is the unique $r > 0$, such that $Q^r(T^{-1}(x))$ lies in $V_* \setminus U_*$, if the renormalization is disjoint, and it lies in $Q^{-1}(V_* \setminus U_*)$ in the case of β-renormalization. So $L(x) > 0$ and $f^{-L(x)}(x)$ belongs either to $V_* \setminus U_*$, or to $Q^{-1}(V_* \setminus U_*)$.

We continue as follows:

$$\widetilde{T}^{-1}(x) = h \circ T^{-1}(x) = \varphi \circ \Pi \circ Q^{-r(x)} \circ f^{-L(x)}(x) =$$

$$= \varphi \circ \widetilde{Q}^{-r(x)} \circ \Pi \circ f^{-L(x)}(x).$$

Observe that every map

$$\Psi = \widetilde{Q}^{-r(x)} \circ \Pi \circ f^{-L(x)}$$

extends from a neighborhood of $x \in W$ to a univalent holomorphic function on either V_* (disjoint type), or U_* (β-type). The argument: this domain (V_*, or U_*) is disjoint with the external rays $R(f^i(0))$, if $f^i(0)$ is not in the central small Julia set. Let us fix the modulus m_* of the annulus A. By this we fix also the maximal dilatation of the map φ and its distortion on the domain $\widetilde{W} = \Pi(W)$. Remind that $\varphi = \Pi^{-1}$ in a neighborhood of $\partial\widetilde{W}$, so that we can extend φ by Π^{-1} outside of $\widetilde{W}$. Since the modulus of the annulus $V_* \setminus W$ is at least m_*, $\Psi(W)$ has bounded distortion (in term of m_*, of cause). Let now the density $\rho(A, U_*)$ of the domains $U \in \Lambda(U_*)$ (i.e. $f^i : U \to U_*$ is an isomorphism, for some $i = i_U$) in the annulus A tend to zero. Then the relative area occupied by the all domains of the form $\Psi(W)$ in $\widetilde{W}$ tends to zero too. For this, we just note that the maps $\widetilde{Q}^{-n}$ are holomorphic on $\Pi(A)$ together with the Koebe space except perhaps one branched point which cannot disturb too much when we consider the area. Thus, for each Ψ the domain $\Psi(W)$ has a small Euclidean diameter and is surrounded by the annulus $\Pi(A)$ (of the modulus m_*). Hence, all maps $\Psi : W \to \widetilde{W}$ are uniformely contractible by a factor $\lambda < 1$, in the hyperbolic metrics of V_* and $\Pi(V_*)$, and λ 0 as $\rho(A, U_*)$ tends to zero. Returning from $\Pi(V_*)$ to V_* by the smooth map φ, we get that all branches of the map $\widetilde{T}^{-1} : W \to W$ are contractible in the hyperbolic metric of V_* by a factor tending to zero with the density $\rho(A, U_*)$. So if the latter density is less than some $\rho_*(m_*)$, we can apply Theorem 6.1.

To end the prove, we show that $\rho_*(m_*) \to 1$ as $m_* \to \infty$. If m_* the modulus of A is large, then the renormalization Q is of disjoint type (see [McM,pp.110-111]). Then Π is close to identity on $Q^{-2}(A) \bigcup Q^{-1}(A) \bigcup A$. Hence, the maximal dilatation and the distortion of φ are close to 1. Moreover, Q and $\widetilde{Q}$ are close on V_*. If now the density $\rho(A, U_*) < 1$ is fixed and $m_* \to \infty$, then $area[\widetilde{T}^{-1}(W)]/area[W]$ cannot be too large since already $Q^{-1}(A)$ gives enough impact in the $area[W]$.

9. Proof of the Theorem 1.2

Fix a renormalization Q. So the number $\rho(W) \in (0, 1)$ and the modulus m_* of the fundamental annulus A are given. Let $d(Q)$ be close to zero. It follows from Sect.2, that $\widetilde{Q}$ is close to Q on V_*, that is Π is close to identity on A (see Remark 9.1 below). Then we can follow the proof of the Theorem 1.1 from the previous section.

Remark 9.1. One should have in mind the following. Let l be any non-tangent access to zero of the polynomial f (it means that l is mapped by the external conjugacy of f to a non-tangent curve converging to a point of the unit circle). Then l is also a non-tangent access to the zero with respect to the quadratic-like map (renormalization) Q. In its turn, the latter curve (in the Q-plane) induces a non-tangent access l_1 to the zero in the plane of the quadratic polynomial $f_{c(Q)}$. So in the uniformization plane of the Mandelbrot set we obtain two curves: one is the external ray $\{w : w = r\exp(2\pi i t(Q)), r > 1\}$, where $t(Q)$ is an external argument of $c(Q)$, and the other one is a non-tangent curve to the same point of the unit circle, namely, this is the curve $B_{c(Q)}(f_{c(Q)}(l_1))$. Now, by a Lindelof's theorem, the limit sets of the parameter $c \in \partial M$ corresponding to these two curves coincide, i.e. this is the limit set of the ray $R^M_{t(Q)}$.

Appendix. A combinatorial relation between a disconnected Julia set and a real polynomial: the R-path

Fix a polynomial f_c outside M and other polynomial f_{c_*} on the intersection of the real line and M. About the latter polynomial we will assume the following:

(A1) f_{c_*} has no attracting and parabolic periodic orbit.

(A2) There is a real decreasing sequence $\{c_n\}_{n=1}^{\infty}$, which tends to c_* and such that c_n is a root of a hyperbolic component of M (that is, f_{c_n} has a periodic orbit with multiplier 1, *or* -1*)*

Denote $\theta_n^{\pm}$ two external arguments of the point $c_n \in \partial M$ [DH1]. They are conjugate, that is the corresponding points $\exp(2\pi i\theta_n^{\pm})$ of the complex plane are conjugated.

Denote $\theta_*^{\pm}$ the limits of the sequences $\theta_n^{\pm}$, and $w_*^{\pm} = \exp(2\pi i\theta_*^{\pm})$. (If M is locally connected, $\theta_*^{\pm}$ are the external arguments of the point $c_* \in \partial M$.)

Let us say that an angle (slope) $\tau \in (0, \pi)$ *corresponds to a point* $w = \exp(2\pi i t)$ *of the unit circle, or simply corresponds to* $t \in [0, 1)$ *iff the curve that passes via the point* w_c *and crosses every circle* $|w| = r, r > 1$ *at the anlge* τ *lands at the point* w.

Denote τ_* a slope that corresponds to one of the points $w_*^{\pm}$, say, to the point w_*^-. The case w_*^+ will differ only by notations. Let C_0 be an analytic curve in the dynamical plane $z \mapsto f_c(z)$ that passes via zero, and crosses every equipotential $\{z : u_c(z) = a\}$, $a < u_c(0)$, at the angle τ_*. (Such a curve is unique as $w_*^{\pm}$ are not periodic points under the map $w \mapsto w^2$.) It lands at two points k_1, k_2 of J_c (it has a finite length because of hyperbolicity of f_c [DH2]). Let J_c^* be the Julia set J_c united with the curve C_0 and all preimages of C_0 under f_c, of all orders [LS]. Then J_c^* is a connected and locally connected compact in the plane. A curve R crossing every equipotential at an angle $\tau \in (0, \pi)$ and joining infinity and the Julia set is said to be a τ-external ray of f_c. It lands at a unique point of J_c and its τ-external argument $t \in \mathbb{T}$ is well defined [DH2], [GM], [LS], [L1]. Namely, the point $\exp(2\pi i t)$ is the end point of the curve, which passes via a point $B_c(z)$, $z \in R$ close to ∞, and crosses every circle $|w| = r$ at the angle τ. In particular, τ_*-ray of zero argument lands at a fixed point of f_c which is called β.

Definition A1. *The R-path, or* $[\beta', \beta]$, *is the unique curve in* J_c^* *which joins the fixed point* β *and its preimage* β' *different from* β. *The positive direction on the R-path is defined from* β' *to* β.

Denote:

$$I_c = I_{c,c_*} = [\beta', \beta] \bigcap J_c$$

is the subset of the Julia set of f_c *on the R-path.*

$$I_{c_*} = \mathbb{R} \bigcap J_{c_F}$$

is the subset of the Julia set of the polynomial f_{c_*} *on the real line, that is simply the closed interval between the fixed point of* f_{c_*} *with positive multiplier,* β_{c_*}, *and its preimage* β'_{c_*} *different from* β_{c_*}.

Remark A1. A point x lies in the R-path if and only if it has at least two τ_*-arguments starting with different digits (in their 2-expansions). In particular, the curve C_0 belongs to I_c since its points have τ_*-arguments $\theta_*/2$ and $\theta_*/2 + 1/2$.

The main result of this section is

Theorem A1.

1. I_c *is invariant under* f_c.

2. There exists a non-decreasing surjective map $H : I_c \to I_{c_*}$ *of the Cantor set* I_c *of the R-path onto the interval* I_{c_*} *such that* $H \circ f_c = f_{c_*} \circ H$, *and* $H(x) = H(y)$, *for some* $x, y \in I_c$ *if and only if* $f_c^r([x, y]) = C_0$, *for some integer* $r \geq 0$, *where* $[x, y]$ *denotes the arc in* $[\beta', \beta]$ *between* x *and* y.

A key ingredient in the proof gives the following

Lemma A1. *Let* $R(c)$ *be the* τ_**-external ray, which contains the critical value* c. *Then* $R(c)$ *lands at a point of the R-path.*

Proof of Lemma A1. It is enough to show that the landing point v of the ray $R(c)$ admits at the same time τ_*-argument θ_*^+ (because $\theta_*^{\pm}$ have different first digits.)

If we will prove, that for every $n = 0, 1, 2, \dots$ the rays $R_n^{\pm}$ of the arguments $\theta_n^{\pm}$ land at the same point x_n, then we will prove the statement (because J_c^* is locally connected and $\theta_n^{\pm} \to \theta_F^{\pm}$). This fact is a particular case of the following known Lemma A2, which can be derived from the Douady-Hubbard-Lavaurs combinatorial theory of the set M (see [Sc]). For the sake of completness we sketch here a proof (cf.[L1]) using a notion of the rotation number, and Douady-Hubbard-Sullivan theorem [D] (on univalence of the multiplier in a hyperbolic component of M).

Lemma A2. *Let t_0 and t_0' be two external arguments of a hyperbolic component Δ of the Mandelbrot set of period d, that is t_0,t_0' are the external arguments of the root c_Δ of Δ. Let us fix c in the complement of M, let $\tau \in (0, \pi)$ be chosen so that τ-ray through w_c lands at a point in the interval (t_0, t_0'). Then, in the dynamical plane of the polynomial f_c, τ-rays of the arguments t_0 and t_0' land at one point, which is a periodic point of f_c of period d.*

Remark A2. To prove Lemma A1, we apply Lemma A2 to the sequence of the hyperbolic components of M with the pairs of external arguments $\theta_n^{\pm}$.

Proof of Lemma A2. (cf. [L1]) Fix a slope τ_0 corresponding to the point $\exp(2\pi i t_0)$. Consider the ray R *of the slope* τ_0 in the dynamical plane f_c. It doubles at a point z_0 such that $f_c^d(z_0) = c$ and then lands at two periodic points z_1 and z_2 of the period d. The points z_1, z_2 belong to different components K_1, K_2 resp. of the set $\{z : u(z) < u(c)/2^d\}$ so that $\overline{K_1} \bigcap \overline{K_2} = \{z_0\}$. The point z_0 admits other τ_0-argument, t_1, and one of the periodic points, say z_2, has other τ_0-argument t_2 so that $2^d(t_2 - t_1) = t_2 - t_0$. We want to show that (a) $t_2 = t_0'$, and (b) *if τ changes between τ_0 and τ_0' then τ-rays of the arguments τ_0, τ_0' always land at the point z_2.* Let the slope τ_2 correspond to t_2. Look at the rotation number $\nu(\tau)$ of z_1 when τ changes from τ_0 to τ_2. Then ν changes monotonically from zero to one. If now (a) is not the case, then there exists another root on the boundary of the hyperbolic component Δ, which is different from c_Δ. This is a contradiction with the Douady-Hubbard-Sullivan theorem. So $t_2 = t_0'$. To prove (b) observe that, for $\tau \in (\tau_0, \tau_0')$, the iterations of the τ-ray $R(c)$ passing via c land inside the component K_1 and its further $d - 1$ iterations. So they don't hit the corresponding iterations of K_2, and, hence, none iteration of $R(c)$ can coincide with the external rays of z_2. Therefore, the τ-arguments of z_2 cannot change (see [GM]). On the other hand, if τ is close enough to and bigger than τ_0, the arguments of z_2 are τ_0, τ_0'. This proves (b).

Remark A3. After Lemma A1, we have the following analogy between the map f_c restricted on the R-path $[\beta', \beta]$ and any real quadratic polynomial f_b, with $-2 \le b \le 0.25$, restricted on the invariant interval I_b of the real line. First of all, a difference is that the R-path is not invariant under the f_c since 0 belongs to it, while its image, c, does not. On the other hand, (and this is the similarity), if k_1, k_2 the landing point of the curve $C_0 \subset [\beta', \beta]$ passing via zero, then, by Lemma A1, the point $v = f_c(k_1) = f_c(k_2)$ lies on $[\beta', \beta]$. Moreover, the images under f_c of the following arcs on the R-path: between β and k_1, and between β' and k_2, coincide with the arc in the R-path between β and v. So the points k_1, k_2 play the role of the "different sides" of the critical point zero of f_b, and the point v plays the role of the critical value b of f_b.

Proof of Theorem A1. Let us continue the preceding remark. By Lemma A1, the point v has arguments $\theta_*^{\pm}$. Then all iterations of v lie on $[\beta', \beta]$ since they have external arguments starting with the different digits. Thus, the kneading sequence $\Theta_c(v)$ of v is well defined and equal to the limit of the kneading sequences $\Theta_c(x_n)$ of the periodic points $x_n = x_n(c)$. Let us show that $\Theta_c(v)$ coincides with the kneading sequence $\Theta_{c_*}(c_*)$ of the critical value of the polynomial f_{c_*} (the kneading invariant of f_{c_*}). Indeed, let us consider again τ_*-rays of the arguments $\theta_n^{\pm}$, $n = 1, 2, \ldots$. For each parameter b in the wake of the corresponding hyperbolic component of M (particularly, for c and c_*) they land at a periodic point $x_n(b)$ of f_b, which is a holomorhpic function of b in the wake (see Lemma A2). Now, take the sequence of real parameters c_n from the definition of c_*. Then the kneading sequence $\Theta_c(x_n)$ equals the kneading sequence $\Theta_{c_n}(c_n)$ of the real map f_{c_n}. But $\Theta_{c_n}(c_n)$ tends to the kneading sequence $\Theta_{c_*}(c_*)$ of the limit map f_{c_*} [MT]. This directly leads to the conclusions of the theorem. (Remind, by Guckenheimer's theorem, the polynomial f_{c_*} does not have wandering intervals, hence, it is determined by its kneading invariant).

Reminder: $\sigma(t) = 2.t(mod 1)$, $t \in [0, 1)$. Then $\sigma^n(t) = 2^n.t(mod 1)$.

Corollary A1. *The set of τ_*-arguments of the points in the R-path consists of those t for which, for all $i > 0$, $\sigma^i(t)$ lies either in $[0, \theta_*^-]$, or in $[\theta_*^+, 1]$.*

Corollary A2. *Fix x in the Julia set J_c and consider the set $\Lambda(x)$ of its τ_*-external arguments. There are only three possibilities:*

1. None iteration of x hits the set I_c, and $\Lambda(x)$ consists of one element $t(x)$.

2. For some $l \geq 0$, the l-th iteration of x hits a point y of I_c, which is not a pre-image of v, and $\Lambda(x)$ consists of two elements $t^{\pm}(x)$ so that

$$t^{\pm}(x) = 0.\epsilon_1, \ldots, \epsilon_l, (t^{\pm}(y)),$$

where $\epsilon_1, \ldots, \epsilon_l$ is a group of digits $0, 1$, and $(t^{\pm}(y))$ denotes the sequence $0, 1$ in the 2-expansion of the external arguments $t^{\pm}(y)$ of the point y.

3. For some $l \geq 0$, the l-th iteration of x hits either k_1, or k_2, and $\Lambda(x)$ consists of three elements $t^{\pm}(x), t^{ad}(x)$ so that

$$t^-(x) = 0.\epsilon_1, \ldots, \epsilon_l, 0, (\theta_*^-),$$

$$t^+(x) = 0.\epsilon_1, \ldots, \epsilon_l, 1, (\theta_*^-),$$

$$t^{ad}(x) = 0.\epsilon_1, \ldots, \epsilon_l, \epsilon, (\theta_*^+),$$

where: $\epsilon_1, \ldots, \epsilon_l$ is a group of digits $0, 1$, ϵ is either 0, or 1 depending on whether the point $f_c^l(x)$ is k_1, or k_2, and (θ) denotes the sequence of $0, 1$ in the 2-expansion of θ.

Proof of Corollary A2.

1. The edges of the R-path have the only external arguments $0, 1/2$.

2. and 3. It is enough to show every point y of I_c has exactly two external arguments whenever this point is not a preimage of v. Assume $t_i, i = 1, 2, 3$ are different arguments of y. Let y_1, y_2 start with the same digit, but differ by a digit

of number $i > 1$. Applying Theorem A1,p.1, we find a preimage of R-path under f_c^i between external rays of arguments t_1 and t_2. It means that f_c^i is not one-to-one in a neighborhood of y, that is y is an i-preimage of v.

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