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SEQUENCES OF INTEGERS GENERATED BY TWO FIXED PRIMES

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ABSTRACT. Let $p$ and $q$ be two distinct fixed prime numbers and $(n_i)_{i \geq 0}$ the sequence of consecutive integers of the form $p^a \cdot q^b$ with $a, b \geq 0$. Tijdeman gave a lower bound (1973) and an upper bound (1974) for the gap size $n_{i+1} - n_i$, with each bound containing an unspecified exponent and implicit constant. We will explicitly bound these four quantities. Earlier Langevin (1976) gave weaker estimates for (only) the exponents.

Given a real number $\alpha > 1$, there exists a smallest number $m$ such that for every $n \geq m$, there exists an integer $n_i$ in $[n, n\alpha)$. Our effective version of Tijdeman’s result immediately implies an upper bound for $m$, which using the Koksma-Erdős-Turan inequality we will improve on. We present a fast algorithm to determine $m$ when $\max\{p, q\}$ is not too large and demonstrate it with numerical material. In an appendix we explain, given $n_i$, how to efficiently determine both $n_i - 1$ and $n_i + 1$, something closely related to work of Bérczes, Dujella and Hajdu.

1. Introduction

Given a set $S = \{p_1, \ldots, p_s\}$ of primes, the numbers $p_1^{e_1} \cdots p_s^{e_s}$ with non-negative exponents are called $S$-units. There are many Diophantine equations involving $S$-units, see for example Evertse et al. [5]. In this note we are interested in the distribution of $S$-units.

In case $p_1, \ldots, p_s$ are the first $s$ prime numbers, the counting function of $S$-units up to $x$ is denoted by $\psi(x, p_1, \ldots, p_s)$, and the $S$-units are called $p_k$-friable or $p_k$-smooth. There is an extensive literature on these numbers, see Hildebrand and Tenenbaum [10] for a nice survey. The gaps between consecutive friable numbers were studied by Heath-Brown [9]. Tijdeman and Meijer [20] studied the distribution of the greatest common divisor of consecutive $S$-units (for general $S$). We will concentrate solely on the case where $S$ has two elements.

Definition 1.1. Let $p, q$ be two primes with $p < q$. We let $(n_i)_{i \geq 0}$ be the sequence of consecutive integers of the form $n = p^a \cdot q^b$ with $a, b \geq 0$.

Put $N_{p,q}(x) := \#\{n_i \leq x\}$. Note that $N_{p,q}(x) = \#\{(e, f) \in \mathbb{Z}_{\geq 0}^2 : e \log p + f \log q \leq \log x\}$, that is it equals the number of lattice points inside a rectangular triangle with sides of length $\log x / \log p$, respectively $\log x / \log q$. The area of this triangle is

\[
\frac{(\log x)^2}{2 \log p \cdot \log q}.
\]

It is known that

\[
N_{p,q}(x) = \frac{\log px \cdot \log qx}{2 \log p \cdot \log q} + o(\log x),
\]

where the error term is worse if $\log px \cdot \log qx$ is replaced by $(\log x)^2$. Interestingly, Ramanujan in his famous first letter to Hardy (Jan. 16th, 1913), claims (in case $p = 2$ and $q = 3$) the latter main
term as an approximation, rather than the trivial (1.1). For more details and references the reader is referred to Moree [14].

Bérczes, Dujella, and Hajdu [2], for any term \( n_i \) determined \( n_{i+1} \), at least in principle, without enumerating all terms of the sequence, and they gave an efficient algorithm to find \( n_{i+1} \) explicitly. They do so by analyzing the behavior of the continued fractions of \( \log p / \log q \) (see Theorems 2.2 and 2.3 of [2]). In the appendix we present a shorter reproof and in addition show how to also efficiently find \( n_{i-1} \) explicitly.

As early as 1908, Thue [17] gave an ineffective proof that the gap size \( n_{i+1} - n_i \) tends to infinity, which was made effective by Cassels [4] in 1960. Tijdeman [18, 19], half a century ago, gave bounds for the gap size. He derived these by making use of estimates of Baker [1] for linear forms in logarithms of algebraic numbers, which had become available a few years earlier.

**Theorem 1.1.** Let \((n_i)_{i \geq 0}\) be as in Definition 1.1. There exist effective constants \( C_1 \) and \( C_2 \) such that

\[
\frac{n_i}{(\log n_i)^{C_1}} \ll_{p,q} n_{i+1} - n_i \ll_{p,q} \frac{n_i}{(\log n_i)^{C_2}}.
\]

The constants \( C_1, C_2 \) and the two implicit constants all may depend on \( p \) and \( q \).

Langevin [12], soon after Tijdeman’s work appeared, gave effective bounds for the constants \( C_1 \) and \( C_2 \).

**Theorem 1.2.** Under the conditions of Theorem 1.1, we have

\[
(2^{126}\log p \cdot \log q \cdot \log \log p)^{-1} < C_2 < C_1 < 2^{126}\log p \cdot \log q \cdot \log \log p.
\]

We will improve on this result and also give explicit bounds for the implicit constants in Theorem 1.1.

**Theorem 1.3.** Assuming \( n_i \geq 3 \), we have

\[
C_3 \frac{n_i}{(\log n_i)^{C_1}} < n_{i+1} - n_i < C_4 \frac{n_i}{(\log n_i)^{C_2}},
\]

where \( C_1 = 2 \cdot 10^9 \log p \cdot \log q \), \( C_2 = C_1^{-1} \), \( C_3 = (\log p)^{C_1} \), \( C_4 = 8q \).

Langevin used a result of van der Poorten [21] to obtain Theorem 1.2. In Section 2 we will use a celebrated result of Matveev (Theorem 2.1) to prove Theorem 1.3. This helps to decrease the coefficient of the bound \( C_2 \) from a 38 digit number to a 9 digit one.

Very recently Stewart [16] derived an analogue of Theorem 1.1 for increasing sequences \((n_i)\) such that the largest prime factor of \( n_i \) is at most \( y(n_i) \), with \( y(x) \) a non-decreasing function slowly tending to infinity.

In Section 3, we consider Bertrand’s Postulate type results for our sequence.

**Definition 1.2.** Given any real number \( \alpha > 1 \), \( n_{p,q}(\alpha) \) is the smallest integer such that every interval \([n, n\alpha)\) with \( n \geq n_{p,q}(\alpha) \) contains an integer of the form \( p^a \cdot q^b \) for every integer \( n \geq n_{p,q}(\alpha) \).

Bertrand’s Postulate type results arise on taking \( \alpha = 2 \). Theorem 1.3 gives right away that if \( \alpha \in (1, p) \) and

\[
n_i > \exp\left(\frac{4q}{\alpha - 1}\right) 2^{10^9 \log p \cdot \log q},
\]

then \( n_{i+1} \in (n_i, \alpha n_i) \). In Section 3 we will (slightly) improve on this lower bound and show that this conclusion already holds if

\[
n_i > \exp\left(2 \log p \left(\frac{60 \log q}{\log \alpha}\right)^{10^9 \log p \cdot \log q}\right).
\]
In Section 4 we present an algorithm for efficiently determining \( n_{p,q}(\alpha) \) and present some of its outputs. In particular, in Section 3.3 we use our algorithm to advance the understanding of the so-called discriminator of an infinite family of second-order recurrent sequences first studied by Faye, Luca and Moree [6] and more recently by Ferrari, Luca and Moree [7].

2. Proof of Theorem 1.3

We need linear forms in logarithms. For any non-zero algebraic number \( \eta \) of degree \( d \) over \( \mathbb{Q} \), whose minimal polynomial over \( \mathbb{Z} \) is \( \prod_{i=1}^{d}(X - \eta^{(i)}) \) (with \( a > 0 \)), we denote by

\[
h(\eta) = \frac{1}{d} \left( \log a + \sum_{i=1}^{d} \log \max(1, |\eta^{(i)}|) \right)
\]

the usual absolute logarithmic height of \( \eta \). If \( \eta_1 \) and \( \eta_2 \) are algebraic numbers, then we have the basic properties

\[
\begin{align*}
    h(\eta_1 \pm \eta_2) &\leq h(\eta_1) + h(\eta_2) + \log 2, \\
    h(\eta_1 \eta_2^\pm) &\leq h(\eta_1) + h(\eta_2), \\
    h(\eta_1^j) &= |j|h(\eta_1),
\end{align*}
\]

where \( j \) is any integer.

We recall Matveev’s main theorem [13] in a version due to Bugeaud, Mignotte and Siksek [3, Thm.9.4]. It applies to algebraic numbers, but we recall it here only for rational numbers. For a rational number \( \gamma = r/s \) with coprime integers \( r \) and \( s > 0 \), let \( h(\gamma) := \max\{\log |r|, \log s\} \) be its naive height.

**Theorem 2.1** (Matveev’s theorem). Let \( \gamma_1, \ldots, \gamma_k \) be positive rational numbers, let \( b_1, \ldots, b_k \) be non-zero integers, and assume that

\[
\Lambda := \gamma_1^{b_1} \cdots \gamma_k^{b_k} - 1,
\]

is non-zero. For every real number \( B \geq \max\{|b_1|, \ldots, |b_k|\} \) we have

\[
\log |\Lambda| > -1.4 \cdot 30^{k+3} \cdot k^{4.5}(1 + \log B) h(\gamma_1) \cdots h(\gamma_k).
\]

Note that \( n_1 = 1, n_2 = p \). Let \( n_i = p^u \cdot q^v \). We assume that \( n_i \geq 3 \) (this holds for \( i = 2 \) in all cases except if \( p = 2 \), in which case we assume that \( i \geq 3 \)). The lower bound on \( n_{i+1} - n_i \) follows right–away from Theorem 2.1. Indeed, let \( n_{i+1} = p^{u'} \cdot q^{v'} \). Note that \( \max\{u, v, u', v'\} \leq 2 \max\{u, v\} \) (since certainly \( n_i^2 > n_i \) is a \( p, q \)-unit, clearly \( n_i^2 \geq n_{i+1} \)). Then

\[
n_{i+1} - n_i = n_i (p^{u'-u}q^{v'-v} - 1).
\]

By Theorem 2.1 the right–hand side is bounded below by

\[
n_i \exp\left(-1.4 \cdot 30^5 \cdot 2^{4.5}(1 + \log(2\max\{u, v\})) \log p \cdot \log q\right).
\]

Since \( \max\{u, v\} \geq 1 \) and for \( x \geq 2 \) we have \( 1 + \log x \leq 2.5 \log x \), we obtain

\[
n_{i+1} - n_i > \frac{n_i}{(2\max\{u, v\})^{1.999\log p \cdot \log q}}.
\]

Since \( n_i = p^u \cdot q^v \leq (pq)^{\max\{u, v\}} \), we get that \( \max\{u, v\} \geq \log n_i / \log(pq) \). Thus,

\[
n_{i+1} - n_i > C_3 \frac{n_i}{(\log n_i)^{C_1}},
\]

where

\[
C_1 = 2 \cdot 10^9 \log p \cdot \log q, \quad C_3 = (0.5 \log pq)^{C_1} > (\log p)^{C_1}.
\]
In order to obtain an upper bound for \( n_{i+1} - n_i \) we proceed as in Langevin [12] and make use of the sequence \((r_k/s_k)_{k \geq 0}\) of convergents of

\[
\theta_{p,q} := \frac{\log p}{\log q}.
\]

Suppose first that \( p^u > q^v \). We assume that \( \ell \) is the index verifying \( s_\ell \leq u < s_{\ell+1} \). Since \( p < q \), it follows that \( a_0 = 0 \) and \( a_1 = \lfloor \log q / \log p \rfloor \). So, if \( \ell = 0 \), then we have \( u < \log q / \log p \), so \( p^u < q \). Thus, \( v = 0 \), \( n_{i+1} \leq q \) and the inequalities hold provided that the multiplicative constant \( C_4 \) implied by the right Vinogradov symbol \( \ll \) is taken to be at least \( q(\log q)^{C_2} \).

Assume next that \( \ell \geq 1 \). One of the rational numbers \( r_\ell / s_\ell \) and \( r_{\ell+1} / s_{\ell+1} \) is at least \( \theta_{p,q} \). Choose \( \ell' \in \{\ell, \ell + 1\} \) such that \( r_{\ell'}/s_{\ell'} > \theta_{p,q} \). By construction

\[
p^{u-s_{\ell'}^*} \cdot q^{v+r_{\ell'}}
\]

is integer larger than \( p^u \cdot q^v \), and so it is at least \( n_{i+1} \). Thus, we obtain

\[
\log \left( \frac{n_{i+1}}{n_i} \right) \leq |\Lambda_{\ell'}|,
\]

where

\[
\Lambda_k := s_k \log p - r_k \log q.
\]

By [8, Thm. 171] we have

\[
|\Lambda_k| < \frac{\log q}{s_{k+1}}.
\]

Combining (2.2) and (2.3) with \( k = \ell' = \ell \) gives

\[
\log \left( \frac{n_{i+1}}{n_i} \right) < \frac{\log q}{s_{\ell+1}}.
\]

Next, assume \( k = \ell' = \ell + 1 \). Now (2.3) gives

\[
\frac{s_{\ell+2}}{\log q} < \frac{1}{|\Lambda|},
\]

where for notational convenience we put \( \Lambda = \Lambda_{\ell+1} \). We need to lower bound \( |\Lambda| \). Note that \( |p^{s_{\ell+1}} \cdot q^{-r_{\ell+1}} - 1| = |\exp(\Lambda) - 1| \). Either \( |\Lambda| \geq 1/2 \), or \( |\Lambda| < 1/2 \), in which case

\[
|\exp(\Lambda) - 1| < 2|\Lambda|.
\]

We lower bound the left–hand side above using Theorem 2.1. We get

\[
2|\Lambda| > |p^{s_{\ell+1}} \cdot q^{-r_{\ell+1}} - 1| > \exp(-1.4 \cdot 30^5 \cdot 2^{4.5}(1 + \log \max\{s_{\ell+1}, r_{\ell+1}\}) \log p \cdot \log q).
\]

Note that we can assume that \( s_{\ell+1} > r_{\ell+1} \). Indeed, otherwise we have \( r_{\ell+1} \geq s_{\ell+1} \) and so

\[
|\Lambda| = |(r_{\ell+1} - s_{\ell+1}) \log q + s_{\ell+1}(\log q - \log p)| \geq \log \left( \frac{q}{p} \right) \geq \log \left( 1 + \frac{1}{p} \right) > \frac{1}{2p},
\]

which is a better inequality. In particular, \( s_{\ell+1} \geq 2 \). We thus get

\[
|\Lambda| > \exp(-2 \cdot 10^9 \log s_{\ell+1} \log p \cdot \log q),
\]

where we have used the fact that \( 1 + \log s_{\ell+1} \leq 2.5 \log s_{\ell+1} \), for \( s_{\ell+1} \geq 2 \). Thus, inequality (2.5) gives

\[
\frac{s_{\ell+2}}{\log q} < \frac{1}{|\Lambda|} < s_{\ell+1}^C_1.
\]

Hence, we get

\[
s_{\ell+1} > 0.5 s_{\ell+2}^C_2.
\]
Then, recalling (2.2)-(2.3), we obtain

\[(2.6) \quad \log\left(\frac{n_{i+1}}{n_i}\right) < \frac{\log q}{s_{\ell+2}} \leq \frac{\log q}{s_{\ell+1}} < \frac{2\log q}{s_{\ell+2}^C_2}.\]

We thus get that in all cases ((2.4) and (2.6)) we have

\[(2.7) \quad \log\left(\frac{n_{i+1}}{n_i}\right) < \frac{2\log q}{s_{\ell+2}^C_2} < \frac{2\log q}{u^C_2}.\]

Since by assumption \(p^u > q^v\) we have \(p^u > n_i^{1/2}\) and hence \(u > \frac{\log n_i}{2\log p}\) we obtain

\[\log\left(\frac{n_{i+1}}{n_i}\right) < \frac{2(2\log p)^C_2 \log q}{(\log n_i)^C_2} < \frac{4\log q}{(\log n_i)^C_2}.\]

But \((n_{i+1} - n_i)/n_i \in (0, p - 1)\). In this interval, the image of the function \(\log(1 + x)/x\) is in the interval \([\frac{\log p}{p - 1}, 1]\). Writing \(n_{i+1}/n_i = 1 + x\), where \(x = (n_{i+1} - n_i)/n_i\), we get that

\[n_{i+1} - n_i < \frac{4(p - 1)\log q}{(\log n_i)^C_2 \log p},\]

and we can choose \(C_4\) to be at least

\[\frac{4(p - 1)\log q}{\log p}.\]

But, we also needed that this constant exceeds \(q(\log q)^{C_2}\). Since \((\log q)^{C_2} < 2\), it follows that if we choose \(C_4 \geq 4q\), then everything works out.

We proceed by a similar argument in the remaining case \(q^v > p^u\). Note that \(v \geq 1\). We take \(\ell\) such that \(r_{\ell} \leq v < r_{\ell+1}\). We choose \(\ell' \in \{\ell, \ell + 1\}\) such that \(r_{\ell}/s_{\ell'}\) is smaller than \(\theta_{p,q}\). Note that if \(\ell = 0\), we then have \(r_0/s_0 = 0 < \theta_{p,q}\). So, \(\ell' \geq 0\) is well-defined even if \(\ell = 0\). We consider the number

\[p^{u+r_{\ell'}} \cdot q^{v-r_{\ell'}},\]

which is an integer which is a \(\{p,q\}\)-unit exceeding \(n_i\), so it is at least \(n_{i+1}\). Assuming \(s_{\ell+1} \geq 2\) and reasoning as before we deduce (2.7) and so

\[(2.8) \quad \log\left(\frac{n_{i+1}}{n_i}\right) < \frac{2\log q}{s_{\ell+2}^C_2}.\]

The above inequality assumes that \(s_{\ell+1} \geq 2\) which holds for all \(\ell \geq 0\) except when \(\ell = 0\) and \(q \in (p, p^2)\). But this is impossible since we must also have \(0 = r_0 \leq v < r_1 = 1\), so \(v = 0\), therefore \(i = 1\), which is false. We need to recast the above inequality in terms of \(v\). By (2.3) we have

\[|r_{\ell+2} \log q - s_{\ell+2} \log p| < \frac{\log q}{s_{\ell+2}}.\]

Hence, using that \(s_{\ell+2} \geq s_{\ell+1} \geq 2\) we obtain

\[s_{\ell+2} \log p \geq \left(r_{\ell+2} - \frac{1}{s_{\ell+2}}\right) \log q \geq 0.5 r_{\ell+2} \log q.\]

Thus, we get

\[s_{\ell+2} \geq \frac{\log q}{2\log p} r_{\ell+2}, \quad s_{\ell+2}^C_2 \geq \left(\frac{\log q}{2\log p}\right)^C_2 r_{\ell+2}^C_2 \geq \frac{r_{\ell+2}^C_2}{2}.\]

By (2.8) we get

\[\log\left(\frac{n_{i+1}}{n_i}\right) < \frac{4\log q}{r_{\ell+2}^C_2} < \frac{4\log q}{v^C_2}.\]
So, we get an upper bound in the right–hand side by a factor of at most 2 larger than in the case when \( p^e > q^e \). Following along we obtain that in this case
\[
 n_{i+1} - n_i < C_4 \frac{n_i}{(\log n_i)^{C_2}},
\]
we must have \( C_4 > 8(p-1)^{\log q_{\log p}} \), and so taking \( C_4 = 8q \) suffices.

3. Bertrand’s Postulate for the sequence \( (n_i)_{i \geq 0} \)

3.1. The statement. We will show a way to compute \( n_{p,q}(\alpha) \). Tables 1 and 2 give some examples for \( p = 2 \) and \( \alpha = 5/3 \), respectively \( 3/2 \), some of which are relevant for an application.

Basic results from the theory of Diophantine approximation (cf. Section 4.1), ensure the existence of integers \( e, f, g \), and \( h \) such that
\[
1 < \frac{p^e}{q^f} < \alpha \quad \text{and} \quad 1 < \frac{q^g}{p^h} < \alpha.
\]
We claim that \( n_{p,q}(\alpha) \leq p^e \cdot q^f \). In order to see this, one can observe that any integer \( n := p^e \cdot q^f \geq p^e \cdot q^f \) satisfies either \( k \geq e \), or \( \ell \geq g \). In case \( k \geq e \), we note that the number \( p^{e+f} \cdot q^{k-e} \) lies in \([n, n\alpha)\). In case \( \ell \geq g \), we have \( p^{e-g} \cdot q^{k+h} \in [n, n\alpha) \). Next one tries to find an integer \( m_{\text{new}} := p^e \cdot q^f \in \lfloor p^e \cdot q^f / \alpha \rfloor, p^e \cdot q^f \rfloor \), where \( \lfloor x \rfloor \) denotes the integral part of \( x \). If successful, we continue until we fail, each time considering the interval \([\lfloor n_{\text{new}} / \alpha \rfloor, n_{\text{new}})\). In Section 4.1 we present a much more refined way of determining \( n_{p,q}(\alpha) \).

3.2. An upper bound for \( n_{p,q}(\alpha) \). Recall that the discrepancy \( D_N \) of a sequence \( (a_m)_{m=1}^N \) of real numbers (not necessarily distinct) is defined as
\[
 D_N = \sup_{0 \leq \gamma \leq 1} \left| \frac{\# \{ m \leq N : \{ a_m \} < \gamma \}}{N} - \gamma \right|,
\]
where \( \{ x \} \) denotes the fractional part of a real number \( x \). The Koksma-Erdős-Turán inequality (see, for example, Kuipers and Niederreiter [11, Lemma 3.2]) states that
\[
 D_N \leq \frac{3}{H} + \frac{3}{N} \sum_{m=1}^{H} m \| a_m \|,
\]
where \( \| x \| \) is the distance from \( x \) to the nearest integer and \( H \leq N \) is an arbitrary positive integer. In this section we will improve on the bound (1.2) by applying this inequality to upper bound the discrepancy of the sequence \( (j\theta_{p,q})_{j \geq 1} \), with \( \theta_{p,q} := \log p / \log q \).

**Theorem 3.1.** Let \( p < q \) be primes and \( \alpha \in (1, p] \). Put \( C_5 = 10^9 \log p \cdot \log q \). There are positive integers \( f \) and \( g \) such that
\[
\left\{ \frac{\log q}{\log p} \right\} \in \left(0, \frac{\log \alpha}{\log q}\right), \quad \left\{ \frac{\log p}{\log q} \right\} \in \left(1 - \frac{\log \alpha}{\log q}, 1\right),
\]
and
\[
\max\{f, g\} < \left(\frac{60 \log q}{\log \alpha}\right)^{C_5}.
\]
In particular,
\[
n_{p,q}(\alpha) \leq q^e \cdot p^f < p^{f+g} < \exp \left(2 \log p \left(\frac{60 \log q}{\log \alpha}\right)^{C_5}\right).
\]
Proof. From the definition of $D_N$ we see that the inequality
\[
\#\{m \leq N : \alpha \leq \{a_m\} < \beta\} \geq (\beta - \alpha)N - 2D_NN
\]
holds for all $0 \leq \alpha \leq \beta \leq 1$. We will apply this with $a_m = m\theta_{p,q}$ for all $m = 1, \ldots, N$. Writing
\[
I = \left(0, \frac{\log \alpha}{\log q}\right), \quad J = \left(1 - \frac{\log \alpha}{\log q}, 1\right),
\]
both intervals of length $\log \alpha / \log q$, it follows from (3.4) that
\[
\#\{m \leq N : \{a_m\} \in I\} \geq |I|N - 2D_NN = \left(\frac{\log \alpha}{\log q}\right)N - 2D_NN,
\]
and similarly with $I$ replaced by $J$. In particular, if the right-hand side is positive, then there is $u \leq N$ with $\{a_u\} \in I$. We will now upper bound $D_N$ using (3.1). To bound $\|a_m\|$, note that
\[
\|a_m\| = \left|m\frac{\log p}{\log q} - t\right| = \frac{|\Lambda|}{\log q}, \text{ with } \Lambda := m\log p - t\log q,
\]
for some integer $t$ with $t \leq m(\log p)/(\log q) + 1 < 2m$. Since $\theta_{p,q}$ is irrational we have either $|\Lambda| \geq 1/2$, or $0 \leq |\Lambda| < 1/2$. If $0 \leq |\Lambda| < 1/2$, then
\[
2|\Lambda| > |\exp(\Lambda) - 1|,
\]
and we can apply Matveev’s theorem to get a lower bound on $|\exp(\Lambda) - 1|$ and hence on $|\Lambda|$. In both cases we take in Matveev’s theorem
\[
k = 2, \quad \gamma_1 = p, \quad \gamma_2 = q, \quad b_1 = m, \quad b_2 = -t,
\]
and, noting that we can set $B := 2m$, we get
\[
2|\Lambda| > \exp\left(-c_1(\log p)(1 + \log(2m)) \log q \right),
\]
where $c_1 = 1.4 \cdot 30^5 \cdot 2^{4.5} < 8 \cdot 10^8 - \log 2$. We get
\[
|\Lambda| > \exp\left(-8 \cdot 10^8(1 + \log(2m)) \log p \cdot \log q \right) = q^{-8 \cdot 10^8(1+\log(2m)) \log p} \quad \text{for} \quad m \geq 1.
\]
We thus obtain that, if $H \geq 60$ and $2m \leq H$, then
\[
1 + \log(2m) \leq 1 + \log H \leq 1.245 \log H \quad (H \geq 60),
\]
and so inequality (3.6) leads to
\[
\frac{1}{\|a_m\|} \leq (\log q) q^{(8 \cdot 1.245) \cdot 10^8 \log H \log p} < q^{(10^9 - 3) \log H \log p} = H^{(10^9 - 3) \log p \log q} < H^{C_5 - 2}.
\]
Thus,
\[
D_N \leq 3 \left(\frac{1}{H} + \frac{H^{C_5 - 2}}{N} \sum_{m=1}^{H} \frac{1}{m}\right) < 3 \left(\frac{1}{H} + \frac{H^{C_5 - 1}}{N}\right),
\]
where we trivially bounded the sum by $H$. Choosing $H := \left[\frac{N^{1/C_5}}{C_5}\right]$ we get, assuming still that $H \geq 60$ and therefore that
\[
N^{1/C_5} \geq 60, \quad \text{which is equivalent to} \quad N \geq 60^{C_5},
\]
we obtain
\[
D_N \leq 3 \left(\frac{1}{H} + \frac{H^{C_5 - 1}}{N}\right) \leq 3 \left(\left\lfloor N^{1/C_5}\right\rfloor^{-1} + N^{-1/C_5}\right) \leq 7N^{-1/C_5}.
\]
To derive the final inequality we used the trivial observation that if \( x \geq 60 \), then
\[
\frac{1}{[x]} + \frac{1}{x} \leq \frac{1}{x \left( \frac{1}{1 - \frac{1}{x}} + 1 \right)} \leq \frac{119}{59} \cdot \frac{1}{x} < \frac{7}{3} \cdot \frac{1}{x}.
\]

Turning now our attention to the inequality (3.5), we see that
\[
N \left( \frac{\log \alpha}{\log q} - 2D_N \right) > N \left( \frac{\log \alpha}{\log q} - 14N^{-1/C_5} \right).
\]
Thus, if \( N \geq N_0 \) with
\[
(3.9) \quad N_0 := \left( \frac{60 \log q}{\log \alpha} \right)^{C_5},
\]
and hence \( N_0 \geq 60C_5 \), the right-hand side of (3.8) is at least
\[
(3.10) \quad \frac{23}{30} N \frac{\log \alpha}{\log q}
\]
and hence positive. From what we have seen at the beginning of Section 3.1, we have \( n_{p,q}(\alpha) \leq q^{\varepsilon} \cdot p^{q} < p^{f+q} \). The proof is now completed on invoking (3.3).

A similar proof also appears in Ferrari et al. [7] in the context of bounding the largest exceptional value of the discriminator of certain Lucas sequences.

3.3. Application to the discriminator of Lucas type sequences. Faye et al. [6] considered the sequence \( \{U_n(k)\}_{n \geq 0} \) defined uniquely by
\[
U_{n+2}(k) = (4k + 2)U_{n+1}(k) - U_n(k), \quad U_0(k) = 0, \quad U_1(k) = 1.
\]
For \( k = 1 \), the sequence is
\[
0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, \ldots,
\]
which is A001109 in OEIS. On noting that
\[
U_{n+2}(k) - U_{n+1}(k) = 4kU_{n+1}(k) + U_{n+1}(k) - U_n(k) \geq 1,
\]
one sees that the sequence \( U_n(k) \) consists of strictly increasing non-negative numbers. We can now define its discriminator \( D_k(n) \) as the smallest positive integer \( m \) such that \( U_0(k), \ldots, U_{n-1}(k) \) are pairwise distinct modulo \( m \).

Primes of the form \( 2^n - 1 \) are called Mersenne primes. Note that \( n \) has to be a prime. The first few Mersenne primes are: 3, 7, 31, 127, 8191, 131071, 524287, \ldots The first few with exponent \( n \equiv 1 \mod 4 \) are 31, 8191, 131071, 2305843009213693951, \ldots Choosing \( k \) to be a Mersenne prime, it turns out that \( D_k(n) \) is a \( \{2, k\} \)-unit for all \( n \) large enough.

**Proposition 3.1.** Let \( p \) be an odd prime such that also \( q := 2^p - 1 \) is a prime number. Let \( (n_i)_{i \geq 0} \) be the sequence of consecutive \( S \)-units with \( S = \{2, q\} \). Then
\[
(3.11) \quad D_q(n) \leq \min \{ n_i \geq n \},
\]
with equality if the interval \([n, 3n/2]\) contains an \( S \)-unit \( n_j \). In particular, if \( n \geq n_{2,q}(3/2) \) we have equality in (3.11). If \( p \equiv 1 \mod 4 \) and \( p > 5 \), then we have equality for \( n \geq n_{2,q}(5/3) \) and \( D_q(n) \) is a \( \{2, 5, q\} \)-unit for every \( n \geq 1 \).

**Proof.** The first assertion follows on taking for \( k \) the Mersenne prime \( q \) in Theorem 3 of Faye et al. [6], the second assertion is a consequence of Theorem 3 of Ferrari et al. [6].

Numerical work by Matteo Ferrari never led to inequality in (3.11) in case \( q = 2^p - 1 \) with \( p \equiv 3 \mod 4 \), but in case \( p \equiv 1 \mod 4 \) he found examples, e.g.,
\[
D_{8191}(129) = 250, \quad D_{131071}(129) = 250, \quad D_{131071}(257) = 500.
\]
sequences of integers generated by two fixed primes

<table>
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Table 1. \(n_2,q(5/3); 3 \leq q \leq 100\).

As predicted by Proposition 3.1, these values are indeed \(\{2, 5, 2^p - 1\}\)-units.

One finds that \(n_2,7(3/2) = 131\) and computing \(D_7(n)\) for the integers \(1 \leq n \leq 130\) shows that for \(q = 7\) there is always equality in (3.11), in line with the observations of Ferrari. Unfortunately, we are not able to classify \(D_q(n)\) completely for any Mersenne prime \(q > 7\), as then \(n_2,q(5/3)\) appears to be very large (and hence \(n_2,q(3/2)\) even more so). The Mersenne prime 127, for example, leads to a very large value of \(n_2,127(3/2)\) the algorithm, which we will describe in Section 4.1, starts with the pair of exponents \((615, 1)\) and, after having generated 23 150 left neighbors, terminates with the pair \((6, 36)\). Hence, the final result is \(n_2,127(3/2) = \lceil 2^6 \cdot 127^{36} \cdot 2/3 \rceil\), a number having 78 digits.

For \(n_2,8191(5/3)\) the algorithm starts with the pair \((73 800, 1)\) and, after having generated 195 078 401 left neighbors, terminates with the pair \((12, 1493)\). Hence, the final result is \(n_2,8191(5/3) = \lceil 2^{12} \cdot 8191^{1493} \cdot 3/5 \rceil\), a number having 5 847 digits. The whole computation of this case took about 20 minutes and 30 seconds.

Apart from having sharper estimates for \(n_{\text{start}}\), the only hope to obtain \(n_2,q(3/2)\) or \(n_2,q(5/3)\) in case \(q > 131071\) is a Mersenne prime, is to implement part of this algorithm using the C programming language. This is doable since, once one has gained the knowledge of a sufficiently large number of the \(\theta_{2,q}\)-convergents, the continued fraction part and the first trivial part can
work on exponents only. On the other hand, for the second trivial part the use of multiprecision arithmetic is mandatory.

\[
pq = 2^{p-1} \alpha \quad n_{2,q}(\alpha) \quad \ell(n_{2,q}(\alpha))
\]

<table>
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<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \alpha )</th>
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<td>5/3</td>
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</tr>
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Table 3. Known values of \( n_{2,M_p}(\alpha) \) for Proposition 3.1; \( \ell(n) \) is the number of decimals of \( n \).

4. Computations

In this section, we will use heavy computations to search for \( n_{p,q}(\alpha) \) and to study the extremal and average behaviors of the gaps \( n_{i+1} - n_i \). In both cases, we will use the knowledge of the continued fractions convergents of \( \theta_{p,q} \) to generate left or right neighbors of a given \( S \)-unit \( n_i \).

The first computational problem we address is to search for \( n_{p,q}(\alpha) \) since, as we will see later, its knowledge plays also a role in studying the extremal and average behaviors of the gaps \( n_{i+1} - n_i \).

4.1. Computation of \( n_{p,q}(\alpha) \). We now explain how the algorithm to compute \( n_{p,q}(\alpha) \) works. First of all, thanks to Theorem 3.1, we know that it is possible to identify \( n_{\text{start}} \), a suitable an upper bound for \( n_{p,q}(\alpha) \). After having identified such a point, we will then generate a part of the \( n_i \) sequence by looking for \( S \)-units less than \( n_i \).

We will now describe a way how to generate such a suitable starting point.

4.1.1. Generating the starting point \( n_{\text{start}} \). An easy argument, see Section 3.1, shows that if \( f, e, g, h \geq 0 \) are such that

\[
1 < \frac{p^f}{q^e} < \alpha \quad \text{and} \quad 1 < \frac{q^h}{p^g} < \alpha
\]

then \( n_{p,q}(\alpha) \leq p^g \cdot q^e \). On taking logarithms these inequalities can be rewritten as

\[
0 < \frac{\theta_{p,q} - e}{f} < \frac{\log \alpha}{f \log q} \quad \text{and} \quad 0 < \frac{h}{g} - \theta_{p,q} < \frac{\log \alpha}{g \log q}.
\]

These type of inequalities appear in the theory of continued fractions and suggest to take

\[
\frac{e}{f} = \frac{r_k}{s_k} \quad \text{and} \quad \frac{h}{g} = \frac{r_\ell}{s_\ell},
\]

where \( k \) is even and minimal, \( \ell \) is odd and minimal and \( (r_m/s_m)_{m \geq 0} \) is the sequence of convergents of \( \theta_{p,q} \). Using the inequality

\[
\left| \frac{\theta_{p,q} - r_m}{s_m} \right| < \frac{1}{s_m s_{m+1}},
\]

see, e.g., [8, Thm. 171], we deduce that in both cases \( (m = k \) and \( m = \ell \) it suffices to require that

\[
\frac{1}{s_m s_{m+1}} < \frac{\log \alpha}{s_m \log q},
\]
that is
\begin{equation}
(4.3) \quad s_{m+1} > \frac{\log q}{\log \alpha}.
\end{equation}
Clearly, the larger \( q \) will be, or the closer to 1 the value of \( \alpha \) will be, the larger \( m \) will become. Let \( M \) be the minimal \( m \geq 0 \) such that \( (4.3) \) holds. In practice, to determine \( M \) we need a way to generate the continued fractions convergents of \( \theta_{p,q} \). To do this, we heavily rely on the PARI/GP [15] internal functions. Letting \( \varepsilon \in (0,1) \) be the required accuracy for the computations, the \texttt{contfrac}(x) \ function of PARI/GP returns the list of the partial quotients \([a_0, \ldots, a_n]\) of the continued fraction expansion of \( x > 0 \) so that \(|x - (a_0 + 1/(a_1 + \cdots + 1/a_n))| < \varepsilon\). Using such a continued fraction expansion of \( x \), the \texttt{contfracpnq} function of PARI/GP returns two sorted lists containing \( p_n \in \mathbb{N}^* \) and \( q_n \in \mathbb{N}^* \), \((p_n, q_n) = 1, \ n \in \mathbb{N}\), the numerators and the denominators of the convergents \( p_n/q_n \) of \( x \). In this way, it is relatively easy to obtain both the upper and lower convergents for each fraction \( \theta_{p,q} \), \( 2 \leq p < q \) both primes, we have to work with. Our practical computations are performed with \( \varepsilon = 10^{-19} \).

Having now such a sorted list of the denominators \( s_m \) of the convergents for \( \theta_{p,q} \), a standard dyadic search procedure will quickly provide \( M \). Nevertheless, it is possible to obtain some theoretical information about \( M \); this might be useful in the case one has initially no access to the sorted list of the convergent denominators. Since \( s_{m+1} \geq F_{m+2} \), where \( F_k \) is the \( k \)-th member of the Fibonacci sequence, it is enough to require that \( F_{m+2} \geq \log q / \log \alpha \). Recalling that \( F_{m+2} \geq \phi^m \), where \( \phi = (1 + \sqrt{5})/2 \), it suffices to take
\begin{equation}
(4.4) \quad m \geq \log \log q - \log \log \alpha \over \log \phi > 0.
\end{equation}
Hence \( M \leq \lceil \log \log q - \log \log \alpha \over \log \phi \rceil \). At the cost of complicating our inequality we can actually do a bit better. This improvement often turns out to make a real difference in numerical practice and so is worth the effort; this is due to the fact that both the numerators and the denominators of the continued fractions convergents are exponentially fast increasing sequences. As a consequence, being able to choose a smaller value for \( M \) ensures that much smaller values for the exponents in the definition of \( n_{\text{start}} \) can be chosen, see \( (4.5) \).

Namely, we will use the sharper inequality \( F_k \geq \phi^k \sqrt{5} - {53 \over 500} \) for every \( k \geq 3 \) (this inequality is a consequence of Binet’s formula \( F_k = \phi^k - (-\phi)^{-k} \sqrt{5} \)) and the observation that for \( k \geq 3 \) one has \(|(-\phi)^{-k} - \phi^k \sqrt{5} | < {53 \over 500} \). Hence if
\[ m + 2 > \frac{\log(q/\log \alpha) + {53 \over 500} + \log 5}{\log \phi} \]
for every \( m \geq 1 \), then \( F_{m+2} \geq \log q / \log \alpha \). It is not hard to verify that \( \phi^m \sqrt{5} - {53 \over 500} > \phi^m \) for every \( m \geq 0 \), and so the latter inequality gives a better lower bound for \( m \) than \( (4.4) \), namely
\[ m \geq \left[ \frac{\log(q/\log \alpha) + {53 \over 500} + \log 5}{\log \phi} \right] - 2 \geq M. \]

Once \( M \) is determined with the dyadic search procedure or using the previously described estimates, we can take \( \{k, \ell \} = \{M, M + 1\} \) in \( (4.2) \), and get that
\begin{equation}
(4.5) \quad n_{p,q}(\alpha) \leq p^\theta \cdot q^e \leq n_{\text{start}} := \min(p^{s_{M+1}} \cdot q^{r_M}; p^M \cdot q^{r_{M+1}}).
\end{equation}
In \( (4.1) \) we want to find the solutions with \( p^\theta \) and \( q^e \) minimal. It follows from the basics of continued fractions that the approach with the convergents as described here is actually optimal. So in retrospect our choice in \( (4.2) \) was best possible.
4.1.2. Generating left neighbors. We take \(n_{\text{start}}\) as starting candidate and proceed as follows. Assume that we know \(n_i = n_{\text{start}}\) and we want to obtain an \(S\)-unit \(n_{i-1}\) such that \(n_{i-1} \geq \lfloor n_i/\alpha \rfloor\). Remark that the goal here is not to find \(n_{i-1}\), the predecessor element in the \(n_i\) sequence, but just an \(S\)-unit less than \(n_i\) that verifies the Bertrand Postulate condition. For this reason in this procedure we will always choose, if possible, the \(S\)-unit \(n_{i-1}^*\) having the maximal distance from \(n_i\) compatible with the condition \(n_{i-1}^* \geq \lfloor n_i/\alpha \rfloor\), because this reduces the total amount of computations to be performed to determine \(n_{p,q}(\alpha)\).

In order to achieve this, we combine three different ways of searching for \(n_{i-1}^*\) in the following algorithm.

a) Searching using continued fraction convergents. Letting \(n_i := p^a q^b\), we have two possible choices.

1) We search for \(\ell\) such that \(a \in [s_\ell, s_{\ell+1}]\), where \((r_m/s_m)_{m \geq 0}\) is the sequence of convergents to \(\theta_{p,q}\).

Choose \(l \in \{\ell, \ell - 1\}\) such that \(r_l/s_l < \theta_{p,q}\); this is equivalent to \(x_1 := -s_l \log p + r_l \log q < 0\) and hence \(q^r/p^s < 1\). Define \(N_1 := n_i \exp(x_1) < n_i\).

2) We search for \(l\) such that \(b \in [r_\ell, r_{\ell+1}]\). Choose \(l \in \{\ell, \ell - 1\}\) such that \(r_l/s_l > \theta_{p,q}\); this is equivalent to \(x_2 := s_l \log p - r_l \log q < 0\) and hence \(p^s/q^r < 1\). Define \(N_2 := n_i \exp(x_2) < n_i\).

Both \(N_1\) and \(N_2\) are less than \(n_i\) and thus candidates to be chosen as \(n_{i-1}^*\). We need now select the best of them; i.e., the one whose distance from \(n_i\) is maximal, compatible with the condition \(n_{i-1}^* \geq n_i/\alpha \geq \lfloor n_i/\alpha \rfloor\). We point out that we compare here with \(n_i/\alpha\), rather than with \(\lfloor n_i/\alpha \rfloor\), since \(n_i/\alpha\) allows us, by taking logarithms, to work on the exponents only; in other words, we try to avoid as long as possible the necessity of performing the costly computations of \(n_i\) and \(\lfloor n_i/\alpha \rfloor\).

Let now \(x_{\text{max}} := \max(x_1; x_2)\) and \(x_{\text{min}} := \min(x_1; x_2)\). If \(x_{\text{max}} < -\log \alpha\), then \(N_1\), \(N_2\) are both \(< n_i/\alpha\). In this case we terminate this step with \(n_i\) and we proceed with step b); we also remark that at this point we know that \(n_{p,q}(\alpha) \leq \lfloor n_i/\alpha \rfloor\).

Assume that \(x_{\text{max}} \geq -\log \alpha\); this means that at least one of \(x_1\) and \(x_2\) is \(\geq -\log \alpha\). If both \(x_1\) and \(x_2\) are \(\geq -\log \alpha\), the best choice is to select the smallest one, \(x_{\text{min}}\), since \(n_i \exp(x_{\text{min}})\) has the largest distance from \(n_i\). Hence, if \(x_{\text{min}} \geq -\log \alpha\), we choose \(n_{i-1}^* = n_i \exp(x_{\text{min}}) = \min(N_1; N_2)\); otherwise we are in the case \(x_{\text{min}} < -\log \alpha \leq x_{\text{max}}\), and we are forced to choose \(n_{i-1}^* = n_i \exp(x_{\text{max}}) = \max(N_1; N_2)\). In both cases, we replace \(n_i\) with \(n_{i-1}^*\) and we repeat step a).

b) Searching trivially: first part. Assuming that step a) terminates with \(n_i\), an improved approximation of \(n_{p,q}(\alpha)\) is then obtained by trivially searching for a value of \(x := a \log p + b \log q\) in the range \([\log(n_i/\alpha), \log n_i]\) where \(a, b \in \mathbb{Z}\) run in the intervals \(0 \leq b \leq \lfloor \log n_i/\log q \rfloor\) and \(0 \leq a \leq \lfloor \log(p^n - b \log q) \log p \rfloor\). Moreover, we can also exploit the fact that the search procedure of step a) produced \(\overline{\pi} := \max(N_1; N_2) = p^{\overline{\pi}} \cdot q^{\overline{\pi}}\) such that \(\overline{\pi} < n_i/\alpha\). This means that the value \(\overline{x} := \overline{\pi} \log p + \overline{\pi} \log q\) is smaller than \(\log(n_i/\alpha)\). Since we need to find \(x \geq \log(n_i/\alpha) > \overline{x}\), we must have either \(a > \overline{\pi}\) or \(b > \overline{\pi}\). As a consequence, since \(\log q > \log p\), the best strategy to determine \(x\) is to first search for \(\overline{x} + 1 \leq b \leq \lfloor \log n_i/\log q \rfloor\) and \(0 \leq a \leq \lfloor \log(p^n - b \log q) \log p \rfloor\). If we have no success, we then work with \(0 \leq b \leq \overline{x}\) and \(\overline{x} + 1 \leq a \leq \lfloor \log(p^n - b \log q) \log p \rfloor\).

If in one of the previously described procedures we find a solution \(x \geq \log(n_i/\alpha)\), we have obtained an \(S\)-unit \(n_{i-1}^* := \exp(x)\) such that \(n_i/\alpha \leq n_{i-1}^* < n_i\). In this case, we replace \(n_i\) with \(n_{i-1}^*\) and we start again the search described in step a). If we do not find any solution, we continue with step c).

c) Searching trivially: second part. After steps a)-b) are over, we have not yet determined \(n_{p,q}(\alpha)\), since before, for efficiency reasons, we have replaced the condition \(\log(\lfloor n_i/\alpha \rfloor) < x\) with the sharper, but easier to compute, \(\log(n_i/\alpha) < x\). Hence we perform here another trivial search like the previous one, but using the the correct lower bound for \(x\) mentioned before. If we find such a solution \(x\), we have obtained an \(S\)-unit \(n_{i-1}^* := \exp(x)\) such that \(\lfloor n_i/\alpha \rfloor < n_{i-1}^* < n_i\). In this case we replace \(n_i\) with \(n_{i-1}^*\) and we start again the search described in step a). If we do not find
any solution $x$, this step has determined $n_0$, the smallest generated $S$-unit such that $n_{i-1} \geq \lfloor n_i/\alpha \rfloor$ holds for every $i \geq 1$. In this case, we continue with step d).

d) **Final computation.** In the final step we obtain $n_{p,q}(\alpha) = \lfloor n_0/\alpha \rfloor$.

The search in step a) is clearly the fastest one. Hence, the previously described procedure optimizes the computational cost by minimizing the number of times we are using the much slower trivial searches of steps b)-c). Moreover, except for step c), in which case the presence of the floor function forces us to compute $n_i$, the computations can be directly performed on the exponents $a, b$, rather than with the prime powers involved. This requires far less memory usage and, at the same time, as much smaller numbers are involved, improves the running time of the algorithm.

We ran our program for $\alpha \in \{2, 7/4, 5/3, 3/2, 4/3\}$, $2 \leq p < q \leq 500$, on the cluster located at the Dipartimento di Matematica “Tullio Levi-Civita” of the University of Padova; the running times were respectively 6 hours and 52 minutes ($\alpha = 2$), 9 hours and 39 minutes ($\alpha = 7/4$), 11 hours and 23 minutes ($\alpha = 5/3$), one day, 14 hours and 36 minutes ($\alpha = 3/2$), and 14 days, 20 hours ($\alpha = 4/3$).

To show the importance of working on the exponents only, we report here some data about the computations for $n_{3,83}(5/3)$. Here the starting value is $3^4 \cdot 83^{45}$, a number having 89 digits. After 2230 iterations our algorithm reached the pair (100, 0) and hence

$$n_{3,83}(5/3) = \lfloor 3^{100} \cdot 3/5 \rfloor = 309226512439206798621876677859372763621264513201,$$

a number having 48 digits. Table 4 provides some data for further cases and makes manifestly clear that handling the problem directly - so, without working on the exponents only - would be infeasible.

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<th>$b_{\text{start}}$</th>
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<td>25976</td>
<td>2146</td>
<td>[197$^7 \cdot 419^{812}/2$]</td>
</tr>
<tr>
<td>13</td>
<td>89</td>
<td>7/4</td>
<td>44875</td>
<td>4</td>
<td>6</td>
<td>3266</td>
<td>566222486</td>
<td>49997</td>
<td>6374</td>
<td>[13$^6 \cdot 89^{3266}/4/7$]</td>
</tr>
<tr>
<td>137</td>
<td>311</td>
<td>7/4</td>
<td>5174</td>
<td>6</td>
<td>6</td>
<td>1410</td>
<td>10339605</td>
<td>11071</td>
<td>3528</td>
<td>[137$^6 \cdot 311^{14410}/4/7$]</td>
</tr>
<tr>
<td>89</td>
<td>479</td>
<td>5/3</td>
<td>4415</td>
<td>8</td>
<td>10</td>
<td>290</td>
<td>7066379</td>
<td>8628</td>
<td>797</td>
<td>[89$^{10} \cdot 479^{200}/3/5$]</td>
</tr>
<tr>
<td>293</td>
<td>491</td>
<td>5/3</td>
<td>5221</td>
<td>11</td>
<td>11</td>
<td>55</td>
<td>12553989</td>
<td>12910</td>
<td>176</td>
<td>[293$^{11} \cdot 491^{55}/3/5$]</td>
</tr>
<tr>
<td>79</td>
<td>293</td>
<td>3/2</td>
<td>5170</td>
<td>10</td>
<td>12</td>
<td>296</td>
<td>10276001</td>
<td>9836</td>
<td>753</td>
<td>[79$^{12} \cdot 293^{296}/2/3$]</td>
</tr>
<tr>
<td>313</td>
<td>487</td>
<td>3/2</td>
<td>14</td>
<td>9749</td>
<td>880</td>
<td>12</td>
<td>8056800</td>
<td>26236</td>
<td>2229</td>
<td>[313$^{80} \cdot 487^{12}/2/3$]</td>
</tr>
<tr>
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<td>293</td>
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<td>12</td>
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<td>[101$^{3412} \cdot 293^{12}/3/4$]</td>
</tr>
<tr>
<td>167</td>
<td>367</td>
<td>4/3</td>
<td>29063</td>
<td>13</td>
<td>14</td>
<td>6792</td>
<td>339708048</td>
<td>64633</td>
<td>17451</td>
<td>[167$^{14} \cdot 367^{6792}/3/4$]</td>
</tr>
</tbody>
</table>

**Table 4.** Some data obtained during the computations. In this table $n_0 = p^{a_{\text{end}}} \cdot q^{b_{\text{end}}}$ and $\ell(n)$ is the number of decimals of $n$.

We will now give a detailed description of the determination of $n_{13,89}(7/4)$.

**Example 4.1.** To explain why some cases are harder than others, we consider $p = 13$ and $q = 89$. The continued fractions convergents of $\theta_{13,89}$ have numerators and denominators respectively equal

\footnote{We used a machine having 2 x Intel(R) Xeon(R) CPU E5-2630L v3 1.80GHz and 192GB of RAM (but we used up to 128GB of RAM in our computations).}
Hence \( r_0/s_0 = 0, r_1/s_1 = 1, r_2/s_2 = 1/2, r_3/s_3 = 3/5, r_4/s_4 = 4/7, r_5/s_5 = 25643/44875 \) and so on. If \( \alpha = 2 \), we obtain \( \lceil \frac{\log 89}{\log 2} \rceil = 7 \) and hence we have \( M = 3, s_{M+1} = 7, r_{M+1} = 4, s_M = 5, r_M = 3 \) in (4.5). In this case the algorithm starts with the information that \( n_{13,89}(2) \leq 13^5 \cdot 89^4 = 23295754887613 \), a number having 14 digits. On the other hand \( \lceil \frac{\log 89}{\log (7/4)} \rceil = 9, M = 4, s_{M+1} = 44875, r_{M+1} = 25643, s_M = 7, r_M = 4 \). So the algorithm starts just with the information that \( n_{13,89}(7/4) \leq 13^{44875} \cdot 89^4 \), a number having 49997 digits. The huge difference between the height \( h(r_4/s_4) \) and \( h(r_5/s_5) \) is responsible for \( n_{13,89}(7/4) \) being much harder to compute than \( n_{13,89}(2) \).

The situation for \( p = 2 \) and \( q = 8191 \) is similar: in this case \( (r_2, s_2) = (1, 13), (r_3, s_3) = (5677, 73800), \( \lceil \frac{\log 8191}{\log 2} \rceil = 13 \) and \( \lceil \frac{\log 8191}{\log (5/3)} \rceil = 23 \). Hence in order to establish that \( n_{2,8191}(5/3) = [2^{12} \cdot 8191^{1493} \cdot 3/5] \), the algorithm starts with \( n_{\text{start}} = 273800 \cdot 8191 \), a number having 22220 digits, while, to obtain \( n_{2,8191}(2) = 1 \), it is sufficient to work with the much smaller starting number \( 2^{12} \cdot 8191 = 33550336 \).

### 4.2. Extremal behavior of the gaps

We now show how to analyze the behavior of \( D_1 \) and \( D_2 \) implicitly defined in the inequalities

\[
\frac{n_i}{(\log n_i)^{D_2}} < n_{i+1} - n_i < \frac{n_i}{(\log n_i)^{D_1}},
\]

where \( n_i \geq 3 \). We further define \( \rho_i = \rho_i(p, q) \) implicitly by

\[
\rho_i = -\frac{\log \left( \frac{n_{i+1}}{n_i} - 1 \right)}{\log \log n_i} = \frac{\log \left( \frac{n_i}{n_{i+1} - n_i} \right)}{\log \log n_i},
\]

so that, if \( n_i \geq 3 \), then \( D_1 = D_1(p, q) = \max_i \rho_i(p, q) \) and \( D_2 = D_2(p, q) = \min_i \rho_i(p, q) \). We note that if \( n_{j+1} - n_j > n_j \) for some \( j \) (that is the Bertrand’s Postulate property does not hold for \( n_j \)), then \( \rho_j(p, q) \) is negative and hence \( D_2(p, q) < 0 \). As this occurs for at most finitely many \( j \), we can allow ourselves to disregard these outliers. Hence we will evaluate \( \rho_i(p, q) \) only for \( n_i \geq n_{p,q}(2) \).

We remark that the smallest \( D_2 \) is zero and this value is reached when \( n_{i+1} = 2n_i \); moreover, the largest \( D_1 \) are usually obtained with small powers of primes and, in fact, the maximal \( D_1 \) we got is for \( n_i = 3, n_{i+1} = 4 \), and is \( \log 3/\log \log 3 \approx 11.681421 \ldots \)

We also remark that \( D_j = C_j - \log(C_{j+2})/\log \log n_i \), \( j = 1, 2 \), where \( C_1, C_2, C_3, C_4 \) are defined in Theorem 1.3. Recalling \( C_3 = (\log p)^2/C_1 \) and \( C_4 = 8q \), this means that

\[
C_1(p, q) = D_1(p, q) \frac{\log \log n_i}{\log \log n_i - \log \log p} \quad \text{and} \quad C_2(p, q) = D_2(p, q) + \frac{\log (8q)}{\log \log n_i},
\]

where, in the first case, we also have to assume \( n_i \neq p \).

As we have just explained, in order to have meaningful results we need to work with \( \rho_i \geq 0 \), which is equivalent with \( n_{i+1} \leq 2n_i \). We will also require that \( n_i := p^a \cdot q^b \geq 3 \) and

\[
n_i \geq N := \exp(\log p \cdot \log q) \quad (= p^{\log q} = q^{\log p}).
\]

Since sometimes we do not know the value of \( n_{p,q}(2) \), the best strategy is then to start working with \( n_{\text{start}} \) as identified in Section 4.1.1 and use a modified form of the left neighbor search (explained in Section 4.1.2) to generate \( n_{i-1} \), until we have produced \( L \) left neighbors, or we have reached \( n_{p,q}(2) \). To do this, we need to slightly modify the search procedure in Section 4.1.2, since in this case the issue is to determine \( n_{i-1} \), rather than to get as close as possible to \( n_{p,q}(2) \). Hence in step a) of Section 4.1.2 we will always choose the value of \( x_{\text{max}} \) and in steps b)-c) of the same section
we will search for the maximal value of the form $x := a \log p + b \log q$ in the ranges there defined, where $a, b \in \mathbb{Z}$ run in the intervals $0 \leq b \leq \lfloor \log n_i / \log q \rfloor$ and $0 \leq a \leq \lfloor (\log n_i - b \log q) / \log p \rfloor$. These changes to the search procedures increase their computational cost; unfortunately, there is no way to work differently since in this case we cannot halt the procedure as soon as we have found an $S$-unit less than $n_i$ in the prescribed interval, but we need to be sure that such a point is the maximal $S$-unit less than $n_i$, or, in other words, that such a point is the left neighbor $n_{i-1}$ of $n_i$.

If we have reached $n_{p,q}(2)$ without having generated $L$ left neighbors, we start to generate right neighbors from $n_{\text{start}}$ until we have obtained a total number of $L$ neighbors. For the generation of the right neighbor $n_{i+1}$ of $n_i$, we used the continued fraction approach described in [2, Theorem 2.2]. The theoretical justification for the neighbor search procedure is provided in the Appendix.

In each step of the previously described algorithm, as soon as we have determined one of the neighbors of $n_i$, we can evaluate $\rho_i$, defined in (4.7), the constants defined in (4.6) and (4.8).

The same remarks we made in Section 4.1 about using the exponents only in the computations apply here as well with a single exception: in evaluating $\log(n_{i+1}/n_i - 1)$ in (4.7), we are in fact forced either to generate both $n_i$ and $n_{i+1}$, or to use

$$\log(n_{i+1}/n_i - 1) = \log(\exp(u \log p + v \log q) - 1),$$

in which we assume $u \log p + v \log q$, where $u, v \in \mathbb{Z}$ are such that $n_{i+1}/n_i = p^u \cdot q^v$, to be known. This is one of the most computationally costly steps of the whole procedure, but unfortunately there is no other way to compute $\rho_i$.

In this way we were able to collect, for every $2 \leq p < q$, with $p$ and $q$ both primes, all the values of $C_1, C_2, C_3, C_4, D_1, D_2$ and of their averaged values (defined in the next section). This is a heavy computation and the largest $P$ we were able to work with was $P = 10^4$. The data in Table 5 were obtained as a part of computation having an accuracy of 19 decimal digits (but the results are here truncated at 10 digits) performed for every $2 \leq p < q < 10^4$, and using $10^4$ neighbors, a computation which required about 2 days and 18 hours on the Dell Optiplex machine mentioned before.

4.3. **Average behavior of the gaps.** Define

$$\mu(p, q; k) := \frac{1}{k^+} \sum_{i=1}^{k^+} \rho_i(p, q), \quad \text{with } k^+ := \#\{1 \leq i \leq k : \rho_i(p, q) \geq 0\}.$$

**Lemma 4.1.** Let $p, q$ be fixed and $\epsilon > 0$. Recall that $C_1 = 2 \cdot 10^3 \log p \cdot \log q$. There exists an integer $k_{p,q}(\epsilon)$ such that

$$C_1^{-1} - \epsilon < \mu(p, q; k) < C_1 \quad \text{for every } k \geq k_{p,q}(\epsilon).$$

**Proof.** By Theorem 1.3 we have

$$C_1^{-1} - \frac{\log C_4}{\log \log n_i} < \rho_i < C_1 - \frac{\log C_3}{\log \log n_i}.$$

The proof follows from these two inequalities, $C_3 > 0$ and the observation that $(\log \log n_i)^{-1}$ tends to zero.

We cannot answer the following natural question.

**Question 4.1.** Does $\lim_{k \to \infty} \mu(p, q; k)$ exist?

The numerical work presented here suggests that the answer is yes. In this case we write $\mu(p, q)$ for the limit.
Table 5. Computed values of $D_1(p, q), D_2(p, q), \mu(p, q; k)$ with $2 \leq p \leq 5, p < q \leq 97$ and having generated 104 neighbors for each case.

Remark 4.1. Instead of $\mu(p, q; k)$ one can consider $\frac{1}{k} \sum_{i=1}^{k} \rho_i(p, q)$, which has also limit $\mu(p, q)$, if the limit exists. However, our preference is to work with $\mu(p, q; k)$, as numerically it seems to behave more regularly.

The programs and the results here described are available at the address: www.math.unipd.it/~languas/Units.html.

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The discriminator values reported on were kindly computed by Matteo Ferrari. The integers $n_{p,q}(\alpha)$ were computed on a machine of the cluster located at the Dipartimento di Matematica “Tullio Levi-Civita” of the University of Padova, see https://hpc.math.unipd.it. Languasco is grateful for having had such computing facilities at his disposal.

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**Appendix A. On efficient left and right neighbors searches**

Here we establish two theorems on determining the neighbors of \( n_i \). We will just make use of [2, Lemma 3.1], which characterizes the best approximants of a real number \( \theta \) in terms of the principal and intermediate convergents generated by its continued fraction. In line with [2] we will use the notation \( s_i \) instead of \( n_i \). The sought for neighbors are then \( s_{i-1} \) and \( s_{i+1} \).

**Theorem A.1 (Left neighbor search).** Suppose \( k \geq 1 \) and that we are given \( s_k = p^c \cdot q^d, (c_k, d_k) \in \mathbb{Z}^2_{\geq 0}, (c_k, d_k) \neq (0, 0) \). Then we can compute its left neighbor \( s_{k-1} \) in the following way:

i) Let \( u_1/v_1 > 0, (u_1, v_1) \in \mathbb{Z}^2_{\geq 0}, \) be the upper convergent of \( \theta_{p,q} \) with maximal numerator for which \( u_1 \leq d_k \) holds.

ii) Let \( u_2/v_2 > 0, (u_2, v_2) \in \mathbb{Z}^2_{\geq 0}, \) be the lower convergent of \( \theta_{p,q} \) with maximal denominator for which \( v_2 \leq c_k \) holds.

iii) Put \( x := |v_1 \log p - u_1 \log q| - |v_2 \log p - u_2 \log q| \) and

\[
\begin{align*}
    c_{k-1} = \begin{cases} 
    c_k + v_1 & \text{if } x < 0, \\
    c_k - v_2 & \text{if } x > 0, 
    \end{cases} \\
    d_{k-1} = \begin{cases} 
    d_k - u_1 & \text{if } x < 0, \\
    d_k + u_2 & \text{if } x > 0. 
    \end{cases}
\end{align*}
\]

Then we have \( s_{k-1} = p^{c_{k-1}} \cdot q^{d_{k-1}} \).

**Proof.** Let \( x < 0 \). Then we have \( c_{k-1} = c_k + v_1, d_{k-1} = d_k - u_1 \). Since \( u_1/v_1 \) is an upper convergent of \( \theta_{p,q} = \log p/\log q \), we have \( v_1 \log p - u_1 \log q < 0 \), and we define \( S := s_k p^{v_1} \cdot q^{-u_1} < s_k \). Now we prove that \( S = s_{k-1} \).

By contradiction, if \( S \neq s_{k-1} \) there exists \( (a, b) \in \mathbb{Z}^2_{\geq 0}, (a, b) \neq (0, 0), \) such that \( S = p^{c_k+v_1} \cdot q^{d_k-u_1} < p^a \cdot q^b < p^c \cdot q^d = s_k \). This is equivalent to

\[
(\text{A.1}) \quad 0 < \frac{d_k - b}{\theta_{p,q}} - (a - c_k) < \frac{u_1}{\theta_{p,q}} - v_1.
\]
If $b = d_k$, we must have $a < c_k$. Moreover, we will obtain that $u_1/(v_1 + c_k - a)$ is an upper convergent of $\theta_{p,q}$ having the same numerator of $u_1/v_1$. This implies $a = c_k$, which is impossible because $p^a \cdot q^b < s_k$ by definition.

So we have that $b \neq d_k$. Now, thanks to [2, Lemma 3.1], from (A.1) we obtain $d_k - b > u_1$. If $b > d_k$, it follows that $d_k > b + u_1 > d_k + u_1$ and hence $u_1 < 0$, which is a contradiction. Assume now $0 \leq b < d_k$. Then $\overline{\sigma} := d_k - b > 0$ and $\overline{\sigma} > u_1$. Again using (A.1), we have $\overline{\sigma} := a - c_k > (\pi - u_1)/\theta_{p,q} + v_1 > 0$ and $\overline{\sigma}/\overline{\tau}$ is an upper convergent for $\theta_{p,q}$. But this is a contradiction, since $u_1$ is maximal between the numerators $\leq d_k$. This proves that $s_{k-1} = p^{c_k+v_1} \cdot q^{d_k-u_1}$. The case $x > 0$ can be proved analogously. \hfill $\square$

In Theorem A.1 we used the same notation used in Theorem 2.2 in [2]; we think that this would help the reader to spot their differences more easily. Remark that the choices of the convergents of $\theta_{p,q}$ in this theorem are precisely the ones in Section 4.1.2, point a), and hence this $s_{k-1}$ corresponds to the choice of $x_{\text{max}}$ there.

We also include an alternative, and shorter, proof of Theorem 2.2 in [2].

**Theorem A.2 (Right neighbor search).** Suppose $k \geq 0$ and that we are given $s_k = p^{c_k} \cdot q^{d_k}$, $(c_k,d_k) \in \mathbb{Z}^2_{\geq 0}$. Then we can compute its right neighbor $s_{k+1}$ in the following way:

1. Let $u_1/v_1 > 0$, $(u_1,v_1) \in \mathbb{Z}^2_{\geq 0}$, be the upper convergent of $\theta_{p,q}$ with maximal denominator for which $v_1 \leq c_k$ holds.
2. Let $u_2/v_2 > 0$, $(u_2,v_2) \in \mathbb{Z}^2_{\geq 0}$, be the lower convergent of $\theta_{p,q}$ with maximal numerator for which $u_2 \leq d_k$ holds.
3. Put $x := |v_1 \log p - u_1 \log q| - |v_2 \log p - u_2 \log q|$ and

$$c_{k+1} = \begin{cases} c_k - v_1 & \text{if } x < 0, \\ c_k + v_2 & \text{if } x > 0, \end{cases}$$

$$d_{k+1} = \begin{cases} d_k + u_1 & \text{if } x < 0, \\ d_k - u_2 & \text{if } x > 0. \end{cases}$$

Then we have $s_{k+1} = p^{c_{k+1}} \cdot q^{d_{k+1}}$.

**Proof.** Assume $x < 0$. Then $|v_1 \log p - u_1 \log q| < |v_2 \log p - u_2 \log q|$ and we choose $c_{k+1} = c_k - v_1$, $d_{k+1} = d_k + u_1$. Since $u_1/v_1$ is an upper convergent of $\theta_{p,q} = \log p/\log q$, we have $u_1 \log q - v_1 \log p > 0$, and we define $S := s_k p^{v_1} \cdot q^{u_1} > s_k$. Now we prove that $S = s_{k+1}$.

By contradiction, if $S \neq s_{k+1}$ there exists $a, b \geq 0$, $(a,b) \neq (0,0)$, such that $p^{c_k-v_1} \cdot q^{d_k+u_1} = S > p^a \cdot q^b > s_k = p^{c_k} \cdot q^{d_k}$. This is equivalent to

$$0 < (b - d_k) - (c_k - a)\theta_{p,q} < u_1 - v_1 \theta_{p,q}. \tag{A.2}$$

If $a = c_k$, we must have $b > d_k$. Moreover, we will obtain that $(d_k - b + u_1)/v_1$ is an upper convergent of $\theta_{p,q}$ having the same denominator of $u_1/v_1$. This implies $b = d_k$, which is impossible because $p^a \cdot q^b > s_k$ by definition.

So we have that $a \neq c_k$. Now, thanks to [2, Lemma 3.1], from (A.2) we obtain $c_k > a + v_1$. If $a > c_k$, we obtain $c_k > a + v_1 > c_k + v_1$ and hence $v_1 < 0$, which is a contradiction. Assume now $0 \leq a < c_k$. Then $\overline{\sigma} := c_k - a > 0$ and $\overline{\sigma} > v_1$. Again using (A.2), we have $\overline{\sigma} := b - d_k > \overline{\sigma} \theta_{p,q} > 0$ and $\overline{\sigma}/\overline{\tau}$ is an upper convergent for $\theta_{p,q}$. But this is a contradiction, since $v_1$ is maximal between the denominators $\leq c_k$. This proves that $s_{k+1} = p^{c_k+v_1} \cdot q^{d_k+u_1}$. The case $x > 0$ can be proved analogously. \hfill $\square$
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