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JORDAN GROUPS AND GEOMETRIC PROPERTIES OF MANIFOLDS

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1. Introduction

The aim of this note is to draw attention to the so called *Jordan property* of groups that was recently actively studied. The property was explicitly formulated by Jean-Pierre Serre and Vladimir Popov in this century, and the name goes back to a classical result of Camille Jordan (1878) about finite subgroups of matrix complex groups. Though defined for arbitrary groups, in special situations it bears a strong geometric meaning. A more detailed review on this topic one may find in [BZ22a].

We will use the standard notation $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ for the set of positive integers, the ring of integers, the fields of rational and complex numbers, respectively. If q is a prime (or a prime power) then we write $\mathbb{F}_q$ for the (finite) q-element field. In this note we consider the following groups.

- $\mathrm{Bir}(X)$ of all birational self-maps of an irreducible complex algebraic variety X;
- $\mathrm{Bim}(X)$ of all bimeromorphic self-maps of a connected complex manifold X;
- $\mathrm{Diff}(X)$ of all diffeormorphisms of a smooth real manifold X;
- $\mathrm{Aut}_{an}(X)$ and $\mathrm{Aut}_{al}(X)$ of all automorphisms of complex or algebraic variety, respectively.

Remark 1.1. If X is a projective variety over the field of complex numbers then $\mathrm{Bim}(X) = \mathrm{Bir}(X)$. In addition, $\mathrm{Aut}_{an}(X) = \mathrm{Aut}_{al}(X)$; we will denote both groups as $\mathrm{Aut}(X)$ when no confusion can arise.

Sometimes these groups are finite; for example, $\mathrm{Bim}(X)$ is finite if X is a compact connected complex manifold of general type (i.e., it

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has maximal possible Kodaira dimension $\varkappa(X) = \dim X$ [KO]. However, in general, the groups $\mathrm{Bim}(X)$ may be infinite and non-algebraic. One of the most interesting and important examples of such groups in birational geometry is the *Cremona group* $\mathrm{Cr}_n = \mathrm{Bir}(\mathbb{P}^n)$ where $\mathbb{P}^n$ is the $n-$dimensional complex projective space. If $n \geq 2$ then Cr_n is a huge non-abelian non-algebraic group. To understand the structure of such groups one is tempted to consider their less complicated subgroups: finite, abelian or their combinations. This is where the Jordan properties come in.

Definition 1.2. A group G is called *Jordan* if there is a positive integer J such that every finite subgroup B of G contains an abelian subgroup A that is normal in B and such that the index $[B : A] \leq J$. The smallest such J is called the *Jordan constant* of G, denoted by J_G ([Se09, Question 6.1], [Po11, Definition 2.1]).

The study of Jordan properties was inspired by the following fundamental results of Jordan and Serre (see [Jor], [Se16, Theorem 9.9], and [Se09, Theorem 5.3] respectively).

Theorem 1.3 (Theorem of Jordan)**.** *Let $\mathbb{C}$ be the field of complex numbers. Then $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ is Jordan.*

Example 1.4. It follows from Theorem 1.3 that every linear algebraic group over any field of characteristic zero is Jordan. Moreover, every connected real (or complex) Lie group is Jordan (Popov, [Po18]).

Example 1.5. It is well known that GL_n contains a subgroup of of order $(n+1)!$ that is isomorphic to the full symmetric group $\mathbf{S}_{n+1}$ of permutations on $(n+1)$ letters. Indeed, permutations of the coordinates in $(n+1)$-dimensional coordinate vector space $\mathbb{C}^{n+1}$ leaves invariant the hyperplane $H = \{\sum_1^{n+1} x_i = 0\} \cong \mathbb{C}^n$. If $n \geq 4$ then $n+1 \geq 5$ and $\mathbf{S}_{n+1}$ is a nonabelian group that does not contain a proper abelian normal subgroup. (Actually, its only proper normal subgroup is the alternating group $\mathbf{A}_{n+1}$ that is simple nonabelian.) This implies that if $n \geq 4$ then

$$J_{\mathrm{GL}_n} \geq J_{\mathbf{S}_{n+1}} = (n+1)!. \tag{1}$$

The equality holds if $n \geq 71$ or $n = 63, 65, 67, 69$ [Col].

Example 1.6. Finite subgroups of the group $\mathrm{GL}_2 = \mathrm{GL}_2(\mathbb{C})$ are classified [Klein]. In particular, it contains a subgroup of order 120 that is isomorphic to $\mathrm{SL}(2, \mathbb{F}_5)$. Its largest abelian normal subgroup C consists of two scalars $\{1, -1\}$ (see below) and the corresponding quotient $\mathrm{SL}(2, \mathbb{F}_5)/C$ is isomorphic to the simple nonabelian alternating group $\mathbf{A}_5$. It follows that $J_{\mathrm{GL}_2(\mathbb{C}))} \geq 60$. Actually, $J_{\mathrm{GL}_2(\mathbb{C}))} = 60$.

Remark 1.7. The precise values of Jordan constants for groups GL_n are known for all n, see the paper of M.J. Collins [Col, Theorems **A** and **B**]). In particular,

(1) $J_{\mathrm{GL}_n} = (n+1)!$ if $n \geq 71$ or $n = 63, 65, 67, 69$.
(2) $J_{\mathrm{GL}_n} = 60^r \cdot r!$ if $n = 2r$ or $2r+1$ and either $20 \leq n \leq 62$ or $n = 64, 66, 68, 70$.

Example 1.8. [Example of a non-Jordan group] Let p be a prime and $\overline{\mathbb{F}}_p$ an algebraic closure of the field $\mathbb{F}_p$. Then $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$ is **not** Jordan.

Indeed, if m is a positive integer and $q = p^m \geq 4$, then $\mathrm{SL}(2, \mathbb{F}_q) \subset \mathrm{SL}(2, \overline{\mathbb{F}}_p)$.

Recall that $\mathrm{SL}(2, \mathbb{F}_q)$ is a finite noncommutative group of order $(q^2 - 1)q$ such that its every proper normal subgroup $C \subsetneq \mathrm{SL}(2, \mathbb{F}_q)$ consists of one or two scalars.

Thus the values of indices

$$[\mathrm{SL}(2, \mathbb{F}_q) : C] = (q^2 - 1)q/2 \text{ or } (q^2 - 1)q$$

are unbounded when m tends to infinity. Hence $\mathrm{SL}(2, \overline{\mathbb{F}_p})$ is *not* Jordan.

Theorem 1.9 (Theorem of Serre)**.** *The Cremona group* $\mathrm{Cr}_2 = \mathrm{Bir}(\mathbb{P}^2)$ *is Jordan,* $J_{\mathrm{Cr}_2} \leq 2^{10}3^4 5^2 7$.

The exact value $J_{\mathrm{Cr}_2} = 7200$ was found by E. Yasinsky [Ya].

In his paper [Po11] V.L. Popov asked whether for any algebraic variety X the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ are Jordan. This question originated an intensive and fruitful activity, see Section 2 below.

The following "Jordan properties" of groups are also very useful.

Definition 1.10.
(1) A group G is called *bounded* if the orders of its finite subgroups are bounded by a universal constant that depends only on G ([Po11, Definition 2.9]).
(2) A Jordan group G is called *strongly Jordan* [PS14, BZ17] if there is a positive integer m such that every finite subgroup of G is generated by at most m elements.
(3) A group G is *very Jordan* ([BZ20]) if there exist a commutative normal subgroup G_0 of G and a bounded group F that sit in a short exact sequence

$$1 \to G_0 \to G \to F \to 1. \tag{2}$$

Example 1.11 (Examples of bounded groups)**.** The matrix group $\mathrm{GL}(n, \mathbb{Q})$ and its subgroup $\mathrm{GL}(n, \mathbb{Z})$ are bounded.

This is a celebrated result of Hermann Minkowski (1887), see [Se16, Section 9.1]. Actually, Minkowski gave an explicit upper bound $M(n)$ for the orders of finite subgroups of $\mathrm{GL}(n, \mathbb{Q})$ (ibid).

Example 1.12. The multiplicative group $\mathbb{C}^*$ of the field $\mathbb{C}$ is commutative, (hence, Jordan) but not bounded. The same is valid for the group of translations of any complex torus of positive dimension.

Remark 1.13. 1) Every finite group is bounded, Jordan, and very Jordan.

2) Every commutative group is Jordan and very Jordan.

3) Every finitely generated commutative group is bounded. Indeed, such a group is isomorphic to a finite direct sum with every summand isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ where n is positive integer.

4) A subgroup of a Jordan group is Jordan. A subgroup of a very Jordan group is very Jordan.

5) "Bounded" implies "very Jordan", "very Jordan" implies "Jordan".

6) "Bounded" implies "strongly Jordan." On the other hand, "very Jordan" does not imply "strongly Jordan." For example, a direct sum of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$ is commutative but has finite subgroups with any given minimal number of generators.

2. Jordan properties of groups $\mathrm{Aut}(X), \mathrm{Bir}(X), \mathrm{Bim}(X)$, and $\mathrm{Diff}(X)$.

In this section we sketch certain facts, methods and tools related to the study of the Jordan properties of groups arising from complex geometry.

Example 2.1. Let X be a smooth irreducible projective curve (Riemann surface) of genus g. Then $\mathrm{Aut}(X) = \mathrm{Bir}(X) = \mathrm{Bim}(X)$. We have:

- If $g > 1$ then $\mathrm{Aut}(X)$ is finite, hence bounded and Jordan.
- If $g = 0$ then $\mathrm{Aut}(X) = \mathrm{PGL}(2, \mathbb{C})$ is Jordan (by the Jordan's Theorem), strongly Jordan, but not bounded and not very Jordan.
- If $g = 1$, i.e., X is an elliptic curve, then it is a commutative algebraic group that acts on itself by translations. Moreover, $X \subset \mathrm{Aut}(X)$ is a normal commutative subgroup of finite index, namely $[\mathrm{Aut}(X) : X] \le 6$. It follows that $\mathrm{Aut}(X)$ is very Jordan, strongly Jordan, but not bounded.

Example 2.2. J. Winkelmann [W] and V. Popov [Po15] proved the existence of a connected non-compact Riemann surface M such that $\mathrm{Aut}(M)$ contains an isomorphic copy of every finitely presented (in particular, every finite) group G. In particular, $\mathrm{Diff}(M)$ is not Jordan.

Example 2.3. The automorphism group $\mathrm{Aut}(A)$ of an abelian variety A is strongly Jordan and very Jordan. Moreover, if d is a positive integer then there are universal constants $J(d)$ and $R(d)$ that depend only on d and such that if A is a d-dimensional abelian variety then every finite subgroup of $\mathrm{Aut}(A)$ may be generated by $r \le R(d)$ elements and $J_A \le J(d)$.

Proof. Let L_A be a lattice in $\mathbb{C}^d$ such that $A = \mathbb{C}^d/L_A$. Thus A is isomorphic as a group to $(\mathbb{R}/\mathbb{Z})^{2d}$, hence every finite subgroup has at most $2d$ generators.

Let $T_A \subset \mathrm{Aut}(A)$ be the (sub)group of translations

$$t_a : A \to A, \ \to x + a, \ (a \in A).$$

Then T_A is isomorphic to A as a group. There is an exact sequence:

$$0 \to T_A \to \mathrm{Aut}(A) \to \mathrm{Aut}(L_A) \cong \mathrm{GL}(2d, \mathbb{Z}).$$

Since T_A is abelian and the subgroup $\mathrm{Aut}(L) \hookrightarrow \mathrm{GL}(2d, \mathbb{Z})$ is bounded, $\mathrm{Aut}(A)$ is Jordan. □

As of today there are no examples of complex algebraic varieties (compact or non-compact) with non Jordan $\mathrm{Aut}(X)$. If X is a compact complex connected manifold, then $\mathrm{Aut}(X)$ carries the natural structure of a (not necessarily connected) complex Lie group [BM]. The connected identity component $\mathrm{Aut}_0(X)$ of $\mathrm{Aut}(X)$ is Jordan for every compact complex space X [Po18, Theorems 5 and 7].

The group $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ of connected components of $\mathrm{Aut}(X)$ is bounded if X is Kähler [BZ20, Proposition 1.4].

It is known that the group $\mathrm{Aut}(X)$ is Jordan if

- X is projective ([MZ]);
- X is a compact complex Kähler manifold ([Kim]):
- X is a compact complex space in Fujiki's Class $\mathscr{C}$ [MPZ] (see also [PS19] for Moishezon threefolds).

Moreover, $\mathrm{Aut}(X)$ is very Jordan if the Kodaira dimension $\varkappa(X)$ of X is non-negative, or if X is a $\mathbb{P}^1-$bundle over a certain non-uniruled complex manifold [BZ20, BZ22, BZ22a].

Remark 2.4. Recall that the Kodaira dimension $\varkappa(X)$ is a numerical parameter of a variety X that can take on values $-\infty, 0, 2, \ldots, \dim X$. As it was already mentioned, varieties of general type are rigid. At the other side of the spectrum ($\varkappa(X) = -\infty$) are, in particular, uniruled varieties. A compact complex variety X is *uniruled* if there exist a compact complex variety Y, its proper complex closed subspace $Z \subset Y$, and a meromorphic dominant map $f : Y \times \mathbb{P}^1 \to X$ such that $\dim(f(y \times \mathbb{P}^1)) = 1$ for any $y \in Y \setminus Z$. If $\dim X \leq 3$ then $\varkappa(X) = -\infty$ implies that X is uniruled. Any projective space is uniruled.

The structure of the groups $\mathrm{Bir}(X)$ and $\mathrm{Bim}(X)$ of birational and bimeromorphic selfmaps, respectively, are more complicated. It appears that uniruled varieties play a special role with respect to Jordan properties.

There are examples of

- a projective variety X_{pr} with non-Jordan group $\mathrm{Bir}(X_{pr})$, namely

$$X_{pr} := E \times \mathbb{P}^1$$

where E is any elliptic curve [Zar14];

- a non-algebraic connected compact complex manifold X_c with non-Jordan group $\mathrm{Bim}(X_c)$:
$$X_c := T \times \mathbb{P}^1,$$
where T is any non-algebraic complex torus of positive algebraic dimension [Zar19];
- a smooth compact real manifold M with non-Jordan group $\mathrm{Diff}(M)$ with M being the direct product of 2-dimensional real torus by 2-dimensional sphere (B. Csikós, L. Pyber, E.Szabó, [CPS]).
  Note that $\mathbb{P}^1$ is a 2-dimensional sphere as a real manifold.

All these examples are essentially the same. Let us note their main features: all those objects are

- uniruled (covered by rational curves);
- direct products with a torus T;
- a torus T carries no rational curves and the group $\mathrm{Aut}(T)$ is an algebraic, commutative, not bounded group.

It seems that the Jordan property (or rather its absence) of the groups $\mathrm{Bir}(X)$, or $\mathrm{Bim}(X)$ for a complex manifold (or projective varietiy) X correlate with such geometric features as being uniruled over a non-uniruled positive dimensional base or being a direct product.

Let us start with surfaces.

Theorem 2.5 ([Po11])**.** *If X is an irreducible projective surface then $\mathrm{Bir}(X)$ is Jordan unless X is birational to a product $E \times \mathbb{P}^1$ of an elliptic curve E and $\mathbb{P}^1$.*

Let us sketch the ideas involved in the proof. They are basic for this theory and, in a more sophisticated form, are widely used.

We will restrict to the smooth situation. Recall that a smooth surface X has a minimal model X_m (that is smooth and contains no (-1) curves , see, e.g., [Sha]).

Case 1. $\varkappa(X) \geq 0$. Then $\mathrm{Bir}(X) = \mathrm{Bir}(X_m) = \mathrm{Aut}(X_m)$.

Every automorphism $f \in \mathrm{Aut}(X_m)$ induces the automorphism $\psi(f)$ of the Néron-Severi group $\mathrm{NS}(X_m)$ (the group of connected components of $\mathrm{Pic}(X)$.) Let $G_i := \ker(\psi)$. This is a complex Lie group that may be included into the exact sequence:

$$0 \longrightarrow G_i \xrightarrow{i} \mathrm{Aut}(X_m) \xrightarrow{\psi} \mathrm{Aut}(\mathrm{NS}(X)). \tag{3}$$

It is known that

- $G_i := \ker(\psi)$ has finitely many connected components;
- the identity component of G_i is a connected algebraic group;
- every connected algebraic group is Jordan;
- there is a finite subgroup $F \subset \mathrm{Aut}(\mathrm{NS}(X))$ (consisting of all torsion elements of $\mathrm{NS}(X)$) such that the quotient $\mathrm{Aut}(NS(X))/F \subset \mathrm{GL}(\rho, \mathbb{Q})$, where ρ is the Picard number of X. As a subgroup

of a bounded group, $\mathrm{Aut}(\mathrm{NS}(X))/F)$, it is bounded. Now the finiteness of F implies that $\mathrm{Aut}(\mathrm{NS}(X))$ is bounded as well.

Now Equation (3) implies that $\mathrm{Bir}(X) = \mathrm{Aut}(X_m)$ is *Jordan*.

Case 2. $\varkappa(X) = -\infty$

As was already mentioned, the case of $\mathrm{Cr}_2(\mathbb{C}) = \mathrm{Bir}(\mathbb{P}^2)$ is due to Serre (see Theorem 1.9 above).

If the surface is birational to a direct product $X_m; = B \times \mathbb{P}^1$ of a curve B of genus $g \geq 1$ and the projective line then every birational automorphism $f \in \mathrm{Bir}(X_m) \cong \mathrm{Bir}(X)$ is fiberwise. It means that it can be included into the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi\downarrow & & \downarrow\pi \\ B & \xrightarrow{\tau(f)} & B \end{array} \quad .$$

Here $\pi : X \to B$ is the natural projection and $\tau(f) \in \mathrm{Aut}(B)$.

The subgroup $G_0 = \{f \in \mathrm{Aut}(X_m) | \tau(f) = \mathrm{id}\} \subset \mathrm{PSL}(2, K)$, where $K = \mathbb{C}(B)$ is the field of rational functions on B, is Jordan.

Once more we have an exact sequence

$$0 \longrightarrow G_0 \xrightarrow{i} \mathrm{Aut}(X_m) \xrightarrow{\tau} G_B \tag{4}$$

where $G_B = \psi(\mathrm{Aut}(X_m)) \subset \mathrm{Aut}(B)$ is finite if genus $g > 1$.

Thus in case $g(B) > 1$ Equation (4) implies that $\mathrm{Bir}(X_m) \cong \mathrm{Bir}(X)$ is Jordan.

The special case: X is birational to $E \times \mathbb{P}^1$ where E is an elliptic curve, is left.

Theorem 2.6 ([Zar14])**.** *If X is birational to $E \times \mathbb{P}^1$ then* $\mathrm{Bir}(X)$ *is not Jordan.*

The proof of this Theorem is done in two steps. First, for every $N \in \mathbb{N}$ a certain group $\mathfrak{G}_N$ is constructed and its Jordan number is shown to be N. Then for every $N \in \mathbb{N}$ build is a surface S_N such that

- it is birational to $E \times \mathbb{P}^1$;
- $\mathrm{Aut}(S_N)$ contains a group $G_N \cong \mathfrak{G}_N$.

It follows that $\mathrm{Bir}(E \times \mathbb{P}^1)$ contains a subgroup G_N with $J_{G_N} = N$ for every $N \in \mathbb{N}$ thus is not Jordan. Let us give some details.

Step 1: Group $\mathfrak{G}_N$: analogues of the Heisenberg groups that were used by D. Mumford [Mum66]. Let

- $\mathbf{K}$ be a finite commutative group of order $N > 1$;
- $\mu_N \subset \mathbb{C}^*$ be the multiplicative group of Nth roots of unity;
- $\hat{\mathbf{K}} = \mathrm{Hom}(\mathbf{K}, \mu_N)$ - the dual of $\mathbf{K}$.

The Mumford's Theta group $\mathfrak{G}_{\mathbf{K}}$ for $\mathbf{K}$ is a group of matrices of the type

$$\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha \in \hat{\mathbf{K}},\ \gamma \in \mathbb{C}^*$, and $\beta \in \mathbf{K}$. The product $\alpha(\beta) \in \mathbb{C}^*$ of $\alpha \in \hat{\mathbf{K}}$ and $\beta \in \mathbf{K}$ is used in order to define a certain natural non-degenerate alternating bilinear form $e_{\mathbf{K}}$ on $\mathbf{H}_{\mathbf{K}} = \mathbf{K} \times \hat{\mathbf{K}}$ with values in $\mathbb{C}^*$ [Zar14, p. 302]. This group may be included into a short exact sequence

$$1 \to \mathbb{C}^* \to \mathfrak{G}_{\mathbf{K}} \to \mathbf{H}_{\mathbf{K}} \to 1$$

where the image of $\mathbb{C}^*$ is the center of $\mathfrak{G}_{\mathbf{K}}$. These groups are Jordan and

$$J_{\mathfrak{G}_{\mathbf{K}}} = \sqrt{\#(\mathbf{H}_{\mathbf{K}})} = N = \#(\mathbf{K}).$$

In particular, let $\mathfrak{G}_N = \mathfrak{G}_{\mathbb{Z}/N\mathbb{Z}}$, i.e., $K = \mathbb{Z}/N\mathbb{Z}$. Then $J_{\mathfrak{G}_N} = N$.

Step 2: Constructing S_N.

Fix $P \in E$. Denote by $[P]$ and L_P the corresponding divisor and holomorphic line bundle on E, respectively. Choose an integer $N > 1$. Let L_N be the total body of the line bundle L_{NP}. Let $S_N = \overline{L_N}$ be its projective closure, i.e., $L_N \cup S$, where S is the "infinite" section of L_{NP}. Actually, $\overline{L_N}$ is the $\mathbb{P}^1$-bundle over E that is the projectivization of the rank two vector bundle $L_N \oplus \mathbf{1}_E$, where $\mathbf{1}_E = E \times \mathbb{C}$ is the trivial line bundle over E. Thus, S_N is a ruled surface birational to $E \times \mathbb{P}^1$.

Let $G(N)$ be a subgroup of all those $f \in \operatorname{Aut}(S_N)$ that may be included into the following commutative digram:

$$\begin{array}{ccc} \overline{L_N} & \xrightarrow{f} & \overline{L_N} \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ E & \xrightarrow{T_Q} & E \end{array}$$

Here $p : S_N \to E$ is a natural projection, $E(N)$ stands for the subgroup of points in E of order dividing N, point $Q \in E(N)$ of order N, and $T_Q : E \to E$ is a translation $e \to e + Q$. Moreover, f induces $\mathbb{C}-$linear isomorphisms between fibers of p over e and $e + Q$.

On $E \times \mathbb{P}^1$ elements of the group $G(N)$ induce birational maps and form a subgroup $G_N \subset \operatorname{Bir}(E \times \mathbb{P}^1)$ that may be described as follows.

$G(N) = \{(Q, f), Q \in E(N), f \in \mathbb{C}(E)^*$ such that $(f) = N[P + Q] - N[P]\}$ is acting as

$$(y, t) \in (E \times \mathbb{P}^1) \longrightarrow (Q, f)(y, t) = (Q + y, f(y)t).$$

Here f is the divisor of a rational function f.

By a theorem of D. Mumford, the group G_N is isomorphic to $\mathfrak{G}_N$; hence $J_{G_N} = N$. Thus, $J_{\operatorname{Bir}(E\times\mathbb{P}^1)} \geq J_{G_N} = N$ for all N, i.e., $\operatorname{Bir}(E \times \mathbb{P}^1)$ is *not* Jordan.

Based on the proof of the non-Jordaness of $\mathrm{Bir}(E \times \mathbb{P}^1)$ B. Csikós, L. Pyber, E.Szabó [CPS] constructed a counterexample to

Conjecture of E. Ghys (1997) If M is a connected compact smooth real manifold then $\mathrm{Diff}(M)$ is Jordan.

From the *real* point of view, $\mathbb{P}^1$ is the two-dimensional sphere $\mathbb{S}^2$, E is the two-dimensonal real torus $\mathbb{T}^2$, and S_N is an oriented $\mathbb{S}^2$-bundle over $\mathbb{T}^2$.

As a smooth manifold S_N is diffeomorphic to the $\mathbb{T}^2 \times \mathbb{S}^2$ if and only if N is even. Therefore for each even N we have $G_N \hookrightarrow \mathrm{Diff}(\mathbb{T}^2 \times \mathbb{S}^2)$.

Since the set J_{G_N} for positive even integers N is unbounded, $\mathrm{Diff}(\mathbb{T}^2 \times \mathbb{S}^2)$ is not Jordan.

Remark 2.7. If X is a complex compact surface with non-negative Kodaira dimension then $\mathrm{Bir}(X)$ is even bounded unless it is one of the following [PS20, Theorem1.1] :

- a complex torus (in particular an abelian surface);
- a bielliptic surface;
- S_{K1} - a surface of Kodaira dimension 1;
- S_K - a Kodaira surface (it is not a Kähler surface). [PS20, Theorem1.1].

Moreover, [PS18], if X is a projective threefold then $\mathrm{Bir}(X)$ is not Jordan if and only if X is birational to a direct product $E \times \mathbb{P}^2$ or $S \times \mathbb{P}^1$, where a surface S is one of surfaces listed above in this Remark.

For complex projective varieties Yu. Prokhorov and C. Shramov, and C. Birkar proved the following

Theorem 2.8. *Let X be a projective irreducible variety of dimension n. Then the following hold.*

(i) *The group* $\mathrm{Bir}(X)$ *is bounded provided that X is non-uniruled and has irregularity* $q(X) = 0$ [PS14, Theorem 1.8].

(ii) *The group* $\mathrm{Bir}(X)$ *is Jordan provided that X is non-uniruled* [PS14, Theorem 1.8].

(iii) *The group* $\mathrm{Bir}(X)$ *is Jordan provided that X has irregularity* $q(X) = 0$ *(*[PS14, Theorem 1.8]*,* [Bi]*).*

Here $q(X) = \dim_{\mathbb{C}} H^1(X, \mathscr{O}_X)$ is the irregularity of X. In particular, the Cremona group Cr_n of any rank n is Jordan ([PS16, Bi]).

The group $\mathrm{Diff}(M)$ of all diffeomorphisms of a smooth manifold M also appeared to be Jordan for certain classes of manifolds.

B. Zimmerman [Zim] proved that if M is compact and $\dim(M) \leq 3$ then $\mathrm{Diff}(M)$ is Jordan. The Jordan properties of $\mathrm{Diff}(M)$ were studied by I. Mundet i Riera. In particular, he proved [MR18] that $\mathrm{Diff}(M)$ is Jordan if M is one of the following:

(1) open acyclic manifolds,

(2) compact manifolds (possibly with boundary) with nonzero Euler characteristic,

(3) homology spheres.

So, in high dimensions the situation is very similar: the group $\mathrm{Bim}(X)$ or $\mathrm{Bir}(X)$ is mostly Jordan, and the worst case from the Jordan properties point of view is the following: a uniruled variety X with $q(X) > 0$ (or fibered over a non-uniruled base) that has many sections (such as a direct product). A typical example of such a variety X is a $\mathbb{P}^1$-bundle over a complex torus T of positive dimension.

The need of "many sections" may be demonstarted by the case of projective non-trivial conic bundles.

Definition 2.9. A regular surjective map $f : X \to Y$ of smooth irreducible projective complex varieties is a *conic bundle* over Y if there is a Zariski-open dense subset $U \subset Y$ such that the fiber $f^{-1}(y) \sim \mathbb{P}^1$ for all $y \in U$.

The *generic fiber* of f is an irreducible smooth projective curve $\mathscr{X}_f$ over the field $K := \mathbb{C}(Y)$ such that its field of rational functions $K(\mathscr{X}_f)$ coincides with $\mathbb{C}(X)$. For a conic bundle $\mathscr{X}_f$ has genus 0. A $K-$ point in $\mathscr{X}_f$ corresponds to a rational section of the conic bundle $f : X \to Y$. If such a K point exists, then $\mathscr{X}_f$ is isomorphic over K to $\mathbb{P}^1(K)$ and X is birational to $Y \times \mathbb{P}^1$.

Theorem 2.10. ([BZ17]) *Let X be a conic bundle over a non-uniruled smooth irreducible projective variety Y with $\dim(Y) \geq 2$. If X is not birational to $Y \times \mathbb{P}^1$ then $\mathrm{Bir}(X)$ is Jordan.*

Let us sketch the proof.

If $f : X \to Y$ is a conic bundle and Y is non-uniruled then every $\phi \in \mathrm{Bir}(X)$ is fiberwise. It means that there is a homomorphism $\tilde{\tau}$: $\mathrm{Bir}(X) \to \mathrm{Bir}(Y)$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ f\downarrow & & \downarrow f \\ Y & \xrightarrow{\tilde{\tau}(\phi)} & Y \end{array} .$$

It follows that there is an exact sequence of groups:

$$0 \to \mathrm{Bir}_{\mathbb{C}(Y)}(\mathscr{X}_f) \to \mathrm{Bir}(X) \to \mathrm{Bir}(Y); \tag{5}$$

Since Y is non-uniruled, the group $\mathrm{Bir}(Y)$ is Jordan thanks to Theorem 2.8 . Moreover, it is strongly Jordan (see [BZ17, Cor. 3.8 and its proof]). Let us compute $\mathrm{Bir}_K(\mathscr{X}_f)$. We have

1. $\mathrm{Bir}(\mathscr{X}_f) = \mathrm{Aut}(\mathscr{X}_f)$ since $\dim(\mathscr{X}_f) = 1$.
2. Since X is not birational to $Y \times \mathbb{P}^1$, the genus 0 curve $\mathscr{X}_f$ has **no** K-points and therefore there exists a ternary quadratic form
$$q(T) = a_1T_1^2 + a_2T_2^2 + a_3T_3^2$$
over K such that

— all a_i are nonzero elements of K;
— $q(T) = 0$ if and only if $T = (0, 0, 0)$) (this means that q is *anisotropic*);
— $\mathcal{X}_f$ is biregular over K to the plane projective quadric

$$\mathbf{X}_q := \{(T_1 : T_2 : T_3) \mid q(T) = 0\} \subset \mathbb{P}^2_K.$$

3. K is a field of characteristic zero that contains all roots of unity.

Now we can use the following fact that was proven in [BZ17] (see also [ShVb]).

Theorem 2.11. *(*[BZ17]*) Suppose that K is a field of characteristic zero that contains all roots of unity, $d \geq 3$ an odd integer, V a d-dimensional K-vector space and let $q : V \to K$ be a quadratic form such that $q(v) \neq 0$ for all nonzero $v \in V$. Let us consider the projective quadric $X_q \subset \mathbb{P}(V)$ defined by the equation $q = 0$, which is a smooth projective irreducible $(d-2)$-dimensional variety over K. Let* $\mathrm{Aut}(X_q)$ *be the group of biregular automorphisms of X_q. Let G be a finite subgroup in* $\mathrm{Aut}(X_q)$. *Then G is commutative, all its non-identity elements have order* 2 *and the order of G divides* 2^{d-1}.

Thus if G is a nontrivial finite subgroup of $\mathrm{Aut}(\mathcal{X}_f)$ then either $G \cong \mathbb{Z}/2\mathbb{Z}$ or $G \cong (\mathbb{Z}/2\mathbb{Z})^2$.
Applying Equation (5), we get that $\mathrm{Bir}(X)$ is Jordan.

We summarize now what we know about the Jordan properties when X a $\mathbb{P}^1$-bundle over a complex torus T of positive dimension n. First, let us recall basic facts about complex tori [BL].

For a complex torus T there exists its *algebraic model* T_0 such that:

- T_0 is an abelian variety;
- there is a holomorphic surjective homomorphism $p : T \to T_0$ with connected kernel;
- the field $\mathbb{C}(T)$ of meromorphic functions on T coincides with $p^*(\mathbb{C}(T_0))$, i.e., every meromorphic function on T is the lift of a rational function on T_0;
- algebraic dimension $a(T)$ is defined as $\dim_{\mathbb{C}} T_0$.

Now we are ready to state our summary.

1. We may consider T as a real manifold T_r. Then, as we have seen in the counterexample to the Ghys Conjecture,
if $\dim_{\mathbb{R}}(T_r) \geq 2$ and $X = \mathbb{S}^2 \times T_r$ then $\mathrm{Diff}(X)$ is not Jordan.

2. Since T is a complex torus, it is a connected compact Kähler manifold.

2.1 Suppose that $a(T) = \dim(T) = n$. This means that T is algebraic, i.e., is an abelian variety. If $X = \mathbb{P}^1 \times T$ then $\mathrm{Bir}(X)$ is not Jordan (see Theorem 2.6). If X is not birational to $\mathbb{P}^1 \times T$ then $\mathrm{Bir}(X)$ is Jordan (see Theorem 2.10).

2.2 Suppose that $0 < a(T) < n$. Then T is a non-algebraic torus and $n > 1$. (In dimension 1 all complex tori are algebraic - they are famous elliptic curves.) If $X = \mathbb{P}^1 \times T$ (or has at least three sections) then $\mathrm{Bim}(X)$ is not Jordan [Zar19].

2.3 Suppose that $a(T) = 0$. Then $n \geq 2$ and T is non-algebraic. This is a "very general" case: in a "versal" family [BL] of all complex tori of a given dimension $n \geq 2$ the subset of tori with algebraic dimension zero is dense. (See [BZ20] for explicit examples of such tori in all dimensions $n \geq 2$.) If $a(T) = 0$ then any $\mathbb{P}^1-$bundle X over T that is not biholomorphic to the direct product $\mathbb{P}^1 \times T$ has at most two sections and $\mathrm{Bim}(X) = \mathrm{Aut}(X)$ is Jordan [BZ20].

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