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ON A VARIETY OF RIGHT-SYMMETRIC ALGEBRAS

NURLAN ISMAILOV AND UALBAI UMIRBAEV

Abstract. We construct a finite-dimensional metabelian right-symmetric algebra over an arbitrary field that does not have a finite basis of identities.

1. Inroduction.

We say that a variety of algebras has *the Specht property* or is *Spechtian* if any of its subvarieties has a finite basis of identities. In other words, a variety of algebras is Spechtian if the set of all its subvarieties satisfies the descending chain condition with respect to inclusion. In 1950 Specht [31] formulated a problem on the Specht property for the variety of all associative algebras over a field of characteristic zero.

Specialists extended the study of this problem for any varieties of algebras over fields of any characteristic. In 1970 Vaughan-Lee [36] constructed an example of a finite-dimensional Lie algebra over a field of characteristic $p = 2$ that does not have a finite basis of identities. In 1974 Drensky [8] extended this result to fields of any positive characteristic $p > 0$. In 1978 Medvedev [25] showed that varieties of metabelian Malcev, Jordan, alternative, and $(-1, 1)$ algebras are Spechtian. In 1984 Umirbaev [33] proved that the variety of metabelian binary Lie algebras over a field of characteristic $\neq 3$ has the Specht property. In 1980 Medvedev [26] also constructed an example of a variety of solvable alternative algebras over a field of characteristic 2 with an infinite basis of identities. In 1985 Umirbaev [34] proved that the varieties of solvable alternative algebras over a field of characteristic $\neq 2, 3$ have the Specht property. Pchelintsev [27] constructed an almost Spechtian variety of alternative algebras over a field of characteristic 3. The Specht property of so-called bicommutative algebras is proven in [9].

In 1976 Belkin [1] proved that the variety of metabelian right-alternative algebras does not have the Specht property. In 1978 L'vov [24] constructed a six-dimensional nonassociative algebra over an arbitrary field satisfying the identity $x(yz) = 0$ with an infinite basis of identities. In 1986 Isaev [15] adapted L'vov's methods for right-alternative algebras and constructed a finite-dimensional metabelian right-alternative algebra over an arbitrary field with an infinite basis of identities. In 2008 Kuz'min [22] gave a sufficient condition for the varieties of metabelian right-alternative algebras over a field of characteristic $\neq 2$ to be Spechtian.

In 1988 Kemer [16, 17] positively solved the famous Specht problem [31] and proved that every variety of associative algebras over a field of characteristic zero has a finite basis of identities. Later the Specht problem was negatively solved for the variety of associative algebras over fields of positive characteristic $p > 0$ [2, 13, 28]. It is also

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known that the varieties of Lie algebras generated by a finite-dimensional algebra over a field of characteristic zero have the Specht property [14, 18]. Despite the efforts of many specialists in this field, the question of whether the variety of Lie algebras over a field of characteristic zero has the Specht property remains open.

This paper is devoted to the study of the Specht property for the variety of right-symmetric algebras. Recall that an algebra A over a field $\mathbb{F}$ is called *right-symmetric* if it satisfies the identity

$$(a, b, c) = (a, c, b), \tag{1}$$

where $(a, b, c) = (ab)c - a(bc)$ is the associator of $a, b, c \in A$.

Right-symmetric algebras are Lie admissible, that is, any right-symmetric algebra with respect to the commutator $[x, y] = xy - yx$ is a Lie algebra. Very often right-symmetric (or left-symmetric) algebras are called pre-Lie algebras and play an important role in the theory of operads [23]. Right-symmetric algebras arise in many different areas of mathematics and physics [3].

In 1994 Segal [30] constructed a basis of free right-symmetric algebras. Chapoton and Livernet [5] and, independently, Löfwall and Dzhumadil'daev [11] gave other bases of free right-symmetric algebras in terms of rooted trees. The identities of right-symmetric algebras were studied by Filippov [12], and he proved that any right-nil right-symmetric algebra over a field of characteristic zero is right nilpotent. An analogue of the PBW basis Theorem for the universal (multiplicative) enveloping algebra of a right-symmetric algebra was given in [19]. The Freiheitssatz and the decidability of the word problem for one-relator right-symmetric algebras were proven in [20]. Recently, Dotsenko and Umirbaev [7] determined that the variety of right-symmetric algebras over a field of characteristic zero is Nielson-Schreier, that is, every subalgebra of a free right-symmetric algebra is free.

A right-symmetric algebra with an additional identity

$$a(bc) = b(ac)$$

is called a Novikov algebra. The class of Novikov algebras is an important and well-studied subclass of right-symmetric algebras. Recently there was great progress in the study of identities, solvability, and nilpotency [38, 12, 10, 29, 35, 32]. In 2022 Dotsenko, Ismailov, and Umirbaev [6] proved that (a) every Novikov algebra satisfying a nontrivial polynomial identity over a field of characteristic zero is right-associator nilpotent and (b) the variety of Novikov algebras over a field of characteristic zero has the Specht property.

In this paper, we continue the study of the identities of right-symmetric algebras. Namely, using the constructions and methods of L'vov [24] and Isaev [15], we construct a finite-dimensional metabelian right-symmetric algebra over an arbitrary field that does not have a finite basis of identities. In fact, our algebra belongs to the variety of algebras $\mathcal{R}$ defined by the identities

$$[[a, b], c] = 0, \tag{2}$$

$$(ab)a = 0, \tag{3}$$

and

$$(ab)(cd) = 0. \tag{4}$$

We determine some identities and operator identities of the variety $\mathcal{R}$ in Section 2. In Section 3, a series of algebras P_n of this variety is constructed. A linear basis of free algebras of the variety $\mathcal{R}$ is constructed in Section 4. Section 5 is devoted to the study of the relationships between the polynomial identities and the operator identities of the algebras P_n. The main result of the paper is given in Section 6 and says that the algebra P_2 does not have any finite basis of identities.

2. A variety of right-symmetric algebras

Let $\mathbb{F}$ be an arbitrary fixed field. In what follows, all vector spaces are considered over $\mathbb{F}$. As above, $\mathcal{R}$ denotes the variety of algebras defined by the identities (2), (3), and (4).

Lemma 2.1. *Every algebra of the variety $\mathcal{R}$ is right-symmetric and right nilpotent of index* 4.

Proof. The linearization of (3) gives

$$(ab)c + (cb)a = 0. \tag{5}$$

This identity and (4) imply that

$$((ab)c)d = -(dc)(ab) = 0. \tag{6}$$

Using (3) and(2) one can also get

$$\begin{aligned}(a, b, c) - (a, c, b) &= (ab)c - a(bc) - (ac)b + a(cb)\\ &= -(cb)a - a(bc) + (bc)a + a(cb)\\ &= [b, c]a - a[b, c] = [[b, c], a] = 0,\end{aligned}$$

i.e., $\mathcal{R}$ is a variety of right-symmetric algebras. □

Let A be an arbitrary algebra of the variety $\mathcal{R}$. Recall that for any $x \in A$ the operators of right multiplication R_x and left multiplication L_x on A are defined by

$$aR_x = ax \quad \text{and} \quad aL_x = xa,$$

respectively. Set also $V_{x,y} = L_xR_y$.

Lemma 2.2.

$$V_{x,x} = 0, \qquad V_{x,y} = -V_{y,x}. \tag{7}$$

$$xR_yL_zL_t = yV_{x,z}L_t - xR_yV_{t,z}. \tag{8}$$

$$xR_yL_z = xV_{z,y} + yR_xL_z - yV_{z,x}. \tag{9}$$

$$xR_yV_{z,t} = yR_xV_{z,t}. \tag{10}$$

$$V_{x,y}R_z = 0. \tag{11}$$

$$V_{x,y}(L_zL_t + V_{t,z}) = 0. \tag{12}$$

Proof. The identities (3) and (6) immediately imply (7). By (1) and (4) we get

$$xR_yL_zL_t = t(z(xy))$$
$$= (tz)(xy) + t((xy)z) - (t(xy))z = yV_{x,z}L_t - xR_yV_{t,z}.$$

From the identity (2) follows (9).

Then (1) and (6) give that

$$xR_yV_{z,t} = (z(xy))t$$
$$= ((zx)y)t + (z(yx))t - ((zy)x)t = yR_xV_{z,t}.$$

By (6) we obtain $tV_{x,z}R_t = ((xt)z)t = 0$, and, therefore, $V_{x,y}R_z = 0$. Set $v = uV_{x,y}$. Then (6), (1), and (4) imply that

$$vL_zL_t = t(zv) - t(vz) = (tz)v - (tv)z = -vV_{t,z}.$$

□

3. Algebras P_n

For each natural n we define the algebra P_n with a linear basis

$$a_{ij},\ b_{ij},\ c_i,\ d_{ij},\ e_{ij},$$

where $i, j \in \{1, 2, \dots, n\}$, and with the product defined by

$$a_{ij}c_i = d_{ij}, \quad b_{ij}c_i = e_{ij},$$
$$a_{ij}e_{ij} = e_{ij}a_{ij} = -b_{ij}d_{ij} = -d_{ij}b_{ij} = c_j,$$

where all zero products are omitted.

Set

$$A_n = \mathrm{Span}\{a_{ij},\ b_{ij} \mid 1 \le i, j \le n\}$$

and

$$D_n = \mathrm{Span}\{c_i,\ d_{ij},\ e_{ij} \mid 1 \le i, j \le n\},$$

where $\mathrm{Span}\, X$ denotes the linear span of X. Then A_n is a subalgebra of P_n and D_n is an ideal of P_n. Moreover, P_n is a direct sum of the vector spaces A_n and D_n. Set also

$$C_n = \mathrm{Span}\{c_i \mid 1 \le i \le n\}, \quad \overline{C}_n = \mathrm{Span}\{d_{ij},\ e_{ij} \mid 1 \le i, j \le n\}.$$

Then

$$P_n^2 = D_n, \quad A_n^2 = D_n^2 = 0, \quad D_n = C_n \oplus \overline{C}_n, \tag{13}$$
$$D_nP_n = C_n, \quad P_nC_n = \overline{C}_n, \quad C_nP_n = 0, \quad P_n\overline{C}_n = C_n.$$

Lemma 3.1. *The algebra P_n belongs to the variety $\mathcal{R}$.*

Proof. Obviously the space of commutators $[P_n, P_n]$ coincides with $\overline{C}_n$, which is in the center of P_n, i.e., (2) holds.

In order to verify the identity (3), it is sufficient to check the identities (3) and (5) for all elements of the basis of P_n. Let us begin with (3). Since $A_n^2 = D_n^2 = (D_nA_n)D_n = 0$, we may assume that $a \in A_n$ and $b \in D_n$. Consider all nonzero products of the space A_nD_n. If $a = a_{ij}$ and $b = c_i$, then

$$(a_{ij}c_i)a_{ij} = d_{ij}a_{ij} = 0.$$

If $a = a_{ij}$ and $b = e_{ij}$, then
$$(a_{ij}e_{ij})a_{ij} = c_j a_{ij} = 0.$$
The other cases can be verified similarly.

Now let's verify (5). Since $(D_nP_n)P_n = 0$, the product $(ab)c$ is nonzero only if $a = a_{ij}$, $b = c_i$, $c = b_{ij}$ or $a = b_{ij}$, $b = c_i$, $c = a_{ij}$. Thus,
$$(ab)c + (cb)a = -c_j + c_j = 0.$$
From the relations $P_n^2 = D_n$ and $D_n^2 = 0$ immediately follow the identity (4). □

Lemma 3.2. *For all $x, y \in P_n$, $d \in D_n$ we have*
$$(A_n + \overline{C}_n)V_{x,y} = 0, \quad V_{d,y} = V_{y,d} = 0.$$

Proof. The relations (13) give that $(P_nA_n)P_n \subseteq C_nP_n = 0$ and $(P_n\overline{C}_n)P_n \subseteq C_nP_n = 0$, i.e., the first equality of the lemma holds. Similarly, by noting that $(D_nP_n)P_n \subseteq C_nP_n = 0$ and $(P_nP_n)D_n \subseteq D_nD_n = 0$, we can deduce the second equality of the lemma. □

Denote by Ann_lP_n the space of left annihilators of P_n.

Lemma 3.3. $\mathrm{Ann}_lP_n = C_n$.

Proof. Assume that $x \in (A_n + \overline{C}_n + C_n) \cap \mathrm{Ann}_lP_n$ and express it as
$$x = \sum_{i,j}(\alpha_{ij}a_{ij} + \beta_{ij}b_{ij} + \gamma_{ij}d_{ij} + \delta_{ij}e_{ij} + \epsilon_i c_i),$$
where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \epsilon_i \in \mathbb{F}$. Then we have
$$xc_i = \sum_j(\alpha_{ij}d_{ij} + \beta_{ij}e_{ij}), \quad xa_{ij} = \delta_{ij}c_j, \quad xb_{ij} = -\gamma_{ij}c_j.$$
From these equations, it can be deduced that $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = \delta_{ij} = 0$. Therefore, we can conclude that $x = \sum_i \epsilon_i c_i$. Consequently, $\mathrm{Ann}_lP_n = C_n$. □

4. Structure of free algebras of $\mathcal{R}$

Let $F(X)$ be the free algebra of the variety $\mathcal{R}$ generated by an infinite countable set $X = \{x_1, x_2, \ldots, x_n, \ldots\}$.

Proposition 4.1. *The set of elements $\mathcal{B}$ of $F(X)$ of the forms*
$$x_i, \quad x_i\hat{R}_{x_j}L_{x_s}, \quad x_i\hat{R}_{x_j}V_{x_{p_1},x_{q_1}}\cdots V_{x_{p_k},x_{q_k}}\hat{L}_{x_s},$$
where $i < j$ and $p_r < q_r$ for all $r = 1, 2, \ldots, k$, $k \geq 1$, and $\hat{T}_x$ denotes that the operator T_x might not occur, is a basis of $F(X)$.

Proof. In order to show that $\mathcal{B}$ linearly spans $F(X)$ it is sufficient to verify that, for any $v \in \mathcal{B}$, the elements vR_{x_i} and vL_{x_i} belong to the linear span of $\mathcal{B}$. This is easy to do using the identities (1), (4), and Lemma 2.2. For example, let
$$v = x_i\hat{R}_{x_j}V_{x_{p_1},x_{q_1}}\cdots V_{x_{p_k},x_{q_k}}L_{x_s}.$$
Then
$$vR_{x_r} = x_i\hat{R}_{x_j}V_{x_{p_1},x_{q_1}}\cdots V_{x_{p_k},x_{q_k}}L_{x_s}R_{x_r} = x_i\hat{R}_{x_j}V_{x_{p_1},x_{q_1}}\cdots V_{x_{p_k},x_{q_k}}V_{x_s,x_r}.$$

By (12), we get

$$vL_{x_r} = x_i\hat{R}_{x_j}V_{x_{p_1},x_{q_1}}\cdots V_{x_{p_k},x_{q_k}}L_{x_s}L_{x_r} = -x_i\hat{R}_{x_j}V_{x_{p_1},x_{q_1}}\cdots V_{x_{p_k},x_{q_k}}V_{x_r,x_s}.$$

Applying (7) we can express vR_{x_r} and vL_{x_r} as a linear combination of elements of $\mathcal{B}$.

It remains to prove the linear independence of elements of $\mathcal{B}$. Suppose that $f = f(x_1, x_2, \ldots, x_n) \in F(X)$ is a nontrivial linear combination of elements of $\mathcal{B}$. Suppose that $v \in \mathcal{B}$ and $\deg_{x_i}(v) = k$. Let's write $v = v(x_i, \ldots, x_i)$ in order to differ the presence of x_i in different places. To linearize v in x_i we use new variables $y_1, \ldots, y_k \in X$ and, after renumeration, we can assume that $y_r < x_j$ if $i < j$ and $x_j < y_r$ if $j < i$ for all $1 \leq r \leq k$. Notice that every word $v(y_{\sigma(1)}, \ldots, y_{\sigma(k)})$, where $\sigma \in S_k$ and S_k is the symmetric group in k symbols, is an element of $\mathcal{B}$. Then the full linearization of v in x_i is a linear combination of basis elements $v(y_{\sigma(1)}, \ldots, y_{\sigma(k)})$. Therefore, by linearizing a nontrivial element f, we obtain a nontrivial element that is a linear combination of multilinear elements from $\mathcal{B}$. Substituting zeroes instead of some variables, if necessary, we can make f linear in each variable. Therefore, we can assume that f is a multilinear nontrivial identity in the variables $x_1, \ldots, x_n$. Let

$$f = \sum_{i=1} \alpha_i u_i,$$

where $\alpha_i \in \mathbb{F}$ and $u_i \in \mathcal{B}$. Suppose, for example, that

$$u_1 = x_iR_{x_j}V_{x_{p_1},x_{q_1}}\cdots V_{x_{p_k},x_{q_k}}L_{x_s}.$$

Set $x_i = d_{1,2}$, $x_j = -b_{1,2}$, $x_{p_r} = a_{r+1,r+2}$, $x_{q_r} = -b_{r+1,r+2}$ for all $r = 1, 2, \ldots k$, $x_s = a_{k+2,k+3}$. We have $c_iV_{a_{ij},-b_{ij}} = c_j$ for all i, j. Then the value of u_1 under this substitution is $d_{k+2,k+3}$ and the value of any other u_i is 0. Consequently, the value of f is $\alpha_1 d_{k+2,k+3} \neq 0$. Thus, f is not an identity for $\mathcal{R}$.

If L_{x_s} does not appear in u_1, then we perform the same substitutions for the variables. If R_{x_j} does not appear in u_1, then we simply set $x_i = c_2$ and perform the same substitutions for the rest of the variables as described above. In both cases the value of f is nonzero. This completes our proof. □

Let $M = M(F(X))$ be the multiplication algebra of the algebra $F(X)$. Denote by E_0 the subalgebra (without identity) of M generated by the operators $V_{i,j} = V_{x_i,x_j}$ with $i < j$ for all $i, j = 1, 2, \ldots$. Set also

$$E_1 = \sum_{j\geq 1} E_0L_{x_j}, \quad E_2 = \sum_{i\geq 1} R_{x_i}E_0, \quad E_3 = \sum_{i,j\geq 1} R_{x_i}E_0L_{x_j},$$

and

$$R_k = \sum_{i\geq 1} x_iE_k, \quad \text{for } k = 0, 1, 2, 3.$$

According to Proposition 4.1, the space $F(X)$ is the direct sum of the subspaces R_k and the linear span of elements of $\mathcal{B}$ of degrees less than or equal to 3.

Lemma 4.2. *An identity $zf(x_1, \ldots, x_m) = 0$, where $f \in E_0$, is a consequence of a system of identities*

$$tg_j(x_1, \ldots, x_l) = 0, \quad g_j \in E_0, \quad j \in \mathcal{J}, \tag{14}$$

in the variety $\mathcal{R}$, where $\mathcal{J}$ is any set of indices, if and only if the operator $f(x_1,\dots,x_m)$ belongs to the ideal of the associative algebra E_0 generated by the set G of all operators $\varphi(g_j)$, where φ runs over the set of all linear endomorphisms $\varphi : X \to \mathbb{F}X = \sum_{i\geq 1}\mathbb{F}x_i$ and $j \in \mathcal{J}$.

Proof. Suppose that f belongs to the ideal of E_0 generated by G. Then

$$f = \sum_{r=1}^{t} u_r g_{j_r}^{\varphi_r} v_r,$$

for some linear endomorphisms φ_r and $u_r, v_r \in E_0$. Therefore,

$$zf = \sum_{r=1}^{t} (zu_r) g_{j_r}^{\varphi_r} v_r$$

and $zf = 0$ is a consequence of the system of identities (14).

Let's describe all the consequences of the identities (14). Let $\varphi : F(X) \to F(X)$ be an arbitrary endomorphism and set $\varphi(x_i) = y_i + h_i$, where $y_i \in \mathbb{F}X$ and $h_i \in F(X)^2$ for all i. Since $g_j \in E_0$, using (4) and (6), we get

$$t\varphi(g_j) = tg_j(y_1,\dots,y_l) = 0.$$

Thus, a general form of consequences of the identities (14) can be expressed as

$$\sum_{r=1}^{t} u_r g_{j_r}^{\varphi_r} v_r,$$

where $u_r \in F(X)$, $v_r \in M(F(X))$, and φ_r are linear endomorphisms. We know that $g_{j_r}^{\varphi_r} \in E_0$. We also claim that u_r and v_r can be represented in the forms

$$x_i \hat{R}_{x_j} V_{x_{p_1},x_{q_1}} \cdots V_{x_{p_k},x_{q_k}}, \qquad V_{x_{p'_1},x_{q'_1}} \cdots , V_{x_{p'_k},x_{q'_k}} \hat{L}_{x_s},$$

respectively, where $i < j$, $p_l < q_l$, $p'_l < q'_l$ and $k = 0, 1, \dots$.

Suppose that u_r is a basis element that ends with L_{x_s}. Then, by (11) and (12), we can derive that

$$V_{x_i,x_j} L_{x_s} V_{x_k,x_t} = V_{x_i,x_j} L_{x_s} L_{x_k} R_{x_t} = -V_{x_i,x_j} V_{x_k,x_s} R_{x_t} = 0.$$

Consequently, we have $V_{x_i,x_j} L_{x_s} E_0 = 0$.

If $u_r = x_i R_{x_j} L_{x_k}$, then by (8) and (11) we get

$$u_r V_{x_s,x_t} = x_i R_{x_j} L_{x_k} L_{x_s} R_{x_t} = x_j V_{x_i,x_k} L_{x_s} R_{x_t} - x_i R_{x_j} V_{x_s,x_k} R_{x_t} = x_j V_{x_i,x_k} V_{x_s,x_t}.$$

So, we can conclude that u_r has the claimed form.

Now, let's consider the case when v_r is a basis element that starts with R_y. According to (11), we have $E_0 R_y = 0$. If v_r starts with $L_{x_i} L_{x_j}$, then by using (12), we find

$$V_{x_k,x_s} L_{x_i} L_{x_j} = -V_{x_k,x_s} V_{x_j,x_i}.$$

Hence, v_r also has the claimed form.

If $zf = 0$ is a consequence of the identities (14), then we get an equality of the form

$$x_{m+1} f(x_1,\dots,x_m) = \sum_{r=1}^{t} \lambda_r x_{i_r} w_r g_{j_r}^{\varphi_r} v_r,$$

where $x_{i_r}w_r = u_r$, $w_r \in E_0 + E_2$ and $v_r \in E_0 + E_1$. Notice that every element $x_{i_r}w_r g_{j_r}^{\varphi_r} v_r$ belongs to $\mathcal{B}$. Consequently, we may assume that $x_{i_r} = x_{m+1}$, $w_r, v_r \in E_0$, and

$$f(x_1, \ldots, x_m) = \sum_{r=1}^{t} \lambda_r w_r g_{j_r}^{\varphi_r} v_r.$$

□

5. Identities of P_n.

In this section, we study the connections between the identities and the operator identities of P_n for $n \geq 2$.

Lemma 5.1. *If $f \in F(X)$ and $f = 0$ is an identity of P_n for $n \geq 2$, then*

$$f = f_0 + f_1 + f_2 + f_3 \in F(X), \quad f_k \in R_k, \tag{15}$$

and $f_k = 0$ is an identity of P_n for all $k = 0, 1, 2, 3$.

Proof. Let

$$f = \sum_{i=1}^{m} \lambda_i x_i + \sum_{i,j=1}^{m} \lambda_{ij} x_i x_j + \sum_{i,j,k=1, i<j}^{m} \lambda_{ijk} x_i R_{x_j} L_{x_k} + f',$$

where f' is a linear combination of elements from $\mathcal{B}$ of degree ≥ 4.

We first show that $\lambda_i = \lambda_{ij} = \lambda_{ijk} = 0$ for all $i, j, k = 1, \ldots, m$. For any fixed i the substitution $x_i = c_1$ and $x_j = 0$ for all $j \neq i$ gives that $\lambda_i c_1 = 0$, which implies $\lambda_i = 0$.

If $i \neq j$ then the substitution $x_i = a_{11}$, $x_j = c_1$, and $x_k = 0$ for all $k \neq i, j$, makes the value of f equal to $\lambda_{ij} d_{11} = 0$. The same value we get if $i = j$ under the substitution $x_i = x_j = a_{11} + c_1$ and $x_k = 0$ for all $k \neq i, j$. This gives $\lambda_{ij} = 0$ in both cases.

Assume that $i < j > k$. If $i \neq k$, then the substitution $x_i = b_{11}$, $x_j = d_{11}$, $x_k = a_{12}$, and $x_t = 0$ for all $t \neq i, j, k$, makes the value of f equal to $-\lambda_{ijk} d_{12}$. This gives that $\lambda_{ijk} = 0$. If $i = k$, then the substitution $x_i = b_{11}$, $x_j = d_{11}$, and $x_t = 0$ for all $t \neq i, j$, gives that $-\lambda_{iji} e_{11} = 0$ and $\lambda_{iji} = 0$. If $i < j = k$, then the substitution $x_i = d_{11}$, $x_j = b_{11}$, and $x_t = 0$ for all $t \neq i, j$, gives that $-\lambda_{ijj} e_{11} = 0$ and $\lambda_{ijj} = 0$. Finally, if $i < j < k$, then the substitution $x_i = d_{11}$, $x_j = x_k = b_{11}$, and $x_t = 0$ for all $t \neq i, j$, gives that $-\lambda_{ijk} e_{11} - \lambda_{ikj} e_{11} = 0$, i.e., $\lambda_{ijk} = -\lambda_{ikj} = 0$.

Thus, f is a linear combination of elements of $\mathcal{B}$ of degree ≥ 4. Suppose that f is written as in (15). Taking into account the relations $D_n P_n \subseteq C_n$ and $P_n C_n \subseteq \overline{C}_n$ it can be observed that the images of $F_0 = f_0 + f_2$ and $F_1 = f_1 + f_3$ belong to C_n and $\overline{C}_n$, respectively. Therefore, if $f = 0$ is an identity of P_n, then $F_0 = 0$ and $F_1 = 0$ are also identities of P_n.

Suppose that

$$f_k(x_1, \ldots, x_m) = \sum_{i=1}^{m} x_i g_i^{(k)}(x_1, \ldots, x_m),$$

where $g_i^{(k)} \in E_k$ and $k = 0, 1, 2, 3$. Let $p_1, \ldots, p_m \in P_n$ with $p_s = v_s + \overline{v}_s + a_s$, where $v_s \in C_n$, $\overline{v}_s \in \overline{C}_n$, $a_s \in A_n$. By Lemma 3.2 we can obtain that

$$p_i V_{p_j,p_k} = v_i V_{p_j,p_k} = v_i V_{v_j+\overline{v}_j+a_j, v_k+\overline{v}_k} + v_i V_{v_j+\overline{v}_j, a_k} + v_i V_{a_j,a_k} = v_i V_{a_j,a_k}.$$

Then we can write as

$$f_0(p_1,\dots,p_m)=\sum_{i=1}^{m} v_i g_i^{(0)}(a_1,\dots,a_m)=f_0(v_1+a_1,\dots,v_m+a_m).$$

It is easy to note that $(A_n+C_n)R_{a_i+v_i}V_{a_j+v_j,a_s+v_s}=0$ for all $a_i,a_j,a_s\in A_n$ and $v_i,v_j,v_s\in C_n$. It follows that

$$f_2(v_1+a_1,\dots,v_m+a_m)=0.$$

Thus,

$$\begin{aligned}&f_0(p_1,\dots,p_m)\\ &=f_0(v_1+a_1,\dots,v_m+a_m)+f_2(v_1+a_1,\dots,v_m+a_m)\\ &=F_0(v_1+a_1,\dots,v_m+a_m)=0.\end{aligned}$$

Therefore, we can conclude that $f_0=0$ and $f_2=0$ are identities of P_n. Similarly, we can establish that $f_1=0$ and $f_3=0$ are also identities of P_n. □

Lemma 5.2. *If $f=f(x_1,\dots,x_m)\in R_1+R_3$, then $fx_{m+1}\in R_0+R_2$ and if $f(x_1,\dots,x_m)x_{m+1}=0$ is an identity of P_n, then $f=0$ is an identity of P_n as well.*

Proof. We have $fx_{m+1}\in R_0+R_2$ by the definition of the spaces R_i, where $0\le i\le 3$. If $fx_{m+1}=0$ is an identity of P_n, then all values of f in P_n belong to $C_n=\mathrm{Ann}_l(P_n)$ by Lemma 3.3. However, since f is an element of R_1+R_3, the values of f must belong to $\overline{C}_n$. Consequently, $f=0$ is an identity of P_n. □

Recall an exact formal definition of the linearization of identities [37, Chapter 1]. Let $\mathcal{V}$ be an arbitrary variety of algebras and $\mathbb{F}\langle X\rangle$ be its free algebra over $\mathbb{F}$ generated by $X=\{x_1,x_2,\dots\}$. Let $y\in\mathbb{F}\langle X\rangle$ be an arbitrary fixed element. For a nonnegative integer k, we define the linear mapping $\Delta_{x_i}^k(y)$ on $\mathbb{F}\langle X\rangle$ as follows:

- $\Delta_{x_i}^0(y)$ is the identity mapping;
- $x_s\Delta_{x_i}^k(y)=0$, if either $k>1$ or $k=1$, $i\neq s$;
- $x_i\Delta_{x_i}^1(y)=y$;
- $(uv)\Delta_{x_i}^k(y)=\sum_{r+s=k}(u\Delta_{x_i}^r(y))(v\Delta_{x_i}^s(y))$,

where $x_i\in X$ and u,v are any monomials in $\mathbb{F}\langle X\rangle$. We also write $\Delta_{x_i}(y)$ instead of $\Delta_{x_i}^1(y)$.

Lemma 5.3. *Suppose that $f=f(x_1,\dots,x_m)\in R_2$. Then $f\Delta_i(x_{m+1}x_{m+2})\in R_0$ for all $1\le i\le m$. Moreover, $f=0$ is an identity of P_n if and only if P_n satisfies the following system of identities*

$$f(x_1,\dots,x_m)\Delta_i(x_{m+1}x_{m+2})=0,\quad 1\le i\le m. \tag{16}$$

Proof. Let $w=xR_yV_{z_1,t_1}\cdots V_{z_r,t_r}\in\mathcal{B}$ and $u,v\in X$. We have

$$(xR_y)\Delta_x(uv)=(uv)R_y=vV_{u,y}.$$

By (1), (4) and (6), we get

$$\begin{aligned}(xR_yV_{z_1,t_1})\Delta_y(uv)&=(z_1(x(uv)))t_1=((z_1x)(uv)-(z_1(uv))x+z_1((uv)x))t_1\\ &=-((z_1(uv))x)t_1+(z_1((uv)x))t_1=(z_1((uv)x))t_1=vV_{u,x}V_{z_1,t_1}.\end{aligned}$$

By (4) one can get that $w\Delta_{z_i}(uv) = w\Delta_{t_i}(uv) = 0$ for any $i = 1, \ldots, r$. Thus, if $f(x_1, \ldots, x_m) \in R_2$, then $f(x_1, \ldots, x_m)\Delta_i(x_{m+1}x_{m+2}) \in R_0$.

If $p_1, \ldots, p_m \in P_n$ and $v_1, \ldots, v_m \in D_n$, then we have

$$f(p_1 + v_1, \ldots, p_m + v_m) = f(p_1, \ldots, p_m) + \sum_{i=1}^{m} f(p_1, \ldots, p_m)\Delta_i(v_i). \tag{17}$$

In fact, by Lemma 1.3 from [37], the relation

$$f(x_1 + y_1, \ldots, x_m + y_m) = \sum_{i_1,\ldots,i_m \geq 0} f\Delta_1^{i_1}(y_1)\cdots\Delta_m^{i_m}(y_m)$$

$$= f(x_1, \ldots, x_m) + \sum_{i=1}^{m} f(x_1, \ldots, x_m)\Delta_i(y_i) + g,$$

where $y_1, \ldots, y_m \notin \{x_1, \ldots, x_m\}$ are distinct variables and the degree of g in the variables $y_1, \ldots, y_m$ is greater than one, holds in $\mathbb{F}\langle X\rangle$. By substituting $x_i = p_i, y_i = v_i$ and using the fact that $D_n^2 = 0$, one can obtain the relation (17).

If $f = 0$ is an identity of P_n, then the relation (17) implies that

$$f(p_1, \ldots, p_m)\Delta_i(v) = f(p_1, \ldots, p_i + v, \ldots, p_m) - f(p_1, \ldots, p_m) = 0$$

for all $p_i \in P_n$ and $v \in D_n$. In other words, the algebra P_n satisfies the system of identities (16).

Conversely, suppose that the system of identities (16) holds in P_n. Assume that $p_1, \ldots, p_m \in P_n$ of the form $p_i = a_i + v_i$, where $a_i \in A_n$ and $v_i \in D_n$. Then using the relation (17), we have

$$f(p_1, \ldots, p_m) = f(a_1 + v_1, \ldots, a_m + v_m)$$

$$= f(a_1, \ldots, a_m) + \sum_{i=1}^{m} f(p_1, \ldots, p_m)\Delta_i(v_i) = f(a_1, \ldots, a_m).$$

Considering $A_n^2 = 0$ and $f \in R_2 \subseteq F(X)^2$, we can conclude that $f(a_1, \ldots, a_m) = 0$. Consequently, $f(p_1, \ldots, p_m) = 0$. □

Lemma 5.4. *If $f = f(x_1, \ldots, x_m) \in R_0$ and $f = 0$ is an identity of P_n of the form*

$$f = \sum_{i=1}^{m} x_i g_i,$$

where $g_i \in E_0$, then $x_{m+1}g_i = 0$ is an identity of P_n.

Proof. For a fixed i set $x_i = v + a_i$ and $x_j = a_j$ for all $j \neq i$, where $v \in D_n$ and $a_j \in A_n$. Taking into account the relations $A_n^2 = D_n^2 = 0$ and Lemma 3.2, one can have

$$f(x_1, \ldots, x_m) = vg_i(a_1, \ldots, a_m) = 0.$$

Hence, $x_{m+1}g_i = 0$ is an identity of P_n. □

Proposition 5.5. *For an arbitrary polynomial* $f = f(x_1,\dots,x_m) \in F(X)$ *there exist* $t(m) = 2m(m+3)$ *polynomials* $g_i(x_1,\dots,x_{m+3}) \in E_0$*, where* $i = 1,\dots,t(m)$*, such that* $f(x_1,\dots,x_m) = 0$ *is an identity of* P_n *for* $n \geq 2$ *if and only if* P_n *satisfies the system of identities*

$$zg_i(x_1,\dots,x_{m+3}) = 0, \quad 1 \leq i \leq t(m).$$

Proof. Let $f = f(x_1,\dots,x_m) \in F(X)$ and suppose that $f = 0$ is an identity of P_n. Then by Lemma 5.1 we obtain

$$f = f_0 + f_1 + f_2 + f_3, \quad f_k \in R_k,$$

and $f_k = 0$ is an identity of the algebra P_n.

By Lemma 5.4, the identity $f_0 = \sum_{i=1}^{m} x_i g_i = 0$ is equivalent to the system of m identities $x_{m+1}g_i = 0$ of P_n, where $1 \leq i \leq m$.

By Lemma 5.2, the identity $f_1 = 0$ is equivalent to $f_1x_{m+1} = 0$ and $f_1x_{m+1} \in R_0$. Moreover, if

$$f_1x_{m+1} = \sum_{i=1}^{m} x_i g_i, \quad g_i \in E_0,$$

then, by Lemma 5.4, the identity $f_1x_{m+1} = 0$ is equivalent to the system of m identities $x_{m+2}g_i = 0$ of P_n, where $1 \leq i \leq m$.

By Lemma 5.3, the identity $f_2 = 0$ is equivalent to the system of m identities $f_2(x_1,\dots,x_m)\Delta_i(x_{m+1}x_{m+2}) = 0$, where $i = 1,\dots,m$, and we have $f_2\Delta_i(x_{m+1}x_{m+2}) \in R_0$. Hence, by Lemma 5.4, it is equivalent to a system of $m(m+2)$ identities of the form $x_{m+3}g_i = 0$, where $g_i(x_1,\dots,x_{m+2}) \in E_0$ and $i = 1,\dots,m(m+2)$.

By Lemma 5.2, the identity $f_3 = 0$ is equivalent to $f_3x_{m+1} = 0$ and $f_3x_{m+1} \in R_2$. The identity (4) implies that $(f_3x_{m+1})\Delta_{m+1}(x_{m+2}x_{m+3}) = 0$. Then, by Lemma 5.3, $f_3 = 0$ is equivalent to the system of m identities $0 = (f_3x_{m+1})\Delta_i(x_{m+2}x_{m+3}) \in R_0$, where $i = 1,\dots,m$. Moreover, by Lemma 5.4, it is equivalent to a system of $m(m+2)$ identities of the form $x_{m+4}g_j = 0$, where $g_j(x_1,\dots,x_{m+3}) \in E_0$ and $1 \leq j \leq m(m+2)$.

Thus, $f = 0$ is equivalent to a system of $t(m) = 2m(m+3)$ identities of the form $zg_i(x_1,\dots,x_{m+3}) = 0$, where $g_i(x_1,\dots,x_{m+3}) \in E_0$ and $i = 1,\dots,t(m)$. □

6. V-identities of P_n.

Let B be an arbitrary algebra in $\mathcal{R}$. We define $E_0(B)$ as the algebra of operators generated by V_{b_1,b_2} for all $b_1, b_2 \in B$, that acts on the algebra B. Denote by $T(E_0(B))$ the ideal of E_0 defined as the intersection of the kernels of all possible homomorphisms from $F(X)$ to B. The elements of $T(E_0(B))$ are called *V-identities* of B.

Lemma 6.1. $E_0(P_n) \cong M_n(\mathbb{F})$*, where* $M_n(\mathbb{F})$ *is algebra of* $n \times n$ *matrices.*

Proof. According to Lemma 3.2, $E_0(P_n)$ annihilates the subspace $A_n + \overline{C}_n$, and C_n is an invariant subspace of P_n under its action. Consequently, $E_0(P_n)$ is isomorphic to a subalgebra L of the algebra $End_{\mathbb{F}}C_n$. Furthermore, the operator $V_{b_{ij},a_{ij}} \in E_0(P_n)$ sends the element c_i to c_j, and c_k to zero if $k \neq i$, resembling the action of a unit matrix. Therefore, the subalgebra L coincides with the entire algebra $End_{\mathbb{F}}(C_n) \cong M_n(\mathbb{F})$. □

Proposition 6.2. *If the algebra P_n has a finite basis of identities for $n \geq 2$, then the ideal $T = T(E_0(P_n))$ is generated by polynomials of bounded degrees.*

Proof. Suppose that P_n has a finite basis of identities for $n \geq 2$. By Proposition 5.5, modulo (1), (3), and (4), every identity is equivalent to a finite system of identities of (14). Consequently, by Lemma 4.2, there exists a finite set of elements $G \subseteq T$ such that the identities $tg = 0$, where $g \in G$, form a basis of identities of P_n. Let m be the maximum of the degrees of polynomials in G. By the same Lemma 4.2, the ideal T is generated by all $\varphi(g)$, where $g \in G$ and φ is linear. Consequently, T is generated by elements of degrees $\leq m$. □

7. Identities of P_2.

We are going to prove that P_2 does not have a finite basis of identities. First, let's construct some important examples of algebras.

Proposition 7.1. *For any $s > 5$ there exists an algebra $B \in \mathcal{R}$ with the following two properties:*

1. *B is generated by a set $Q = \{q_1, \ldots, q_{s+3}\}$ such that $T \nsubseteq T(E_0(B))$.*
2. *Let C be a subalgebra of B generated by any subset Q' of Q with s elements. Then*
$$tg(c_1, \ldots, c_k) = 0$$
for all $g(x_1, \ldots, x_k) \in T$, $c_1, \ldots, c_k \in C$, and $t \in B$.

Proof. Set $n = s-5 \geq 1$. Let H be the free algebra with identity in the variety of algebras generated by the field $\mathbb{F}$ with free generators $\{h_1, \ldots, h_n\}$. Denote by W the subspace of H, spanned by all words in $h_1, \ldots, h_n$, including the unit element 1, that do not contain at least one h_i. Then $W \neq H$. By Theorem 1.6 from [37], the algebra $A = H \otimes_{\mathbb{F}} P_3$ belongs to $\mathcal{R}$. Consider the subalgebra L of A generated by the following set of elements:

$$\{1 \otimes c_1, 1 \otimes a_{11}, 1 \otimes b_{11}, 1 \otimes a_{12}, 1 \otimes b_{12}, h_i \otimes a_{22}, 1 \otimes b_{22}, 1 \otimes a_{23}, 1 \otimes b_{23}\} \tag{18}$$

where $i = 1, \ldots, n$.

We note that
$$1 \otimes c_2 = -(1 \otimes b_{12})((1 \otimes a_{12})(1 \otimes c_1)),$$
$$h_j \otimes c_2 = -(1 \otimes b_{22})((h_j \otimes a_{22})(1 \otimes c_2)),$$
$$h_i h_j \otimes c_2 = -(1 \otimes b_{22})((h_i \otimes a_{22})(h_j \otimes c_2)).$$
Thus, by induction on the length of h, one can derive that $h \otimes c_2 \in L$ for any word h in $h_1, \ldots, h_n$. In addition, $h \otimes c_3 \in L$ since
$$h \otimes c_3 = -(1 \otimes b_{23})((1 \otimes a_{23})(h \otimes c_2)).$$
Note that $h \otimes c_3$ is a two-sided annihilator of L since
$$L \subseteq H \otimes (D_3 + \sum_{i \leq j \leq 3, (i,j) \neq (3,3)} (\mathbb{F}a_{ij} + \mathbb{F}b_{ij})).$$

Consequently, $N = H \otimes c_3$ and $N' = W \otimes c_3$ are ideals of L. Set $B = L/N'$ and let's show that it satisfies the properties (1) and (2) of the proposition.

Verification of Property (1). Denote by $q_1, \dots, q_{s+3}$ the images of generators of (18) under the natural projection $L \to L/N'$. By Lemma 6.1, the algebra $E_0(P_2)$ satisfies the well-known Hall's identity

$$[[\overline{f}_1, \overline{f}_2] \circ [\overline{f}_3, \overline{f}_4], \overline{f}_5] = 0,$$

for all $\overline{f}_i \in E_0(P_2)$, where $1 \le i \le 5$ and $a \circ b = ab + ba$. It follows that $S = [[f_1, f_2] \circ [f_3, f_4], f_5] \in T$ for all $f_i \in E_0$.

It is easy to choose $f_1, \dots, f_5 \in E_0$ and $\varphi : F(X) \to L$ such that

$$f_1^{\varphi} = V_{1\otimes b_{12}, 1\otimes a_{12}} \prod_{i=1}^{n} V_{1\otimes b_{22}, h_i \otimes a_{22}}, \quad f_2^{\varphi} = f_5^{\varphi} = V_{1\otimes b_{11}, 1\otimes a_{11}},$$

$$f_3^{\varphi} = V_{1\otimes b_{22}, 1\otimes a_{22}}, \quad f_4^{\varphi} = V_{1\otimes b_{23}, 1\otimes a_{23}}.$$

The actions of the operators $f_1^{\varphi}, f_2^{\varphi}, f_3^{\varphi}, f_4^{\varphi}$ on L give us

$$f_1^{\varphi} f_2^{\varphi} = 0, \quad f_2^{\varphi} f_1^{\varphi} = V_{1\otimes b_{12}, v\otimes a_{12}},$$

$$f_3^{\varphi} f_4^{\varphi} = V_{1\otimes b_{23}, 1\otimes a_{23}}, \quad f_4^{\varphi} f_3^{\varphi} = 0,$$

and we have

$$S^{\varphi} = [-V_{1\otimes b_{12}, v\otimes a_{12}} \circ V_{1\otimes b_{23}, 1\otimes a_{23}}, V_{1\otimes b_{11}, 1\otimes a_{11}}] = V_{1\otimes b_{12}, v\otimes a_{12}} V_{1\otimes b_{23}, 1\otimes a_{23}},$$

where $v = h_1 \cdots h_n$. Since

$$(1 \otimes c_1) S^{\varphi} = v \otimes c_3 \neq 0 (mod\, N'),$$

we obtain $S \notin T(E_0(B))$ and therefore $T \nsubseteq T(E_0(B))$.

Verification of Property (2). Let L' be a subalgebra of L generated by a subset of the set (18) that contains no more than s elements. Assume that $f(x_1, \dots, x_k) \in T$. Let M be the set of all elements of the form $(1 \otimes c_1) f(l_1, \dots, l_k)$, where $l_i \in L'$. We claim that

$$M \cap N \subseteq N'. \tag{19}$$

Let's assume that (19) does not hold. In other words, there is an element

$$g = (h_1 \cdots h_n \otimes c_3)(h' \otimes c_3) + h'' \otimes c_3 \in M \cap N$$

for some nonzero $h' \in H$ and some $h^{''} \in W$.

Note that

$$(1 \otimes c_1) V_{1\otimes b_{12}, 1\otimes a_{12}} = 1 \otimes c_2, \quad (1 \otimes c_2) V_{1\otimes b_{22}, h_i\otimes a_{22}} = h_i \otimes c_2,$$

$$(h_i \otimes c_2) V_{1\otimes b_{22}, h_j\otimes a_{22}} = h_i h_j \otimes c_2, \; (h_1 \cdots h_n \otimes c_2) V_{1\otimes b_{23}, 1\otimes a_{23}} = h_1 \cdots h_n \otimes c_3.$$

Without using $1 \otimes c_1$ and all of the operators

$$V_{1\otimes b_{12}, 1\otimes a_{12}}, \quad V_{1\otimes b_{22}, h_i\otimes a_{22}}, \quad V_{1\otimes b_{23}, 1\otimes a_{23}},$$

we cannot get elements containing the product $h_1 \dots h_n$. This means that $M \cap N$ contains g if and only if the following $s+1$ elements appear in our calculations:

$$1 \otimes c_1, 1 \otimes a_{12}, 1 \otimes b_{12}, h_i \otimes a_{22}, 1 \otimes b_{22}, 1 \otimes a_{23}, 1 \otimes b_{23} \quad (i = 1, \dots, n).$$

It is impossible since L' is generated by s elements. This contradiction establishes the inclusion (19).

Set $\overline{f} = f(l_1, \ldots, l_k)$ for some fixed $l_1, \ldots, l_k \in L'$. We show that $L\overline{f} = 0 \,(mod\, N')$. By Lemma 3.2, one can have that $(H \otimes (\overline{C}_3 + A_3 + \mathbb{F}c_3))E_0(L) = 0$. Then, because of (19), it is sufficient to prove that

$$(1 \otimes c_1)\overline{f} = 0(mod\, N), \quad (H \otimes c_2)\overline{f} = 0. \tag{20}$$

If $l_i = l'_i + l''_i$, where

$$l'_i \in H \otimes \sum_{i \leq j \leq 2} (\mathbb{F}a_{ij} + \mathbb{F}b_{ij}), \quad l''_i \in H \otimes (\mathbb{F}a_{23} + \mathbb{F}b_{23}),$$

then

$$(1 \otimes c_1)\overline{f} = (1 \otimes c_1)f(l'_1, \ldots, l'_k)(mod\, N). \tag{21}$$

We have

$$f(x_1 + y_1, \ldots, x_k + y_k) = f(x_1, \ldots, x_k) + g(x_1, \ldots, x_k, y_1, \ldots, y_k),$$

where every monomial of $g \in E_0$ involve at least one variable from $y_1, \ldots, y_k$. Since $LV_{l''_i,L} \subseteq N$, we obtain

$$(1 \otimes c_1)g(l'_1, \ldots, l'_k, l''_1 \ldots, l''_k) = 0(mod\, N),$$

which implies (21).

Consequently, to prove the first relation of (20), without loss of generality we can assume that $l_i \in H \otimes \sum_{i \leq j \leq 2}(\mathbb{F}a_{ij} + \mathbb{F}b_{ij})$. In this case, the elements $c_1, l_1, \ldots, l_k$ generate a subalgebra of $H \otimes P_2$. The algebra H can be embedded into the Cartesian product $\mathbb{F}^\alpha$ of the algebra $\mathbb{F}$. So, $H \otimes P_2$ can be embedded into $\mathbb{F}^\alpha \otimes P_2 \cong P_2^\alpha$ and satisfies all the identities of P_2. Consequently, it satisfies all V-identities from T. Thus, the first relation of (20) is established. The second relation of (20) can be established similarly using the equality

$$H \otimes (\sum_{j \leq 2} (\mathbb{F}a_{1j} + \mathbb{F}b_{1j}))c_2 = 0.$$

In this case, we can assume that

$$l_i \in H \otimes (\sum_{2 \leq i, j \leq 3} (\mathbb{F}a_{ij} + \mathbb{F}b_{ij})).$$

Then the elements $c_2, l_1, \ldots, l_k$ generate an algebra isomorphic to a subalgebra of $H \otimes P_2$. Thus, we have $L\overline{f} = 0(mod\, N')$. The factorization by N' completes the proof of Proposition 7.1. □

Lemma 7.2. *Let Σ be a set of generators of the ideal $T = T(E_0(P_2))$ of E_0. Then for any natural number s, there exists a polynomial $f_s \in \Sigma$ that depends on more than s variables.*

Proof. Suppose, contrary, that Σ consists of polynomials that depend on $\leq s$ variables. Let B be the algebra satisfying the conditions of Proposition 7.1. Consider the epimorphism $\tau : F(X) \to B$ defined by

$$\tau(x_i) = \begin{cases} q_i & \text{if} \quad i \leq s + 3, \\ 0 & \text{if} \quad i > s + 3. \end{cases}$$

This induces the epimorphism $\tilde{\tau} : E_0 \to E_0(B)$ defined by

$$\tilde{\tau} g(x_1, \ldots, x_k) = g(\tau(x_1), \ldots, \tau(x_k)).$$

If $g(x_1, \ldots, x_k) \in \Sigma$, then $k \leq s$ and $c_i = \tau(x_i)$ belong to a subalgebra of B generated by $\leq s$ elements. By Proposition 7.1(2), $g(c_1, \ldots, c_k) = 0$. Thus, $g \in Ker\, \tilde{\tau}$. So $Ker \tilde{\tau}$ contains Σ and, consequently, T as well.

Now let $f(x_1, \ldots, x_m) \in T$. For any $b_1, \ldots, b_m \in B$ there exist $r_i \in F(X)$ such that $b_i = \tau(r_i)$ for all $i = 1, \ldots, m$. Since $f(r_1, \ldots, r_m) \in T$, we have $f(b_1, \ldots, b_m) = \tilde{\tau} f(r_1, \ldots, r_m) = 0$. This proves that every element of T is a V-identity of B. This contradicts Proposition 7.1(1). □

Theorem 7.3. *Algebra P_2 over an arbitrary field $\mathbb{F}$ does not have a finite basis of identities.*

Proof. If P_2 has a finite basis of identities, then the ideal $T = T(E_0(P_2))$ is generated by polynomials of bounded degree by Proposition 6.2. This contradicts Lemma 7.2. □

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