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BIELLIPTIC SHIMURA CURVES $X_0^D(N)$ WITH NONTRIVIAL LEVEL

OANA PADURARIU AND FREDERICK SAIA

Abstract. We work towards completely classifying all bielliptic Shimura curves $X_0^D(N)$ with nontrivial level N, extending a result of Rotger that provided such a classification for level one. Combined with prior work, this allows us to determine the list of all pairs (D, N) for which $X_0^D(N)$ has infinitely many degree 2 points, up to 2 explicit possible exceptions. As an application, we use these results to make progress on determining which curves $X_0^D(N)$ have sporadic points.

1. Introduction

The study of low degree points on classical families of modular curves over $\mathbb{Q}$, including $X_0(N)$ and $X_1(N)$, is a subject of great modern interest in number theory, by virtue of its relationship to the study of rational isogenies and torsion points of elliptic curves over number fields. In this work, we are interested in quadratic points on the family $X_0^D(N)$ of Shimura curves over $\mathbb{Q}$.

The $D = 1$ case of $X_0^1(N) \cong Y_0(N)$ recovers the elliptic modular curve setting. Whereas the rational points on $Y_0(N)$ were the subject of careful study in [Maz78], we have that $X_0^D(N)(\mathbb{R}) = \emptyset$ when $D > 1$ [Shi75, Thm. 0]. Therefore, one first asks about the degree two points on these curves, and in this work we are specifically interested in which curves $X_0^D(N)$ have *infinitely many* quadratic points. We set the following notation:

Definition 1.1. Let X be a curve over a number field F. The **arithmetic degree of irrationality** of X is the positive integer

$$\text{a.irr}_F(X) := \min\left\{ d : \left(\bigcup_{[L:F]=d} X(L) \right) \text{ is infinite} \right\}.$$

We are interested in which pairs (D, N) have $\text{a.irr}_{\mathbb{Q}}(X_0^D(N)) \leq 2$. A curve can have infinitely many rational points only if it is of genus at most one by Faltings' Theorem [Fal83]. Recall the following definition:

Definition 1.2. For a curve X over a number field F, we say that X is **bielliptic (over F)** if there exists an elliptic curve E over F and a degree 2 map $X \to E$ over F. We say that X is **geometrically bielliptic** if it is bielliptic over some finite extension of F.

A theorem of Harris–Silverman [HS91] states that if $g(X) \geq 2$ and $\text{a.irr}_F(X) = 2$, then X is either hyperelliptic or is bielliptic with a degree 2 map to an elliptic curve over F of positive rank.[1]

In the $D = 1$ case, all hyperelliptic modular curves $X_0(N)$ of genus at least 2 were determined by Ogg [Ogg74], and all bielliptic modular curves $X_0(N)$ were determined

[1]The analogue for $d = 3$ was proven by Abramovich–Harris [AH91]: if $\text{a.irr}_F(X) = 3$, then X is trigonal or is trielliptic with a degree 3 map to an elliptic curve of positive rank. This pattern fails for $\text{a.irr}_F(X) > 3$; see [DF93], and see [KV22] for further results in this vein.

by Bars [Bar99]. Further, Bars determined which bielliptic curves $X_0(N)$ have a bielliptic quotient of positive rank over $\mathbb{Q}$, and thus determined all curves $X_0(N)$ with $\mathrm{a.irr}_{\mathbb{Q}}(X_0(N)) = 2$ [Bar99, Thm. 4.3].

We therefore restrict ourselves to the $D > 1$ case from this point onwards. Voight [Voi09] listed all (D, N) for which $X_0^D(N)$ has genus zero:

$$\{(6,1),(10,1),(22,1)\},$$

and genus one:

$$\{(6,5),(6,7),(6,13),(10,3),(10,7),(14,1),(15,1),(21,1),(33,1),(34,1),(46,1)\}.$$

Work of Ogg [Ogg83] and of Guo–Yang [GY17] determines all (D, N) for which $X_0^D(N)$ is hyperelliptic over $\mathbb{Q}$:

$$\begin{aligned}\{&(6,11),(6,19),(6,29),(6,31),(6,37),(10,11),(10,23),(14,5),(15,2),\\ &(22,3),(22,5),(26,1),(35,1),(38,1),(39,1),(39,2),(51,1),(55,1),(58,1),(62,1),\\ &(69,1),(74,1),(86,1),(87,1),(94,1),(95,1),(111,1),(119,1),(134,1),\\ &(146,1),(159,1),(194,1),(206,1)\}.\end{aligned}$$

Rotger [Rot02, Theorem 7] determined all of the bielliptic Shimura curves $X_0^D(1)$. Those which are of genus at least 2 and are not also hyperelliptic are as follows:

$$\begin{aligned}D \in \{&57,65,77,82,85,106,115,118,122,129,143,\\ &166,178,202,210,215,314,330,390,462,510,546\}.\end{aligned}$$

Moreover, Rotger determined which of these curves have bielliptic quotients of positive rank over $\mathbb{Q}$, and thus completed the determination of Shimura curves $X_0^D(1)$ with $\mathrm{a.irr}_{\mathbb{Q}}(X_0^D(1)) = 2$ [Rot02, Theorem 9]. Exclusing the curves that are either hyperelliptic or of genus at most one, we are left with

$$D \in \{57,65,77,82,106,118,122,129,143,166,210,215,314,330,390,510,546\}.$$

In this work, we extend Rotger's results to the $N > 1$ case. The following result settles the determination of the bielliptic curves $X_0^D(N)$, aside from four possible exceptions:

Theorem 1.3. *Let $X_0^D(N)$ be a Shimura curve with $N > 1$. If $X_0^D(N)$ is geometrically bielliptic then the bielliptic involution is one of the Atkin–Lehner involutions w_m, and (D, N, m) is listed in Table 2 or Table 3, possibly except when (D, N) is one the following pairs:*

$$(6,25),(10,9),(33,4),(34,7).$$

For $(6, 25)$ and $(10, 9)$, although there are bielliptic involutions of Atkin–Lehner type, we remain unsure of whether the Shimura curve $X_0^D(N)$ has any bielliptic involutions that are not Atkin–Lehner involutions. For $(33, 4)$ and $(34, 7)$, the curve $X_0^D(N)$ does not admit any bielliptic involutions of Atkin–Lehner type, but it may have bielliptic involutions that are not of Atkin–Lehner type.

To reach this result, we begin with background on Shimura curves and their local points in §2 and §3, and relevant algebro-geometric results in §4. The proof then comes in §5, with the following structure:

- We reduce to finitely many candidate curves, using an explicit lower bound on the gonality of $X_0^D(N)$ based on an explicit lower bound on $g(X_0^D(N))$ and a result of Abramovich (see Theorem 4.1) that relates the gonality and the genus of a Shimura curve.
- For most of these candidates, we are able in §5.1 to use results on $\mathrm{Aut}(X_0^D(N))$ to restrict consideration to Atkin–Lehner involutions of $X_0^D(N)$.

- We then determine which candidate curves have a genus one Atkin–Lehner quotient, and work to decide which such quotients are elliptic curves over $\mathbb{Q}$.

In the course of our proof of Theorem 1.3, we also aim to determine which genus one Atkin–Lehner quotients are elliptic curves *of positive rank* over $\mathbb{Q}$. We answer this question for all but three potential exceptions:

Theorem 1.4. *Suppose that $N > 1$, that $g(X_0^D(N)) \geq 2$ and that $X_0^D(N)$ is not hyperelliptic. Then if $\text{a.irr}_{\mathbb{Q}}(X_0^D(N)) = 2$ (necessarily by virtue of $X_0^D(N)$ being bielliptic with a degree 2 map to an elliptic curve of positive rank over $\mathbb{Q}$), we have*

$$(D, N) \in \{(6, 17), (6, 23), (6, 25), (6, 41), (6, 71), (10, 13), (10, 17), (10, 29), (22, 7), (22, 17), (34, 7)\}.$$

For all pairs in this set we have $\text{a.irr}_{\mathbb{Q}}(X_0^D(N)) = 2$, except possibly for $(6, 25)$ and $(34, 7)$.

Up to determining whether the pairs $(6, 25)$ and $(34, 7)$ have

$$\text{a.irr}_{\mathbb{Q}}(X_0^D(N)) = 2,$$

this completes the determination of curves $X_0^D(N)$ with infinitely many degree 2 points. (See also the rephrasing of Theorem 1.4 to this aim as Theorem 6.3.)

As an application of our main results, in §6 we improve on a result of the second author from [Sai22, §10] (see Theorem 6.2) concerning sporadic points on the Shimura curves $X_0^D(N)$ and $X_1^D(N)$. An abridged version of our main result from this section is as follows; see Theorem 6.4 for the full statement.

Theorem 1.5. *(1) For all but at most 148 pairs (D, N) with $D > 1$ and $\gcd(D, N) = 1$, the Shimura curve $X_0^D(N)$ has a sporadic CM point. For at least 73 of these pairs, this curve has no sporadic points.*
(2) For all but at most 380 pairs (D, N) with $D > 1$ and $\gcd(D, N) = 1$, the Shimura curve $X_1^D(N)$ has a sporadic CM point. For at least 58 of these pairs, this curve has no sporadic points.

All computations described in this paper were performed using the Magma computer algebra system [BCP97], and all relevant code can be found in [PS24].

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2. Some background on Shimura curves

Let D be an indefinite rational quaternion discriminant, i.e., a product of an even number of distinct prime numbers, and let B_D denote the unique (up to isomorphism) quaternion algebra over $\mathbb{Q}$ of discriminant D. The curve $X_0^D(N)$ can then be described up to isomorphism as the coarse space for either of the following moduli problems:

- tuples (A, ι, λ, Q), where A is an abelian surface, $\iota : \mathcal{O} \hookrightarrow \text{End}(A)$ is an embedding of a maximal order $\mathcal{O} \subseteq B_D$ into the endomorphism ring of A, λ is a principal polarization of A which is compatible[2] with ι and $Q \leq A[N]$ is a cyclic $\mathcal{O}$-submodule of rank 2 as a module over $\mathbb{Z}/N\mathbb{Z}$.

[2] See [Sai22, §2.1] for details.

- triples (A, ι, λ), where A is an abelian surface, $\iota : \mathcal{O}_N \hookrightarrow \mathrm{End}(A)$ is an embedding of an Eichler order $\mathcal{O}_N \subseteq B_D$ of level N and λ is a principal polarization compatible with ι.

Similar to the first interpretation above, the curve $X_1^D(N)$ parametrizes triples (A, ι, λ, P) where (A, ι, λ) is as in the first interpretation above and $P \in A[N]$ is of order N. We call the data of (A, ι) as in any of the interpretations above a **QM abelian surface**; Shimura curves parametrize polarized QM abelian surfaces with additional structure.

There is a natural covering map $X_1^D(N) \to X_0^D(N)$ of degree $\frac{\phi(N)}{2}$. On the level of moduli, this is described as $[(A, \iota, \lambda, P)] \mapsto [(A, \iota, \lambda, \iota(\mathcal{O}) \cdot P)]$, where $\iota(\mathcal{O}) \cdot P$ is the $\mathcal{O}$-cyclic subgroup of $A[N]$ generated by P. While the curves $X_0^D(N)$ will be the main interest in this work, the curves $X_1^D(N)$ will also come into play when we study sporadic points on both families in §6.

2.1. **CM points and embedding numbers.** In this work, we will mainly be concerned with the arithmetic of these Shimura curves as algebraic curves over $\mathbb{Q}$, and not specifically with their moduli interpretations. The place where the moduli interpretation will be most relevant will be in our discussion of CM points.

Definition 2.1. Let K be an imaginary quadratic number field. A point $x \in X_0^D(N)$ or $X_1^D(N)$ is a K**-CM point** if it is induced by a QM abelian surface (A, ι) with either of the following equivalent properties:

- A is isogeneous to E^2, where E is an elliptic curve with K-CM.
- The ring $\mathrm{End}(A, \iota)$ of $\iota(\mathcal{O})$-equivariant endomorphisms of A is an order in K.

We call x a **CM point** if it is a K-CM point for some imaginary quadratic field K.

Remark 2.2. CM points are algebraic points [Shi75]. More specifically, for any K-CM point $x \in X_0^D(N)$, the corresponding residue field $\mathbb{Q}(x)$ is either a ring class field of the CM field K, or an index 2 subfield of a ring class field (necessarily totally complex if $D > 1$), as follows from the main results of [Sai22] in this generality.

As for the modular curves $X_0(N)$, one often seeks to attach a specified imaginary quadratic order R of K to a K-CM point on $X_0^D(N)$, and it seems that our definition above provides a clear way to do so: take $R := \mathrm{End}(A, \iota)$. The catch here is that if $x \in X_0^D(N)$, for example, then we have provided two moduli interpretations: should we take ι to be a QM structure by a maximal order (such that our moduli datum also has the information of a certain $\mathcal{O}$-cyclic subgroup of $A[N]$), or by an Eichler order of level N? Both are reasonable choices, and they provide the same set of K-CM points, but it is important to distinguish between these choices as they alter the notion of an R-CM point for a fixed R (see [Sai22, Remark 2.13] for a related remark).

We will need to assign imaginary quadratic orders to CM points in two places in this paper: when applying results of Ogg [Ogg83] (see Theorem 2.6 and Theorem 2.7) to count the fixed points of Atkin–Lehner involutions, and when applying a result of González–Rotger [GR06, Cor 5.14] on the residue fields of CM points on Atkin–Lehner quotients of $X_0^D(N)$. All these results use the correspondence between R-CM points on $X_0^D(N)$ and optimal embeddings of orders in quadratic number fields into Eichler orders in B_D – see Definition 2.4 and Theorem 2.5. This particular correspondence makes use of the second moduli interpretation we gave for $X_0^D(N)$, involving Eichler orders of level N. Thus, when we specify CM orders in this work, we will mean according to this interpretation.

We recall here the genus formula for $X_0^D(N)$, which can be found, for example, in [Voi21, Thm. 39.4.20]. First, some notation: Let φ and ψ be the multiplicative functions such that

$$\varphi(p^k) = p^k - p^{k-1}, \quad \psi(p^k) = p^k + p^{k-1},$$

and let $\left(\frac{\cdot}{\cdot}\right)$ denote the Kronecker quadratic symbol

Proposition 2.3. *For D an indefinite rational quaternion algebra, N a positive integer coprime to D and $k \in \{3, 4\}$, define*

$$e_k(D, N) := \prod_{p|D} \left(1 - \left(\frac{-k}{p}\right)\right) \prod_{q\|N} \left(1 + \left(\frac{-k}{q}\right)\right) \prod_{q^2|N} \delta_q(k),$$

where

$$\delta_q(k) = \begin{cases} 2 & \text{if } \left(\frac{-k}{q}\right) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $D > 1$,

$$g(X_0^D(N)) = 1 + \frac{\varphi(D)\psi(N)}{12} - \frac{e_4(D, N)}{4} - \frac{e_3(D, N)}{3}.$$

We next recall a result of Eichler which relates local embedding numbers of quadratic orders into quaternion orders to global embedding numbers. These will be relevant both in results related to local points on Shimura curves and to fixed points of Atkin–Lehner involutions. We first set relevant definitions and notation.

Definition 2.4. Let $\mathcal{O}$ be an Eichler order in B_D, let K be a quadratic number field and let R be an order in K. An **optimal embedding** of R into $\mathcal{O}$ is an injection $\iota_K : K \hookrightarrow B_D$ such that $\iota_K^{-1}(\mathcal{O}) = R$.

Two optimal embeddings ι_1 and ι_2 are **equivalent** if there is some $\gamma \in \mathcal{O}^\times$ such that $\iota_2(\alpha) = \gamma\iota_1(\alpha)\gamma^{-1}$ for all $\alpha \in K$.

For the remainder of the paper, let $\mathcal{O}_N$ denote a fixed Eichler order of level N in B_D. For p a rational prime, we let $(B_D)_p$ denote the localization of B_D at p and let $(\mathcal{O}_N)_p$ be the localization of $\mathcal{O}_N$ at p. We let $\nu(R, \mathcal{O}_N)$ denote the number of inequivalent optimal embeddings of R into $\mathcal{O}_N$, and let $\nu_p(R, \mathcal{O}_N)$ denote the number of inequivalent optimal embeddings of R_p into $(\mathcal{O}_N)_p$.

Theorem 2.5 (Eichler)**.** *Let R be an order in a quadratic number field, and let $h(R)$ be the class number of R. Then*

$$\nu(R, \mathcal{O}_N) = h(R) \prod_{p|DN} \nu_p(R, \mathcal{O}_N).$$

More precisely, suppose we are given for each $p \mid DN$ an equivalence class of optimal embeddings $R_p \hookrightarrow (\mathcal{O}_N)_p$. Then there are exactly $h(R)$ inequivalent optimal embeddings $R \hookrightarrow \mathcal{O}_N$ which are in the given local classes.

The following result of Ogg allows us to compute these local embedding numbers, and hence to compute global embedding numbers.

Theorem 2.6. [Ogg83, Theorem 2] *Let f be the conductor of R. Then $\nu_p := \nu_p(R, \mathcal{O}_N)$ is given below, according to various cases; divisibility is to be understood as in $\mathbb{Z}_p$, and ψ_p is the multiplicative function with $\psi_p(p^k) = p^k(1 + 1/p)$ and $\psi_p(n) = 1$ if $p \nmid n$.*

(i) If $p \mid D$, then $\nu_p = 1 - \left(\frac{R}{p}\right)$.

(ii) If $p \| N$, then $\nu_p = 1 + \left(\frac{R}{p}\right)$.

(iii) Suppose $p^2 \mid N$.

(a) *If* $(pf)^2 \mid N$, *then*

$$\nu_p = \begin{cases} 2\psi_p(f) & \text{if } \left(\frac{L}{p}\right) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If* $pf^2 \parallel N$ *(say* k *is such that* $p^k \parallel f$*), then*

$$\nu_p = \begin{cases} 2\psi_p(f) & \text{if } \left(\frac{L}{p}\right) = 1, \\ p^k & \text{if } \left(\frac{L}{p}\right) = 0, \\ 0 & \text{if } \left(\frac{L}{p}\right) = -1. \end{cases}$$

(c) *If* $f^2 \parallel N$ *(say* k *is such that* $p^k \parallel f$*), then*

$$\nu_p = p^{k-1}\left(p + 1 + \left(\frac{L}{p}\right)\right).$$

(d) *If* $pN \mid f^2$, *then*

$$\nu_p = \begin{cases} p^k + p^{k-1} & \text{if } N \parallel p^{2k}, \\ 2p^k & \text{if } N \sim p^{2k+1}. \end{cases}$$

2.2. **The Atkin–Lehner group.** Consider the group of norm 1 units in our Eichler order

$$\mathcal{O}_N^1 := \{\gamma \in \mathcal{O}_N \mid \mathrm{nrd}(\gamma) = 1\}.$$

Elements of the normalizer subgroup

$$N_{B_D{}^{\times}_{>0}}(\mathcal{O}_N^1) := \{\alpha \in B_D{}^{\times} \mid \mathrm{nrd}(\alpha) > 0 \text{ and } \alpha^{-1}\mathcal{O}_N^1\alpha = \mathcal{O}_N^1\}$$

naturally act on $X_0^D(N)$ via action on the QM-structure ι, with $\mathbb{Q}^\times \hookrightarrow B_D{}^\times$ acting trivially. The full **Atkin–Lehner group** is the group

$$W_0(D,N) := N_{B_D{}^{\times}_{>0}}(\mathcal{O}_N^1)/\mathbb{Q}^\times\mathcal{O}_N^1 \subseteq \mathrm{Aut}(X_0^D(N)).$$

The group $W_0(D,N)$ is a finite abelian 2-group, with an **Atkin–Lehner involution** w_m associated to each Hall divisor of DN. That is,

$$W_0(D,N) = \{w_m \mid m \parallel DN\} \cong (\mathbb{Z}/2\mathbb{Z})^{\omega(DN)},$$

where $\omega(DN)$ denotes the number of prime divisors of DN.

The following result says that fixed points of Atkin–Lehner involutions correspond to optimal embeddings of specific imaginary quadratic orders, i.e., to CM points by specific orders.

Theorem 2.7. [Ogg83, p. 283] *The fixed points by the Atkin–Lehner involution of level* $m \parallel DN$ *acting on* $X_0^D(N)$ *are* CM *points by*

$$R = \begin{cases} \mathbb{Z}[i], \mathbb{Z}[\sqrt{-2}] & \text{if } m = 2; \\ \mathbb{Z}\left[\frac{1+\sqrt{-m}}{2}\right], \mathbb{Z}[\sqrt{-m}] & \text{if } m \equiv 3 \pmod 4; \\ \mathbb{Z}[\sqrt{-m}] & \text{otherwise.} \end{cases}$$

The number of fixed points corresponding to any of the previous orders R *is given by*

$$h(R) \prod_{p \mid \frac{DN}{m}} \nu_p(R, \mathcal{O}_N).$$

Consequently, the Fricke involution w_{DN} always has fixed points:

Corollary 2.8. *Assume $D > 1$. Then the number of fixed points of $X_0^D(N)$ by the Fricke involution is*

$$\#X_0^D(N)^{w_{DN}} = \begin{cases} h\left(\mathbb{Z}\left[\frac{1+\sqrt{-DN}}{2}\right]\right) + h(\mathbb{Z}[\sqrt{-DN}]) & \text{if } DN \equiv 3 \pmod 4, \\ h(\mathbb{Z}[\sqrt{-m}]) & \text{otherwise.} \end{cases}$$

3. Local points

To prove that $X_0^D(N)/\langle w_m \rangle$ has no $\mathbb{Q}$-rational points, it is sufficient to prove the non-existence of points over $\mathbb{R}$ or over $\mathbb{Q}_p$ for some prime p. We will use such arguments later in determining which genus one Atkin–Lehner quotients are in fact elliptic curves over $\mathbb{Q}$, so in this section we recall results on local points on these quotients.

3.1. **Real points.** We mentioned in the introduction that when $D > 1$, we have

$$X_0^D(N)(\mathbb{R}) = \emptyset.$$

In [Ogg83], Ogg completes a study of real points on quotients $X_0^D(N)/\langle w_m \rangle$ for $m \parallel DN$. As $\left(X_0^D(N)/\langle w_m \rangle\right)(\mathbb{R})$ is a real manifold of dimension one, it is a disjoint union of circles. The number of connected components is related to the number of classes of certain optimal embeddings of orders in the real quadratic field $\mathbb{Q}(\sqrt{m})$ into $\mathcal{O}_N$; we summarize Ogg's results here:

Theorem 3.1. [Ogg83, Proposition 1, Theorem 3] *Let $D > 1$ and $m \parallel DN$, and set*

$$\nu(m) = \sum_R h(R) \prod_{p \mid \frac{DN}{m}} \nu_p(R, \mathcal{O}_N),$$

where R ranges over $\mathbb{Z}[\sqrt{m}]$, and also $\mathbb{Z}\left[\frac{1+\sqrt{m}}{2}\right]$ if $m \equiv 1 \pmod 4$.

Let $\#(m)$ denote the number of connected components of $\left(X_0^D(N)/\langle w_m \rangle\right)(\mathbb{R})$. If m is a square, then $\#(m) = 0$, i.e., this quotient has no real points. If m is not a square, then

$$\#(m) = \nu(m)/2,$$

unless $\nu(m) > 0, i \in \mathcal{O}_N, DN = 2t$, with t odd, $m = t$ or $2t$, and $x^2 - my^2 = \pm 2$ is solvable with $x, y \in \mathbb{Z}$, in which case

$$\#(m) = \left(\nu(m) + 2^{\omega(DN)-2}\right)/2.$$

3.2. **p-adic points.** We next recall results on $\mathbb{Q}_p$ points on $X_0^D(N)$ and certain Atkin–Lehner quotients thereof, coming from work of Ogg [Ogg85] and thesis work of Clark [Cla03].

Theorem 3.2. [Ogg85, Théorème, p. 206] *Let p be a prime dividing D, and let $\mathcal{O}_N$ be an Eichler order of level N in B_D. Suppose that $m \parallel \frac{DN}{p}$ and $m > 1$.*

Now let $\widehat{B}$ denote the definite (ramified at infinity) quaternion algebra over $\mathbb{Q}$ of discriminant D/p, let $\widehat{\theta}$ be a choice of Eichler order of level N in $\widehat{B}$ and let $h = h\left(\widehat{\theta}\right)$ be the class number of $\widehat{\theta}$. Let $\widehat{\theta}_1, \ldots, \widehat{\theta}_h$ denote the inequivalent Eichler orders of level N in $\widehat{B}$.

- *(i) $X_0^D(N)(\mathbb{Q}_p)$ is non-empty if and only if $p = 2$ and $\sqrt{-1} \in \mathcal{O}_N$, or $p \equiv 1 \pmod 4$, $N = 1$ and $D = 2p$.*
- *(ii) If $X_0^D(N)(\mathbb{Q}_p)$ is empty, then $(X_0^D(N)/\langle w_m \rangle)(\mathbb{Q}_p)$ is non-empty if and only if one of the following holds:*
  - *a) $p = 2$, $m = \frac{DN}{p}$ and $\sqrt{-2} \in \mathcal{O}_N$,*

b) $p > 2$, $\sqrt{-p} \in \mathcal{O}_N$, $\left(\frac{-m}{p}\right) = 1$, $\frac{DN}{p} \in \{m, 2m\}$ *and* $\big(8|(p+1)(m+1)$ *if* $\frac{DN}{p} = 2m$ *and* $2|N\big)$, *or*

c) $p \equiv 1 \pmod 4$, $\sqrt{-1} \in \widehat{\theta}_i$ *for some* $1 \le i \le h$, $\frac{DN}{p} \in \{m, 2m\}$ *and* $\sqrt{-pm} \in \mathcal{O}_N$.

(iii) $(X_0^D(N)/\langle w_p \rangle)(\mathbb{Q}_p)$ *is non-empty if and only if there is an index* $1 \le i \le h$ *such that* $\widehat{\theta}_i$ *contains* $\sqrt{-p}$ *or a root of unity not equal to* ± 1.

(iv) If $X_0^D(N)(\mathbb{Q}_p)$ *is empty, then* $(X_0^D(N)/\langle w_{pm} \rangle)(\mathbb{Q}_p)$ *is non-empty if and only if there is some* $1 \le i \le h$ *with* $\sqrt{-m} \in \widehat{\theta}_i$, *or* $\sqrt{-1} \in \widehat{\theta}_i$ *in the case of* $m = 2$.

Theorem 3.3. [Cla03, Main Theorem 3a)] *Assume* $D = pq$ *is a product of two primes and that* N *is a prime number. Then* $(X_0^{pq}(N)/\langle w_{pq} \rangle)(\mathbb{Q}_p)$ *is non-empty if and only if* N *is not inert in* $\mathbb{Q}(\sqrt{-q})$.

Remark 3.4. In [Cla03], it is claimed that $(X_0^{pq}(N)/\langle w_{pq} \rangle)(\mathbb{Q}_p)$ is nonempty if and only if N is a norm from $\mathbb{Q}(\sqrt{-q})$. After corresponding with the author, we concluded that the condition should be "N is not inert in $\mathbb{Q}(\sqrt{-q})$" as written above. The proof for this corrected statement is as given in [Cla03] for the original statement, with the modification of changing each statement about an *element* with a specified norm to the same statement but regarding an *ideal* of that specified norm.

4. Algebraic geometry tools

For a field F and a curve C/F, the F-**gonality** $\mathrm{gon}_F(C)$ is defined to be the least degree of a non-constant morphism $f : C \to \mathbb{P}^1$ defined over F. For $\overline{F}$ an algebraic closure of F, the $\overline{F}$-gonality is also called the **geometric gonality**. We will use the following result to obtain an upper bound on the genus of $X_0^D(N)$:

Theorem 4.1. [Abr96, Theorem 1.1] *For a Shimura curve* $X_0^D(N)$, *we have*

$$g(X_0^D(N)) \le \frac{200}{21}\,\mathrm{gon}_{\mathbb{C}}(X_0^D(N)) + 1.$$

Proposition 4.2. *Let* $f : X \to Y$ *be a non-constant morphism of curves over* F. *Then*

$$\mathrm{gon}_F(X) \le \deg(f)\,\mathrm{gon}_F(Y).$$

Proposition 4.3. [HS91] *If* X *is a bielliptic curve, and if* $X \to Y$ *is a finite map, then* Y *is either hyperelliptic, bielliptic, or it has genus at most one.*

Corollary 4.4. *The gonality of a bielliptic curve is at most* 4.

Lemma 4.5. [BKX13, Lemma 4.3] *Consider a Galois cover* $\phi : X \to Y$ *of degree* d *between two non-singular projective curves of genus* $g_X \ge 2$ *and* g_Y, *respectively. Suppose that* $g_Y \ge 2$ *or* d *is odd.*

(1) Suppose that $2g_X + 2 > d(2g_Y + 2)$. *Then* X *is not hyperelliptic.*

(2) Denote by $\#Y^\sigma$ *the number of fixed points of an involution* σ *of* Y. *Suppose* $2g_X - 2 > d \cdot \#Y^\sigma$ *for any involution* σ *on* Y. *Then, if* $g_X \ge 6$, X *is not bielliptic.*

(3) Suppose $2g_X - 2 > d(2g_Y + 2)$. *Then, if* $g_X \ge 6$, X *is not bielliptic.*

Lemma 4.6. [Rot02, Lemma 5.(2)] *Let* C/K, *char*$(K) \ne 2$, *be a bielliptic curve of genus* g *with* $\mathrm{Aut}(C) \cong C_2^s$ *for some* $s \ge 1$.

- *If* g *is even, then* $s \le 3$.
- *If* g *is odd, then* $s \le 4$.

The following result follows from the Riemann–Hurwitz formula:

Proposition 4.7. *Let σ be any involution on a smooth projective curve X over an algebraically closed field F of characteristic 0, and let $\#X^\sigma$ denote the number of fixed points of σ. Then, we have the following genus formula:*

$$g(X/\langle\sigma\rangle) = \frac{1}{4}(2g(X) + 2 - \#X^\sigma).$$

Remark 4.8. A curve of genus g is geometrically bielliptic if and only if there is an involution with $2g-2$ fixed points.

Lemma 4.9. [BKS23, Lemma 4.3] *Let σ be an involution of X with more than 8 fixed points. Then either σ is a bielliptic involution or X is not bielliptic.*

Lemma 4.10. [BKS23, Proposition 4.8] *Let X be a curve of genus g at least 6. Assume that $\mathrm{Aut}(X)$ has a subgroup H of order 2^t such that $2^t \nmid 2(g-1)$. Then either the bielliptic involution of X is contained in H or X is not bielliptic.*

The group $W_0(D,N)$ is a subgroup of $\mathrm{Aut}(X_0^D(N))$ of order $2^{\omega(DN)}$. We therefore have the following corollary:

Corollary 4.11. *Suppose that $g(X_0^D(N)) \geq 6$ and that $X_0^D(N)$ is geometrically bielliptic. If $g(X_0^D(N)) \not\equiv 1 \pmod{2^{\omega(DN)-1}}$, then the bielliptic involution is an Atkin–Lehner involution.*

5. Proof of Theorem 1.3 and Theorem 1.4

From Theorem 4.1 and Corollary 4.4, we find that a bielliptic Shimura curve $X_0^D(N)$ must have genus $g(X_0^D(N)) \leq 39$.

Lemma 5.1. [Sai22, Lemma 10.6] *For $D > 1$ an indefinite rational quaternion discriminant and $N \in \mathbb{Z}^+$ relatively prime to D, we have*

$$g(X_0^D(N)) > 1 + \frac{DN}{12}\left(\frac{1}{e^\gamma \log\log(DN) + \frac{3}{\log\log 6}}\right) - \frac{7\sqrt{DN}}{3}.$$

With Lemma 5.1, we find that if $DN > 78530$ then we have that $g(X_0^D(N)) > 39$ and thus $X_0^D(N)$ is not bielliptic.

If $X_0^D(N)$ is bielliptic, then $X_0^D(1)$ must be bielliptic, hyperelliptic, or of genus $g(X_0^D(1)) \leq 1$ by Proposition 4.3. By prior results of Voight [Voi09] (for genus at most one), Ogg [Ogg83] (for hyperelliptic of genus at least 2), and Rotger [Rot02] (for bielliptic), it follows that

$$D \in \{6, 10, 14, 15, 21, 22, 26, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 86, 87, 94, 95, 106, 111, 118, 119, 122, 129, 134, 143, 146, 159, 166, 194, 206, 210, 215, 314, 330, 390, 510, 546\}.$$

Using this and our genus bound Lemma 5.1, we arrive at 350 candidate pairs (D,N). All of these pairs have $\omega(D) = 2$, and N is squarefree for all but 55 of these pairs.

5.1. Automorphisms of candidate pairs.

In this section, we prove for certain candidate pairs (D,N) that $\mathrm{Aut}(X_0^D(N)) = W_0(D,N)$ is the group of Atkin–Lehner involutions. This will help us determine when this Shimura curve is *not* bielliptic (over $\mathbb{Q}$), as for such pairs we do not have the possibility that there could be a bielliptic involution not in $W_0(D,N)$.

Our main tools come from [KR08], which extends work of [Rot02]. In particular, we make use of the following result:

Lemma 5.2. *Suppose that $X_0^D(N)$ is a Shimura curve with N is squarefree and that $g := g(X_0^D(N)) \geq 2$. If any of the following statements holds:*

(1) $e_3(D,N) = e_4(D,N) = 0$*,*

(2) $2 \mid DN$*, for all primes* $p \mid N$ *we have* $\left(\frac{-4}{p}\right) \neq -1$ *and for at most one prime* $p \mid D$ *we have* $\left(\frac{-4}{p}\right) = 1$*.*

(3) $3 \mid DN$*, for all primes* $p \mid N$ *we have* $\left(\frac{-3}{p}\right) \neq -1$ *and for at most one prime* $p \mid D$ *we have* $\left(\frac{-3}{p}\right) = 1$*.*

(4) $\omega(DN) = \mathrm{ord}_2(g-1) + 2$*,*

then $\mathrm{Aut}(X_0^D(N)) = W_0(D,N)$*.*

Proof. The first part is [KR08, Thm 1.6 (i)], while the second and third parts are [KR08, Thnm 1.7 (i)]. The fourth part follows from [KR08, Thm 1.6 (iii)]. □

Lemma 5.3. *Suppose that (D,N) is a candidate pair with N squarefree and $g(X_0^D(N)) \geq 2$. If either*

- *g is even and $\omega(DN) = 3$, or*
- *g is odd and $\omega(DN) = 4$*

and $X_0^D(N)$ is bielliptic, then $\mathrm{Aut}(X_0^D(N)) = W_0(D,N)$.

Proof. Our hypotheses on N and the genus imply $\mathrm{Aut}(X_0^D(N)) \cong (\mathbb{Z}/2\mathbb{Z})^s$ for some $s \geq r$ by [KR08, Prop 1.5]. The result then follows from Lemma 4.6. □

Using Lemma 5.2 and Lemma 5.3 and explicit computations, we find that if $X_0^D(N)$ is bielliptic then $\mathrm{Aut}(X_0^D(N)) = W_0(D,N)$ for all of our candidate pairs (D,N) with N squarefree and $g(X_0^D(N)) \geq 2$, except for possibly the following 25 pairs:

$$\{(10,19),(10,31),(10,43),(10,67),(10,79),(10,103),(21,5),(21,17),$$
$$(21,29),(22,7),(22,31),(33,5),(33,17),(34,7),(34,19),(46,7),$$
$$(55,7),(57,5),(58,7),(69,5),(77,5),(82,7),(94,7),(106,7),(118,7)\}.$$

The genera of the curves $X_0^D(N)$ for the 25 pairs in this set are among the set

$$\{5,9,13,17,21,25,29,33,37\}.$$

We can exclude $(D,N) \in \{(10,31),(33,5)\}$ from further consideration, as the two corresponding curves $X_0^D(N)$ have a bielliptic Atkin–Lehner involution (see Table 2), which must be unique given that $g(X_0^D(N)) \geq 6$ for both.

Remark 5.4. By [KMV11], a curve of genus 5 can have $0,1,2,3$ or 5 bielliptic involutions. The genus 5 curves with 5 biellliptic involutions are called **Humbert curves** and form a 2-dimensional family. Their automorphism groups have order 160.

For $(D,N) \in \{(21,5),(22,7)\}$, there are three bielliptic involutions of Atkin–Lehner type (see Table 2). Thus either $X_0^D(N)$ is a Humbert curve, or it has exactly three bielliptic involutions. Since a Humbert curve has 160 automorphisms and $\mathrm{Aut}(X_0^D(N))$ is a 2-group, it follows that the bielliptic involutions of $X_0^D(N)$ are of Atkin–Lehner type.

5.2. **Squarefree level N.** We have 295 candidate pairs (D,N) where N is squarefree. For all except for 55 of these pairs, there exists $m|DN$ such that

$$\#X_0^D(N)^{w_m} \neq 2g(X_0^D(N)) - 2 \quad \text{and} \quad \#X_0^D(N)^{w_m} > 8.$$

Thus by Lemma 4.9, we can conclude that these 240 pairs do not correspond to bielliptic Shimura curves. We know that 41 of the remaining 55 admit (at least) a bielliptic Atkin–Lehner involution by explicit genus computations. We attempt using Lemma 4.5(3) to prove that the remaining 14 are not bielliptic, which works in all cases except $(D, N) = (34, 7)$.

For a triple (D, N, m) for which the quotient $X_0^D(N)/\langle w_m \rangle$ has genus one, we may list out all of the quadratic CM points on $X_0^D(N)$ using the results of [GR06] or of [Sai22]. For N squarefree, we can then use [GR06, Cor 5.14] to determine, for each quadratic point P on $X_0^D(N)$, the residue field of the image of P on $X_0^D(N)/\langle w_m \rangle$ under the natural quotient map. Doing so, we determine all of the rational CM points of genus one quotients $X_0^D(N)/\langle w_m \rangle$ with N squarefree:

Proposition 5.5. *Let $D > 1$ be an indefinite rational quaternion discriminant, let $N > 1$ squarefree and relatively prime to D and let $m \parallel DN$ such that $X_0^D(N)/\langle w_m \rangle$ has genus one. If $X_0^D(N)/\langle w_m \rangle$ has a $\mathbb{Q}$-rational K-CM point for an imaginary quadratic field K, then this information is listed in Table 1.*

For each triple (D, N, m) appearing in this table, we therefore have that $X_0^D(N)/\langle w_m \rangle$ is an elliptic curve over $\mathbb{Q}$, and hence $X_0^D(N)$ is bielliptic over $\mathbb{Q}$.

Remark 5.6. If N is not squarefree, then the work of [Sai22] can still provide a full list of quadratic CM points on $X_0^D(N)$. The squarefree restriction on this method comes from the determination in [GR06, Cor. 5.14] of the field of moduli of the image of a CM point on $X_0^D(N)$ under an Atkin–Lehner quotient. With the squarefree restriction on N, the work of [GR06] is enough to list all of the quadratic CM points on $X_0^D(N)$.

Table 1: Rational CM points on genus one quotients $X_0^D(N)/\langle w_m \rangle$ with $N > 1$ squarefree

(D, N, m)	Δ_K such that $X_0^D(N)/\langle w_m \rangle$ has a $\mathbb{Q}$-rational K-CM point
$(6, 5, 15)$	-4
$(6, 7, 14)$	-3
$(6, 13, 26)$	-3
$(6, 13, 39)$	-4
$(6, 17, 51)$	-4
$(6, 17, 102)$	$-4, -19, -43, -67$
$(6, 23, 138)$	$-19, -43, -67$
$(6, 41, 246)$	$-4, -43, -163$
$(6, 71, 426)$	$-67, -163$
$(10, 3, 10)$	-3
$(10, 3, 15)$	-8
$(10, 13, 130)$	$-3, -43$
$(10, 17, 170)$	$-8, -43, -67$
$(10, 29, 290)$	-67
$(14, 3, 21)$	-8
$(14, 3, 42)$	$-8, -11$
$(14, 5, 35)$	-4
$(14, 5, 70)$	$-4 - 11$
$(14, 13, 182)$	$-4, -43$
$(14, 19, 266)$	$-8, -67$
$(15, 2, 30)$	-7

$(15, 7, 105)$	$-3, -7$
$(15, 13, 195)$	$-3, -43$
$(15, 17, 255)$	$-43, -67$
$(21, 2, 21)$	$-4, -7$
$(21, 2, 42)$	$-4, -7$
$(21, 5, 105)$	-4
$(21, 11, 231)$	$-7, -43$
$(22, 7, 154)$	-3
$(22, 17, 374)$	$-4, -67$
$(33, 2, 66)$	$-3, -4$
$(33, 7, 231)$	-3
$(35, 2, 35)$	-7
$(46, 5, 230)$	-4

In the thesis work of Nualart-Riera [NR15], one finds defining equations for the Atkin–Lehner quotients of $X_0^D(N)$ for (D, N) in

$$\{(6,5),(6,7),(6,11),(10,3),(10,7),(10,9),(22,5),(22,7)\}.$$

As defining equations for Shimura curves and their quotients are difficult to compute, for squarefree level N one may use Ribet's isogeny to compute $\mathrm{rk}(\mathrm{Jac}(X_0^D(N)/\langle w_m\rangle)(\mathbb{Q}))$:

Theorem 5.7. [Rib80],[BD96] *Let $X_0^D(N)$ be a Shimura curve with squarefree level N. Then there exists an isogeny defined over $\mathbb{Q}$*

$$\psi : J_0(DN)^{D-\mathrm{new}} \to \mathrm{Jac}(X_0^D(N)), \tag{1}$$

such that, for each $w_m(D, N) \in W_0(D, N)$, we have

$$\psi^*(w_m(D,N)) = (-1)^{\omega(\gcd(D,m))} w_m(1, DN) \in \mathrm{Aut}_{\mathbb{Q}}(J_0(DN)). \tag{2}$$

As we usually do not have defining equations for Shimura curves, we are going to use Ribet's isogeny in the following way. Magma is able to decompose

$$J_0(DN) = (J_0(DN))^{m+} \oplus (J_0(DN))^{m-}.$$

We pick one of the two subspaces according to the parity of $\omega(\gcd(D, m))$, and after taking the D-new part we obtain an abelian variety isogenous to $\mathrm{Jac}(X_0^D(N)/\langle w_m\rangle)$. In the cases we are left to study, if we know that $X_0^D(N)/\langle w_m\rangle$ is an elliptic curve over $\mathbb{Q}$, we are able to use Magma to compute $\mathrm{rk}_{\mathbb{Q}}(\mathrm{Jac}(X_0^D(N)/\langle w_m\rangle))$.

In summary, we determine all 41 triples (D, N, m) with relatively prime $D, N > 1$ and N squarefree such that $X_0^D(N)/\langle w_m\rangle$ has genus one. We then work to determine, for each triple, whether the corresponding quotient is an elliptic curve over $\mathbb{Q}$: we use CM points or the results of [NR15],[PS23] to answer this in the affirmative, and use the results on local points from §3 or the results of [NR15] to answer this in the negative. When this answer is positive, we use Ribet's isogeny to determine ranks over $\mathbb{Q}$.

Our results are summarized in Table 2. Reasonings for whether the quotient is rationally bielliptic provided in the table should not be taken to be exhaustive; in some places we provide multiple reasonings, but more arguments of the types we use than just those listed may be successful. The only triple for which we are unable to determine whether the quotient has a rational point is $(6, 23, 69)$, but we still provide the rank of $\mathrm{Jac}\left(X_0^6(23)/\langle w_{69}\rangle\right)$ over $\mathbb{Q}$.

Table 2: Squarefree N and bielliptic Atkin–Lehner involutions

(D,N,m)	$g(X_0(D,N))$	bielliptic over $\mathbb{Q}$?	reason
$(6,5,3)$	1	no	[NR15]
$(6,5,5)$	1	no	[NR15]
$(6,5,15)$	1	yes (rank 0)	[NR15], CM-points
$(6,7,2)$	1	no	[NR15],[Ogg83]
$(6,7,7)$	1	yes (rank 0)	[NR15]
$(6,7,14)$	1	yes (rank 0)	[NR15], CM-points
$(6,11,6)$	3	no	[Cla03],[NR15],[PS23]
$(6,11,22)$	3	no	[NR15],[PS23]
$(6,11,33)$	3	no	[NR15],[PS23]
$(6,13,6)$	1	no	[Cla03]
$(6,13,26)$	1	yes (rank 0)	CM-points
$(6,13,39)$	1	yes (rank 0)	CM-points
$(6,17,2)$	3	no	[PS23]
$(6,17,51)$	3	yes (rank 0)	[PS23]
$(6,17,102)$	3	yes (rank 1)	[PS23]
$(6,19,3)$	3	no	[PS23]
$(6,19,19)$	3	no	[PS23]
$(6,19,57)$	3	no	[PS23]
$(6,23,46)$	5	no	[Ogg85]
$(6,23,69)$	5	**not known** (rank 0)	N/A
$(6,23,138)$	5	yes (rank 1)	CM-points
$(6,41,246)$	7	yes (rank 1)	CM-points
$(6,43,129)$	7	no	[Ogg83]
$(6,47,94)$	9	no	[Ogg85]
$(6,71,426)$	13	yes (rank 1)	CM-points
$(10,3,6)$	1	no	[NR15]
$(10,3,10)$	1	yes (rank 0)	[NR15], CM-points
$(10,3,15)$	1	yes (rank 0)	[NR15], CM-points
$(10,7,2)$	1	no	[NR15],[Ogg83]
$(10,7,7)$	1	no	[NR15]
$(10,7,14)$	1	no	[NR15]
$(10,13,10)$	3	no	[Cla03], [PS23]
$(10,13,13)$	3	no	[PS23]
$(10,13,130)$	3	yes (rank 1)	[PS23], CM-points
$(10,17,170)$	7	yes (rank 1)	CM-points
$(10,29,290)$	11	yes (rank 1)	CM-points
$(10,31,62)$	9	no	[Ogg85]
$(14,3,2)$	3	no	[PS23]
$(14,3,21)$	3	yes (rank 0)	[PS23], CM-points
$(14,3,42)$	3	yes (rank 0)	[PS23], CM-points
$(14,5,2)$	3	no	[PS23]
$(14,5,35)$ [3]	3	yes (rank 0)	[PS23], CM-points,

[3]There is a typo in [GY17]. The formula of $w_{35} \in W_0(14,5)$ has an extra minus sign in the second coordinate.

$(14,5,70)$	3	yes (rank 0)	[PS23], CM-points
$(14,13,182)$	7	yes (rank 0)	CM-points
$(14,19,266)$	11	yes (rank 0)	CM-points
$(15,2,3)$	3	yes (rank 0)	[PS23]
$(15,2,10)$	3	no	[PS23]
$(15,2,30)$	3	yes (rank 0)	[PS23], CM-points
$(15,7,3)$	5	no	[Ogg85]
$(15,7,7)$	5	no	[Ogg85]
$(15,7,105)$	5	yes (rank 0)	CM-points
$(15,11,55)$	9	no	[Ogg85]
$(15,13,195)$	9	yes (rank 0)	CM-points
$(15,17,255)$	13	yes (rank 0)	CM-points
$(21,2,2)$	3	no	[Cla03], [PS23]
$(21,2,21)$	3	yes (rank 0)	[PS23]
$(21,2,42)$	3	yes (rank 0)	[PS23]
$(21,5,15)$	5	no	[Ogg85]
$(21,5,21)$	5	no	[Cla03]
$(21,5,105)$	5	yes (rank 0)	CM-points
$(21,11,231)$	13	yes (rank 0)	CM-points
$(22,3,3)$	3	no	[NR15],[PS23]
$(22,3,11)$	3	no	[NR15],[PS23]
$(22,3,33)$	3	no	[NR15],[PS23]
$(22,7,14)$	5	no	[NR15]
$(22,7,77)$	5	yes (rank 0)	[NR15]
$(22,7,154)$	5	yes (rank 1)	CM-points
$(22,17,374)$	15	yes (rank 1)	CM-points
$(26,5,26)$	7	no	[Cla03]
$(33,2,66)$	5	yes (rank 0)	CM-points
$(33,5,55)$	9	no	[Ogg85]
$(33,7,231)$	13	yes (rank 0)	CM-points
$(34,3,17)$	5	no	[Ogg83]
$(35,2,35)$	7	yes (rank 0)	CM-points
$(35,3,35)$	9	no	[Cla03]
$(38,3,38)$	7	no	[Cla03]
$(46,5,230)$	11	yes (rank 0)	CM-points

5.3. **Non-squarefree level** N**.** We have 55 candidate pairs (D,N) where N is not squarefree. For all these pairs except

$$(6,25),(10,9),(14,9),(15,8),(21,4),(22,9),(33,4),(39,4)$$

there exists $m \parallel DN$ such that

$$\#X_0^D(N)^{w_m} \neq 2g(X_0^D(N))-2, \quad \text{and} \quad \#X_0^D(N)^{w_m} > 8,$$

thus, by Lemma 4.5 (3), $X_0^D(N)$ is not bielliptic.

For $(D,N) \in \{(14,9),(21,4),(22,9),(33,4)\}$ we have $g \in \{7,11\}$, and using Corollary 4.11 we obtain that $X_0^D(N)$ can only have a bielliptic involution of Atkin–Lehner type.

For $(D,N) \in \{(15,8),(21,4),(39,4)\}$ we have that $X_0^D(N)$ has a bielliptic Atkin–Lehner involution. As $g(X_0^D(N)) > 6$, it follows that this bielliptic involution is unique.

For $(D,N) \in \{(6,25),(10,9)\}$ the curve $X_0^D(N)$ has genus 5, so while this curve has exactly one bielliptic involution of Atkin–Lehner type we are not sure whether it also admits a bielliptic involution that is not of Atkin–Lehner type.

Remark 5.8.
- We know that $g(X_0^{39}(4)) = 13 > 6$ and that w_{13} is a bielliptic involution, thus it is the only one. The prime 13 is split in $\mathbb{Q}(i)$, so $\mathbb{Q}(i)$ does not split B_{39}. Then [Ogg85] says that $X_0^{39}(4)(\mathbb{Q}_2) = \emptyset$, and because $13 \neq 39$ part ii says $\left(X_0^{39}(4)/\langle w_{13}\rangle\right)(\mathbb{Q}_2) = \emptyset$.
- We know that $g(X_0^{15}(8)) = 9 > 6$ and that w_{15} is a bielliptic involution, thus it is the only one.
- $X_0^{15}(4)$ is a genus 5 geometrically hyperelliptic curve [GY17]. Since a hyperelliptic curve of genus $g \geq 4$ cannot be bielliptic, it follows that $X_0^{15}(4)$ is not bielliptic.
- The curve $X_0^{33}(4)$ has genus 11, and while it does not have a bielliptic involution of Atkin–Lehner type we remain unsure of whether it is geometrically bielliptic. That said, this curve is a cover of $X_0^{33}(2)$, which does not have infinitely many quadratic points, and hence $X_0^{33}(4)$ does not have infinitely many quadratic points; if $X_0^{33}(4)$ is geometrically bielliptic, it does not have a bielliptic quotient which is an elliptic curve of positive rank over $\mathbb{Q}$.

Remark 5.9. For $(10,9)$ we have equations for Atkin–Lehner quotients. Using [NR15], we know that $\mathrm{Jac}(X_0^{10}(9))$ is a product of 5 distinct rank 0 elliptic curves with the following Cremona references: "30a1","30a2","90a3","90b3","2880e1". Thus $X_0^{10}(9)$ cannot have infinitely many quadratic points, so $\mathrm{a.irr}_{\mathbb{Q}}(X_0^{10}(9)) > 2$.

Table 3: Non-squarefree N and bielliptic Atkin–Lehner involutions

(D,N,m)	$g(X_0(D,N))$	bielliptic over $\mathbb{Q}$?	reason
$(6,25,150)$	5	**not known**	N/A
$(10,9,90)$	5	yes (rank 0)	[NR15]
$(15,8,15)$	9	no	[NR15] and [Ogg85]
$(21,4,7)$	7	no	[NR15] and [Ogg83]
$(39,4,39)$	13	no	[NR15] and [Ogg85]

6. Sporadic points on $X_0^D(N)$

We begin by recalling the definition of a sporadic point:

Definition 6.1. Let X be a curve over a number field F. A point $x \in X$ is **sporadic** if $\deg(x) := [F(x) : F] < \mathrm{a.irr}_F(X)$. In other words, x is sporadic if there are only finitely many points $y \in X$ with $\deg(y) \leq \deg(x)$.

In [Sai22, §10], the author pursued the question of whether the curves $X_0^D(N)$ and $X_1^D(N)$ with $D > 1$ have a sporadic point, following the pursuit of the same question in the $D = 1$ case in [CGPS22, §8]. For both of these families of curves, it is proven using a combination of Theorem 4.1 and a result of Frey [Fre94], which bounds $\mathrm{a.irr}_{\mathbb{Q}}(X_0^D(N))$ above and below in terms of $\mathrm{gon}_{\mathbb{Q}}(X_0^D(N))$, that $X_0^D(N)$ and $X_1^D(N)$ have sporadic CM points for DN sufficiently large.

This work then narrows down the list of pairs (D,N) for which it remains to be proven whether $X_0^D(N)$ has a sporadic point (and similarly for $X_1^D(N)$) using known results on $\mathrm{a.irr}_{\mathbb{Q}}(X_0^D(N))$ and $\mathrm{a.irr}_{\mathbb{Q}}(X_1^D(N))$ and computations of the least degrees of CM points on these curves. In the case of $D > 1$, the following main result is reached:

Theorem 6.2. [Sai22, Thm 10.9]

(1) For all but at most 455 *explicit pairs* (D, N)*, consisting of a rational quaternion discriminant* $D > 1$ *and a positive integer* N *coprime to* D*, the Shimura curve* $X_0^D(N)$ *has a sporadic CM point. For at least* 64 *of these pairs,* $X_0^D(N)$ *has no sporadic points.*

(2) For all but at most 456 *explicit pairs* (D, N)*, consisting of a rational quaternion discriminant* $D > 1$ *and a positive integer* N *coprime to* D*, the Shimura curve* $X_1^D(N)$ *has a sporadic CM point. For at least* 54 *of these pairs,* $X_1^D(N)$ *has no sporadic points.*

In this section, we apply our results on bielliptic Shimura curves $X_0^D(N)$ to improve on Theorem 6.2.

6.1. **Shimura curves with infinitely many quadratic points.** The theorem of Harris–Silverman [HS91] mentioned in the introduction states that if $\mathrm{a.irr}_F(X) = 2$, then X is either hyperelliptic or is bielliptic with a degree 2 map to an elliptic curve over F of positive rank. We recalled in §1 the full list of hyperelliptic curves $X_0^D(N)$ of genus at least 2, and the list of D such that $X_0^D(1)$ is not hyperelliptic and has infinitely many quadratic points. Combining this with our study of bielliptic curves $X_0^D(N)$, we immediately obtain the following result.

Theorem 6.3. *If* $\mathrm{a.irr}_{\mathbb{Q}}(X_0^D(N)) = 2$ *with* $D > 1$ *and* $(D, N) = 1$*, then the pair* (D, N) *is among the following set:*

$$\{(6,1),(6,5),(6,7),(6,11),(6,13),(6,17),(6,19),(6,23),(6,25),(6,29),(6,31),(6,37),(6,41),(6,71),(10,1),(10,3),(10,7),(10,11),(10,13),(10,17),(10,23),(10,29),(14,1),(14,5),(15,1),(15,2),(21,1),(22,1),(22,3),(22,5),(22,7),(22,17),(26,1),(33,1),(34,1),(34,7),(35,1),(38,1),(39,1),(39,2),(46,1),(51,1),(55,1),(57,1),(58,1),(62,1),(65,1),(69,1),(74,1),(77,1),(82,1),(86,1),(87,1),(94,1),(95,1),(106,1),(111,1),(118,1),(119,1),(122,1),(129,1),(134,1),(143,1),(146,1),(159,1),(166,1),(194,1),(206,1),(210,1),(215,1),(314,1),(330,1),(390,1),(510,1),(546,1)\}.$$

For all pairs in this set except possibly for $(6, 25)$ *and* $(34, 7)$*, we have* $\mathrm{a.irr}_{\mathbb{Q}}(X_0^D(N)) = 2$.

6.2. **Sporadic points.** In [Sai22, §10], the author describes and implements computations of the least degree of a CM point on $X_0^D(N)$ or $X_1^D(N)$ for a fixed pair (D, N) with $D > 1$ and $(D, N) = 1$. We denote these quantities by $d_{\mathrm{CM}}(X_0^D(N))$ and $d_{\mathrm{CM}}(X_1^D(N))$ for the respective curves. Using such computations and Theorem 6.3, we arrive at the following:

Theorem 6.4. *(1) For the pairs* (D, N) *in the following list, the curve* $X_0^D(N)$ *has no sporadic points:*

$$\{(6,17),(6,23),(6,41),(6,71),(10,13),(10,17),(10,29),(22,7),(22,17)\}.$$

(2) For $(D, N) \in \{(6,23),(6,71),(10,17),(10,29)\}$*, the curve* $X_1^D(N)$ *has no sporadic CM points.*

(3) Of the 382 *pairs for which we remain unsure of whether* $X_0^D(N)$ *has a sporadic CM point following Theorem 6.2 and part (1), all but at most* 75 *have a sporadic CM point. We list these* 75 *pairs in Table 4.*

(4) *Of the* 398 *pairs for which we remain unsure of whether* $X_1^D(N)$ *has a sporadic CM point following Theorem 6.2 and part (2), all but at most* 322 *have a sporadic CM point. These pairs comprise the union of those in Table 4 and those in Table 5.*

Proof. (1) For these pairs we have proven that $\text{a.irr}_{\mathbb{Q}}(X_0^D(N)) = 2$ by virtue of having a bielliptic quotient with positive rank over $\mathbb{Q}$. We know that $X_0^D(N)(\mathbb{Q}) = \emptyset$ for $D > 1$, so these curves cannot have sporadic points.

(2) For these pairs, we have $\text{a.irr}(X_0^D(N)) = 2$, giving the inequality

$$\text{a.irr}(X_1^D(N)) \leq 2 \cdot \deg\left(X_1^D(N) \to X_0^D(N)\right) = 2 \cdot \max\{1, \phi(N)/2\}.$$

We compute for each that

$$\max\{2, \phi(N)\} \leq d_{\text{CM}}(X_1^D(N)),$$

and thus there are no sporadic CM points.

(3) For these 391 unknown pairs, we compute that $d_{\text{CM}}(X_0^D(N)) = 2$ for all but 73. For all of these pairs except for $(6, 25)$ and $(34, 7)$, the curve $X_0^D(N)$ is not among those listed in Theorem 6.3, and thus any CM point of degree 2 is sporadic.

(4) If $2 < \text{a.irr}(X_0^D(N)) \leq \text{a.irr}(X_1^D(N))$ and $d_{\text{CM}}(X_1^D(N)) = 2$, then each quadratic CM point on $X_1^D(N)$ is sporadic. Of the remaining 402 pairs, there are 76 that satisfy these conditions and thus have a sporadic CM point, and there are 4 for which proved there are no sporadic points in part (2).

□

Table 4: The 75 pairs (D, N) with $D > 1$ and $\gcd(D, N) = 1$ for which we remain unsure of whether $X_0^D(N)$ has a sporadic point

D	N	$g(X_0^D(N))$	$d_{\text{CM}}(X_0^D(N))$	D	N	$g(X_0^D(N))$	$d_{\text{CM}}(X_0^D(N))$
6	25	5	2	51	5	17	4
	155	33	4		10	49	4
	203	41	4		20	97	6
	287	57	4	55	8	41	4
	295	61	4	62	15	61	4
	319	61	4	69	11	45	4
	335	69	4	74	15	73	4
	355	73	4	77	6	61	4
	371	73	4	86	7	29	4
	407	77	4	87	8	57	4
10	69	33	4	95	3	25	4
	77	33	4		6	73	4
	87	41	4		9	73	4
	119	49	4	111	2	19	4
	141	65	4		4	37	4
	159	73	4		8	73	4
	161	65	4	119	6	97	6
	191	65	4	122	7	41	4
14	39	29	4	129	7	57	4
	87	61	4	134	3	23	4
	95	61	4		9	67	4

15	34	37	4	143	2	31	4
	68	73	4		4	61	4
21	38	61	4	146	7	49	4
	55	73	4	183	5	61	4
22	35	41	4	185	4	73	4
	51	61	4	194	3	33	4
26	21	33	4	215	2	43	4
33	16	41	4		3	57	4
34	7	9	2	219	5	73	4
	29	41	4	274	5	69	4
	35	65	4	326	3	55	4
35	12	49	4	327	2	55	4
38	21	49	4		4	109	6
39	10	37	4	335	2	67	4
	20	73	4	390	7	65	4
	31	65	4	546	5	73	4
46	15	45	4				

Table 5: The 247 pairs (D, N) with $D > 1$ and $\gcd(D, N) = 1$ which are not included in Table 4 for which we remain unsure of whether $X_1^D(N)$ has a sporadic point

$(6,5)$	$(6,7)$	$(6,13)$	$(6,17)$	$(6,19)$	$(6,29)$	$(6,31)$	$(6,35)$
$(6,37)$	$(6,41)$	$(6,43)$	$(6,47)$	$(6,49)$	$(6,53)$	$(6,55)$	$(6,59)$
$(6,61)$	$(6,65)$	$(6,67)$	$(6,73)$	$(6,77)$	$(6,79)$	$(6,83)$	$(6,85)$
$(6,89)$	$(6,91)$	$(6,95)$	$(6,97)$	$(6,101)$	$(6,103)$	$(6,107)$	$(6,109)$
$(6,113)$	$(6,115)$	$(6,119)$	$(6,121)$	$(6,125)$	$(6,127)$	$(6,131)$	$(6,133)$
$(6,137)$	$(6,139)$	$(6,143)$	$(6,145)$	$(6,149)$	$(6,151)$	$(6,157)$	$(6,161)$
$(6,163)$	$(6,167)$	$(6,169)$	$(6,173)$	$(6,179)$	$(6,181)$	$(6,185)$	$(6,187)$
$(6,191)$	$(6,193)$	$(6,197)$	$(6,199)$	$(6,211)$	$(6,223)$	$(6,227)$	$(6,229)$
$(6,233)$	$(6,241)$	$(10,7)$	$(10,9)$	$(10,13)$	$(10,19)$	$(10,21)$	$(10,27)$
$(10,31)$	$(10,33)$	$(10,37)$	$(10,39)$	$(10,41)$	$(10,43)$	$(10,47)$	$(10,49)$
$(10,51)$	$(10,53)$	$(10,57)$	$(10,59)$	$(10,61)$	$(10,63)$	$(10,67)$	$(10,71)$
$(10,73)$	$(10,79)$	$(10,81)$	$(10,83)$	$(10,89)$	$(10,91)$	$(10,97)$	$(10,101)$
$(10,103)$	$(10,107)$	$(10,109)$	$(10,113)$	$(14,5)$	$(14,9)$	$(14,11)$	$(14,13)$
$(14,15)$	$(14,17)$	$(14,19)$	$(14,23)$	$(14,25)$	$(14,27)$	$(14,29)$	$(14,31)$
$(14,33)$	$(14,37)$	$(14,41)$	$(14,43)$	$(14,45)$	$(14,47)$	$(14,51)$	$(14,53)$
$(14,55)$	$(14,59)$	$(14,61)$	$(14,67)$	$(14,71)$	$(14,73)$	$(15,8)$	$(15,11)$
$(15,13)$	$(15,14)$	$(15,16)$	$(15,17)$	$(15,19)$	$(15,22)$	$(15,23)$	$(15,26)$
$(15,28)$	$(15,29)$	$(15,31)$	$(15,32)$	$(15,37)$	$(15,41)$	$(15,43)$	$(15,47)$
$(15,49)$	$(15,53)$	$(21,8)$	$(21,11)$	$(21,13)$	$(21,16)$	$(21,17)$	$(21,19)$
$(21,20)$	$(21,22)$	$(21,23)$	$(21,25)$	$(21,29)$	$(21,31)$	$(21,37)$	$(22,5)$
$(22,7)$	$(22,9)$	$(22,13)$	$(22,15)$	$(22,17)$	$(22,19)$	$(22,21)$	$(22,23)$
$(22,25)$	$(22,27)$	$(22,29)$	$(22,31)$	$(22,37)$	$(22,41)$	$(22,43)$	$(26,5)$
$(26,7)$	$(26,9)$	$(26,11)$	$(26,15)$	$(26,17)$	$(26,19)$	$(26,23)$	$(26,25)$
$(26,27)$	$(26,29)$	$(26,31)$	$(26,37)$	$(33,8)$	$(33,13)$	$(33,17)$	$(33,19)$
$(34,5)$	$(34,9)$	$(34,11)$	$(34,13)$	$(34,15)$	$(34,19)$	$(34,23)$	$(35,8)$

$(35,9)$	$(35,11)$	$(35,13)$	$(35,17)$	$(38,7)$	$(38,9)$	$(38,11)$	$(38,13)$
$(38,15)$	$(38,17)$	$(38,23)$	$(39,5)$	$(39,7)$	$(39,8)$	$(39,11)$	$(39,17)$
$(46,9)$	$(46,11)$	$(46,13)$	$(46,17)$	$(46,19)$	$(51,8)$	$(51,11)$	$(51,13)$
$(57,7)$	$(57,8)$	$(57,11)$	$(58,5)$	$(58,9)$	$(58,11)$	$(58,13)$	$(62,7)$
$(62,9)$	$(62,11)$	$(62,13)$	$(65,7)$	$(74,5)$	$(74,7)$	$(74,9)$	$(74,11)$
$(82,5)$	$(87,5)$	$(91,5)$	$(106,5)$	$(111,5)$	$(122,5)$	$(146,5)$	

7. A note on trigonal Shimura curves

Shimura curves with $D > 1$ have no real points, so, in particular, they have no odd-degree points. If X is such a curve and X has a degree d map to either $\mathbb{P}^1_{\mathbb{Q}}$ or an elliptic curve E over $\mathbb{Q}$, then it follows that d is even. On the other hand, this does not preclude the existence of odd-gonality or odd-degree maps to elliptic curves geometrically.

There do indeed exist geometrically trielliptic Shimura curves. For example, if $D > 1$ is odd and $X_0^D(1)$ has genus 1 (so, is a pointless genus 1 curve over $\mathbb{Q}$), then $X_0^D(2)$ is a degree 3 cover of $X_0^D(1)$ and hence is trielliptic over a degree 2 extension. This applies to $D \in \{15, 21, 33\}$. Similarly, $X_0^{10}(9)$ is geometrically trielliptic with a degree 3 map to $X_0^{10}(3)$. We learn from these examples that $\text{a.irr}_{\mathbb{Q}}(X_0^{10}(9) \in \{4, 6\}$ for $(D, N) \in \{(10, 9), (21, 1), (33, 2)\}$.

We show in this section, however, that there are no geometrically trigonal Shimura curves $X_0^D(N)$.

Definition 7.1. A curve X of genus $g \geq 2$ over a number field F is (geometrically) **trigonal** if it has geometric gonality 3, i.e., if there exists a non-constant morphism

$$f : X \otimes_{\text{Spec}F} \text{Spec}\overline{\mathbb{Q}} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}}$$

of degree 3.

If $X_0^D(N)$ is trigonal, then by Theorem 4.1 we must have $g(X_0^D(N)) \leq 29$. There are 260 pairs (D, N) such that $g(X_0^D(N)) \leq 29$. Applying the following results proves that none of these pairs corresponds to a trigonal curve:

Lemma 7.2. [Sch15, Lemma 3.4] *Let X be a trigonal curve of genus g and σ an involution on X.*
(a) If g is odd, then σ has exactly 4 fixed points.
(b) If g is even, then σ has 2 or 6 fixed points.

Proposition 7.3. [Sch15, Corollary 3.5] *Let X be a curve of genus $g \equiv 1 \pmod 4$. If $\text{Aut}(X)$ has a subgroup H isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then X cannot be trigonal.*

By checking all the possible cases, we can conclude that

Proposition 7.4. *There are no trigonal Shimura curves $X_0^D(N)$.*

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