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Affine groups acting properly discontinuously

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Abstract. In 1964 L. Auslander conjectured that every subgroup Γ of an affine group $\mathrm{Aff}(\mathbb{R}^n)$ that acts properly discontinuously on $\mathbb{R}^n$ such that $\mathbb{R}^n/\Gamma$ compact is virtually solvable, i.e. contains a solvable subgroup of finite index. We prove the Auslander conjecture for $n < 7$. The proof is based mainly on dynamic arguments.
We prove that if an affine group Γ acts properly discontinuously on $\mathbb{R}^n, n \leq 5$ and the semisimple part of the Zariski closure of Γ does not contain $SO(2,1)$ as a normal subgroup then Γ is virtually solvable.

1 Introduction

Let X be a topological space and Γ be a subgroup of the group of homeomorphisms of X. A subgroup Γ is said to act *properly discontinuously* on X if for every compact subset K of X the set $\{g \in \Gamma : gK \cap K \neq \emptyset\}$ is finite. A subgroup Γ is called *crystallographic* if Γ acts properly discontinuously on X and the orbit space X/Γ is compact.

The study of crystallographic groups has a long history. The crystallographic groups of hyperbolic transformations have been investigated by Fricke and Klein in the lectures on the theory of automorphic functions [FK]. In 3-dimensional Euclidean space Fedorov [F], Schoenflies [Sc], and later Rohn [Ro] have shown that there are only a finite number of essentially different kinds of euclidean crystallographic groups. The 3-dimensional Euclidean crystallographic groups are the symmetry groups of crystalline structures and so are of fundamental importance in the science of crystallography.

D. Hilbert wrote in his famous lecture delivered on the IMC at Paris, 1900 ([Hil], 18th Problem) :

" Now, while the results and methods of proof applicable to elliptic and hyperbolic space hold directly for n-dimensional space also, the generalization of the theorem for Euclidean space seems to offer decided difficulties. The investigation of the following question is therefore desirable:

Is there in n-dimensional Euclidean space also only a finite number of essentially different kinds of groups of motions with a fundamental region? "

In response to this problem Bieberbach showed in a series of papers that this was so. The key result is the following famous theorem of Bieberbach. A group Γ acting isometrically on the n–dimensional Euclidean space $\mathbb{R}^n$ with compact quotient $\mathbb{R}^n/\Gamma$ contains a subgroup of a finite index consisting of translations. In particular, such a group Γ is virtually abelian, i.e. Γ contains an abelian subgroup of finite index. Moreover, in [B1, B2, B3] Bieberbach proved that a group Γ acting isometrically and properly discontinuously on the n–dimensional Euclidean space is virtually abelian.

A natural way to generalize the classical problem is to broaden the class of allowed motions and consider crystallographic groups of affine transformations. This raises the question of the group-theoretic properties of affine crystallographic groups.

Let $n > 1$ and let $V = \mathbb{R}^n$ be the vector space. We can and will consider $\mathbb{R}^n$ as an affine space. Let $GL(V)$ (resp. $\mathrm{Aff}(\mathbb{R}^n)$) be the group of all linear (res. affine) transformations. Let us recall that the group of affine transformations $\mathrm{Aff}(\mathbb{R}^n)$ is the semidirect product

$GL(V) \ltimes V$ where V is identified with the group of its translations. Let $l : \mathrm{Aff}(\mathbb{R}^n) \to GL(V)$ be the natural homomorphism. Then $l(g)$ is called the *linear part* of the affine transformation g. Let $X \subseteq \mathrm{Aff}(\mathbb{R}^n)$ then the set $l(X) = \{l(x), x \in X\}$ is called the *linear part* of X. Let $\Gamma < \mathrm{Aff}(\mathbb{R}^n)$ and let G be the Zariski closure of $l(\Gamma)$ in $GL(V)$.
Auslander proposed the following conjecture in [Au].

The Auslander Conjecture. *Every crystallographic subgroup Γ of Aff$(\mathbb{R}^n)$ is virtually solvable i.e. contains a solvable subgroup of finite index.*

The proof in [Au] of this conjecture is unfortunately incorrect, but the conjecture is still an open and central problem. In [FD] the Auslander conjecture was proved for dimensions ≤ 3. D. Fried and W. Goldmann deduce the proof from the following key statement:t if Γ is a crystallographic subgroup of G_B where B is a non-degenerate quadratic form of signature $(2, 1)$ then $l(\Gamma)$ is not Zariski dense in $SO(2, 1)$. In [To3] the author attempts to prove the Auslander conjecture for dimensions 4 and 5. Unfortunately, the proof there is incomplete for dimension 4 and incorrect for dimension 5. The proof has been corrected and completed in [To4]. The Auslander conjecture for dimensions 4 and 5 was proved in [AMS5].

The Auslander conjecture was proven for some special cases. In [GK] it was proved in the case where the linear part of a crystallographic group Γ is a subgroup of $SO(n, 1)$. Then F. Grunewald and G. Margulis [GM] proved that if the linear part of Γ is a subgroup of a simple Lie group of real rank 1, then Γ is virtually solvable. This result was generalized in [To1]. It was proved that if the semisimple part of G is a simple group of real rank 1, then Γ is virtually solvable. Finally, in [S2] and [To2], it was proved, that if the linear part of Γ is a subgroup of a semisimple Lie group G and every non-commutative simple subgroup of G has real rank ≤ 1 then Γ is virtually solvable. Let us remark, that all papers [FG], [GK], [GM], [S2] and [To1,2] where the Auslander conjecture was proved basically use the same idea which was first introduced in [FG]. We call this idea "the cohomological

argument" because it is based on using the virtual cohomological dimension of Γ.
In [AMS4] the Auslander conjecture was proved for an affine group Γ leaving a non-degenerated quadratic from B of signature $(n-2,2)$ invariant. By contrast, [AMS4] and [M] are based on a completely different approach, namely on dynamical ideas (see also [AMS1,2,3]).

We prove the following theorem

Theorem A *Let Γ be a crystallographic subgroup of Aff(A) and $n < 7$, then Γ is virtually solvable.*

The proof of this theorem is based mainly on dynamical arguments. In some cases, we use the cohomological argument to shorten the proofs .We would like to admit that the first proof of this theorem was published in [AMS5].

There is an additional geometric interest in properly discontinuous groups since they can be represented as fundamental groups of manifolds with certain geometric structures, namely complete flat affine manifolds. If M is a complete flat affine manifold, its universal covering manifold is isomorphic to $\mathbb{R}^n$. It follows that its fundamental group $\Gamma = \pi_1(M)$ is in a natural way a properly discontinuous torsion-free subgroup of $\text{Aff}(\mathbb{R}^n)$ Conversely, if Γ is a properly discontinuous torsion-free subgroup of $\text{Aff}(\mathbb{R}^n)$, then $\mathbb{R}^n/\Gamma$ is a complete flat affine manifold M with $\pi_1(M) = \Gamma$.

In 1977 J.Milnor asked if the fundamental group $\pi(M)$ of a complete locally flat affine manifold M contains a free non-commutative subgroup.
The Tits' alternative implies, that if the answer to Milnor's question is negative then the fundamental group $\pi(M)$ is virtually solvable. Thus the negative answer to Milnor's question means that the Auslander conjecture is true without the assumption that M is compact. It can be stated as the following question

Question 1. *Does an affine group Γ that acts properly discontinuously on the affine* *space $\mathbb{R}^n$ virtually solvable?*

In comments to his question Milnor wrote: "I do not know if such a manifold exists even in dimension 3" and proposed "to construct a Lorentz-flat example by starting with a discrete subgroup $\mathbb{Z}*\mathbb{Z} \leq SO(2,1)$ then adding translation components to obtain a group of isometries of Lorentz 3-space, but it seems difficult to decide whether the resulting group action is properly discontinuous" [Mi2, p. 184].

G. Margulis gave a positive answer to Milnor's question in dimension 3 in [M]. He constructed a free non-commutative subgroup Γ of isometries of Lorentz 3-space acting properly discontinuously on $\mathbb{R}^3$. In order to study the dynamics of an affine action, Margulis introduced the concept of the sign of an affine transformation for an affine group $\Gamma, l(\Gamma) \subseteq SO(2,1)$. This example came as the surprise and is sometimes called "the Margulis' phenomenon".

We show that the Margulis phenomenon is the reason that an answer to Milnor's question is positive for an affine space $\mathbb{R}^n, n \leq 5$. We prove the following theorem

Theorem B. *Let Γ be an affine group acting properly discontinuously on the affine space $\mathbb{R}^n, n \leq 5$. Assuming that the semisimple part of the algebraic closure of Γ does not contain $SO(2,1)$ as a normal subgroup then Γ is virtually solvable.*

Together with the Margulis phenomenon, this leads us to the following conclusion. Let Γ be an affine group acting on the affine space $\mathbb{R}^n, n \leq 5$. Then Γ contains a free subgroup that acts properly discontinuously, if and only if the semisimple part of the Zariski closure of Γ contains $SO(2,1)$ as a normal subgroup. Note that this is not true for $n = 6$ [DGK].

Let us give a short description of the paper. As the first step in section 2, we obtain a list of all possible semisimple groups S which might be a semisimple part of the Zariski closure of an affine crystallographic group for $n \leq 6$. In section 3 we give a list of all possible semisimple groups S which might be a semisimple part of the Zariski closure of group Γ that acts properly discontinuously for $n \leq 5$ and does not have $SO(2,1)$ as a normal subgroup. Using these lists of possible linear parts we prove the Auslander conjecture for dim ≤ 6. We prove the Auslander conjecture in dimensions 4 and 5 in section 4. In section 5 we show that the semisimple part S of the Zariski closure of $l(\Gamma)$ cannot be $SO(3,2)$ or $SO(3) \times SL_3(\mathbb{R})$. The proof is based on the cohomological argument we have mentioned above. Namely, we will compare the virtual cohomological dimension of Γ and the dimension of the symmetric space S/K, where K is a maximal compact subgroup of S. We will prove that none of these cases is possible.

The most difficult part is to show that the semisimple part of the Zariski closure of $l(\Gamma)$ is not $SO(2,1) \times SL_3(\mathbb{R})$. This is done in section 6. We show that it is possible to change the sign of a hyperbolic element (see Main Lemma 6.7) in this case. Thus, by Lemma 6.5, we conclude that the semisimple part of the Zariski closure of $l(\Gamma)$ cannot be $SO(2,1) \times SL_3(\mathbb{R})$. Hence none of the possible non-trivial semisimple groups can be the semisimple part of the Zariski closure of Γ. Therefore the semisimple part of the Zariski closure of Γ is trivial. Hence Γ is virtually solvable.
In section 8 based on the results we obtain, in section 3 we prove Theorem B. The most difficult case here is to show that the semisimple part of the Zariski closure of $\ell(\Gamma)$ is not $SL_2(\mathbb{R}) \times SO(3)$.

2 Linear parts of crystallographic groups

2.1. Notation and terminology. In this section we introduce the terminology we will use throughout the paper. Let $V = \mathbb{R}^n, n > 1$ be a vector space and let $GL(V)$ be the group of all linear transformation of V. Let $\text{Aff}(\mathbb{R}^n)$ be the group of affine transformation of an affine space $\mathbb{R}^n$. Since the group $\text{Aff}(\mathbb{R}^n)$ is the semidirect product $GL(V) \ltimes V$ every element $g \in \text{Aff}(\mathbb{R}^n)$ is a pair $g = (l(g), v_g)$ where $l(g) \in GL(V), v_g \in V$. The linear transformation $l(g)$ is called the linear part of g and v_g is called a translational vector. Let $[l(g)]$ be the matrix of $l(g)$ and let $[v_g]$ be the coordinate of v_g in the same basic. Thus we obtain a group isomorphism

$$\phi(g) = \begin{pmatrix} [l(g)] & [v_g] \\ 0 & 1 \end{pmatrix} \qquad (*)$$

between $\mathrm{Aff}(\mathbb{R}^n)$ and the subgroup of $GL_{n+1}(\mathbb{R})$.

Denote by l the natural homomorphism $l : \mathrm{Aff}(\mathbb{R}^n) \to GL(V)$. The set $l(X)$ where $X \subseteq \mathrm{Aff}(\mathbb{R}^n)$ is called the linear part of X.

Proposition 2.2 *Let* Γ *be an affine group acting properly discontinuously. Let* g *be an element of the connected component of the Zariski closure of* Γ. *Then* $l(g)$ *has* 1 *as an eigenvalue.*

Proof It is easy to see if the linear part $l(g)$ of $g \in \mathrm{Aff}(\mathbb{R}^n)$ does not have 1 as an eigenvalue then g has a fixed point. Thus every element of an affine torsion free group acting properly discontinuously has one as an eigenvalue. Let G be the Zariski closer of Γ and let G^0 be the connected component of G. It is well known, that exists a finitely generated subgroup Γ_0 of Γ such that the Zariski closure of Γ and Γ_0 coincide. Since G^0 is an open and closed subgroup of G a finite index subgroup $\Gamma_1 = \Gamma_0 \cap G^0$ of Γ_0 is a finitely generated group which is dense in G^0. By Selberg's lemma we conclude that there exists a torsion free subgroup $\Gamma_2 \leq \Gamma_0$ of finite index. Hence the linear part $l(g)$ for every $g \in \Gamma_2$ has one as an eigenvalue, because Γ_2 acts properly discontinuously. Consequently the same is true for every element of the Zariski closure of Γ_2. Obviously Γ_2 is Zariski dense in G^0. This proves the statement.

Let Γ be an affine crystallographic group and let G be the Zariski closure of Γ. Let S be a semisimple part of G. Clearly, S is a semisimple part of the connected component of the linear part $l(G)$ of G. The goal of this section is to give a complete list of all possible non trivial semisimple subgroups $S, S < GL(V)$ which might be a semisimple part of $l(G) < GL(V)$. The possible semisimple subgroups of $l(G)$, which occur in our list fulfil the following assumptions .

(A1) $S < GL(V)$, $dim\, V \leq 6$.

Let $S = \prod_{1 \leq i \leq k} S_k$ be the decomposition of the semisimple part into an almost direct product of simple groups. If $rank_{\mathbb{R}}(S_i) \leq 1$ for all $1 \leq i \leq k$ then Γ is not crystallographic [S2], [To2]. Therefore from now on unless otherwise noted we will assume that in case $S \neq 1$ we have

$$\max_{1 \leq i \leq k} rank_{\mathbb{R}}(S_i) \geq 2.$$

Hence we will assume that

(A2) *There is a simple normal subgroup* $S_1 \leq S$ *with* $rank_{\mathbb{R}}(S_1) \geq 2$.

By Proposition 2.2 every element of the connected component $l(G)^0$ of $l(G)$ has one as an eigenvalue. Therefore we add to our assumptions the following one.

(A3) *Every element* $g \in l(G)^0$ *has one as an eigenvalue.*

We call the group G an $\boldsymbol{A}$–group if G fulfils the assumptions (A1), (A2) and (A3).

The main steps to establish our list are the following. For a semisimple group S satisfying the properties (A1) -(A3) we shall see that there are at most two non-trivial irreducible components $V_i, i \leq 2$, of the representation of the complexification $\overline{S}$ of S on the complexification $\overline{V}$ of V and that the image $\overline{S}_i$ of $\overline{S}$ in $GL(\overline{V_i})$ is a simple group for every non-trivial irreducible component $\overline{V}_i$, see 2.6. Furthermore it does not happen that $\overline{V}$ contains two non-trivial irreducible components $\overline{V}_1$ and $\overline{V}_2$ such that $\overline{S}_1$ and $\overline{S}_2$ are isomorphic, see 2.4. It follows, that if the real Lie group S is simple then also $\overline{S}$ is simple, see 2.5. Note that there are several ways to satisfy (A3). Let V_0 be the subspace of V

such that S acts trivially on V_0. Let W be an S-invariant subspace such that $V = V_0 \oplus W$. If there exists an element $s \in S$ with no eigenvalue one on W then $\dim V_0 > 0$ and $\dim W \leq 5$.

We will assume from now on that G is an $\boldsymbol{A}$–group. If the dimension of every simple normal subgroup of a semisimple part S is ≤ 6 then $(A2)$ does not hold. Thus there exists a simple normal subgroup of S with dimension > 6. Let us now recall a list [PV, pp 260-261] of all possible complex representations ρ of a simple complex Lie group S with $\dim \rho \leq 6 \leq \dim S$. In the first column the symbols SL_n, Sp_{2n}, SO_n denote the corresponding simple Lie (algebraic) group in their simplest representation. The symbol $S^m H$ (resp. $\wedge^m H$) denotes the m^{th} symmetric (resp. exterior) power of a linear group, and $S_0^m H$ (resp. $\wedge_0^m H$) is the highest (Cartan) irreducible component of this representation.

Table 1

S	$\dim \rho$	n
$SL_n, n \geq 3$	n	$n = 3, 4, 5$
$SO_n, n \neq 4, n \geq 3$	n	$n = 3, 5, 6$
Sp_{2n}	$2n$	$2, 3$
$AdSL_n$	$n^2 - 1$	$n = 2$
$S^2 Sl_n$	$n(n+1)/2$	$n = 2, 3$
$\wedge^2 SL_n, n \geq 4$	$n(n-1)/2$	$n = 4$
$\wedge^2 SO_n, n \geq 3, n \neq 4$	$n(n-1)/2$	$n = 3$
$\wedge_0^2 Sp_{2n}, n \geq 2$	$(n-1)(2n+1)$	$n = 2$

Our next goal is to provide a list of all possible real simple linear groups S which might be a semisimple part of G and which are possible as a factor of a semisimple part of an $\boldsymbol{A}$–group. We will use the following notation. Let $\overline{V} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V and let $\overline{S}$ be the complex Lie group, such that S is a real form of $\overline{S}$.

If the group $\overline{S}$ is simple then the group S is a simple real Lie group. Assume that the space $\overline{V}$ is irreducible. Then $\overline{S}$ is a group listed in Table 1.

2.3 Simple and irreducible. Thus using [OV] we have the following list of all real simple groups S which are a real form of a simple complex Lie group $\overline{S}$ listed in the Table 1 no matter if they are $\boldsymbol{A}$–groups or not.

Table 2

S	$\dim \rho$
$SL_k(\mathbb{R}), 2 \leq k \leq n$	k
$SO(3,2)$	5
$SO(2,1)$	3
$SO_3(\mathbb{R})$	3
$Sp_4(\mathbb{R})$	6

2.4. Simple and reducible. Here we assume that the space $\overline{V}$ is reducible and the complex group $\overline{S}$ is simple. There exists an $\overline{S}$-invariant non-trivial subspace $W \leq \overline{V}$ with non-trivial representation of $\overline{S}$. Obviously $\dim W \geq 2$. If $\dim W = 2$ then $rank S \leq 1$. If $\dim W = 3$ the real form of $\overline{S}$ does not have one as an eigenvalue if $rank S \geq 2$. Thus there is no simple $\boldsymbol{A}$–group such that $\overline{V}$ is $\overline{S}$ -reducible.

2.5. Semisimple not simple Let S be a simple Lie group, such that $\overline{S}$ is not simple and $\overline{V}$ is irreducible. There exists a complex structure on S. Namely, there is a complex simple Lie group $\tilde{S}$ such that $S = \tilde{S}(\sigma(\mathbb{C}))$, where $\sigma : \mathbb{C} \longrightarrow M_2(\mathbb{R})$ is the natural embedding of the field $\mathbb{C}$, see [OV]. In this case $\overline{S}$ considered as real Lie group is isomorphic to $\tilde{S} \times \tilde{S}$. The possible groups of this type are listed in the following table 3.

Table 3

S	$\dim \rho$
$SL_k(\sigma(\mathbb{C})), k = 2, 3$	$2k$
$SO_3(\sigma(\mathbb{C}))$	6

Note that non of the groups listed in Table 3 can be a normal subgroup of the semisimple part S of an $\boldsymbol{A}$–group G. Indeed, let S_1 be a normal subgroup of S. Suppose that $S_1 = SL_2(\sigma(\mathbb{C}))$. Then $V = V_1 \oplus V_2$. We can assume that S acts on V_1 as $S_1 = SL_2(\sigma(\mathbb{C}))$. Then $\dim V_2 = 2$. Since the real rank of $SL_2(\sigma(\mathbb{C}))$ is 1, then the semisiple part of G does not fulfil (A2). If $S_1 = SO_3(\sigma(\mathbb{C}))$ then $S = S_1$ and the semisiple part of G does not fulfil (A2). If $S_1 = SL_3(\sigma(\mathbb{C}))$ then $S = S_1$ and the semisiple part of G does not fulfil (A1).

2.6. General case. The semisimple group $\overline{S}$ is the almost direct product of simple groups $\overline{S} = \prod_{1 \leq i \leq k} \overline{S}_i, k \geq 2$. Let $W_0 = \{v \in \overline{V} : sv = v \,\forall s \in \overline{S}\,\}$. There exists a unique $\overline{S}$–invariant subspace $\overline{W}$ of the space $\overline{V}$ such that $\overline{V}$ is the direct sum of W_0 and $\overline{W}$. If the restriction $\overline{S}|_{\overline{W}}$ is an irreducible representation of $\overline{S}$, then it is the tensor product of $\overline{S}_i$-irreducible representations for all $i = 1, \dots, k$. Thus if $\dim \overline{V} \leq 6$ it follows immediately that this is impossible for an $\boldsymbol{A}$–group. Therefore $\overline{W}$ is the direct sum of $\overline{S}$–invariant non-trivial irreducible subspaces $W_i, i = 1, \dots, k$ such the restriction $\overline{S_j}|_{W_i}$ is trivial for every $i \neq j, i, j = 1, \dots, k$. As we know, every element of G^0 has one as an eigenvalue. Thus it follows from $(A3)$ that if the subspace W_0 is trivial, there exists an $i_0, 1 \leq i_o \leq k$, such that every element $s \in \overline{S}_{i_0}$ has one as an eigenvalue. Since for every $i = 1, \dots, k$ the group $\overline{S}_i$ is an irreducible subgroup of $GL(W_i)$, we can and will again use Table 1 and Table 2. This will lead us to a complete list of all possible cases under the assumption $\boldsymbol{A}$.

2.7. Linear parts and decompositions. Let us summarize all we did in 2.3, 2.4, 2.5 and list all cases we have to consider. Let V_0 be the maximal subspace in $V = \mathbb{R}^n$ such that S acts trivially on V_0. Let V_1 be the unique S–invariant subspace such that $\mathbb{R}^n = V_0 \oplus V_1$. Let $\pi_S : G \longrightarrow S$ be the projection. We will use these notations throughout the rest of the paper. Recall that G is an $\boldsymbol{A}$–group.

Case 1 Assume that for every regular element $s \in S$ the restriction $s|_{V_1}$ does not have 1 as an eigenvalue. Thus $V_0 \neq 0$. Consider the inclusion $i_s : S \longrightarrow GL(V_1)$ as a representation of the semisimple Lie group S. Assume first that S is a simple group.It follows from 2.5 that the complexification of $i_s(S)$ is a simple irreducible group. Thus it follows from the Table 2 that all possible semisimple part of G which has property (A2) are:

(1) $S = SL_l(\mathbb{R}), V_1 = \mathbb{R}^l, l < n, 3 \le l \le 5, 4 \le n \le 6.$

(2) $S = Sp_4(\mathbb{R}), V_1 = \mathbb{R}^4, n = 5, 6.$

Suppose that the group S is semisimple, but not simple. As we show in this case $i_s(S)$ is the direct product of two simple groups such that their complexifications are simple complex groups. It follows from Table 2 that all possible semisimple parts in this case which have property (A2) are:

(3) $S = SL_2(\mathbb{R}) \times SL_3(\mathbb{R}), V_1 = \mathbb{R}^5, n = 6.$

Case 2. Assume that for every regular element $s \in S$ the restriction $s|_{V_1}$ has 1 as an eigenvalue. Suppose that S is a simple group. It follows from 2.4 and 2.5 and Table 2 that the group $\overline{S}$ is simple. Therefore $S = SO(3,2)$ and $\dim V = 5, 6$. If S is a semisimple but not a simple group, we show above (see 2.5) that S is the almost direct product of two simple group S_1 and S_2 such that their complexifications $\overline{S}_1$ and $\overline{S}_2$ are simple complex groups. Since S is an $\boldsymbol{A}$–group it follows from Table 2 and (A2) and (A3) that $\dim V = 6$ and $S = SL_3(\mathbb{R}) \times SO(2,1)$, or $S = SL_3(\mathbb{R}) \times SO(3)$. Therefore we conclude that in this case

(1) $S = SO(3,2), \dim V_1 = 5, n = 5, 6.$

(2) $S = SO(3) \times SL_3(\mathbb{R}), n = 6.$

(3) $S = SO(2,1) \times SL_3(\mathbb{R}), n = 6.$

Case 1 and 2 give us the complete list of all possible semisimple parts of the $\boldsymbol{A}$–group G.

Let Γ be a crystallographic group and let G be the Zariski closure of Γ. Then $l(G)$ is an $\boldsymbol{A}$–group for $\dim V \leq 6$. Therefore if the semisimple part S is non-trivial it is one of the groups listed in Case 1 and Case 2. Our strategy is to show case by case that none of the semisimple groups listed in Case 1 and Case 2 is a semisimple part G. Thus, $S = 1$ and Γ is virtually solvable.

Remark 2.8. The lists above shows if $\dim V \leq 5$ and S is semisimple part of $l(G)$, then S is a simple group.

3 The linear parts of affine groups groups acting properly discontinuously

Let Γ be an affine group and let G be the Zariski closure of Γ. Let S be a semisimple part of G. Clearly, S is a semisimple part of the connected component of the linear part $l(G)$ of G. The goal of this section is to give a complete list of all possible non- trivial semisimple subgroups $S, S < GL(V), \dim V \leq 5$ that might be a semisimple part of an affine group which acts properly discontinuously. The semisimple subgroups of $l(G)$, which occur in our list have to fulfill the following assumptions$(P1), (P2)$ and $P(3)$ below.

(P1) *$S < GL(V)$, $dim\, V \leq 5$.*

(P2) *S does not contain $SO(2,1)$ as a normal subgroup*

(P3) *Every element $g \in l(G)^0$ has one as an eigenvalue.*

The motivations for $(P1)$ and $(P2)$ are obvious. The justification for $(P3)$ follows from Proposition 2.2 . We will follow along the way we used in the previous chapter taking into account that property (A1) is not valid. If the semismiple part S is compact then Γ

is virtually abelian [B2]. Hence we will assume that S is not compact.

Let $l(G)$ be a subgroup of $GL(V), \dim V \leq 5$. Let V_0 be the maximal subspace in V such that S acts trivially on V_0. Let V_1 be the unique S–invariant subspace such that $V = V_0 \oplus V_1$.

Case 1 Assume that for every regular element $s \in S$ the restriction $s|_{V_1}$ does not have 1 as an eigenvalue. Thus $V_0 \neq 0$. Consider the inclusion $i_s : S \longrightarrow GL(V_1)$ as a representation of the semisimple Lie group S.

Assume first that S is a simple group. It follows from remarks 2.8, 2.4 and 2.5 that all possible semisimple parts of G which have the property (P2) are:

(1) $S = SL_l(\mathbb{R}), V_1 = \mathbb{R}^l, 2 \leq l < 5, 2 < n \leq 5, l < n,$

(2) $S = Sp_4(\mathbb{R}), V_1 = \mathbb{R}^4$

(3) $S = SL_2(\sigma(\mathbb{C})), V_1 = \mathbb{R}^4$

where $\sigma : \mathbb{C} \to M_2(\mathbb{R})$ is the standard embedding.

Suppose that the group S is semisimple, but not simple. It follows from 2.5 that all possible semisimple parts in this case that have the property (P3) are:

(4) $S = SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), V_1 = \mathbb{R}^4, n = 5.$

Case 2. Assume that for every regular element $s \in S$ the restriction $s|_{V_1}$ has 1 as an eigenvalue. It follows from 2.4 and 2.5 that in this case

(1) $S = SO(3,2), \dim V_1 = 5$

(2) $S = SO(4,1), \dim V_1 = 5.$

(3) $S = SO(3) \times SL_2(\mathbb{R}),\ n = 5.$

Case 1 and 2 give us a complete list of all possible semisimple parts of G that have properties (P1),(P2) and)P3).

As for crystallographic groups our strategy for affine groups acting properly discontinuously is to show case by case that none of the semisimple groups listed in Case 1 and Case 2 is a semisimple part of G. Thus, or the semisimple part of G contains $SO(2,1)$ as a normal subgroup or Γ is virtually solvable.

4 The dynamic of action of an affine group.

Let V be a vector space of dimension n over $\mathbb{R}$ and let $g \in GL(V)$ be a linear transformation. Let $F_g(x) \in \mathbb{R}[x]$ be the characteristic polynomial of g. Let $\Omega(g) = \{a_1, \dots a_n\}$ be the set of all root of $F_g(x)$. Set $\Omega_\alpha^+(g) = \{a_i, a_i \in \Omega(g) : |a_i| > \alpha\}$ (resp.$\Omega_\alpha(g) = \{a_i, a_i \in \Omega(g) : |a_i| \geq \alpha\}$) Suppose that $a \in \Omega(g)$ and $a \notin \mathbb{R}$ then $\overline{a} \in \Omega(g)$. Since $|a| = |\overline{a}|$ we conclude that $F_\alpha^+(x) = \prod_{a_i \in \Omega_\alpha^+(g)}(x - a_i) \in \mathbb{R}[x]$and $F_\alpha(x) = \prod_{a_i \in \Omega_\alpha(g)}(x - a_i) \in \mathbb{R}[x]$. For $\alpha = 1$ we have $F_1^+(x) = \prod_{a_i \in \Omega_1^+(g)}(x - a_i) \in \mathbb{R}[x]$, $F_1(x) = \prod_{a_i \in \Omega_1(g)}(x - a_i) \in \mathbb{R}[x]$. Thus we have two linear endomorphisms $F_1^+(g) : V \to V$ and $F_1(g) : V \to V$. We define the following g-invariant subspaces of V. Set $A^+(g) = \ker F_1^+(g)$, $D^+(g) = \ker F_1(g)$, $A^-(g) = A^+(g^{-1})$, $D^-(g) = D^+(g^{-1})$ and $A^0(g) = D^+(g) \cap D^-(g)$. Roughly speaking, $A^+(g)$ (resp. $A^-(g)$) is a subspace of V spanned by eigenvectors of g with eigenvalue modulus > 1, (resp. < 1); $D^+(g)$ (resp. $D^-(g)$) is a subspace of V spanned by eigenvectors of g with eigenvalue modulus ≥ 1, (resp. ≤ 1)

4.1. Let $g \in GL(V)$. Set $V_g^0 = \{v \in V; gv = v\}$. Let G be a subgroup of $GL(V)$. A semisimple element $g \in G$ is called *regular* in G if

$$\dim V_g^0 = \min\{\dim V_t^0 | \, t \in G, t \ \textit{ semisimple } \}$$

Let us remark that the set of regular elements of an algebraic group is Zariski open. Let $g \in G$ be a semisimple element. such that

$$\dim(A^0(g)) = \min\{\dim A^0(t) | t \in G, t \textit{ semisimple,}\}$$

then g is called $\mathbb{R}$-***regular*** in G. Let G be an affine group, $G <\mathrm{Aff}\mathbb{R}^n$. An affine transformation $g \in G$ is called regular (respectively $\mathbb{R}$-regular) if $l(g)$ is a regular (respectively $\mathbb{R}$-regular) element of $l(G)$.

Our definition of $\mathbb{R}$-regular element slightly differs from that of [P] were it was first introduced. Note that the set of $\mathbb{R}$-regular elements in an algebraic group G need not be Zariski open in G. Nevertheless under some conditions a Zariski dense subgroup of an algebraic group G contains an $\mathbb{R}$-regular element [P],[AMS1],[AMS4]. For example this is true if $G = SO(B)$ where B is a non degenerate form of signature (p, q) and Γ is a Zariski dense subgroup of G. Note that in case $p = 2, q = 1$ every hyperbolic element is regular and $\mathbb{R}$-regular.

4.2. If we use topological concepts and do not specify the topology, we mean the Zariski topology. If we refer to the Euclidean topology we mention this explicitly and use expressions like Euclidean-open, Euclidean-connected, etc.

Let $\|\,.\,\|$ and d denote the norm and metric on $\mathbb{R}^n$ corresponding to a inner product on $\mathbb{R}^n$. Let $\|g\|_-$ be the norm of the restriction $g|_{A^-(g)}$. Denote by $\|g\|_+ = \|g^{-1}\|_-$ and put $s(g) = \max\{\|g\|_+, \|g\|_-\}$. A regular element g is called *hyperbolic* if $s(g) < 1$. It is clear that for $\mathbb{R}$- regular element g of a non compact connected semisimple Lie group there exists a number N such that for $n > N$ the element g^n is hyperbolic.

Let $P = \mathbb{P}(\mathbb{R}^n)$ be the projective space corresponding to $\mathbb{R}^n$. Let $\pi : \mathbb{R}^n \setminus \{0\} \to P$ be the natural projection. For a subset X of $\mathbb{R}^n$ we denote $\pi(X) = \pi(X \setminus \{0\})$.

The metric $\|\,.\,\|$ on $\mathbb{R}^n$ induces the standard metric $\widehat{d}$ on the projective space $P =$

$\mathbb{P}(\mathbb{R}^n)$ by the formula (see [T])

$$\widehat{d}(p,q) = \frac{\|v \wedge w\|}{\|v\| \cdot \|w\|}, p = \pi(v), q = \pi(w).$$

Thus for any point $p \in P$ and any subset $A \subseteq P$, we can define $\widehat{d}(p,A) = \inf_{a \in A} \widehat{d}(p,a)$. Let A_1 and A_2 be two subsets of P. We define

$$\underline{d}(A_1, A_2) = \inf_{a_1 \in A_1, a_2 \in A_2} \widehat{d}(a_1, a_2)$$

$$\widehat{d}(A_1, A_2) = \sup_{a_1 \in A_1} \inf_{a_2 \in A_2} \widehat{d}(a_1, a_2)$$

For two subspaces $W_1 \subseteq \mathbb{R}^n$ and $W_2 \subseteq \mathbb{R}^n$ we put $\widehat{d}(W_1, W_2) = \widehat{d}(\pi(W_1), \pi(W_2))$ and $\underline{d}(W_1, W_2) = \underline{d}(\pi(W_1), \pi(W_2))$ A hyperbolic element g is called ε-*hyperbolic* if

$$\underline{d}(A^+(g), D^-(g)) \geq \varepsilon$$

and

$$\underline{d}(A^-(g), D^+(g)) \geq \varepsilon.$$

Two different hyperbolic elements g_1 and g_2 are called *transversal* if

$$A^{\pm}(g_1) \cap D^{\mp}(g_2) = \{0\})$$

and

$$A^{\pm}(g_2) \cap D^{\mp}(g_1) = \{0\}.$$

Two different hyperbolic elements g_1 and g_2 are called ε-*transversal* if

$$\underline{d}(A^{\pm}(g_1), D^{\mp}(g_2)) \geq \varepsilon$$

and

$$\underline{d}(A^{\pm}(g_2), D^{\mp}(g_1)) \geq \varepsilon.$$

Obviously, two different hyperbolic elements g_1 and g_2 are *transversal* (resp. ε-*transversal*) if and only if g_1^{-1} and g_2^{-1} are *transversal* (resp. ε-*transversal*) The following notions were first introduced in [BG]. Two transversal elements g_1 and g_2 are very transversal if g_1 and g_2^{-1} are transversal. Therefore if g_1 and g_2 are very transversal then g_2 and g_1^{-1} are transversal. Two ε-transversal elements g_1 and g_2 are very $\varepsilon-$ transversal if g_1 and g_2^{-1} are ε–transversal. Hence if g_1 and g_2 are very $\varepsilon-$ transversal then g_2 and g_1^{-1} are ε-transversal.

Let B be a non degenerate quadratic form defined on V and let $g \in SO(B) \leq GL(V)$ be a $\mathbb{R}$-regular element in $SO(B)$. Since $A^+(g)$ (resp.$A^-(g)$) is the unique maximal isotropic subspace of $D^+(g)$ (resp. $D^-(g)$) it is easy to see that two hyperbolic $\mathbb{R}$-regular elements g_1 and g_2 of $SO(B)$ are transversal if and only if $A^+(g_1) \cap A^-(g_2) = \{0\}$ and $A^+(g_2) \cap A^-(g_1) = \{0\}$.

Clearly g and g^{-1} are not transversal for any regular element g. Nevertheless it is quite important to be able to find an element t of a given linear group G such that g and $tg^{-1}t^{-1}$ are transversal. It is possible for example for $G = SO(B)$.

Let $g_1 \in SO(B)$ and $g_2 \in SO(B)$ be two hyperbolic transversal elements.

Then

(1) for every ε there exists $\delta = \delta(\varepsilon)$ such that ifg_1 and g_2 are ε-transversal then $\underline{d}(A^+(g_1), A^-(g_2)) > \delta$ and $\underline{d}(A^-(g_1), A^+(g_2)) > \delta$;

(2) for every δ there exists $\varepsilon = \varepsilon(\delta)$ such that if $\underline{d}(A^+(g_1), A^-(g_2)) > \delta$ and $\underline{d}A^-(g_1), A^+(g_2)) > \delta$ then the two hyperbolic elements g_1 and g_2 are ε -transversal.

Clearly

(3) There exists ε such that g_1 and g_2 are ε-transversal if and only if g_1^{-1} and g_2^{-1} are ε-transversal.

Let Γ be an affine group acting properly discontinuously. Let G be the Zariski closure of Γ. Obviously, Γ is a crystallographic group if and only if every finite index subgroup of Γ is crystallographic. Thus, since the connected component of a Zariski closed group is a subgroup of finite index, we will assume from now on that the Zariski closure of Γ is connected. Therefore every element of Γ has 1 as an eigenvalue by Proposition 2.2. Hence for any semisimple element g of Γ there exists a g-invariant line L_g. The restriction of g to L_g is the translation by a non- zero vector t_g. Set $v_0(g) = t_g/\|t_g\|$. Let us note that all such lines are parallel and the vector t_g does not depend on the choice of L_g. We take for g the g-invariant line L_g that is closest to the origin. Let us define the following affine subspaces: $E_g^+ = D^+(g) + L_g,\ E_g^- = D^-(g) + L_g,\ E_g^+ \cap E_g^- = C_g$. Let $p \in L_g$ be a point. Then $t_g = \overrightarrow{p\,gp}$. Clearly $t_g = -t_{g^{-1}}$, $L_g = L_{g^{-1}}$. Let s be an affine transformation. Then

$$L_h = sL_g,\, t_h = l(s)t_g \tag{$**$}$$

for $h = sgs^{-1}$. Denote by $o(g)$ the restriction of g onto C_g. Let g be a ε-hyperbolic element $g \in G$. Assume that $x \in E_g^-$ and $y \in L_g$ such that $\overrightarrow{xy} \in D^-(g)$. Then there exists a constant $c(\varepsilon)$ such that for $n \in \mathbb{Z}, n > 0$. we have

$$d(g^n(x), g^n(y)) \leq c(\varepsilon)s(g)^n d(x, y).$$

Definition 4.3. *Let $g \in G \subseteq GL(V)$ be a $\mathbb{R}$-regular element such that* $\dim A^+(g) \geq \dim A^-(g)$. *We will say that g can be transformed into a transversal pair inside G if there exists an element $t \in G$ and a subspace $W \subset A^+(g)$ such that $V = tW \oplus D^+(g)$.*

Remark. It is easy to see that an element $g \in G$ can be transformed into a transversal pair inside G if and only if there exists an element $t \in G$ such that $D^+(g) + tA^+(g) = V$. Suppose that g can be transformed into a transversal pair let $h = tg^{-1}t^{-1}$.

The next proposition shows that this property depends only on the Zariski closure $\overline{G}$ of a group G, and thus G can be safely ignored in most of what we do.

Proposition 4.4. *Let $\overline{G}$ be the Zariski closure of $G \subseteq SL(V)$. Assume that $\overline{G}$ is connected non-solvable group. Let $g \in G$ be a regular element of $\overline{G}$ which can be transformed into a transversal pair inside $\overline{G}$. Thus there exist a subspace W of $A^+(g)$ and $t \in \overline{G}$ such that $V = D^+(g) \oplus tW$. Then*

(1) The set $\Omega(g) = \left\{t \in \overline{G}, V = D^+(g) \oplus tW\right\}$ is non-empty and open,

(2) Let $\Omega_{ab}(g)$ be the set of all $t \in \overline{G}$ such that g and t do not commute. Then the set $\Omega_{ab}(g)$ is open.

(3) If the set $\Omega_{ab}(g)$ is non-empty there exists $t \in G \cap \Omega(g) \cap \Omega_{ab}(g)$. Therefore we have $V = D^+(g) \oplus tW$ and the group generated by g and tgt^{-1} is not commutative.

(4) The set

$$\Omega = \left\{(t, g), t \in \overline{G}, g \in \overline{G} : tA^+(g) + D^+(g) = V, g \textit{ is a regular elment of} \overline{G}\right\}$$

is non- empty and open in $\overline{G} \times \overline{G}$.

Proof. The sets $\Omega(g)$ and $\Omega_{ab}(g)$ are Zariski open since their complement is determined by algebraic equations. From Definition 2.5 follows that $\Omega(g) \neq \varnothing$. The semisimple part of $\overline{G}$ is not trivial, therefore the set $\Omega_{ab}(g) \neq \varnothing$. This proves (1) and (2). Clearly Ω is the intersection of two open subsets of $\overline{G} \times \overline{G}$. Thus Ω is an open subset. Since there exists a regular element of $\overline{G}$ which can be transformed into a transversal pair inside $\overline{G}$ we conclude that $\Omega \neq \varnothing$. Note that G is dense and $\Omega(g)$ and $\Omega_{ab}(g)$ are open subsets in $\overline{G}$. Hence the set $\Omega(g) \cap \Omega_{ab}(g) \cap G$ is non-empty. This proves the proposition.

Proposition 4.5. *Let $G \subset GL(V)$ be the Zariski closure of the linear part of an* *affine group Γ. Let S be a semisimple part of G and let U be the unipotent radical of G. Assume that G is a connected group and V is the direct sum of two non-trivial S-invariant subspaces V_0 and V_1 with the following properties.*

(1) $gv = v$ for all $g \in G, v \in V_0$ and the induced action $g : V/V_0 \to V/V_0$ is trivial for all $g \in U$.

(2) The restriction $g|_{V_1}$ for one (then for every) regular element g of S does not have 1 as an eigenvalue.

(3) Every regular element $s \in S$ can be transformed into a transversal pair inside S.

Then Γ does not act properly discontinuously

Proof We can and will assume that the solvable radical of G is unipotent. Indeed let

$\Gamma_1 = [\Gamma, \Gamma]$ and $G_1 = [G, G]$. Let R be the solvable radical of G. It is well known that $[G, R] \subseteq U$. Hence the solvable radical of G_1 is unipotent. Obviously G_1 is the Zariski closure of $l(\Gamma_1)$ and fulfills all requirements of the proposition. Thus if Γ_1 does not act properly discontinuously then the same is true for Γ.
Let $\tilde{S}$ be a maximal reductive subgroup of a connected group G whose solvable radical R is unipotent. Then $\tilde{S} \cap R = \{1\}$. Thus S is a maximal reductive subgroup of G. Consequently, every regular element g of G is conjugate to an element of S. Let $\sigma : G \to S$ be the projection. The set of regular elements in S is Zariski open. Since Γ is Zariski dense in G there exists an element $g \in \Gamma$ such that $\sigma(l(g))$ is a regular element of S. Let $x = l(g)$ and let $x = x_s x_u$ be the Jordan decomposition of x. Thus $\sigma(x)$ is a regular element of S. Therefore $\sigma(x_u) = 1$ and consequently $x_u \in U$. By the arguments above, x_s is conjugate to an element of S. Hence we can and will assume that $x_s \in S$. As a result we have $\sigma(x) = x_s$ and $x_u \in U$. From (1) and(2) follows that $l(g) = x_s \in S$. Indeed, by (1), $x_u v - v \in V_0$ for every $v \in V$. By direct calculations from $x_s x_u = x_u x_s$ and (2) we

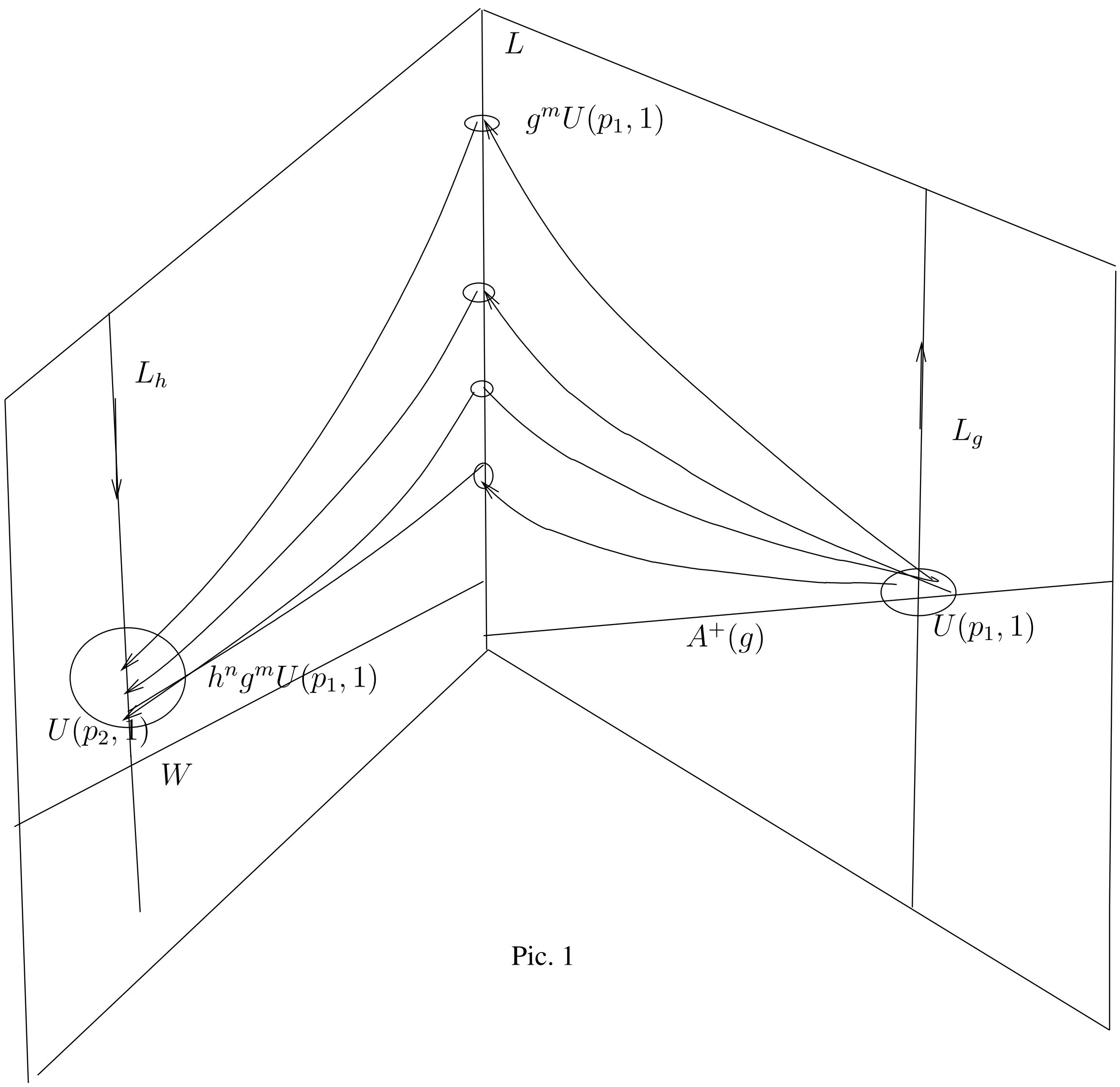

Figure 1: Transversal pairs

conclude that $x_u = 1$. Hence, $l(g)$ is a regular element in S. By (3) the element $l(g)$ can be transformed into a transversal pair inside G. The set $\Omega_{ab}(l(g))$ is clearly not empty. By Proposition 4.4 the element $g \in \Gamma$ can be transformed into a transversal pair by an element $t \in \Gamma$, such that the elements g and tgt^{-1} do not commute.

Set $h = tg^{-1}t^{-1}$. Clearly $A^-(h) = l(t)A^+(g)$. Since $t_g \in V_0$ it follows from (1) and chapter 4.2, (**) that $t_h = -t_g$. In particular the lines L_g and L_h are parallel. Set $v = t_g$.

By definition 4.3 there exists a subspace $W \subseteq A^+(g)$ such that $l(t)W \oplus D^+(g) = V$. Put $\tilde{W} = l(t)W$. Clearly, $\tilde{W} \subset A^-(h)$. Obviously, the intersection $(L_h + \tilde{W}) \cap E_g^+ = L$ is a one dimensional affine space. Moreover, since L_g and L_h are parallel, L is parallel to each of them. Since h and g do not commute we conclude that $L_g \cap L_h = \varnothing$. Otherwise Γ does not act properly discontinuously. There exists a constant $c = c(g, h)$ such that the distance $d(L_g, L) \leq c$.

Fix a point $p_1 \in L_g$. There exists a point $p \in L$ such that the vector $\overrightarrow{pp_1}$ is in $D^+(g)$. Let p_2 be a point in L_h such that $\overrightarrow{pp_2} \in \tilde{W}$. Let $U_d(p_1)$ be the ball in $D^+(g)$ of radius d with the center at p_1 and let $U_a(p_2)$ be the ball in $L_h + A^+(h)$ of radius a with the center at p_2. We can and will assume that $U_d(p_1) \cap U_a(p_2) = \varnothing$. It is easy to see, that there exist $N \in \mathbb{Z}, N > 0$, such that for every point $x_n = p + nv$ we have $g^{-n}x_n \in U_d(p_1)$ and $h^n x_n \in U_a(p_2)$ for every $n > N$ (see Pic.1). Thus for every $n > N$ there exists a point $y_n \in U_d(p_1)$ such that $h^n g^n y_n \in U_a(p_2)$. Hence $h^n g^n \neq 1$ and $h^n g^n U_d(p_1) \cap U_a(p_2) \neq \varnothing$ for all $n > N, n \in \mathbb{N}$. Thus

(1) $h^{Nm_1} g^{Nm_1} \neq h^{Nm_2} g^{Nm_2}$ for all $m_1 \neq m_2, m_1, m_2 \in \mathbb{N}$.

(2) $h^{Nm} g^{Nm} U_d(p_1) \cap U_a(p_2) \neq \varnothing$ for all $m \in \mathbb{N}$.

Therefore the group Γ does not act properly discontinuously.

We will prove a slightly more general statement.

Proposition 4.6. *Let $G \subset GL(V)$ be the Zariski closure of the linear part of an affine group Γ. Let S be a semisimple part of G and let U be the unipotent radical of G. Assume there exists a chain of $l(G)$-invariant subspaces $0 \subseteq V_0 \subset V_1 \subseteq V_2 = V$ such that the following conditions hold.*

(1) the induced representations of S on V_2/V_1 and V_0 and the induced representation of U on V_1/V_0 are trivial.

(2) Let $i : S \to SL(V_1/V_0)$ be the induced representation of S. Then for one (then for every) regular element g of S the element $i(g)$ does not have one as an eigenvalue.

(3) Every regular element $s \in S$ can be transformed into a transversal pair inside S.

Then Γ does not act properly discontinuously.

Proof. Let $\Gamma_1 = [\Gamma, \Gamma]$ and let G_1 be the Zariski closure of Γ_1. From (1) follows that the

solvable radical R of $l(G_1)$ is a unipotent subgroup of $GL(V)$. Let $\Gamma_{m+1} = [\Gamma_1, \Gamma_m]$ and let G_m be the Zariski closure of $\Gamma_m, m \geq 1, m \in \mathbb{Z}$. It is well known that $G_{m+1} = [G_1, G_m]$. There exists an $N \in \mathbb{N}$ such that for all $m \geq N, m \in \mathbb{Z}$, the restriction of $l(G_m)$ to V_0 and the induced action of $l(G_m)$ on V_2/V_1 are trivial. Since a semisimple part of G is also a semisimple part of G_m for all $m \in \mathbb{N}$ we conclude that Γ_m fulfils all requirements of the proposition. Assume that $m \in \mathbb{Z}, m > N$. We will show that the group Γ_m does not act properly discontinuously.
Indeed, since the induced representation $l(G_m)$ on V_2/V_1 is trivial it follows from 2.1 $(*)$, that the affine subspace $\overline{V}_1 = V_1 + 0$ is G_{m+1} invariant. Denote by $\overline{\Gamma}$ (resp. $\overline{G}$) the restriction of Γ_{m+1} (resp. G_{m+1}) to $\overline{V}$.

If $V_0 = 0$ then for every regular element $\gamma \in \Gamma_{m+1}$ there exists a fixed point q_γ. Hence Γ_{m+1} does not act properly discontinuously. Since $\Gamma_{m+1} \leq \Gamma$ the group Γ does not act properly discontinuously. Assume that $V_0 \neq 0$. Obviously $\overline{\Gamma}$ and $\overline{G}$ fulfil the hypotheses of Proposition 4.5. Hence $\overline{\Gamma}$ does not act properly discontinuously. Hence by the same argument as above we conclude that Γ does not act properly discontinuously. This proves the proposition.

5 The Auslander conjecture in dimensions 4 and 5

5.1. In this section we will prove the Auslander conjecture in dimensions 4 and 5.

Let Γ be a discrete subgroup of an affine group and let G be the Zariski closure of Γ. It follows from Remark 2.8 that the semisimple part S of G is a simple group. Let us now recall the list of all possible cases for $n = \dim V \leq 6$. It follows from 3.5 that all possible cases are

$$S = SL_l(\mathbb{R}), V_1 = \mathbb{R}^l, 3 \leq l \leq 5, l < n, 4 \leq n \leq 6 \qquad (s_1)$$

$$S = Sp_4(\mathbb{R}), V_1 = \mathbb{R}^4, SO(3,2), n = 5, 6 \qquad (s_2)$$

We will deal with the case $S = SO(3,2), \dim V = 6$, in the next chapter. Therefore in the next proposition we will assume that if $S = SO(3,2)$ then $\dim V = 5$.

Proposition 5.2. *Let Γ be a discrete subgroup of an affine group and let G be the Zariski closure of Γ. Assume that the simple part S of G is as in (s_1) or (s_2). Then the group Γ does not act properly discontinuously..*

Proof. ***Case 1*** $S = SO(3,2), \dim V = 5$. By Theorem B [AMS3] the group Γ does not act properly discontinuously.

Case 2 $S \neq SO(3,2)$. Obviously S fulfils all requirements of Proposition 2.10. Thus Γ does not act properly discontinuously. This proves the proposition.

Proposition 5.3. *Let $\Gamma \subseteq Aff(\mathbb{R}^n), n \leq 5$ be a crystallographic group. Then Γ is virtually solvable.*

Proof Let G be the Zariski closure of Γ. Since the connected component G^0 is a finite index subgroup of G we conclude that $\Gamma \cap G^0$ is a finite index subgroup of Γ. Clearly $\Gamma \cap G^0$ is a crystallographic group. Thus we shall and will assume that the group G is connected. As we explained above , the group $l(G)$ is an $\boldsymbol{A}$–group. Assume that $l(G)$ has a non-trivial semisimple part S. By 5.1 for $n \leq 5$ the group S is a simple group listed in (s_1) or (s_2). Thus by Proposition 5.2 Γ does not act properly discontinuously. Therefore, the semisimple part S is trivial. Hence the group Γ is virtually solvable.

6 The Auslander conjecture in dimension 6. The cohomological argument.

We start use the same notations as in Chapter 5. The goal of this chapter is to show that if Γ is a crystallographic group and $\dim V = 6$ then the semisimple part of the Zariksi closure of Γ can not be one of the groups listed in Case 1 and Case 2 (1), (2). The groups of Case 2 (3) will be dealt with in the next section 7. We will start with the following

Proposition 6.1. *Let Γ be an affine group and let G be the Zariski closure of Γ. Assume that the semisimple part S of G is as in the Case 1 (1), (2). Then the group Γ does not act properly discontinuously*

Proof. The proof follows immediately from Proposition 4.5.

Proposition 6.2 *Let Γ be an affine group and let G be the Zariski closure of Γ. Assume that the semisimple part S of G is as in Case 1 (3) $S = SL_2(\mathbb{R}) \times SL_3(\mathbb{R})$. Then the group Γ does not act properly discontinuously.*

Proof. We have a chain $0 \subseteq W_0 \subset W_1 \subseteq W_2 = V$ of $l(G)$-invariant subspaces. There are three possible cases

(i) $\dim W_0 = 1$,

(ii) $\dim W_1/W_0 = 1$,

(iii) $\dim W_2/W_1 = 1$.

Cases (i) and (iii). It follows from (A3) that in case (i) we have

$l(G)|_{W_0} = 1$ and $\dim W_2/W_1 = 0$. In case (iii) the induced representation $l(G) \to GL(W_2/W_1)$ is trivial and $\dim W_0 = 0$. Hence Γ does not act properly discontinuously by Proposition 4.5.

Cases (ii). The induced representation $l(G) \to GL(W_1/W_0)$ is trivial as follows again

from (A3). Roughly speaking the space of S-fixed vectors is "in between". Set $U_0 = W_0$. There exist S–invariant spaces U_1 and U_2 such that $W_1 = U_0 \oplus U_1$, and $V = U_0 \oplus U_1 \oplus U_2$,

We will prove the statement of the proposition assuming that $S|_{U_0} = SL_3(\mathbb{R})$, $S|_{U_1} = I$ and $S|_{U_2} = SL_2(\mathbb{R})$. The proof in case $S|_{W_0} = SL_2(\mathbb{R})$, $S|_{U_1} = I$ and $S|_{U_2} = SL_3(\mathbb{R})$ is a verbatim repetition.

There exists a $g \in \Gamma$ such that $l(g)$ is an $\mathbb{R}$–regular element in $l(G)$ ([AMS1], [P]). We can and will assume that $l(g) \in S$. Let $g_0 = l(g)|_{U_0} \in SL_3(\mathbb{R})$, $g_1 = l(g)|_{U_1} = 1$ and $g_2 = l(g)|_{U_2} \in SL_2(\mathbb{R})$. We can assume that $\dim A^-(g_0) < \dim A^+(g_0)$. Thus $\dim A^+(g_0) = 2$. Note that $\dim A^+(g_2) = 1$ and $A^0(g) = U_1$. Let U be a one dimensional $l(g)$–invariant subspace of $A^+(g_0)$. Then there exists $t \in S$ such that $l(t)U \notin A^+(g_0) \cup A^-(g_0)$ and $l(t)A^+(g_2) \notin A^+(g_2) \cup A^-(g_2)$ and $l(t)U \oplus A^+(g_0) = U_0$ and $l(t)A^+(g_2) \oplus A^+(g_2) = U_2$. Set $A(t) = l(t)U + l(t)A^+(g_2)$. Then $A(t) \oplus D^+(g) = V$ since $D^+(g) = A^+(g_0) + U_1 + A^+(g_2)$ Let $\sigma : G \to S$ be the projection. Clearly $\sigma(\Gamma)$ is Zariski dense in S. Therefore we can and will assume the $t \in \Gamma$. Put $h = tg^{-1}t^{-1} \in \Gamma$. Clearly $A(t) \subseteq A^-(h)$. Remark, that $0 \neq u \in U$ is an eigenvector of h but not an eigenvector of g. Therefore $h^n \neq g^m$ for all $n, m \in \mathbb{Z}, n \neq 0, m \neq 0$. Let $A = l(t)U_1 + A(t), A \subseteq D^-(h)$ and let $D = A + L_h$. Clearly, D, is an h–invariant affine space in E_h^- and $\dim D \cap E_g^+ = 1$. Let $L = D \cap E_g^+$. We have the projections $\pi_1 : \mathbb{R}^6 \longrightarrow L_g$ of an affine space $\mathbb{R}^6$ onto L_g along $A^+(g) + A^-(g)$

and $\pi_2 : \mathbb{R}^6 \longrightarrow L_h$ along $A^+(h) + A^-(h)$. The restriction $\overline{\pi}_i = \pi_i|_L, i = 1, 2$ is an affine isomorphism. Set $\theta = \overline{\pi}_2^{-1} \circ \overline{\pi}_1$. Then $\theta : L_g \longrightarrow L_h$ is an affine isomorphism. Since $g_1 = 1$ we conclude $l(t)t_g - t_g \in U_0$. Combining this with chapter 4.2, (**) we obtain $\theta(t_g) = -t_h$.

Let $p_1 \in L_g$ and let $p_2 \in L_h$. There exists a point $p \in L$ such that the vector $\overrightarrow{pp_1} \in A^+(g)$ and $\overrightarrow{pp_2} \in A^-(h)$. Consider a ball $U_1(p_1)$of radius 1 and the center at p_1 and a ball $U_1(p_2)$of radius 1 and the center at p_2. there exists a point $q \in L$ such that the vector $\pi_1(\overrightarrow{pq}) = t_g$. Set $x_k = p + kv, k \in \mathbb{N}, k > 0$. Then there exists a positive $N, N \in \mathbb{Z}$ such that for $m > N$ we have $g^{-m}x_m \in U_1(p_1)$ and $h^m x_m \in U_1(p_2)$. As in the proof of Proposition 2.9 we conclude that for $m > N$ we have $h^m g^m U_1(p_1) \cap U_1(p_2) \neq \varnothing$. Since $h^n \neq g^m$ for all $n, m \in \mathbb{Z}, n \neq 0, m \neq 0$ the group Γ does not act properly discontinuously.

Proposition 6.3. *Let Γ be an affine group and let G be the Zariski closure of Γ. Assume that G is connected and the semisimple part S of G is as in Case 2 (1), (2). Then the group Γ is not a crystallographic group.*

Proof . Let us first explain the main idea of the proof. Since the subgroup $\Gamma \subseteq G_n$ is a crystallographic group, the virtual cohomological dimension $vcd(\Gamma)$ of Γ is $\dim \mathbb{R}^n = n$. Hence $vcd(\Gamma) = 6$. As a first step we will show that $vcd(\Gamma) \leq \dim(S/K)$, where S/K is the symmetric space of S. Then we compare $\dim S/K$ and $vcd(\Gamma)$ in the cases $S = SO(3) \times SL_3(\mathbb{R})$, $S = SO(3, 2)$ and come to the conclusion that $\dim S/K \geq vcd(\Gamma)$. This will lead to a contradiction. We actually show that the projection $p : G \to S$ restricts to an isomorphism of Γ onto a discrete subgroup of S. In case $S = SO(3) \times SL_3(\mathbb{R})$ the dimension of S/K is 5 and so $vcd(\Gamma) \leq dim(S/K)$ is impossible. In case $S = SO(3, 2)$ the dimension of S/K is 6 and so $p(\Gamma)$ would be a cocompact lattice in S and we will get a contradiction using the Margulis rigidity theorem.

Let us first show that $vcd(\Gamma) \leq \dim(S/K)$. Let R be the solvable radical and U be the unipotent radical of G. Recall that G acts trivially on the factor-group R/U. Thus it is easy to see that in Case 2 (2) we have $R = U$. Let $\Gamma_r = R \cap \Gamma$ and let R_1 be the Zariski closure of Γ_r. Then the group R_1 is a normal solvable subgroup in G since Γ_r is a normal solvable subgroup in Γ. Let T_1 be a maximal torus and let U_1 be the unipotent radical of R_1. Since $\tilde{S} = ST_1$ is a reductive subgroup of G there exists a point q_0 such that $\tilde{S}q_0 = q_0$ [see (2.1)]. Set $W = R_1 q_0$. Since $T_1 q_0 = q_0$ we conclude that $W = U_1 q_0$. For every $g \in R_1$ there are unique elements $t \in T_1$ and $u \in U_1$ such that $g = tu$. Define a map $\pi : R_1 \to U$ by $\pi(g) = u$. Then $\pi(\Gamma_r)$ contains a uniform lattice of U_1 [S2, Proposition 2]. Since $W = U_1 q_0$ we conclude that $\Gamma_r \setminus W$ is compact.

Since $sq_0 = q_0$ and R_1 is a normal subgroup of G then obviously $sW = W$ for every $s \in S$. Let $\rho : S \to GL(T_{q_0})$ be the representation of S on the tangent space T_{q_0} of W at q_0. It is clear that the only possible numbers for $\dim(T_{q_0})$ are $\{0, 3, 6\}$ if $S = SO(3) \times SL_3(\mathbb{R})$ and $\{0, 1, 5, 6\}$ if $S = SO(3, 2)$. Let us show that in each case $\dim(T_{q_0}) = 0$. Assume that $\dim(T_{q_0}) = 6$. Then $W = R_1 q_0 = \mathbb{R}^6$. As we show above $\Gamma_r \setminus W$ is compact. Thus Γ_r is a crystallographic group. On the other hand Γ_r is a subgroup of a crystallographic group Γ which acts on the same affine space. Thus the index $|\Gamma/\Gamma_r|$ is finite. On the other hand the index $|\Gamma/\Gamma_r|$ is infinite. Otherwise the Zariski closure of the solvable group Γ_r would contain the connected component of the Zariski closure of Γ which is impossible. We thus have shown that $\dim(T_{q_0}) < 6$.

We will treat the two cases $S = SO(3) \times SL_3(\mathbb{R})$ and $SO(3, 2)$ separately.

Let $S = SO(3) \times SL_3(\mathbb{R})$ and $\dim(T_{q_0}) = 3$. Then G is a subgroup of the following group $\widetilde{G} = \{X : X \in GL_7(\mathbb{R})\}$, where

$$X = \begin{pmatrix} A & B & v_1 \\ 0 & C & v_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where $A \in SO(3), C \in SL_3(\mathbb{R}), v_1, v_2 \in \mathbb{R}^3$, or

$$X = \begin{pmatrix} A & B & v_1 \\ 0 & C & v_2 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2)$$

where $A \in SL_3(\mathbb{R}), C \in SO(3), B \in M_3(\mathbb{R}), v_1, v_2 \in \mathbb{R}^3$.

We claim that $\dim T_{q_0} = 0$. We will prove this for (1). The proof for (2) will go along the same lines.

Since the semisimple part of a group has to commute with one maximal reductive subgroup of its solvable radical the solvable radical of G is unipotent. Therefore for a $X \in R$ we have

$$X = \begin{pmatrix} I_3 & B(X) & v_1(X) \\ 0 & I_3 & v_2(X) \\ 0 & 0 & 1 \end{pmatrix}.$$

Assume that there is an element X of the unipoten U radical of G such that $B(X) \neq 0$. Since $l(U)$ is a normal subgroup of $l(G)$ by direct calculations we show that for every $B \in M_3(\mathbb{R})$ there exists $X \in U$ such that

$$l(X) = \begin{pmatrix} I_3 & B \\ 0 & I_3 \end{pmatrix} \qquad (3).$$

Otherwise $B(X) = 0$ for every element $X \in U$.

The unipotent group R_1 is a normal connected subgroup of G and $R_1 \leq U$. There are three connected proper nontrivial normal unipotent subgroups of G, namely,
$R_1 = \{X \in U, v_2(X) = 0\}$, $R_1 = \{X \in U, B(X) = 0, , v_2(X) = 0\}$ and
$R_1 = \{X \in U, B(X) = 0, v_1(X) = 0\}$. We conclude that in these cases $W = R_1 q_0$ is an affine G–invariant subspace. Thus we have a nontrivial G–invariant affine space W where Γ and Γ_r act as crystallographic groups. By the same argument we used in case $\dim T_{q_0} = 6$ we conclude that the subgroup R_1 is trivial. By Auslander's theorem [R], $\pi_S(\Gamma)$ is a

discrete subgroup of S. Since the intersection $\Gamma \cap R$ is trivial, $\pi_S(\Gamma)$ and Γ are isomorphic. Hence $vcd(\Gamma) = vcd(\pi_S(\Gamma)) \leq \dim S/K$, where K is a maximal compact subgroup in S. Thus $vcd(\Gamma) \leq 5$. On the other hand, $vcd(\Gamma) = 6$, a contradiction.

Let us now show that Case 2 (1) is also impossible. We will use the notation introduced in 3.5. Recall that $V = V_0 \oplus V_1$ where the restriction $S|_{V_0}$ gives a trivial representation and the restriction $S|_{V_1} = SO(3,2)$. Assume that V_1 is $l(G)$-invariant. Then it follows from the linear representation(*) in 2.1, that the affine space $V_1 + q_0$ is Γ_1 -invariant, where $\Gamma_1 = [\Gamma, \Gamma]$. Obviously $\dim V_1 = 5$ and $l(\Gamma_1)|_{V_1} \leq SO(3,2)$. It follows from Proposition 4.2, case 1, that Γ_1 does not act properly discontinuously on $V_1 + q_0$. Therefore Γ is not a crystallographic group. Thus we can and will assume that V_0 is $l(G)$ invariant. We will prove first that $\dim W = 0$. Recall that $W = R_1 q_0$ and $l(G)q_0 = q_0$. We have the following matrix representation of G. Let $X \in G$ then

$$X = \begin{pmatrix} \lambda(X) & w(X) & a(X) \\ 0 & A(X) & v(X) \\ 0 & 0 & 1 \end{pmatrix},$$

where $A(X) \in SO(3,2), w(X), v(X) \in \mathbb{R}^5, \lambda(X), a(X) \in \mathbb{R}$. As we concluded above, there are three possible cases for $\dim W$, namely, $\dim W = 0, 1, 5$. Our goal is to show $\dim W \neq 1, 5$.

Assume that $\dim W = 1$. The representation ρ of S on T_{q_0} is trivial. Clearly, $S = SO(3,2)$ is an irreducible subgroup of $GL(\mathbb{R}^5)$. Therefore we conclude that if X is an element in the normal subgroup R_1 of G, then $v(X) = 0$. Thus for every $X \in R_1$ we have

$$X = \begin{pmatrix} 1 & w & a \\ 0 & I_5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $w \in \mathbb{R}^5, a \in \mathbb{R}$. Hence W is an affine $\Gamma-$ invariant subspace in $\mathbb{R}^6$. Therefore we have a natural homomorphism $\theta : \Gamma \longrightarrow \text{Aff}(\mathbb{R}^6/W)$. By [S2, Lemma 4], $\Gamma/\Gamma_r = \theta(\Gamma)$ is a crystallographic subgroup in $\mathrm{Aff}(\mathbb{R}^6/W)$. Obviously the semisimple part of the Zariski closure of $\theta(\Gamma)$ is $SO(3,2)$ and $\mathbb{R}^6/W = \mathbb{R}^5$. By [AMS3, Theorem A] this is impossible.

Assume that $\dim W = 5$. Again consider the space of orbits $\widehat{R} = \{gW, g \in G\}$. Recall that the unipotent radical U acts transitively on $\widehat{R}$ [S2]. It is clear that $\widehat{R}$ is a one dimensional manifold. As in [S2] we have a representation ρ of $l(G)$ on the tangent space T_W of $\widehat{R}$ at W. We show in [S2, Theorem A] that one is an eigenvalue of $\rho(g)$ for every element $g \in l(G)$. Hence the representation ρ is trivial. Note that this implies that $\lambda(X) = 1$ for every $X \in R_1$. Thus R_1 is a unipotent group. Since $\dim W = 5$ there exists an $X \in R_1$ such that $v(X) \neq 0$. On the other hand looking at the representation of $SO(3,2)$ on R_1 we conclude that if there exists $X \in R_1$ such that $v(X) \neq 0$ then for every $v \in \mathbb{R}^5$ there exists $X \in R_1$ such that $v(X) = v$. Therefore for every $g \in \Gamma$ there exists $r \in R_1$ such that for $\widehat{g} = gr^{-1}$ we have $v(\widehat{g}) = 0$. Obviously

$$\widehat{g} = \begin{pmatrix} 1 & w(\widehat{g}) & a(\widehat{g}) \\ 0 & S(\widehat{g}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (4).$$

Let $g_1, g_2 \in \Gamma$ be two elements on Γ. There exist r_1, r_2 such that for $\widehat{g}_i r_i = g_i, i = 1,2$ we have $v(\widehat{g}_i) = 0$. Clearly, we have $[\widehat{g}_1, \widehat{g}_2] \in l(G)$ and $[g_1, g_2] = [\widehat{g}_1, \widehat{g}_2]r_0$ where $r_0 \in R_1$. Therefore $[g_1, g_2]W = [\widehat{g}_1, \widehat{g}_2]r_0 R_1 q_0 = R_1 q_0 = W$. Set $\Gamma/\Gamma_r = \theta(\Gamma)$. By [S2, Lemma 4], $\Gamma/\Gamma_r = \theta(\Gamma)$ acts as a crystallographic group on $\widehat{R}$. Therefore $Stab_{\theta(\Gamma)}(W)$ is a finite set. From $[\Gamma, \Gamma]W = W$ and $\Gamma_r W = W$ follows that $[\theta(\Gamma), \theta(\Gamma)] \leq Stab_{\theta(\Gamma)}(W)$. So the group $[\theta(\Gamma), \theta(\Gamma)]$ is finite. Consequently $\theta(\Gamma)$ is a virtually abelian group. Therefore Γ is a virtually solvable group. This is a contradiction. Thus we conclude that $W = 0$. Hence $R_1 = \{e\}$ and the restriction of the homomorphism $\pi_S : G \longrightarrow S = G/R$ onto Γ is an isomorphism. By Auslander's theorem [R], the projection $\pi_S(\Gamma)$ is a discrete subgroup in S and $vcd(\pi_S(\Gamma)) = vcd(\Gamma) = 6$. On the other hand $vcd(\pi_S(\Gamma)) \leq \dim S/K$, where K is a maximal subgroup in S. Obviously, $\dim S/K = 6$. Hence $vcd(\pi_S(\Gamma)) = \dim S/K$.

Therefore $\pi_S(\Gamma)$ is a co-compact lattice in S. We can apply the Margulis rigidity theorem, since $rank_{\mathbb{R}}(S) = 2$ and conclude that there exists a $g \in \Gamma$ such that $\tilde{\Gamma} = g\Gamma g^{-1} \cap S$ is a subgroup of finite index in Γ. Since $\tilde{\Gamma} \leqslant S$ we have $\tilde{\Gamma} p_0 = p_0$. Thus Γ does not act properly discontinuously. Hence Γ is not a crystallographic

Remark A more sophisticated arguments based on dynamical ideas and results from

[AMS4] enable one to prove that Γ does not act properly discontinuously under the assumption of Proposition 6.3.

7 The Auslander conjecture in dimension 6*. Dynamical arguments*

7.1. Orientation. The dynamical approach we have used in [AMS3] and will use here

is based on the so called ***Margulis sign*** of an affine transformation. The case $S = SO(2,1) \times SL_3(\mathbb{R})$ needs other tools, namely a new version of the Margulis sign. We will need to introduce it for the natural representation of S which goes roughly saying by ignoring the $SL_3(\mathbb{R})$–factor. We then have a lemma similar to the cases of $SO(k+1,k)$, namely lemma 7.7, which says that if a group acts properly discontinuously, then opposite signs are impossible.

Now we will recall the important definition of the sign of an affine transformation. This definition was first introduced by G. Margulis [M] for $n = 3$. Then it was generalized in [AMS3] for the case in which the signature of the quadratic form is $(k+1,k)$ and finally for an arbitrary quadratic form in [AMS4]. Our presentation will follow along the lines of [AMS4]. Let B be a quadratic form of signature (p,q), $p \geq q$, $p+q = n$. Let v be a vector in $\mathbb{R}^n$, $v = x_1v_1 + \cdots + x_pv_p + y_1w_1 + \cdots + y_qw_q$, where $v_1, v_2, \ldots, v_p$, $w_1, w_2, \ldots, w_q$

is a basis of $\mathbb{R}^n$. We can and will assume that

$$B(v,v) = x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - y_q^2.$$

Consider the set Φ of all maximal B–isotropic subspaces. Let X be the subspace spanned by $\{v_1, v_2, \ldots, v_p\}$ and Y be the subspace spanned by $\{w_1, w_2, \ldots, w_q\}$. It is clear that $\mathbb{R}^n = X \oplus Y$. Define the cone

$$Con_B = \{v \in \mathbb{R}^n : B(v,v) < 0\}.$$

Clearly $Y \backslash \{0\} \subset Con_B$. We have the two projections

$$\pi_X : \mathbb{R}^n \longrightarrow X \text{ and } \pi_{\mathrm{Y}} : \mathbb{R}^{\mathrm{n}} \longrightarrow \mathrm{Y}$$

along Y and X, respectively. The restriction of π_Y to $W \in \Phi$ is a linear isomorphism $W \longrightarrow Y$. Hence if we fix an orientation on Y, then we have also fixed an orientation on each $W \in \Phi$. For $W \in \Phi$, let us denote the B–orthogonal subspace of W by $W^\perp = \{z \in \mathbb{R}^n \ ; \ B(z, W) = 0\}$. We have $W \subset W^\perp$ since W is B–isotropic. We also have

$$\dim W^\perp = \dim W + (p - q) = p.$$

The restriction of π_X to $W^\perp$ is a linear isomorphism $W^\perp \longrightarrow X$. Hence if we fix an orientation on X, then we have also fixed an orientation on $W^\perp$ for each $W \in \Phi$. Thus we have orientations on both W and $W^\perp$ and we have naturally induced an orientation on any subspace $\widehat{W}$, such that $W^\perp = W \oplus \widehat{W}$. If $V_1 \in \Phi$ and $V_2 \in \Phi$ are transversal, then $V_0 = V_1^\perp \cap V_2^\perp$ is a subspace that is transversal to both V_1 and V_2; therefore $V_0 \oplus V_1 = V_1^\perp$ and $V_0 \oplus V_2 = V_2^\perp$. So there are two orientations ω_1 and ω_2 on V_0, where ω_i is defined if we consider V_0 as a subspace in $V_i^\perp$. We have [see AMS3, Lemma 2.1]

Lemma 7.2. *The orientations defined above on V_0 are the same if q is even and they are opposite if q is odd, i.e. $\omega_1 = (-1)^q \omega_2$.*

Example 7.3. Let $p = k+1, q = k$. Let V_1 and V_2 be the maximal isotropic subspaces spanned by the vectors $\{w_1 + v_1, \dots, w_k + v_k\}$ and $\{w_1 - v_1, \dots, w_k - v_k\}$ respectively. Since for every $i = 1, \dots, k$ we have $\pi_Y(w_i \pm v_i)) = w_i, i = 1, \dots, k$, we conclude that $w_1 + v_1, \dots, w_k + v_k$ (resp. $w_1 - v_1, \dots, w_k - v_k$) is a positively oriented basis of V_1 (resp. V_2). Then $V_1^\perp \cap V_2^\perp$ is spanned by the vector v_{k+1}. Let $v^0(V_1^\perp) \in V_1^\perp \cap V_2^\perp$ and $v^0(V_2^\perp) \in V_1^\perp \cap V_2^\perp$ such that $\{w_1 + v_1, \dots, w_k + v_k, v^0(V_1^\perp)\}$ (resp. $\{w_1 - v_1, \dots, w_k - v_k, v^0(V_2^\perp)\}$) is a positively oriented basis of $V_1^\perp$ (resp. $V_2^\perp$.) We have $v^0(V_1^\perp) = (-1)^k v^0(V_2^\perp)$ since $\pi_X(w_i + v_i) = v_i$ and $\pi_X(w_i - v_i) = -v_i$ for all $i, i = 1, \dots, k$. In particular, $v^0(V_1^\perp) = -v^0(V_2^\perp)$ when $k = 1$.

7.4 Margulis's sign. Let us recall now the definition of the Margulis sign (or for short *sign*) of an affine element [AMS3]. Let $g \in \mathrm{Aff}\mathbb{R}^n$ be an $\mathbb{R}$-regular element with $l(g) \in SO(B)$ where B is a non-degenerate form on $\mathbb{R}^n$ of signature $(k+1, k)$. Note, that in this case $l(g)$ is a regular element of $SO(B)$. Obviously, the subspaces $A^+(g)$ and $A^-(g)$ are maximal B–isotropic subspaces, $D^+(g) = A^+(g)^\perp$, $D^-(g) = A^-(g)^\perp$ and $\dim A^0(g) = 1$. Following the procedure above for the element g we choose and fix a vector $v_+ \in A^0(g)$ with the following property $B(v_+, v^0(D^+(g))) > 0$ Thus we fix an orientation on this line by the choice of the orientation on $A^+(g)$ and $D^+(g)$. Likewise we fix an orientation on $A^0(g^{-1})$ by the choice of the orientations on $A^+(g^{-1}) = A^-(g)$ and $D^+(g^{-1}) = D^-(g)$. We will denote the corresponding vector in $A^0(g^{-1})$ by v_-. Recall that $A^0(g) = A^0(g^{-1})$.Therefore we have two orientations on the same one-dimensional space.

Let $q \in \mathbb{R}^n$. Set

$$\alpha(g) = B(gq - q, v_+)/B(v_+, v_+)^{1/2}.$$

It is clear that $\alpha(g)$ does not depend on the point $q \in \mathbb{R}^n$ and we have $\alpha(g) = \alpha(x^{-1}gx)$ for every $x \in \mathrm{Aff}\mathbb{R}^n$ such that $l(x) \in SO(B)$. Consider now any $\mathbb{R}$-regular element g and let us show that $\alpha(g^{-1}) = (-1)^k \alpha(g)$. Indeed by Example 6.3, $v^0(D^+(g^{-1})) = v^0(D^-(g)) = (-1)^k v^0(D^+(g))$. Hence $v_- = (-1)^k v_+$. We have $\alpha(g^{-1}) = B(g^{-1}q - q, v_-)/B(v_-, v_-)^{1/2}$

$= (-1)^k B(g^{-1}q - q, v_+)/B(v_+, v_+)^{1/2} = (-1)^{k+1} B(q - g^{-1}q, v_+)/B(v_+, v_+)^{1/2}$. Put $p = g^{-1}q$. Hence $\alpha(g^{-1}) = (-1)^{k+1}\alpha(g)$. Note that $\alpha(g) = \alpha(g^{-1})$ if $k = 1$. We call $\alpha(g)$ the

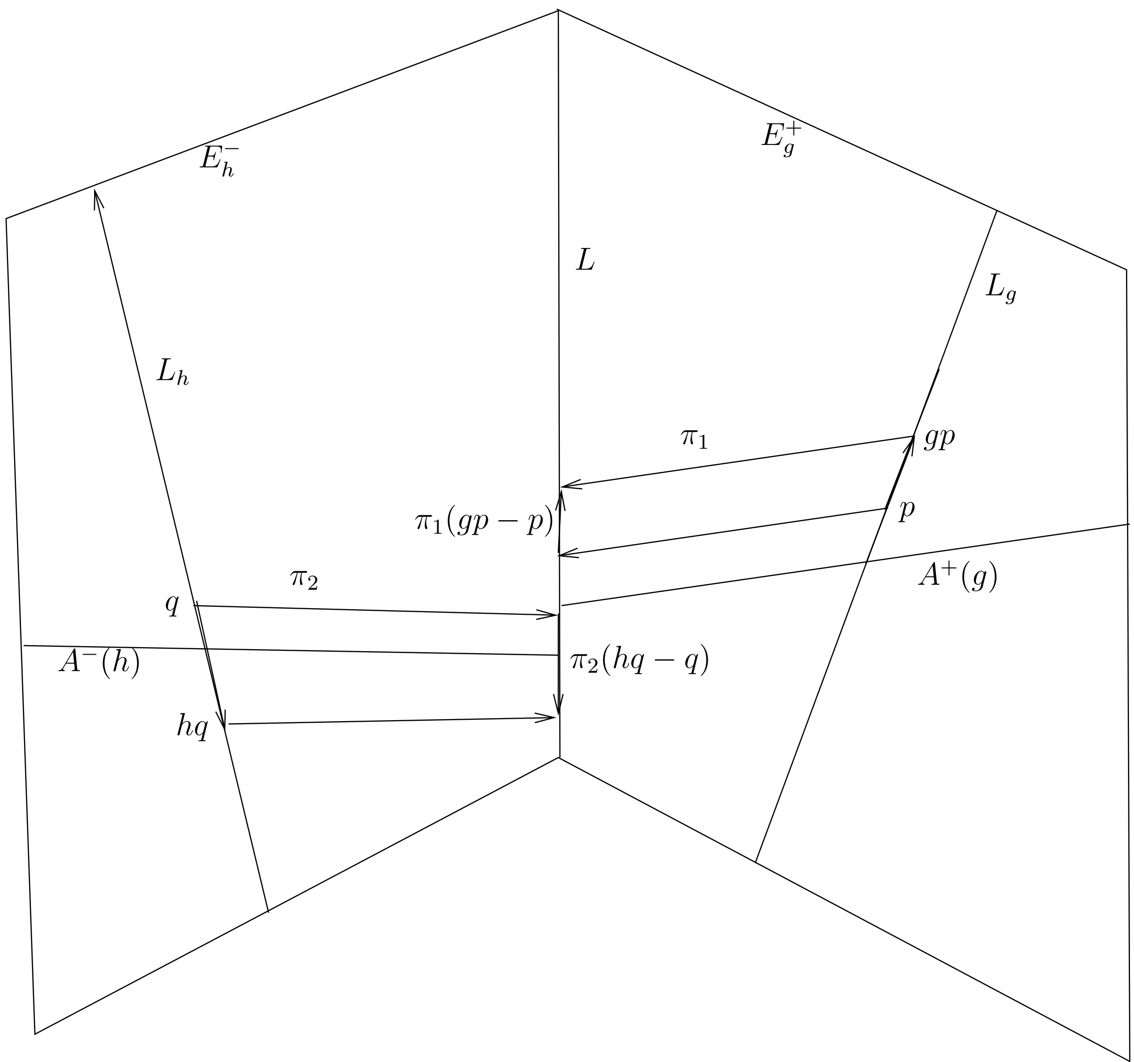

Figure 2: Positive and negative parts, illustration 1

sign of g. Although $\alpha(g)$ is a non-zero real number, since we are only interested in its sign and not in its absolute value we call $\alpha(g)$ a sign.

From now on unless otherwise stated we assume that the semisimple part S of the

Zariski closure G of Γ is $SO(2,1) \times SL_3(\mathbb{R})$ Clearly, that $V = V_1 \oplus V_2$, $S|_{V_1} = SO(2,1)$ and $S|_{V_2} = SL_3(\mathbb{R})$. Hence we have two natural homomorphisms: $\theta_1 : G \to SO(2,1) \subseteq GL(V_1)$ and $\theta_2 : G \to SL_3(\mathbb{R}) \subseteq GL(V_2)$. It is easy to see that the unipotent radical of G is an abelian group. We will also assume that our standard inner product (see 3.2) is chosen so that the subspaces V_1 and V_2 are orthogonal. Let $g \in S$ be a regular element. Let $\widehat{g}$ be the restriction $g|_{V_1}$. Obviously $A^0(g) = A^0(\widehat{g})$. We set $v_g = v_+/B(v_+, v_+)^{1/2}$. Let $g \in G$ be a regular element. There exists a unique $u \in G$ such that $l(h) = l(ugu^{-1}) \in S$. Set $v_h = l(u)(v_g)$. There is a simple geometrical explanation of this definition. Let $\pi : V \longrightarrow V_1$ be the natural projection onto V_1 along V_2. We have the corresponding homomorphism $\widehat{\pi} : G \longrightarrow SO(2,1)$. It is easy to see that the restriction of π to $A^0(g)$ gives an isomorphism onto $A^0(\widehat{\pi}(g))$ and $\pi(v_g) = v_{\widehat{\pi}(g)}$. Let $\tau_g : V \longrightarrow L_g$ be the natural projection of the affine space V onto the line L_g along the subspace $A^+(g) \oplus A^-(g)$, where g is a regular element . There exists a unique $\alpha \in \mathbb{R}$ such that $\tau_g(p) - p = \alpha v_g$. We set $\alpha(g) = \alpha$. Clearly, since $\pi(v_g) = v_{\widehat{\pi}(g)}$ and B is the form of signature $(2,1)$ on V_1 fixed by $SO(2,1)$, we have $\alpha(g) = B(\pi(\tau_g(p) - p), \pi(v_g)) = \alpha(\widehat{\pi}(g))$. Obviously $\alpha(g)$ does not depend on the chosen point therefore we have $\alpha(g^{-1}) = \alpha(g)$ and $\alpha(g^n) = |n|\alpha(g)$. It is clear that, $\alpha(g) = \alpha(hgh^{-1})$ for any $h \in G$. For more details see [AMS4, p.5].

A regular element $g \in G$ is called hyperbolic, if $\theta_1(g)$ and $\theta_2(g)$ are hyperbolic. Let us now explain the main application of the sign. Let g and h be two hyperbolic transversal elements. Then $A^-(h) \oplus D^+(g) = V$ and $\dim(D^-(h) \cap D^+(g)) = 1$. Let $V_{g,h} = D^-(h) \cap D^+(g)$. Clearly,the line $L = E^+_g \cap E^-_h$ is parallel to $V_{g,h}$. Let $\pi_1 : L_g \longrightarrow L$ be the projection of L_g onto L along $A^+(g)$ and let $\pi_2 : L_h \longrightarrow L$. be the projection of L_h onto L along $A^-(h)$ (see Fig.2). By the above arguments for $p \in L_g, q \in L_h$ the vectors $\pi_2(hq - q)$ and $\pi_1(gp - p)$ have opposite directions if $\alpha(g)\alpha(h) < 0$. Then as in the proof of Theorem A [AMS3], we conclude that there exist infinitely many positive numbers n, m and two balls $B(p,1)$ and $B(q,1)$ such that $h^m g^n B(p,1) \bigcap B(q,1) \neq \emptyset$. Thus we conclude

Lemma 7.5. *If there exist two hyperbolic transversal elements g and h of Γ such that $\alpha(g)\alpha(h) < 0$ then Γ does not act properly discontinuously.*

7.6 Let v_1, v_2, w_1 be a basis of V_1 such that for any vector $v \in V_1, v = x_1v_1 + x_2v_2 + y_1w_1$ we have $B(v, v) = x_1^2 + x_2^2 - y_1^2$ and $(v, v) = x_1^2 + x_2^2 + y_1^2$. We will use the notations and definitions from 7.1. Let ∂Con_B be the boundary of Con_B. Let U (see Fig.3) be a maximal B-isotropic subspace of V_1 and let v be the vector of U such that $\pi_Y(v) = w_1$. Clearly, U is spanned by v. Let v_0 be the vector in $U^\perp \cap X$ such that $B(v_0, v_0) = 1$ and the basis $\pi_X(v), v_0$ has the same orientation as v_1, v_2. Let W be a maximal B-isotropic subspace of V_1 and suppose $W \neq U$. Then $\dim(U^\perp \cap W^\perp) = 1$. There exists a unique vector $w_0(W)$ in $U^\perp \cap W^\perp$ and $\widehat{v} \in U$ such that $w_0(W) = v_0 + \widehat{v}$. Obviously there exists a unique number $\alpha(W)$ such that $\widehat{v} = \alpha(W)v$. Set $\Phi_U^+ = \{W \in \Phi |\, \alpha(W) > 0\}$ and $\Phi_U^- = \{W \in \Phi |\, \alpha(W) < 0\}$. We have $B(v_0, w_1) = 0$ since $v_0 \in X$. Therefore $B(w_0(W)), w_1) = \alpha(W)B(v, w_1) = -\alpha(W)$. Let $\widehat{U}$ be the sum of the two subspaces U and $< w_1 >$. Then Φ_U^+ and Φ_U^- are two different connected components of the set $\partial Con_B \backslash \widehat{U}$. Obviously $\partial Con_B \backslash \widehat{U} = \Phi_U^+ \cup \Phi_U^-$. We conclude :

(1) For every $W \in \Phi_U^+$ (resp. $W \in \Phi_U^-$) we have $B(w_0(W), w_1) < 0$ (resp. $B(w_0(W), w_1) > 0$).

(2) Let W_1, W_2, W_3, W_4 be maximal B–isotropic subspaces of V_1 such that $w_1 \in (W_1 + W_2) \bigcap (W_3 + W_4)$ and $\{W_1, W_2, W_3, W_4\} \subset \partial Con_B \backslash \widehat{U}$. It is easy to see that W_1 and W_2 belong to different connected components of the set $\partial Con_B \setminus \widehat{U}$. Indeed, since $w_1 \in W_1 + W_2$ we have $\alpha(W_1) = -\alpha(W_2)$. The same is true for W_3, W_4.

(2_a) It follows from (2) that for every maximal B–isotropic subspace U of V_1 if $W_i \in \Phi_U^\pm$ then $W_{i+1} \in \Phi_U^\mp$ where $i = 1, 3$.

(2_b) Let $d = \min_{1 \leq i \neq j \leq 4} \{d(W_i, W_j)\}$. Let U be a maximal B–isotropic subspace of V. It follows from (2_a) that there exists $\delta = \delta(d)$ such that for every four maximal B–isotropic subspaces $\widehat{W_i}, i = 1, 2, 3, 4$ of V with $d(\widehat{W_i}, W_i) \leq \delta$ for $1 \leq i \leq 4$ there exists an $i_0 \in$

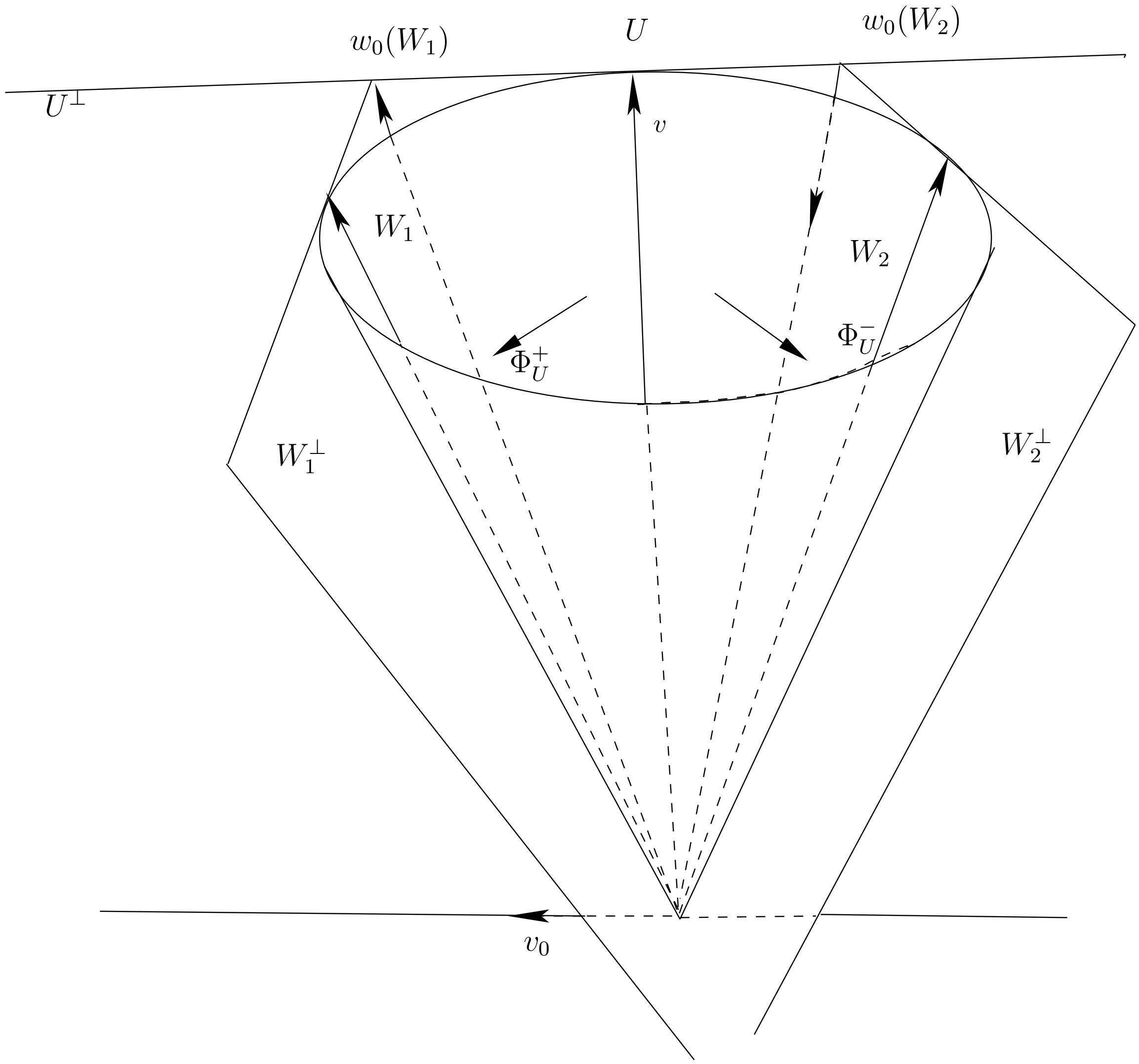

Figure 3: Positive and negative parts, illustration 2

$\{1,3\}$ such that $\widehat{W}_{i_0} \in \Phi_U^-$ and

$\widehat{W}_{i_0+1} \in \Phi_U^+$.

(3) Assume first that $W_1 \in \Phi_U^+$ and $W_2 \in \Phi_U^-$. Then there exists a δ such that for a maximal B–isotropic subspaces $\widehat{U}$, $\widehat{W}_1, \widehat{W}_2$ with $d(\widehat{U}, U) < \delta$, $d(\widehat{W}_1, W_1) < \delta$ and $d(\widehat{W}_2, W_2) < \delta$

we have $\widehat{W_1} \in \Phi^+_{\widehat{U}}$ and $\widehat{W_2} \in \Phi^-_{\widehat{U}}$.

Directions. Let us explain the motivation for the above construction. We show that if the group Γ is crystallographic, then there are two hyperbolic transversal elements in Γ with an opposite sign and conclude that case 2 (3) is impossible because of Lemma 7.5.

Let $S \subseteq \mathrm{Aff}(\mathbb{R}^n)$ be an infinite subset. We will say that S is unbounded if for every compact set K, $K \subset \mathbb{R}^n$ there exists an $s \in S$ such that $K \cap sK = \emptyset$. Every infinite subset S of Γ is unbounded since Γ acts properly discontinuously.

We will say that a non-zero vector $v \in \mathbb{R}^6$ is the direction of an unbounded subset S on a compact K_0 if there exists an infinite sequence $\{\gamma_n, n \in \mathbb{N}\}$ of S and a sequence $\{x_n, n \in \mathbb{N}\}$ in K_0 such that

$$\frac{\gamma_n x_n - x_n}{d(\gamma_n x_n, x_n)} \to \frac{v}{||v||}$$

for $n \to \infty$.

Denote by $V(S, K_0)$ the set of all directions of S on K_0.

Since Γ is a crystallographic group there exists a compact subset K of $\mathbb{R}^6$ such that $\Gamma K = \mathbb{R}^6$. Let us show that $V(\Gamma, K) = \{v, v \in \mathbb{R}^6, \|v\| = 1\}$. Indeed, let v be a norm one vector in $\mathbb{R}^6$. Let $L^+(v) = \{tv, t \in \mathbb{R}, t > 0\}$ be a directed line. Clearly, for every point $x \in L^+(v)$ there exists a point $k_x \in K$ and $\gamma_x \in \Gamma$ such that $\gamma_x k_x = x$. Obviously, for $x \to \infty$ we have

$$\frac{\gamma_x k_x - k_x}{d(\gamma_x k_x, k_x)} \to v$$

The fact that $V(\Gamma, K) = \{v, v \in \mathbb{R}^6, \|v\| = 1\}$ is crucial. Let us admit that if Γ acts properly discontinuously but not cocompact this is not true.

The key point is to show that there are two hyperbolic transversal elements in Γ with an opposite sign. In order to show this we construct two sequences S_1 and S_2 such that $w_1 \in V(S_1, K)$ and $w_1 \in V(S_2, K)$. Then we show that there are two hyperbolic elements $\gamma_+ \in S_1$ and $\gamma_- \in S_2$ such that for $A^{(1)}_+ = A^+(\gamma_+) \cap V_1$, $A^{(1)}_- = A^-(\gamma_+) \cap V_1$ (resp. $A^{(2)}_+ =$

$A^+(\gamma_-) \cap V_1,\ A_-^{(2)} = A^-(\gamma_-) \cap V_1)$ we have $A_-^{(1)} \in \Phi^+_{A_+^{(1)}}$ (resp. $A_-^{(2)} \in \Phi^-_{A_-^{(2)}}$). Hence by ***7.6*** (1) we conclude that these elements have an opposite sign.

Let us end with a small remark. We need not only to have hyperbolic elements with an opposite sign, but also ensure that they are transversal. The difficulty which comes up here is the following. We first contract a sequence S such that $w_1 \in V(S, K)$. Since we do not know a priory the dimension of $\dim A^-(\theta_2(\gamma)), \gamma \in S$ we "prepare" two sets S_i and T_i which fulfill (4) of Lemma 7.8 below. We start with the following simple lemma

Lemma 7.7 *Let $\widehat{\Gamma} \subset GL(V_1)$ be a Zariski dense subgroup of $SO(2,1)$. Then there exist four transversal hyperbolic elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ in $\widehat{\Gamma}$ such that we have $B(v, v) < 0$ for every non- zero vector*

$$v \in (A^+(\gamma_1) + A^+(\gamma_2)) \cap (A^+(\gamma_3) + A^+(\gamma_4))$$

Proof Since $\widehat{\Gamma}$ is Zariski dense in $SO(2,1)$ there are four transversal hyperbolic elements [AMS1]. It is enough now to order of these four elements such that vectors vectors $A^+(\gamma_3)$ and $A^+(\gamma_4)$ belongs to the different connected components of $\mathbb{R}^3 \setminus V$ where V is a subspace of $\mathbb{R}^3$ spanned by vectors $A^+(\gamma_1), A^+(\gamma_2)$. Then any non-zero vector $v \in (A^+(\gamma_1) + A^+(\gamma_2)) \bigcap (A^+(\gamma_3) + A^+(\gamma_4))$ will be inside the cone Con_B. Thus $B(v, v) < 0$ for any non-zero $v \in (A^+(\gamma_1) + A^+(\gamma_2)) \bigcap (A^+(\gamma_3) + A^+(\gamma_4))$ which proves the lemma.

To make products hyperbolic and transversal, one needs the following quantitative version of hyperbolicity and transversality.

Lemma 2.7. [AMS3] *There exists $s(\varepsilon) < 1$ and $c(\varepsilon)$ such that for any two ε-hyperbolic ε-transversal elements $g, h \in GL(V)$ with $s(g) < s(\varepsilon)$ and $s(h) < s(\varepsilon)$, for all $n, m \in \mathbb{Z}, n > 0, m > 0$ we have*

(1) $\widehat{d}(A^+(g^n h^m), A^+(g)) \leq c(\varepsilon) s(g)^n$;

(2) $\widehat{d}(A^-(g^n h^m), A^-(h)) \leq c(\varepsilon) s(h)^m$;

(3) the element $g^n h^m$ *is* $\varepsilon/2$*-hyperbolic and is* $\varepsilon/2$*-transversal to both* g *and* h*;*

(4) $s(g^n h^m) \leq c(\varepsilon) s(g)^n s(h)^m$.

Note that therefore an element $g^n h^m$ is $\varepsilon/2$–hyperbolic for sufficiently big n, m and ε-very hyperbolic elements g, h.

Let $\gamma_i, i = 1,2,3,4$ be elements of Γ that fulfill conditions and conclusions of Lemma 7.7. Obviously we can conjugate Γ. Hence without loss of generality we will assume that $w_1 \in (A^-(\theta_1(\gamma_1)) + A^-(\theta_1(\gamma_2))) \cap (A^-(\theta_1(\gamma_3)) + A^-\theta_1((\gamma_4)))$. Set $A_i = A^-(\theta_1(\gamma_i))$, for $i = 1,2,3,4$. Let $d = \min_{1 \leq i \neq j \leq 4} \widehat{d}(A_i, A_j)$

Lemma 7.8 *There exist sets* $S_i(n,m) = \{g_{ik}(n,m), k = 1,2,3, i = 1,2,3,4, n,m > 0, n,m \in \mathbb{Z}\ \}$*,*$T_i(n,m) = \{h_{ik}(n,m), k = 1,2,3, i = 1,2,3,4, n,m > 0, n,m \in \mathbb{Z}\}$ *, a positive real numbers* $\varepsilon, q, \varepsilon > 0, 0 < q < 1$ *such that for every positive* δ *there exists* $N, N > 0, N \in \mathbb{Z}$ *such that for* $n, m > N$ *we have*

(1) $\widehat{d}(A^-(\theta_1(g_{ik}(n,m))), A_i) < \delta$ *and* $\widehat{d}(A^-(\theta_1(h_{ik})(n,m)), A_i) < \delta$;

(2) $g_{ik}(n,m)$ *and* $h_{ik}(n,m)$ *are* ε*-hyperbolic for* $k = 1,2,3$;

(3) $\max_{1 \leq i \leq 4, 1 \leq k \leq 3}\{s(g_{ik}(n,m), s(h_{ik}(n,m))\} < q^n$;

(4) let i *be an index with* $1 \leq i \leq 4$*. Then for every* $k = 1,2,3$ *we have* $\dim A^-(\theta_2(g_{ik}(n,m))) = 2$ *and* $\dim A^-(\theta_2(h_{ik}(n,m))) = 1$;

(5) for every index i *with* $1 \leq i \leq 4$ *we have* $\bigcap_{1 \leq k \leq 3} A^-(\theta_2(g_{ik}(n,m)) = \{0\}$;

(6) for every index i *with* $1 \leq i \leq 4$ *we have* $\dim(A^-(\theta_2(h_{i1}(n,m))) + A^-(\theta_2(h_{i2}(n,m))) + A^-(\theta_2(h_{i3}(n,m)))) = 3$.

Proof Obviously it is enough to prove the statement for one subspace. Let us do it for A_1. It is easy to show that there exists a hyperbolic element γ of Γ such that

i) $\theta_1(\gamma)$ and $\theta_1(\gamma^{-1})$ are transversal to $\theta_1(\gamma_1)$;

ii) any proper $\theta_2(\gamma)$–invariant subspace does not contain a proper $\theta_2(\gamma_1)$–invariant subspace;

iii) any proper $\theta_2(\gamma_1)$– invariant subspace does not contain a proper $\theta_2(\gamma)$–invariant subspace.

(iv) $\theta_1(\gamma)$ (resp. $\theta_2(\gamma)$) is $\mathbb{R}$–regular element in $SO(2,1)$ (resp. $SL_3(\mathbb{R})$) [AMS1]

Put $\gamma_n = \gamma_1^{-n}\gamma\gamma_1^n$. Since γ is a hyperbolic $\mathbb{R}$–regular element then $\dim A^-(\gamma) = 2$ or $\dim A^-(\gamma) = 3$.

We can and will assume that $\dim A^-(\gamma) = 3$ otherwise we will consider γ^{-1} instead of γ. Let us first show that for some positive numbers n_1, n_2, n_3 we have $\cap_{1\leq i\leq 3} A^-(\theta_2(\gamma_{n_i})) = \{0\}$. Since for $n \neq m$ we have $A^-(\theta_2(\gamma_n)) \neq A^-(\theta_2(\gamma_m))$ there are positive numbers n_1 and n_2 such that $\dim A^-(\theta_2(\gamma_{n_1})) \cap A^-(\theta_2(\gamma_{n_2}) = 1$. Let v be a non -zero vector of this intersection. If $\theta_2(\gamma_1)^{-n}v \in A^-(\theta_2(\gamma))$ for infinitely many positive n then the proper $\theta_2(\gamma)$–invariant subspace $A^-(\theta_2(\gamma))$ contains a $\theta_2(\gamma_1)$– invariant subspace. This contradicts our assumptions. Thus, by the choice of γ and γ_1 there exists an n_3 such that $\theta_2(\gamma_1)^{-n_3}v \notin A^-(\theta_2(\gamma))$ Therefore $v \notin \theta_2(\gamma_1)^{n_3} A^-(\theta_2(\gamma)) = A^-(\theta_2(\gamma_{n_3}))$.

Clearly, $A^-(\theta_2(\gamma_{n_1+m}))\cap A^-(\theta_2(\gamma_{n_2+m}))\cap A^-(\theta_2(\gamma_{n_3+m})) = \{0\}$ for all positive numbers m. Set $\gamma_{1i}(m) = \gamma_{n_i+m}, i = 1,2,3$. Remark that for all m we have

$$A^-(\theta_2(\gamma_{1,1}(m)) \cap A^-(\theta_2(\gamma_{1,2}(m)) \cap A^-(\theta_2(\gamma_{13}(m)) = \{0\} \qquad (0)$$

Since the projective space PV is compact we can and will assume that $A^+(\gamma_{1i}(m)) \longrightarrow X_i^+$, $A^-(\gamma_{1i}(m)) \longrightarrow X_i^-$. Note, that

$$A^-(\theta_1(\gamma_{1i}(m))) \xrightarrow[n\to\infty]{} A_1 \qquad (1)$$

for $i=1,2,3$. Since $l(\Gamma)$ is Zariski dense in $SO(2,1)\times SL_3(\mathbb{R})$ we conclude that there exists a hyperbolic element γ_0 such that $A^+(\gamma_0)\cap X_i^\pm=0$ and $A^-(\gamma_0)\cap X_i^\pm=0$. Hence $\widehat{d}(A^-(\gamma_0),X_i^\pm)>0$, and
$\widehat{d}(A^-(\gamma_0),X_i^\pm)>0$, for all $i=1,2,3$.
Let $\widehat{\varepsilon}=\min_{1\leq i\leq 3}\{\widehat{d}(A^+(\gamma_0),X_i^\pm),$. Thus there exists an $M\in\mathbb{N}$ such that for $m\geq M$ the elements γ_0 and $\gamma_{1,i}(m)$ are $\widehat{\varepsilon}/2$–transversal. Let $q_1=\min\{s(\gamma_0),\ s(\gamma)\}$. It follows from [MS] and [AMS1] that there exists a positive number N such that for $m,n>N$ we have

$$\widehat{d}(A^+(\gamma_0^n\gamma_{1i}^n(m),A^+(\gamma_0))\leq q_1^{n/2} \tag{2}$$

$$\widehat{d}(A^-(\gamma_0^n\gamma_{1i}^n(m)),A^-(\gamma_{1,i}(m)))\leq q_1^{n/2} \tag{3}$$

and

$$s(\gamma_0^n\gamma_{1i}^n(m))\leq q_1^{n/2} \tag{4}$$

for $i=1,2,3$. Since γ_0 and $\gamma_{1,i}(m)$ are $\widehat{\varepsilon}/2$–transversal then for sufficiently big n from (2), (3) and (4) follows that element $\gamma_0^n\gamma_{1i}^n(m))$ is $\widehat{\varepsilon}/4$–hyperbolic. Set $\varepsilon=\widehat{\varepsilon}/4$
From (3) follows that

$$A^-(\gamma_0^n\gamma_{1i}^n(m))\xrightarrow[n\to\infty]{}A^-(\gamma_{1i}(m)).$$

Obviously

$$A^-(\theta_1(\gamma_0^n\gamma_{1i}^n(m)))\xrightarrow[n\to\infty]{}A^-(\theta_1(\gamma_{1i}(m))).$$

Clearly,

$$A^-(\theta_1(\gamma_{1i}(m)))\xrightarrow[n\to\infty]{}A_1.$$

It follows from (3) that

$$A^-(\theta_2(\gamma_0^n\gamma_{1i}^n(m))\xrightarrow[n\to\infty]{}A^-(\theta_2(\gamma_{1i}(m))).$$

Since the projective space is compact we can and will assume that the sequence $\{A^-(\theta_2(\gamma_{1i}(m)))\}$ converges to the subspace A_{1i}^- when $m\to\infty$. Hence

$$A^-(\gamma_{1i}(m))\xrightarrow[m\to\infty]{}A_1\oplus A_{1i}^- \tag{5}$$

It immediately follows from (2) that for every m

$$A^+(\gamma_0^n\gamma_{1i}^n(m)) \xrightarrow[n\to\infty]{} A^+(\gamma_0) \qquad (6).$$

Since $\dim A^-(\theta_2(\gamma_{1i}^n(m)))) = 2$ we conclude by (2) that $\dim A^-(\theta_2(\gamma_0^n\gamma_{1i}^n(m))) = \dim A^-(\theta_2(\gamma_{1i}^n(m)) = 2$ for $i = 1, 2, 3$ and for $n > N_2$ we have $\cap_{1\le i\le 3} A^-(\theta_2(\gamma_0^n\gamma_{1i}^n(m))) = \{0\}$. Take n, m such that $\min n, m > \max\{N_1, N_2\}$ and set $S_1(n,m) = \{g_{1k}(n,m) = \gamma_0^n\gamma_{1k}^n(m), k = 1, 2, 3\}$. Then for every pair n, m the set $S_1(n,m)$ fulfills the requirements of Lemma 7.8. Using the same arguments starting with a hyperbolic element γ, such that $\dim(A^-(\theta_2(\gamma)) = 1$ one can show that there are sets $T_i(n,m) = \{h_{i1}(n,m), h_{i2}(n,m), h_{i3}(n,m)\}$,

$i = 1, 2, 3, 4$ with properties 1-4, 6. This proves Lemma 7.8.

Limit subspaces. We will use the notations of Lemma 7.8 in this chapter. Let's summarize what is proved. Recall that $g_{ik}(n,m) = \gamma_0^n\gamma_{ik}^n(m)$, $i = 1, 2, 3, 4, k = 1, 2, 3$.

$$A^-(\theta_1(g_{ik}(n,m)) \xrightarrow[n\to\infty]{} A^-(\theta_1(\gamma_{ik}(m)) \xrightarrow[m\to\infty]{} A_i^- \qquad (\boldsymbol{LS}_1)$$

,

$$A^-(\theta_2(g_{ik}(n,m)) \xrightarrow[n\to\infty]{} A^-(\theta_2(\gamma_{ik}(m)) \xrightarrow[m\to\infty]{} A_{ik}^- \qquad (\boldsymbol{LS}_2)$$

Then by Lemma 7.8 (6), $(\boldsymbol{LS}_1), (\boldsymbol{LS}_2)$

$$A^-(g_{ik}(n,m)) \xrightarrow[n\to\infty]{} A^-(\gamma_{ik}(m) \xrightarrow[m\to\infty]{} A_i \oplus A_{ik}^- \qquad (\boldsymbol{LS}_3)$$

$$A^+(g_{ik}(n,m)) \xrightarrow[n\to\infty]{} A^+(\gamma_0) \qquad (\boldsymbol{LS}_4)$$

For every $1 \le i \le 4$ and m we have

$$A^-(\theta_2(\gamma_{i1}(m)) \cap A^-(\theta_2(\gamma_{i2}(m)) \cap A^-(\theta_2(\gamma_{i3}(m)) = 0$$

From (0) and $(\boldsymbol{LS}_1)$ follows that there exists a positive integer N such that for every $1 \le i \le 3$ and m for all $n > N$ we have

$$A^-(\theta_2(g_{i1}(n,m)) \cap A^-(\theta_2(g_{i2}(n,m)) \cap A^-(\theta_2(g_{i3}(n,m)) = 0 \qquad (\boldsymbol{LS}_5)$$

We can and will assume that ($\boldsymbol{LS}_5$) holds for all n, m. Let U be one dimensional subspace of V_1. Then from ($\boldsymbol{LS}_2$) and ($\boldsymbol{LS}_5$) follows that

$$d(i,m) = \inf_{U \subset V_1, n \in \mathbb{N}} \sum_{1 \le k \le 3} \widehat{d}(U, A^-(\theta_2(g_{ik}(n,m)))) > 0$$

Set

$$d_2^{(S)}(m) = \min_{1 \le i \le 4} d(i,m).$$

Recall, that for every two different i and j, $1 \le i, j \le 4$ we have $A_i \cap A_j = 0$. Hence by ($\boldsymbol{LS}_1$) we conclude that there exists $N, N \in \mathbb{N}$ such that for $n, m > N$ we have

$$A^-(\theta_1(g_{ik}(n,m)) \cap A^-(\theta_1(g_{jr}(n,m)) = 0 \qquad (\boldsymbol{LS}_6)$$

Thus we can and will assume that ($\boldsymbol{LS}_6$) holds for all n, m. Let U be a one dimensional subspace of V_1. Set

$$d_{ij} = \inf_{U,n,m,k,t,i \ne j} \widehat{d}(U, A^-(\theta_1(g_{ik}(n,m)))) + \widehat{d}(U, A^-(\theta_1(g_{jt}(n,m))))$$

It follows from ($\boldsymbol{LS}_1$) that for we have for $1 \le i \ne j \le 4$

$$d_{ij} \ge \widehat{d}(U, A_i) + \widehat{d}(U, A_j)/2 \ge d/4 \qquad (\boldsymbol{LS}_7)$$

Set

$$d_1^{(S)} = \min_{i \ne j, 1 \le i,j \le 4} d_{ij}$$

Clearly, $d_1^{(S)} \ge d/4$.

By the same arguments we prove that there exist a positive constants $d_2^{(T)}(m)$ and $d_1^{(T)}$. The only one difference is that to define the constant $d_2^{(T)}(m)$ we have to consider a subspace $U$of V_2 of dimension two because $\dim A^-(\theta_2(h_{ij}(n,m)) = 1$.

Main Lemma 7.9 *There are two hyperbolic elements of the group* Γ *such that* $\alpha(g)\alpha(h) < 0$.

Proof. By [AMS1] for a $q < 1$ and every $t, t > 0, t \in \mathbb{Z}$ there exist a finite subset $S_t(\Gamma) = \{g_{1,t}, \dots, g_{m,t}\}$ of Γ such that for every $\gamma \in \Gamma$ there exists $g_{i,t} \in S_t(\Gamma)$ such that the element $\gamma g_{i,t}$ is ε–hyperbolic, where $\varepsilon = \varepsilon(\Gamma)$ and $s(\gamma g_{i,t}) < q^t$. Set $S_t = S_t(\Gamma)$.

Let K be a compact subset of V such that $\Gamma K = V$. Following along the same arguments we have used in the chapter ***Directions***, we conclude, that there exists a sequence $Q_0 = \{\gamma_n\}_{n\in\mathbb{N}}$ of elements of Γ such that $w_1 \in V(Q_0, K)$. Since the set S_t is finite the set $K_t = S_t^{-1}K$ is compact. Set $R_t = Q_0 S_t$. Let us show that $w_1 \in V(R_t, K_t)$ for every t. Indeed. By definition, there exists a set $\{k_n\}$ of points in K such that $\gamma_n k_n - k_n / d(\gamma_n k_n, k_n) \to w_1$. Let $\gamma \in S_t$. Then $\widehat{k}_n = \gamma^{-1} k_n \in K_t$ and $\gamma_n \gamma \widehat{k}_n - \widehat{k}_n = \gamma_n k_n - k_n + (k_n - \gamma^{-1} k_n)$. Obviously, that for every t there is a constant C_t such that
$\|k_n - \gamma^{-1} k_n\| < C_t$. On the other hand $\|\gamma_n k_n - k_n\| \to \infty$ when $n \to \infty$ Therefore

$$\frac{g_n \gamma \widehat{k}_n - \widehat{k}_n}{\|g_n \gamma \widehat{k}_n - \widehat{k}_n\|} \to w_1$$

So, we conclude that for every t there exists a sequence $Q_t, Q_t \subset R_t$ and a compact set K_t such that

(*) $w_1 \in V(Q_t, K_t)$

(**) every element $\gamma \in Q_t$ is ε–hyperbolic and $s(\gamma) < q^t$.

The projective space PV is compact. Thus we can and will assume that the sequences $\{A^+(\gamma_{n,t})\}, \gamma_{n,t} \in Q_t$ and $\{A^-(\gamma_{n,t})\}, \gamma_{n,t} \in Q_t$ converge when $n \to \infty$. Let A_t^+ (resp.A_t^-) , be a subspace of $\mathbb{R}^6$ such that $A^+(\gamma_{n,t}) \xrightarrow[n\to\infty]{} A_t^+$ (resp. $A^-(\gamma_{n,t}) \xrightarrow[n\to\infty]{} A_t^-$.) We can and will assume, since the projective space is compact, that there are two subspaces A^+, A^- such that $A_t^+ \xrightarrow[t\to\infty]{} A^+$ and $A_t^- \xrightarrow[t\to\infty]{} A^-$. Recall that element $\gamma_{n,t}$ are ε–hyperbolic, for all n and t. Clearly, $\widehat{d}(A^+, A^-) \geq \varepsilon$. Set $A_{\theta_1}^+ = V_1 \cap A^+$, $A_{\theta_1}^- = V_1 \cap A^-$, $A_{\theta_2}^+ = V_2 \cap A^+$, $A_{\theta_2}^- = V_2 \cap A^-$.

There are two cases.

(i) $\dim A^+_{\theta_2} = 1$, $\dim A^-_{\theta_2} = 2$

(ii) $\dim A^+_{\theta_2} = 2$, $\dim A^-_{\theta_2} = 1$

It is enough to prove Main Lemma in case (i) because in case (ii) the proof follows along the same way but using the sets T_i, instead of sets S_i.

There exists a hyperbolic element $\tilde{\gamma} \in \Gamma$ such that $\tilde{\gamma}$ and $\tilde{\gamma}^{-1}$ are proximal (see [AMS1)]. In particula,r all eigenvalues of $\theta_2(\tilde{\gamma})$ differ. Clearly, we can and will assume that $\dim A^-(\theta_2(\tilde{\gamma})) = 2$. Since Γ is Zariski dense subgroup in $SO(2,1) \times SL_3(\mathbb{R})$ by standard arguments ([BG], [MS],) , we can choose an element $\tilde{\gamma} \in \Gamma$ such that

$$A^-(\tilde{\gamma}) \cap A^+(\gamma_0) = 0, A^-(\tilde{\gamma}) \cap A^+ = 0, A^+(\tilde{\gamma}) \cap A^- = 0$$

Hence there exist positive ε_1 such that

$$\widehat{d}(A^-(\tilde{\gamma}), A^+) \geq \varepsilon_1, \widehat{d}(A^+(\tilde{\gamma}), A^-) \geq \varepsilon_1, \widehat{d}(A^+(\gamma_0), A^-(\tilde{\gamma})) \geq \varepsilon_1 \qquad (1)$$

Then there exists a positive number N_1 such that for $t > N_1$ we have $\widehat{d}(A^-_t, A^+) \geq \varepsilon_1/4, \widehat{d}(A^+_t, A^-) \geq \varepsilon_1./4$. Since $A^+(\gamma_{n,t}) \underset{n\to\infty}{\longrightarrow} A^+_t$, $A^-(\gamma_{n,t}) \underset{n\to\infty}{\longrightarrow} A^-_t$.) when $n \to \infty$ there exists N_2 such that for $t > N_1$, $n > N_2$ we have

$$\widehat{d}(A^+(\gamma_{n,t}), A^-(\tilde{\gamma})) \geq \varepsilon_1/2, \widehat{d}(A^-(\gamma_{n,t}), A^+(\tilde{\gamma})) \geq \varepsilon_1/2 \qquad (2)$$

By Lemma 2.7 , we have

(1) $\widehat{d}(A^+(\gamma_{n,t}\tilde{\gamma}^m), A^+(\gamma_{n,t})) \leq q^{t/2}$;

(2) $\widehat{d}(A^-(\gamma_{n,t}\tilde{\gamma}^m)), A^-(\tilde{\gamma})) \leq s(\tilde{\gamma})^{m/2}$;

(3) $s(\gamma_{n,t}\tilde{\gamma}^m) \leq q^{t/2} s(\tilde{\gamma})^{m/2}$.

Let $q_1 = \max q^{1/2} s(\tilde{\gamma})^{1/2}$ and let $A^+_{t,m}$ be a subspace of $\mathbb{R}^6$ such that

$$A^+(\gamma_{n,t}\tilde{\gamma}^m) \underset{n\to\infty}{\longrightarrow} A^+_{t,m}. \qquad (2)$$

Clearly, we have $\widehat{d}(A^+_{t,m}, A_t) \le q_1^t$ for every m. Hence for every m we have

$$A^+_{t,m} \xrightarrow[t\to\infty]{} A^+ \tag{3}$$

It follows from *(2)* that for every n, t, such that $t > N_1, n > N_2$ we have

$$A^-(\gamma_{n,t}\tilde{\gamma}^m) \xrightarrow[m\to\infty]{} A^-(\tilde{\gamma}) \tag{4}$$

Set $\tilde{Q}_{t,m} = Q_t\tilde{\gamma}^m$ and $\tilde{K}_{t.m} = \tilde{\gamma}^{-m}K_t$. By the same arguments as above, we see that for every m, we have $w_1 \in V(\tilde{Q}_{t,m}, \tilde{K}_{t.m})$

V_1***-part*** It follows from Lemma 7.7 that if $A_i \in \Phi^+_{A^+_{\theta_1}}$ then $A_{i+1} \in \Phi^-_{A^+_{\theta_1}}, i = 1, 2$. Since $\widehat{d}(A_i, A_j) > d_1^{(S)}, i \ne j$ there are three points $A_{i_1}, A_{i_2}, A_{i_3}$ such that $\widehat{d}(A^+_{\theta_1}, A_{j_k}) > d_1^{(S)}/4, k = 1, 2, 3$. Therefore there are two different spaces A_i and A_j such that, one belongs to $\Phi^+_{A^+_{\theta_1}}$ and the second one to $\Phi^-_{A^+_{\theta_1}}$ and $\widehat{d}(A^+_{\theta_1}, A_k) > d_1^{(S)}/4, k = i, j$. Without loss of generality we can and will assume that $A_1 \in \Phi^+_{A^+_{\theta_1}}$ and $A_2 \in \Phi^-_{A^+_{\theta_1}}$. We show in 7.6 *(3)* that there exists a positive δ such that for every one dimensional subspaces W, W_1, W_2 of V_1 such that $\widehat{d}(W, A^+_{\theta_1},) < \delta$, $\widehat{d}(W_1, A_1) < \delta$, $\widehat{d}(W_2, A_2) < \delta$, we have $W_1 \in \Phi^+_W$ and $W_1 \in \Phi^-_W$. We can and will additionally assume that $\delta < d_1^{(S)}/100$. Then if W, W_1, W_2 are one dimensional subspaces of V_1 such that $\widehat{d}(W, A^+_{\theta_1},) < \delta$, $\widehat{d}(W_1, A_1) < \delta$, $\widehat{d}(W_2, A_2) < \delta$, then

$$\widehat{d}(W, W_1) \ge d_1^{(S)}/5, \;\; \widehat{d}(W, W_2) \ge d_1^{(S)}/5. \tag{5}$$

By ($\boldsymbol{LS}_1$) for sufficiently big n and all m we obviously have

$$\widehat{d}(A^-(\theta_1(g_{ik}(n, m)), A^-(\theta_1(\gamma_{ik}(m))) < \delta/2$$

and for sufficiently big m we have

$$\widehat{d}(A^-(\theta_1(\gamma_{ik}(m)), A_i^-) < \delta/2$$

where $i = 1, 2, 3, 4, k = 1, 2, 3$. Consequently, there exist m_0 and N_0 such that for $g_{ik}(n) = g_{ik}(n, m_0)$, all $n > N_0$, $i = 1, 2, 3, 4, k = 1, 2, 3$ we have

$$\widehat{d}(A^-(\theta_1(g_{ik}(n)), A_i^-) \le \delta, i = 1, 2, k = 1, 2, 3. \tag{6}$$

Denote $d_2^{(S)}(m_0) = d_2^{(S)}$

It is easy to conclude from (2), (3) that there exists $\tilde{N}_0$ such that for $n, t > \tilde{N}_0$ and all m we have

$$\widehat{d}(A^+(\theta_1(\gamma_{n,t}\tilde{\gamma}^m), A^+) < \delta \tag{7}$$

Let $\overline{Q}_t = \gamma_{n,t}, n > N_0$. Set $\overline{Q}_{t,m} = \overline{Q}_t\tilde{\gamma}^m$. Then $w_1 \in V(\overline{Q}_{t,m}, \tilde{K}_{t.m})$. Therefore we will assume that for every $\gamma, \gamma \in \tilde{Q}_{t,m}$, t, m we have $\widehat{d}(A^+(\theta_1(\gamma), A^+) < \delta$. Hence for all integers $n, n > 0, k = 1, 2, 3$ we have

$$A^-(\theta_1(g_{1k}(n))) \in \Phi^+_{\theta_1(\gamma)}, A^-(\theta_1(g_{2k}(n))) \in \Phi^-_{\theta_1(\gamma)} \tag{8}$$

It follows from (5), that for all n, t, m, $i = 1, 2, k = 1, 2, 3$ and every $\gamma, \gamma \in \tilde{Q}_{t,m}$ we have

$$\widehat{d}(A^-(\theta_1(g_{ik}(n))), A^+(\theta_1(\gamma))) > d_1^{(S)}/5 \tag{9*}$$

It follows from (1) that

$$\widehat{d}(A^+(\theta_1(\gamma_0)), A^-(\theta_1(\tilde{\gamma}))) \geq \varepsilon_1$$

then for sufficiently big n we have by $(\boldsymbol{LS}_3)$

$$\widehat{d}(A^+(\theta_1(g_{ik}(n))), A^-(\theta_1(\tilde{\gamma}))) \geq \varepsilon_1/2$$

This and (4) lead us to the conclusion that for every $\gamma, \gamma \in \tilde{Q}_{t,m}$ we have

$$\widehat{d}(A^+(\theta_1(g_{ik}(n))), A^-(\theta_1(\gamma))) \geq \varepsilon_1/4 \tag{9**}$$

V_2***-part.*** The goal of this section is to show that for all n, t, m every two elements $\theta_2(g_{ik}(n))$ and $\theta_2(\gamma), \gamma \in \tilde{Q}_{t,m}$, are $\tilde{\varepsilon}_2$–transversal for some $\tilde{\varepsilon}_2$ that does not depends on n, t, m.

For a one dimensional subspace $A^+_{\theta_2}$ for all n $i = 1, 2$ we have

$$\Sigma_{1 \leq i \leq 3}\widehat{d}(A^+_{\theta_2}, A^-(\theta_2(g_{ik}(n)))) > d_2^{(S}, i = 1, 2$$

Then there exist $1 \leq k_1, k_2 \leq 3$ such that for all n we have

$$\widehat{d}(A^+_{\theta_2}, A^-(\theta_2(g_{1k_1}(n)))) > d_2^{(S}/3, \widehat{d}(A^+_{\theta_2}, A^-(\theta_2(g_{1k_2}(n)))) > d_2^{(S}/3$$

Set $g_1(n) = g_{1,k_1}(n)$ and $g_2(n) = g_{2,k_2}(n)$. Thus

$$\widehat{d}(A^+_{\theta_2}, A^-(\theta_2(g_1(n)))) > d_2^{(S}/3, \widehat{d}(A^+_{\theta_2}, A^-(\theta_2(g_2(n)))) > d_2^{(S}/3 \tag{10}$$

On the other hand, for every δ_1 there exists $N(\delta_1)$ such that for $n, t > N(\delta_1)$ and all m we have

$$\widehat{d}(A^+_{\theta_2}, A^+(\theta_2(\gamma_{t,n}\tilde{\gamma}^m))) < \delta_1$$

Assume that $\delta_1 < \frac{d^{(S})_2}{100}$ then for $i = 1, 2$ and for all $n, t > N(\delta_1)$ and all positive numbers r, m we have

$$\widehat{d}(A^-(g_i(r)), A^+(\theta_2(\gamma_{t,n}\tilde{\gamma}^m))) > \frac{d_2^{(S}}{4} \tag{11}$$

As above we conclude from (1)

$$\widehat{d}(A^+(\theta_2(\gamma_0)), A^-(\theta_2(\tilde{\gamma}))) \geq \varepsilon_1$$

This inequality together with (4), ($\boldsymbol{LS}_3$) and ($\boldsymbol{LS}_4$) leads us to the following follows conclusion: there exists $N(\varepsilon_1)$ such that for $r, m > N(\varepsilon_1)$ we have

$$\widehat{d}(A^+(\theta_2(g_i(r))), A^-(\theta_2(\gamma_{n,t}\tilde{\gamma}^m))) \geq \varepsilon_1/2 \tag{12}$$

Denote by $\tilde{\varepsilon}_1 = \min\{d_1^{(S)}/5, \varepsilon_1/4\}$. Thus for every n two elements $\theta_2(g_{ik}(n)))$ and $\theta_2(\gamma), \gamma \in \tilde{Q}_{t,m}$ are $\tilde{\varepsilon}_2$–transversal where $i = 1, 2, 3, 4, k = 1, 2, 3$ Hence we have two elements $\tilde{\varepsilon}$ – transversal elements $g_i(r)$ $i = 1, 2$ and $\gamma, \gamma \in \tilde{Q}_{t,m}$. where $\tilde{\varepsilon} = min(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2)$

It is obvious that for $n, t > N(\delta_1)$, $r, m > N(\varepsilon_1)$ elements $g_1(r)$ and $\gamma_{n,t}\tilde{\gamma}^m$, $g_2(r)$ and $\gamma_{n,t}\tilde{\gamma}^m$ are $\tilde{\varepsilon}$– transversal. By Lemma 2.7 there exists $N(\tilde{\varepsilon})$ such that for $r, m > N(\tilde{\varepsilon})$, $n, t > N(\delta_1)$

(1) $\gamma_{n,t}\tilde{\gamma}^m g_1(r)$ and $\gamma_{n,t}\tilde{\gamma}^m g_2(r)$ are $\tilde{\varepsilon}/2$-hyperbolic

(2) $\gamma_{n,t}\tilde{\gamma}^m g_1(r)$ and $\gamma_{n,t}\tilde{\gamma}^m g_2(r)$ are $\tilde{\varepsilon}/2$-transversal

Set $\tilde{N} = \max(N(\tilde{\varepsilon}), N(\delta_1))$ By definition of $\Phi_U^{\pm}$ from (8) follows that if $r, m > \tilde{N}$ and $n, t > \tilde{N}$ then the sign $\alpha(\gamma_{n,t}\tilde{\gamma}^m g_1(r))$ is positive, and the sign $\alpha(\gamma_{n,t}\tilde{\gamma}^m g_2(r))$ is negative. This proves the lemma.

Proposition 7.10 *The group Γ is not crystallographic.*

Proof Assume that the group Γ is a crystallographic subgroup of Aff $\mathbb{R}^6$. It follows from the Main Lemma 7.9 that there are two hyperbolic transversal elements with opposite sign. Thus Γ does not act properly discontinuously by Lemma 7.5. Contradiction that proves Proposition 7.10.

Theorem A *Let Γ be a crystallographic subgroup of Aff$(\mathbb{R}^n)$ and $n < 7$. Then Γ is virtually solvable.*

Proof . Let G be the Zariski closure of the group Γ. Let $\dim V \leq 5$. Then Γ is virtually solvable by Proposition 5.3. Let $\dim V = 6$. Assume that the semisimple part S of G is not trivial. It follows from [S2], [To2] that the real rank of at least one simple factor group of S is ≥ 2 if Γ is crystallographic. Therefore S is one of groups listed in Case 1 and 2. Thus Γ is not crystallographic by Propositions 5.3, 6.3 and 7.10. This contradiction shows that S must be the trivial group. Hence the group Γ is virtually solvable.

8 The dynamics of an affine group action

Let Γ be an affine group acting properly discontinuously on $\mathbb{R}^n$. Let G be the Zariski closure of Γ. Obviously Γ acts properly discontinuously if a subgroup of a finite index of

Γ acts properly discontinuously. Therefore from now on we will assume that the linear part $l(G)$ of G is a connected group, $l(G) < GL(\mathbb{R}^n)$. We denote by $o(g)$ the restriction of g to C_g. Let A_1 and A_2 be two subsets of P. Recall that

$$\underline{d}(A_1, A_2) = \inf_{a_1 \in A_1, a_2 \in A_2} \widehat{d}(a_1, a_2)$$

$$\widehat{d}(A_1, A_2) = \sup_{a_1 \in A_1} \inf_{a_2 \in A_2} \widehat{d}(a_1, a_2)$$

Let $\{g_0, h_1, \ldots, h_m\} \subset G$ be ε–hyperbolic elements, pairwise very ε– transversal. Set $s = \max\{s(g_0), s(h_1), \ldots, s(h_m)\}$ and $s_0 = s^{1/2}$. Let $g_\ell = h_{i_\ell}^{n_\ell} \cdot \cdots \cdot h_{i_1}^{n_1} \cdot g_0$, $1 \leq i_k \leq m$, $i_k \neq i_{k+1}, n_k \in \mathbb{Z}, 1 \leq k \leq (l-1)$, and $M_\ell = |n_1| + \cdots + |n_\ell|$. From Lemma 1.3 [AMS2] follows then that there exists a constant $s(\varepsilon) < 1$ such that if $s_0 < s(\varepsilon)$,

$$s(g_\ell) \leq s_0^{M_\ell + 1} \tag{1}$$

$$\widehat{d}(A^+(g_{\ell-1}), A^+(g_\ell)) \leq \frac{\varepsilon}{2} s_0^{M_{\ell-1}} \tag{2}$$

$$\widehat{d}(A^-(g_0)A^-(g_\ell)) \leq \frac{\varepsilon}{2} s_0 \tag{3}$$

$$\widehat{d}(A^+g_\ell), A^+(h_{i_\ell})) \leq \frac{\varepsilon}{2} s_0^{i_\ell} \tag{4}$$

$$\underline{d}(A^+(g_\ell), A^+(h_i) \cup A^-(h_i)) \geq \frac{\varepsilon}{2}, i \neq i_\ell \tag{5}$$

$$\underline{d}(A^+(g_\ell), A^-(g_\ell)) > \varepsilon/2 \tag{6}$$

It is well known that there exists a positive constant $s_1(\varepsilon)$ such that for $s_0 \leq s_1(\varepsilon)$ the group G_1 generated by $g_0, h_1, \ldots, h_m$ is free with free generators $g_0, h_1, \ldots, h_m$. There is a choice of $g_0, h_1, \ldots, h_m$ such that the group generated by $g_0, h_1, \ldots, h_m$ is Zariski dense in G. The proof is based on the so-called Ping-Pong Lemma. For details see [AMS1], [AMS2].

Let $q_0 \in \mathbb{R}^n$ be the origin.Let q_ℓ be the point of C_{g_ℓ} such that $d(q_0, q_\ell) = d(q_0, C_{g_\ell})$. Recall that $d_{g_\ell} = d(q_\ell, g_\ell q_\ell)$ From Lemma 1.6 [AMS2] follows that there exist a constants $s_2(\varepsilon)$, $d_1(\varepsilon)$ and $d_2(\varepsilon)$ such that for $s_0 < \min\{s(\varepsilon), s_2(\varepsilon)\}$ we have

$$d(q_0, C_{g_\ell}) < d_1(\varepsilon) \tag{7}$$

and

$$d_{g_\ell} \leq d_2(\varepsilon)|M_l| \tag{8}$$

The identification procedure. Let g and h be two hyperbolic, transversal elements of G. Following [AMS2, chapter 3] we consider the following subspaces and projections. Let $C_{h,g} = E_h^+ \cap E_g^-$ and $C_{g,h} = E_h^- \cap E_g^+$. Set $\pi_h^- : C_{g,h} \to C_h$ along $A^-(h)$ $\pi_h^+ : C_h \to C_{h,g}$ along $A^+(h)$, $\pi_g^- : C_{h,g} \to C_g$ along $A^-(g)$ and $\pi_g^+ : C_g \to C_{g,h}$ Define the following transformation $\overline{o}(gh)$ of $C_{g,h}$ as $\overline{o}(gh) = \pi_g^+ \overline{o}(g) \pi_g^- \pi_h^+ \overline{o}(h) \pi_h^-$. Obviously, $\overline{o}(g^n h^m) = \pi_g^+ \overline{o}(g)^n \pi_g^- \pi_h^+ \overline{o}(h)^m \pi_h^-$ for positive $n, m \in \mathbb{Z}$.
The reasons for this definition are the following. The map $\overline{o}(g^n h^m)$ of $C_{g,h}$ approximates $g^n h^m$ in the following sense. For positive integers n, m such that $n \to \infty, m \to \infty$ we have $E^+_{g^n h^m} \to E_g^+$ and $E^-_{g^n h^m} \to E_h^-$. Therefore $C_{g^n h^m} \to C_{g,h}$. For a given $q \in C_{g,h}$ and $\overline{q} = \overline{o}(g^n h^m) q$ for every positive numbers ε_k such that $\varepsilon_k \to 0$, there exists $\delta_k, \delta_k > 0$, $\delta_k \to 0$, positive numbers N_k, $N_k \to \infty$ and $q_k \in U(q, \delta_k)$ such that for $n_k, m_k > N_k$ we have $d(\overline{o}(g^{n_k} h^{m_k}) q, g^{n_k} h^{m_k} q_k) < \varepsilon_k$. We can thus approximate $g^n h^m$ for certain points near $C_{g,h}$ by the orthogonal map $\overline{o}(g^n h^m)$ for sufficiently big n, m.

Let $\{g_0, h_1, \ldots, h_m\} \subset G$ be ε–hyperbolic elements, pairwise very ε–transversal. and let $g_\ell = h_{i_\ell}^{n_\ell} \cdot \cdots \cdot h_{i_1}^{n_1} \cdot g_0$, $1 \leq i_k \leq m$, $i_k \neq i_{k+1}, n_k \in \mathbb{Z}, 1 \leq k \leq (l-1)$, and $M_\ell = |n_1| + \cdots + |n_\ell|$. Set

$$\overline{o}(g_\ell) = \pi^+_{h_{i_l}} \overline{o}(h_{i_\ell}^{n_\ell}) \pi^-_{h_{i_l}} \ldots \pi^+_{h_{i_1}} \overline{o}(h_{i_1}^{n_1}) \pi^-_{h_{i_1}} \pi^+_{g_0} \overline{o}(g_0) \pi^-_{g_0} =$$

$$\pi^+_{h_{i_l}} \overline{o}(h_{i_\ell})^{n_\ell} \ldots \overline{o}(h_{i_1})^{n_1} \pi^-_{h_{i_1}} \pi^+_{g_0} \overline{o}(g_0) \pi^-_{g_0}$$

and let $\pi_\ell = \pi^+_{h_{i_l}} \pi^-_{h_{i_l}} \ldots \pi^+_{h_{i_1}} \pi^+_{g_0} \pi^-_{g_0}$.

From now on unless otherwise stated we will assume that Γ is an affine group such that the linear part $l(\Gamma)$ is Zariski dense in $SL_2(\mathbb{R}) \times SO(3)$. Hence $l(G) = SL_2(\mathbb{R}) \times SO(3)$. In this case for a $\mathbb{R}$–regular element $g \in G$ we have $\dim A^+(g) = \dim A^-(g) = 1$, $\dim A^0(g) = 3$ and the restriction $l(g) \mid_{A^0(g)} \in SO(3)$. Let V_1 and V_2 be two $l(G)$–invariant subspaces of $\mathbb{R}^5$ such that $\mathbb{R}^5 = V_1 \oplus V_2$ and $l(G)|_{V_1} = SL_2(\mathbb{R})$ and $l(G)|_{V_2} = SO(3)$. Denote by π_i the map $\pi_i : l(\Gamma)) \to l(G)|_{V_i}, i = 1, 2$. Let $g \in SO(3)$ be an element of infinite order. Then there exists an eigenvector $v_0(g) \in \mathbb{R}^3$ with eigenvalue 1. Let $V_0(g)$ be the one- dimensional subspace of $\mathbb{R}^3$ spanned by $v_0(g)$. Let p_g be the set $V_0(g) \cap S^2$. Let $g, h \in SO(3)$ be two elements of infinite order which do not commute. Let P be the subspace of $\mathbb{R}^3$ spanned by $v_0(g)$ and $v_0(h)$. Obviously, $\dim P = 2$.

Lemma 8.1 Let $g, h \in SO(3)$ be two non-commuting elements of infinite order. Let $g(t)$ and $h(s)$ be the one parameter subgroups, such that $g(1) = g$ and $h(1) = h$. Let P be the subspace of $\mathbb{R}^3$ spanned by $v_0(g)$ and $v_0(h)$. Then for every vector $v \in \mathbb{R}^3 \backslash P$ there exist $t, s \in \mathbb{R}, t, s > 0$ such that $g(t)h(s)v = v$.

Proof Let σ be the reflection in P. Then there exist two rotations $g(t)$ and $h(s)$ such that $h(s)v = \sigma v$ and $g(t)\sigma v = v$. Thus, $g(t)h(s)v = v$.

Let $\gamma_a, \gamma_b \in \Gamma$ be two $\mathbb{R}$-regular elements. Denote by $V_0(\pi_2(l(\gamma_a^m \gamma_b^n)))$ the space spanned by $v_0(\pi_2(l(\gamma_a^m \gamma_b^n)))$ and put $p_{(n,m)} = V_0(\pi_2(l(\gamma_a^m \gamma_b^n))) \cap S^2$

Proposition 8.2. There exist two very transversal hyperbolic elements $\gamma_a, \gamma_b \in \Gamma$ such that the set $\{p_{(n,m)}, n, m \in \mathbb{Z}, n > 0, m > 0\}$, is dense in S^2.

Proof. Let γ_a and γ_b be two very transversal elements. Then the group Γ_1 generated by $l(\gamma_a)$ and $l(\gamma_b)$ contains the free group generated by $l(\gamma_a^n)$ and $l(\gamma_b^n)$ for some enough big n. Let us show that the group generated by $\pi_2(l(\gamma_a))$ and $\pi_2(l(\gamma_b))$ is dense in $SO(3)$. Indeed, if the subgroup generated by $\pi_2(l(\gamma_a))$ and $\pi_2(l(\gamma_b))$ is not dense in $SO(3)$ then it is virtually abelian. Therefore there exists G_1 a subgroup of finite index in G and nonzero vector $v, v \in V_2$ such that $\pi_2(l(g))v = v$ for every $g \in G_1$. Assume that V_1 is $l(G)$–invariant. Then L_{g_a} and L_{g_b} are parallel. Hence by the same arguments we use in the proof of Proposition 2.9, [AMS3] we conclude that Γ does not act properly discontinuously. Assume that V_2 is $l(G)$–invariant. Since the restriction $l(G)|_{V_2}$ is virtually abelian, the infinite group $[G_1, G_1]$ acts trivially on V_2. Hence $[G_1, G_1]$ has a fixed point in $\mathbb{R}^5$ that is impossible because an infinite subgroup $\Gamma \cap G_1$ acts properly discontinuously. Thus we will assume that elements $\pi_2(l(\gamma_a))$ and $\pi_2(l(\gamma_b))$ fulfill the requirements of Lemma 8.1. Let $\overline{\gamma_a} = \pi_2(\gamma_a)$ and $\overline{\gamma_b} = \pi_2(\gamma_b)$ and $\overline{\gamma_a}(t)$ and $\overline{\gamma_b}(t)$ be one parameter subgroups such that $\overline{\gamma_a}(1) = \overline{\gamma_a}$ and $\overline{\gamma_b}(1) = \overline{\gamma_b}$. The semigroup generated by $\overline{\gamma_a}$ (resp. $\overline{\gamma_b}$) is dense in $\overline{\gamma_a}(t)$ (resp. $\overline{\gamma_b}(t)$). Therefore by lemma 8.1 the set $p_{(n,m)}$ is dense in S^2.

Remark Let us recall that from (1)-(4) follows that $A^+(\gamma_a^n\gamma_b^m) \to A^+(\gamma_a)$, $A^-(\gamma_a^n\gamma_b^m) \to A^-(\gamma_b)$, $E^+_{\gamma_a^n\gamma_b^m} \to E^+_{\gamma_a}$ and $E^-_{\gamma_a^n\gamma_b^m} \to E^-_{\gamma_b}$. when $m, n \to \infty$.

There exist ε and a set of ε–hyperbolic, pairwise very ε–transversal elements $\{\gamma_0, \gamma_1, \dots, \gamma_m\} \subset \Gamma$, such that the group generated by the set $\{l(\gamma_0), l(\gamma_1), l(\gamma_2) \dots, l(\gamma_m)\}$ is a free Zariski dense subgroup of $l(G)$ freely generated by$\{l(\gamma_0), l(\gamma_1), \dots, l(\gamma_m)\}$ (see [AMS1, Proposition 3.7]). Denote by Γ_0 the group generated by the set $\{\gamma_1, \dots, \gamma_m\}$ and put $\Gamma_n = \Gamma_0\gamma_0^n, n \in \mathbb{Z}, n > 0$. Recall that any element $\gamma \in \Gamma_n, n \geq 1$ is $\varepsilon/2$–hyperbolic.

Let q_0 be the point of origin. By (8) there exists a constant $d^* = d(\varepsilon)$ such that

$$d_\Gamma = \max_{n \in \mathbb{Z}, n>0} \{d(q_0, C_\gamma), \gamma \in \Gamma_n, n \geq 1\} \leq d^*. \tag{11}$$

By definition above, $d_\gamma = d(q_\gamma, \gamma q_\gamma)$, where $q_\gamma \in C_\gamma$ such that $d(q_0, C_\gamma) = d(q_0, q_\gamma)$. Obviously $\{\gamma_0^n, n \in \mathbb{Z}, n \neq 1\} \cap \Gamma_1 = \emptyset$ and $\Gamma_n \cap \Gamma_m = \emptyset$ for $n \neq m$. Since Γ acts properly discontinuously, from (11) follows that for every Γ_n there exists an element $\gamma_n \in \Gamma_n$ such that $d_{\gamma_n} = \min\{d_\gamma, , \gamma \in \Gamma_n\}$. Set $d_n = d_{\gamma_n}$ and $I_M = \{m, m > 0, m \in \mathbb{Z} | d_m < M\}$

Lemma 8.3 . For every $M \in \mathbb{Z}, M > 0$ the set $I_M = \{m, m > 0, m \in \mathbb{Z} | d_m < M\}$ is finite.

Proof. Suppose that there exists a positive number M such that the set $I_M = \{m, m > 0, m \in \mathbb{Z} | d_m < M\}$ is infinite. It is obvious that $d(q_0, \gamma_m q_{\gamma_m}) \leq d_\Gamma + M$. Hence for all $\gamma_m, m \in I_M$ we have $B(q_0, d_\Gamma + M) \cap \gamma_m B(q_0, d_\Gamma + M) \neq \emptyset$. This is a contradiction since Γ acts properly discontinuously.

From Lemma 8.3 follows that there exists an infinite sequence $\{\gamma_m, \gamma_m \in \Gamma_m\}$ such that $d_m = d_{\gamma_m} \to \infty$ when $m \to \infty$.

Recall that $A^-(\gamma_m) \to A^-(\gamma_0)$ and $E^-_{\gamma_m} \to E^-_{\gamma_0}$ when $m \to \infty$. Since the projective space is compact we can and will assume that there are two subspaces A^+ and E^+ such that $A^+(\gamma_m) \to A^+$ and $E^+_{\gamma_m} \to E^+$ when $m \to \infty$.

Proposition 8.4. If $l(\Gamma)$ is Zariski dense in $SL_2(\mathbb{R}) \times SO(3)$ then Γ does not act properly discontinuously.

Proof. Our proof follows the same strategy that we used in the proof of [Lemma 5.1 AMS2.] Namely, we will show that there exists a constant $C = C(\varepsilon)$ such that if $d_m > C$ there exist an element γ of the group generated by $\gamma_a, \gamma_b \in \Gamma_0$ and positive number t such that $d_{\gamma^t \gamma_m} < d_{\gamma_m} = d_m$. Since, $\gamma^t \in \Gamma_0$ we will have $\gamma^t \gamma_m \in \Gamma_m$. This will contradict the definition $d_{\gamma_m} = \min\{d_\gamma, \gamma \in \Gamma_m\}$.

Using the notations from the Remark above we set $E^+_s = C_{\gamma_s} \oplus A^+$, $C_s(n,m) = E^+_s \cap E^-_{g_{(n,m)}}$, where $\gamma_{(n,m)} = \gamma_a^n \gamma_b^m$ and $C_{s,n,m} = (A^-(\gamma_0) \oplus C^-_{\gamma_s}) \cap E^+_{\gamma_{(n,m)}}$, $C_{\gamma_{(n,m)}} = E^-_{\gamma_{(n,m)}} \cap (C_{\gamma_m} \oplus A^+.)$ Let us set the following projections $\pi^-_s : C_{s,n,m} \to C_{\gamma_s}$ along $A^-(\gamma_s)$, $\pi^+_s : C_{\gamma_s} \to C_s(n,m)$ along A^+, $\pi^-_{\gamma(n,m)} : C_s(n,m) \to C_{\gamma_{(n,m)}}$ along $A^-(\gamma_{(n,m)})$ and $\pi^+_{(n,m)} : C_{\gamma_{(n,m)}} \to C_{s,n,m}$. Since elements $\gamma_{(n,m)}, \gamma_s$ are ε–transversal and ε–hyperbolic all these projections are $L(\varepsilon)$– Lipschitz. From Proposition 8.2 follows that for every positive θ there exist a finite subset $S^* \subseteq \{\gamma_a^n \gamma_b^m, n, m \in \mathbb{Z}\}$ such that $\Pi = \{p_{(n,m)}, \gamma_a^n \gamma_b^m \in S^*\}$ is a

θ–net of the sphere $S^2 \subset \mathbb{R}^3$. Namely, for every vector of norm one in V_2 there exists an element $\gamma \in S^*$ such that $|\sin \angle(v, \tau_\gamma)| < \theta$. We choose θ such that

$$\theta L(\varepsilon) < 1/4 \tag{12}$$

Let $q_{s,n,m}$ be a point in $C_{s,n,m}$ such that $\pi^-_{\gamma_s}(q_{s,n,m}) = q_s$. Then

$$q_{s,n,m}(k) = \pi^+_{\gamma(n,m)} o(\gamma_{(n.m)})^k \pi^-_{\gamma(n,m)} \pi^+_s o(\gamma_s) \pi^-_s(q_{s,n,m}) \in C_{s,n,m}$$

and $\pi^-_{\gamma(n,m)}\pi^+_s o(\gamma_s)(q_s) - \pi^-_{\gamma(n,m)}\pi^+_s(q_s) = \pi^-_{\gamma(n,m)}\pi^+_s \gamma_s q_s - \pi^-_{\gamma(n,m)}\pi^+_s(q_s) = \pi^-_{\gamma(n,m)}\pi^+_s(\gamma_s q_s - q_s)$.

Set $\pi_k : C_{s,n,m} \to C_{\gamma^k_{(n,m)}\gamma_s}$ along $A^+(\gamma^k_{(n,m)}\gamma_s) \oplus A^-(\gamma^k_{(n,m)}\gamma_s)$. Let $q_1 = \pi_k(q_{s,n,m})$, $q_2 = \pi_k(\gamma^k_{(n,m)}\gamma_s q_1)$. Then $q_2 = \gamma^k_{(n,m)}\gamma_s q_1$ It is easy to see that if the scalar product $(\tau_{\gamma_{(m,n)}}, \pi_{s,n,m}(\tau_{\gamma_s})) > 0$ then the scalar product $(\tau_{\gamma_{(-m,-n)}}, \pi_{s,-n,-m}(\tau_{\gamma_s})) < 0$. Thus we can and will assume that we take an element $\gamma_{(m,n)} \in S^*$ such that the scalar product is negative. Using the same argument we used in the proof of Lemma 5.7 [AMS2] we conclude from (12) that there exists an element $\gamma_{(n,m)} \in S^*$, a positive number $k = k(\gamma_s)$, and constants $c(\varepsilon)$ and $c(S^*)$ such that we have

$$d_{\gamma^k_{(n,m)}\gamma_s} \leq \frac{1}{4} d_{\gamma_s} + c(\varepsilon) + c(S^*)$$

Therefore if $d_{\gamma_s} > 2[c(\varepsilon) + c(S^*)]$ then $d_{\gamma^k_{(n,m)}\gamma_s} < d_{\gamma_s}$. Since $\gamma_{(n,m)} \in \Gamma_0$ this contradicts the definition of d_{γ_s} and proves the proposition.

Theorem B. *Let Γ be an affine group acting properly discontinuously on the affine space $\mathbb{R}^n, n \leq 5$. Assume that the semisimple part of the algebraic closure of Γ does not contain $SO(2,1)$ as a normal subgroup then Γ is virtually solvable.*

Proof. Let G be the Zariski closure of Γ. Assume that Γ acts properly discontinuously and the semisimple part of G is not trivial. Then the possible cases for the linear realization of $l(G)$ are listed in Case 1, (1) -(4) and Case 2, (1)-(3). By Proposition 5.2 we conclude that

Case 1, (1) -(4) are impossible. Let $l(G)$ be as in Case 2. If $l(G) = SO(3,2)$ then by [AMS 1] Γ does not act properly discontinuously. Assume that $l(G) = SO(4,1)$. Then G leaves invariant a form of signature $p = 4, q = 1$. Since $p - q > 2$ then Γ does not act properly discontinuously by [AMS 1] . In case 2 (3) Γ does not act properly discontinuously by Proposition 8.4 . This proves the statement.

Corollary. Let Γ be a crystallographic group, $\Gamma < \mathrm{Aff}\mathbb{R}^n, n \leq 5$. Then Γ is virtually solvable.

Proof. Let G be the Zariski closure of Γ. Assume that $l(G)$ does not contain $SO(2,1)$ as a normal subgroup,. Then Γ is virtually solvable by Theorem B. Assume that $l(G)$ contains $SO(2,1)$ as a normal subgroup. Then the space $\mathbb{R}^5$ is the direct sum of two $l(G)$–invariant subspace $\mathbb{R}^5 = V_1 \oplus V_2, \dim V_1 = 3, \dim V_2 = 2$. Then the real rank of every simple subgroup of $l(G)$ is ≤ 1. Hence Γ is virtually solvable [S],[To].

9 A geometric version of the Auslander conjecture.

The classical problem stated by Hilbert on Euclidean crystallographic groups. The groups that leave a positively definite quadratic form invariant. Thus it is natural state the following conjecture.

Conjecture *Let Γ be a crystallographic affine group $\Gamma \subseteq Aff(\mathbb{R}^n)$ leaving a non degenerated quadratic form invariant, then Γ is virtually solvable.*

Based on our recent (unpublished results) we think that the essential step toward the proof of this conjecture is to show that an answer to the question below is negative.

Problem *Does there exist a crystallographic group $\Gamma \subseteq Aff(\mathbb{R}^{2n+1})$ such that $l(\Gamma)$ is Zariski dense in $SO(n+1,n)$, n is odd ?*

We think that this is very difficult problem. The cohomological argument does not work here. Note that $\alpha(\gamma) = \alpha(\gamma^{-1})$ by 7.4. Thus there is no simple way to change the sign of a hyperbolic element of Γ and conclude that Γ does not act properly discontinuously.

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References

[A] Abels, H., *Properly discontinuous groups of affine transformations. A survey*, Geom. Dedicata 87 (2001), 309–333.

[AMS1] Abels, H., Margulis G.A., Soifer G.A.: *Semigroups containing proximal linear maps*, Israel J. of Math. 91 (1995), 1–30.

[AMS2] —: *Properly discontinuous groups of affine transformations with orthogonal linear part*, Comptes Rendus Acad. Sci. Paris 324 I (1997), 253–258.

[AMS3] —: *On the Zariski closure of the linear part of a properly discontinuous group of affine transformations*, Journal of Diff. Geom. 60, 2 (2003), 314-344.

[AMS4] —: *The linear part of an affine group acting properly discontinuously and leaving a quadratic form invariant*, Geom Dedicata (2011) 153: 1-46.

[AMS5] —: *The Auslander conjecture for dimension* < 7, https://arxiv.org/abs/1211.2525v4

[Au] Auslander, L.: *The structure of complete locally affine manifolds*, Topology 3 Suppl. 1., (1964), 131-139.

[B1] Bieberbach, L.:*Über die Bewegungsgruppen des n-dimensionalen euklidischen Raumes mit einem endlichen Fundamentalbereich*, Nachr. Königl. Ges. Wiss. Göttingen. Math. phys. Klasse (1910), 75-84.

[B2] Bieberbach, L.:*Über die Bewegungsgruppen der Euklidischen Räume (Erste Abhandlung)*,Math. Ann. 70 (1911), 297-336.

[B3] Bieberbach, L.:*Über die Bewegungsgruppen der Euklidischen Räume (Zweite Abhandlung) Die Gruppen mit einem endlichen Fundamentalbereich*, Math. Ann. 72 (1912), 400-412.

[BG] Breuillard, E, Gelander,T.: *A topological Tits alternative* , Annals of Math., 166 (2007), 427-474.

[CDGM] Charette, V., Drumm, T., Goldman, W., Morrill, M. : *Flat Affine and Lorentzian Manifolds*, Geometriae Dedicata 97, (2003), 187 -198.

[F] E. Fedorov: *Symmetrie der regelmassigen Systeme von Figuren*, l890.

[FK] R. Fricke and F. Klein: *Vorlesungen uber die Theoire der automorphen Functionen*, Teubner, Leipzig, 1897, Abschnitt I, Chapters 2 and 3.

[FG] Fried, D., Goldman W.: *Three–dimensional affine crystallographic groups*, Adv. in Math. 471 (1983), 1–49.

[DGK] Danciger, J., Gueritaud, F., Kassel, F.*Proper affine action for right-angel Coxeter groups*, Duke.Math.J. 169 (2020), 2231-2280.

[DG1] Drumm, T., Goldman, W:*Complete flat Lorenz 3–manifolds with free fundamental group*, Int. J. of Math. 1 (1990), 149-161.

[DG2] —, *The geometry of crooked planes*, Topology 38 (1999), 323-351.

[DI] Dekimpe, K., Igodt, P.: *Polycyclic-by-finite groups admit a bounded-degree polynomial structure*, Invent. Math. 129 (1997), 121-140

[GK] Goldman, W., Kamishima Y.: *The fundamental group of a compact flat Lorentz space form is virtually polycyclic*, J. Differential Geom. 19 (1984), 233–240.

[GrM] Grunewald, F., Margulis G.A.: *Transitive and quasitransitive actions of affine groups preserving a generalized Lorentz-structure*, J. Geom. Phys. 5 (1989), 493–531.

[Hil] Hilbert, D.: *Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900*. Göttingen, 1900.

[M1] Margulis, G.A.: *Complete affine locally flat manifold with a free fundamental group*, J.Soviet Math. 134 (1987), 129–139.

[M2] Margulis, G.A.: *Discrete subgroups of Lie groups*, J.Soviet Math. 134 (1987), 129–139.

[MS] Margulis, G.A., Soifer G.A.: *Maximal subgroups of infinite index of linear groups*, J of Algebra, 1981, 1, 1-31

[Me] Mess, G.: *Lorentz spacetimes of constant curvature*, Geometriae Dedicata, 126 (2007), 3-45.

[Mi1] Milnor, J.: *On fundamental groups of complete affinely flat manifolds*, Adv. Math. 25 (2) (1977), 178-187.

[Mi2] Milnor, J.: *Hilbert's problem 18: On crystallographic groups, fundamental domains, and on sphere packing* Mathematical Developments Arising from Hilbert Problems, A.M.S. Proceedings of Symposia in Pure Mathematics 28, 491-506, Amer. Math. Soc., Providence, R. I., 1976.

[MO] Mostow, G.D.: *On the fundamental group of homogeneous spaces*, Ann. of Math. 66, 1957, 249-255.

[OV] Onishchik, A.L., Vinberg E.B.: *Lie groups and algebraic groups*, Springer Verlag, Berlin and New York, 1990.

[PV] Popov, V., Vinberg E.: *Invariant Theory*, Encyclopaedia of Mathematical Science 55 (1994), 123-278.

[P] Prasad, G.: $\mathbb{R}$*-regular elements in Zariski dense subgroups*, Quart. J. Math. Oxford (2) 45 (1994), 541-545

[R] Raghunathan, M.S.: *Discrete subgroups of Lie groups*, Springer= Verlag, Berlin and New York, 1972.

[Ro] K. Rohn: *Einige Sätze über regelmässige Punktgruppen*, Math. Annalen, 53 (1900), 440-449.

[Sc] A.Schoenflies: *Krystallsysteme und Krystallstruktur*, Teubner, Leipzig, 1891.

[S1] Soifer, G.: *Affine semigroup acting properly discontinuously on a hyperbolic space*, Israel Journal of Math. 192 (2012),1-20.

[S2] Soifer, G.: *Affine Crystallographic Groups*, Amer. Math. Soc. Transl. (2) 163 No. 4 (1995), 165-170.

[T] Tits, J.: *Free subgroups in linear groups*, Journal of Algebra 20 (1972), 250–270.

[To1] Tomanov, G.M.: *The fundamental group of a generalized Lorentz space form is virtually solvable* , Preprint, Tata Institute of Fundamental Research, Bombay, 1989.

[To2] Tomanov, G.: *The virtual solvability of the fundamental group of a generalized Lorentz space form*, Journal of Diff. Geom. 32, (1990), 2, 539-547.

[To3] Tomanov, G. M.: *On a conjecture of L. Auslander*, Comptes rendus de l'Academie bulgare des Sciences. 43, (1990), 2, 9-12.

[To4] Tomanov, G.: *Properly discontinuous group actions on affine homogeneous spaces*, Tr. Mat. Inst. Steklova 292, no. Algebra, Geoemtriya i Teoriya Chisel, 268-279, Reprinted in Proc. Steklov Inst. Math. 292 (2016), no. 1, 260-271.