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ON A VARIETY OF RIGHT-SYMMETRIC ALGEBRAS

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Abstract. We construct a finite-dimensional metabelian right-symmetric algebra over an arbitrary field that does not have a finite basis of identities.

1. Introduction.

We say that a variety of algebras has the Specht property or is Spechtian if any of its subvarieties has a finite basis of identities. In other words, a variety of algebras is Spechtian if the set of all its subvarieties satisfies the descending chain condition with respect to inclusion. In 1950 Specht [31] formulated a problem on the Specht property for the variety of all associative algebras over a field of characteristic zero.

Specialists extended the study of this problem for any varieties of algebras over fields of any characteristic. In 1970 Vaughan-Lee [36] constructed an example of a finite-dimensional Lie algebra over a field of characteristic $p = 2$ that does not have a finite basis of identities. In 1974 Drensky [8] extended this result to fields of any positive characteristic $p > 0$. In 1978 Medvedev [25] showed that varieties of metabelian Malcev, Jordan, alternative, and $(-1,1)$ algebras are Spechtian. In 1984 Umirbaev [33] proved that the variety of metabelian binary Lie algebras over a field of characteristic $\neq 3$ has the Specht property. In 1980 Medvedev [26] also constructed an example of a variety of solvable alternative algebras over a field of characteristic $2$ with an infinite basis of identities. In 1985 Umirbaev [34] proved that the varieties of solvable alternative algebras over a field of characteristic $\neq 2, 3$ have the Specht property. Pchelintsev [27] constructed an almost Spechtian variety of alternative algebras over a field of characteristic $3$. The Specht property of so-called bicommutative algebras is proven in [9].

In 1976 Belkin [1] proved that the variety of metabelian right-alternative algebras does not have the Specht property. In 1978 L’vov [24] constructed a six-dimensional nonassociative algebra over an arbitrary field satisfying the identity $x(yz) = 0$ with an infinite basis of identities. In 1986 Isaev [15] adapted L’vov’s methods for right-alternative algebras and constructed a finite-dimensional metabelian right-alternative algebra over an arbitrary field with an infinite basis of identities. In 2008 Kuz’min [22] gave a sufficient condition for the varieties of metabelian right-alternative algebras over a field of characteristic $\neq 2$ to be Spechtian.

In 1988 Kemer [16, 17] positively solved the famous Specht problem [31] and proved that every variety of associative algebras over a field of characteristic zero has a finite basis of identities. Later the Specht problem was negatively solved for the variety of associative algebras over fields of positive characteristic $p > 0$ [2, 13, 28]. It is also

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known that the varieties of Lie algebras generated by a finite-dimensional algebra over a field of characteristic zero have the Specht property [14, 18]. Despite the efforts of many specialists in this field, the question of whether the variety of Lie algebras over a field of characteristic zero has the Specht property remains open.

This paper is devoted to the study of the Specht property for the variety of right-symmetric algebras. Recall that an algebra $A$ over a field $F$ is called right-symmetric if it satisfies the identity

\[(a, b, c) = (a, c, b),\]

where $(a, b, c) = (ab)c - a(bc)$ is the associator of $a, b, c \in A$.

Right-symmetric algebras are Lie admissible, that is, any right-symmetric algebra with respect to the commutator $[x, y] = xy - yx$ is a Lie algebra. Very often right-symmetric (or left-symmetric) algebras are called pre-Lie algebras and play an important role in the theory of operads [23]. Right-symmetric algebras arise in many different areas of mathematics and physics [3].

In 1994 Segal [30] constructed a basis of free right-symmetric algebras. Chapoton and Livernet [5] and, independently, Löfwall and Dzhumadil’daev [11] gave other bases of free right-symmetric algebras in terms of rooted trees. The identities of right-symmetric algebras were studied by Filippov [12], and he proved that any right-nil right-symmetric algebra over a field of characteristic zero is right nilpotent. An analogue of the PBW basis Theorem for the universal (multiplicative) enveloping algebra of a right-symmetric algebra was given in [19]. The Freiheitssatz and the decidability of the word problem for one-relator right-symmetric algebras were proven in [20]. Recently, Dotsenko and Umirbaev [7] determined that the variety of right-symmetric algebras over a field of characteristic zero is Nielson-Schreier, that is, every subalgebra of a free right-symmetric algebra is free.

A right-symmetric algebra with an additional identity

\[a(bc) = b(ac)\]

is called a Novikov algebra. The class of Novikov algebras is an important and well-studied subclass of right-symmetric algebras. Recently there was great progress in the study of identities, solvability, and nilpotency [38, 12, 10, 29, 35, 32]. In 2022 Dotsenko, Ismailov, and Umirbaev [6] proved that (a) every Novikov algebra satisfying a nontrivial polynomial identity over a field of characteristic zero is right-associator nilpotent and (b) the variety of Novikov algebras over a field of characteristic zero has the Specht property.

In this paper, we continue the study of the identities of right-symmetric algebras. Namely, using the constructions and methods of L’vov [24] and Isaev [15], we construct a finite-dimensional metabelian right-symmetric algebra over an arbitrary field that does not have a finite basis of identities. In fact, our algebra belongs to the variety of algebras $\mathcal{R}$ defined by the identities

\[(2) \quad [[a, b], c] = 0,\]

\[(3) \quad (ab)a = 0,\]
and
\[(4) \quad (ab)(cd) = 0.\]

We determine some identities and operator identities of the variety \( \mathcal{R} \) in Section 2. In Section 3, a series of algebras \( P_n \) of this variety is constructed. A linear basis of free algebras of the variety \( \mathcal{R} \) is constructed in Section 4. Section 5 is devoted to the study of the relationships between the polynomial identities and the operator identities of the algebras \( P_n \). The main result of the paper is given in Section 6 and says that the algebra \( P_2 \) does not have any finite basis of identities.

2. A VARIETY OF RIGHT-SYMMETRIC ALGEBRAS

Let \( \mathbb{F} \) be an arbitrary fixed field. In what follows, all vector spaces are considered over \( \mathbb{F} \). As above, \( \mathcal{R} \) denotes the variety of algebras defined by the identities (2), (3), and (4).

**Lemma 2.1.** Every algebra of the variety \( \mathcal{R} \) is right-symmetric and right nilpotent of index 4.

**Proof.** The linearization of (3) gives
\[(5) \quad (ab)c + (cb)a = 0.\]
This identity and (4) imply that
\[(6) \quad ((ab)c)d = -(dc)(ab) = 0.\]
Using (3) and (2) one can also get
\[(a, b, c) - (a, c, b) = (ab)c - a(bc) - (ac)b + a(cb)\]
\[= -(cb)a - a(bc) + (bc)a + a(cb)\]
\[= [b, c]a - a[b, c] = [[b, c], a] = 0,\]
i.e., \( \mathcal{R} \) is a variety of right-symmetric algebras. \( \square \)

Let \( A \) be an arbitrary algebra of the variety \( \mathcal{R} \). Recall that for any \( x \in A \) the operators of right multiplication \( R_x \) and left multiplication \( L_x \) on \( A \) are defined by
\[aR_x = ax \quad \text{and} \quad aL_x = xa,\]
respectively. Set also \( V_{x,y} = L_xR_y \).

**Lemma 2.2.**
\[(7) \quad V_{x,x} = 0, \quad V_{x,y} = -V_{y,x}.\]
\[(8) \quad xR_yL_zL_t = yV_{x,z}L_t - xR_yV_{t,z}.\]
\[(9) \quad xR_yL_z = xV_{z,y} + yR_zL_x - yV_{x,z}.\]
\[(10) \quad xR_yV_{z,t} = yR_zV_{z,t}.\]
\[(11) \quad V_{x,y}R_z = 0.\]
\[(12) \quad V_{x,y}(L_zL_t + V_{t,z}) = 0.\]
Proof. The identities (3) and (6) immediately imply (7). By (1) and (4) we get
\[ xR_yL_zL_t = t(z(xy)) \]
\[ = (tz)(xy) + t((xy)z) - (t(xy))z = yV_{x,z}L_t - xR_yV_{t,z}. \]
From the identity (2) follows (9).

Then (1) and (6) give that
\[ xR_yV_{z,t} = (z(xy))t = ((zx)y)t + (z(yx))t = yV_{x,y}R_t, \]
By (1), we obtain
\[ tV_{x,z}R_t = ((xt)z)t = 0, \]
and, therefore,
\[ V_{x,y}R_z = 0. \]
Set 
\[ V_{x,y}R_z = 0. \]
Then
\[ (a_{ij}c_i)a_{ij} = d_{ij}a_{ij} = 0. \]

3. Algebras \( P_n \)

For each natural \( n \) we define the algebra \( P_n \) with a linear basis
\[ a_{ij}, b_{ij}, c_i, d_{ij}, e_{ij}, \]
where \( i, j \in \{1, 2, \ldots, n\} \), and with the product defined by
\[ a_{ij}c_i = d_{ij}, \quad b_{ij}c_i = e_{ij}, \]
\[ a_{ij}e_{ij} = e_{ij}a_{ij} = -b_{ij}d_{ij} = -d_{ij}b_{ij} = c_j, \]
where all zero products are omitted.

Set
\[ A_n = \text{Span}\{a_{ij}, b_{ij} | 1 \leq i, j \leq n\} \]
and
\[ D_n = \text{Span}\{c_i, d_{ij}, e_{ij} | 1 \leq i, j \leq n\}, \]
where \( \text{Span} X \) denotes the linear span of \( X \). Then \( A_n \) is a subalgebra of \( P_n \) and \( D_n \) is an ideal of \( P_n \). Moreover, \( P_n \) is a direct sum of the vector spaces \( A_n \) and \( D_n \). Set also
\[ C_n = \text{Span}\{c_i | 1 \leq i \leq n\}, \quad \overline{C}_n = \text{Span}\{d_{ij}, e_{ij} | 1 \leq i, j \leq n\}. \]

Then
\[ P_n^2 = D_n, \quad A_n^2 = D_n^2 = 0, \quad D_n = C_n \oplus \overline{C}_n, \]
\[ D_nP_n = C_n, \quad P_nC_n = \overline{C}_n, \quad C_nP_n = 0, \quad P_n\overline{C}_n = C_n. \]

Lemma 3.1. The algebra \( P_n \) belongs to the variety \( \mathcal{R} \).

Proof. Obviously the space of commutators \([P_n, P_n]\) coincides with \( \overline{C}_n \), which is in the center of \( P_n \), i.e., (2) holds.

In order to verify the identity (3), it is sufficient to check the identities (3) and (5) for all elements of the basis of \( P_n \). Let us begin with (3). Since \( A_n^2 = D_n^2 = (D_nA_n)D_n = 0 \), we may assume that \( a \in A_n \) and \( b \in D_n \). Consider all nonzero products of the space \( A_nD_n \). If \( a = a_{ij} \) and \( b = c_i \), then
\[ (a_{ij}c_i)a_{ij} = d_{ij}a_{ij} = 0. \]
Lemma 3.2. For all $a, b \in A$ and $c \in B$, the relations (13) give that $(ab)c = (a(bc))$. The other cases can be verified similarly.

Proof. Assume that $a = a_{ij}$, $b = c_i$, $c = b_{ij}$ or $a = b_{ij}$, $b = c_i$, $c = a_{ij}$. Thus,

$$(ab)c + (cb)a = -c_i + c_j = 0.$$ 

From the relations $P_n^2 = D_n$ and $D_n^2 = 0$ immediately follow the identity (4). □

Lemma 3.3. Ann$_iP_n$ is the space of left annihilators of $P_n$.

Proof. Assume that $x \in (A_n + C_n + C_n) \cap$ Ann$_iP_n$ and express it as

$$x = \sum_{i,j} (\alpha_{ij} a_{ij} + \beta_{ij} b_{ij} + \gamma_{ij} d_{ij} + \delta_{ij} e_{ij} + \epsilon_i c_i),$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \epsilon_i \in F$. Then we have

$$xc_i = \sum_{j} (\alpha_{ij} d_{ij} + \beta_{ij} e_{ij}), \quad xa_{ij} = \delta_{ij} c_j, \quad xb_{ij} = -\gamma_{ij} c_j.$$ 

From these equations, it can be deduced that $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = \delta_{ij} = 0$. Therefore, we can conclude that $x = \sum_i \epsilon_i c_i$. Consequently, Ann$_iP_n = C_n$. □

4. Structure of free algebras of $R$

Let $F(X)$ be the free algebra of the variety $R$ generated by an infinite countable set $X = \{x_1, x_2, \ldots, x_n, \ldots\}$.

Proposition 4.1. The set of elements $B$ of $F(X)$ of the forms

$$x_i, \quad x_i \hat{R}_{x_j} L_{x_s}, \quad x_i \hat{R}_{x_j} V_{x_{p_1},x_{q_1}} \cdots V_{x_{p_k},x_{q_k}} \hat{L}_{x_s},$$

where $i < j$ and $p_r < q_r$ for all $r = 1, 2, \ldots, k$, $k \geq 1$, and $\hat{T}_x$ denotes that the operator $T_x$ might not occur, is a basis of $F(X)$.

Proof. In order to show that $B$ linearly spans $F(X)$ it is sufficient to verify that, for any $v \in B$, the elements $v R_{x_r}$ and $v L_{x_r}$ belong to the linear span of $B$. This is easy to do using the identities (1), (4), and Lemma 2.2. For example, let

$$v = x_i \hat{R}_{x_j} V_{x_{p_1},x_{q_1}} \cdots V_{x_{p_k},x_{q_k}} L_{x_s}.$$ 

Then

$$v R_{x_r} = x_i \hat{R}_{x_j} V_{x_{p_1},x_{q_1}} \cdots V_{x_{p_k},x_{q_k}} L_{x_s} R_{x_r} = x_i \hat{R}_{x_j} V_{x_{p_1},x_{q_1}} \cdots V_{x_{p_k},x_{q_k}} V_{x_s,x_r}.$$
By (12), we get
\[ vL_{x_i} = x_i \tilde{R}_{x_j} V_{x_{p_1}, x_{q_1}} \cdots V_{x_{p_k}, x_{q_k}} L_{x_s} = -x_i \tilde{R}_{x_j} V_{x_{p_1}, x_{q_1}} \cdots V_{x_{p_k}, x_{q_k}} V_{x_r, x_s}. \]

Applying (7) we can express \( vR_{x_i} \) and \( vL_{x_i} \) as a linear combination of elements of \( B \).

It remains to prove the linear independence of elements of \( B \). Suppose that \( f = f(x_1, x_2, \ldots, x_n) \in F(X) \) is a nontrivial linear combination of elements of \( B \). Suppose that \( v \in B \) and \( \deg_x(v) = k \). Let’s write \( v = v(x_i, \ldots, x_j) \) in order to differ the presence of \( x_j \) in different places. To linearize \( v \) in \( x_j \), we use new variables \( y_1, \ldots, y_k \in X \) and, after renumbering, we can assume that \( y_r < x_j \) if \( i < j \) and \( x_j < y_r \) if \( j < i \) for all \( 1 \leq r \leq k \).

Notice that every word \( v(y_{\sigma(1)}, \ldots, y_{\sigma(k)}) \), where \( \sigma \in S_k \) and \( S_k \) is the symmetric group in \( k \) symbols, is an element of \( B \). Then the full linearization of \( v \) in \( x_i \) is a linear combination of basis elements \( v(y_{\sigma(1)}, \ldots, y_{\sigma(k)}) \). Therefore, by linearizing a nontrivial element \( f \), we obtain a nontrivial element that is a linear combination of multilinear elements from \( B \). Substituting zeroes instead of some variables, if necessary, we can make \( f \) linear in each variable. Therefore, we can assume that \( f \) is a multilinear nontrivial identity in the variables \( x_1, \ldots, x_n \). Let
\[ f = \sum_{i=1}^{n} \alpha_i u_i, \]
where \( \alpha_i \in F \) and \( u_i \in B \). Suppose, for example, that
\[ u_1 = x_i R_{x_j} V_{x_{p_1}, x_{q_1}} \cdots V_{x_{p_k}, x_{q_k}} L_{x_s}. \]

Set \( x_i = d_{1,2}, x_j = -b_{1,2}, x_{p_r} = a_{r+1, r+2}, x_{q_r} = -b_{r+1, r+2} \) for all \( r = 1, 2, \ldots, k \), \( x_s = a_{k+2, k+3} \). We have \( c_i V_{a_{ij}, -b_{ij}} = c_j \) for all \( i, j \). Then the value of \( u_1 \) under this substitution is \( d_{k+2, k+3} \) and the value of any other \( u_i \) is 0. Consequently, the value of \( f \) is \( \alpha_1 d_{k+2, k+3} \neq 0 \). Thus, \( f \) is not an identity for \( R \).

If \( L_{x_s} \) does not appear in \( u_1 \), then we perform the same substitutions for the variables. If \( R_{x_j} \) does not appear in \( u_1 \), then we simply set \( x_i = c_2 \) and perform the same substitutions for the rest of the variables as described above. In both cases the value of \( f \) is nonzero. This completes our proof. \( \square \)

Let \( M = M(F(X)) \) be the multiplication algebra of the algebra \( F(X) \). Denote by \( E_0 \) the subalgebra (without identity) of \( M \) generated by the operators \( V_{i, j} = V_{x_i, x_j} \) with \( i < j \) for all \( i, j = 1, 2, \ldots \). Set also
\[ E_1 = \sum_{j \geq 1} E_0 L_{x_j}, \quad E_2 = \sum_{i \geq 1} R_{x_i} E_0, \quad E_3 = \sum_{i, j \geq 1} R_{x_i} E_0 L_{x_j}, \]
and
\[ R_k = \sum_{i \geq 1} x_i E_k, \quad k = 0, 1, 2, 3. \]

According to Proposition 4.1, the space \( F(X) \) is the direct sum of the subspaces \( R_k \) and the linear span of elements of \( B \) of degrees less than or equal to 3.

Lemma 4.2. An identity \( z f(x_1, \ldots, x_m) = 0 \), where \( f \in E_0 \), is a consequence of a system of identities
\[ (14) \quad t g_j(x_1, \ldots, x_i) = 0, \quad g_j \in E_0, \quad j \in J, \]
in the variety $\mathcal{R}$, where $\mathcal{J}$ is any set of indices, if and only if the operator $f(x_1, \ldots, x_m)$ belongs to the ideal of the associative algebra $E_0$ generated by the set $G$ of all operators $\varphi(g_j)$, where $\varphi$ runs over the set of all linear endomorphisms $\varphi : X \to \mathbb{F}X = \sum_{i \geq 1} \mathbb{F}x_i$ and $j \in \mathcal{J}$.

Proof. Suppose that $f$ belongs to the ideal of $E_0$ generated by $G$. Then

$$f = \sum_{r=1}^{t} u_r g_{j_r}^{r} v_r,$$

for some linear endomorphisms $\varphi_r$ and $u_r, v_r \in E_0$. Therefore,

$$zf = \sum_{r=1}^{t} (zu_r) g_{j_r}^{r} v_r$$

and $zf = 0$ is a consequence of the system of identities (14).

Let's describe all the consequences of the identities (14). Let $\varphi : F(X) \to F(X)$ be an arbitrary endomorphism and set $\varphi(x_i) = y_i + h_i$, where $y_i \in \mathbb{F}X$ and $h_i \in F(X)^2$ for all $i$. Since $g_j \in E_0$, using (4) and (6), we get

$$t\varphi(g_j) = tg_j(y_1, \ldots, y_t) = 0.$$

Thus, a general form of consequences of the identities (14) can be expressed as

$$\sum_{r=1}^{t} u_r g_{j_r}^{r} v_r,$$

where $u_r \in F(X)$, $v_r \in M(F(X))$, and $\varphi_r$ are linear endomorphisms. We know that $g_{j_r}^{r} \in E_0$. We also claim that $u_r$ and $v_r$ can be represented in the forms

$$x_i \hat{R}_x, V_{x_1} x_1, V_{x_2} x_2, \ldots, V_{x_k} x_k,$$

respectively, where $i < j$, $p_i < q_i$, $p_i' < q_i'$ and $k = 0, 1, \ldots$.

Suppose that $u_r$ is a basis element that ends with $L_{x_k}$. Then, by (11) and (12), we can derive that

$$V_{x_1} x_1 L_{x_1} V_{x_k} x_k = V_{x_1} x_1 L_{x_1} L_{x_k} R_{x_k} = -V_{x_1} x_1 V_{x_k} x_k R_{x_k} = 0.$$

Consequently, we have $V_{x_1} x_1 L_{x_1} E_0 = 0$.

If $u_r = x_i R_{x_j} L_{x_k}$, then by (8) and (11) we get

$$u_r V_{x_1} x_1 = x_i R_{x_j} L_{x_k} R_{x_k} = x_j V_{x_1} x_k L_{x_k} R_{x_k} - x_i R_{x_j} V_{x_1} x_k R_{x_k} = x_j V_{x_1} x_k V_{x_2} x_k.$$

So, we can conclude that $u_r$ has the claimed form.

Now, let’s consider the case when $v_r$ is a basis element that starts with $R_y$. According to (11), we have $E_0 R_y = 0$. If $v_r$ starts with $L_{x_1} L_{x_j}$, then by using (12), we find

$$V_{x_1} x_1 L_{x_1} L_{x_j} = -V_{x_1} x_1 V_{x_j} x_j.$$

Hence, $v_r$ also has the claimed form.

If $zf = 0$ is a consequence of the identities (14), then we get an equality of the form

$$x_{m+1} f(x_1, \ldots, x_m) = \sum_{r=1}^{t} \lambda_r x_i, w_r g_{j_r}^{r} v_r,$$
where \( x_i, w_r = u_r, w_r \in E_0 + E_2 \) and \( v_r \in E_0 + E_1 \). Notice that every element \( x_i, w_r g_{jk}^{fr} v_r \) belongs to \( \mathcal{B} \). Consequently, we may assume that \( x_i = x_{m+1}, w_r, v_r \in E_0 \), and

\[
f(x_1, \ldots, x_m) = \sum_{r=1}^{t} \lambda_r w_r g_{jk}^{fr} v_r.
\]

\( \Box \)

5. IDENTITIES OF \( P_n \).

In this section, we study the connections between the identities and the operator identities of \( P_n \) for \( n \geq 2 \).

**Lemma 5.1.** If \( f \in F(X) \) and \( f = 0 \) is an identity of \( P_n \) for \( n \geq 2 \), then

\[
f = f_0 + f_1 + f_2 + f_3 \in F(X), \quad f_k \in R_k,
\]

and \( f_k = 0 \) is an identity of \( P_n \) for all \( k = 0, 1, 2, 3 \).

**Proof.** Let

\[
f = \sum_{i=1}^{m} \lambda_i x_i + \sum_{i,j=1}^{m} \lambda_{ij} x_i x_j + \sum_{i,j,k=1}^{m} \lambda_{ijk} x_i R_{ij} L_{jk} + f',
\]

where \( f' \) is a linear combination of elements from \( \mathcal{B} \) of degree \( \geq 4 \).

We first show that \( \lambda_i = \lambda_{ij} = \lambda_{ijk} = 0 \) for all \( i, j, k = 1, \ldots, m \). For any fixed \( i \) the substitution \( x_i = c_1 \) and \( x_j = 0 \) for all \( j \neq i \) gives that \( \lambda_i c_1 = 0 \), which implies \( \lambda_i = 0 \).

If \( i \neq j \) then the substitution \( x_i = a_{11}, x_j = c_1, \) and \( x_k = 0 \) for all \( k \neq i, j \), makes the value of \( f \) equal to \( \lambda_{ij} d_{11} = 0 \). The same value we get if \( i = j \) under the substitution \( x_i = x_j = a_{11} + c_1 \) and \( x_k = 0 \) for all \( k \neq i, j \). This gives \( \lambda_{ij} = 0 \) in both cases.

Assume that \( i < j < k \). If \( i \neq k \), then the substitution \( x_i = b_{11}, x_j = d_{11}, \) \( x_k = a_{12}, \) and \( x_t = 0 \) for all \( t \neq i, j, k \), makes the value of \( f \) equal to \( -\lambda_{ijk} d_{12} \). This gives that \( \lambda_{ijk} = 0 \). If \( i = k \), then the substitution \( x_i = b_{11}, x_j = d_{11}, \) \( x_t = 0 \) for all \( t \neq i, j \), gives that \( -\lambda_{ij} e_{11} = 0 \) and \( \lambda_{ij} = 0 \). If \( i < j = k \), then the substitution \( x_i = d_{11}, x_j = b_{11}, \) and \( x_t = 0 \) for all \( t \neq i, j \), gives that \( -\lambda_{ij} e_{11} = 0 \) and \( \lambda_{ij} = 0 \). Finally, if \( i < j < k \), then the substitution \( x_i = d_{11}, x_j = x_k = b_{11}, \) and \( x_t = 0 \) for all \( t \neq i, j \), gives that \( -\lambda_{ijk} e_{11} = 0 \), i.e., \( \lambda_{ijk} = -\lambda_{ikj} = 0 \).

Thus, \( f \) is a linear combination of elements of \( \mathcal{B} \) of degree \( \geq 4 \). Suppose that \( f \) is written as in (15). Taking into account the relations \( D_n P_n \subseteq C_n \) and \( P_n C_n \subseteq \overline{C}_n \) it can be observed that the images of \( F_0 = f_0 + f_2 \) and \( F_1 = f_1 + f_3 \) belong to \( C_n \) and \( \overline{C}_n \), respectively. Therefore, if \( f = 0 \) is an identity of \( P_n \), then \( F_0 = 0 \) and \( F_1 = 0 \) are also identities of \( P_n \).

Suppose that

\[
f_k(x_1, \ldots, x_m) = \sum_{i=1}^{m} x_i g_i^{(k)}(x_1, \ldots, x_m),
\]

where \( g_i^{(k)} \in E_k \) and \( k = 0, 1, 2, 3 \). Let \( p_1, \ldots, p_m \in P_n \) with \( p_s = v_s + \overline{v}_s + a_s \), where \( v_s \in C_n, \overline{v}_s \in \overline{C}_n, a_s \in A_n \). By Lemma 3.2 we can obtain that

\[
p_p V_p, p_k = v_i V_{p_j, p_k} = v_i V_{v_j + \overline{v}_j + a_j, v_k + \overline{v}_k} + v_i V_{v_j + \overline{v}_j, a_k} + v_i V_{a_j, a_k} = v_i V_{a_j, a_k}.
\]
Then we can write as

\[ f_0(p_1, \ldots, p_m) = \sum_{i=1}^{m} v_i g_i^{(0)}(a_{1}, \ldots, a_{m}) = f_0(v_1 + a_1, \ldots, v_m + a_m). \]

It is easy to note that \((A_n + C_n)R_{a_i + v_j}V_{a_j + v_j, a_s + v_s} = 0\) for all \(a_i, a_j, a_s \in A_n\) and \(v_i, v_j, v_s \in C_n\). It follows that

\[ f_2(v_1 + a_1, \ldots, v_m + a_m) = 0. \]

Thus,

\[ f_0(p_1, \ldots, p_m) = f_0(v_1 + a_1, \ldots, v_m + a_m) + f_2(v_1 + a_1, \ldots, v_m + a_m) \]

\[ = F_0(v_1 + a_1, \ldots, v_m + a_m) = 0. \]

Therefore, we can conclude that \(f_0 = 0\) and \(f_2 = 0\) are identities of \(P_n\). Similarly, we can establish that \(f_1 = 0\) and \(f_3 = 0\) are also identities of \(P_n\). \(\square\)

**Lemma 5.2.** If \(f = f(x_1, \ldots, x_m) \in R_1 + R_3\), then \(f x_{m+1} \in R_0 + R_2\) and if \(f(x_1, \ldots, x_m) x_{m+1} = 0\) is an identity of \(P_n\), then \(f = 0\) is an identity of \(P_n\) as well.

**Proof.** We have \(f x_{m+1} \in R_0 + R_2\) by the definition of the spaces \(R_i\), where \(0 \leq i \leq 3\). If \(f x_{m+1} = 0\) is an identity of \(P_n\), then all values of \(f\) in \(P_n\) belong to \(C_n = Ann_P(\mathbb{F})\) by Lemma 3.3. However, since \(f\) is an element of \(R_1 + R_3\), the values of \(f\) must belong to \(C_n\). Consequently, \(f = 0\) is an identity of \(P_n\). \(\square\)

Recall an exact formal definition of the linearization of identities [37, Chapter 1]. Let \(\mathcal{V}\) be an arbitrary variety of algebras and \(\mathbb{F}(X)\) be its free algebra over \(\mathbb{F}\) generated by \(X = \{x_1, x_2, \ldots\}\). Let \(y \in \mathbb{F}(X)\) be an arbitrary fixed element. For a nonnegative integer \(k\), we define the linear mapping \(\Delta_{x_i}^k(y)\) on \(\mathbb{F}(X)\) as follows:

- \(\Delta_{x_i}^0(y)\) is the identity mapping;
- \(x_i \Delta_{x_i}^k(y) = 0\), if either \(k > 1\) or \(k = 1\), \(i \neq s\);
- \(x_i \Delta_{x_i}^1(y) = y\);
- \((uv) \Delta_{x_i}^k(y) = \sum_{r+s=k} (u \Delta_{x_i}^r(y))(v \Delta_{x_i}^s(y))\),

where \(x_i \in X\) and \(u, v\) are any monomials in \(\mathbb{F}(X)\). We also write \(\Delta_{x_i}(y)\) instead of \(\Delta_{x_i}^1(y)\).

**Lemma 5.3.** Suppose that \(f = f(x_1, \ldots, x_m) \in R_2\). Then \(f \Delta_{i}(x_{m+1}x_{m+2}) \in R_0\) for all \(1 \leq i \leq m\). Moreover, \(f = 0\) is an identity of \(P_n\) if and only if \(P_n\) satisfies the following system of identities

\[ f(x_1, \ldots, x_m) \Delta_i(x_{m+1}x_{m+2}) = 0, \quad 1 \leq i \leq m. \]  

**Proof.** Let \(w = x R_y V_{z_{i_1,t_1}} \cdots V_{z_{i_r,t_r}} \in \mathcal{B}\) and \(u, v \in X\). We have

\[ (x R_y) \Delta_x(x v) = (u v) R_y = v V_{u,y}. \]

By (1), (4) and (6), we get

\[ (x R_y V_{z_{i_1,t_1}}) \Delta_y(x v) = (z_1(x(v)))t_1 = ((z_1(x))(u v) - (z_1(u v)))x + z_1((u v)x))t_1 \]

\[ = -((z_1(u v))x)t_1 + (z_1((u v)x))t_1 = (z_1((u v)x))t_1 = v V_{u,x} V_{z_{i_1,t_1}}. \]
Hence, taking into account the relations \( A \).

Proof. For a fixed \( g \) where \( g \in R_2 \), then \( f(x_1, \ldots, x_m) \Delta_i(x_{m+1}, x_{m+2}) \in R_0 \).

If \( p_1, \ldots, p_m \in P_n \) and \( v_1, \ldots, v_m \in D_n \), then we have

\[
(17) \quad f(p_1 + v_1, \ldots, p_m + v_m) = f(p_1, \ldots, p_m) + \sum_{i=1}^{m} f(p_1, \ldots, p_m) \Delta_i(v_i).
\]

In fact, by Lemma 1.3 from [37], the relation

\[
f(x_1 + y_1, \ldots, x_m + y_m) = \sum_{i_1\ldots,i_m \geq 0} f \Delta_{i_1}^{x_1}(y_1) \cdots \Delta_{i_m}^{x_m}(y_m)
\]

\[
= f(x_1, \ldots, x_m) + \sum_{i=1}^{m} f(x_1, \ldots, x_m) \Delta_i(y_i) + g,
\]

where \( y_1, \ldots, y_m \notin \{x_1, \ldots, x_m\} \) are distinct variables and the degree of \( g \) in the variables \( y_1, \ldots, y_m \) is greater than one, holds in \( F(X) \). By substituting \( x_i = p_i, y_i = v_i \) and using the fact that \( D_n^2 = 0 \), one can obtain the relation (17).

If \( f = 0 \) is an identity of \( P_n \), then the relation (17) implies that

\[
f(p_1, \ldots, p_m) \Delta_i(v) = f(p_1, \ldots, p_i + v, \ldots, p_m) - f(p_1, \ldots, p_m) = 0
\]

for all \( p_i \in P_n \) and \( v \in D_n \). In other words, the algebra \( P_n \) satisfies the system of identities (16).

Conversely, suppose that the system of identities (16) holds in \( P_n \). Assume that \( p_1, \ldots, p_m \in P_n \) of the form \( p_i = a_i + v_i \), where \( a_i \in A_n \) and \( v_i \in D_n \). Then using the relation (17), we have

\[
f(p_1, \ldots, p_m) = f(a_1 + v_1, \ldots, a_m + v_m)
\]

\[
= f(a_1, \ldots, a_m) + \sum_{i=1}^{m} f(p_1, \ldots, p_m) \Delta_i(v_i) = f(a_1, \ldots, a_m).
\]

Considering \( A_n^2 = 0 \) and \( f \in R_2 \subseteq F(X)^2 \), we can conclude that \( f(a_1, \ldots, a_m) = 0 \). Consequently, \( f(p_1, \ldots, p_m) = 0 \).

Lemma 5.4. If \( f = f(x_1, \ldots, x_m) \in R_0 \) and \( f = 0 \) is an identity of \( P_n \) of the form

\[
f = \sum_{i=1}^{m} x_i g_i,
\]

where \( g_i \in E_0 \), then \( x_{m+1} g_i = 0 \) is an identity of \( P_n \).

Proof. For a fixed \( i \) set \( x_i = v + a_i \) and \( x_j = a_j \) for all \( j \neq i \), where \( v \in D_n \) and \( a_j \in A_n \). Taking into account the relations \( A_n^2 = D_n^2 = 0 \) and Lemma 3.2, one can have

\[
f(x_1, \ldots, x_m) = v g_i(a_1, \ldots, a_m) = 0.
\]

Hence, \( x_{m+1} g_i = 0 \) is an identity of \( P_n \).
Proposition 5.5. For an arbitrary polynomial \( f = f(x_1, \ldots, x_m) \in F(X) \) there exist \( t(m) = 2m(m+3) \) polynomials \( g_i(x_1, \ldots, x_{m+3}) \in E_0 \), where \( i = 1, \ldots, t(m) \), such that \( f(x_1, \ldots, x_m) = 0 \) is an identity of \( P_n \) for \( n \geq 2 \) if and only if \( P_n \) satisfies the system of identities

\[
zg_i(x_1, \ldots, x_{m+3}) = 0, \quad 1 \leq i \leq t(m).
\]

Proof. Let \( f = f(x_1, \ldots, x_m) \in F(X) \) and suppose that \( f = 0 \) is an identity of \( P_n \). Then by Lemma 5.1 we obtain

\[
f = f_0 + f_1 + f_2 + f_3, \quad f_k \in R_k,
\]

and \( f_k = 0 \) is an identity of the algebra \( P_n \).

By Lemma 5.4, the identity \( f_0 = \sum_{i=1}^{m} x_ig_i = 0 \) is equivalent to the system of \( m \) identities \( x_{m+1}g_i = 0 \) of \( P_n \), where \( 1 \leq i \leq m \).

By Lemma 5.2, the identity \( f_1 = 0 \) is equivalent to \( f_1x_{m+1} = 0 \) and \( f_1x_{m+1} \in R_0 \).

Moreover, if

\[
f_1x_{m+1} = \sum_{i=1}^{m} x_ig_i, \quad g_i \in E_0,
\]

then, by Lemma 5.4, the identity \( f_1x_{m+1} = 0 \) is equivalent to the system of \( m \) identities \( x_{m+2}g_i = 0 \) of \( P_n \), where \( 1 \leq i \leq m \).

By Lemma 5.3, the identity \( f_2 = 0 \) is equivalent to the system of \( m \) identities \( f_2(x_1, \ldots, x_m)\Delta_i(x_{m+1}x_{m+2}) = 0 \), where \( i = 1, \ldots, m \), and we have \( f_2\Delta_i(x_{m+1}x_{m+2}) \in R_0 \).

Hence, by Lemma 5.4, it is equivalent to a system of \( m(m+2) \) identities of the form \( x_{m+3}g_i = 0 \), where \( g_i(x_1, \ldots, x_{m+2}) \in E_0 \) and \( i = 1, \ldots, m(m+2) \).

By Lemma 5.2, the identity \( f_3 = 0 \) is equivalent to \( f_3x_{m+1} = 0 \) and \( f_3x_{m+1} \in R_2 \). The identity (4) implies that \( (f_3x_{m+1})\Delta_{m+1}(x_{m+2}x_{m+3}) = 0 \). Then, by Lemma 5.3, \( f_3 = 0 \) is equivalent to the system of \( m \) identities \( 0 = (f_3x_{m+1})\Delta_i(x_{m+2}x_{m+3}) \in R_0 \), where \( i = 1, \ldots, m \).

Moreover, by Lemma 5.4, it is equivalent to a system of \( m(m+2) \) identities of the form \( x_{m+4}g_j = 0 \), where \( g_j(x_1, \ldots, x_{m+3}) \in E_0 \) and \( 1 \leq j \leq m(m+2) \).

Thus, \( f = 0 \) is equivalent to a system of \( t(m) = 2m(m+3) \) identities of the form \( zg_i(x_1, \ldots, x_{m+3}) = 0 \), where \( g_i(x_1, \ldots, x_{m+3}) \in E_0 \) and \( i = 1, \ldots, t(m) \).

6. \( V \)-identities of \( P_n \).

Let \( B \) be an arbitrary algebra in \( \mathcal{R} \). We define \( E_0(B) \) as the algebra of operators generated by \( V_{b_1,b_2} \) for all \( b_1,b_2 \in B \), that acts on the algebra \( B \). Denote by \( T(E_0(B)) \) the ideal of \( E_0 \) defined as the intersection of the kernels of all possible homomorphisms from \( F(X) \) to \( B \). The elements of \( T(E_0(B)) \) are called \( V \)-identities of \( B \).

Lemma 6.1. \( E_0(P_n) \cong M_n(\mathbb{F}) \), where \( M_n(\mathbb{F}) \) is algebra of \( n \times n \) matrices.

Proof. According to Lemma 3.2, \( E_0(P_n) \) annihilates the subspace \( A_n + \overline{C}_n \), and \( C_n \) is an invariant subspace of \( P_n \) under its action. Consequently, \( E_0(P_n) \) is isomorphic to a subalgebra \( L \) of the algebra \( End_\mathbb{F}C_n \). Furthermore, the operator \( V_{b_1,b_2} \in E_0(P_n) \) sends the element \( c_i \) to \( c_j \), and \( c_k \) to zero if \( k \neq i \), resembling the action of a unit matrix.

Therefore, the subalgebra \( L \) coincides with the entire algebra \( End_\mathbb{F}(C_n) \cong M_n(\mathbb{F}) \). \( \square \)
Proposition 6.2. If the algebra \( P_n \) has a finite basis of identities for \( n \geq 2 \), then the ideal \( T = T(E_0(P_n)) \) is generated by polynomials of bounded degrees.

Proof. Suppose that \( P_n \) has a finite basis of identities for \( n \geq 2 \). By Proposition 5.5, modulo (1), (3), and (4), every identity is equivalent to a finite system of identities of (14). Consequently, by Lemma 4.2, there exists a finite set of elements \( G \subseteq T \) such that the identities \( tg = 0 \), where \( g \in G \), form a basis of identities of \( P_n \). Let \( m \) be the maximum of the degrees of polynomials in \( G \). By the same Lemma 4.2, the ideal \( T \) is generated by all \( \varphi(g) \), where \( g \in G \) and \( \varphi \) is linear. Consequently, \( T \) is generated by elements of degrees \( \leq m \). \( \square \)

7. Identities of \( P_2 \).

We are going to prove that \( P_2 \) does not have a finite basis of identities. First, let’s construct some important examples of algebras.

Proposition 7.1. For any \( s > 5 \) there exists an algebra \( B \in \mathcal{R} \) with the following two properties:

1. \( B \) is generated by a set \( Q = \{ q_1, \ldots, q_{s+3} \} \) such that \( T \not\subseteq T(E_0(B)) \).
2. Let \( C \) be a subalgebra of \( B \) generated by any subset \( Q' \) of \( Q \) with \( s \) elements. Then
   \[ tg(c_1, \ldots, c_k) = 0 \]
   for all \( g(x_1, \ldots, x_k) \in T, c_1, \ldots, c_k \in C, \) and \( t \in B \).

Proof. Set \( n = s - 5 \geq 1 \). Let \( H \) be the free algebra with identity in the variety of algebras generated by the field \( \mathbb{F} \) with free generators \( \{ h_1, \ldots, h_n \} \). Denote by \( W \) the subspace of \( H \), spanned by all words in \( h_1, \ldots, h_n \), including the unit element 1, that do not contain at least one \( h_i \). Then \( W \neq H \). By Theorem 1.6 from [37], the algebra \( A = H \otimes_{\mathbb{F}} P_3 \) belongs to \( \mathcal{R} \). Consider the subalgebra \( L \) of \( A \) generated by the following set of elements:

\[
(18) \quad \{ 1 \otimes c_1, 1 \otimes a_{11}, 1 \otimes b_{11}, 1 \otimes a_{12}, 1 \otimes b_{12}, h_i \otimes a_{22}, 1 \otimes b_{22}, 1 \otimes a_{23}, 1 \otimes b_{23} \}
\]

where \( i = 1, \ldots, n \).

We note that

\[
1 \otimes c_2 = -(1 \otimes b_{12})((1 \otimes a_{12})(1 \otimes c_1)),
\]

\[
h_j \otimes c_2 = -(1 \otimes b_{22})((h_j \otimes a_{22})(1 \otimes c_2)),
\]

\[
h_i h_j \otimes c_2 = -(1 \otimes b_{22})((h_i \otimes a_{22})(h_j \otimes c_2)).
\]

Thus, by induction on the length of \( h \), one can derive that \( h \otimes c_2 \in L \) for any word \( h \) in \( h_1, \ldots, h_n \). In addition, \( h \otimes c_3 \in L \) since

\[
h \otimes c_3 = -(1 \otimes b_{23})((1 \otimes a_{23})(h \otimes c_2)).
\]

Note that \( h \otimes c_3 \) is a two-sided annihilator of \( L \) since

\[
L \subseteq H \otimes (D_3 + \sum_{i \leq j \leq 3, (i, j) \neq (3, 3)}^{(3, 3)} (F a_{ij} + F b_{ij})).
\]

Consequently, \( N = H \otimes c_3 \) and \( N' = W \otimes c_3 \) are ideals of \( L \). Set \( B = L/N' \) and let’s show that it satisfies the properties (1) and (2) of the proposition.
Verification of Property (1). Denote by \( q_1, \ldots, q_{s+3} \) the images of generators of (18) under the natural projection \( L \to L/N' \). By Lemma 6.1, the algebra \( E_0(P_2) \) satisfies the well-known Hall’s identity

\[
[[\overline{f}_1, \overline{f}_2] \circ [\overline{f}_3, \overline{f}_4], \overline{f}_5] = 0,
\]

for all \( \overline{f}_i \in E_0(P_2) \), where \( 1 \leq i \leq 5 \) and \( a \circ b = ab + ba \). It follows that \( S = [[f_1, f_2] \circ [f_3, f_4], f_5] \in T \) for all \( f_i \in E_0 \).

It is easy to choose \( f_1, \ldots, f_5 \in E_0 \) and \( \varphi : F(X) \to L \) such that

\[
f_1^e = V_{1 \oplus b_{12} \oplus a_{12}} \prod_{i=1}^{n} V_{1 \oplus b_{22} \oplus a_{22}}, \quad f_2^e = f_3^e = V_{1 \oplus b_{11} \oplus a_{11}},
\]

\[
f_4^e = V_{1 \oplus b_{22} \oplus a_{22}}, \quad f_5^e = V_{1 \oplus b_{23} \oplus a_{23}}.
\]

The actions of the operators \( f_1^e, f_2^e, f_3^e, f_4^e, f_5^e \) on \( L \) give us

\[
f_1^e f_2^e = 0, \quad f_2^e f_1^e = V_{1 \oplus b_{12} \oplus a_{12}},
\]

\[
f_3^e f_2^e = V_{1 \oplus b_{23} \oplus a_{23}}, \quad f_4^e f_5^e = 0,
\]

and we have

\[
S^e = [-V_{1 \oplus b_{12} \oplus a_{12}} \circ V_{1 \oplus b_{23} \oplus a_{23}}, V_{1 \oplus b_{11} \oplus a_{11}}] = V_{1 \oplus b_{12} \oplus a_{12}} V_{1 \oplus b_{23} \oplus a_{23}},
\]

where \( v = h_1 \cdots h_n \). Since

\[
(1 \oplus c_1) S^e = v \otimes c_3 \neq 0 \text{ (mod } N'),
\]

we obtain \( S \notin T(E_0(B)) \) and therefore \( T \nsubseteq T(E_0(B)) \).

Verification of Property (2). Let \( L' \) be a subalgebra of \( L \) generated by a subset of the set (18) that contains no more than \( s \) elements. Assume that \( f(x_1, \ldots, x_k) \in T \). Let \( M \) be the set of all elements of the form \((1 \oplus c_1) f(l_1, \ldots, l_k)\), where \( l_i \in L' \). We claim that

\[
M \cap N \subseteq N'.
\]

Let’s assume that (19) does not hold. In other words, there is an element

\[
g = (h_1 \cdots h_n \otimes c_3)(h' \otimes c_3) + h'' \otimes c_3 \in M \cap N
\]

for some nonzero \( h' \in H \) and some \( h'' \in W \).

Note that

\[
(1 \oplus c_1) V_{1 \oplus b_{12} \oplus a_{12}} = 1 \otimes c_2, \quad (1 \oplus c_2) V_{1 \oplus b_{22} \oplus h_j \otimes a_{22}} = h_j \otimes c_2,
\]

\[
(h_i \otimes c_2) V_{1 \oplus b_{22} \oplus h_j \otimes a_{22}} = h_i h_j \otimes c_2, \quad (h_1 \cdots h_n \otimes c_2) V_{1 \oplus b_{23} \oplus a_{23}} = h_1 \cdots h_n \otimes c_3.
\]

Without using \( 1 \oplus c_1 \) and all of the operators

\[
V_{1 \oplus b_{12} \oplus a_{12}}, \quad V_{1 \oplus b_{22} \oplus h_i \otimes a_{22}}, \quad V_{1 \oplus b_{23} \oplus a_{23}},
\]

we cannot get elements containing the product \( h_1 \cdots h_n \). This means that \( M \cap N \) contains \( g \) if and only if the following \( s + 1 \) elements appear in our calculations:

\[
1 \otimes c_1, 1 \otimes a_{12}, 1 \otimes b_{12}, h_i \otimes a_{22}, 1 \otimes b_{22}, 1 \otimes a_{23}, 1 \otimes b_{23} \quad (i = 1, \ldots, n).
\]

It is impossible since \( L' \) is generated by \( s \) elements. This contradiction establishes the inclusion (19).
Set $\bar{f} = f(l_1, \ldots, l_k)$ for some fixed $l_1, \ldots, l_k \in L'$. We show that $L \bar{f} = 0 \ (mod \ N')$. By Lemma 3.2, one can have that $(H \otimes (C_3 + A_3 + \mathbb{F}c_3))E_0(L) = 0$. Then, because of (19), it is sufficient to prove that

\begin{equation}
(1 \otimes c_1)\bar{f} = 0 (mod \ N), \quad (H \otimes c_2)\bar{f} = 0.
\end{equation}

If $l_i = l'_i + l''_i$, where

\begin{align*}
l'_i &\in H \otimes \sum_{i \leq j \leq 2} (F_\alpha a_{ij} + F_\beta b_{ij}), \\
l''_i &\in H \otimes (F_\alpha a_{23} + F_\beta b_{23}),
\end{align*}

then

\begin{equation}
(1 \otimes c_1)\bar{f} = (1 \otimes c_1)f(l'_1, \ldots, l'_k) (mod \ N).
\end{equation}

We have

\[ f(x_1 + y_1, \ldots, x_k + y_k) = f(x_1, \ldots, x_k) + g(x_1, \ldots, x_k, y_1, \ldots, y_k), \]

where every monomial of $g \in E_0$ involve at least one variable from $y_1, \ldots, y_k$. Since $LV_{l''_i} \subseteq N$, we obtain

\[ (1 \otimes c_1)g(l'_1, \ldots, l'_k, l''_1, \ldots, l''_k) = 0 (mod \ N), \]

which implies (21).

Consequently, to prove the first relation of (20), without loss of generality we can assume that $l_i \in H \otimes \sum_{i \leq j \leq 2} (F_\alpha a_{ij} + F_\beta b_{ij})$. In this case, the elements $c_1, l_1, \ldots, l_k$ generate a subalgebra of $H \otimes P_2$. The algebra $H$ can be embedded into the Cartesian product $\mathbb{F}^a$ of the algebra $F$. So, $H \otimes P_2$ can be embedded into $\mathbb{F}^a \otimes P_2 \cong P_2^a$ and satisfies all the identities of $P_2$. Consequently, it satisfies all $V$-identities from $T$. Thus, the first relation of (20) is established. The second relation of (20) can be established similarly using the equality

\[ H \otimes (\sum_{j \leq 2} (F_\alpha a_{1j} + F_\beta b_{1j}))c_2 = 0. \]

In this case, we can assume that

\[ l_i \in H \otimes (\sum_{2 \leq i, j \leq 3} (F_\alpha a_{ij} + F_\beta b_{ij})). \]

Then the elements $c_2, l_1, \ldots, l_k$ generate an algebra isomorphic to a subalgebra of $H \otimes P_2$. Thus, we have $L \bar{f} = 0 (mod \ N')$. The factorization by $N'$ completes the proof of Proposition 7.1. \hfill \Box

**Lemma 7.2.** Let $\Sigma$ be a set of generators of the ideal $T = T(E_0(P_2))$ of $E_0$. Then for any natural number $s$, there exists a polynomial $f_s \in \Sigma$ that depends on more than $s$ variables.

**Proof.** Suppose, contrary, that $\Sigma$ consists of polynomials that depend on $\leq s$ variables. Let $B$ be the algebra satisfying the conditions of Proposition 7.1. Consider the epimorphism $\tau : F(X) \to B$ defined by

\[ \tau(x_i) = \begin{cases} 
q_i & \text{if } i \leq s + 3, \\
0 & \text{if } i > s + 3.
\end{cases} \]
This induces the epimorphism $\tilde{\tau} : E_0 \to E_0(B)$ defined by
$$\tilde{\tau} g(x_1, \ldots, x_k) = g(\tau(x_1), \ldots, \tau(x_k)).$$
If $g(x_1, \ldots, x_k) \in \Sigma$, then $k \leq s$ and $c_i = \tau(x_i)$ belong to a subalgebra of $B$ generated by $\leq s$ elements. By Proposition 7.1(2), $g(c_1, \ldots, c_k) = 0$. Thus, $g \in \text{Ker} \tilde{\tau}$. So $\text{Ker} \tilde{\tau}$ contains $\Sigma$ and, consequently, $T$ as well.

Now let $f(x_1, \ldots, x_m) \in T$. For any $b_1, \ldots, b_m \in B$ there exist $r_i \in F(X)$ such that $b_i = \tau(r_i)$ for all $i = 1, \ldots, m$. Since $f(r_1, \ldots, r_m) \in T$, we have $f(b_1, \ldots, b_m) = \tilde{\tau} f(r_1, \ldots, r_m) = 0$. This proves that every element of $T$ is a $V$-identity of $B$. This contradicts Proposition 7.1(1).

Theorem 7.3. Algebra $P_2$ over an arbitrary field $F$ does not have a finite basis of identities.

Proof. If $P_2$ has a finite basis of identities, then the ideal $T = T(E_0(P_2))$ is generated by polynomials of bounded degree by Proposition 6.2. This contradicts Lemma 7.2.

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