Basic points in the moduli spaces of PEL-type

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§1. Introduction

In [15] Rapoport-Zink proved that the formal completion of a Shimura variety of PEL-type $\mathcal{A}$ along its basic locus Z admits a uniformization by a formal scheme. Rapoport conjectures ([12, p. 427 Remark 2.2 (i)], [15, p. xvii]) that the basic locus is non-empty . In this paper we prove that excluding the cases of 0-dimensional quaternionic Shimura varieties, basic points do exist in every irreducible component of the special fiber; see Theorem 3.4 and Corollary 3.5. This confirms that theory of Rapoport-Zink about the tubular neighborhood of the basic loci is indeed non-empty.

The basic loci of Shimura varieties, or concretely of PEL-type, also have deep connection with the theory of automorphic forms. The idea may be traced back to Deligne who proves that the (part of) Langlands correspondence for GL(2) can be realized in the cohomology of modular curves. This global method is employed in the proof of the local Langlands correspondence for GL(n) in the work of Harris and Taylor [4]. Extending Boyer's localization principle [1], Harris conjectures that if $\Pi = \Pi_\infty \otimes \Pi_p \otimes \Pi_f^p$ is an automorphic representation so that Π_∞ is of cohomological type and Π_p is supercuspidal, then the Π-isotypic component of the cohomology of the Shimura variety is contributed from that of the basic locus (see [5, Conjecture 5.2], [6, Conjecture 8.3.4]). The conjecture is verified in several cases of PEL-type by Fargues [3]. The non-emptiness of the basic locus sets up the framework for this theory. Indeed, this will imply that such an automorphic representation occurs in the cohomology of the Shimura variety.

The existence of basic points in the Siegel moduli spaces or Hilbert-Blumenthal moduli spaces, those exactly are supersingular, is well-known. Some cases in type C or type A of good reduction are certainly known. The cases where the defining group G is connected and the moduli space has good reduction seems to be proved in Fargues [3]. Our proof is different from previous ones; the Honda-Tate theory or the Kottwitz invariant [8] is not used. The main ingredient is using the Hecke orbits. In fact, we modified slightly the definition (Definition 3.2) of basic points and show that a prime-to-p Hecke orbit is finite if and only if it contains basic points (Proposition 4.8). Then using the specialization argument we conclude the existence of basic points in each irreducible component.

We work on slightly more general moduli spaces of PEL-type than ones considered in Rapoport and Zink [15] where the determinant condition and separability condition are imposed. However, it turns out to be convenient to work on this more general setting. We think this formulation should be considered when one wants to study the inseparable isogenies and correspondences.

The paper is organized as follows. Sections 2 and 3 contains the definition of the moduli spaces of PEL-type, the basic points and the statements of our main results. Section 4 introduces the Hecke orbits and leaves, and gives the proof of Theorem 3.4. Section 5 compares our definition of basic points and Kottwitz's one.

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§2. Preliminaries

(2.1) A *rational PEL-datum* $\mathcal{D}$ is a 4-tuple $(B, *, V, \psi)$, where

- B is a finite-dimensional semi-simple algebra $\mathbb{Q}$ with a positive involution $*$;
- V is a finite faithful B-module together with a non-degenerate $\mathbb{Q}$-valued skew-Hermitian form ψ. That

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is, $\psi : V \times V \to \mathbb{Q}$ is a non-degenerate alternating form such that $\psi(bx, y) = \psi(x, b^*y)$ for all $x, y \in V$ and $b \in B$.

The datum $\mathcal{D}$ gives rise to two reductive groups over $\mathbb{Q}$ as usual:

$$G(\mathcal{D})(\mathbb{Q}) = \{x \in \mathrm{End}_B(V) \,|\, x^*x \in \mathbb{Q}^\times\},$$
$$G_1(\mathcal{D})(\mathbb{Q}) = \{x \in \mathrm{End}_B(V) \,|\, x^*x = 1\},$$

where x^* is the adjoint of x with respect to ψ. We shall write G for $G(\mathcal{D})$ and G_1 for $G_1(\mathcal{D})$ when the PEL datum $\mathcal{D}$ is chosen.

(2.2) Lemma *Let $\mathcal{D}$ be a rational PEL datum.*

(1) *There is an $\mathbb{R}$-linear $*$-homomorphism h from $\mathbb{C}$ to $\mathrm{Hom}_{B\otimes\mathbb{R}}(V_\mathbb{R})$ such that the symmetric bilinear form $(x, y) := \langle x, h(i)y\rangle$ is positive definite.*

(2) *If $h' : \mathbb{C} \to \mathrm{Hom}_{B\otimes\mathbb{R}}(V_\mathbb{R})$ is another homomorphism satisfying the same property, then there is an element $g \in G_1(\mathbb{R})$ such that $h' = \mathrm{Int}(g) \circ h$.*

Proof. (2) is proved in Kottwitz [8, Lemma 4.3]. (1) should be well-known and it is implicit in Sec. 4 (p. 386-387) of loc. cit. We indicate a proof for the convenience of the reader. It suffices to construct a complex structure $J \in \mathrm{End}_B(V_\mathbb{R})$ satisfying the Riemann condition, and define h by $h(a + bi) := a + bJ$. Using Morita's equivalence, one reduces to the three cases:

(C) (V, ψ) is a non-degenerate symplectic real space;

(A) V is a $\mathbb{C}$-vector space together with an $\mathbb{R}$-valued skew-Hermitian form ψ;

(D) V is an $\mathbb{H}$-vector space together with an $\mathbb{R}$-valued skew-Hermitian form ψ.

(C) Choose a symplectic basis $\{e_i, f_i\}$ for V with respect to ψ. Define $J(e_i) = f_i$ and $J(f_i) = -e_i$ for $1 \le i \le n$. Then $\{e_i, f_i\}$ is an orthonormal basis for $\psi_J(x, y) = \psi(x, Jy)$.

(A) There is a unique lifting $\psi' : V \times V \to \mathbb{C}$ such that $\psi(x, y) = \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\frac{i}{2}\psi'(x, y))$. Then there is an orthogonal basis such that

$$\psi'(\sum z_i e_i, \sum z'_i e_i) = \sum_{i=1}^{p} z_i \bar{z}'_i - \sum_{i=p+1}^{n} z_i \bar{z}'_i.$$

Define

$$J(e_j) = \begin{cases} ie_j & 1 \le j \le p, \\ -ie_j & p+1 \le j \le n. \end{cases}$$

Then $\psi_J(\sum z_i e_i, \sum z'_i e_i) = \sum_{1\le i\le n} z_i \bar{z}'_i$.

(D) We lift ψ to $\psi' : V \times V \to \mathbb{H}$ such that $\psi = \mathrm{Trd}_{\mathbb{H}/\mathbb{R}}\,\psi'$. We can choose an orthogonal basis $\{e_i\}$ for V over $\mathbb{H}$ with respect to ψ'. Put $\alpha_i = \psi'(e_i, e_i)$, and we have

$$\psi'(\sum \lambda_i e_i, \sum \lambda'_i e_i) = \sum_{i=1}^{n} \lambda_i \alpha_i \bar{\lambda}'_i, \quad \bar{\alpha}_i = -\alpha_i.$$

Let $J(e_i) = \alpha_i/|\alpha_i|\, e_i$, and one checks $J^2 = -I$ and

$$\psi'(\sum \lambda_i e_i, J(\sum \lambda'_i e_i)) = \sum \lambda_i \bar{\lambda}'_i.$$

It follows that ψ_J is positive definite. ∎

Let h be a homomorphism as above. One also calls $(B, *, V, \psi, h)$ a rational PEL datum, although h is uniquely determined by $\mathcal{D}$ up to conjugation of $G_1(\mathbb{R})$.

Let X be the $G(\mathbb{R})$-conjugacy class of h; it has a natural HSD structure on which $G(\mathbb{R})$ acts biholomorphically. The pair (G, X) satisfies the axioms for defining Shimura varieties. For each open compact subgroup U of $G(\mathbb{A}_f)$, one has a quasi-polarized algebraic variety $\mathrm{Sh}_U(G, X)$ over the reflex field E. The Shimura variety $\mathrm{Sh}(G, X)$ consists of the tower of algebraic varieties $\{\mathrm{Sh}_U(G, X)\}_U$ together with the right action of $G(\mathbb{A}_f)$.

(2.3) Let p be a fixed rational prime. Let O_B be an order of B stable under $*$ such that $O_B \otimes \mathbb{Z}_p$ is a maximal order. Choose an $O_B \otimes \hat{\mathbb{Z}}^{(p)}$-lattice V^p in $V \otimes \mathbb{A}_f^{(p)}$ such that $\psi(V^p, V^p) \subset \hat{\mathbb{Z}}^{(p)}$. The choice of V^p gives rise to a $\hat{\mathbb{Z}}^{(p)}$-structure of G and that of G_1. Let $\mathbf{n} \geq 3$ be a prime-to-p integer, and let $U_{\mathbf{n}}$ be the open compact subgroup $\ker(G_1(\hat{\mathbb{Z}}^{(p)}) \to G_1(\hat{\mathbb{Z}}^{(p)}/\mathbf{n}\hat{\mathbb{Z}}^{(p)}))$ of $G_1(\mathbb{A}_f^{(p)})$. Let $g := \frac{1}{2}\dim_{\mathbb{Q}} V$.

Recall a *polarized abelian O_B-variety* is a triple $\underline{A} = (A, \lambda, \iota)$, where A is an abelian variety, $\lambda : A \to A^t$ is a polarization, $\iota : O_B \to \mathrm{End}(A)$ is a ring monomorphism such that $\lambda\iota(b^*) = \iota(b)^t\lambda$ for all $b \in O_B$.

By a $(V^p, U_{\mathbf{n}})$*-level structure* on a g-dimensional polarized abelian O_B-variety $\underline{A}$, we mean an $U_{\mathbf{n}}$-orbit $\bar{\eta}$ of isomorphisms, compatible with the additional structures,

$$\eta : V^p \simeq \mathrm{H}_1(A, \hat{\mathbb{Z}}^{(p)})$$

such that the pull-pack of the Weil pairing is ψ, for a suitable trivialization $\hat{\mathbb{Z}}^{(p)} \simeq \hat{\mathbb{Z}}^{(p)}(1)$ modulo $(1 + \mathbf{n}\hat{\mathbb{Z}}^{(p)})^\times$. (Note that $\mathrm{Isom}(\hat{\mathbb{Z}}^{(p)}, \hat{\mathbb{Z}}^{(p)}(1))$ is a $(\hat{\mathbb{Z}}^{(p)})^\times$-torsor. Any isomorphism η as above determines uniquely a trivialization so that the pull-back of the Weil pairing is the ψ. A $U_{\mathbf{n}}$-orbit $\overline{\eta}$ of isomorphisms then determines a $(1 + \mathbf{n}\hat{\mathbb{Z}}^{(p)})^\times$-orbit of trivializations.) We will assume for simplicity that $\mathbf{n}$ is prime to the discriminant of ψ on V^p as well. In this case, the level structure above will reduce to a usual O_B-linear symplectic level-$\mathbf{n}$ structure $\eta : V^p/\mathbf{n}V^p \simeq A[\mathbf{n}]$ for a suitable trivialization $\mathbb{Z}/\mathbf{n}\mathbb{Z} \simeq \mu_{\mathbf{n}}$. Let us choose a primitive $\mathbf{n}$-th root of unity $\zeta_{\mathbf{n}} \in \overline{\mathbb{Q}} \subset \mathbb{C}$, and choose an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$; hence we fix the trivialization $\mathbb{Z}/\mathbf{n}\mathbb{Z} \simeq \mu_{\mathbf{n}}$ in both characteristic 0 and p. Note that for a polarized abelian O_B-scheme $\underline{A}$ over a connected $\mathbb{Z}_{(p)}$-scheme S, the map $s \mapsto [\mathrm{H}_1(A_{\bar{s}}, \hat{\mathbb{Z}}^{(p)})]$ from S to the set of isomorphism classes of skew-Hermitian $O_B \otimes \hat{\mathbb{Z}}^{(p)}$-lattices is constant. This follows from the invariance of étale cohomologies under specialization. Now we can formulate a general PEL-type moduli space.

Keep the notation as above; we write $\mathcal{D}_f^p := (B, *, O_B, V, \psi, V^p, U_{\mathbf{n}}, \zeta_{\mathbf{n}})$. Let $\mathcal{M}[\mathcal{D}_f^p]$ denote the moduli space over $\mathbb{Z}_{(p)}[\zeta_{\mathbf{n}}]$ of polarized abelian O_B-varieties with a $(V^p, U_{\mathbf{n}})$-level structure. An object in $\mathcal{M}[\mathcal{D}_f^p]$ is denoted by $\underline{A} = (A, \lambda, \iota, \eta)$. The moduli scheme is locally of finite type and we have the forgetful morphism

$$f : \mathcal{M}[\mathcal{D}_f^p] \to \mathcal{A}_{g,dp^*,\mathbf{n}} := \coprod_{m \geq 0} \mathcal{A}_{g,dp^m,\mathbf{n}}, \quad (A, \lambda, \iota, \eta) \mapsto (A, \lambda, \eta) \tag{1}$$

for which the induced morphism $f_m : \mathcal{M}[\mathcal{D}_f^p]_m \to \mathcal{A}_{g,dp^m,\mathbf{n}}$ is finite if the source scheme is non-empty, where $d^2 = \mathrm{disc}(\psi)$.

The moduli space we are interested is $\mathcal{M}[\mathcal{D}_f^p]_{\bar{\mathbb{F}}_p}$, the special fiber of $\mathcal{M}[\mathcal{D}_f^p]$ (at the chosen place corresponding to $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$). For simplicity, we shall write $\mathcal{M}_{\mathcal{D}}$ for it, keeping in mind some other auxiliary items are chosen.

(2.4) Remark

(1) We do not require the existence of a self-dual $O_B \otimes \mathbb{Z}_p$-lattice for the choice of the datum $\mathcal{D}$. Therefore the moduli space $\mathcal{M}[\mathcal{D}_f^p]_0$ could be empty and the moduli space $\mathcal{M}_{\mathcal{D}}$ may not contain separably polarized objects.

(2) We do not impose the usual determinant condition on our moduli problem. Hence it is defined over $\mathbb{Z}_{(p)}[\zeta_{\mathbf{n}}]$ in stead of over $O_{E,\mathfrak{p}}[\zeta_{\mathbf{n}}]$, and the moduli space is not flat over $\mathbb{Z}_{(p)}[\zeta_{\mathbf{n}}]$ in general. However, our main theorem works well for its reduction $\mathcal{M}_{\mathcal{D}}$ mod p.

(3) The definition of $\mathcal{M}[\mathcal{D}_f^p]$ only uses the induced $\mathbb{A}_f^{(p)}$-structure of V, not its $\mathbb{Q}$-structure. Therefore the moduli space also includes the abelian varieties of additional structures coming from other data $(\mathcal{D}')_f^p = (B, *, O_B, V', \psi', V'^p, U'_{\mathbf{n}}, \zeta_{\mathbf{n}})$ such that the prime-to-p (V'^p, O_B, ψ') is isomorphic to (V^p, O_B, ψ).

(4) It is more flexible to work with $\mathcal{M}_{\mathcal{D}}$ than a moduli space on which a determinant condition and the separable polarization condition both are imposed. Because these two conditions are not preserved under an inseparable isogeny, and it is our interest to allow performing the correspondences from isogenies of p-power degrees. Another reason we do not wish to impose a determinant condition even for smooth cases is

that inseparable isogenies can "connect" points in two moduli spaces with different determinant conditions imposed. For example, let O_K be the integral ring of an imaginary quadratic field K in which p is inert. Then an abelian O_K-fourfold whose Lie algebra has signature type $(1,3)$ in characteristic p can be isogenous to another O_K-fourfold with signature type $(2,2)$.

§3. Statement of main theorems

(3.1) Let k be an algebraically closed field of characteristic p, W the ring of Witt vectors over k, and let $L := \mathrm{Frac}(W)$ and σ the Frobenius map on L. Keep the notation as in the previous section.

In practice, we can assume that B is a division algebra, as the moduli space $\mathcal{M}_{\mathcal{D}}$ is essentially a finite product of the moduli spaces $\mathcal{M}_{\mathcal{D}'}$ of this simpler kind (and it is so when O_B is a maximal order).

(3.1.1) If B is a division algebra, the neutral component G^0 of G is commutative if

(CM) $B = K$ is a CM field and $\dim_K V = 1$, or

(0-dim D) B is a totally definite quaternion algebra over a totally real field F and $\dim_B V = 1$.

These are the cases where the Shimura variety $\mathrm{Sh}(G, X)$ is 0-dimensional.

Let $\underline{A} = (A, \lambda, \iota)$ be a g-dimensional polarized abelian O_B-variety over k, where $2g = \dim_{\mathbb{Q}} V$. Let $\underline{H}$ be the associated p-divisible group and $\underline{M}$ the attached covariant Dieudonné module.

(3.2) Definition Suppose that B is a division algebra.

(1) (*Excluding Case* (0-dim D)) The object $\underline{A}$ is called *basic* if it satisfies the following conditions

$$(a)\ \ \dim_L \mathrm{End}_{B\otimes L}(M \otimes \mathbb{Q}_p) = \dim_{\mathbb{Q}_p} \mathrm{End}^0_B(H),$$
$$(b)\ \ \dim_{\mathbb{Q}_p} \mathrm{End}^0_B(H) = \dim_{\mathbb{Q}} \mathrm{End}^0_B(A).$$

(2) (*Case* (0-dim D)) Any object $\underline{A}$ is called *basic* unconditionally.

Note that in Case (CM), any object $\underline{A}$ is basic as the conditions in (1) of Definition 3.2 are satisfied.

(3.2.1) One can similarly define the basic objects for the case where B is semi-simple. Write $B = \oplus M_{n_i}(B_i)$ into simple factors, where each B_i is a division algebra. Then the object $\underline{A}$ is isogenous to $\oplus \underline{A}_i^{n_i}$, where each $\underline{A}_i$ is a polarized abelian O_{B_i}-variety. Define $\underline{A}$ to be basic if each member $\underline{A}_i$ is basic as in (3.2). Then one can easily check that if (B, V) has no factor of type (0-dim D), $\underline{A}$ is basic if and only if the conditions (a) and (b) above are satisfied.

The notion of basic points (or classes) is due to Kottwitz [7]. It plays the similar role in Shimura varieties in positive characteristic as supersingular abelian varieties do in the Siegel moduli spaces. We include Kottwitz's definition for convenience of discussion. Note that when G is not connected, the definition for the basic classes is not given explicitly in [9] but the notion should be clear (cf. [15, p. 291, 6.25]).

(3.3) Definition (Kottwitz)

(1) Let (V_p, ψ_p) be a $\mathbb{Q}_p$-valued non-degenerate skew-Hermitian B_p-module, where $B_p := B \otimes \mathbb{Q}_p$. An object $\underline{A}$ is said to be *related to* (V_p, ψ_p) if there is a $B_p \otimes L$-linear isomorphism $\alpha : M(\underline{A}) \otimes_W L \simeq (V_p, \psi_p) \otimes L$ which preserves the pairings for a suitable identification $L(1) \simeq L$.

Let $G' := \mathrm{GAut}_{B_p}(V_p, \psi_p)$ be the algebraic group of B_p-linear similitudes. A choice α gives rise to an element $b \in G'(L)$ so that one has an isomorphism of isocrystals with additional structures $M(\underline{A}) \otimes L \simeq (V_p \otimes L, \psi_p, b(\mathrm{id} \otimes \sigma))$. The decomposition of $V_p \otimes L$ into isoclinic components induces a $\mathbb{Q}$-graded structure, and thus defines a (slope) homomorphism $\nu_{[b]} : \mathbf{D} \to G'$ over some finite extension $\mathbb{Q}_{p^s}$ of $\mathbb{Q}_p$, where $\mathbf{D}$ is the pro-torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$.

(2) An object $\underline{A}$ is called *basic with respect to* (V_p, ψ_p) if
   (i) $\underline{A}$ is related to (V_p, ψ_p), and
   (ii) the slope homomorphism ν is central.

(3) An object $\underline{A}$ is called *basic* if it is basic with respect to (V_p, ψ_p) for some skew-Hermitian space (V_p, ψ_p).

The main result of this paper is the following

(3.4) Theorem *Let $\mathcal{M}_{\mathcal{D}}$ be as in (2.3). Then any irreducible component of $\mathcal{M}_{\mathcal{D}}$ contains a basic point in the sense of (3.2).*

Our definition is not the same as Kottwitz's when the datum $\mathcal{D}$ contains a factor of type (0-dim D). It is not hard to see that the condition (b) of (3.2) is equivalent to that for Kottwitz's (Lemma 5.3). Therefore, Theorem 3.4 particularly implies

(3.5) Corollary *If $\mathcal{D}$ does not contain a factor of type* (0-dim D)*, then any irreducible component of $\mathcal{M}_{\mathcal{D}}$ contains a basic point (in the sense of Kottwitz).*

When the rational PEL datum $\mathcal{D}$ has no factor of type (0-dim), Corollary 3.5 confirms a conjecture of Rapoport [12, p. 427 Remark 2.2 (i)] on the existence of basic points in the moduli space of PEL-type. We remark our result is stronger than Rapoport's conjecture. We do not assume the existence of self-dual lattices for $\mathcal{D}$ nor impose a determinant condition as well as the separability condition for the moduli spaces $\mathcal{M}_{\mathcal{D}}$; and basic points do exist in each component not just in the reduction of a Shimura variety of PEL-type, which consists of finitely many irreducible components.

For the case (0-dim D) (in fact for cases of type (C) and (D)), Kottwitz's basic points are exactly supersingular points. In this case, one can construct directly a supersingular point in the Shimura variety. See an example in (3.7).

In Theorem 3.4, we prove the existence of points which are seemingly stronger than Kottwitz's basic ones. However, like supersingular points, Kottwitz's basic points should also share the properties (a) and (b) in (3.2). We further prove that it is indeed the case. The proof we give, which is much simpler than our original one, is communicated to the author by C.-L. Chai.

(3.6) Theorem *An object $\underline{A}$ satisfies the conditions (a) and (b) in (3.2) if and only if it is basic in the sense of Kottwitz.*

(3.7) Example Let D be a definite quaternion algebra over $\mathbb{Q}$. It is known that the canonical involution is the unique positive one. Take an ordinary elliptic curve E over a finite field k such that $\mathrm{End}(E)$ is the maximal order of $K := \mathrm{End}(E) \otimes \mathbb{Q}$. Note that p splits in K.

Assume that D splits over K. This means that for every place v of $\mathbb{Q}$ for which D is ramified, K_v is a field. This particularly implies that p is unramified in D.

Let $A = E \oplus E$. Then we have an embedding $D \to D \otimes K \simeq \mathrm{End}(A) \otimes \mathbb{Q} = M_2(K)$ so that a maximal order O_D is contained in $\mathrm{End}(A)$. One can choose a principal O_K-linear polarization on A. Then it induces the positive involution on D. Then we get an ordinary principally polarized abelian O_D-surface.

However, we also can construct a principally polarized abelian O_D-surface which is superspecial. For simplicity, assume that D is ramified exactly at $\{\infty, \ell\}$. Let D' be the quaternion algebra over $\mathbb{Q}$ which is ramified at $\{\ell, p\}$. Then $D \otimes D' \simeq M_2(D'') = \mathrm{End}(A) \otimes \mathbb{Q}$ for a superspecial abelian surface A on which O_D-acts. Again one can choose a principal polarization on A which preserves D, and thus construct a desire one.

We find a definite quaternion algebra D, together with a maximal order O_D, in which p is unramified such that the moduli space over $\overline{\mathbb{F}}_p$ of principally polarized abelian O_D-surfaces is non-empty and contains both superspecial points and ordinary points.

§4. Hecke orbits and proof of Theorem 3.4

(4.1) Keep the notation as in Section 2. Let $x = \underline{A} = (A, \lambda, \iota, \eta) \in \mathcal{M}_{\mathcal{D}}(k)$. Let G_x denote the automorphism group scheme over $\mathbb{Z}$ associated to x; for any commutative ring R, its group of R-points is

$$\{ g \in \mathrm{End}_{O_B}(A) \otimes R;\ g'g = 1 \},$$

where $g \mapsto g'$ is the Rosati involution induced by λ.

Denote by $\mathcal{C}(x, \mathcal{M}_{\mathcal{D}})$ the *leaf* passing through x in $\mathcal{M}_{\mathcal{D}}$ ([11, Section 3],[17]); it is a reduced subscry heme of $\mathcal{M}_{\mathcal{D}}$ over k that is characterized by the property

$$\mathcal{C}(x, \mathcal{M}_{\mathcal{D}})(K) = \{ y \in \mathcal{M}_{\mathcal{D}}(K);\ \exists\, \varphi : \underline{A}_y[p^\infty] \simeq \underline{A}[p^\infty] \text{ over } K \}$$

for any algebraically closed field K containing k. Write $\mathcal{C}_x$ for the irreducible component of $\mathcal{C}(x, \mathcal{M}_{\mathcal{D}})$ containing x. We will also write $\mathcal{C}(x)$ for $\mathcal{C}(x, \mathcal{M}_{\mathcal{D}})$ when the moduli space $\mathcal{M}_{\mathcal{D}}$ is fixed.

(4.2) Proposition *Notation as above.*

(1) *For any $x \in \mathcal{M}_{\mathcal{D}}(k)$, $\mathcal{C}(x)$ is quasi-affine, smooth and of equi-dimensional.*

(2) *If $\dim \mathcal{C}(x) > 0$, then the Zariski closure $\overline{\mathcal{C}}_y$ of any irreducible component $\mathcal{C}_y$ in $\mathcal{M}_{\mathcal{D}}$ is strictly larger than $\mathcal{C}_y$.*

PROOF. (1) The quasi-affineness follows from the finiteness of the forgetful morphism from $\mathcal{M}_{\mathcal{D}}$ to $\mathcal{A}_{g,dp^*,\mathbf{n}}$ and a result of Oort that any leaf in a Siegel moduli space is quasi-affine [11]. It follows from the Serre-Tate theorem and Corollary 1.7 in [11] that for any two points x_1, x_2 in $\mathcal{C}(x)$, one has $\mathcal{C}(x)^\wedge_{x_1} \simeq \mathcal{C}(x)^\wedge_{x_2}$ (cf. [11, Theorem 3.13]). Then the remaining two properties follow.

(2) (TO BE FILLED) ∎

Let $\mathcal{H}^{(p)}(x)$ denote the *prime-to-p Hecke orbit* of x in $\mathcal{M}_{\mathcal{D}}(k)$; it is the subset of $\mathcal{M}_{\mathcal{D}}(k)$ that consists of objects $\underline{A}_y$ such that there is a prime-to-p O_B-linear quasi-isogeny $\varphi : A_y \to A$ which preserves the polarizations. Clearly, one has $\mathcal{H}^{(p)}(x) \subset \mathcal{C}(x)(k)$.

(4.3) Proposition *There is a natural isomorphism between $\mathcal{H}^{(p)}(x)$ and $G_x(\mathbb{Z}_{(p)}) \backslash G_1(\mathbb{A}_f^p)/U_{\mathbf{n}}$.*

PROOF. For any two objects $\underline{A}_i = (A_i, \lambda_i, \iota_i, \eta_i) \in \mathcal{M}_{\mathcal{D}}(k)$ $(i = 1, 2)$ and any prime $\ell \neq p$, we let

- $\mathrm{Isom}(\underline{A}_1, \underline{A}_2)$ (resp. $\mathrm{Isom}(\underline{A}_1[\ell^\infty], \underline{A}_2[\ell^\infty])$) be the set of O_B-linear isomorphisms from A_1 to A_2 (resp. from $A_1[\ell^\infty]$ to $A_2[\ell^\infty]$) which preserve the polarizations and the level structures, and
- $\text{Q-isom}^{(p)}(\underline{A}_1, \underline{A}_2)$ (resp. $\text{Q-isom}(\underline{A}_1[\ell^\infty], \underline{A}_2[\ell^\infty])$) be the set of prime-to-p O_B-linear quasi-isogenies from A_1 to A_2 (resp. from $A_1[\ell^\infty]$ to $A_2[\ell^\infty]$) which preserve the polarizations.

One has, and via $\bar{\eta}$ has

$$\text{Q-isom}^{(p)}(\underline{A}, \underline{A}) = G_x(\mathbb{Z}_{(p)}), \quad \text{Q-isom}(\underline{A}[\ell^\infty], \underline{A}[\ell^\infty]) = G_1(\mathbb{Q}_\ell),$$

$$\prod_{\ell \neq p} \mathrm{Isom}(\underline{A}[\ell^\infty], \underline{A}[\ell^\infty]) = U_{\mathbf{n}}, \quad G_x(\mathbb{Q}_\ell) \subset G_1(\mathbb{Q}_\ell).$$

Given an element $\underline{A}_1 \in \mathcal{H}^{(p)}(x)$, consider the natural map

$$m(\underline{A}_1) : \text{Q-isom}^{(p)}(\underline{A}_1, \underline{A}) \times \prod_{\ell} \mathrm{Isom}_k(\underline{A}[\ell^\infty], \underline{A}_1[\ell^\infty]) \to {\prod_{\ell \neq p}}' \, \text{Q-isom}(\underline{A}[\ell^\infty], \underline{A}[\ell^\infty]) = G_1(\mathbb{A}_f^p) \qquad (2)$$

which sends $(\phi, (\alpha_\ell)_\ell)$ to $(\phi\alpha_\ell)_\ell$. Clearly if c_1 is an element in the image $c(\underline{A}_1)$ of $m(\underline{A}_1)$, then $c(\underline{A}_1)$ equals to the double coset $G_x(\mathbb{Z}_{(p)})\, c_1\, U_{\mathbf{n}}$. Thus, $c(\underline{A}_1)$ defines an element in $G_x(\mathbb{Z}_{(p)}) \backslash G(\mathbb{A}_f^p)/U_{\mathbf{n}}$.

If we have $\underline{A}_1, \underline{A}_2 \in \mathcal{H}^{(p)}(x)$ such that $c(\underline{A}_1) = c(\underline{A}_2)$. Write $c(\underline{A}_1) = [(\phi_1\alpha_\ell)_\ell]$ and $c(\underline{A}_2) = [(\phi_2\alpha'_\ell)_\ell]$. Then there exist $b \in G_x(\mathbb{Z}_{(p)})$ and $k_\ell \in U_{\mathbf{n},\ell}$ for all $\ell \neq p$ such that $b\phi_1\alpha_\ell k_\ell = \phi_2\alpha'_\ell$. Then

$$(b\phi_1)^{-1}\phi_2 = \alpha_\ell k_\ell(\alpha'_\ell)^{-1} \in \text{Q-isom}^{(p)}(\underline{A}_2, \underline{A}_1) \cap \prod_{\ell \neq p} \text{Isom}(\underline{A}_2[\ell^\infty], \underline{A}_1[\ell^\infty]) = \text{Isom}(\underline{A}_2, \underline{A}_1).$$

Thus $\underline{A}_2 \simeq \underline{A}_1$ and this shows the injectivity of c.

Given $[(\phi_\ell)_\ell]$ in $G_x(\mathbb{Z}_{(p)})\backslash G_1(\mathbb{A}_f^p)/U_\mathbf{n}$, choose an positive prime-to-p integer N so that $f_\ell := N\phi_\ell^{-1}$ is an isogeny for all $\ell \neq p$. Let H be the product of the kernels of $N\phi_\ell^{-1}$; it is a finite subgroup scheme of A invariant under the O_B-action. Take $A_1 := A/H$ and let $\pi : A \to A_1$ be the natural projection; A_1 is equipped with a natural action by O_B so that π is O_B-linear. Let $\lambda_1 \in \text{Hom}(A_1, A_1^t) \otimes \mathbb{Z}_{(p)}$ be the fractional polarization on A_1 such that $(N^{-1}\pi)^*\lambda_1 = \lambda$; it is O_B-linear as π is so. As π_ℓ and f_ℓ have the same kernel, there is an element $\alpha_\ell \in \text{Isom}_k(\underline{A}[\ell^\infty], \underline{A}_1[\ell^\infty])$ such that $\alpha_\ell f_\ell = \pi_\ell$. This shows that λ_1 is actually in $\text{Hom}_{O_B}(A, A^t)$ and one obtains $\underline{A}_1 \in \mathcal{H}^{(p)}(x)$. Put $\phi := (N^{-1}\pi)^{-1} \in \text{Q-isom}^{(p)}(\underline{A}_1, \underline{A})$. One checks

$$\phi\alpha_\ell = N\pi_\ell^{-1}\alpha_\ell = Nf_\ell^{-1} = \phi_\ell.$$

This shows $c(\underline{A}_1) = [(\phi_\ell)_\ell]$ and the surjectivity of c.

(4.4) Remark A special case of Proposition 4.3 is proved in [2, Section 1] for the Siegel moduli spaces.

(4.5) Lemma *Let H be an algebraic group over $\mathbb{Q}$. Then $H^0(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/U$ is finite, where U is an open compact subgroup of $H(\mathbb{A}_f)$ and H^0 is the neutral component of H.*

Proof. Choose a faithful representation $H \to \text{GL}_n$ and fix an integral structure of H. Then it suffices to show that $H^0(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/H(\hat{\mathbb{Z}})$ is finite, or equivalently that $H^0(\mathbb{Q}_\ell)\backslash H(\mathbb{Q}_\ell)/H(\mathbb{Z}_\ell)$ is trivial for almost all ℓ. One has

$$H(\mathbb{Z}_\ell)/H^0(\mathbb{Z}_\ell) \subset H(\mathbb{Q}_\ell)/H^0(\mathbb{Q}_\ell) \subset H/H^0(\mathbb{Q}_\ell) = H/H^0(\mathbb{Z}_\ell).$$

By Lang's Theorem, $H(\mathbb{Z}_\ell)/H^0(\mathbb{Z}_\ell) = H/H^0(\mathbb{Z}_\ell)$ for almost all ℓ. Hence $H^0(\mathbb{Q}_\ell)\backslash H(\mathbb{Q}_\ell)/H(\mathbb{Z}_\ell)$ is trivial for almost all ℓ. ∎

(4.6) Lemma *If each factor of $\mathcal{D}$ is of type* (0-dim D)*, then $\mathcal{H}^{(p)}(x)$ is finite.*

Proof. By Proposition 4.3, we want to show that $G_x(\mathbb{Z}_{(p)})\backslash G_1(\mathbb{A}_f^p)/U_1^p$, for any open compact subgroup U_1^p of $G_1(\mathbb{A}_f^p)$, is finite. Let K be a field of finite type over $\mathbb{F}_p$ for which $\underline{A}$ is defined. As G^0 is commutative, the image of the ℓ-adic representations of $\text{Gal}(K^{\text{sep}}/K)$ is commutative after replacing K by a finite extension. By Zarhin's theorem [18], A is of CM-type, and hence we may assume, by a theorem of Grothendieck, that K is finite, after replacing $\underline{A}$ by an isogeny. By Tate's theorem on endomorphisms of abelian varieties, we proved

$$G_x^0(\mathbb{A}_f^p) = G_1^0(\mathbb{A}_f^p) \tag{3}$$

By Lemma 4.5, $G_1^0(\mathbb{A}_f^p)\backslash G_1(\mathbb{A}_f^p)/U_1^p$ is finite. Let $c_1, \dots, c_h$ be a set of complete representatives for $G_1^0(\mathbb{A}_f^p)\backslash G_1(\mathbb{A}_f^p)/U_1^p$. Then we have

$$G_1(\mathbb{A}_f^p)/U_1^p \simeq \coprod_{i=1}^h G_1^0(\mathbb{A}_f^p)/U_i, \quad U_i := c_i U_1^p c_i^{-1} \cap G_1^0(\mathbb{A}_f^p).$$

Using (3), it remains to prove the finiteness of $G_x^0(\mathbb{Z}_{(p)})\backslash G_x^0(\mathbb{A}_f^p)/U_i$. But this follows from the finiteness theorem of Borel, and thus the proof is complete. ∎

(4.7) Lemma *If $\mathcal{D}$ has no factor of type* (0-dim D)*, then $\underline{A}$ is basic if and only if $\mathcal{H}^{(p)}(x)$ is finite.*

Proof. If x is basic, then $G_x(\mathbb{A}_f^p) = G_1(\mathbb{A}_f^p)$ by the conditions (a) and (b) of (3.2). By Proposition 4.3 and the finiteness theorem of Borel, $\mathcal{H}^{(p)}(x)$ is finite.

Suppose that x is not basic. This implies that G_x^0 is a proper reductive subgroup of G_1^0 over $\mathbb{Q}_\ell$ for all $\ell \neq p$. Using the proof of Lemma 4.6, the finiteness of $G_x(\mathbb{Z}_{(p)})\backslash G_1(\mathbb{A}_f^p)/U_1^p$ is the same as that of $G_x^0(\mathbb{Z}_{(p)})\backslash G_1^0(\mathbb{A}_f^p)/U_1'^p$, where $U_1'^p$ is an open compact subgroup of $G_1^0(\mathbb{A}_f^p)$. As the space $G_x^0(\mathbb{Z}_{(p)})\backslash G_x^0(\mathbb{A}_f^p)/U_x^p$ is finite, the finiteness is also the same as that of $G_x^0(\mathbb{A}_f^p)\backslash G_1^0(\mathbb{A}_f^p)/U_1'^p$. As $G_x^0(\mathbb{Q}_\ell)\backslash G_1^0(\mathbb{Q}_\ell)/G_1^0(\mathbb{Z}_\ell)$ is included in the latter double coset space for some ℓ and it is not finite, $G_x^0(\mathbb{A}_f^p)\backslash G_1^0(\mathbb{A}_f^p)/U_1'^p$ is not finite. Indeed, choose a parabolic subgroup P of G_1^0 which contains G_x^0 as a Levi factor and write $P = G_x^0 U$ (U=the unipotent radical of P), then $G_x^0(\mathbb{Q}_\ell)\backslash P(\mathbb{Q}_\ell)/P(\mathbb{Z}_\ell) = U(\mathbb{Q}_\ell)/U(\mathbb{Z}_\ell)$ is not finite. Therefore, the proof is complete. ∎

By Lemmas 4.6 and 4.7, we proved

(4.8) Proposition *An object $x = \underline{A}$ is basic if and only if $\mathcal{H}^{(p)}(x)$ is finite.*

A special case of Proposition 4.8 is proved in [2, Proposition 1] for Siegel moduli spaces by a different proof. The proof presented here is elementary and does not rely on Wang's generalization of the Borel density theorem ([16, Cor. 1.4]).

(4.9) Proof of Theorem 3.4. Let $\mathcal{M}'$ be an irreducible component of $\mathcal{M}_{\mathcal{D}}$. Pick any point x on $\mathcal{M}'$. If x is basic, then we are done. Suppose that x is not basic, then $\mathcal{C}(x, \mathcal{M}_{\mathcal{D}})$ has positive dimension. By (2) of Proposition 4.2, there is a point y in $\overline{\mathcal{C}_x} \setminus \mathcal{C}_x$ in $\mathcal{M}'$ with $\dim \mathcal{C}(y, \mathcal{M}_{\mathcal{D}})$ smaller. Continuing this process we achieve a basic point in $\mathcal{M}'$. This completes the proof.

§5. Properties of basic points

(5.1) The goal of this section is to prove Theorem 3.6. We follow closely Chapter 6 of Rapoport-Zink [15]. For the convenience of the reader, we include some results in loc. cit. on basic abelian varieties in question. The only new result is Proposition 5.6.

Keep the notation as in Section 2. Let F be the center of B and F_0 be the subfield fixed by the involution on F, which we will denote by $a \mapsto \bar{a}$. Let Σ_p be the set of primes of F over p, and for a prime $\mathbf{p}|p$, write $\mathrm{ord}_{\mathbf{p}}$ the corresponding p-adic valuation normalized so that $\mathrm{ord}_{\mathbf{p}}(p) = 1$. Let $F_p := F \otimes \mathbb{Q}_p = \prod_{\mathbf{p}|p} F_{\mathbf{p}}$ be the decomposition as a product of local fields. For each isocrystal N with an F_p-linear action, let

$$N = \oplus_{\mathbf{p}|p} N_{\mathbf{p}},$$

be the decomposition with respect to the F_p-action.

In this section, basic points will mean those in the sense of Kottwitz (3.3).

(5.2) Lemma

(1) *The center of G is the algebraic group over $\mathbb{Q}$ whose group of $\mathbb{Q}$-rational points is*

$$\{g \in F^\times ; g\bar{g} \in \mathbb{Q}^\times \}.$$

(2) *Let N be an isocrystal with additional structures and suppose that it is related to $(V \otimes \mathbb{Q}_p, \psi)$ (3.3). Then N is basic with respect to $(V \otimes \mathbb{Q}_p, \psi)$ if and only if each component $N_{\mathbf{p}}$ is isotypic. In particular, if N is basic, then $N_{\mathbf{p}}$ is supersingular for primes $\mathbf{p}$ with $\mathbf{p} = \bar{\mathbf{p}}$.*

Proof. This is proved in 6.25 of loc. cit.

(5.3) Lemma *Let H be a quasi-polarized p-divisible O_{B_p}-group. Suppose that the associated isocrystal $N := M \otimes \mathbb{Q}_p$ is related to $(V \otimes \mathbb{Q}_p, \psi)$. Then the condition (a) of (3.2) is satisfied for H if and only if each component $N_{\mathbf{p}}$ is isotypic. Therefore, N is basic if and only if it satisfies the condition (a).*

PROOF. This follows immediately from the dimensions of the endomorphism algebras. ∎

(5.4) Lemma *Given any set $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ of rational numbers with $0 \le \lambda_{\mathbf{p}} \le 1$ and $\lambda_{\mathbf{p}} + \lambda_{\bar{\mathbf{p}}} = 1$, then there is a positive integer s and $u \in O_F[\frac{1}{p}]^\times$ such that*

$$u\overline{u} = q, \quad \text{and} \quad \operatorname{ord}_{\mathbf{p}} u = s\lambda_{\mathbf{p}}, \ \forall\, \mathbf{p} \in \Sigma_p,$$

where $q = p^s$.

PROOF. Consider the map

$$\operatorname{ord} : O_F[\frac{1}{p}]^\times \to \mathbb{Z}^{\Sigma_p}, \quad u \mapsto (\operatorname{ord}_{\mathbf{p}}(u))_{\mathbf{p}\in\Sigma_p}.$$

By Dirichlet's unit theorem, the image is of finite index. Therefore, there is a positive integer s such that there is an element $u \in O_F[\frac{1}{p}]^\times$ so that $\operatorname{ord}_{\mathbf{p}}(u) = s\lambda_{\mathbf{p}} =: r_{\mathbf{p}}$ for all $\mathbf{p} \in \Sigma_p$. Let $q = p^s$ and $u' := qu/\bar{u}$, then one computes

$$\operatorname{ord}_{\mathbf{p}} u' = 2r_{\mathbf{p}}, u'\overline{u}' = q^2.$$

Replacing u by u' and q by q^2, one gets the desire results. ∎

(5.5) Lemma *Fix $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$, $q = p^s$ as in Lemma 5.4. Then there is a positive integer N such that for any basic polarized abelian O_B-variety $\underline{A}$ over a finite extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ with slopes $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$, the N-th power of relative Frobenius morphism π_A^N lies in $\iota(F)$.*

PROOF. We first prove that the statement holds for one such object $\underline{A}$. Let M be the Dieudonné module of $\underline{A}$. Within the isogeny class, we can choose $\underline{A}$ so that $F^s M_{\mathbf{p}} = p^{r_{\mathbf{p}}} M_{\mathbf{p}}$ for all $\mathbf{p} \in \Sigma_p$, where $r_{\mathbf{p}} = s\lambda_{\mathbf{p}}$. Let u be as in Lemma 5.4, then $\iota(u)^{-m}\pi_A$ as an automorphism of A that preserves the polarization. Therefore, a power of it is the identity.

Let $C := \operatorname{End}^0_B(A)$. By the result we just proved, C is independent of $\underline{A}$ and has center $\iota(F)$. Therefore, there is a positive integer N so that any roots of unity ζ in C satisfies $\zeta^N = 1$.

Repeat the same proof above and we get $\pi_A^N \in \iota(F)$ for all such objects $\underline{A}$. ∎

(5.6) Proposition *Let $\underline{A}$ be a basic polarized abelian O_B-variety over k. Then there exist a polarized abelian O_B-variety $\underline{A}'$ over a finite field and an O_B-linear isogeny $\varphi : A' \to A$ over k that preserves the polarizations.*

PROOF. It suffices to show that A is of CM-type. Then there exist an abelian O_B-variety (A', ι') over a finite field and an O_B-linear isogeny $\varphi : A' \to A$ over k. Then take the pull-back polarization λ' on A', which particularly is defined over a finite field.

Let $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$ be the set of slopes for $\underline{A}$. Let $q = p^s$ and N be as in Lemmas 5.4 and 5.5. Let K be a field of finite type of $\mathbb{F}_q$ that $\underline{A}$ is defined. The $\underline{A}$ extends to a polarized abelian O_B-scheme $\underline{\mathbf{A}}$ over a subring R of K with $\operatorname{Frac}(R) = K$, smooth and of finite type over $\mathbb{F}_q$. Let $S = \operatorname{Spec} R$. Let s be a closed point of S and η be the generic point. By Grothendieck's specialization theorem, the special fiber $\underline{\mathbf{A}}_s$ over s also has the same slopes $\{\lambda_{\mathbf{p}}\}_{\mathbf{p}|p}$, and hence is basic.

We identify the endomorphism rings $\operatorname{End}_R(\mathbf{A}) = \operatorname{End}_K(A) \subset \operatorname{End}(\mathbf{A}_{\bar{s}})$, and write ι for the O_B-actions on these abelian varieties. Let

$$\rho_\ell : \pi_1(S, \bar{\eta}) \to \operatorname{Aut}(T_\ell(A_{\bar{\eta}}))$$

be the associated ℓ-adic representation. The action of $\mathrm{Gal}(\bar\eta/\eta)$ on $T_\ell(A_{\bar\eta})$ factors through ρ_ℓ. Again we identify the Tate modules $T_\ell(\mathbf{A}_{\bar s}) = T_\ell(\mathbf{A}_{\widetilde{S}_{\bar s}}) = T_\ell(A_{\bar\eta})$, where $\widetilde{S}_{\bar s}$ is the (strict) Henselization of S at $\bar s$.

Let π_{A_s} be the relative Frobenius morphism on $\mathbf{A}_s$ and Frob_s the geometric Frobenius element in $\pi_1(S,\bar\eta)$ corresponding to the closed point s. We have

(i) $\pi_{A_s}^N \in \iota(F) \subset \mathrm{End}(T_\ell(\mathbf{A}_{\bar s}))$, by Lemma 5.5;

(ii) $\rho_\ell(\mathrm{Frob}_s^N) = \pi_{A_s}^N$ lies in the center $Z(\mathbb{Q}_\ell)$ of $\mathrm{GAut}_{B_\ell}(T_\ell(A_{\bar\eta}), \langle\,,\rangle)$, by identifying the Tate modules and (i);

(iii) the Frobenius elements Frob_s for all closed points generate a dense subgroup of $\pi_1(S,\bar\eta)$.

Let $G_\ell := \rho_\ell(\pi_1(S,\bar\eta))$ be the ℓ-adic monodromy group. Let $m_N : G_\ell \to G_\ell$ be the multiplication by N. It is an open mapping and the image of m_N contains an open subgroup U of G_ℓ. Clearly U lies in the center $Z(\mathbb{Q}_\ell)$ by (ii) and (iii). Replacing K by a finite extension, we have $G_\ell \subset Z(\mathbb{Q}_\ell)$. Let $\mathbb{Q}_\ell[\pi]$ be the (commutative) subalgebra of $\mathrm{End}(T_\ell(A_{\bar\eta}))$ generated by G_ℓ. By Zarhin's theorem [18], $\mathbb{Q}_\ell[\pi]$ is semi-simple and commutative, and $\mathrm{End}_{\mathbb{Q}_\ell[\pi]}(T_\ell(A_{\bar\eta})) = \mathrm{End}(A) \otimes \mathbb{Q}_\ell$. This shows that any maximal commutative semi-simple subalgebra of $\mathrm{End}^0(A)$ has degree $2g$. This completes the proof. ∎

Theorem 3.6 follows from Lemmas 5.3, 5.5 and Proposition 5.6.

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