Max-Planck-Institut für Mathematik
Bonn

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Max-Planck-Institut für Mathematik
Preprint Series 2023 (27)

Date of submission: December 5, 2023

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MPIM 23-27

SPECTRAL INSTABILITY OF COVERINGS

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Abstract. We study the behaviour of eigenvalues, below the bottom of the essential spectrum, of the Laplacian under finite Riemannian coverings of complete and connected Riemannian manifolds. We define spectral stability and instability of such coverings. Among others, we provide necessary conditions for stability or, equivalently, sufficient conditions for instability.

1. Introduction

Recently Magee et al. [17, 13] have initiated a study of the spectrum of the Laplacian of a random Riemannian cover of a fixed hyperbolic (i.e., curvature $= -1$) surface. Broadly speaking, the main results obtained by them say that asymptotically almost surely (with respect to the uniform measure on the space of n-sheeted covers) the spectrum of a covering surface does not acquire a new eigenvalue below a specific threshold $< 1/4$. What is even more interesting is that this threshold is independent of the bottom surface, but only depends on its type; see below.

A classical result of Randol [22] is quite opposite to the results of Magee et al. Namely, for any hyperbolic metric on a *closed* (i.e., *compact and connected with empty boundary*) surface S of genus $g \geq 2$, any natural number ℓ and any $\varepsilon > 0$, there is a finite Riemannian covering $p\colon S' \to S$ such that S' has at least ℓ eigenvalues in $[0, \varepsilon)$. (See [3, Theorem 4.1] for an elementary proof.)

One motivation for our studies in this paper is that, although the above mentioned results of Magee et al. show that asymptotically almost all n-sheeted covers of the surface in question are *spectrally stable* in a specific range, they do not provide any necessary or sufficient condition for this to happen. Among others, we provide, in this paper, some necessary conditions.

To set the stage, let M be a complete and connected Riemannian manifold of dimension m. Denote by $\tilde{M}$ the universal covering space of M, endowed with the lifted Riemannian metric, and let Γ be the fundamental group of M, viewed as the group of covering transformations on $\tilde{M}$.

Denote by $\lambda_0(M) \leq \lambda_{\mathrm{ess}}(M)$ the bottom of the spectrum and the essential spectrum (of the Laplacian Δ) of M, respectively. Recall that $\lambda_0(M)$ need not vanish in general, that $\lambda_0(M) = 0$ and $\lambda_{\mathrm{ess}}(M) = \infty$ in the case where M is closed (that is, compact and connected without boundary) and that the spectrum of M below $\lambda_{\mathrm{ess}}(M)$ consists of a locally finite set of eigenvalues of finite multiplicity. We assume throughout that

$$\lambda_0(M) < \lambda_{\mathrm{ess}}(M) \tag{1.1}$$

Date: December 5, 2023.

2010 *Mathematics Subject Classification.* 35P15, 58J50, 53C20.

Key words and phrases. Laplace operator, eigenvalues, spectrum, stability.

Acknowledgments. We are grateful to the Max Planck Institute for Mathematics and the Hausdorff Center for Mathematics in Bonn for their support and hospitality. The second named author would like to thank C.S. Rajan and C. Drutu for helpful discussions.

and enumerate the eigenvalues of M in $[0, \lambda_{\mathrm{ess}}(M))$ according to their size and multiplicity as

$$0 \le \lambda_0(M) < \lambda_1(M) \le \lambda_2(M) \le \dots, \tag{1.2}$$

where we recall that the eigenvalue $\lambda_0(M)$ has multiplicity one since its eigenfunctions do not change sign. For $\lambda \ge 0$, we denote by $N_M(\lambda)$ and $N_M(\lambda-)$ the number of $\lambda_k(M)$ in $[0,\lambda]$ and $[0,\lambda)$, respectively. More generally, for any subset $I \subseteq [0, \lambda_{\mathrm{ess}}(M))$, we denote by $N_M(I)$ the number of $\lambda_k(M)$ in I.

Let $p\colon M' \to M$ be a finite Riemannian covering of complete and connected Riemannian manifolds. Then

$$\lambda_0(M) = \lambda_0(M') \quad \text{and} \quad \lambda_{\mathrm{ess}}(M) = \lambda_{\mathrm{ess}}(M'), \tag{1.3}$$

see (2.5). Since the lifts $p^*\varphi = \varphi \circ p$ of eigenfunctions φ of M are eigenfunctions of M', we always have

$$\lambda_k(M') \le \lambda_k(M). \tag{1.4}$$

Likewise, for any subset $I \subseteq [0, \lambda_{\mathrm{ess}}(M))$,

$$N_M(I) \le N_{M'}(I). \tag{1.5}$$

We say that p is *I-stable* if

$$N_M(I) = N_{M'}(I). \tag{1.6}$$

With respect to this terminology, the results of Magee et al. say that,
1) for any orientable, convex cocompact, non-compact hyperbolic surface S with Hausdorff dimension of its limit set $\delta > 1/2$ and any $\sigma \in (3\delta/4, \delta)$, any finite Riemannian cover $p\colon S' \to S$ is asymptotically almost surely $[\delta(1-\delta), \sigma(1-\sigma)]$-stable as $|p| \to \infty$; see [17]. Note that here $\lambda_0(S) = \delta(1-\delta) < 1/4 = \lambda_{\mathrm{ess}}(S)$.
2) for any orientable and compact hyperbolic surface S and any $\varepsilon > 0$, any finite Riemannian cover $p\colon S' \to S$ is asymptotically almost surely $[0, 3/16 - \varepsilon]$-stable as $|p| \to \infty$; see [18].
3) for any orientable and non-compact hyperbolic surface S of finite area and any $\varepsilon > 0$, any finite Riemannian cover $p\colon S' \to S$ is asymptotically almost surely $[0, 1/4 - \varepsilon]$-stable as $|p| \to \infty$; see [13]. Here $\lambda_0(S) = 0 < 1/4 = \lambda_{\mathrm{ess}}(S)$.

Clearly, I-stability means that lifting yields an isomorphism between eigenspaces of M and M' for all eigenvalues of M and M' in I. In particular,
1) if $J \subseteq [0, \infty)$ is a further subset and $I \subseteq J$, then

$$I\text{-instability of } p \text{ implies } J\text{-instability of } p. \tag{1.7}$$

2) if $q\colon S'' \to S'$ is a further finite Riemannian covering of complete and connected Riemannian manifolds, then

$$I\text{-instability of } p \text{ or } q \text{ implies } I\text{-instability of } p \circ q. \tag{1.8}$$

For $k > 0$, we say that p is *λ_k-stable* if

$$N_M(\lambda) = N_{M'}(\lambda), \tag{1.9}$$

where $\lambda = \lambda_k(M) < \lambda_{\mathrm{ess}}(M)$. By definition, λ_k-instability $N_M(\lambda) < N_{M'}(\lambda)$ can occur in two ways: Either p is *strictly λ_k-unstable*, that is, $N_M(\lambda-) < N_{M'}(\lambda-)$, or else p is *weakly λ_k-unstable*, that is, $N_M(\lambda-) = N_{M'}(\lambda-)$, but the multiplicity of λ as an eigenvalue increases, $\mu(\lambda, M) < \mu(\lambda, M')$. By (1.7),

$$\lambda_k\text{-instability implies } \lambda_\ell\text{-instability, for any } 1 \le k \le \ell, \tag{1.10}$$

where $\lambda_\ell(M) < \lambda_{\mathrm{ess}}(M)$. In particular, λ_1-instability implies λ_k-instability for all $k \ge 1$. For that reason, our main focus is on λ_1-stability and instability.

For an eigenfunction φ on a Riemannian manifold, the set $\mathcal{Z}(\varphi) = \{\varphi = 0\}$ is called the *nodal set of* φ and the connected components of $\{\varphi \neq 0\}$ are called *nodal* *domains of* φ. (In general, we use the term *domain* to indicate *connected open sets*.) One of our main arguments uses connectedness of preimages in M' of nodal sets in M.

Theorem A. *If* $p\colon M' \to M$ *is a* λ_k*-stable finite Riemannian covering of complete and connected Riemannian manifolds, where* $\lambda_0(M) < \lambda_k(M) < \lambda_{\mathrm{ess}}(M)$*, then the preimage* $p^{-1}(U)$ *of any nodal domain* U *of any* λ*-eigenfunction* φ *on* M *is connected, for any* $\lambda_0(M) \le \lambda \le \lambda_k(M)$*. In fact, if* U *is any nodal domain of any* λ*-eigenfunction* φ *on* M *for any such* λ *and* $j \ge 1$ *denotes the number of connected components of* $p^{-1}(U)$*, then*

$$N_{M'}(\lambda-) \ge N_M(\lambda-) + j - 1.$$

Theorem A is a special case of Theorem 4.2. An easy application of the main results of [15] and [9] and Theorem A yields the following

Corollary B. *A closed manifold* M *carries a Riemannian metric* g*, such that* p *is strictly* λ_1*-unstable with respect to* g*, for any non-trivial finite covering* $p\colon M' \to M$ *of closed and connected manifolds, where* M' *is endowed with the lifted Riemannian metric* g'*. In fact, for an appropriate choice of* g*,*

$$N_{M'}(\lambda-) \ge |p| \quad (= N_M(\lambda-) + |p| - 1),$$

where $\lambda = \lambda_1(M, g)$ *and* $|p|$ *denotes the degree (number of sheets) of* p*.*

Proof. By [15, Main Theorem] and [9, Theorem 1.1], M carries a Riemannian metric g, which has a topological ball U as a nodal domain. (Note that the proof in [15] also works in the non-orientable case.) Since balls are simply connected, $p^{-1}(U)$ has $|p|$ disjoint lifts. Now the claim follows from Theorem A. □

Corollary C. *The orientable closed surface* S *of genus two carries a hyperbolic metric such that any non-trivial Riemannian covering* $p\colon S' \to S$*, that is not generated by one element, is strictly* λ_1*-unstable.*

Here we say that a covering $p\colon M' \to M$ of connected manifolds is *generated by* k *elements* if, for one–or any–point $x \in M$, there are k loops in M at x such that any two points in $p^{-1}(x)$ can be connected by lifts to M' of concatenations of these loops and their inverses; see also Section 3.1. This property is independent of the choice of x.

Proof of Corollary C. By [20], there exists a hyperbolic metric on S which has a $\lambda_1(S)$-eigenfunction φ such that one of its nodal domains U is either a disc or an annulus. In the first case, the preimage of U is disconnected with $|p|$ components and the assertion is a consequence of Theorem A. In the second csae, if the preimage of U is connected and $x \in U$, then any two points of $p^{-1}(x)$ can be connected by a lift of an iterate of any loop in U at x which generates $\pi_1(U, x)$; see also Lemma 3.1. This shows the assertion in the second case. □

Theorem D. *Suppose that* M *is complete and connected with* $\lambda_1(M) < \lambda_{\mathrm{ess}}(M)$ *and carries a* $\lambda_1(M)$*-eigenfunction* φ *such that its nodal set is not connected.*

1. *Then there is a two-sheeted Riemannian covering of* M *which is strictly* λ_1*-unstable.*
2. *If* M *is orientable, then* M *carries an* n*-sheeted cyclic Riemannian covering which is strictly* λ_1*-unstable, for any* $n \ge 2$*.*

Theorem D is a special case of Theorem 4.6.

Let S be a complete and connected Riemannain surface. (To avoid misunderstandings: a Riemannian surface is a surface with a Riemannian metric.) Recall that S is said to be of *finite type* if it is diffeomorphic to the interior of a compact surface with (possibly empty) boundary.

For a domain U in S and a point $x \in U$, we identify Γ with $\pi_1(S, x)$ and denote the *image of* $\pi_1(U, x)$ *in* Γ by Γ_U. Corresponding assertions will be independent of the choice of x.

Theorem E. *Assume that S is of finite type with $\chi(S) < 0$, and let φ be a λ-eigenfunction, where $\lambda_0(S) < \lambda < \lambda_{\mathrm{ess}}(S)$. Then φ has $\nu \geq 2$ nodal domains and at least one, U, such that $\chi(S)/\nu \leq \chi(U) \leq 1$. For any such U, Γ admits a surjective homomorphism I to $\mathbb{Z}_2^\mu$, respectively $\mathbb{Z}^\mu$ if S is orientable, where $\mu \geq -(\nu - 1)\chi(S)/\nu$, such that $\Gamma_U \subseteq \ker I$. In particular, if $\Gamma' \subseteq \Gamma$ is a finite index subgroup containing Γ_U, then the corresponding Riemannian covering $p\colon S' \to S$ is strictly λ-unstable. More precisely,*

$$N_{S'}(\lambda-) \geq N_S(\lambda-) + |p| - 1,$$

where $|p| = |\Gamma' \backslash \Gamma|$.

Theorem E is a special case of Theorem 5.19. It applies, for example, to non-compact hyperbolic surfaces S of finite area with $0 < \lambda < 1/4$ since, for them, $\lambda_{\mathrm{ess}}(S) = 1/4$. The number μ is determined in the proof of Theorem 5.19.

Remark 1.11 (Weyl's law)**.** Let $p\colon M' \to M$ be a non-trivial finite Riemannian covering of closed Riemannian manifolds. Then, by Weyl's law,

$$\lim_{\lambda \to \infty} \frac{N_M(\lambda)}{\lambda^{m/2}} = C_m \operatorname{Vol} M \quad \text{and} \quad \lim_{\lambda \to \infty} \frac{N_{M'}(\lambda)}{\lambda^{m/2}} = C_m \operatorname{Vol} M',$$

where C_m equals the volume of the ball of radius $1/2\pi$ in $\mathbb{R}^m$. Since $\operatorname{Vol} M' = |p| \operatorname{Vol} M$ and $|p| \geq 2$, we get that $N_{M'}(\lambda) > N_M(\lambda)$ for all sufficiently large λ. Therefore stability of p can only hold in a bounded range of λ.

A sequence of Riemannian coverings of Riemannian manifolds,

$$\cdots \to M_k \to M_{k-1} \to \cdots \to M_1 \to M_0 = M,$$

is called a *tower of Riemannan coverings.* If the M_k are connected, then the universal covering $\tilde{M}$ of M sits above the tower. We endow it with the lifted Riemannian metric. The next result asserts that we should not expect stability beyond the bottom of the spectrum of $\tilde{M}$.

Theorem F. *For a tower of finite Riemannian coverings of complete and connected Riemannian manifolds, suppose that the degree of the coverings $M_k \to M$ tends to infinity and that $\lambda_0(\tilde{M}) < \lambda_{\mathrm{ess}}(M)$. Then, for any $\lambda_0(\tilde{M}) < \lambda < \lambda_{\mathrm{ess}}(M)$ and $l \geq 1$, M_k has at least l eigenvalues below λ, for all sufficiently large k; in short,*

$$\limsup_{k \to \infty} \lambda_l(M_k) \leq \lambda_0(\tilde{M}).$$

Theorem F is a special case of Theorem 6.2. It is motivated by [26, Proposition 6] and [6, Theorem 2]. Note that we do not need residual finiteness of Γ respectively that $\cap \Gamma_k = \{1\}$, since our proof relies on the push down construction from [3].

The assumption $\lambda_0(\tilde{M}) < \lambda_{\mathrm{ess}}(M)$ is satisfied if M is compact since the essential spectrum of closed Riemannian manifolds is empty. On the other hand, if $\lambda_0(\tilde{M}) = \lambda_{\mathrm{ess}}(M)$, the argument in the proof of Theorem F still applies, but the assertion might be meaningless for any $\lambda > \lambda_{\mathrm{ess}}(M)$. At least the spectral projection corresponding to $[0, \lambda)$ would then have infinite rank.

2. Setup and preliminaries

Let M be a complete and connected Riemannian manifold of dimension m. Let Δ denote the Laplace-Beltrami operator, acting on the space of smooth functions $C^\infty(M)$ on M. Recall that Δ is *essentially self-adjoint* on $C^\infty(M) \subseteq L^2(M)$. Its closure will also be denoted by Δ. It has domain $H^1(M)$, and its spectrum, depending on the context denoted by

$$\sigma(M, \Delta) = \sigma(\Delta) = \sigma(M),$$

can be decomposed into two sets,

$$\sigma(M) = \sigma_d(M) \sqcup \sigma_{\mathrm{ess}}(M),$$

the *discrete spectrum* and the *essential spectrum*. Recall that $\sigma_d(M)$ consists of isolated eigenvalues of Δ of finite multiplicity and that $\sigma_{\mathrm{ess}}(M)$ consists of those $\lambda \in \mathbb{R}$ for which $\Delta - \lambda$ is not a Fredholm operator. By elliptic regularity theory, $\sigma(M) = \sigma_d(M)$ if M is compact. By the above characterization of the discrete spectrum, $\sigma(M) = \sigma_{\mathrm{ess}}(M)$ if M is homogeneous and non-compact.

Denote by $\lambda_0(M) \leq \lambda_{\mathrm{ess}}(M)$ the bottom of $\sigma(M)$ and $\sigma_{\mathrm{ess}}(M)$, respectively. If M is compact, then $\lambda_0(M) = 0 < \lambda_{\mathrm{ess}}(M) = \infty$. Furthermore, 0 is an eigenvalue of Δ of multiplicity one with constant functions as eigenfunctions. In general, $\lambda_0(M)$ may be positive and may belong to $\sigma_d(M)$ or we may have $\lambda_0(M) = \lambda_{\mathrm{ess}}(M)$. We shall be interested in the case where

$$\lambda_0(M) < \lambda_{\mathrm{ess}}(M). \tag{2.1}$$

Then $\lambda_0(M)$ is an eigenvalue of Δ of multiplicity one with eigenfunctions which do not change sign. Moreover, by the above, we have

$$\sigma(M) \cap [0, \lambda_{\mathrm{ess}}(M)) \subseteq \sigma_d(M). \tag{2.2}$$

We enumerate the eigenvalues of Δ in $[0, \lambda_{\mathrm{ess}}(M))$ by size,

$$\lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \cdots < \lambda_{\mathrm{ess}}(M), \tag{2.3}$$

where repetitions account for multiplicities. In general, the number of eigenvalues of Δ below $\lambda_{\mathrm{ess}}(M)$ might be infinite, even if $\lambda_{\mathrm{ess}}(M) < \infty$. On the other hand, we have the *variational characterization* of $\lambda_k(M) < \lambda_{\mathrm{ess}}(M)$ by

$$\lambda_k(M) = \inf_F \max_{0 \neq \varphi \in F} \operatorname{Ray} \varphi, \tag{2.4}$$

where F runs over all subspaces of $H^1(M)$ of dimension $k+1$ and $\operatorname{Ray} \varphi$ denotes the *Rayleigh quotient* of φ. The infimum is achieved by the linear span of the $\lambda_j(M)$-eigenfunctions, where $0 \leq j \leq k$.

2.1. **Spectrum under finite Riemannian coverings.**

Unless otherwise specified, Riemannian manifolds are assumed to be complete and connected. Similarly, $p\colon M' \to M$ will denote a finite Riemannian covering of Riemannian manifolds with $|p|$ sheets and group Γ of covering transformations. Recall that Γ is transitive on the fibers of p if and only if p is normal. Since p is finite, we have

$$\lambda_0(M') = \lambda_0(M) \quad \text{and} \quad \lambda_{\mathrm{ess}}(M') = \lambda_{\mathrm{ess}}(M). \tag{2.5}$$

To show the first equality, recall that λ_0 is the supremum of the *positive spectrum*, that pull back and averaging are inverses to each other on the respective spaces of positive functions on M and M', and that both, pull back and averaging, are compatible with the Laplacian; see [25, Theorem 2.1]. As for the second equality, recall the well known fact that λ_{ess} is the $\limsup$ of λ_0 on the family of neighborhoods of infinity of the corresponding manifold and that the first equality does not require completeness; see e.g. [5, Proposition 4.8].

As indicated in (2.1), we assume throughout that

$$\lambda_0(M) < \lambda_{\mathrm{ess}}(M). \tag{2.6}$$

Then $\lambda_0(M)$ is an eigenvalue of Δ on M' and M of multiplicity one with unique positive eigenfunctions φ_0' and φ_0 of respective L^2-norms equal to one.

For any functions φ' on M' and φ on M, let $p_*\varphi'$ be the function on M such that $(p_*\varphi')(x)$ is the average of the $\varphi'(y)$, $y \in p^{-1}(x)$, and $p^*\varphi = \varphi \circ p$ be the pull-back of φ to M'. Say that φ' is p-invariant if φ' is constant along the fibers of p. Obviously, this holds if and only if there is a function φ on M such that $\varphi' = p^*\varphi$ or, equivalently, if and only if $\varphi' = p^*p_*\varphi'$. Clearly, p_* and p^* preserve all the standard regularity and integrability conditions. For $\varphi' \in L^2(M')$ and $\varphi \in L^2(M)$, we have

$$\langle \varphi', p^*\varphi\rangle_{M'} = |p|\langle p_*\varphi', \varphi\rangle_M, \tag{2.7}$$

where the indices M' and M indicate the scalar products in $L^2(M')$ and $L^2(M)$, respectively. Furthermore,

$$L^2(M') = \operatorname{im} p^* \oplus \ker p_*. \tag{2.8}$$

For any $\lambda \geq 0$, let E_λ' and E_λ be the λ-eigenspaces of Δ on M' and M, respectively. Since lifts of λ-eigenfunctions on M are λ-eigenfunctions on M', p^*E_λ is equal to the space of p-invariant functions in E_λ'.

Proposition 2.9. *For any $\lambda \geq 0$,*

1. *$E_\lambda' = p^*E_\lambda \oplus \ker p_*$ is an L^2-orthogonal splitting;*
2. *$\sqrt{|p|}p_* \colon p^*E_\lambda \to E_\lambda$ is an orthogonal isomorphism with inverse $p^*/\sqrt{|p|}$.*

Note here that λ-eigenspaces of Δ are closed subspaces of $L^2(M)$ and $L^2(M')$, even if λ belongs to the essential spectrum of M respectively M'.

3. Connectedness under coverings

Fix a point $x \in M$. For $x' \in p^{-1}(x)$ and a loop $c\colon [0,1] \to M$ at x, let $c_{x'}$ be the lift of c to M' starting at x'. Then $x'[c] = c_{x'}(1)$ defines a right action of $\Gamma = \pi_1(M,x)$ on the fiber $p^{-1}(x)$ of p over x, where $[c] \in \Gamma$ denotes the homotopy class of c.

Lemma 3.1. *Let $U \subseteq M$ be a connected open subset containing x. Then the connected components of $p^{-1}(U)$ are in canonical one-to-one correspondence with the orbits of Γ_U on $p^{-1}(x)$, where Γ_U denotes the image of $\pi_1(U,x)$ in Γ.*

Proof. Let $c\colon [0,1] \to U$ be a loop at x, $x' \in p^{-1}(x)$, and $c_{x'}$ be the lift of c starting at x'. Then the image of $c_{x'}$ is contained in $p^{-1}(U)$, and hence $x'[c]$ belongs to the component of $p^{-1}(U)$ containing x'. Conversely, if $x'' \in p^{-1}(x)$ belongs to the same component of $p^{-1}(U)$ as x', then there is a path $c'\colon [0,1] \to p^{-1}(U)$ from x' to x''. Then $c' = c_{x'}$ and, therefore, $x'' = c_{x'}(1)$, where $c = p \circ c'$ is a loop at x. □

Fixing a point $x' \in p^{-1}(x)$, we get a canonical identification $p^{-1}(x) = \Gamma'\backslash\Gamma$, where Γ' denotes the image of $\pi_1(M',x')$ in Γ. With respect to this identfication, the right action of Γ on $p^{-1}(x)$ corresponds to the right action of Γ on $\Gamma'\backslash\Gamma$.

Corollary 3.2. *After the choice of a point $x' \in p^{-1}(x)$, the connected components of $p^{-1}(U)$ are in canonical one-to-one correspondence with the elements of $\Gamma'\backslash\Gamma/\Gamma_U$, the space of orbits of the right action of Γ_U on $\Gamma'\backslash\Gamma$.*

When it comes to the existence of λ_k-unstable coverings, the case $\Gamma_U \subseteq \Gamma'$ is of interest. Our next result corresponds to the lifting property of covering projections.

Lemma 3.3. *If $\Gamma_U \subseteq \Gamma'$, then the right action of Γ_U on $\Gamma'\backslash\Gamma$ fixes $\Gamma' e$. Hence the action has more than one orbit unless $\Gamma' = \Gamma$. If the normalizer $N_\Gamma(\Gamma_U)$ of Γ_U is contained in Γ', then the right action of Γ_U on $\Gamma'\backslash\Gamma$ is trivial. In this case, the action has $|p| = |\Gamma'\backslash\Gamma|$ orbits.*

Proof. If $g \in \Gamma_U$, then $\Gamma' eg = \Gamma' g = \Gamma'$ since $g \in \Gamma'$. Under the second assumption, if $g \in \Gamma_U$ and $h \in \Gamma$, then $\Gamma' hg = \Gamma' g' h = \Gamma' h$ since $g' \in N_\Gamma(\Gamma_U) \subseteq \Gamma'$. □

Example 3.4 (Abelian coverings)**.** For a domain $U \subseteq M$ and a point $x \in U$, consider the Hurewicz homomorphism

$$H_x \colon \pi_1(M,x) \to \pi_1(M,x)/[\pi_1(M,x), \pi_1(M,x)] = H_1(M)$$

and the projection

$$H_1(M) \to H_1(M)/i_*(H_1(U)) =: A,$$

where H_1 indicates first homology groups with coefficients in $\mathbb{Z}$ and $i \colon U \to M$ denotes the inclusion. Under their composition, the preimage of $0 \in A$ in $\pi_1(M,x)$ equals Γ_U. Hence the preimage Γ' of any finite index subgroup A' of A is a normal subgroup of Γ containing Γ_U such that $\Gamma'\backslash\Gamma \cong A'\backslash A$ is a finite Abelian group. A question, among others adressed in Theorem 4.6 and Theorem 5.19, is whether A is trivial.

3.1. **Minimal number of generators.** Say that $\Gamma'\backslash\Gamma$ is *generated by k elements* if there is a subset G of Γ with $|G| = k$ such that $\Gamma' \cup G$ generates Γ; then the elements of G are also called *generators of* $\Gamma'\backslash\Gamma$. In the case where Γ' is a normal subgroup of Γ, this terminology coincides with the usual one for the group $\Gamma'\backslash\Gamma$. The *minimal number of generators* of $\Gamma'\backslash\Gamma$ is denoted by $\mu(\Gamma'\backslash\Gamma)$.

Lemma 3.5. *If $\mu(\Gamma'\backslash\Gamma) = k$ and $\Delta \subseteq \Gamma$ is a subgroup generated by ℓ elements, then the right action of Δ on $\Gamma'\backslash\Gamma$ has at least $k - \ell + 1$ orbits.*

Proof. The claim is true for $\ell \geq k$. Suppose now, by induction, that it is true for $\ell + 1 \leq k$. Then the right action of the subgroup Δ_g of Γ generated by Δ and any additional element $g \in \Gamma$ has at least $k - \ell$ orbits in $\Gamma'\backslash\Gamma$. If all of these would be orbits of Δ already, for any choice of $g \in \Gamma$, then the Δ-orbits would be invariant under Γ. However, that cannot be because Γ has only one orbit. Hence there is a choice of a $g \in \Gamma$ which decreases the number of orbits by one. □

Remark 3.6. In general, the calculation of the minimal number of generators of quotients $\Gamma'\backslash\Gamma$ is a difficult problem. However, in the case where Γ' is a normal subgroup of Γ such that $A = \Gamma'\backslash\Gamma$ is a finite Abelian group,

$$A = \mathbb{Z}/k_1 \times \cdots \times \mathbb{Z}/k_n,$$

then $\mu(A)$ equals the maximal number of k_i which share a common divisor.

3.2. **Asymptotic estimate of minimal number of generators.** Pyber [21] conjectures that almost all finite groups are nilpotent. It is therefore interesting to get estimates on the minimal number of generators of finite nilpotent groups.

For a prime p, a finite group G is called a *p-group* if the orders of all elements of G are divisible by p. This holds if and only if the order of G is a power of p; that is, $|G| = p^\alpha$ for some positive integer α. The number of isomorphism classes $f(n)$ of groups of order $n = p^\alpha$ is given by

$$f(n) = n^{(2/27 + o(1)) \cdot \alpha^2}, \tag{3.7}$$

by Higman [12, Theorem 3.5] and Sims [24, Proposition 1.1]. Because any p-group is nilpotent and hence solvable, the number $f(d,n)$ of groups of order $n = p^\alpha$ with a generating set of at most d elements satisfies

$$f(d,n) \le n^{(d+1)\cdot\alpha}, \tag{3.8}$$

by Mann [19, Theorem 2]. Combining (3.7) and (3.8), we conclude that the proportion of groups of order $n = p^\alpha$ with a generating set of at most d elements tends to zero as α tends to infinity.

For a non-trivial finite group G, a Sylow p-subgroup is a non-trivial subgroup P of G such that $|P|$ is the highest power of p dividing $|G|$. If G is nilpotent, then G is the product of its p_i-Sylow subgroups P_i,

$$G = P_1 \times \cdots \times P_k, \tag{3.9}$$

where the p_i run through the primes dividing $|G|$; see [8, Theorem 3, Chapter 6]. We conclude

Theorem 3.10. *Let n_i be a sequence of natural numbers such that the maximal exponent of the prime factors of n_i tends to infinity with i. Then the proportion of d generator nilpotent groups of order n_i among all nilpotent groups of order n_i tends to zero with i.*

Thus it is asymptotically unlikely that a random nilpotent group of order n_i has a generating set with at most d elements. We observe that it is necessary to assume, in Theorem 3.10, that the maximal exponent of the prime factors of n_i tends to infinity with i. This follows from a result of Guralnick [10] and Lucchini [16] that says that a finite group is generated by at most $d+1$ elements if each of its Sylow subgroups is generated by at most d elements.

Let M be a complete and connected Riemannian manifold of dimension m. Say that two normal Riemannian coverings $p_1\colon M_1 \to M$ and $p_2\colon M_2 \to M$ are *coarsely equivalent* if their groups of covering transformations are isomorphic. By Theorem 3.10, it is unlikely that random coarse equivalence classes of covering transformations of n_i-sheeted normal nilpotent coverings of a surface S have at most d generators as i tends to infinity. Thus asymtotic λ_1-stability is unlikely as well as $n_i \to \infty$.

4. A basic argument and applications

Let $\lambda < \lambda_{\mathrm{ess}}$, $\varphi \in E_\lambda$, and $(U_i)_{i\in I}$ be the family of pairwise different nodal domains of φ. For each $i \in I$, let $(U_{ij})_{j\in J_i}$ be the family of pairwise different nodal domains of $p^*\varphi$ over U_i. For each $i \in I$ and $j \in J_i$, let φ_{ij} be the function on M' which coincides with $p^*\varphi/k_{ij}$ on U_{ij}, vanishes on all other U_{ik}, and is equal to $p^*\varphi/|p|$ on the rest of M', where k_{ij} is the degree of the covering $p\colon U_{ij} \to U_i$. The set J of pairs ij with $i \in I$ and $j \in J_i$ labels the set of all nodal domains U_{ij} of $p^*\varphi$, sorted by the nodal domains U_i of φ. For any function ψ on M, we have

$$\int_{M'} \varphi_{ij} p^*\psi = \int_M \varphi\psi. \tag{4.1}$$

Notice the similarity, and difference, between (2.7) and (4.1). The definition of the φ_{ij} is adapted to what is needed in the comparison of N_M and $N_{M'}$. Recall here that $N_M(\lambda)$, $N_{M'}(\lambda)$ and $N_M(\lambda-)$, $N_{M'}(\lambda-)$ denote the number of eigenvalues of M and M' in $[0,\lambda]$ and $[0,\lambda)$, respectively.

Theorem 4.2. *There are at least $|J| - |I|$ linearly independent eigenfunctions on M' with eigenvalues in $(0,\lambda)$, which are perpendicular to $p^*(L^2(M))$. In particular,*

$$N_{M'}(\lambda-) \ge N_M(\lambda-) + \sum_{i\in I}(|J_i| - 1) = N_M(\lambda-) + \sum_{i\in I}|J_i| - |I|.$$

Note that $|J_i| \geq 1$ for all i so that the summands in the middle, the *contributions* $|J_i| - 1$ *of the* U_i, are all non-negative.

Proof of Theorem A. The contribution of U to the estimate in Theorem 4.2 is $j-1$. Since the contributions of the other nodal domains are non-negative, the claim follows. □

Proof of Theorem 4.2. Let X be the space spanned by the φ_{ij} and $Y = p^*(\varphi^\perp)$. By (4.1), X and Y satisfy the assumptions of Lemma A.2, applied to the quadratic form Q associated to the operator $A = \Delta' - \lambda$. Namely, $Q \leq 0$ on X. Furthermore, by (4.1), X and Y are perpendicular in $H = L^2(M')$. Finally, since φ is an eigenfunction of Δ, $PY \subseteq Y$, where P is the spectral projection of Δ' associated to $[0, \lambda)$. It remains to clarify the dimensions of X and $X \cap H_0$, where here $H_0 = E'_\lambda$.

To determine $\dim X$, suppose that $\sum \alpha_{ij}\varphi_{ij} = 0$. Let $i \in I$ and $j \in J_i$. Then on U_{ij}, the φ_{ik}, for $k \neq j$, vanish. Hence, on U_{ij},

$$\alpha_{ij}\varphi_{ij} = -\sum_{\substack{k\in I\setminus\{i\}\\ l\in J_k}} \alpha_{kl}\varphi_{kl}.$$

Now on U_{ij}, φ_{ij} is equal to $p^*\varphi/k_{ij}$ and each φ_{kl} on the right to $p^*\varphi/|p|$. Since $p^*\varphi \neq 0$ on U_{ij}, we get

$$\frac{\alpha_{ij}}{k_{ij}} = -\sum_{\substack{k\in I\setminus\{i\}\\ l\in J_k}} \frac{\alpha_{kl}}{|p|} =: \alpha_i.$$

Hence, on $p^{-1}(U_i)$,

$$\sum_{j\in J_i} \alpha_{ij}\varphi_{ij} = \alpha_i p^*\varphi.$$

We also get that $\alpha_{ij} = \alpha_i k_{ij}$. Since $\sum_{j\in J_i} k_{ij} = |p|$ for all $i \in I$, we infer that $\sum_{j\in J_i} \alpha_{ij} = \alpha_i|p|$. Hence the above displayed equality also holds on the rest of M'. Therefore the α_i satisfy the linear equation

$$\sum_{i\in I} \alpha_i = 0,$$

which has $|I| - 1$ independent solutions. In conclusion,

$$\dim X = \sum_{i\in I} |J_i| - |I| + 1.$$

To determine $\dim X \cap E'_\lambda$, note first that $X \cap E'_\lambda$ contains $p^*\varphi$. Conversely, any linear combination of the φ_{ij} is a multiple of $p^*\varphi$ on any of the nodal domains U_{kl} of $p^*\varphi$. Hence, by the unique continuation property, any smooth function in X is a multiple of $p^*\varphi$. In particular, $X \cap E'_\lambda$ consists of multiples of $p^*\varphi$. Therefore $X \cap E'_\lambda$ has dimension one. □

Corollary 4.3. *If k of the nodal domains of φ are simply connected, then*

$$N_{M'}(\lambda-) \geq N_M(\lambda-) + k(|p| - 1).$$

Proof. The preimage under p of any of the simply connected nodal domains has $|p|$ components. Hence their contribution to the estimate in Theorem 4.2 is $k(|p| - 1)$. Since the contributions of the other nodal domains are non-negative, the claim follows. □

In Corollary 4.4, for any nodal domain U of φ, we let $\Gamma' = p_*\pi_1(M, x')$, for any given $x \in U$ and $x' \in p^{-1}(x)$ (and Γ_U as usual). The assertions are independent of the choice of x and x'. Generalizing Corollary 4.3, we have

Corollary 4.4. *If k of the nodal domains U of φ satisfy*

(1) $\Gamma_U \subseteq \Gamma'$, *then* $N_{M'}(\lambda-) \geq N_M(\lambda-) + k$;
(2) $N_\Gamma(\Gamma_U) \subseteq \Gamma'$, *then* $N_{M'}(\lambda-) \geq N_M(\lambda-) + k(|p| - 1)$.

Proof. For any U as in the first assertion, $p^{-1}(U)$ has at least two, in the second $|p|$ components, by Lemma 3.3. □

Remark 4.5. Since liftings of eigenfunctions from M to M' are eigenfunctions on M' with the same eigenvalues, an inequality $N_{M'}(\lambda-) \geq N_M(\lambda-) + C$ implies that $N_{M'}(\kappa-) \geq N_M(\kappa-) + C$ and $N_{M'}(\kappa) \geq N_M(\kappa) + C$, for any $\kappa \geq \lambda$.

Theorem 4.6. *Suppose that M is complete and connected with $\lambda_1(M) < \lambda_{\mathrm{ess}}(M)$ and carries a $\lambda_1(M)$-eigenfunction φ such that its nodal set $\mathcal{Z}(\varphi)$ has at least $\mu+1 \geq 2$ components. Let U be one of the two nodal domains of φ and $x \in U$. Then there is a surjective homomorphism $I_x\colon \Gamma \to \mathbb{Z}_2^\mu$ such that $\Gamma_U \subseteq \ker I_x$. If M is orientable, there is a surjective homomorphism $I_x\colon \Gamma \to \mathbb{Z}^\mu$ such that $\Gamma_U \subseteq \ker I_x$. In both cases, if Γ' is a finite index subgroup of Γ containing Γ_U, then the corresponding Riemannian covering $p\colon M' \to M$ is strictly λ_1-unstable. More precisely, with $|p| = |\Gamma' \backslash \Gamma|$,*

$$N_{M'}(\lambda_-) \geq N_M(\lambda_-) + \mu(|p| - 1).$$

Proof. Suppose first that 0 is a regular value of φ. Then $\mathcal{Z}(\varphi)$ is a smooth manifold. Since φ is a $\lambda_1(M)$ eigenfunction, it has exactly two nodal domains, $\{\varphi < 0\}$ and $\{\varphi > 0\}$.

By assumption, $\mathcal{Z}(\varphi)$ has at least $\mu + 1$ connected components, $Z_1, \dots, Z_{\mu+1}$. Let $Z = Z_j$ for some $1 \leq j \leq \mu$, $Z' = Z_{\mu+1}$, and $z \in Z$ and $z' \in Z'$ be points. Since $\{\varphi < 0\}$ and $\{\varphi > 0\}$ are connected, there exist paths c_- in $\{\varphi \leq 0\}$ and c_+ in $\{\varphi \geq 0\}$ between z and z'. Their union c is a closed loop in M that has intersection number one with Z. In particular, intersection with the different $Z = Z_j$ defines a non-trivial homomorphism i_Z from Γ to $\mathbb{Z}^\mu$ if M is oriented and to $\mathbb{Z}_2^\mu$ otherwise.

Let now $U = \{\varphi < 0\}$ and $x \in U$. Then Γ_U is contained in the kernel Γ' of i_Z. Now Lemma 3.3 applies and shows the claim.

If 0 is not a regular value of φ, the above argument still applies in principle, but we need some preparation to define intersection numbers. To that end, write

$$\begin{aligned}\mathcal{Z}(\varphi) &= \{z \in \mathcal{Z}(\varphi) \mid d\varphi(z) \neq 0\} \cup \{z \in \mathcal{Z}(\varphi) \mid d\varphi(z) \neq 0\} \\ &= \mathcal{Z}(\varphi)_{\mathrm{reg}} \cup \mathcal{Z}(\varphi)_{\mathrm{sing}},\end{aligned}$$

the *regular* and *singular part* of $\mathcal{Z}(\varphi)$. Let Z and Z' be as above and set

$$Z_{\mathrm{reg}} = Z \cap \mathcal{Z}(\varphi)_{\mathrm{reg}}, \quad Z_{\mathrm{sing}} = Z \cap \mathcal{Z}(\varphi)_{\mathrm{sing}}.$$

Recall from the (elementary) proof of [11, Lemma 1.9] that any point in Z_{sing} is contained in an open ball B in M such that $Z_{\mathrm{sing}} \cap B$ is contained in a finite union of embedded submanifolds of dimension $\dim M - 2$. Let now $U = \{\varphi < 0\}$ and $x \in U$ as above.

Claim 1) Any loop in M at x is homotopic to a loop which does not meet Z_{sing} and meets Z_{reg} transversally in at most finitely many points.

To show 1), let c be a loop in M at x. Then $I = c^{-1}(Z_{\mathrm{sing}})$ is compact. Hence I can be covered by finitely many consecutive intervals such that the image of each of these intervals is contained in a ball B as above. Since the codimension of the corresponding embedded submanifolds as above is two, c can be deformed consecutively to a loop at x which does not meet Z_{sing}. This shows the first assertion. The second follows from standard transversality theory, applied to the smooth hypersurface Z_{reg}.

Claim 2) For any two homotopic loops in M at x, which do not meet Z_{sing} and meet Z_{reg} transversally in at most finitely many points, the (oriented respectively non-oriented) intersection numbers with Z coincide.

To show 2), let c_1 and c_2 be two such homotopic loops. Since they do not meet Z_{sing} and intersect Z_{reg} transversally in at most finitely many points, there is an $\varepsilon > 0$, which is a regular value of φ such that $\{\varphi = \varepsilon\}$ has a component Z_ε such that the intersections of c_1 with Z and Z_ε are in one-to one correspondence to each other, and similarly for c_2. Now the intersection numbers of c_1 and c_2 with Z_ε are well-defined and agree with the intersection numbers with Z_{reg}, by what we said. Since c_1 and c_2 are homotopic, their intersection numbers with Z_ε agree, hence also the ones with Z_{reg}.

Now 1) and 2) show that intersection numbers with the different $Z = Z_j$ respectively $Z_{j,\mathrm{reg}}$ define a homomorphism Γ to $\mathbb{Z}^\mu$ in the oriented case and $\mathbb{Z}_2^\mu$ otherwise. The rest of the proof is as in the regular case. □

Remark 4.7. Under appropriate assumptions, the conclusion of Theorem D holds with λ_k-unstable in place of λ_1-unstable. More precisely, if there is a $\lambda_k(M)$-eigenfunction φ on M such that there are components of $\mathcal{Z}(\varphi)$ together with loops in M which intersect exactly once, then the above arguments apply and show λ_k-instability. This may indicate that λ_k-instability becomes more likely, the more nodal domains $\lambda_k(M)$-eigenfunctions have.

4.1. **Absolute estimate.** In the above, we estimated $N_{M'}$ against N_M. That is what is behind the definition of the φ_{ij} in the beginning of the section. An easier approach leads to an estimate of $N_{M'}$ without comparing it with N_M. The point is as follws: Let $(D_i)_{i\in I}$ be a family of domains in M and, for each $i \in I$, $(D_{ij})_{j\in J_i}$ be the connected components of $p^{-1}(D_i)$. Let $\lambda \geq 0$ and φi be a smooth function on D_i with compact support and Rayleigh quotient $\leq \lambda$. For each $i \in I$ and $j \in J_i$, let φ_{ij} now be the function which equals $p^*\varphi_i$ on D_{ij} and vanishes outside of D_{ij}. Then the φ_{ij} are pairwise L^2-orthogonal and have Rayleigh quotient $\leq \lambda$. Hence

$$N_{M'}(\lambda) \geq \sum_{i\in I} |J_i|. \tag{4.8}$$

Clearly, $N_M(\lambda) \geq |I|$, but that does not lead to an inequality as in Theorem 4.2.

In the following discussion, we use (4.8) only in the case of one domain in M, that is, $|I| = 1$; cf. Remark 4.13. Set

$$\sigma(M) = \inf_D \lambda_0(D), \tag{4.9}$$

where the infimum is taken over all simply connected domains $D \subset M$. By monotonicity, $\lambda_0(M) \leq \sigma(D)$. Recall also that $\lambda_0(M') = \lambda_0(M)$.

In general, (4.9) poses the optimization problem of the existence of a simply connected domain D in M such that $\sigma(M) = \lambda_0(D)$ and of a Dirichlet eigenfunction φ on D for $\lambda_0(D)$.

Proposition 4.10. *If $\sigma(M) < \lambda_{\mathrm{ess}}(M)$, then M' has at least $|p|$ eigenvalues in $[0, \sigma(M)]$.*

The assumption $\sigma(M) < \lambda_{\mathrm{ess}}(M)$ is satisfied if M is closed since the essential spectrum of closed Riemannian manifolds is empty. Recall also that $\lambda_{\mathrm{ess}}(M') = \lambda_{\mathrm{ess}}(M)$ since p is finite.

Proof of Proposition 4.10. For any $\sigma(M) < \lambda < \lambda_{\mathrm{ess}}(M)$, let $D \subseteq M$ be a simply connected domain such that there is a $\varphi \in C_c^\infty(M)$ with support in D with Rayleigh quotient $< \lambda$. There are precisely $|p|$ lifts of D to M', and they are pairwise disjoint. For any such lift C, let $\varphi_C \in C_c^\infty(M')$ be the function with support in C such that

$\varphi_C = \varphi \circ p$ on C. Then the φ_Cs and their gradients are pairwise L^2-orthogonal and have the same Rayleigh quotient as φ. □

For $\ell \geq 1$, let $\sigma_\ell(M) = \inf_D \lambda_0(D)$, where the infimum is taken over all domains $D \subset M$ such that the fundamental group of D is generated by at most ℓ elements. Notice that $\sigma(M) = \sigma_0(M)$.

Proposition 4.11. *If the minimal number of generators of* $\Gamma'\backslash\Gamma$ *is* k *and* $\sigma_\ell(M) < \lambda_{\mathrm{ess}}(M)$ *for some* $\ell \leq k$ *, then* M' *has at least* $k - \ell + 1$ *eigenvalues in* $[0, \sigma_\ell(M)]$.

Proof. For any $\sigma_\ell(M) < \lambda < \lambda_{\mathrm{ess}}(M)$, there is a domain D in M with $\sigma_\ell(M) \leq \lambda_0(D) < \lambda$ such that the fundamental group of D is generated by at most ℓ elements. Hence the preimage of D under p has at least $k-\ell+1$ components, by Corollary 3.2 and Lemma 3.5. Therefore M' has at least $k - \ell + 1$ eigenvalues in $[0, \lambda)$. □

Remark 4.12. For a Riemannian surface S, $\sigma_1(S)$ coincides with the *analytic systol of* S, introduced in [1]. Recall that S has at most $-\chi(S)$ eigenvalues in $[0, \sigma_1(S)]$, by [2, Theorem 1.5]. Here we get that the covering surface S' has at least two eigenvalues in $[0, \sigma_1(S)]$, provided that the minimal number of generators of $\Gamma'\backslash\Gamma$ is at least two.

Remark 4.13. As in Theorem 4.2, we can also consider families of pairwise disjoint domains to get a more general estimate than the one in Proposition 4.11. Namely, if $(D_i)_{i\in I}$ is a finite family of pairwise disjoint domains in M such that the fundamental group of D_i is generated by at most ℓ_i elements and such that $\lambda = \max \lambda_0(D_i) < \lambda_{\mathrm{ess}}(M)$, then M has at least $\sum_{i\in I}(k - \ell_i + 1)$ eigenvalues in $[0, \lambda)$. The point is that the different lifts of functions $\varphi_i \in C_c^\infty(D_i)$ to the different components of $p^{-1}(D_i)$ are pairwise L^2-perpendicular.

5. Coverings of surfaces

Let S be a connected surface. Assume that S is of finite type, that is, S is diffeomorphic to the interior of a compact surface $\bar{S}$ with boundary. Equivalently, the Euler characteristic $\chi(S) > -\infty$. The connected components of $\bar{S} \setminus S$ consist of circles, called *holes* or *circles at infinity*.

Suppose that S is endowed with a complete Riemannian metric which has a square-integrable eigenfunction φ_0 at the bottom λ_0 of its spectrum. Recall that the multiplicity of λ_0 is one and that φ_0 does not change sign, hence can be chosen to be positive. If the area $|S| < \infty$, then φ_0 is constant and $\lambda_0 = 0$. Eigenfunctions of S for eigenvalues $\lambda > \lambda_0$ are perpendicular to φ_0 and hence change sign. In particular, they have at least two nodal domains. The structure of the nodal set of such an eigenfunction φ was clarified in [7, Theorem 2.5]:

Theorem 5.1 (Cheng)**.** *The nodal set* $\mathcal{Z}(\varphi)$ *of* φ *is a locally finite graph in* S. *Moreover,* $z \in \mathcal{Z}(\varphi)$ *has valence* $2n$ *if and only if* φ *vanishes to order* n *at* z. *The opening angles between the edges at* z *are equal to* π/n. *Furthermore,* $\mathcal{Z}(\varphi)$ *is a locally finite union of immersed circles and lines.*

We will need the following topological result for the study of nodal domains.

Lemma 5.2. *Let* S *be a connected surface of finite type and* $U \subset S$ *an open domain with piecewise smooth boundary. Assume that the complement* U^c *of* U *in* S *contains only finitely many components which are discs or annuli. Then* U *has finite type.*

Proof. The proof rests on the fact that a surface (orientable or not) has finite type if and only if any family of simple closed curves, which are not null-homotopic, pairwise disjoint, and pairwise not freely homotopic, is finite; cf. [23, pp. 259–260].

Assuming that U is not of finite type, there is an infinite family $\mathcal{F}$ of simple closed curves in U, which satisfies these conditions with respect to U.

If c is a member of $\mathcal{F}$ and c is null-homotopic in S, then c bounds an embedded disc D in S, $c = \partial D$. Now $\partial U \cap D$ cannot contain components of ∂U which are line segments and hence consists of finitely many simple closed curves, which bound discs in D which belong to U^c. There are only finitely many such discs, by assumption. Hence the union U' of such discs with U is of finite type if and only if U is. Therefore we can assume from now on that the complement of U does not contain components which are discs.

Let now c_0 and c_1 be members of $\mathcal{F}$ which are freely homotopic in S. Then there is an embedded annulus A in S such that $c_0 \cup c_1 = \partial A$. Now $\partial U \cap A$ cannot contain components of ∂U which are line segments, nor can it contain closed curves which are homotopic to zero, by assumption. Hence it consists of two boundary curves $\hat{c}_0$ and $\hat{c}_1$, such that the parts of A between c_0 and $\hat{c}_0$ respectively c_1 and $\hat{c}_1$ belong to U and the rest, $\hat{A}$, to U^c. Now $\hat{A}$ is an annulus. Hence there are only finitely many such, by assumption. Since S is of finite type, we arrive at a contradiction to the assumption that $\mathcal{F}$ is infinite. □

Lemma 5.3. *A λ-eigenfunction φ on S with $\lambda < \lambda_{\text{ess}}(S)$ has only finitely many nodal domains.*

Proof. Let $\mathcal{N}$ be the family of different nodal domains of φ. For any $U \in \mathcal{N}$, let φ_U be the function which coincides with φ on U and vanishes otherwise. Then the φ_U are pairwise L^2-perpendicular, are in $H^1(S)$, and have Rayleigh quotient $\le \lambda$. Since $\lambda < \lambda_{\text{ess}}(S)$, the variational characterization of eigenvalues implies that $\mathcal{N}$ is finite. □

Corollary 5.4. *For $\lambda < \lambda_{\text{ess}}(S)$, the nodal domains of λ-eigenfunctions on S are geometrically finite.*

Proof. By Theorem 5.1, any nodal domains U of φ is a domain in S with piecewise smooth boundary. (If ∂U has a self-intersection at a critical point x of φ, push U a bit inside, away from x, to get ∂U embedded.) Now Lemma 5.2 and Lemma 5.3 imply the assertion. □

5.1. **On the topology of nodal domains.** Assume from now on that $\chi(S) \le 0$ and let φ be an eigenfunction of S perpendicular to φ_0.

Lemma 5.5. *If none of the nodal domains of φ is a disc, then each nodal domain U of φ is a domain of finite type; in particular $\chi(U) > -\infty$. Moreover $\chi(U) < 0$ unless U is a disc or an annulus or a Möbius band.*

Proof. Let U be a nodal domain of φ. If a component D of U^c is a closed disc, then a component of the open set $D \setminus \mathcal{Z}(\varphi)$ is a disc, a contradiction to the assumption. Now Lemma 5.2 implies the assertion. □

Lemma 5.6. *If φ is an eigenfunction of S perpendicular to φ_0, then φ has a nodal domain U such that $\chi(S)/2 \le \chi(U) \le 1$. More generally, if $\ell \ge 2$ denotes the number of nodal domains of φ, then φ has a nodal domain U such that*

$$\chi(S)/\ell \le \chi(U) \le 1.$$

Proof. Since $\chi(S) \le 0$, we can assume that no nodal domain of φ is a disc. Then all nodal domains of φ are domains of finite type, in particular with finite Euler characteristic. By [7, Theorem 2.5], the nodal set Z of φ is a graph with vertices of (even) order at least two. Therefore $\chi(Z) \le 0$ and hence

$$\sum \chi(U) \ge \chi(Z) + \sum \chi(U) = \chi(S),$$

where the sum is over all nodal domains U of φ. Hence there is at least one nodal domain U of φ such that $\chi(S)/\ell \le \chi(U) \le 1$. □

Let Γ be the fundamental group of S. If S is closed, $S = \bar{S}$, the minimal number of generators of Γ is

$$\nu = \nu(\Gamma) = \nu(S) = 2 - \chi(S). \tag{5.7}$$

If S is non-compact, then Γ is a free group and the minimal number of generators of Γ is

$$\nu = \nu(\Gamma) = \nu(S) = 1 - \chi(S). \tag{5.8}$$

Lemma 5.9. *Let φ be an eigenfunction of S with $\ell \ge 2$ nodal domains.*

(1) *If $\ell \ge \nu(S)$, at least one of the nodal domains is a disc or an annulus or a Möbius band.*
(2) *If $\ell < \nu(S)$, then φ has a nodal domain with minimal number of generators of its fundamental group at most $1 - \chi(S)/\ell$.*

Proof. If all nodal domains of φ have negative Euler characteristic, then $\ell \le -\chi(S)$. The first claim now follows from (5.7) and (5.8). As for the second, we may assume that all nodal domains of φ have negative Euler characteristic. Hence, by Lemma 5.6, φ has a nodal domain U with $-\chi(S)/\ell \ge -\chi(U)$. But then $\nu(U) = 1 - \chi(U) \le 1 - \chi(S)/\ell$ by (5.8). □

Corollary 5.10. *There is a nodal domain U of φ such that the fundamental group of U admits a system with at most $\nu/2$ generators if S is closed and $(\nu+1)/2$ generators otherwise, where $\nu = \nu(S)$.*

Consider now a finite Riemannian covering $p \colon S' \to S$ of complete and connected Riemannian surfaces of finite type. Write $S = \Gamma\backslash\tilde{S}$ and $S' = \Gamma'\backslash\tilde{S}$, where the fundamental groups $\Gamma \supseteq \Gamma'$ of S and S' are viewed as groups of covering transformations of the universal covering surface $\tilde{S}$ of S and S'.

Proposition 5.11. *If p is λ_1-stable, where $\lambda_1(S) < \lambda_{\mathrm{ess}}$, then the minimal number of generators of $\Gamma'\backslash\Gamma$ is at most $\nu/2$ if S is closed and $(\nu+1)/2$ otherwise, where $\nu = \nu(S)$.*

Proof. Let φ be a $\lambda_1(S)$-eigenfunction. By Corollary 5.10, φ has a nodal domain U such that the fundamental group of U admits a system with at most $\nu/2$ respectively $(\nu+1)/2$ generators. If the minimal number of generators for $\Gamma'\backslash\Gamma$ is strictly bigger than $\nu/2$ respectively $(\nu+1)/2$, then the right action of $\pi_1(U)$ on $\Gamma'\backslash\Gamma$ cannot be transitive. By Corollary 3.2 together with Theorem 4.2, we get a contradiction to λ_1-stability. □

5.2. **Intersection numbers and coverings.** Let U be a nodal domain of an eigenfunction φ of S perpendicular to φ_0. Then the complement U^c of U is a surface of finite type, and we let V be a component of U^c. Then V has $k \ge 1$ boundary circles in S, that we call *doors* $d_1, \dots, d_k$, through which it is connected to U and $\ell \ge 0$ boundary circles in $\bar{S} \setminus S$, that we call *exits* $e_1, \dots, e_\ell$ *at infinity*. We draw the first kind in green, meaning that we may enter U through them, and the latter kind in red, indicating that we exit S through them eventually.

We now view V as a regular plane polyhedron P in the orientable case, respectively Q in the non-orientable case, with the colored holes in its interior and with the standard identifications of its edges, indicated by the labellings

$$\begin{aligned} &P_0 \colon aa^{-1}, && \text{where } g = 0,\\ &P_g \colon a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}, && \text{where } g \ge 1,\\ &Q_g \colon a_1a_1\cdots a_ga_g, && \text{where } g \ge 1. \end{aligned}$$

In terms of (orientable respectively non-orientable) *genus* g and numbers k and ℓ of holes, the negative of the Euler characteristic of V is

$$-\chi(V) = 2g + k + \ell - 2 \quad \text{and} \quad -\chi(V) = g + k + l - 2 \tag{5.12}$$

in the orientable (P_g) and non-orientable (Q_g) case, respectively.

If $\ell \geq 2$, we draw disjoint segments $h_2, \dots, h_\ell$ from e_1 to the other red circles $e_2, \dots, e_\ell$. We get a homomorphism

$$I = I_V \colon H_1(S) \to \mathbb{Z}^\mu \quad \text{respectively} \quad I = I_V \colon H_1(S) \to \mathbb{Z}_2^\mu, \tag{5.13}$$

the *intersection homomorphism*, where

$$\begin{aligned} \mu = \mu(V) &= 2g + k - 1 + (\ell - 1) \vee 0 \quad \text{for } P_g, \\ \mu = \mu(V) &= g + k - 1 + (\ell - 1) \vee 0 \quad \text{for } Q_g, \end{aligned} \tag{5.14}$$

by taking,

(1) in the oriented case, oriented intersection numbers with the circles in V coming from the edges of P labeled $a_1, b_1, \dots, a_g, b_g$, the boundary circles $d_2 \dots, d_k$, and the segments $f_2 \dots, f_\ell$.
(2) in the non-orientable case, intersection numbers modulo two with the circles in V coming from the edges of Q labeled $a_1, \dots, a_g$, the boundary circles $d_2 \dots, d_k$, and the segments $f_2 \dots, f_\ell$.

From (5.12) and (5.14), we conclude

Lemma 5.15. *Irrespective of whether V is orientable or not, we have*

$$\mu(V) = -\chi(V) + \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell \geq 1 \end{cases} \geq -\chi(V).$$

Remark 5.16. We have $\mu = \mu(V) \geq 1$ unless (g, k, ℓ) equals $(0, 1, 0)$ or $(0, 1, 1)$, and then V is a disc with one green or an annulus with one green and one red boundary circle, respectively. There are also only a few cases with $\mu = 1$. Using the above (g, k, l) notation, if V is orientable, then it is one of the following:

(0, 2, 0) an annulus, boundary circles green; $\chi(V) = 0$,
(0, 2, 1) a pair of pants, two boundary circles green, one red,
(0, 1, 2) a pair of pants, one boundary circle green, two red.

If V is non-orientable, it is one of the following:

(1, 1, 0) a *Möbius band*, boundary circle green; $\chi(V) = 0$.
(1, 1, 1) a projective plane with two holes, boundary circles green and red.

Lemma 5.17. *Under the Hurewicz homomorphism (see Example 3.4) from Γ to $H_1(S)$ respectively $H_1(S; \mathbb{Z}_2)$, Γ_U is contained in the kernel* $\ker I_V$.

Proof. By definition, homotopy classes of loops in Γ_U have representatives which are contained in U and hence have empty intersection with curves in $V \subseteq U^c$. □

Theorem 5.18. *Let $p \colon S' \to S$ be a finite Riemannian covering of complete and connected Riemannian surfaces and φ be a λ-eigenfunction of M, where $\lambda_0(S) < \lambda < \lambda_{\text{ess}}(S)$. Let k be the number of nodal domains U of φ such that, for some component V of U^c, $\ker I_V \subseteq \Gamma'$, where Γ' is the image of $\pi_1(S', x')$ in $\pi_1(S, x)$ under $p_\#$ for some $x' \in p^{-1}(x)$ (see Section 3). Then*

$$N_{S'}(\lambda -) \geq N_S(\lambda -) + k(|p| - 1).$$

Proof. By Lemma 5.17 and since $\ker I_V$ is a normal subgroup of Γ, $N_\Gamma(\Gamma_U)$ is contained in $\ker I_V$, for any nodal domain U and any component V of its complement as in the assertion. Hence $p^{-1}(U)$ has $|p|$ components for any such U, by Lemma 3.3. Now Theorem 4.2 implies the assertion. □

Theorem 5.19. *Assume that S is of finite type with $\chi(S) < 0$, and let φ be a λ-eigenfunction, where $\lambda_0(S) < \lambda < \lambda_{\mathrm{ess}}(S)$. Then φ has $\nu \geq 2$ nodal domains and at least one, U, such that $\chi(S)/\nu \leq \chi(U) \leq 1$. For any such U, Γ admits a surjective homomorphism I to $\mathbb{Z}_2^\mu$, respectively $\mathbb{Z}^\mu$ if S is orientable, where $\mu \geq -(\nu-1)\chi(S)/\nu$, such that $\Gamma_U \subseteq \ker I$. In particular, if $\ker I \subseteq \Gamma' \subseteq \Gamma$ is a finite index subgroup, then the corresponding Riemannian covering $p\colon S' \to S$ is strictly λ-unstable. More precisely,*

$$N_{S'}(\lambda-) \geq N_S(\lambda-) + |p| - 1,$$

where $|p| = |\Gamma' \backslash \Gamma|$.

Proof. By Corollary 5.10, φ has a nodal domain U such that $\chi(U) \geq \chi(S)/\nu$. Then, by Theorem 5.1,

$$\chi(U^c) = \chi(S) - \chi(U) \leq (\nu - 1)\chi(S)/\nu < 0.$$

Therefore the components V_j of U^c with $\chi(V_j) < 0$ satisfy

$$\sum_j \chi(V_j) \leq (\nu - 1)\chi(S)/\nu. \tag{5.20}$$

For each i, we have

$$\mu(V_j) \geq -\chi(V_j),$$

by Lemma 5.15. Therefore, if μ equals the sum of the $\mu(V_j)$, then the sum

$$I = \oplus_j I_{V_j} \colon \ \oplus_j H_1(V_j) \to \oplus_j \operatorname{im} I_{V_j}$$

is a homomorphism to $\mathbb{Z}^\mu$ respectively $\mathbb{Z}_2^\mu$ as asserted, except that we compose it with the corresponding Hurewicz homomorphism to have it defined on Γ. □

6. Towers of coverings

Let $\cdots \to M_k \to M_{k-1} \to \cdots \to M_1 \to M_0 = M$ be a tower of Riemannian coverings of complete and connected Riemannian manifolds. Assume that the degrees $|p_k|$ of the coverings $p_k\colon M_k \to M$ tend to infinity as k tends to infinity.

We say that a sequence of points $x_k \in M_k$ is a *tower of points* above $x = x_0 \in M_0$ if they lie above each other. Given $x \in M$, we choose a tower (x_k) of points above x and get the fundamental groups $\Gamma = \pi_1(M, x)$ and $\Gamma_k = \pi_1(M_k, x_k)$. By choosing a point $\tilde{x}$above the x_k in the universal covering space $\tilde{M}$, we identify Γ and the Γ_k with the corresponding groups of covering transformations of $\tilde{M}$ such that

$$\cdots \subseteq \Gamma_k \subseteq \Gamma_{k-1} \subseteq \cdots \subseteq \Gamma_1 \subseteq \Gamma_0 = \Gamma$$

and $M_k = \Gamma_k \backslash \tilde{M}$, $M = \Gamma \backslash \tilde{M}$.

Lemma 6.1. *For any $x_0 \in M$, $\operatorname{diam} p_k^{-1}(x_0) \to \infty$ as $k \to \infty$.*

Proof. Let $r > 0$ be the injectivity radius of M at x_0. Then the geodesic ball $B(x_0, r)$ is evenly covered by any Riemannian covering of M.

Let $\tilde{p}\colon \tilde{M} \to M$ be the universal covering. Endowed with the lifted Riemannian metric, $\tilde{M}$ is complete and the coverings $\tilde{M} \to M_k$ are Riemannian.

Let $k \geq 1$ and $y_1, \dots, y_l$ be the points in $p_k^{-1}(x_0)$, where $l = |p_k|$. Then $\operatorname{diam} p_k^{-1}(x_0)$ is realized by the minimal $\operatorname{diam}\{z_1, \dots, z_l\}$, where the minimum is taken over all choices of lifts z_i of the y_i to $\tilde{M}$. The minimum is attained since $\tilde{p}^{-1}(x_0)$ is a discrete subset of $\tilde{M}$. Now the assertion follows by the same reason and since $|p_k| \to \infty$ as $k \to \infty$. □

For a tower of Riemannian coverings as above, we say that a covering $\hat{p}\colon \hat{M} \to M$ is a *roof* of the tower if $\hat{p}$ is a Riemannian covering of complete and connected Riemannian manifolds which factors through the p_k. The universal covering of M is the highest roof. The lowest roof is given by $\cap\Gamma_k$.

The following result extends Theorem F.

Theorem 6.2. *For a tower of finite Riemannian coverings together with a roof $\hat{M}$ as above, suppose that the covering $\hat{p}\colon \hat{M} \to M$ is normal and that $\lambda_0(\hat{M}) < \lambda_{\text{ess}}(M)$. Then, for any $\lambda_0(\hat{M}) < \lambda < \lambda_{\text{ess}}(M)$ and $l \geq 1$, M_k has at least l eigenvalues below λ, for all sufficiently large k; in short,*

$$\limsup_{k\to\infty} \lambda_l(M_k) \leq \lambda_0(\hat{M}).$$

Proof. For λ as in the assertion, there is a smooth function φ on $\hat{M}$ with support contained in some geodesic ball $B(x,r) \subset \hat{M}$ and with Rayleigh quotient $\operatorname{Ray}\varphi < \lambda$.

Let $x_0 \in M$ be the point under x. Since the degrees of the coverings $p_k\colon M_k \to M$ tend towards ∞, the same is true for the diameter of the fibers F_k of p_k over x_0, by Lemma 6.1. In particular, there is a first $k_2 \geq 1$ such that F_{k_2} contains two points $y_{k_2,1}, y_{k_2,2}$ of distance $d_2 \geq 2r$. Then any fiber F_k with $k \geq k_2$ contains two points $y_{k,1}, y_{k,2}$ of the same distance d_2. There is then a first $k_3 \geq k_2$ such that F_{k_3} contains a further point $y_{k_3,3}$ of distance $\geq 2r$ to $y_{k_3,1}$ and $y_{k_3,2}$. Then any fiber F_k with $k \geq k_3$ contains a further point $y_{k,3}$ with distances

$$d_3 = d(y_{k,1}, y_{k,3}) = d(y_{k_3,1}, y_{k_3,3}) \quad \text{and} \quad d(y_{k,2}, y_{k,3} \geq 2r.$$

Proceeding in this way, given any $l \geq 1$, there is $k_l \geq 1$ such that F_k, for $k \geq k_l$, contains l points of pairwise distance $\geq 2r$. The pushdown, in the sense of [4, Section 4], of translates of φ to preimages of these points in $\hat{M}$ then yields functions with compact support in the balls of radius r about these points and with Rayleigh quotients $< \lambda$. Since the supports of these functions in M_k are disjoint, they and their gradients are pairwise L^2-orthogonal. Hence $\lambda_l(M_k) < \lambda$, by the variational characterization of eigenvalues. □

Corollary 6.3. *Under the assumptions of Theorem F, if the group of covering transformations of $\hat{p}$ is amenable, then $\lim_{k\to\infty} \lambda_l(M_k) = \lambda_0(M)$, for any $l \geq 1$.*

Proof. Since the group of covering transformations of $\hat{p}$ is amenable, we have $\lambda_0(\hat{M}) = \lambda_0(M)$, by [4, Theorem 1.2]. The asserted equality follows now from $\lambda_0(M_k) \geq \lambda_0(M)$, which is a consequence of the characterization of λ_0 as the supremum of the positive spectrum. □

Appendix A. A remark about spectral theory

Let A be an unbounded operator with dense domain $\mathcal{D}$ in a Hilbert space H. Consider the polar decomposition $A = U|A|$ of A and the associated orthogonal decomposition

$$H = H_- \oplus H_0 \oplus H_+ \tag{A.1}$$

with $Ux = \pm x$ for $x \in H_\pm$ and $H_0 = \ker A = \ker U$ as in [14, Section VI.7]. Since A and $|A|$ commute with U, the decomposition of H is invariant under A and $|A|$. In particular,

$$\mathcal{D} = (\mathcal{D} \cap H_-) \oplus H_0 \oplus (\mathcal{D} \oplus H_+)$$

and, similarly,

$$\mathcal{D}_Q = (\mathcal{D}_Q \cap H_-) \oplus H_0 \oplus (\mathcal{D}_Q \oplus H_+)$$

for the domain $\mathcal{D}_Q \supseteq \mathcal{D}$ of the quadratic form $Q = Q(x,y) = \langle Ax, y\rangle$ in H associated to A. Since $\ker A = \ker |A| = H_0$, we have $Q < 0$ on $\mathcal{D}_Q \cap H_-$ and $Q > 0$ on $\mathcal{D}_Q \cap H_+$.

The following is a refined version of the usual variational characterization of eigenvalues of A in the case where the spectrum of A is discrete.

Lemma A.2. *Let $X \subseteq \mathcal{D}_Q$ and $Y \subseteq H$ be subspaces such that $Q \le 0$ on X, $X \perp Y$, and $PY \subseteq Y$, where P denotes the orthogonal projection of H onto H_-. Then*

$$X \cap \ker P = X \cap H_0 \quad \textit{and} \quad PX \perp Y.$$

In particular, if $\dim X < \infty$*, then*

$$\dim(H_- \ominus Y) \ge \dim PX = \dim X - \dim(X \cap H_0).$$

Proof. Write $x \in X$ as $x = x_- + x_0 + x_+$ according to (A.1), where $Px = x_-$. Now we have

$$Q(x,x) = Q(x_-, x_-) + Q(x_+, x_+) \le 0$$

and hence

$$Q(x_+, x_+) \le -Q(x_-, x_-).$$

Since $Q > 0$ on H_+, $x_- = 0$ implies that $x_+ = 0$ so that then $x = x_0 \in H_0$. This shows the first assertion. As for the second, we have

$$\langle PX, Y\rangle = \langle X, PY\rangle = 0$$

since P is orthogonal, $PY \subseteq Y$, and $X \perp Y$. □

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