DOMAINS OF HOLOMORPHY FOR IRREDUCIBLE UNITARY REPRESENTATIONS OF SIMPLE LIE GROUPS

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1. **Introduction**

Let us consider a unitary irreducible representation $(\pi, \mathcal{H})$ of a simple, non-compact and connected Lie group G. Let us denote by K a maximal compact subgroup of G. According to Harish-Chandra, the Lie algebra submodule $\mathcal{H}_K$ of K-finite vectors of π consists of analytic vectors of the representation. We determine, and in full generality, their natural domain of definition as holomorphic functions (see Theorem 6.1 below):

Theorem 1.1. *Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G. Let $v \in \mathcal{H}$ be a non-zero K-finite vector and*

$$f_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

the corresponding orbit map. Then there exists a maximal $G \times K_{\mathbb{C}}$-invariant domain $D_\pi \subseteq G_{\mathbb{C}}$, independent of v, to which f_v extends holomorphically. Explicitely:

(i) *$D_\pi = G_{\mathbb{C}}$ if π is the trivial representation.*
(ii) *$D_\pi = \Xi^+ K_{\mathbb{C}}$ if G is Hermitian and π is a non-trivial highest weight representation.*
(iii) *$D_\pi = \Xi^- K_{\mathbb{C}}$ if G is Hermitian and π is a non-trivial lowest weight representation.*
(iv) *$D_\pi = \Xi K_{\mathbb{C}}$ in all other cases.*

Let us explain the objects Ξ, Ξ^+ and Ξ^- in the statement. We form $X = G/K$, the associated Riemann symmetric space, and view X as a totally real submanifold of its affine complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. The natural G-invariant complexification of X, the crown domain, is denoted by Ξ ($\subseteq X_{\mathbb{C}}$). For a domain $D \subseteq X_{\mathbb{C}}$ we denote by $DK_{\mathbb{C}}$ its preimage in $G_{\mathbb{C}}$.

In [4] we observed that a $G \times K_{\mathbb{C}}$-invariant domain of definition of f_v, say $D_v \subseteq G_{\mathbb{C}}$, must be such that G acts properly on $D_v/K_{\mathbb{C}} \subseteq X_{\mathbb{C}}$. By our work with Robert J. Stanton we know that we can choose D_v such that $D_v \supseteq \Xi K_{\mathbb{C}}$ (see [5], [6]). Therefore it is useful to classify all G-domains $\Xi \subseteq D \subseteq X_{\mathbb{C}}$ with proper action. As it turns out, they allow a simple description. We extract from Theorem 4.1 and Theorem 5.2 below:

Theorem 1.2. *Let $\Xi \subseteq D \subseteq X_{\mathbb{C}}$ be a G-invariant domain on which G acts properly. Then:*

(i) *If G is not of Hermitian type, then $D = \Xi$.*
(ii) *If G is of Hermitian type, then either $D \subseteq \Xi^+$ or $D \subseteq \Xi^-$ with Ξ^+ and Ξ^- two explicite maximal domains for proper G-action.*

Finally, let us emphasize that proofs in this paper are modelled after $G = \mathrm{Sl}(2,\mathbb{R})$ which was dealt with earlier in [4].

Acknowledgment: I am happy to point out that this paper is part of an ongoing project with Eric M. Opdam [4]. Also I would like to thank Joseph Bernstein who, over the years, helped me with his comments to understand the material much better.

2. **Notation**

Throughout this paper G shall denote a connected simple non-compact Lie group. We denote by $G_\mathbb{C}$ the universal complexfication of G and request:

- $G \subseteq G_\mathbb{C}$;
- $G_\mathbb{C}$ is simply connected.

We fix a maximal compact subgroup $K < G$ and form

$$X = G/K\,,$$

the associated Riemannian symmetric space of the non-compact type. The universal complexification $K_\mathbb{C}$ of K naturally realizes as a subgroup of $G_\mathbb{C}$. We set

$$X_\mathbb{C} = G_\mathbb{C}/K_\mathbb{C}$$

and call $X_\mathbb{C}$ the *affine complexification* of X. Note that

$$X \hookrightarrow X_\mathbb{C}, \quad gK \mapsto gK_\mathbb{C}$$

defines a G-equivariant embedding which realizes X as a totally real form of the Stein symmetric space $X_\mathbb{C}$. We write $x_0 = K_\mathbb{C} \in X_\mathbb{C}$ for the standard base point in $X_\mathbb{C}$ (obviously $x_0 = K \in X$ as well).

However, the natural complexification of X is not $X_\mathbb{C}$, but the *crown domain* $\Xi \subsetneq X_\mathbb{C}$ whose definition we recall now.

We shall provide the standard definition of Ξ, see [1]. To begin with let us fix a choice of horospherical coordinates on X:

$$N \times A \xrightarrow{\simeq} X, \quad (n,a) \mapsto na\cdot x_0\,.$$

Lie algebras of subgroups $L < G$ will be denoted by the corresponding lower case altdeutsche Frakturschrift, i.e. $\mathfrak{l} < \mathfrak{g}$; complexifications of Lie algebras are marked with a $\mathbb{C}$-subscript. We write $\Sigma = \Sigma(\mathfrak{g},\mathfrak{a})$ for the associated restricted root system, $\mathcal{W}$ for its Weyl group and let Σ^+

be the choice of positive system associated to $\mathfrak{n}$. Within $\mathfrak{a}$ we declare a $\mathcal{W}$-invariant relative compact set

$$\Omega = \{Y \in \mathfrak{a} \mid \alpha(Y) < \pi/2 \ \forall \alpha \in \Sigma\}$$

and define the crown domain by

$$\Xi = G\exp(i\Omega) \cdot x_0 \, . \tag{2.1}$$

As all flats $\mathfrak{a}$ are conjugate under $\mathrm{Ad}(K)$, Ξ is independent of the choice of $\mathfrak{a}$ and intrinsically attached to X.

3. Remarks on G-invariant domains in $X_\mathbb{C}$ with proper action

One defines elliptic elements in $X_\mathbb{C}$ by

$$X_{\mathbb{C},\mathrm{ell}} = G\exp(i\mathfrak{a}) \cdot x_0 \, .$$

The main result of [1] was to show that Ξ is a maximal domain in $X_{\mathbb{C},\mathrm{ell}}$ with G-action proper. In particular G acts properly on Ξ.

In [4] it was found that Ξ in general is not a maximal domain in $X_\mathbb{C}$ for proper G-action. To know all maximal domains is important for the theory of representations [4], Sect. 4.

That Ξ in general is not maximal for proper action is related to the unipotent model for the crown which was discovered in [4]. To be more precise, we showed that there exists a domain $\Lambda \subseteq \mathfrak{n}$ containing 0 such that

$$\Xi = G\exp(i\Lambda) \cdot x_0 \, . \tag{3.1}$$

Now there is a big difference between the unipotent parametrization (3.1) and the elliptic parametrization (2.1): If we enlarge Ω the result is no longer open; in particular, $X_{\mathbb{C},\mathrm{ell}}$ is not a domain. On the other hand, if we enlarge the open set Λ the resulting set is still open; in particular $X_{\mathbb{C},\mathrm{u}} := G\exp(i\mathfrak{n}) \cdot x_0$ is a domain. Thus, if there to exist a bigger domain than Ξ with proper action, then it is likely by enlargement of Λ.

We need some facts on the boundary of Ξ.

3.1. Boundary of Ξ

Let us denote by $\partial\Xi$ the topological boundary of Ξ in $X_\mathbb{C}$. One shows that

$$\partial_{\mathrm{ell}}\Xi := G\exp(i\partial\Omega) \cdot x_0 \subseteq \partial\Xi$$

(cf. [6]) and calls $\partial_{\rm ell}\Xi$ the *elliptic part* of $\partial\Xi$. We define the *unipotent part* $\partial_{\rm u}\Xi$ of $\partial\Xi$ to be the complement to the elliptic part:

$$\partial_{\rm u}\Xi = \partial\Xi \setminus \partial_{\rm ell}\Xi \,.$$

The relevance of $\partial_{\rm u}\Xi$ is as follows. Let $\Xi \subseteq D \subseteq X_{\mathbb C}$ denote a G-domain with proper G-action. Then $D \cap \partial_{\rm ell}\Xi = \emptyset$ by the above cited result of [1]. Thus if $D \supsetneq \Xi$ one has

$$D \cap \partial_{\rm u}\Xi \neq \emptyset \,.$$

We need to describe $\partial_{\rm u}\Xi$. Results of Matsuki (exploited in [2]) give the following simple characterization of $\partial_{\rm ell}\Xi$ and $\partial_{\rm u}\Xi$:

- $\partial_{\rm ell}\Xi = \{z \in \partial\Xi \mid G\cdot z \text{ is closed}\}$,
- $\partial_{\rm u}\Xi = \{z \in \partial\Xi \mid G\cdot z \text{ is non-closed}\}$.

Let us now describe $\partial_{\rm u}\Xi$ in more detail. Let $Y \in \Omega$ and set $t := \exp(iY)$. Define a reductive subalgebra of $\mathfrak{g}_{\mathbb C}$ by

$$\mathfrak{g}_{\mathbb C}[t] = \{Z \in \mathfrak{g}_{\mathbb C} \mid \mathrm{Ad}(t^{-2})\circ\sigma(Z) = Z\}$$

with σ the Cartan involution on $\mathfrak{g}_{\mathbb C}$ which fixes $\mathfrak{k} + i\mathfrak{p}$. Then there is a partial result on $\partial_{\rm u}\Xi$, for instance stated in [2],

$$\partial_{\rm u}\Xi \subseteq \{G\exp(e)\exp(iY)\cdot x_0 \mid Y \in \partial\Omega, \tag{3.2}$$

$$0 \neq e \in \mathfrak{g}_{\mathbb C}[t] \cap i\mathfrak{g} \text{ nilpotent}\} \tag{3.3}$$

If Y is such that only one root, say α, attains the value $\pi/2$, then we call Y and as well the elements in the boundary orbit $G\exp(e)\exp(iY)\cdot x_0$ *regular*. Define the *regular unipotent* boundary by

$$\partial_{\rm u,reg}\Xi = \{z \in \partial_{\rm u}\Xi \mid z \text{ regular}\} \,.$$

Note that $\mathfrak{g}_{\mathbb C}[t]$ is of especially simple form for regular Y, namely

$$\mathfrak{g}_{\mathbb C}[t] = i\mathfrak{a} + \mathfrak{m} + \mathfrak{g}[\alpha]^{-\theta} + i\mathfrak{g}[\alpha]^{\theta}$$

where $\mathfrak{g}[\alpha] = \mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha}$ and $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ as usual.

Hence, in the regular situation, one can choose e above to be in $i\mathfrak{g}[\alpha]^{\theta} + i\mathfrak{a}$. By $\mathfrak{sl}(2)$-reduction one can show that for $0 \neq e' \in i\mathfrak{g}[\alpha]^{\theta} = i\mathfrak{a}$ that $\exp(\epsilon e')\exp(iY)\cdot x_0 \in \partial_{\rm u}\Xi$ for $\epsilon = 1$ or $\epsilon = -1$ (see [4]).

We thus proved:

Proposition 3.1. *Let $\Xi \subsetneq D \subseteq X_{\mathbb C}$ be a G-invariant domain with proper G-action. Then:*

(i) $D \cap \partial_{\rm u,reg}\Xi \neq \emptyset$.

(ii) *Let* $Y \in \partial\Omega$ *be regular and* $\alpha \in \Sigma$ *the unique root with* $\alpha(Y) = \pi/2$*. Then there exists* $0 \neq e \in i\mathfrak{g}[\alpha]^\theta + i\mathfrak{a}$*, nilpotent such that*

$$\exp(e)\exp(iY)\cdot x_0 \in \partial_{\mathrm{u,reg}}\Xi\,.$$

4. Complex crowns which are maximal for proper action

The aim of this section is to prove the following theorem:

Theorem 4.1. *Suppose that* G *is not of Hermitian type. Then* Ξ *is a maximal* G*-invariant domain with proper* G*-action.*

Remark 4.2. *If* G *is not of Hermitian type, then the nature of the action of* $M = Z_K(A)$ *on the various root spaces is different in nature. This will be exploited in the sequel. For the necessary information on* M*-groups one may consult Appendix C of the monography* [3]*.*

Proof. Suppose that G is not of Hermitian type. Let $D \supsetneq \Xi$ be a G-invariant Stein domain with proper G-action. We shall show that D does not exist.

According to Proposition 3.1 we find a regular $Y \in \partial\Omega$ and a nilpotent $e \in \mathfrak{g}_\mathbb{C}[\exp(iY)] \cap i\mathfrak{g}$ such that

$$\exp(e)\exp(iY)\cdot x_0 \in \partial_{\mathrm{u,reg}}\Xi\,.$$

If $\alpha \in \Sigma$ is the root corresponding to Y , then $\mathfrak{sl}(2)$-reduction in conjunction with Proposition 7.3 implies the existence of $E_\alpha \in \mathfrak{g}^\alpha$ such that:

- $\{E_\alpha, \theta(E_\alpha), [E_\alpha, \theta(E_\alpha)]\}$ is an $\mathfrak{sl}(2)$-triplet,
- $\exp(iE_\alpha)\exp(iY')\cdot x_0 \in \partial_{\mathrm{u,reg}}\Xi \cap D$,
- $Y' \in \Omega$ such that $\alpha(Y') = 0$.

Now, as G is not of Hermitian type, there exists an element $m \in M$ such that $\mathrm{Ad}(m)E_\alpha = -E_\alpha$ (this can be extracted from App. C in [3]). Hence

$$\exp(-iE_\alpha)\exp(iY')\cdot x_0 \in \partial_{\mathrm{u,reg}}\Xi$$

as well. But this, via $\mathfrak{sl}(2)$-reduction, contradicts Proposition 7.1(i) below. □

5. Groups of Hermitian type

The goal of this section is to give a classification of all maximal G-domains $\Xi \subseteq D \subseteq X_\mathbb{C}$ with proper G-action under the assumption that G is of Hermitian type.

Let $G \subseteq P^- K_\mathbb{C} P^+$ be a Harish-Chandra decomposition of G in $G_\mathbb{C}$. We define flag varieties

$$F^+ = G_\mathbb{C}/K_\mathbb{C}P^+ \quad \text{and} \quad F^- = G_\mathbb{C}/K_\mathbb{C}P^-$$

and inside of them we declare the flag domains

$$D^+ = GK_\mathbb{C}P^+/K_\mathbb{C}P^+ \quad \text{and} \quad D^- = GK_\mathbb{C}P^-/K_\mathbb{C}P^- .$$

Then

$$X_\mathbb{C} \hookrightarrow F^+ \times F^-, \quad gK_\mathbb{C} \mapsto (gK_\mathbb{C}P^+, gK_\mathbb{C}P^-) \tag{5.1}$$

identifies $X_\mathbb{C}$ as a Zariski open affine piece of $F^+ \times F^-$. In more detail: As G is of Hermitian type, there exist $w_0 \in N_{G_\mathbb{C}}(K_\mathbb{C})$ such that $w_0 P^\pm w_0^{-1} = P^\mp$. In turn, this element induces a $G_\mathbb{C}$-equivariant biholomorphic map:

$$\phi : F^+ \to F^-, \quad gK_\mathbb{C}P^+ \mapsto gw_0K_\mathbb{C}P^- .$$

With that the embedding (5.1) gives the following identification for $X_\mathbb{C}$:

$$X_\mathbb{C} = \{(z, w) \in F^+ \times F^- \mid \phi(z) \top w\} , \tag{5.2}$$

where $\top$ stands for the transversality notion in the flag variety F^-.

The description of Ξ is quite simple:

$$\Xi = D^+ \times D^-$$

(see [6]).

We now define

$$\Xi^+ = (D^+ \times F^-) \cap X_\mathbb{C} ,$$
$$\Xi^- = (F^+ \times D^-) \cap X_\mathbb{C} .$$

Proposition 5.1. *The following assertions hold:*

(i) *G acts properly on Ξ^+ and Ξ^-; moreover, both Ξ^+ and Ξ^- are maximal in $X_\mathbb{C}$ for proper G-action.*

(ii) *Ξ^+ and Ξ^- are Stein manifolds.*

Proof. (i) As the G-action is proper on D^+ and D^-, it follows that G acts properly on both Ξ^+ and Ξ^-. That Ξ^+ and Ξ^- are in fact maximal domains in $X_\mathbb{C}$ for proper G-action will be a consequence of Theorem 5.2 below.

(ii) Let us deal with the $+$-case only. By the definition of Ξ^+ and the characterization of $X_\mathbb{C}$ we conclude that $\Xi^+ \to D^+$ is a holomorphic fiber bundle with fiber isomorphic to the complex vector space P^+. Now, a holomorphic fiber bundle is Stein if and only if base and fiber are Stein. We conclude that Ξ^+ is Stein. □

We state main result of the section.

Theorem 5.2. *Suppose that G is of Hermitian type. If $\Xi \subseteq D \subsetneq X_{\mathbb{C}}$ is a G-invariant domain with proper G-action, then $D \subseteq \Xi^+$ or $D \subseteq \Xi^-$.*

We postpone the proof of the theorem to the end of this section.

5.1. The structure of Ξ^+ and Ξ^-

We now devote ourselves to the stucture theory of the domains Ξ^+ and Ξ^-.

If $D \subseteq X_{\mathbb{C}}$ is a subset, then we write $DK_{\mathbb{C}}$ for its preimage in $G_{\mathbb{C}}$ under the canonical projection $G_{\mathbb{C}} \to X_{\mathbb{C}}$. Standard reasoning yields the following characterization of Ξ^+ and Ξ^- in terms of the preimage.

Proposition 5.3. *The following assertions hold:*

(i) $\Xi^+ K_{\mathbb{C}} = GK_{\mathbb{C}}P^+$,
(ii) $\Xi^- K_{\mathbb{C}} = GK_{\mathbb{C}}P^-$.

Next we realize Ξ^+ and Ξ^- as cone bundles over X. For that some terminology is needed.

According to Harish-Chandra, Σ is of type C_n or BC_n. Hence we find a subset $\{\gamma_1, \ldots, \gamma_n\}$ of long strongly orthogonal roots. We fix $E_j \in \mathfrak{g}^{\gamma_j}$ such that

$$\{E_j, \theta(E_j), [E_j, \theta E_j]\}$$

becomes an $\mathfrak{sl}(2)$-triplet. Set $T_j := 1/2[E_j, \theta E_j]$ and note that

$$\Omega = \bigoplus_{j=1}^{n} (-1,1)T_j \,.$$

We set $V = \bigoplus_{j=1}^n \mathbb{R} \cdot E_j$ and declare a cube inside V by

$$\Lambda = \bigoplus_{j=1}^{n} (-1,1)E_j \,.$$

In [4], Sect. 8, we have shown that

$$\Xi = G \exp(i\Lambda) \cdot x_0 \,.$$

In this parametrization of Ξ the unipotent boundary piece has a simple description:

$$\partial_{\mathrm{u}} \Xi = G \exp(i\partial\Lambda) \cdot x_0 \,. \tag{5.3}$$

The strategy now is to enlarge Ξ by enlarging Λ hereby maintaining that the object stays a domain on which G acts properly. But now we have to be a little bit careful with our choice of E_j. Replacing E_j by $-E_j$ has no effect for the matters cited above. But for the sequel. Notice that $V \subseteq \mathfrak{p}^+ = \log P^+$ and we request that E_j lies in the closure of the positive cone of the associated Jordan algebra (this makes the sign unique). We set

$$\Lambda^+ = \bigoplus_{j=1}^n (-1, \infty) E_j \quad \text{and} \quad \Lambda^- = \bigoplus_{j=1}^n (-\infty, 1) E_j \,.$$

Then, generalizing [4], Prop. 4.5, we immediately get:

Proposition 5.4. *The following assertions hold:*

(i) $\Xi^+ = G \exp(i\Lambda^+) \cdot x_0$,
(ii) $\Xi^- = G \exp(i\Lambda^-) \cdot x_0$.

Write $\mathcal{N} \subseteq \mathfrak{g}$ for the nilcone and note that

$$\mathcal{N} = \mathrm{Ad}(G)\mathfrak{n} = \mathrm{Ad}(K)\mathfrak{n} \,.$$

Set

$$\Lambda^{++} = \bigoplus_{j=1}^n [0, \infty) E_j \quad \text{and} \quad \Lambda^{--} = \bigoplus_{j=1}^n (-\infty, 0] E_j$$

and define cones in $\mathcal{N}$ by

$$\mathcal{N}^+ = \mathrm{Ad}(K)\Lambda^{++} \quad \text{and} \quad \mathcal{N}^- = \mathrm{Ad}(K)\Lambda^{--} \,.$$

Theorem 5.5. *The maps*

$$G \times_K \mathcal{N}^\pm \to \Xi^\pm, \quad [g, Y] \mapsto g \exp(iY) \cdot x_0$$

are homeomorphism. In particular,

(i) *G acts properly on $\Xi^\pm$.*
(ii) *$\Xi^\pm$ is contractible.*

Proof. We may restrict ourselves to the +-case.

A simple induction on the rank shows that the map

$$G \times V \to X_{\mathbb{C}}, \quad (g, Y) \mapsto g \exp(iY) \cdot x_0$$

is open. This and Proposition 5.4 imply that it is sufficient to show that the map

$$G \times_K \mathcal{N}^+ \to \Xi^+, \quad [g, Y] \mapsto g \exp(iY) \cdot x_0$$

is injective. In other words: for $Y, Y' \in \Lambda^+$ and $g \in G$ we have to show that

$$\exp(iY)\cdot x_0 = g\exp(iY')\cdot x_0 \tag{5.4}$$

implies $g = k \in K$ and $\mathrm{Ad}(k)Y' = Y$.

We first show that Y equals Y', up to a permutation of coordinates. For that consider a finite dimensional representations (π, W) of $G_{\mathbb{C}}$ (as a real group) which is both G and $K_{\mathbb{C}}$-spherical. We let v_G and $v_{K_{\mathbb{C}}}$ be vectors in W fixed under G, resp. $K_{\mathbb{C}}$. We form the real polynomial functions on $G_{\mathbb{C}}$

$$f_\pi(g) = \langle \pi(g)v_{K_{\mathbb{C}}}, v_G\rangle \qquad (g \in G_{\mathbb{C}})$$

and with that the functions

$$F_\pi(s_1, \ldots, s_n) = f_\pi(\exp(i\sum_{j=1}^n s_jE_j))\,.$$

It follows from finite dimensional representation theory in conjunction with (7.1) below that

$$\langle F_\pi \mid \pi \text{ is } (G, K_{\mathbb{C}}) - \text{spherical}\rangle = \mathbb{C}[s_1^2, \ldots, s_n^2]^{S_n}\,.$$

We conclude from (5.4) that there is no loss of generality to assume that $Y = Y'$. Set $n = \exp(iY)$. It remains to show that $n = gnk$ for some $k \in K_{\mathbb{C}}$ implies $g \in Z_K(n) < K$. We look at the equivalent identity

$$ngn^{-1} = k \tag{5.5}$$

for $g \in G$, $n \in \exp(iV)$ and $k \in K_{\mathbb{C}}$ and show that (5.5) forces $g \in Z_K(n)$.

Matters can be embedded into the symplectic group and hence it is sufficient to deal with this case. For $G = \mathrm{Sp}(n, \mathbb{R})$ we have

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \mid u, v \in M(n, \mathbb{C}); uu^t - vv^t = 1, uv^t - v^tu = 0 \right\}\,.$$

So let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $n = \begin{pmatrix} 1 & is \\ 0 & 1 \end{pmatrix} \in \exp(iV)$, and $k = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$. Note that the $n \times n$-matrices a, b, c, d, s are all real. We write (5.5) out and get that

$$\begin{pmatrix} 1 & is \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & -is \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

which is

$$\begin{pmatrix} a+isc & (a+isc)(-is)+(b+isd) \\ c & -ics+d \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

As a, b, c, d, s are all real, we arrive at the following conditions

$$a = d\,, \tag{5.6}$$

$$sc = -cs\,, \tag{5.7}$$

$$as = sa\,, \tag{5.8}$$

$$c = -scs - b\,. \tag{5.9}$$

Up to this point we have not used the fact at all that Y is "positive". This will come now. Notice that $s = \mathrm{diag}(s_1, \ldots, s_n)$ with all $s_i \geq 0$. We may assume that there is j such that $s_i > 0$ for $i \leq j$ and $s_i = 0$ for $i > j$. From (5.7) we now conclude that $sc = cs = 0$. Hence (5.9) implies that $b = -c$. As a result $g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in G$ which means that $g \in K$. Also we have shown that g commutes with n. This completes the proof of the theorem. □

Remark 5.6. *For the above proof our uniform choice of the signs of E_j mattered. Otherwise the theorem would not be true. It is sufficient to consider the smallest example in rank 2, namely $G = \mathrm{Sp}(2, \mathbb{R})$. Set*

$$\Lambda^{+-} = [0, \infty)E_1 \times (-\infty, 0]E_2$$

and set

$$\Xi^{+-} = G \exp(i\Lambda^{+-}) \cdot x_0\,.$$

Then Ξ^{+-} is a G-domain in $X_{\mathbb{C}}$ but the action fails to be proper. In fact, let $\mathbf{s} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$, for $s > 1$, $\mathbf{a} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and set $\mathbf{c} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$. Assume that

$$a^2 - (s^2 - 1)c^2 = 1 \tag{5.10}$$

is fullfilled and let

$$g = \begin{pmatrix} \mathbf{a} & (s^2-1)\mathbf{c} \\ \mathbf{c} & \mathbf{a} \end{pmatrix}\,.$$

Then $g \in G$ and conditions (5.6) - (5.9) are satisfied in addition. Hence $gn \cdot x_0 = n \cdot x_0$ for $n = \begin{pmatrix} 1 & i\mathbf{s} \\ 0 & 1 \end{pmatrix}$. Now for $s > 1$ equation (5.10) has non-bounded solutions a, c and hence the stabilizer in G of $n \cdot x_0$ is not compact.

5.2. **Proof of Theorem 5.2**

It remains to prove the main rersult of the section. We begin with a weaker statement which is actually sufficient for the application in this paper.

Lemma 5.7. *Let $\Xi \subsetneq D \subset X_{\mathbb{C}}$ be a G-invariant domain with proper G-action. Then we either have $D \cap \partial\Xi \subset \Xi^+$ or $D \cap \partial\Xi \subset \Xi^-$, but not both.*

Proof. As $D \supsetneq \Xi$, it follows that $D \cap \partial\Xi \neq \emptyset$. Moreover, as G acts properly, it follows that

$$D \cap \partial\Xi = D \cap \partial_{\mathrm{u}}\Xi\,.$$

We recall from (5.3) that

$$\partial_{\mathrm{u}}\Xi = G\exp(i\partial\Lambda)\cdot x_0\,.$$

Thus if $D \cap \partial\Xi \not\subseteq \Xi^+$ nor $D \cap \partial\Xi \not\subseteq \Xi^-$, then there exists an element $e = \sum_{j=1}^n x_j E_j \in \overline{\Lambda}$ with $x_i = 1$ and $x_k = -1$ for some $1 \leq i \neq k \leq n$ and such that $\exp(ie)\cdot x_0 \in D \cap \partial_{\mathrm{u}}\Xi$. We are allowed to permute i and k and, via $\mathfrak{sl}(2)$-reduction, obtain a contradiction to Proposition 7.1(i) in the Appendix. By the same reason the alternative is exclusive. □

Proof. (of Theorem 5.2) Let $D \subsetneq \Xi$ be a domain with proper G-action. In view of the preceeding lemma, we have $\Xi \subsetneq D \cap \Xi^+$ or $\Xi \subsetneq D \cap \Xi^-$. We may assume that we are in the situation of "+". The theorem will be proved, if we can show that $D \subseteq \Xi^+$.

If $D \not\subseteq \Xi^+$, then $D \cap \partial\Xi^+ \neq \emptyset$. Observe that

$$\partial\Xi^+ = (\partial D^+ \times F^-) \cap X_{\mathbb{C}}\,.$$

We decompose this boundary into three pieces:

$$\partial\Xi^+ = \underbrace{(\partial D^+ \times \partial D^-) \cap X_{\mathbb{C}}}_{=\partial_{\mathrm{ell}}\Xi} \amalg \underbrace{(\partial D^+ \times D^-) \cap X_{\mathbb{C}}}_{=\partial\Xi\cap\Xi^-}$$
$$\amalg\, (\partial D^+ \times F^- \setminus \overline{D^-}) \cap X_{\mathbb{C}}\,.$$

As G acts properly, the first two components are not allowed to contribute to $D \cap \partial\Xi^+$ (cf. Lemma 5.7 and Subsection 3.1). Therefore

$$D \cap (\partial D^+ \times F^- \setminus \overline{D^-}) \neq \emptyset\,.$$

But this contradicts the fact that G acts properly. In order to see this we allow ourselves to suppress standard technicalities and focus on the essential case of $G = \mathrm{Sl}(2,\mathbb{R})$. We use the terminolgy of the appendix, i.e. $F^+ = F^- = \mathbb{P}^1(\mathbb{C})$ and D^+, resp. D^-, the upper, resp. lower, half plane. It is no loss of generality to assume that $(i,0) \in D \cap \partial\Xi^+$. Now, define an anti-holomorphic, G-equivariant automorphism

$$\tau : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \quad (z,w) \mapsto (z,\overline{w})\,.$$

Then consider $D' = \tau(D)$ which again is a G-domain with proper action. But D' has now the property that it intersects both $\partial\Xi \cap \Xi^+$ and $\partial\Xi \cap \Xi^-$ (use that τ fixes $(i,0)$). This contradicts Proposition 7.1(i) from below. □

6. Application to representation theory

Let $(\pi, \mathcal{H})$ be a unitary representation of G and $\mathcal{H}_K$ the underlying Harish-Chandra module of K-finite vectors. Notice that $\mathcal{H}_K$ is naturally a module for $K_\mathbb{C}$.

We say that $(\pi, \mathcal{H})$ is a highest, resp. lowest, weight representation if G is of Hermitian type and $\mathfrak{p}^+ = \mathrm{Lie}(P^+)$, resp. $\mathfrak{p}^-$, acts on $\mathcal{H}_K$ in a finite manner.

We turn to the main result of this paper.

Theorem 6.1. *Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G. Let $v \in \mathcal{H}$ be a non-zero K-finite vector and*

$$f_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

the corresponding orbit map. Then there exists a maximal $G \times K_\mathbb{C}$-invariant domain $D_\pi \subseteq G_\mathbb{C}$, independent of v, to which f_v extends holomorphically. Explicitely:

(i) *$D_\pi = G_\mathbb{C}$ if π is the trivial representation.*
(ii) *$D_\pi = \Xi^+ K_\mathbb{C}$ if G is Hermitian and π is a non-trivial highest weight representation.*
(iii) *$D_\pi = \Xi^- K_\mathbb{C}$ if G is Hermitian and π is a non-trivial lowest weight representation.*
(iv) *$D_\pi = \Xi K_\mathbb{C}$ in all other cases.*

Proof. If π is trivial, then the assertion is clear. So let us assume that π is non-trivial in the sequel. Fix a nonzero K-finite vector v and consider the orbit map $f_v : G \to \mathcal{H}$. We recall the following two facts:

- f_v extends to a holomorphic G-equivariant map $f_v : \Xi K_\mathbb{C} \to \mathcal{H}$ (see [6], Th. 1.1).
- If $D_v \subseteq G_\mathbb{C}$ is a $G \times K_\mathbb{C}$-invariant domain to which f_v extends holomorphically, then G acts properly on $D_v/K_\mathbb{C}$ (see [4], Th. 4.3)

We begin with the case where G is not of Hermitian type. Here the assertion follows from the bulleted items above in conjunction with Theorem 4.1.

So we may assume for the remainder that G is of Hermitian type. If π is a highest weight representation, then it is clear that f_v extends to a holomorphic map $GK_{\mathbb{C}}P^+ \to \mathcal{H}$. Thus, in this case $\Xi^+K_{\mathbb{C}} = GK_{\mathbb{C}}P^+$ (cf. Proposition 5.3) is a maximal domain of definition for f_v by Proposition 5.1 and the second bulleted item form above. Likewise, if $(\pi, \mathcal{H})$ is a lowest weight representation, then $\Xi^-K_{\mathbb{C}}$ is a maximal domain of definition of f_v. In view of Theorem 5.2 it remains to show:

- If f_v extends to a holomorphic map on a domain D such that $\Xi \subsetneq D \subseteq \Xi^+$, then $(\pi, \mathcal{H})$ is a highest weight representation.
- If f_v extends to a holomorphic map on a domain D such that $\Xi \subsetneq D \subseteq \Xi^-$, then $(\pi, \mathcal{H})$ is a lowest weight representation.

It is sufficient to deal with the first case. So suppose that f_v extends to a bigger domain $\Xi \subsetneq D \subseteq \Xi^+$. Taking derivatives and applying the fact that $d\pi(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))v = \mathcal{H}_K$, we see that f_u extends to D for all $u \in \mathcal{H}_K$. Let $1 \leq j \leq n$ be such that $\exp(iE_j)\exp(iY)\cdot x_0 \in D$ for some $Y \in \Omega$ with $\gamma_j(Y) = 0$. Let $G_j < G$ be the analytic subgroup corresponding to the $\mathfrak{sl}(2)$-triplet $\{E_j, \theta(E_j), [E_j, \theta(E_j)]\}$. Basic representation theory of type I-groups in conjunction with [4], Th. 4.7, yields that $\pi\big|_{G_j}$ breaks into a direct sum of highest weight representations. Applying the Weyl group, this holds for any other G_k as well. But this means that π is a highest weight representation and completes the proof of the theorem. $\square$

Remark 6.2. *The domains Ξ, Ξ^+ and Ξ^- are independent of the choice of the connected group G. Accordingly, the above theorem holds for all simple connected non-compact Lie groups G, i.e. we can drop the assumption that $G \subseteq G_{\mathbb{C}}$ and $G_{\mathbb{C}}$ simply connected.*

7. **Appendix on** $\mathrm{Sl}(2, \mathbb{R})$

In this section we summarize our results from [4] for $G = \mathrm{Sl}(2, \mathbb{R})$.

$K = \mathrm{SO}(2, \mathbb{R})$ is the standard choice for the maximal compact and $X = G/K$ naturally identifies with the upper half plane $D^+ := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$. Further

$$X_{\mathbb{C}} = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \mathrm{diag}[\mathbb{P}^1(\mathbb{C})]$$

with $G_{\mathbb{C}}$ acting diagonally by fractional linear transformations and

$$\Xi = D^+ \times D^- \subseteq X_{\mathbb{C}}\,.$$

The G-embedding of $X = D^+$ into $X_{\mathbb{C}}$ is given by

$$z \mapsto (z, \overline{z}) \in \Xi .$$

Furthermore

$$\Xi^+ = D^+ \times \mathbb{P}^1(\mathbb{C}) \setminus \mathrm{diag}[\mathbb{P}^1(\mathbb{C})] ,$$
$$\Xi^- = \mathbb{P}^1(\mathbb{C}) \times D^- \setminus \mathrm{diag}[\mathbb{P}^1(\mathbb{C})] .$$

With $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ our choices for $\mathfrak{a}$ and $\mathfrak{n}$ are

$$\mathfrak{a} = \mathbb{R} \cdot T \quad \text{and} \quad \mathfrak{n} = \mathbb{R} \cdot E .$$

In [4], Sect. 3 and 4, it was established:

Proposition 7.1. *Let $-\infty \leq a < b \leq \infty$ with $-a, b \geq 0$. Then*

$$\Xi_{a,b} := G \exp(i(a,b)E) \cdot x_0$$

is a G-domain in $X_{\mathbb{C}}$ and the following holds:

(i) *G acts properly on $\Xi_{a,b}$ if and only if* $\min\{-a, b\} \leq 1$.
(ii) $\Xi = \Xi_{-1,1}$.
(iii) $\Xi^+ = \Xi_{-1,\infty}$.
(iv) $\Xi^- = \Xi_{-\infty,1}$.

The relationship between elliptic and unipotent parametrization is as follows (sse [4], Lemma 3.3):

Proposition 7.2. *For $-\pi/4 < \varphi < \pi/4$ one has*

$$G \exp(i\varphi T) \cdot x_0 = G \exp(i \sin(2\varphi)E) \cdot x_0 . \tag{7.1}$$

Finally we recall the description of $\partial\Xi$ as a fiber bundle over the affine symmetric space G/H where $H = \mathrm{SO}_e(1,1)$. Notice that H is the stabilizer of the boundary point

$$z_H := (1, -1) \in \partial\Xi .$$

Write τ for the involution on G, resp. $\mathfrak{g}$, fixing H, resp. $\mathfrak{h}$, and denote by $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the corresponding eigenspace decomposition. The $\mathfrak{h}$-module $\mathfrak{q}$ breaks into two eigenspaces $\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^-$ with

$$\mathfrak{q}^\pm = \mathbb{R} \cdot e^\pm \quad \text{where} \quad e^\pm = \begin{pmatrix} 1 & \mp 1 \\ \pm 1 & -1 \end{pmatrix} .$$

Finally write

$$\mathcal{C} = \mathbb{R}_{\geq 0} \cdot e^+ \cup \mathbb{R}_{\geq 0} \cdot e^-$$

and $\mathcal{C}^\times = \mathcal{C} \setminus \{0\}$. Note that both $\mathcal{C}$ and $\mathcal{C}^\times$ are H-stable. We cite [4], Th. 3.1:

Proposition 7.3. *The map*

$$G \times_H \mathcal{C} \to \partial\Xi, \quad [g, e] \mapsto g \exp(ie) \cdot z_H$$

is a G-equivariant homeomorphism. Moreover,

(i) $\partial_{\rm ell}\Xi = G \cdot z_H \simeq G/H$,
(ii) $\partial_{\rm u}\Xi = G \exp(i\mathcal{C}^\times) \cdot z_H \simeq G \times_H \mathcal{C}^\times$,
(iii) $\partial_{\rm u}\Xi = G \exp(iE) \cdot x_0 \amalg G \exp(-iE) \cdot x_0$.

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