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ENERGY-MINIMIZING MAPPINGS OF COMPLEX PROJECTIVE SPACES

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Abstract. We determine the infimum of the energy in all homotopy classes of mappings from complex projective spaces to Riemannian manifolds, by showing that the infimum in each homotopy class is proportional to the infimal area in the homotopy class of mappings of the 2-sphere representing its action on the second homotopy group. We then establish a family of optimal lower bounds, for mappings from real and complex projective spaces to Riemannian manifolds, for a larger class of energy functionals. We also give a new proof of a theorem of Ohnita that stable harmonic mappings of complex projective spaces are pluriharmonic and establish several properties of these maps.

1. Introduction

The first goal of this paper is to determine the infimum of the energy in homotopy classes of mappings from complex projective spaces to Riemannian manifolds:

Theorem 1.1. *Let $(\mathbb{C}P^N, g_0)$ be the complex projective space with its canonical Riemannian metric normalized so the maximum of its sectional curvature is 4. Let Φ be a homotopy class of mappings from $(\mathbb{C}P^N, g_0)$ to a Riemannian manifold (M, g) and φ the homotopy class of mappings $S^2 \to (M, g)$ represented by composing the inclusion $\mathbb{C}P^1 \subseteq \mathbb{C}P^N$ with $F \in \Phi$. Let $A^\star$ be the infimum of the areas of mappings in φ, that is:*

$$A^\star = \inf_{f \in \varphi} \int_{S^2} \sqrt{det(df^T \circ df)} dA. \tag{1.1}$$

Then, letting $E_2(F)$ be the energy of a mapping F and $C_N = \frac{\pi^{N-1}}{(N-1)!}$,

$$\inf_{F \in \Phi} E_2(F) = C_N A^\star. \tag{1.2}$$

After proving Theorem 1.1, we will discuss conditions under which this infimum is realized by a continuous mapping. One source of examples is the theorem of Lichnerowicz that holomorphic and antiholomorphic mappings between compact Kähler manifolds minimize energy in their homotopy classes, in [Li70]. We will explain in Remark 4.2 how, for energy minimizing mappings of complex projective spaces, the following theorem of Ohnita can be thought of as a partial converse to Lichnerowicz's result:

Theorem 1.2. (Ohnita [Oh87]) *Let $F : (\mathbb{C}P^N, g_0) \to (M, g)$ be a stable harmonic mapping from complex projective space to a Riemannian manifold. Then F is pluriharmonic; that is, letting J be the complex structure of $\mathbb{C}P^N$ and α_F the second fundamental form of F (as in Definition 2.1 below),*

$$\alpha_F(JV, JW) = -\alpha_F(V, W). \tag{1.3}$$

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We will give a new proof of Ohnita's theorem, based on a result about stable harmonic mappings of compact Hermitian symmetric spaces in Lemma 2.6. Before proving Theorem 1.1, we will also show via a direct calculation in Proposition 3.6 that (1.2) gives the energy of any holomorphic or antiholomorphic mapping from $(\mathbb{C}P^N, g_0)$ to a compact Kähler manifold.

The infimum in Theorem 1.1 coincides with a lower bound for the energy of mappings from complex projective spaces to Riemannian manifolds which Croke established in [Cr87]. We will extend Croke's result to a sharp lower bound for a larger family of energy functionals:

Theorem 1.3. *Let $F : (\mathbb{C}P^N, g_0) \to (M, g)$ be a Lipschitz mapping from complex projective space to a Riemannian manifold. Let $A^\star$ be the invariant associated to the homotopy class of F in Theorem 1.1 and $E_p(F)$ its p-energy (as in Definition 3.2 below). Then for all $p > 2$,*

$$E_p(F) \geq \frac{\pi^N}{2N!}\left(\frac{2N}{\pi}A^\star\right)^{\frac{p}{2}}. \tag{1.4}$$

Equality for at least one $p > 2$ implies that F is a homothety onto a pluriharmonically immersed minimal submanifold, and in particular that equality holds in (1.4) for all $p \geq 2$.

Special cases of Theorems 1.1 and 1.3 state that the identity mapping of complex projective space minimizes p-energy in its homotopy class for $p \geq 2$. In [Wh86], White studied homotopy-theoretic conditions under which there are positive infima for a very general class of energy functionals in a homotopy class of mappings between Riemannian manifolds. His results imply that for $1 \leq p < 2$, the infimum of the p-energy in any homotopy class of mappings from complex projective space to a closed Riemannian manifold is 0. In this sense, the statement that the identity mapping minimizes p-energy in its homotopy class for $p \geq 2$ is optimal. For $p > 2$, Theorem 1.3 implies that the only p-energy minimizing mappings in its homotopy class are isometries, however for $p = 2$ Lichnerowicz's theorem cited above implies that all biholomorphic mappings of $(\mathbb{C}P^N, g_0)$ minimize energy in their homotopy class. In the homotopy class of the identity, this includes projective linear transformations which are not isometries.

White's results in [Wh86] also imply that in all homotopy classes of mappings from quaternionic projective spaces $(\mathbb{H}P^N, g_0)$, the Cayley projective plane $(\mathcal{C}aP^2, g_0)$ and round spheres (S^n, g_0) of dimension $n \geq 3$ to Riemannian manifolds, the infimum of the energy is 0. Together with Theorem 1.1, these results determine the infimum of the energy for all homotopy classes of mappings of compact rank-one Riemannian symmetric spaces other than real projective spaces, and in particular for all simply connected spaces of this type. For mappings of real projective spaces, we have some results similar to our results for complex projective spaces above, but we will show there are also some potentially significant differences:

Croke proved in [Cr87] that the identity mapping of real projective space minimizes energy in its homotopy class. We will extend this result to a sharp lower bound for the full range of p-energy functionals in real projective space:

Theorem 1.4. (See [Cr87] for $p = 2$) *Let $(\mathbb{R}P^n, g_0)$, $n \geq 2$, be the real projective space with its Riemannian metric of constant curvature 1. Let $F : (\mathbb{R}P^n, g_0) \to (M, g)$ be a Lipschitz mapping to a Riemannian manifold, $E_p(F)$ its p-energy, and $L^\star$ the infimum of the lengths of closed curves freely homotopic to $F_*(\gamma)$, where γ is a generator of $\pi_1(\mathbb{R}P^n)$. Then for all $p \geq 1$,*

$$E_p(F) \geq \frac{\sigma(n)}{4}\left(\frac{\sqrt{n}}{\pi}L^\star\right)^p, \tag{1.5}$$

where $\sigma(n)$ is the volume of the unit n-sphere. Equality for at least one $p > 1$ implies F is a homothety onto a totally geodesic submanifold, and in particular that equality holds for all $p \geq 1$. If F is a smooth immersion, equality for $p = 1$ also implies these conditions.

As with Theorems 1.1 and 1.3 for complex projective space, Theorem 1.4 implies that the identity mapping of real projective space minimizes p-energy in its homotopy class for all $p \geq 1$. Unlike complex projective space however, in which the "threshold" value $p = 2$ admits a larger family of energy minimizing mappings than the p-energy for $p > 2$, in real projective space the characterization of p-energy minimizing mappings is equally rigid for all $p \geq 1$. For $p > 1$ this follows from an adaptation of Croke's proof of the case $p = 2$ in [Cr87], however a key step in this proof is not valid for $p = 1$. We will therefore give a different argument which draws on the proof of the Blaschke conjecture by Berger, Green, Kazdan and Yang, cf. [Gn62, Be12].

Although Theorem 1.4 gives a lower bound for the energy of mappings of real projective spaces which is similar to the lower bound for complex projective spaces implied by Theorem 1.1, determining the infimum of the energy in a homotopy class of mappings of $(\mathbb{R}P^n, g_0)$ may be more complicated: let g be a Riemannian metric on $\mathbb{R}P^2$ of area $A(\mathbb{R}P^2, g)$ and ψ the homotopy class of the identity mapping of $\mathbb{R}P^2$, viewed as a collection of mappings from $(\mathbb{R}P^2, g_0)$ to $(\mathbb{R}P^2, g)$. A basic property of the energy is that for mappings of surfaces, the energy is bounded below by the area of the image, with equality precisely for conformal mappings, cf. Lemma 3.3 below. Therefore, for any $f \in \psi$, $E_2(f) \geq A(\mathbb{R}P^2, g)$. However, Pu's inequality (Theorem 4.3 below) implies that, in the notation of Theorem 1.4, $A(\mathbb{R}P^2, g) \geq \frac{2}{\pi} {L^\star}^2$, with equality only if g has constant curvature. Therefore, for any g which is not isometric to a rescaling of g_0, the infimum of the energy in ψ is strictly greater than the lower bound in Theorem 1.4.

In fact, it follows from the uniformization theorem that in this situation, $A(\mathbb{R}P^2, g)$ is the minimum energy of mappings in ψ. This, together with the $p = 2$ case of Theorem 1.4 in [Cr87], gives an alternate proof of Pu's inequality. More generally, the uniformization theorem implies that in any homotopy class of mappings from $(\mathbb{R}P^2, g_0)$ to a Riemannian manifold, the infimum of the energy is equal to the infimum of area, cf. Lemma 4.1. But although this determines the infimum of the energy in homotopy classes of mappings of $(\mathbb{R}P^2, g_0)$, the result of Lemma 4.4 suggests that, as a consequence of Pu's inequality, the infimum of the energy in homotopy classes of mappings of $(\mathbb{R}P^n, g_0)$, $n \geq 3$ is also usually greater than the lower bound in Theorem 1.4 and may be more difficult to determine.

In contrast with these observations, though, in Theorem 4.6 we will give a two-sided estimate for the infimum of the energy in a homotopy class of mappings of $(\mathbb{R}P^3, g_0)$. This result suggests this infimum may be determined in part by the same geometric data as in our result for complex projective spaces above. We intend to take up this problem in future work, along with analogous problems for quaternionic projective spaces and the Cayley plane.

Outline and Notation: In Section 2, we will prove a slight generalization of Theorem 1.2, in Proposition 2.7, as a consequence of a result we will establish for stable harmonic mappings of compact Hermitian symmetric spaces in Lemma 2.6. We will also review some background related to the energy functional and harmonic mappings. In Section 3 we will prove Theorems 1.3 and 1.4, after reviewing the definition of the p-energy of a mapping. In Sections 2 and 3 we will also establish several properties of pluriharmonic mappings of complex projective spaces. In Section 4 we will prove Theorem 1.1, and we will discuss some of the issues involved in finding the infimum of the energy in homotopy classes of mappings of real projective spaces. We will also establish upper and lower bounds for the infimum of the energy in homotopy classes of mappings from $(\mathbb{R}P^3, g_0)$ to Riemannian manifolds in Theorem 4.6. Throughout the paper, we will write $\sigma(k)$ for the volume of the unit sphere in $\mathbb{R}^{k+1}$. We will write J for the complex structure of any complex manifold.

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2. Stable Harmonic and Pluriharmonic Mappings

In this section we will prove a slightly more general version of Theorem 1.2, in Proposition 2.7, as a corollary of a property we will establish for stable harmonic mappings of compact Hermitian symmetric spaces in Lemma 2.6. We will also discuss some properties of pluriharmonic mappings of complex projective spaces, in Theorems 2.8 and 2.10.

The energy of a Lipschitz map $F : (N^n, h) \to (M^m, g)$ of Riemannian manifolds is:

$$E_2(F) = \frac{1}{2}\int_N |dF_x|^2 dVol_h, \tag{2.1}$$

where $|dF_x|$ is the Euclidean norm of $dF : T_xN \to T_{F(x)}M$ at a point $x \in N$ at which F is differentiable. There are many equivalent ways to define this invariant, discussed by Eells and Sampson in [ES64]. In this work, they initiated the study of mappings which are critical for the energy, known as harmonic mappings. The Euler-Lagrange equation for harmonic mappings can be formulated in terms of the second fundamental form of a mapping:

Definition 2.1. *Let $F : (N, h) \to (M, g)$ be a smooth mapping of Riemannian manifolds, ∇^h and ∇^g the Levi-Civita connections of (N, h) and (M, g), F^*TM the pullback of the tangent bundle of M via F, and $F^*\nabla^g$ the induced connection in F^*TM. Let $F_* : TN \to F^*TM$ be the mapping induced by the differential dF of F. The second fundamental form α_F of F is the symmetric F^*TM-valued 2-tensor on N which, for vector fields V, W, satisfies:*

$$\alpha_F(V, W) = F^*\nabla^g_V F_*W - F_*(\nabla^h_V W). \tag{2.2}$$

The section of F^*TM given by taking the trace of the second fundamental form is called the tension vector field of the mapping F:

Theorem 2.2. (Eells, Sampson [ES64]) *Let $F : (N, h) \to (M, g)$ be a smooth mapping of Riemannian manifolds and $Tr(\alpha_F)$ its tension field. Then F is harmonic if and only if $Tr(\alpha_F) = 0$.*

Continuous, weakly harmonic maps are smooth, cf. [Aub13, Ch.10]. In particular, a continuous mapping which minimizes energy in its homotopy class is a smooth harmonic map. However a homotopy class may not contain a continuous, energy minimizing map. One instance of this is in the homotopy class of the identity mapping of $(S^n g_0)$, $n \geq 3$: the infimum of the energy is 0, but any mapping $F \simeq Id_{S^n}$ is nonconstant and therefore has $E_2(F) > 0$.

A stable harmonic mapping is a harmonic map for which the second variation of the energy is nonnegative, for all variations as in Lemma 2.3 below. Xin showed in [Xin80] that round spheres of dimension $n \geq 3$ do not admit any nonconstant stable harmonic mappings to Riemannian manifolds. Ohnita then showed in [Oh86] that this is also the case for quaternionic projective spaces and the Cayley plane. Smith studied the second variation formula for the energy of a harmonic map in [Sm75]. A special case of this formula which we will use appears in Ohnita's work:

Lemma 2.3. (Ohnita [Oh87, p.563–564, (1.4)]) *Let $F : (N, h) \to (M, g)$ be a harmonic mapping between closed Riemannian manifolds. Let $\mathcal{J}_F$ be the Jacobi operator of F; that is, for a variation F_t of $F = F_0$ with $\frac{\partial F}{\partial t}|_{t=0} = W$,*

$$\frac{d^2}{dt^2}(E_2(F_t))|_{t=0} = \int_N g(\mathcal{J}_F(W), W)\ dVol_N. \tag{2.3}$$

Then for a vector field V on N,

$$\mathcal{J}_F(F_*V) = -F_*\Big(Tr(\nabla^h\nabla^h V) + Ric^N(V)\Big) - 2Tr(\alpha_F \circ \nabla^h V), \tag{2.4}$$

where $Tr(\alpha_F \circ \nabla^h V) = \sum_{i=1}^{n} \alpha_F(\nabla^h_{e_i} V, e_i)$ for an orthonormal frame $\{e_1, \dots, e_n\}$ for TN.

A pluriharmonic map F from a Kähler manifold (X, h) with complex structure J to a Riemannian manifold (M, g) is a harmonic mapping with the property that $\alpha_F(JV, JW) = -\alpha_F(V, W)$. Holomorphic and antiholomorphic mappings between Kähler manifolds are pluriharmonic, but a pluriharmonic mapping between Kähler manifolds may not be either holomorphic or antiholomorphic – for example, all harmonic mappings of orientable surfaces are pluriharmonic. The condition of being pluriharmonic is related to the condition of holomorphicity by the following:

Lemma 2.4. *For a smoooth map $F : (X, h) \to (M, g)$ from a Kähler manifold to a Riemannian manifold, the following are equivalent:*

- **i.)** *F is pluriharmonic.*
- **ii.)** *For any holomorophic mapping $G : (Y, \widetilde{h}) \to (X, h)$, $F \circ G$ is pluriharmonic.*
- **iii.)** *For any germ of a complex curve $\Sigma \subseteq X$, $F|_\Sigma$ is a harmonic map.*

Proof. That ii.) implies i.) and iii.) is immediate. That iii.) implies i.) is a theorem of Rawnsley, see [BBdeBR89, Section 4]. To see that i.) implies ii.), we calculate the second fundamental form of $F \circ G$: letting $\widetilde{F}_* : G^*TX \to (F \circ G)^*TM$ be the bundle homomorphism induced by $F_* : TX \to F^*TM$,

$$\alpha_{F\circ G}(V, W) = \alpha_{\widetilde{F}}(G_*V, G_*W) + \widetilde{F}_*\alpha_G(V, W). \tag{2.5}$$

The pluriharmonicity of F and the holomorphicity of G then imply that $\alpha_{F\circ G}(JV, JW) = -\alpha_{F\circ G}(V, W)$. □

The basis for our proof of Theorem 1.2 is a formula for the second variation of energy of a harmonic map of a Kähler manifold along the flow generated by a holomorphic vector field:

Lemma 2.5. *Let (X, h) be a Kähler manifold and $F : (X, h) \to (M, g)$ a harmonic mapping to a Riemannian manifold. Let $\mathcal{J}_F$ be the Jacobi operator of F, as in Lemma 2.3. If V is a holomorphic vector field on X, then:*

$$\mathcal{J}_F(F_*V) = -2Tr(\alpha_F \circ \nabla^h V), \tag{2.6}$$

where $Tr(\alpha_F \circ \nabla^h V)$ is as in Lemma 2.3.

Proof. Let R be the curvature tensor of (X, h), with $R(V, W)U = \nabla^h_V\nabla^h_W U - \nabla^h_W\nabla^h_V U - \nabla^h_{[V,W]}U$, and let V be a holomorphic vector field on (X, h). By extending a unit tangent vector $e \in T_xX$ to a locally defined holomorphic vector field E, we have:

$$\begin{aligned}&\nabla^h\nabla^h V(e,e)+\nabla^h\nabla^h V(Je,Je)+R(V,e)e+R(V,Je)Je\\&=\nabla^h_E[E,V]+\nabla^h_{JE}[JE,V]-\nabla^h_{[V,E]}E-\nabla^h_{[V,JE]}JE=0,\end{aligned}\tag{2.7}$$

which implies $Tr(\nabla^h\nabla^h V)+Ric(V)=0$. By Lemma 2.3 this implies the identity (2.6). □

For a compact Hermitian symmetric space, Lemma 2.5 implies:

Lemma 2.6. *Let (Z_0,h_0) be a compact Hermitian symmetric space, $\mathfrak{g}$ the Lie algebra of Killing vector fields on (Z_0,h_0), and $\widetilde{\mathfrak{g}}$ the space of holomorphic vector fields on Z_0 of the form $\{JV : V\in\mathfrak{g}\}$. Let $F:(Z_0,h_0)\to(M,g)$ be a harmonic mapping to a Riemannian manifold, $\mathcal{J}_F$ the Jacobi operator of F as in Lemma 2.5, and II the bilinear form on $\widetilde{\mathfrak{g}}$ defined by:*

$$II(\widetilde{V},\widetilde{W})=\int_{Z_0} g(\mathcal{J}_F(F_*\widetilde{V}),F_*\widetilde{W})dVol_{h_0}.\tag{2.8}$$

Then $Tr(II)=0$. In particular, if F is a stable harmonic mapping, then for any Killing vector field V on (Z_0,h_0), letting $\widetilde{V}=JV$, we have $F_*\widetilde{V}\in\ker(\mathcal{J}_F)$.

Proof. Let $V_1,V_2,\ldots,V_r$ be an orthonormal basis for $\mathfrak{g}$ relative to the negative of the Killing form and $\widetilde{V_i}=JV_i$. Then we have:

$$Tr(II)=\int_{Z_0}\sum_{i=1}^r g(\mathcal{J}_F(F_*\widetilde{V_i}),F_*\widetilde{V_i})dVol_{h_0}.\tag{2.9}$$

The pointwise value of the integrand $\sum_{i=1}^r g(\mathcal{J}_F(F_*\widetilde{V_i}),F_*\widetilde{V_i})$ in (2.9) is independent of the orthonormal basis V_i for $\mathfrak{g}$. Because (Z_0,h_0) is a symmetric space, for each $z\in Z_0$ we have an orthogonal decomposition $\mathfrak{g}=\mathfrak{p}_z\oplus\mathfrak{k}_z$, where $\mathfrak{p}_z$ is the space Killing fields V with $\nabla^{h_0}V=0$ at z and $\mathfrak{k}_z$ is the space of V with $V=0$ at z. Choosing an orthonormal basis for $\mathfrak{g}$ such that $V_1,\ldots,V_n$ is a basis for $\mathfrak{p}_z$ and $V_{n+1},\ldots,V_r$ is a basis for $\mathfrak{k}_z$, Lemma 2.5 implies that:

$$g(\mathcal{J}_F(F_*\widetilde{V_i}),F_*\widetilde{V_i})=-2g(Tr(\alpha_F\circ\nabla^{h_0}\widetilde{V_i}),F_*\widetilde{V_i}).\tag{2.10}$$

For $i=1,\ldots,n$ we have $g(\mathcal{J}_F(F_*\widetilde{V_i}),F_*\widetilde{V_i})=0$ at z because $\nabla^{h_0}\widetilde{V_i}=J\circ\nabla^{h_0}V_i=0$ at z, and for $i=n+1,\ldots,r$ we have $g(\mathcal{J}_F(F_*\widetilde{V_i}),F_*\widetilde{V_i})=0$ because $\widetilde{V_i}=0$ at z. The integrand in (2.9) therefore vanishes identically and $Tr(II)=0$. If F is a stable harmonic map, let $0=\lambda_0<\lambda_1<\cdots<\lambda_j<\cdots$ be the distinct eigenvalues of $\mathcal{J}_F$ acting on sections of F^*TM. Let $\widetilde{V}_1,\ldots,\widetilde{V}_r$ be an orthonormal basis for $\widetilde{\mathfrak{g}}$, and let $\widetilde{V}_i^j$ be the component of $F_*\widetilde{V_i}$ belonging to the j^{th} eigenspace of $\mathcal{J}_F$. By (2.8), we then have:

$$0=Tr(II)=\sum_{i=1}^r\int_{Z_0} g(\mathcal{J}_F(F_*\widetilde{V_i}),F_*\widetilde{V_i})=\sum_{j=0}^\infty\lambda_j\left(\sum_{i=1}^r\int_{Z_0}|\widetilde{V}_i^j|^2 dVol_{h_0}\right),\tag{2.11}$$

which implies $\widetilde{V}_i^j=0$ for all $j\geq 1$ and $i=1,\ldots,r$, and therefore that $F_*\widetilde{V_i}$ is in the 0-eigenspace of $\mathcal{J}_F$. □

For mappings of complex projective space, Lemma 2.6 implies:

Proposition 2.7. *Let $F : (\mathbb{C}P^N, g_0) \to (M, g)$ be a harmonic mapping from complex projective space to a Riemannian manifold and $\mathcal{J}_F$ its Jacobi operator. Suppose that for all Killing vector fields V on $(\mathbb{C}P^N, g_0)$, letting $\widetilde{V} = JV$, we have $F_* \widetilde{V} \in \ker(\mathcal{J}_F)$.*

Then F is pluriharmonic. In particular, stable harmonic mappings of $(\mathbb{C}P^N, g_0)$ are pluriharmonic.

Proof. By Lemma 2.5, for all Killing vector fields V on $(\mathbb{C}P^N, g_0)$, letting $\widetilde{V} = JV$, we have:

$$\mathcal{J}_F(\widetilde{V}) = -2Tr(\alpha_F \circ \nabla^{g_0} \widetilde{V}) = 0. \tag{2.12}$$

At $x \in \mathbb{C}P^N$, the Lie algebra $\mathfrak{k}_x$ of Killing vector fields which vanish at x is isomorphic to $\mathfrak{u}(N)$, and the identification $V \mapsto \nabla^{g_0} V$ gives an isomorphism of $\mathfrak{k}_x$ with the algebra of skew-Hermitian linear transformations of $T_x \mathbb{C}P^N$. Given any unit tangent vector e to $\mathbb{C}P^N$ at x, there is therefore a Killing vector field $V \in \mathfrak{k}_x$ with $\nabla^{g_0}_e V = Je$, $\nabla^{g_0}_{Je} V = -e$ and $\nabla^{g_0}_{e'} V = 0$ for e' orthogonal to the real 2-plane spanned by $\{e, Je\}$. Letting $\widetilde{V} = JV$, we then have $\nabla^{g_0} \widetilde{V} = J \nabla^{g_0} V$, and by (2.12), we therefore have:

$$\begin{aligned} 0 = Tr(\alpha_F \circ \nabla^{g_0} \widetilde{V}) &= \alpha_F(\nabla^{g_0}_e \widetilde{V}, e) + \alpha_F(\nabla^{g_0}_{Je} \widetilde{V}, Je) \\ &= -\left[\alpha_F(e, e) + \alpha_F(Je, Je)\right], \end{aligned} \tag{2.13}$$

so that $\alpha_F(Je, Je) = -\alpha_F(e, e)$. By the polarization identity, the bilinear form α_F then satisfies the identity $\alpha_F(JV, JW) = -\alpha_F(V, W)$ and F is pluriharmonic. By Lemma 2.6, this is the case for stable harmonic mappings of $(\mathbb{C}P^N, g_0)$. □

Pluriharmonic mappings of complex projective spaces retain some properties of holomorphic mappings, even when the target is only assumed to be a Riemannian manifold. First, note that harmonic mappings of $\mathbb{C}P^1$ are conformal branched immersions:

Theorem 2.8. (Lemaire [Lem78, Theorem 2.8]) *A harmonic mapping $f : (S^2, g_0) \to (M, g)$ from the 2-sphere to a Riemannian manifold is a conformal branched immersion. In particular, $f^* g = \mu(x) g_0$ for a smooth, nonnegative function μ on S^2.*

Together with Lemma 2.4, Theorem 2.8 implies that pluriharmonic mappings of complex projective space have complex-linear derivatives, in the sense of Lemma 2.9 below. Later we will quote a result of Ohnita, in Theorem 2.10, which subsumes this result, but Lemma 2.9 is sufficient for several of our purposes and follows from what we have already established by a short, direct proof. We therefore include this argument below:

Lemma 2.9. *Let $F : (\mathbb{C}P^N, g_0) \to (M, g)$ be a pluriharmonic mapping from complex projective space to a Riemannian manifold. Then $F^* g$ is a Hermitian bilinear form on $\mathbb{C}P^N$; that is:*

$$F^* g(JV, JW) = F^* g(V, W). \tag{2.14}$$

Proof. Let F be a pluriharmonic mapping of $(\mathbb{C}P^N, g_0)$. By Lemma 2.4, $F|_{\mathbb{C}P^1}$ is harmonic for all degree-1 curves $\mathbb{C}P^1 \subseteq \mathbb{C}P^N$. Every unit tangent vector $\vec{u}$ to $\mathbb{C}P^N$ is tangent to a unique such curve, which we will denote $\mathsf{T}(\vec{u})$, to which $J\vec{u}$ is also tangent. By Theorem 2.8 applied to $F|_{\mathsf{T}(\vec{u})}$, $|dF(\vec{u})| = |dF(J\vec{u})|$. The polarization identity then implies the bilinear form $F^* g$ is Hermitian. □

In fact, we have:

Theorem 2.10. (Ohnita [Oh87, Proposition 4.2], see also [BBdeBR89, Lemmas 4 and 5]) *Let X be a closed complex manifold which admits a Kähler-Einstein metric with positive Ricci curvature. Let h be any Kähler metric on X, and let $F:(X,h)\to(M,g)$ be a pluriharmonic mapping to a Riemannian manifold. Then F^*g is a Hermitian bilinear form on (X,h), as in Lemma 2.9. Moreover, the 2-form $\omega^*(V,W)=F^*g(JV,W)$ on X is closed.*

3. Lower Bounds for Energy Functionals of Mappings

In this section, we will prove Theorems 1.3 and 1.4, and we will establish more properties of pluriharmonic mappings of complex projective spaces, in Proposition 3.6 and Lemma 3.7. In proving Theorem 1.3, we will also show that the quantity $C_N A^\star$ associated to the homotopy class Φ in Theorem 1.1 is a lower bound for the energy of mappings $F\in\Phi$. We will then complete the proof of Theorem 1.1 in Section 4 by showing that it is the infimum in each homotopy class. The starting point for all these results is the following formula for the energy of a mapping due to Croke:

Lemma 3.1. (Croke [Cr87, Proposition 1]) *Let $F:(N^n,h)\to(M^m,g)$ be a Lipschitz mapping of Riemannian manifolds and $x\in N$ a point at which F is differentiable. Then:*

$$|dF_x|^2=\frac{n}{\sigma(n-1)}\int\limits_{U_x(N,h)}|dF(\vec{u})|^2d\vec{u}. \tag{3.1}$$

In particular,

$$E_2(F)=\frac{n}{2\sigma(n-1)}\int\limits_{U(N,h)}|dF(\vec{u})|^2d\vec{u}, \tag{3.2}$$

where $U(N,h)$ is the unit tangent bundle of (N,h) and $U_x(N,h)$ its fibre at x.

The energy $E_2(F)$ fits naturally into a 1-parameter family of functionals:

Definition 3.2 (cf. [Wh86, HL87, Wh88, We98])**.** *Let $F:(N^n,h)\to(M^m,g)$ be a Lipschitz mapping of Riemannian manifolds. For $p\geq 1$, the p-energy $E_p(F)$ of F is:*

$$\frac{1}{2}\int\limits_N|dF_x|^p dVol_h. \tag{3.3}$$

Note that for $p\neq 2$, continuous maps which are critical for $E_p(F)$ need not be smooth, or even $C^{1,1}$, cf. [HL87]. Note also that the definition of p-energy in some papers differs by a constant from the formula in (3.3).

At all $x\in N$ at which the mapping F in Definition 3.2 is differentiable, F^*g is a positive semidefinite, symmetric bilinear form on T_xN which can be diagonalized relative to h. Letting $e_1,e_2,\cdots,e_n$ be an orthonormal basis for T_xN (relative to h) of eigenvectors for F^*g, we then have:

$$|dF_x|^2=\sum_{i=1}^n|dF(e_i)|^2. \tag{3.4}$$

Lemma 3.1 can be derived from this identity, which also gives the following elementary lower bound for the p-energy of $F:(N^n,h)\to(M^m,g)$ for $p\geq\dim(N)$:

Lemma 3.3. *Let (N^n, h) be a finite volume Riemannian manifold and $F : (N^n, h) \to (M^m, g)$ a Lipschitz mapping, and define $Vol_h(N, F^*g)$ to be:*

$$\int\limits_N \sqrt{det(dF_x^T \circ dF_x)} dVol_h. \tag{3.5}$$

Then for $p \geq n$,

$$E_p(F) \geq \frac{n^{\frac{p}{2}} Vol_h(N, F^*g)^{\frac{p}{n}}}{2\ Vol(N,h)^{\frac{p-n}{n}}}. \tag{3.6}$$

For $p = n$, equality holds if and only if dF_x is a homothety at almost all $x \in N$. For $p > n$, equality holds if and only if dF_x is a homothety, by a constant factor κ_F, at almost all $x \in N$.

Proof. For $p = n$ Lemma 3.3 follows from the inequality $|dF_x|^p \geq n^{\frac{p}{2}}(\sqrt{det(dF_x^T \circ dF_x)})^{\frac{p}{n}}$, which follows from (3.4) and the arithmetic-geometric mean inequality for the eigenvalues of F^*g relative to h. For $p > n$, Lemma 3.3 follows from this pointwise inequality together with Hölder's inequality. □

Note that in Lemma 3.3, equality for at least one $p > n$ implies equality for all $p \geq n$, and in fact that (3.6) is an equality for all $p \geq 1$. If F is smooth, the equality condition for $p = n$ in Lemma 3.3 says that F is a semiconformal mapping, that is $F^*g = \mu(x)h$ for a nonnegative function μ on N, and the equality condition for $p > n$ says that F is a homothety, i.e. F^*g is a rescaling of h. This generalizes the well-known fact that for mappings of surfaces, the energy is pointwise bounded below by the area of the image, with equality precisely where the mapping is conformal. Also, note that if F is a pluriharmonic mapping of $(\mathbb{C}P^N, g_0)$, then by Lemma 2.9 we can diagonalize F^*g as in (3.4) by a unitary basis $e_1, e_2 = Je_1, \cdots, e_{2N} = Je_{2N-1}$, with $|dF(e_i)| = |dF(Je_i)|$.

In proving Theorems 1.1, 1.3 and 1.4, we will work with the following measure spaces associated to the canonical Riemannian metrics on real and complex projective space:

Definition 3.4. A. *Let $\mathcal{G}$ be the space of oriented geodesics γ in $(\mathbb{R}P^n, g_0)$; that is, the quotient of the unit tangent bundle $U(\mathbb{R}P^n, g_0)$ by the geodesic flow. Let $dVol_U$ be the canonical measure on $U(\mathbb{R}P^n, g_0)$, $\zeta : U(\mathbb{R}P^n, g_0) \to \mathcal{G}$ the quotient map, $\zeta_\# dVol_U$ the push-forward of $dVol_U$ via ζ and $d\gamma$ the measure on $\mathcal{G}$ which is given by $\frac{1}{\pi}\zeta_\# dVol_{U(\mathbb{R}P^n)}$.*

B. *Let $\mathcal{L}$ be the family of linearly embedded subspaces $\mathbb{C}P^1 \subseteq \mathbb{C}P^N$, which we define as the quotient of the unit tangent bundle $U(\mathbb{C}P^N, g_0)$ by the quotient mapping T which sends $\vec{u} \in U(\mathbb{C}P^N, g_0)$ to the unique degree-1 curve $\mathbb{C}P^1 \subseteq \mathbb{C}P^N$ to which $\vec{u}$ is tangent, as in the proof of Lemma 2.9. Let $dVol_U$ be the canonical measure on $U(\mathbb{C}P^N, g_0)$, $\mathsf{T}_\# dVol_U$ its push-foward by T and $d\mathcal{P}$ the measure on $\mathcal{L}$ given by $\frac{1}{2\pi^2}\mathsf{T}_\# dVol_U$.*

The total volumes of $\mathcal{G}$ and $\mathcal{L}$ in the measures $d\gamma$ and $d\mathcal{P}$ in Definition 3.4 are $\frac{\sigma(n)\sigma(n-1)}{2\pi}$ and $\frac{\pi^{2N-2}}{N!(N-1)!}$ respectively. The normalizations for these measures are chosen so that, for example, if η is an integrable function on $U(\mathbb{R}P^n, g_0)$, then by Fubini,

$$\int\limits_{U(\mathbb{R}P^n, g_0)} \eta(\vec{u})\ d\vec{u} = \int\limits_{\mathcal{G}} \int\limits_{\gamma} \eta(\gamma'(t)) dt d\gamma. \tag{3.7}$$

The identity in (3.1), together with an identity as in (3.7), is the basis for the following formula for the energy of a mapping of complex projective space:

Lemma 3.5. *Let $F : (\mathbb{C}P^N, g_0) \to (M, g)$ be a Lipschitz mapping from complex projective space to a Riemannian manifold. Let $\mathcal{L}$ and $d\mathcal{P}$ be as in Definition 3.4.B, and for $\mathcal{P} \in \mathcal{L}$ let $E_2(F|_{\mathcal{P}})$ be the energy of the mapping $\mathcal{P} \to (M, g)$ given by composing the inclusion $\mathcal{P} \subseteq \mathbb{C}P^N$ with F. Then:*

$$E_2(F) = \frac{N!}{\pi^{N-1}} \int\limits_{\mathcal{L}} E_2(F|_{\mathcal{P}}) d\mathcal{P}. \tag{3.8}$$

Proof. By Lemma 3.1, we have:

$$E_2(F) = \frac{N}{\sigma(2N-1)} \int\limits_{U(\mathbb{C}P^N, g_0)} |dF(\vec{u})|^2 d\vec{u}.$$

Letting $U(\mathcal{P}, g_0)$ be the unit tangent bundle of $\mathcal{P} \in \mathcal{L}$ in the metric $g_0|_{\mathcal{P}}$, by Fubini's theorem as in (3.7),

$$E_2(F) = \frac{N}{\sigma(2N-1)} \int\limits_{\mathcal{L}} \int\limits_{U(\mathcal{P}, g_0)} |dF(\vec{u})|^2 \, d\vec{u} \, d\mathcal{P}. \tag{3.9}$$

For each $\mathcal{P} \in \mathcal{L}$, the mapping $F|_{\mathcal{P}}$ is also Lipschitz and has a well-defined energy which can be calculated using Lemma 3.1, so (3.9) is equal to $\frac{N!}{\pi^{N-1}} \int\limits_{\mathcal{L}} E_2(F|_{\mathcal{P}}) d\mathcal{P}$. $\square$

Proposition 3.6. *Let $F : (\mathbb{C}P^N, g_0) \to (M, g)$ be a pluriharmonic mapping to a Riemannian manifold and $\mathcal{L}$ as in Definition 3.4.B.*

Then there is a constant $\mathcal{A}$ such that $E_2(F|_{\mathcal{P}}) = |F(\mathcal{P})| = \mathcal{A}$ for all $\mathcal{P} \in \mathcal{L}$, and $E_2(F) = C_N \mathcal{A}$, where C_N is as in Theorem 1.1. In particular, if F is a holomorphic or antiholomorphic mapping to a compact Kähler manifold, then $E_2(F) = C_N A^\star$, where $A^\star$ is as in (1.1).

Proof. Because F is pluriharmonic, by Lemma 2.4, $F|_{\mathcal{P}}$ is harmonic for all $\mathcal{P} \in \mathcal{L}$. To see that $E_2(F|_{\mathcal{P}})$ is the same for all $\mathcal{P} \in \mathcal{L}$, note that any two elements $\mathcal{P}_0$, $\mathcal{P}_1$ of $\mathcal{L}$ can be joined by a 1-parameter family $\{\mathcal{P}_t\}_{0 \le t \le 1}$, by composing the inclusion $\mathcal{P}_0 \subseteq \mathbb{C}P^N$ with a 1-parameter family of isometries of $(\mathbb{C}P^N, g_0)$. Because $F|_{\mathcal{P}_t}$ is harmonic for all $0 \le t \le 1$, the total energy of $F|_{\mathcal{P}_t}$ is constant in t, so $E_2(F|_{\mathcal{P}_0}) = E_2(F|_{\mathcal{P}_1})$. By Theorem 2.8, for all $\mathcal{P} \in \mathcal{L}$, $F|_{\mathcal{P}}$ is conformal, so that the area of its image is equal to its energy, and by Lemma 3.5, $E_2(F) = C_N \mathcal{A}$, where $\mathcal{A}$ is the common value of $E_2(F|_{\mathcal{P}})$ for $\mathcal{P} \in \mathcal{L}$.

If F is a holomorphic or antiholomorphic mapping to a compact Kähler manifold (X, h), then for $\mathcal{P} \in \mathcal{L}$, $F(\mathcal{P})$ is a closed complex curve in (X, h) and has the minimum area of any cycle representing its homology class. In particular, its area is minimal among mappings in its homotopy class, so that $\mathcal{A} = A^\star$. $\square$

For nonconstant pluriharmonic mappings of $(\mathbb{C}P^N, g_0)$, we have:

Lemma 3.7. *Let $F : (\mathbb{C}P^N, g_0) \to (M, g)$ be a nonconstant pluriharmonic mapping from complex projective space to a Riemannian manifold. Then F is an immersion on an open, dense subset of $\mathbb{C}P^N$.*

Proof. The subset of $\mathbb{C}P^N$ on which $rk(dF) = 2N$ is defined by the condition $\sqrt{\det(dF^T \circ dF)} \neq 0$ and is therefore open. To see that it is dense, suppose $x \in \mathbb{C}P^N$ is a point with $rk(dF_x) < 2N$. By Lemma 2.9, $\ker(dF_x)$ and $\ker(dF_x)^\perp$ are complex subspaces of $T_x\mathbb{C}P^N$, so $rk(dF_x) = 2k = \dim(\ker(dF_x)^\perp)$. Let $\vec{u} \in \ker(dF_x)$, and let $\mathsf{T}(\vec{u}) \in \mathcal{L}$ be the degree-1 curve to which $\vec{u}$ is tangent, as in the proof of Lemma 2.9. By Proposition 3.6, $F|_{\mathsf{T}(\vec{u})}$ is nonconstant, and by Lemma 2.4 it is harmonic. By Theorem 2.8, $F|_{\mathsf{T}(\vec{u})}$ is therefore a branched conformal immersion. The structure of $F|_{\mathsf{T}(\vec{u})}$ near x is that of a branch point, and for $y \neq x$ in a sufficiently small neighborhood of x in $\mathsf{T}(\vec{u})$, the rank of $dF|_{\mathsf{T}(\vec{u})}$ at y is 2 and the rank of dF on $T_y\mathbb{C}P^N$ is therefore at least $2k+2$. Any open subset of $\mathbb{C}P^N$ containing x must therefore contain points y with $rk(dF_y) \geq 2k+2$. Repeating this argument, the set of points z with $rk(dF_z) = 2N$ must intersect all open subsets of $\mathbb{C}P^N$. □

Proof of Theorem 1.3. Let $F : (\mathbb{C}P^N, g_0) \to (M, g)$ be a Lipschitz mapping. By Lemma 3.5 and the fact that for all $\mathcal{P} \in \mathcal{L}$ we have $E_2(F|_{\mathcal{P}}) \geq |F(\mathcal{P})| \geq A^\star$, we have:

$$E_2(F) \geq \frac{\pi^{N-1}}{(N-1)!} A^\star, \tag{3.10}$$

in other words, the quantity $C_N A^\star$ associated to the homotopy class of F in Theorem 1.1 is a lower bound for $E_2(F)$. For $p > 2$, Hölder's inequality then implies:

$$\begin{aligned} E_p(F) = \frac{1}{2} \int_{\mathbb{C}P^N} |dF|^p dVol_{g_0} &\geq \frac{1}{2} Vol(\mathbb{C}P^N, g_0)^{1-\frac{p}{2}} \left(\int_{\mathbb{C}P^N} |dF|^2 dVol_{g_0} \right)^{\frac{p}{2}} \\ &= \left(\frac{\pi^N}{2N!} \right)^{1-\frac{p}{2}} E_2(F)^{\frac{p}{2}}, \end{aligned} \tag{3.11}$$

which together with (3.10) implies the inequality in Theorem 1.3.

Suppose $p > 2$ and equality holds for $E_p(F)$ in Theorem 1.3. Equality then holds in (3.10), which implies F is energy minimizing in its homotopy class and is therefore a stable harmonic map, in particular smooth, and pluriharmonic by Theorem 1.2. We also have equality in Hölder's inequality in (3.11), which implies $|dF|$ is constant and therefore that equality holds in Theorem 1.3 for all $p \geq 2$. Letting ω^* be the closed 2-form associated to the Hermitian form F^*g as in Theorem 2.10, for $\mathcal{P} \in \mathcal{L}$ we have:

$$\int_{\mathcal{P}} \omega^* = |F(\mathcal{P})| = A^\star. \tag{3.12}$$

Letting ω_0 be the Kähler form of g_0, this implies ω^* is cohomologous to $\frac{A^\star}{\pi}\omega_0$, and therefore that, letting $Vol_{g_0}(\mathbb{C}P^N, F^*g)$ be as defined in Lemma 3.3,

$$Vol_{g_0}(\mathbb{C}P^N, F^*g) = \frac{1}{N!} \int_{\mathbb{C}P^N} {\omega^*}^N = \frac{{A^\star}^N}{N!}. \tag{3.13}$$

This implies that for $p \geq 2N$, the lower bound in Theorem 1.3 coincides with the lower bound in Lemma 3.3, and therefore that F realizes equality in Lemma 3.3 for all $p \geq 2N$. Because F realizes equality in Lemma 3.3 for $p > 2N$ it is a homothety. □

In our result for mappings of $(\mathbb{R}P^n, g_0)$ in Theorem 1.4, the equality condition implies that a mapping $F : (\mathbb{R}P^n, g_0) \to (M, g)$ which minimizes p-energy in its homotopy class is a homothety onto a totally geodesic submanifold; that is, the second fundamental form α_F vanishes identically. Totally geodesic mappings of Kähler manifolds are pluriharmonic, so in this sense, the condition that a mapping is totally geodesic in Theorem 1.4 is stronger than the condition that a mapping is pluriharmonic in Theorem 1.3. The homogeneous quadric curve in $\mathbb{C}P^2$, described in homogeneous coordinates $[z_0 : z_1 : z_2]$ by $z_0^2 + z_1^2 + z_2^2 = 0$, shows that a mapping need not be totally geodesic for equality to hold in Theorem 1.3.

Proof of Theorem 1.4. Let $F : (\mathbb{R}P^n, g_0) \to (M, g)$ be a Lipschitz map. By (3.1) and the Cauchy-Schwarz inequality,

$$E_1(F) = \frac{1}{2} \int_{\mathbb{R}P^n} |dF_x| dVol_{g_0} = \frac{1}{2} \int_{\mathbb{R}P^n} \left(\frac{n}{\sigma(n-1)} \int_{U_x(\mathbb{R}P^n, g_0)} |dF(\vec{u})|^2 d\vec{u} \right)^{(\frac{1}{2})} dVol_{g_0}$$
$$\geq \frac{\sqrt{n}}{2\sigma(n-1)} \int_{U(\mathbb{R}P^n, g_0)} |dF(\vec{u})| d\vec{u}. \tag{3.14}$$

Letting the space $\mathcal{G}$ and measure $d\gamma$ be as in Definition 3.4.A, and writing $\gamma : [0, \pi] \to \mathbb{R}P^n$ for a unit-speed parametrization of $\gamma \in \mathcal{G}$ and $F \circ \gamma : [0, \pi] \to M$ for the associated parametrization of $F \circ \gamma$,

$$|F \circ \gamma| = \int_0^\pi |(F \circ \gamma)'(t)| dt. \tag{3.15}$$

By (3.14), (3.15), and Fubini, we then have:

$$E_1(F) \geq \frac{\sqrt{n}}{2\sigma(n-1)} \int_{\mathcal{G}} |F \circ \gamma| d\gamma. \tag{3.16}$$

Because $|F \circ \gamma| \geq L^\star$ for all $\gamma \in \mathcal{G}$, this implies the inequality in Theorem 1.4 for $p = 1$. For $p > 1$, by Hölder's inequality,

$$E_p(F) = \frac{1}{2} \int_{\mathbb{R}P^N} |dF_x|^p dVol_{g_0} \geq \left(\frac{4}{\sigma(n)} \right)^{p-1} E_1(F)^p, \tag{3.17}$$

which gives the inequality for $p > 1$.

Suppose equality holds for $p = 1$.

Supposing only that F is Lipschitz, this implies equality holds in the Cauchy-Schwarz inequality in (3.14) for a.e. $x \in \mathbb{R}P^n$. For all x at which F is differentiable and for which this equality holds, $|dF_x(\vec{u})|$ depends only on x. This implies that F^*g is a.e. equal to $\mu(x)g_0$, where μ is a nonnegative function on $\mathbb{R}P^n$. Because all $\gamma \in \mathcal{G}$ map to rectifiable curves $F \circ \gamma$ in (M, g) with well-defined lengths, equality also implies that for almost all $\gamma \in \mathcal{G}$, $|F \circ \gamma| = L^\star$. Because $|F \circ \gamma| \geq L^\star$ and $|F \circ \gamma|$ is a lower semicontinuous function of $\gamma \in \mathcal{G}$, we in fact have $|F \circ \gamma| = L^\star$ for all γ. The image via F of each geodesic γ is therefore a closed geodesic in (M, g), of minimal length $L^\star$ in its free homotopy class, although we have not inferred that $t \mapsto (F \circ \gamma)(t)$ is a constant speed parametrization or even locally injective.

If in addition F is a smooth immersion, then because each geodesic γ in $(\mathbb{R}P^n, g_0)$ maps to a closed geodesic in (M, g), the image of F is a totally geodesic submanifold of (M, g). Because $F \circ \gamma$ is of minimal length in its free homotopy class in (M, g), F^*g is a Blaschke metric on $\mathbb{R}P^n$, cf. Remark 3.8 below. By the Berger-Green-Kazdan-Yang proof of the Blaschke conjecture in [Gn62, Be12], F^*g therefore has constant curvature. This does not yet imply that the mapping F is an isometry or a homothety. However after rescaling F^*g if necessary and letting ι be a diffeomorphism of $\mathbb{R}P^n$ which gives an isometry from $(\mathbb{R}P^n, F^*g)$ to $(\mathbb{R}P^n, g_0)$, we have $\iota^* g_0 = F^*g = \mu(x) g_0$, where μ is the semiconformal factor as above and is in fact a conformal factor, i.e. is everywhere-defined and positive, because F is a smooth immersion. As a conformal diffeomorphism of $(\mathbb{R}P^n, g_0)$, ι must in fact be an isometry of $(\mathbb{R}P^n, g_0)$, so μ is constant and F is an isometry or a homothety.

Now suppose $p > 1$ and equality holds for p.

Assuming only that F is Lipschitz, this implies all the conditions which hold for Lipshitz mappings which realize equality for $p = 1$ and also implies equality in Hölder's inequality in (3.17), which implies $|dF|$ is constant and equality holds for all $p \geq 1$. Because equality holds for $p = 2$, F is a harmonic mapping and therefore smooth, and because $|dF|$ is constant, the semiconformal factor μ is a constant function. F is therefore either a constant map, if $\mu \equiv 0$, or a homothety if $\mu \equiv$ const. > 0. In the latter case, the equality conditions for $p = 1$ imply that F is totally geodesic. □

Remark 3.8. To see that the conditions for equality when $p = 1$ and F is an immersion imply that F^*g is a Blaschke metric, i.e. that the first conjugate locus of each point x_0 in $(\mathbb{R}P^n, F^*g)$ is a single point, in fact x_0, note that each unit-speed geodesic $c : [0, L^\star] \to (\mathbb{R}P^n, F^*g)$ has a conjugate point at $c(L^\star) = c(0)$, where all geodesics based at $c(0)$ intersect, and that this must be the first conjugate point to $c(0)$ along c because $c([0, L^\star])$ is length minimizing in its homotopy class.

Also, the $p = 1$ case of Theorem 1.4 is false without the stipulation that $n \geq 2$: any diffeomorphism of $\mathbb{R}P^1$ homotopic to the identity has 1-energy equal to the identity. This is essentially the same observation as in the proof of Theorem 1.4 that although Lipschitz maps F for which equality holds for $p = 1$ must map each geodesic γ in $(\mathbb{R}P^n, g_0)$ onto a geodesic of (M, g), we do not know if $(F \circ \gamma)(t)$ is an arclength parametrization. This is why we have assumed F is a smooth immersion in characterizing the equality case in Theorem 1.4 for $p = 1$. It would be interesting to know if this assumption can be removed.

4. Infima of the Energy Functional in Homotopy Classes of Mappings

In this section we will complete the proof of Theorem 1.1. We will also discuss the problem of determining the infimum of the energy in a homotopy class of mappings of real projective space. We will prove a two-sided estimate for the infimum of the energy in a homotopy class of mappings of $(\mathbb{R}P^3, g_0)$ in Theorem 4.6.

In proving Theorem 1.1, we will use the fact that the infimum of the energy in any homotopy class of mappings from the 2-sphere to a Riemannian manifold (M, g) is equal to the infimum of the area (i.e. the theorem as it applies to $\mathbb{C}P^1$). If a homotopy class of mappings of $\mathbb{C}P^1$ contains an energy minimizing mapping, this follows from Theorem 2.8, however a homotopy class of mappings from the 2-sphere to a Riemannian manifold may not contain such a minimizing map. Sacks and Uhlenbeck studied the existence of harmonic and energy minimizing mappings of 2-spheres in [SU81] and discuss this issue. We will show the two infima are equal via the following lemma, which we will prove by an argument adapted from the solution of Plateau's problem as in [Law80]:

Lemma 4.1. *Let $f : (\mathbb{C}P^1, g_0) \to (M, g)$ be a smooth mapping to a Riemannian manifold, and let $A(f)$ be the area of its image, as in the integral expression in (1.1).*

Then for any $\delta > 0$ there is a diffeomorphism $\phi : \mathbb{C}P^1 \to \mathbb{C}P^1$, which we can take to be homotopic to the identity, such that $E_2(f \circ \phi) < A(f) + \delta$. In particular, f is homotopic to a map with energy less than $A(f) + \delta$. If f is antipodally invariant, we can choose ϕ to be antipodally invariant, so that the same conclusion holds for mappings of $(\mathbb{R}P^2, g_0)$.

Proof. Given f as above, let $f_r : (\mathbb{C}P^1, g_0) \to (M, g) \times (S^2, g_{0_r})$ be the product of the mapping f with a homothety to a round sphere (S^2, g_{0_r}) of constant curvature $\frac{1}{r^2}$. f_r is then a smooth immersion of $\mathbb{C}P^1$ into $N \times S^2$. By the uniformization theorem, there is a diffeomorphism $\phi_r : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that $f_r \circ \phi_r$ is a conformal mapping of $(\mathbb{C}P^1, g_0)$. We therefore have:

$$E_2(f_r \circ \phi_r) = |Im(f_r \circ \phi_r)| = |Im(f_r)|. \tag{4.1}$$

An elementary calculation shows that $E_2(f \circ \phi_r) < E_2(f_r \circ \phi_r)$, and that by choosing r small enough, we can ensure that $|Im(f_r)| < |Im(f)| + \delta$ for any $\delta > 0$. Because ϕ_r is homotopic to the identity, f is homotopic to $f \circ \phi_r$. If f is antipodally invariant, then f_r is as well, and the uniformization theorem implies we can choose ϕ_r to be antipodally invariant. □

Proof of Theorem 1.1. Let Φ be a homotopy class of mappings from $(\mathbb{C}P^N, g_0)$ to a Riemannian manifold (M, g), and let φ be the associated homotopy class of mappings $S^2 \to (M, g)$. Let $[z_0 : z_1 : z_2 : \cdots : z_N]$ be homogeneous coordinates for $\mathbb{C}P^N$ and $\mathcal{P}_0$ the following linearly embedded subspace of $\mathbb{C}P^N$ isomorphic to $\mathbb{C}P^1$:

$$\mathcal{P}_0 = \{[z_0 : z_1 : 0 : \cdots : 0]\}. \tag{4.2}$$

Let $F_0 \in \Phi$ and $f_0 = F_0|_{\mathcal{P}_0}$. By the definition of $A^\star$ and Lemma 4.1, for any $\varepsilon > 0$, f_0 is homotopic to a smooth mapping $f_1 : \mathcal{P}_0 \to (M, g)$ with $E_2(f_1) < A^\star + \frac{\varepsilon}{C_N}$, where C_N is the constant in Theorem 1.1. By the homotopy extension property for $\mathbb{C}P^1 \subseteq \mathbb{C}P^N$, any homotopy $f_0 \simeq f_1$ can be extended to a homotopy $F_0 \simeq F_1$. Let $F_1 \in \Phi$ be such a mapping, which we assume to be smooth, with $F_1|_{\mathcal{P}_0} = f_1$. For $\lambda > 0$, let $T_\lambda : \mathbb{C}P^N \to \mathbb{C}P^N$ be the projective linear transformation associated to the following linear transformation of $\mathbb{C}^{N+1}$:

$$(z_0, z_1, z_2, \ldots, z_N) \mapsto (\lambda z_0, \lambda z_1, z_2, \ldots, z_N). \tag{4.3}$$

Let $\mathcal{L}$ be as in Definition 3.4.B. Because $T_\lambda : \mathbb{C}P^N \to \mathbb{C}P^N$ is biholomorphic, $T_\lambda|_{\mathcal{P}}$ is a conformal diffeomorphism for all $\mathcal{P} \in \mathcal{L}$, and for any Lipschitz mapping $F : (\mathbb{C}P^N, g_0) \to (M, g)$ and $\mathcal{P} \in \mathcal{L}$ we therefore have:

$$E_2(F \circ T_\lambda|_{\mathcal{P}}) = E_2(F|_{T_\lambda(\mathcal{P})}). \tag{4.4}$$

Let $\mathcal{C}(\mathcal{P}_0) = \{[0 : 0 : z_2 : \cdots : z_N]\} \subseteq \mathbb{C}P^N$ be the intersection of the cut loci in $(\mathbb{C}P^N, g_0)$ of points in $\mathcal{P}_0$. For all $\mathcal{P} \in \mathcal{L}$ which do not intersect $\mathcal{C}(\mathcal{P}_0)$ in $\mathbb{C}P^N$, as $\lambda \to \infty$, $T_\lambda(\mathcal{P})$ converges to $\mathcal{P}_0$ in the C^1 topology on submanifolds of $\mathbb{C}P^N$. By (4.4), for any C^1 mapping F and $\mathcal{P} \in \mathcal{L}$ disjoint from $\mathcal{C}(\mathcal{P}_0)$, we therefore have:

$$\lim_{\lambda \to \infty} E_2(F \circ T_\lambda|_{\mathcal{P}}) = E_2(F|_{\mathcal{P}_0}). \tag{4.5}$$

Letting $\mathcal{S} \subseteq \mathcal{L}$ be $\{\mathcal{P} : \mathcal{P} \cap \mathcal{C}(\mathcal{P}_0) \neq \emptyset\}$, $\mathcal{S}$ has measure 0 relative to the measure $d\mathcal{P}$. Letting F_1 be the mapping above, by Lemma 3.5, (4.5) and the dominated convergence theorem,

$$\begin{aligned} \lim_{\lambda \to \infty} E_2(F_1 \circ T_\lambda) &= \lim_{\lambda \to \infty} \frac{N!}{\pi^{N-1}} \int_{\mathcal{L}} E_2(F_1 \circ T_\lambda(\mathcal{P})) d\mathcal{P} \\ &= C_N E_2(f_1) < C_N A^\star + \varepsilon. \end{aligned} \tag{4.6}$$

Because $\varepsilon > 0$ was arbitrary, this completes the proof of Theorem 1.1. □

Remark 4.2. Theorems 1.2 and 2.10 and Lemma 3.7 imply that if $F : (\mathbb{C}P^N, g_0) \to (M, g)$ is a nonconstant map which minimizes energy in its homotopy class, then F^*g is a Kähler metric on an open, dense subset Ω of $\mathbb{C}P^N$ on which F is an immersion. This can be thought of as a partial converse, for mappings of complex projective spaces, to Lichnerowicz's theorem in [Li70] that holomorphic and antiholomorphic mappings between compact Kähler manifolds minimize energy in their homotopy classes, in the following two senses: First, (2.5) implies that the second fundamental form of $F(\Omega)$ in (M, g) can be diagonalized by a unitary basis, like the second fundamental form of a Kähler submanifold. In this sense, F is indistinguishable "to second order" along Ω from a holomorphic mapping to a Kähler manifold. Second, letting ω^* be the 2-form associated to F^*g and ω_0 the Kähler form of g_0, ω^* is cohomologous up to scale to ω_0, as in the proof of Theorem 1.3. By the $\partial\overline{\partial}$-lemma, after rescaling if necessary, we have:

$$\omega^* = \omega_0 + i\partial\overline{\partial}\xi, \tag{4.7}$$

where ξ is a smooth, real-valued function on $\mathbb{C}P^N$. By the positivity of the 2-form ω_0 and the nonnegativity of the 2-form ω^*, for all $0 \leq \rho < 1$ the 2-form $\omega_0 + \rho i\partial\overline{\partial}\xi$ is positive and thus induces a Kähler metric g_ρ on $\mathbb{C}P^N$. In this sense, even if F^*g is singular on a nonempty subset Ω^{C} of $\mathbb{C}P^N$, F can be seen as a limit of holomorphic mappings given by the identity mapping $(\mathbb{C}P^N, g_0) \to (\mathbb{C}P^N, g_\rho)$ as $\rho \to 1$.

In some situations, stable harmonic mappings of complex projective spaces must be holomorphic or antiholomorphic: Ohnita proved in [Oh87], as a corollary of Theorem 1.2, that this is the case for stable harmonic mappings between complex projective spaces. Burns, Burstall, de Bartolomeis and Rawnsley extended this result to all stable harmonic mappings from complex projective spaces to compact, simple Hermitian symmetric spaces in [BBdeBR89]. It also follows from the results of Lichnerowicz in [Li70] that if a homotopy class of mappings between compact Kähler manifolds contains a holomorphic mapping then all energy minimizing mappings in that class are holomorphic, with a parallel statement for antiholomorphic mappings. In general however, although energy minimizing mappings of $(\mathbb{C}P^N, g_0)$ resemble holomorphic mappings locally as described above, they may not be globally orientation-preserving. For example, the double covering $S^2 \to \mathbb{R}P^2$ minimizes energy in its homotopy class.

Our last results give some information about the infimum of the energy in homotopy classes of mappings of $(\mathbb{R}P^n, g_0)$. In discussing this problem, we will draw on the result of Pu's systolic inequality:

Theorem 4.3. (Pu [Pu52], see also [CK03]) *Let g be a Riemannian metric on $\mathbb{R}P^2$. Let $A(\mathbb{R}P^2, g)$ be its area and $\mathrm{sys}(g)$ the length of the shortest noncontractible curve in $(\mathbb{R}P^2, g)$, known as a systole. Then:*

$$A(\mathbb{R}P^2, g) \geq \frac{2}{\pi}\,\mathrm{sys}(g)^2. \tag{4.8}$$

Equality holds precisely if g has constant curvature.

The work of Katz-Nowik [KN20] and Katz-Sabourau [KS21] gives quantitative versions of (4.8), similar to Bonnesen's quantitative isoperimetric inequalities in the plane, cf. [Os79]. These results give a precise sense in which Riemannian metrics on $\mathbb{R}P^2$ that are nearly optimal for Pu's inequality must be close to the round metric. The following formula for the energy of a mapping of real projective space can be derived from Lemma 3.1:

Lemma 4.4. *Let $F : (\mathbb{R}P^n, g_0) \to (M, g)$ be a Lipschitz mapping to a Riemannian manifold. Let $\mathcal{H}$ be the set of totally geodesic $\mathcal{Q} \cong \mathbb{R}P^2 \subseteq \mathbb{R}P^n$ and $d\mathcal{Q}$ the measure on $\mathcal{H}$ which is invariant under the action of the isometry group of $(\mathbb{R}P^n, g_0)$, normalized to have total mass $\frac{n\sigma(n)}{8\pi}$. Then:*

$$E_2(F) = \int_{\mathcal{H}} E_2(F|_{\mathcal{Q}})d\mathcal{Q}. \tag{4.9}$$

If Ψ is a homotopy class of mappings from $(\mathbb{R}P^n, g_0)$ to (M, g), ψ the homotopy class of mappings of $\mathbb{R}P^2$ represented by composing the inclusion $\mathbb{R}P^2 \subseteq \mathbb{R}P^n$ with $F \in \Psi$, and $B^\star$ the infimum of the areas of mappings in ψ, Lemma 4.4 implies:

$$\inf_{F \in \Psi} E_2(F) \geq \frac{n\sigma(n)}{8\pi} B^\star. \tag{4.10}$$

If the infimal area $B^\star$ is realized by an immersion $f : \mathbb{R}P^2 \to (M, g)$, then by Theorem 4.3,

$$B^\star \geq \frac{2}{\pi}\,\mathrm{sys}(f^*g)^2 \geq \frac{2}{\pi} {L^\star}^2, \tag{4.11}$$

where $L^\star$ is as in Theorem 1.4. Moreover, (4.11) must be a strict inequality unless $(\mathbb{R}P^2, f^*g)$ is isometric up to a rescaling to $(\mathbb{R}P^2, g_0)$. If (4.11) is a strict inequality, then by (4.10),

$$\inf_{F \in \Psi} E_2(F) > \frac{n\sigma(n)}{4\pi^2} {L^\star}^2, \tag{4.12}$$

so that the infimum is strictly greater than the lower bound in Theorem 1.4. When the infimal area $B^\star$ is not realized by a smooth immersion, one can show by considering minimizing sequences for the area of mappings $f \in \psi$, perhaps together with the proof of Lemma 4.1, that the strict inequality in (4.12) holds unless there is a minimizing sequence with a fairly rigid geometry, as indicated by the results in [KN20, KS21] cited above.

Our final result gives upper and lower bounds for the infimum of the energy in homotopy classes of mappings of $(\mathbb{R}P^3, g_0)$ in terms of the induced homotopy class of mappings of $\mathbb{R}P^2$. The following, which is an immediate corollary of [Wh86, Corollary 1] in the results of White cited above, implies that this infimum is determined by the induced class of mappings of $\mathbb{R}P^2$ in the following sense:

Proposition 4.5. (see [Wh86, Corollary 1]) *Let Ψ be a homotopy class of mappings from $(\mathbb{R}P^n, g_0)$ to a closed Riemannian manifold (M,g), and let $\widetilde{\Psi}$ be the union of all homotopy classes of mappings from $(\mathbb{R}P^n, g_0)$ to (M,g) which give the same homotopy class of mappings of $\mathbb{R}P^2$ as Ψ when composed with the inclusion $\mathbb{R}P^2 \subseteq \mathbb{R}P^n$. Then:*

$$\inf_{F\in\Psi} E_2(F) = \inf_{F\in\widetilde{\Psi}} E_2(F). \tag{4.13}$$

In connection with Theorem 1.1, one can also infer from [Wh86, Corollary 1] that the infimum of the energy in a homotopy class of mappings Φ from $(\mathbb{C}P^N, g_0)$ to (M,g) depends only on the induced homomorphism $\Phi_* : \pi_2(\mathbb{C}P^N) \to \pi_2(M)$. These observations about the homotopy-theoretic dependence of the infimum do not seem to lead to estimates for the infimal energy in most homotopy classes, although in some cases they imply it is 0. They also do not seem to indicate explicitly how the infimum is determined by the geometry and topology of the spaces in question. For homotopy classes of mappings of $(\mathbb{R}P^3, g_0)$ however, the infimum of the energy never exceeds the lower bound in (4.10) by more than a third:

Theorem 4.6. *Let Ψ be a homotopy class of mappings from $(\mathbb{R}P^3, g_0)$ to a Riemannian manifold (M,g), ψ the induced homotopy class $\mathbb{R}P^2 \to (M,g)$ and $B^\star$ the infimal area of mappings $f \in \psi$ as above. Then:*

$$\frac{3\pi}{4} B^\star \leq \inf_{F\in\Psi} E_2(F) \leq \pi B^\star. \tag{4.14}$$

Proof. That $\frac{3\pi}{4}B^\star \leq \inf_{F\in\Psi} E_2(F)$ is (4.10). To show that $\inf_{F\in\Psi} E_2(F) \leq \pi B^\star$, note that any $F_0 \in \Psi$ is homotopic to a mapping F_1 where, for a fixed totally geodesic $\mathcal{Q}_0 \cong \mathbb{R}P^2 \subseteq \mathbb{R}P^3$ and any $\delta > 0$, we have $E_2(F_1|_{\mathcal{Q}_0}) < B^\star + \delta$, as in the proof of Theorem 1.1. Let $\delta > 0$ and consider a mapping $F \in \Psi$ which satisfies the condition $E_2(F|_{\mathcal{Q}_0}) < B^\star + \frac{\delta}{\pi}$. Let $p_0 \in \mathbb{R}P^3$ be the unique point at distance $\frac{\pi}{2}$ from $\mathcal{Q}_0$, let $\widetilde{p}_0 \in S^3$ be a point in the preimage of p_0 via the covering $\tau : S^3 \to \mathbb{R}P^3$ and let $\beta : S^3 \setminus \{-\widetilde{p}_0\} \to \mathbb{R}^3$ be the stereographic projection which takes $\widetilde{p}_0$ to the origin. An elementary calculation shows that, letting $\theta_t : S^3 \to S^3$ be the conformal diffeomorphism of S^3 given by multiplication by t in the coordinates given by β, we have $\lim_{t\to\infty} E_2(\theta_t) = 0$.

Let $D(\widetilde{p}_0) \subseteq S^3$ be the open hemisphere centered at $\widetilde{p}_0$, which we regard as a fundamental domain for the covering τ. Define the mapping $\Theta_t : \mathbb{R}P^3 \to \mathbb{R}P^3$ as follows: for $p \in \mathbb{R}P^3$ with a preimage $\tau^{-1}(p)$ belonging to $\theta_t^{-1}(D(\widetilde{p}_0))$, let $\Theta_t(p) = \tau \circ \theta_t(\tau^{-1}(p))$. For $p \in \mathbb{R}P^3$ with a preimage $\tau^{-1}(p) \in D(\widetilde{p}_0) \setminus \theta_t^{-1}(D(\widetilde{p}_0))$, let Θ_t be the image of p under the radial projection from p_0 onto $\mathcal{Q}_0$, and for $p \in \mathcal{Q}_0$, let $\Theta_t(p) \equiv p$. Because θ_t maps $\theta_t^{-1}(D(\widetilde{p}_0))$ conformally to $D(\widetilde{p}_0)$ and $\lim_{t\to\infty} E_2(\theta_t|_{\theta_t^{-1}(D(\widetilde{p}_0))}) = 0$, the energy of $F \circ \Theta_t|_{\tau(\theta_t^{-1}(D(\widetilde{p}_0)))}$ goes to 0 as $t \to \infty$. We then have:

$$\lim_{t\to\infty} E_2(F\circ\Theta_t) = \lim_{t\to\infty} \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_{\partial B_r(p_0)} |d(F\circ\Theta_t)_x|^2 dx dr = \frac{\pi}{2} \times 2 \times E_2(F|_{\mathcal{Q}_0}) < \pi B^\star + \delta. \tag{4.15}$$

Because $\delta > 0$ was arbitrary, this completes the proof. □

By a similar argument, one can establish a two-sided estimate for the infimum of the energy in homotopy classes of mappings of higher-dimensional real projective spaces.

References

[Aub13] Thierry Aubin: *Some Nonlinear Problems in Riemannian Geometry*, Springer Science and Business Media, 2013.

[Be12] Arthur L. Besse: *Manifolds All of Whose Geodesics are Closed*, Springer Science and Business Media, 2012.

[BBdeBR89] D. Burns, F. Burstall, P. de Bartolomeis and J. Rawnsley: *Stability of Harmonic Maps of Kähler Manifolds*, Journal of Differential Geometry, 30(2) (1989), 579-594.

[Cr87] Christopher B. Croke: *Lower Bounds on the Energy of Maps*, Duke Mathematical Journal 55.4 (1987), 901-908.

[CK03] Christopher B. Croke and Mikhail Katz: *Universal Volume Bounds in Riemannian Manifolds*, Surveys in Differential Geometry 8.1 (2003), 109-137.

[ES64] James Eells and Joseph H. Sampson: *Harmonic Mappings of Riemannian Manifolds*, American Journal of Mathematics 86.1 (1964), 109-160.

[Gn62] Leon W. Green: *Auf Wiedersehensflächen*, Annals of Mathematics 78.2 (1963), 289-299.

[HL87] Robert Hardt and FangHua Lin: *Mappings Minimizing the Lp Norm of the Gradient*, Communications on Pure and Applied Mathematics 40.5 (1987), 555-588.

[KN20] Mikhail Katz and Tahl Nowik: *A Systolic Inequality With Remainder in the Real Projective Plane*, Open Mathematics 18.1 (2020), 902-906.

[KS21] Mikhail Katz and Stéphane Sabourau *A Pu-Bonnesen Inequality*, Journal of Geometry 112.2 (2021), 18.

[Law80] H. Blaine Lawson, Jr.: *Lectures on Minimal Submanifolds, Vol. I*, Publish or Perish. Inc., Berkeley (1980).

[Lem78] Luc Lemaire: *Applications Harmoniques de Surfaces Riemanniennes*, Journal of Differential Geometry 13.1 (1978), 51-78.

[Li70] André Lichnerowicz: *Applications Harmoniques et Variétés Kähleriennes*, Symp. Math. III, Bologna (1970), 341-402.

[Oh87] Yoshihiro Ohnita: *On Pluriharmonicity of Stable Harmonic Maps*, Journal of the London Mathematical Society 2.3 (1987), 563-568.

[Oh86] Yoshihiro Ohnita: *Stability of Harmonic Maps and Standard Minimal Immersions*, Tohoku Math. J., Second Series 38.2 (1986), 259-267.

[Os79] Robert Osserman: *Bonnesen-style Isoperimetric Inequalities*, The American Mathematical Monthly 86.1 (1979), 1-29.

[Pu52] Pao Ming Pu: *Some Inequalities in Certain Nonorientable Riemannian Manifolds*, Pacific J. Math 2.1 (1952), 55-71.

[SU81] Jonathan Sacks and Karen Uhlenbeck: *The Existence of Minimal Immersions of 2-spheres*, Annals of Mathematics 113.1 (1981), 1-24.

[Sm75] Robert T. Smith: *The Second Variation Formula for Harmonic Mappings*, Proceedings of the American Mathematical Society 47.1 (1975), 229-236.

[We98] Shishu W. Wei: *Representing Homotopy Groups and Spaces of Maps by p-Harmonic Maps*, Indiana University Mathematics Journal 47.2 (1998), 625-670.

[Wh86] Brian White: *Infima of Energy Functionals in Homotopy Classes of Mappings*, Journal of Differential Geometry 23.2 (1986), 127-142.

[Wh88] Brian White: *Homotopy Classes in Sobolev Spaces and the Existence of Energy Minimizing Maps*, Acta Mathematica 160.1 (1988), 1-17.

[Xin80] Y.L. Xin: *Some Results on Stable Harmonic Maps*, Duke Math. J. 47.3 (1980), 609-613.

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