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COMPATIBLE POISSON BRACKETS ASSOCIATED WITH ELLIPTIC CURVES IN $G(2,5)$

NIKITA MARKARIAN AND ALEXANDER POLISHCHUK

Abstract. We prove that a pair of Feigin-Odesskii Poisson brackets on $\mathbb{P}^4$ associated with elliptic curves given as linear sections of the Grassmannian $G(2,5)$ are compatible if and only if this pair of elliptic curves is contained in a del Pezzo surface obtained as a linear section of $G(2,5)$.

1. Introduction

We work over an algebraically closed field $\mathbf{k}$ of characteristic 0.

In this paper we continue to study compatible pairs among the Poisson brackets on projective spaces introduced by Feigin-Odesskii (see [1], [10]). Their construction associates with every stable vector bundle $\mathcal{V}$ of degree $n>0$ and rank k on an elliptic curve E, a Poisson bracket on the projective space $\mathbb{P}H^0(E,\mathcal{V})^*$. We refer to such Poisson brackets as FO brackets of type $q_{n,k}$.

Two Poisson brackets are called *compatible* if the corresponding bivectors satisfy $[\Pi_1,\Pi_2]$ (equivalently, any linear combination of these brackets is again Poisson). In [9] Odesskii and Wolf discovered 9-dimensional spaces of compatible FO brackets of type $q_{n,1}$ on $\mathbb{P}^{n-1}$ for each $n\geq 3$. Their construction was interpreted and extended in [3], where the authors showed that one gets compatible FO brackets if the elliptic curves are anticanonical divisors on a surface S and the stable bundles on them are restrictions of a single exceptional bundle on S that forms an exceptional pair with $\mathcal{O}_S$ (see [3, Thm. 4.4]). One can ask whether any two compatible FO brackets of type $q_{n,k}$ on $\mathbb{P}^{n-1}$ appear in this way. In [7] we have shown that this is the case for $k=1$ (for some specific rational surfaces containing normal elliptic curves in projective spaces). In the present work, we consider the case of FO brackets of type $q_{5,2}$ on $\mathbb{P}^4$. Note that the question of finding bihamiltonian structures with brackets of type $q_{5,2}$ was raised by Rubtsov in [11].

Let V be a 5-dimensional vector space. Consider the Plucker embedding

$$G(2,V)\to\mathbb{P}(\bigwedge^2 V).$$

It is well known that for a generic 5-dimensional subspace $W\subset\bigwedge^2 V$ the corresponding linear section

$$E_W:=G(2,V)\cap\mathbb{P}W$$

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is an elliptic curve. Furthermore, if $\mathcal{U} \subset V \otimes \mathcal{O}$ is the universal subbundle on $G(2,V)$, then one can check that the restriction

$$V_W := \mathcal{U}^\vee|_{E_W}$$

is a stable bundle of rank 2 and degree 5 on E_W (see Lemma 2.2.1 below). Thus, we have the corresponding Feigin-Odesskii bracket of type $q_{5,2}$ on $\mathbb{P}H^0(E_W, V_W)^*$.

Furthermore, one can check that the restriction map

$$V^* = H^0(G(2,V), \mathcal{U}^\vee) \to H^0(E_W, V_W)$$

is an isomorphism (see Lemma 2.2.1). Thus, we get a Poisson bracket Π_W on $\mathbb{P}V$ (defined up to a rescaling).

On the other hand, we have a natural $\mathrm{GL}(V)$-invariant map

$$\pi_{5,2} : \bigwedge^5(\bigwedge^2 V) \to H^0(\mathbb{P}V, \bigwedge^2 T) \otimes \det^2(V)$$

constructed as follows.

Note that we have a natural isomorphism $V \simeq H^0(\mathbb{P}V, T(-1))$, hence we get a natural map $V \otimes \mathcal{O}(1) \to T$, and hence, the composed map

$$\phi : W \otimes \mathcal{O}(2) \to \bigwedge^2 V \otimes \mathcal{O}(2) \to \bigwedge^2 T$$

on $\mathbb{P}V$. Taking the 5th exterior power of this map, we get a map

$$\bigwedge^5(\phi) : \det(W) \otimes \mathcal{O}(10) \to \bigwedge^5(\bigwedge^2 T) \simeq (\bigwedge^2 T)^\vee \otimes \det^3(T),$$

where we used the identification $\det(\bigwedge^2 T) \simeq \det^3(T)$. Note that we have a nondegenerate pairing given by the exterior product,

$$\bigwedge^2 T \otimes \bigwedge^2 T \to \det(T),$$

hence, we have an isomorphism $\bigwedge^2 T \simeq (\bigwedge^2 T)^\vee \otimes \det(T)$, and we can rewrite the above map as

$$\det(W) \to \bigwedge^2 T \otimes \det^2(T)(-10) \simeq \bigwedge^2 T \otimes \det^2(V).$$

Theorem A. *For every* 5*-dimensional subspace* $W \subset \bigwedge^2 V$*, such that* $E_W := G(2,V) \cap \mathbb{P}W$ *is an elliptic curve, one has an equality*

$$\pi_{5,2}(\lambda_W) = \Pi_W \otimes \delta,$$

for some trivializations $\lambda_W \in \bigwedge^5 W$ *and* $\delta \in \det^2(V)$.

Theorem B. *(i) For* 5*-dimensional subspaces* $W, W' \subset \bigwedge^2 V$ *such that* E_W *and* $E_{W'}$ *are elliptic curves, the Poisson brackets* Π_W *and* $\Pi_{W'}$ *are compatible if and only if* $\dim W \cap W' \geq 4$.
(ii) For any collection (W_i) *of* 5*-dimensional subspaces in* $\bigwedge^2 V$*, the brackets* (Π_{W_i}) *are pairwise compatible if and only if either there exists a* 6*-dimensional subspace* $U \subset \bigwedge^2 V$ *such that each* W_i *is contained in* U*, or there exists a* 4*-dimensional subspace* $K \subset \bigwedge^2 V$ *such that each* W_i *contains* K.

Corollary C. *The maximal dimension of a linear subspace of Poisson brackets on* $\mathbb{P}(V)$*, where* $\dim V = 5$*, spanned by some FO brackets* Π_W *of type* $q_{5,2}$*, is* 6.

Theorems A and B suggest the following

Conjecture D. *Let* $W \subset \bigwedge^2 V$ *be a* 5*-dimensional subspace such that* E_W *is an elliptic curve. Consider the subspace*

$$T_W := (\bigwedge^4 W) \wedge (\bigwedge^2 V) \subset \bigwedge^5 (\bigwedge^2 V)$$

(the quotient of the latter subspace by $\bigwedge^5 W$ *is exactly the image of the tangent space to the Grassmannian* $G(5, \bigwedge^2 V)$ *under Plücker embedding). Then the subspace of* $\xi \in \bigwedge^5(\bigwedge^2 V)$ *satisfying* $[\pi_{5,2}(\xi), \Pi_W] = 0$ *coincides with* $T_W + \ker(\pi_{5,2})$*.*

Note that we know the inclusion one way: the subspace T_W is spanned by $\bigwedge^5(W')$ such that $\dim(W' \cap W) \geq 4$ and $E_{W'}$ is an elliptic curve, and by Theorems A and B, $[\pi_{5,2}(\bigwedge^5(W')) \wedge \Pi_W] = 0$.

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2. Generalities

2.1. **Feigin-Odesskii Poisson brackets of type $q_{n,k}$.** Let E be an elliptic curve, with a fixed trivialization $\eta : \mathcal{O}_E \to \omega_E$, $\mathcal{V}$ a stable bundle on E of rank k and degree $n > 0$. We consider the corresponding Feigin-Odesskii Poisson bracket $\Pi = \Pi_{E,\mathcal{V}}$ of type $q_{n,k}$ on the projective space $\mathbb{P}H^1(E, \mathcal{V}^\vee)$ defined as in [10].

We will need the following definition of Π in terms of triple Massey products. For nonzero $\phi \in H^1(E, \mathcal{V}^\vee)$, we denote by $\langle\phi\rangle$ the corresponding line, and we use the identification of the cotangent space to $\langle\phi\rangle$ with $\langle\phi\rangle^\perp \subset H^0(E, \mathcal{V})$ (where we use the Serre duality $H^0(E, \mathcal{V}) \simeq H^1(E, \mathcal{V}^\vee)^*$).

Lemma 2.1.1. *(*[3, Lem. 2.1]*) For* $s_1, s_2 \in \langle\phi\rangle^\perp$ *one has*

$$\Pi_\phi(s_1 \wedge s_2) = \langle \phi, MP(s_1, \phi, s_2)\rangle,$$

where MP *denotes the triple Massey product for the arrows*

$$\mathcal{O} \xrightarrow{s_2} \mathcal{V} \xrightarrow{\phi} \mathcal{O}[1] \xrightarrow{s_1} \mathcal{V}[1].$$

2.2. **Formula for a family of complete intersections.** Let X be a smooth projective variety of dimension n, $C \subset X$ a connected curve given as the zero locus of a regular section F of a vector bundle N of rank $n-1$, such that $\det(N)^{-1} \simeq \omega_X$. Then the normal bundle to C is isomorphic to $N|_C$, so by the adjunction formula, ω_C is trivial, so C is an elliptic curve. Assume that P is a vector bundle on X, such that the following cohomology vanishing holds:

$$H^i(X, \bigwedge^i N^\vee \otimes P) = H^{i-1}(X, \bigwedge^i N^\vee \otimes P) \quad \text{for } 1 \leq i \leq n-1. \tag{2.1}$$

We have the following Koszul resolution for $\mathcal{O}_C$:

$$0 \to \bigwedge^{n-1} N^\vee \to \ldots \to \bigwedge^2 N^\vee \xrightarrow{\delta_2(F)} N^\vee \xrightarrow{\delta_1(F)} \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

which induces a map $e_C : \mathcal{O}_C \to \bigwedge^{n-1} N^\vee[n-1]$ in the derived category of X. Here the differential $\delta_i(F)$ is given by the contraction with $F \in H^0(X, N)$, so it depends linearly on F.

Lemma 2.2.1. *(i) The natural restriction map* $H^0(X, P) \to H^0(C, P|_C)$ *and the map*

$$\operatorname{Ext}^1(P, \mathcal{O}_C) \xrightarrow{e_C} \operatorname{Ext}^n(P, \bigwedge^{n-1} N^\vee) \simeq \operatorname{Ext}^n(P, \omega_X)$$

are isomorphisms. These maps are dual via the Serre duality isomorphisms

$$\operatorname{Ext}^1(P|_C, \mathcal{O}_C) \simeq H^0(C, P|_C)^*, \quad \operatorname{Ext}^n(P, \omega_X) \simeq H^0(X, P)^*.$$

(ii) Assume in addition that $\operatorname{End}(P) = \mathbf{k}$ *and we have the following vanishing:*

$$\operatorname{Ext}^i(P, \bigwedge^i N^\vee \otimes P) = \operatorname{Ext}^{i-1}(P, \bigwedge^i N^\vee \otimes P) = 0 \quad \text{for } 1 \le i \le n-1. \tag{2.2}$$

Then the bundle $P|_C$ *is stable.*

Proof. (i) This is obtained from the Koszul resolution of $\mathcal{O}_C$.
(ii) Computing $\operatorname{Hom}(P|_C, P|_C) = \operatorname{Hom}(P, P|_C)$ using the Koszul resolution of $P|_C = P \otimes \mathcal{O}_C$, we get that it is 1-dimensional. Hence, $P|_C$ is stable. □

Now we can rewrite the formula of Lemma 2.1.1 for the FO-bracket $\Pi_{C,P|_C}$ on $\mathbb{P}H^1(C, P^\vee|_C) \simeq \mathbb{P}\operatorname{Ext}^n(P, \omega_X)$ in terms of higher products on X (obtained by the homological perturbation from a dg-enhancement of $D^b(\operatorname{Coh}(X))$).

Proposition 2.2.2. *For nonzero* $\phi \in \operatorname{Ext}^n(P, \omega_X) \simeq \operatorname{Ext}^1_C(P|_C, \mathcal{O}_C)$*, and* $s_1, s_2 \in \langle \phi \rangle^\perp \subset H^0(X, P)$*, one has*

$$\Pi_{C,P|_C,\phi}(s_1 \wedge s_2) = \pm \langle \phi, \sum_{i=1}^{n} (-1)^i m_{n+2}(\delta_1(F), \ldots, \delta_{i-1}(F), s_1, \delta_i(F), \ldots, \delta_{n-1}(F), \phi, s_2) \rangle.$$

Proof. The computation is completely analogous to that of [8, Prop. 3.1], so we will only sketch it. First, one shows that our Massey product can be computed as the triple product m_3 for the arrows

$$\mathcal{O}_X \to P \xrightarrow{[1]} \mathcal{O}_C \to P|_C$$

given by s_2, ϕ and s_1. Then we use resolutions $\bigwedge^\bullet N^\vee \to \mathcal{O}_C$ and $\bigwedge^\bullet N^\vee \otimes P \to P|_C$. Thus, we have to calculate the following triple product in the category of twisted complexes:

$$\begin{array}{ccccccc}
\mathcal{O}_X & & & & & & \\
\downarrow{\scriptstyle s_2} & & & & & & \\
P & & & & & & \\
\downarrow{\scriptstyle \phi} & & & & & & \\
\bigwedge^{n-1} N^\vee[n-1] & \xrightarrow{\delta_{n-1}(F)} & \dots & \xrightarrow{\delta_2(F)} & N^\vee[1] & \xrightarrow{\delta_1(F)} & \mathcal{O}_X \\
\downarrow{\scriptstyle s_1} & & & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_1} \\
\bigwedge^{n-1} N^\vee \otimes P[n-1] & \xrightarrow{\delta_{n-1}(F)} & \dots & \xrightarrow{\delta_2(F)} & N^\vee \otimes P[1] & \xrightarrow{\delta_1(F)} & P
\end{array}$$

where we view ϕ as a morphism of degree 1 from P to the twisted complex $\bigoplus \bigwedge^i N^\vee[i]$. Now, the result follows from the formula for m_3 on twisted complexes (see [5, Sec. 7.6]). $\square$

2.3. **Conormal Lie algebra.** Let $\mathcal{V}$ be a stable bundle of positive degree on an elliptic curve E, with a fixed trivialization of ω_E, and consider the corresponding FO bracket Π on the projective space $X = \mathbb{P}H^0(\mathcal{V})^* = \mathbb{P}\operatorname{Ext}^1(\mathcal{V}, \mathcal{O})$. Recall that for every point x of a smooth Poisson variety (X, Π) there is a natural Lie algebra structure on

$$\mathfrak{g}_x := (\operatorname{im}\Pi_x)^\perp \subset T_x^* X,$$

where we consider Π_x as a map $T_x^*X \to T_xX$. We call $\mathfrak{g}_x$ the *conormal Lie algebra*. In the case when Π vanishes on x, we have $\mathfrak{g}_x = T_x^*$.

Let us consider a nontrivial extension

$$0 \to \mathcal{O} \xrightarrow{i} \widetilde{\mathcal{V}} \xrightarrow{p} \mathcal{V} \to 0$$

with the class $\phi \in \operatorname{Ext}^1(\mathcal{V}, \mathcal{O})$. By Serre duality, we have the corresponding hyperplane $\langle\phi\rangle^\perp \subset H^0(\mathcal{V})$, and we have an identification $\langle\phi\rangle^\perp \simeq T_\phi^*\mathbb{P}H^0(\mathcal{V})^*$.

Consider a natural map

$$\operatorname{End}(\widetilde{\mathcal{V}})/\langle \mathrm{id}\rangle \to \langle\phi\rangle^\perp \simeq T_\phi^*\mathbb{P}H^0(\mathcal{V})^* : A \mapsto p \circ A \circ i. \tag{2.3}$$

The following result was proved in [2].

Theorem 2.3.1. *The above map induces an isomorphism of Lie algebras from* $\operatorname{End}(\widetilde{\mathcal{V}})/\langle \mathrm{id}\rangle$ *to the conormal Lie algebra of* Π *at the point* ϕ.

Note that in particular, the subspace $(\operatorname{im}\Pi_x)^\perp \subset \langle\phi\rangle^\perp$ is equal to the image of the map (2.3).

3. FO brackets associated with elliptic curves in $G(2,5)$

3.1. **Proof of Theorem A.**

Lemma 3.1.1. *The subset $Z \subset \mathrm{Gr}(5, \bigwedge^2 V)$ of 5-dimensional subspaces $W \subset \bigwedge^2 V$ such that $\dim(\mathbb{P}W \cap G(2,V)) \geq 2$ has codimension > 1.*

Proof. Let us denote by F the variety of flags $L \subset W \subset \bigwedge^2 V$, where $\dim(L) = 3$, $\dim(W) = 5$, such that $\mathbb{P}L \cap G(2,V) \neq \emptyset$. We claim that F is irreducible of dimension ≤ 30. Note that we have a proper closed subset $\widetilde{Z} \subset F$ consisting of (L,W) such that $\dim(\mathbb{P}W \cap G(2,V)) \geq 2$ (as an example of a point in $F \setminus \widetilde{Z}$, we can take W such that $E_W = \mathbb{P}W \cap G(2,V)$ is an elliptic curve and pick $\mathbb{P}L \subset \mathbb{P}W$ intersecting E_W). Since $\widetilde{Z}$ fibers over Z with fibers $\mathrm{Gr}(3,5)$, our claim would imply that $\dim(\widetilde{Z}) = \dim Z + 6 < 30$, i.e., $\dim Z < 24$, as required.

To estimate the dimension of F, we observe that we have a fibration $F \to Y$ with fibers $G(2,7)$, where $Y \subset \mathrm{Gr}(3, \bigwedge^2 V)$ is the subvariety of 3-dimensional subspaces L such that $\mathbb{P}L \cap G(2,V) \neq \emptyset$. Thus, it is enough to prove that Y is irreducible of dimension ≤ 20. Now we use a surjective map $\widetilde{Y} \to Y$, where $\widetilde{Y}$ is the variety of flags $\ell \subset L \subset \bigwedge^2 V$, where $\dim(\ell) = 1$, $\dim(L) = 3$, such that $\ell \in G(2,V)$. We have a fibration $\widetilde{Y} \to G(2,V)$ with fibers $G(2,9)$, hence $\widetilde{Y}$ is irreducible of dimension $6 + 14 = 20$. Hence, Y is irreducible of dimension ≤ 20. □

Proof of Theorem A. First, we can apply Proposition 2.2.2 to an elliptic curve $E_W \subset X = G(2,V)$. Namely, as a bundle P on X we take $\mathcal{U}^\vee$, the dual of the universal subbundle. We can view the embedding

$$R := W^\perp \to \bigwedge^2 V^* = H^0(X, \mathcal{O}(1)),$$

where $\mathcal{O}(1) = \det(\mathcal{U}^\vee)$, as a regular section $F \in H^0(X,N)$, where $N = R^* \otimes \mathcal{O}(1)$. It is easy to see that we have a $\mathrm{GL}(V)$-invariant identification

$$\omega_X \simeq \det(V)^{-2} \otimes \mathcal{O}(-5).$$

Thus, by adjunction we get an isomorphism

$$\omega_{E_W} \simeq \det(N) \otimes \omega_X|_{E_W} \simeq \det(R^*) \otimes \det(V)^{-2} \otimes \mathcal{O}_{E_W}.$$

Since $\det(R^*) \simeq \det(\bigwedge^2 V) \otimes \det(W^*) \simeq \det(V)^4 \otimes \det(W^*)$, we can rewrite this as

$$\omega_{E_W} \simeq \det(W^*) \otimes \det(V)^2 \otimes \mathcal{O}_{E_W}. \tag{3.1}$$

The vanishings (2.1) and (2.2) in this case follow from the well known vanishings

$$H^*(X, \mathcal{U}^\vee(-i)) = 0, \quad \text{for } 1 \leq i \leq 5,$$

$$\mathrm{Ext}^*(\mathcal{U}^\vee, \mathcal{U}^\vee(-i)) = 0, \text{ for } 1 \leq i \leq 3, \ \mathrm{Ext}^{<6}(\mathcal{U}^\vee, \mathcal{U}^\vee(-4)) = \mathrm{Ext}^{<6}(\mathcal{U}^\vee, \mathcal{U}^\vee(-5)) = 0$$

(see [4]). Thus, Proposition 2.2.2 gives a formula for Π_W.

This shows that the association $W \mapsto \Pi_W$ gives a regular morphism

$$f : \mathrm{Gr}(5, \bigwedge^2 V) \to \mathbb{P}H^0(\mathbb{P}V, \bigwedge^2 T).$$

Furthermore, we claim that

$$f^*\mathcal{O}(1) \simeq \mathcal{O}_{\mathrm{Gr}(5,\bigwedge^2 V)}(1) \otimes \det(V)^{-2}.$$

Indeed, we have a family of Gorenstein curves $\pi : \mathcal{C} \to B = \mathrm{Gr}(5, \bigwedge^2 V) \setminus Z$, where Z was defined in Lemma 3.1.1, such that

$$\omega_{\mathcal{C}/B} \simeq \pi^*(\mathcal{O}(1) \otimes \det(V)^2).$$

Indeed, this is implied by the argument leading to (3.1), which works for any curve (not necessarily smooth) cut out by $\mathbb{P}W$ in $G(2, V)$. Now [3, Prop. 4.1] implies that the relation $f^*\mathcal{O}(1) = \mathcal{O}(1) \otimes \det(V)^{-2}$ holds over $\mathrm{Gr}(5, \bigwedge^2 V) \setminus Z$. Since Z has codimension ≥ 1, it holds over the entire $\mathrm{Gr}(5, \bigwedge^2 V)$.

Next, since $H^0(\mathrm{Gr}(5, \bigwedge^2 V), \mathcal{O}(1)) \simeq \bigwedge^5(\bigwedge^2 V)^*$, the map f is given by a $\mathrm{GL}(V)$-invariant linear map

$$\bigwedge^5(\bigwedge^2 V) \to H^0(\mathbb{P}V, \bigwedge^2 T) \otimes \det(V)^2.$$

To show that this map coincides with $\pi_{5,2}$, up to a constant factor, it remains to show that the space $\mathrm{Hom}_{\mathrm{GL}(V)}(\bigwedge 5(\bigwedge^2 V), H^0(\mathbb{P}V, \bigwedge^2 T) \otimes \det(V)^2)$ is 1-dimensional.

The representation of $\mathrm{GL}(V)$ on $H^0(\mathbb{P}V, \bigwedge^2 T)$ is easy to identify due to the exact sequence

$$0 \to \mathbf{k} \to V \otimes V^* \otimes \bigwedge^2 V \otimes S^2 V^* \to H^0(\mathbb{P}V, \bigwedge^2 T) \to 0.$$

Using the Littlewood-Richardson rule, we deduce

$$H^0(\mathbb{P}V, \bigwedge^2 T) \otimes \det(V^*) \simeq \Sigma^{3,1,1}(V^*),$$

where Σ^λ denotes the Schur functor associated with a partition λ. It follows that

$$H^0(\mathbb{P}V, \bigwedge^2 T) \otimes \det(V)^2 \simeq \Sigma^{3,3,2,2}(V).$$

On the other hand, the decomposition of the plethysm $e_5 \circ e_2$ (see [6, Ex. I.8.6]) shows that $\Sigma^{3,3,2,2}(V)$ appears with multiplicity 1 in the $\mathrm{GL}(V)$-representation $\bigwedge^5(\bigwedge^2 V)$. This implies the claimed assertion about $\mathrm{GL}(V)$-maps. □

3.2. **Rank stratification for a bracket of type $q_{5,2}$.** Let E be an elliptic curve, $\mathcal{V}$ be a stable vector bundle of rank 2 and degree 5. We consider the FO bracket Π on the projective space $\mathbb{P}\,\mathrm{Ext}^1(\mathcal{V}, \mathcal{O}) \simeq \mathbb{P}H^0(\mathcal{V})^*$. We want to describe the corresponding rank stratification of $\mathbb{P}H^0(\mathcal{V})^* = \mathbb{P}^4$. For every point $p \in E$, we consider the subspace $L_p := \mathcal{V}|_p^* \subset H^0(\mathcal{V})^*$ and the corresponding projective line $\mathbb{P}L_p \subset \mathbb{P}H^0(\mathcal{V})^*$.

Recall that the rank of Π at a point corresponding to an extension $\widetilde{\mathcal{V}}$ is equal to $5 - \dim \mathrm{End}(\widetilde{V})$ (see [3, Prop. 2.3]).

Lemma 3.2.1. *(i) The bracket* Π *vanishes at the point of* $\mathbb{P}\operatorname{Ext}^1(\mathcal{V},\mathcal{O})$ *corresponding to an extension*

$$0 \to \mathcal{O} \to \widetilde{\mathcal{V}} \to \mathcal{V} \to 0$$

if and only if this extension splits under $\mathcal{O} \to \mathcal{O}(p)$ *for some point* $p \in E$*, which happens if and only if* $\widetilde{\mathcal{V}} \simeq \mathcal{O}(p) \oplus \mathcal{V}'$*, where* $\mathcal{V}'$ *is semistable of rank* 2 *and degree* 4*. Furthermore, in this case* $\dim \operatorname{End}(\mathcal{V}') = 2$*, so* $\mathcal{V}'$ *is either indecomposable, or* $\mathcal{V}' \simeq L_1 \oplus L_2$*, where* L_1 *and* L_2 *are nonisomorphic line bundles of degree* 2*.*
(ii) The bracket Π *has rank* ≤ 2 *if and only the corresponding extension* $\widetilde{\mathcal{V}}$ *is unstable, or equivalently, there exists a line bundle* L_2 *of degree* 2 *such that the extension splits over the unique embedding* $L_2 \hookrightarrow \mathcal{V}$*. In other words, the extension class comes from a subspace of the form*

$$W_{L_2} := H^0(L_2)^{\perp} \subset H^0(\mathcal{V})^* = V, \tag{3.2}$$

where we use the unique embedding $L_2 \to \mathcal{V}$ *and consider the induced embedding* $H^0(L_2) \hookrightarrow H^0(\mathcal{V})$*.*
(iii) Each plane $\mathbb{P}W_{L_2} \subset \mathbb{P}V$ *is a Poisson subvariety, and there is an embedding of the curve* E *into* $\mathbb{P}W_{L_2}$ *by a degree* 3 *linear system, so that* $\mathbb{P}W_{L_2} \setminus E$ *is a symplectic leaf.*

Proof. (i) Suppose a nontrivial extension

$$0 \to \mathcal{O} \to \widetilde{\mathcal{V}} \to \mathcal{V} \to 0$$

splits under $\mathcal{O} \to \mathcal{O}(p)$. Then $\widetilde{\mathcal{V}}$ is an extension of $\mathcal{O}(p)$ by $\mathcal{V}'$ where $\mathcal{V}' \subset \mathcal{V}$ is the kernel of the corresponding surjective map $\mathcal{V} \to \mathcal{O}_p$. Hence, $\mathcal{V}'$ is semistable of slope 2, which implies that

$$\widetilde{\mathcal{V}} \simeq \mathcal{O}(p) \oplus \mathcal{V}'.$$

It follows that $\dim \operatorname{End}(\mathcal{V}') \ge 2$, and so

$$\dim \operatorname{End}(\widetilde{\mathcal{V}}) = 3 + \dim \operatorname{End}(\mathcal{V}') \ge 5.$$

Hence, Π_E vanishes on the points of the line $\mathbb{P}L_p \subset \mathbb{P}V$, and we have $\dim \operatorname{End}(\mathcal{V}') = 2$, which means that either $\mathcal{V}'$ is indecomposable or $\mathcal{V}' \simeq L_1 \oplus L_2$, for two nonisomorphic line bundles L_1, L_2 of degree 2.

Conversely, assume Π vanishes at the point corresponding to $\widetilde{\mathcal{V}}$, so $\dim \operatorname{End}(\widetilde{\mathcal{V}}) = 5$. Then HN-components of $\widetilde{\mathcal{V}}$ cannot be three line bundles (since they would have to have different positive degrees that add up to 5), so $\widetilde{\mathcal{V}} = L \oplus \mathcal{V}'$ where L is a line bundle and $\mathcal{V}'$ is semistable of rank 2, $\deg(L) > 0$, $0 < \deg(\mathcal{V}')$, $\deg(L) + \deg(\mathcal{V}') = 5$.

The case $\deg(L) = 1$ leads to the locus discussed above. If $\deg(L) = 2$ and $\deg(\mathcal{V}') = 3$ then $\dim \operatorname{Hom}(\mathcal{V}', L) = 1$, so we get $\dim \operatorname{End}(\mathcal{V}') = 3$ which is impossible. If $\deg(L) \ge 3$, then $\deg(\mathcal{V}') \le 2$ and $\dim \operatorname{Hom}(\mathcal{V}', L) \ge 4$, so $\dim \operatorname{End}(\mathcal{V}) > 5$, a contradiction.
(ii) The rank of Π is ≤ 2 at $\widetilde{\mathcal{V}}$ if and only if $\dim \operatorname{End}(\widetilde{\mathcal{V}}) \ge 3$. Clearly, such $\widetilde{\mathcal{V}}$ has to be unstable. Conversely, any unstable $\widetilde{\mathcal{V}}$ would have form $L \oplus \mathcal{V}'$ with either $\operatorname{Hom}(L, \mathcal{V}') \ne 0$ or $\operatorname{Hom}(\mathcal{V}', L) \ne 0$, hence $\dim \operatorname{End}(\widetilde{\mathcal{V}}) \ge 3$.

Note that $\mu(\widetilde{\mathcal{V}}) = 5/3$. Hence, if the extension splits over some $L_2 \subset \mathcal{V}$, then $\widetilde{\mathcal{V}}$ is unstable. Conversely, if $\widetilde{\mathcal{V}}$ is unstable then either it has a line subbundle of degree 2, or a

semistable subbundle $\mathcal{V}'$ of rank 2 and degree ≥ 4. But any such $\mathcal{V}'$ has a line subbundle of degree ≥ 2.
(iii) We can identify $H^0(L_2)^\perp$ with $H^0(L_3)^* \subset H^0(\mathcal{V})^*$, where $L_3 := \mathcal{V}/L_2$. It is easy to see that the intersection of $\mathbb{P}W_{L_2}$ with the zero locus of Π is exactly the image of E under the map given by $|L_3|$.

Given an extension $\widetilde{\mathcal{V}} \to \mathcal{V}$, split over $L_2 \subset \mathcal{V}$, the splitting $L_2 \to \widetilde{\mathcal{V}}$ is unique, and the quotient $\widetilde{\mathcal{V}}/L_2$ is an extension of $L_3 = \mathcal{V}/L_2$ by $\mathcal{O}$. It is well known that for points of $\mathbb{P}W_{L_2} \setminus E$ the latter extension is stable, so $\mathcal{V}_{L_3} = \widetilde{\mathcal{V}}/L_2$ is a stable bundle of rank 2 with determinant L_3. Since $\mathrm{Ext}^1(\mathcal{V}_{L_3}, L_2) = 0$, we deduce that $\widetilde{\mathcal{V}} = \mathcal{V}_{L_3} \oplus L_2$. Now we can calculate the image of the map (2.3). The space $\mathrm{End}(\widetilde{\mathcal{V}})/\langle \mathrm{id} \rangle$ has a basis $\langle \mathrm{id}_{L_2}, e \rangle$, where e is a generator of $\mathrm{Hom}(\mathcal{V}_{L_3}, L_2)$. Their images under (2.3) both factor through $L_2 \to E$, hence the image of (2.3) (which is 2-dimensional) is $H^0(L_2) \subset H^0(\mathcal{V})$. But this is exactly the conormal subspace to the projective plane $\mathbb{P}W_{L_2}$. This shows that $\mathbb{P}W_{L_2} \setminus E$ (and hence $\mathbb{P}W_{L_2}$) is a Poisson subvariety. Since the rank of Π on $\mathbb{P}W_{L_2} \setminus E$ is equal to 2 and $\Pi|_E = 0$, we deduce that $\mathbb{P}W_{L_2} \setminus E$ is a symplectic leaf. □

By Lemma 3.2.1(i) the vanishing locus of Π corresponds to extensions $\mathcal{V}$ by $\mathcal{O}$, which split over $\mathcal{O}(p)$. This is the union S_E of the lines $\mathbb{P}L_p$, where $L_p = \mathcal{V}|_p^* \subset \mathbb{P}H^0(\mathcal{V})^*$, over $p \in E$. The surface S_E is the image of the natural map $\mathbb{P}(\mathcal{V}^\vee) \to \mathbb{P}(V)$, associated with the embedding of bundles $\mathcal{V}^\vee \to V \otimes \mathcal{O}_E$. We will prove that in fact this map induces an isomorphism of the projective bundle $\mathbb{P}(\mathcal{V}^\vee)$ with S_E.

Lemma 3.2.2. *Let $\mathcal{E}$ be a vector bundle over a smooth curve C and let $W \to H^0(C, \mathcal{E})$ be a linear map from a vector space W, such that for any $x \in C$ the composition $p_x : W \to H^0(C, \mathcal{E}) \to \mathcal{E}|_x$ is surjective, so that we have a morphism*

$$f : \mathbb{P}(\mathcal{E}^\vee) \to \mathbb{P}(W^*).$$

Assume that we have a closed subset $Z \subset \mathbb{P}(\mathcal{E}^\vee)$ with the following properties.

- *For every $x, y \in C$, $x \neq y$, consider $p_x(\ker(p_y)) \subset \mathcal{E}|_x$. Then any $\ell \in \mathbb{P}(\mathcal{E}^\vee|_x)$, which is orthogonal to $p_x(\ker(p_y))$, is contained in Z.*
- *For every $x \in C$, consider the map $W \to H^0(\mathcal{E}|_{2x})$ and the induced map*

$$K_x := \ker(W \to \mathcal{E}|_x) \to T_x^* C \otimes \mathcal{E}|_x$$

  (where we use the identification $T_x^* C \otimes \mathcal{E}|_x = \ker(H^0(\mathcal{E}|_{2x}) \to \mathcal{E}|_x)$). Then any $\ell \in \mathbb{P}(\mathcal{E}^\vee|_x)$, which is orthogonal to the image of $K_x \otimes T_x C$, is contained in Z.

Then the map $\mathbb{P}(\mathcal{E}^\vee) \setminus Z \to \mathbb{P}(W^*)$ is a locally closed embedding.

Proof. Assume that for $x \neq y$, we have two nonzero functionals $\phi_x : \mathcal{E}|_x \to k$, $\phi_y : \mathcal{E}|_y \to k$ such that $\phi_x \circ p_x = \phi_y \circ p_y$. Then $(\phi_x \circ p_x)|_{\ker(\phi_y)} = 0$. Hence, ϕ_x vanishes on $p_x(\ker(p_y))$. By assumption, this can happen only when ϕ_x is in Z. Thus, the map from $\mathbb{P}(\mathcal{E}^\vee) \setminus Z$ is set-theoretically one-to-one.

Next, we need to check that our map is injective on tangent spaces. The tangent space to $\mathbb{P}(\mathcal{E}^\vee)$ at a point corresponding to $\ell \subset \mathcal{E}^\vee|_x$ can be described as follows. Consider the

canonical extension
$$0 \to T_x^*C \otimes \mathcal{E}|_x \to H^0(\mathcal{E}|_{2x}) \to \mathcal{E}|_x \to 0.$$
Passing to the dual extension of $T_xC \otimes \mathcal{E}^\vee|_x$ by $\mathcal{E}^\vee|_x$, and restricting it to $T_xC \otimes \ell \subset T_xC \otimes \mathcal{E}^\vee|_x$, we get an extension
$$0 \to \mathcal{E}^\vee|_x \to H_\ell \to T_xC \otimes \ell \to 0$$
Now the quotient $(\ell^{-1} \otimes H_\ell)/\mathbf{k}$, where we use the natural embedding
$$k = \ell^{-1} \otimes \ell \to \ell^{-1} \otimes \mathcal{E}^\vee|_x \to \ell^1 \otimes H_\ell,$$
is identified with the tangent space $T_\ell\mathbb{P}(\mathcal{E}^\vee)$.

The restriction of the map $H^0(\mathcal{E}|_{2x})^\vee \to W^*$, dual to the natural map $W \to H^0(\mathcal{E}|_{2x})$, to H_ℓ, induces a map
$$(\ell^{-1} \otimes H_\ell)/\mathbf{k} \to W^*/\ell,$$
which is exactly the tangent map to f. It is injective if and only if the map $H_\ell \to W^*$ is injective. Equivalently, the dual map $W \to H_\ell^*$ should be surjective. The latter map is compatible with (surjective) projections to $\mathcal{E}|_x$, so this is equivalent to surjectivity of the map
$$K_x = \ker(W \to \mathcal{E}|_x) \to \ker(H_\ell^* \to \mathcal{E}|_x) = T_x^*C \otimes \ell^{-1}.$$
The latter map factors as a composition
$$K_x \to T_x^*C \otimes \mathcal{E}|_x \to T_x^*C \otimes \ell^{-1},$$
so it is surjective (equivalently, nonzero) if and only if ℓ is not orthogonal to the image of $K_x \to T_x^*C \otimes \mathcal{E}|_x$. By assumption, this never happens for points of $\mathbb{P}(\mathcal{E}^\vee) \setminus Z$. □

Lemma 3.2.3. *The map $\mathbb{P}(\mathcal{V}^\vee) \to S_E$ is an isomorphism.*

Proof. We will check the conditions of Lemma 3.2.2. It suffices to check surjectivity of the maps $H^0(\mathcal{V}) \to \mathcal{V}|_x \oplus \mathcal{V}|_y$ for $x \neq y$ and of $H^0(\mathcal{V}) \to H^0(\mathcal{V}|_{2x})$. But this follows from the exact sequence
$$0 \to \mathcal{V}(-D) \to \mathcal{V} \to \mathcal{V}|_D \to 0$$
for any effective divisor D of degree 2 and from the vanishing of $H^1(\mathcal{V}(-D))$ by stability of $\mathcal{V}$. □

By Lemma 3.3.3 the degeneracy locus $\mathcal{D}_E$ of our Poisson bracket (which is a quintic hypersurface) is the union of planes $\mathbb{P}W_{L_2} \subset \mathbb{P}V$ over $L_2 \in \mathrm{Pic}^2(E)$ (see (3.2)). Let us consider the vector bundle $\mathcal{W}$ over $\widetilde{E} := \mathrm{Pic}^2(E)$, such that the fiber of $\mathcal{W}$ over L_2 is W_{L_2}. Note that we have a natural identification $\widetilde{E} \simeq \mathrm{Pic}^3(E) : L_2 \mapsto L_3 := \det(\mathcal{V}) \otimes L_2^{-1}$. In terms of L_3 we have $W_{L_2} = H^0(L_3)^* \subset H^0(\mathcal{V})^*$, where we use a surjection $\mathcal{V} \to L_3$. To define the vector bundle $\mathcal{W}$ precisely, we consider the universal line bundle $\mathcal{L}_3$ of degree 3 over $E \times \widetilde{E} \simeq E \times \mathrm{Pic}^3(E)$, normalized so that the line bundle $p_{2*}\underline{\mathrm{Hom}}(p_1^*\mathcal{V}, \mathcal{L}_3)$ is trivial. We set
$$\mathcal{W} := p_{2*}(\mathcal{L}_3)^\vee.$$
Note that applying p_{2*} to the natural surjection $p_1^*\mathcal{V} \to \mathcal{L}_3$ we get a surjection $H^0(\mathcal{V}) \otimes \mathcal{O} \to p_{2*}(\mathcal{L}_3)$. Passing to the dual, we get a morphism $\mathbb{P}(\mathcal{W}) \to \mathbb{P}V$, whose image is $\mathcal{D}_E$.

Lemma 3.2.4. *The morphism $\mathbb{P}(\mathcal{W}) \to \mathcal{D}_E$ is an isomorphism over $\mathcal{D}_E \setminus S_E$.*

Proof. We need to check two conditions of Lemma 3.2.2 for the morphism $H^0(\mathcal{V})\otimes\mathcal{O} \to \mathcal{W}^\vee$ over $\widetilde{E}$, with $Z \subset \mathbb{P}(\mathcal{W})$ being the preimage of S. Note that the intersection of Z with each plane $\mathbb{P}H^0(L_3)^* \subset H^0(\mathcal{V})^*$ is the elliptic curve E embedded by the linear system $|L_3|$.

To check the first condition, we use the exact sequence

$$0 \to H^0(L_2) \to H^0(\mathcal{V}) \to H^0(L_3) \to 0$$

where $L_2 \otimes L_2 \simeq \mathcal{V}$. If L_3' is different from L_3' then the composed map $L_2 \to \mathcal{V} \to L_3'$ is nonzero, hence, it identifies L_2 with the subsheaf $L_3'(-x)$ for some point $p \in E$. Hence, the image of $H^0(L_2)$ is precisely the plane $H^0(L_3'(-p)) \subset H^0(L_3')$. Hence, the only point of $\mathbb{P}H^0(L_3')^*$ orthogonal to this plane is the point $p \in E \subset \mathbb{P}H^0(L_3')^*$, which lies in Z.

To check the second condition, we need to understand the map $H^0(\mathcal{V}) \to H^0(\mathcal{W}^\vee|_{2x})$ for $x \in \widetilde{E} \simeq \mathrm{Pic}^3(E)$. For this we observe that this map is equal to the composition

$$H^0(\mathcal{V}) \to H^0(E \times \{2x\}, p_1^*\mathcal{V}|_{E\times\{2x\}}) \to H^0(E \times \{2x\}, \mathcal{L}_3|_{E\times\{2x\}}),$$

which is the map induced on H^0 by the morphism of sheaves on E,

$$\alpha : \mathcal{V} \to \mathcal{V} \otimes H^0(\mathcal{O}_{2x}) = p_{1*}(p_1^*\mathcal{V}|_{E\times\{2x\}}) \to p_{1*}(\mathcal{L}_3|_{E\times\{2x\}}).$$

Note that for $x = L_3$, the bundle $F_x := p_{1*}(\mathcal{L}_3|_{E\times\{2x\}})$ on E is an extension of L_3 by $T_x^*\widetilde{E} \otimes L_3$, which gives the Kodaira-Spencer map for the family $\mathcal{L}_3$, so this extension is nontrivial. The composition

$$\mathcal{V} \xrightarrow{\alpha} F_x \to L_3$$

is the canonical surjection with the kernel $L_2 \subset \mathcal{V}$. Hence, α fits into a morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_2 & \longrightarrow & \mathcal{V} & \longrightarrow & L_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha|_{L_2}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & T_x^*\widetilde{E} \otimes L_3 & \longrightarrow & F_x & \longrightarrow & L_3 & \longrightarrow & 0
\end{array}$$

Note that the map $\alpha|_{L_2}$ is nonzero, since otherwise we would get a splitting of the extension $F_x \to L_3$.

Now the kernel of the map $H^0(\mathcal{V}) \to \mathcal{W}^\vee|_x = H^0(L_3)$ is identified with $H^0(L_2)$, and the induced map $H^0(L_2) \to T_x^*\widetilde{E} \otimes H^0(L_3)$ is given by a nonzero map

$$\alpha|_{L_2} : L_2 \to T_x^*\widetilde{E} \otimes L_3 \simeq L_3.$$

Hence, its image is the subspace of the form $H^0(L_3(-p))$, and we again deduce that any point of $\mathbb{P}H^0(L_3)^*$ orthogonal to it lies in Z. □

Corollary 3.2.5. *(i) There is a regular map $\mathcal{D}_E \setminus S_E \to \widetilde{E}$ such that the fiber over L_2 is the symplectic leaf $\mathbb{P}W_{L_2} \setminus E$.*

(ii) Any line contained in $\mathcal{D}_E$ is either contained in S_E or in some plane $\mathbb{P}W_{L_2}$, where $L_2 \in \mathrm{Pic}^2(E)$.

Proof. For (ii) we observe that given a line $L \subset \mathcal{D}_E$ not contained in S_E, the restriction of the map $\mathcal{D}_E \setminus S \to \widetilde{E}$ to $L \setminus S_E \to \widetilde{E}$ is necessarily constant. Hence, L is contained in some plane $\mathbb{P}W_{L_2}$. □

3.3. **Two-dimensional distribution on $G(2,5)$ associated with the elliptic curve.** Let $E \subset G(2,V)$ be the elliptic curve obtained as the intersection with the linear subspace $\mathbb{P}W \subset \mathbb{P}(\bigwedge^2 V)$ in the Plucker embedding, where $\dim W = 5$. Equivalently, E is cut out by the linear subspace of sections $W^\perp \subset \bigwedge^2 V^* \simeq H^0(G(2,V), \mathcal{O}(1))$. As before, we denote by $\mathcal{V}$ the restriction of $\mathcal{U}^\vee$, the dual of the universal bundle. Then $\bigwedge^2(\mathcal{V})$ is the restriction of $\mathcal{O}(1)$, and we have an exact sequence

$$0 \to W^\perp \to \bigwedge^2 V^* \to H^0(E, \bigwedge^2(\mathcal{V})) \to 0.$$

In other words, we can identify the dual map to the embedding $W \hookrightarrow \bigwedge^2 V$ with the natural map

$$\bigwedge^2 H^0(\mathcal{V}) \to H^0(\bigwedge^2 \mathcal{V}).$$

We have a regular map

$$f : G(2,V) \setminus E \to \mathbb{P}^4$$

given by the linear system $|W^\perp| \subset |\mathcal{O}(1)|$.

Then for every point $p \in G(2,V) \setminus E$, we define the subspace $D_p \subset T_pG(2,V)$ as the kernel of the tangent map to f at p. Note that for generic p, one has $\dim D_p = 2$.

We have the following characterization of D_p.

Lemma 3.3.1. *Let $L_p \subset V$ denote the 2-dimensional subspace corresponding to $p \in G(2,V) \setminus E$.*
(i) Under the identification $T_pG(2,V) \otimes \det(L_p) \simeq L_p \otimes V/L$, we have

$$D_p \otimes \det(L_p) = W \cap (L_p \wedge V) = W \cap (L_p \otimes V/L_p),$$

where the second intersection is taken in $\bigwedge^2 V / \bigwedge^2 L_p$.
(ii) For each $v \in L_p$, let us denote by $\pi_v : T_pG(2,V) \to V/L_p$ the natural projection. Assume that $\Pi_{E,v}$ has rank 4, for some nonzero $v \in L_p$. Then D_p is 2-dimensional, and $\pi_v(D_p)$ is the 2-dimensional subspace of V/L_p given as follows:

$$\pi_v(D_p) = \{x \in V/L_p \mid x \wedge \Pi_{E,v}^{norm} = 0\},$$

where $\Pi_{E,v}^{norm} \in \bigwedge^2(V/L_p)$ is the image of $\Pi_{E,v} \in \bigwedge^2(V/v)$.

Proof. (i) The map $d_L f$ is the composition of the Plucker embedding $G(2,V) \to \mathbb{P}(\bigwedge^2 V)$ with the linear projection

$$\mathbb{P}(\bigwedge^2 V) \setminus \mathbb{P}(W) \to \mathbb{P}(\bigwedge^2 V/W).$$

Thus, the tangent map to f at $L \subset W$ is the composition

$$\mathrm{Hom}(L, V/L) \xrightarrow{\alpha} \mathrm{Hom}(\bigwedge^2 L, \bigwedge^2 V/\bigwedge^2 L) \to \mathrm{Hom}(\bigwedge^2 L, \bigwedge^2 V/(\bigwedge^2 L + W)),$$

where $\alpha(A)(l_1 \wedge l_2) = Al_1 \wedge l_2 + l_1 \wedge Al_2 \bmod \bigwedge^2 L$. Equivalently, the map α is the natural map

$$\mathrm{Hom}(L, V/L) \simeq L^* \otimes V/L \simeq \det^{-1}(L) \otimes L \otimes V/L \to \det^{-1}(L) \otimes \bigwedge^2 V/\bigwedge^2 L,$$

given by $l \otimes (v \bmod L) \mapsto l \wedge v \bmod \bigwedge^2 L$.

Now the assertion follows from the identification

$$W = \ker\big(\bigwedge^2 V/\bigwedge^2 L \to \bigwedge^2 V/(\bigwedge^2 L + W)\big).$$

(ii) Our identification of Π_W from Theorem A implies the following property of the bivector $\Pi_{W,v} \in \bigwedge^2(V/v)$. Consider the natural map $\phi_v : W \to \bigwedge^2(V/v)$. Let $S = S_E \subset \mathbb{P}V$ denote the surface, obtained as the union of lines corresponding to $E \subset G(2, V)$. We claim that the map ϕ_v is injective if and only if $\langle v \rangle$ is not in S. Indeed, an element in the kernel of ϕ_v is an element $v \wedge v'$ contained in W, so the plane $\langle v, v' \rangle$ corresponds to a point of E. Hence, this is true when $\Pi_{W,v}$ is nonzero.

Now assume the rank of $\Pi_{W,v}$ is 4. We have a nondegenerate symmetric pairing on $\bigwedge^2(V/v)$ with values in $\det(V/v)$, given by the exterior product. Now our description of Π_W implies that for $\langle v \rangle \notin S$, $\Pi_{W,v}$ is nonzero and

$$\phi_v(W) = \langle \Pi_{W,v} \rangle^\perp.$$

Since $\Pi_{W,v}$ has maximal rank, the skew-symmetric form $(x_1, x_2) = x_1 \wedge x_2 \wedge \Pi_{W,v}$ on V/v is nondegenerate. Hence, the subspace $(L_p/\langle v \rangle) \otimes (V/L_p)$ cannot be contained in $\langle \Pi_{W,v} \rangle^\perp$ (this would mean that $L_p/\langle v \rangle$ lies in the kernel of $(\cdot, \cdot)$). Hence, the intersection

$$I := (L_p/\langle v \rangle) \otimes (V/L_p) \cap \langle \Pi_{W,v} \rangle^\perp$$

is 2-dimensional. Since the subspace $\phi_v(W \cap (L_p \wedge V))$ is contained in I, we deduce that its dimension is ≤ 2, and so $\dim D_p \leq 2$. But we also know that $\dim D_p \geq 2$, hence in fact, we have $\dim D_p = 2$ and $\phi_v(W \cap (L_p \wedge V)) = I$.

The last assertion follows from the fact that under trivialization of $L_p/\langle v \rangle$, the subspace $I \subset V/L_p$ coincides with $\pi_v(D_p)$. □

Definition 3.3.2. We define $\Sigma_E \subset G(2, V)$ as the closed locus of points $p \in G(2, V)$ such that $\dim W \cap (L_p \wedge V) \geq 3$.

Lemma 3.3.3. *One has $\Sigma_E \subset G(2, V) \setminus E$.*

Proof. Let $L = H^0(\mathcal{V}|_p)^* \subset H^0(\mathcal{V})^* = V$ for some $p \in E$. We have to prove that $\dim W \cap (L \wedge V) \leq 2$. We have, $L^\perp = H^0(\mathcal{V}(-p)) \subset H^0(\mathcal{V})$ and so,

$$V/L \simeq H^0(\mathcal{V}(-p))^*.$$

The intersection $W \cap (L \wedge V)$ is the kernel of the composed map

$$W \hookrightarrow \bigwedge^2 V \to \bigwedge^2 (V/L).$$

The dual map can be identified with the composition

$$\bigwedge^2 H^0(\mathcal{V}(-p)) \to \bigwedge^2 H^0(\mathcal{V}) \to H^0(\det \mathcal{V})$$

which also factors as the composition

$$\bigwedge^2 H^0(\mathcal{V}(-p)) \to H^0(\bigwedge^2(\mathcal{V}(-p))) = H^0((\det \mathcal{V})(-2p)) \subset H^0(\det \mathcal{V}).$$

We need to check that this map has corank 2, or equivalently the first arrow is an isomorphism.

Set $\mathcal{V}' = \mathcal{V}(-p)$. This is a stable bundle of rank 2 and degree 3. We need to check that the map

$$\bigwedge^2 H^0(\mathcal{V}') \to H^0(\det \mathcal{V}')$$

is surjective. For any point $p \in E$, we have an exact sequence

$$0 \to H^0(\mathcal{O}(p)) \to H^0(\mathcal{V}') \to H^0((\det \mathcal{V}')(-p)) \to 0$$

and it is easy to see that the restriction of the above map to $H^0(\mathcal{O}(p)) \wedge H^0(\mathcal{V}')$ surjects onto the subspace $H^0((\det \mathcal{V}')(-p)) \subset H^0(\det \mathcal{V}')$. Varying the point p, we get the needed surjectivity. □

Thus, by Lemma 3.3.1(i), Σ_E is exactly the set of points $p \in G(2, V) \setminus E$ where $\dim D_p \geq 3$. We have the following geometric description of Σ_E. Recall that we have a collection of 3-dimensional subspaces $W_q \subset V$, associated with points of $\widetilde{E} = \mathrm{Pic}^2(E)$ (see (3.2)).

Proposition 3.3.4. *For $p \in G(2, V)$, we have $p \in \Sigma_E$ if and only if the corresponding line L_p is contained in some plane $\mathbb{P}W_q$, where $q \in \widetilde{E}$. In other words, $\Sigma_E = \cup_{q \in \widetilde{E}} G(2, W_q)$.*

Proof. Assume first that $p \in \Sigma_E$. As we have seen above, this means that $p \in G(2, V) \setminus E$ and $\dim D_p \geq 3$. By Lemma 3.3.1(ii), this implies that the rank of the Poisson bracket Π_W on points of L_p is ≤ 2. Hence, by Lemma 3.2.1(ii), L_p is contained in the quintic $\mathcal{D}_E$. By Corollary 3.2.5, this implies that L_p is contained in some plane $\mathbb{P}W_q$.

Conversely, assume that we have a 2-dimensional subspace $L \subset H^0(M)^* \subset H^0(\mathcal{V})^* = V$, where $\mathcal{V} \to M$ is a surjection to a degree 3 line bundle M. Then $L = \langle s \rangle^{\perp} \subset H^0(M)^*$ for some 1-dimensional subspace $\langle s \rangle \subset H^0(M)$. Set $P = L^{\perp} \subset H^0(\mathcal{V})$. Then P is the preimage of $\langle s \rangle \subset H^0(M)$ under the projection $H^0(\mathcal{V}) \to H^0(M)$.

By Lemma 3.3.1, the space D_p (where $L = L_p$ for $p \in G(2, V)$) is isomorphic to the kernel of the composed map

$$W \to \bigwedge^2 V \to \bigwedge^2 (V/L).$$

Hence, $\dim(D_p)$ is equal to the corank of the dual map

$$\bigwedge^2 (P) \to \bigwedge^2 H^0(\mathcal{V}) \to H^0(\bigwedge^2 \mathcal{V}). \tag{3.3}$$

Let B denote the divisor of zeroes of s. We claim that the image of (3.3) is contained in the subspace $H^0(\bigwedge^2\mathcal{V}(-B)) \subset H^0(\bigwedge^2\mathcal{V})$. Indeed, we have an exact sequence

$$0 \to N \to \mathcal{V} \to M \to 0$$

where N is a line bundle of degree 2. It is easy to see that the composed map

$$H^0(N) \wedge H^0(\mathcal{V}) \hookrightarrow \bigwedge^2 H^0(\mathcal{V}) \to H^0(\bigwedge^2 \mathcal{V})$$

coincides with the natural multiplication map

$$H^0(N) \wedge H^0(\mathcal{V})/\bigwedge^2 H^0(N) \simeq H^0(N) \otimes H^0(M) \to H^0(N \otimes M) \simeq H^0(\bigwedge^2 \mathcal{V}).$$

The exact sequence

$$0 \to H^0(N) \to P \to \langle s \rangle \to 0$$

shows that $\bigwedge^2 P \subset H^0(N) \wedge H^0(\mathcal{V})$ and its image in $H^0(N) \otimes H^0(M)$ is contained in $H^0(N) \otimes \langle s \rangle$. This proves our claim about the image of the map (3.3). It follows that the corank of this map is ≥ 3, so $p \in \Sigma_E$. □

Lemma 3.3.5. *Let $L_p \subset V$ denote the 2-dimensional subspace corresponding to $p \in G(2, V) \setminus E$.*
(i) For any 3-dimensional subspace $M \subset V$ containing L_p, one has $W \cap \bigwedge^2 M = \bigwedge^2 L_p$.
(ii) Assume that for generic $v \in L_p$, the rank of $\Pi_{E,v}$ is 4. Then the map $D_p \otimes \mathcal{O} \to V/L_p \otimes \mathcal{O}(1)$ over the projective line $\mathbb{P}L_p$ is an embedding of a rank 2 subbundle.

Proof. (i) Since all elements of $\bigwedge^2 M$ are decomposable, the intersection $Q := W \cap \bigwedge^2 M$ is a linear subspace consisting of decomposable elements. But all decomposable elements of W are of the form $\bigwedge^2 L_q$ for some point $q \in E$. Hence, we would get an embedding $\mathbb{P}(Q) \to E$, which implies that Q is 1-dimensional, so $Q = \bigwedge^2 L_p$.
(ii) From part (i) and from Lemma 3.3.1 we get that for any 3-dimensional subspace $M \subset V$ containing L_p, one has $D_p \cap L_p \otimes M/L_p = 0$. Let us set $P = V/L_p$, and let us consider the exact sequence

$$0 \to D_p \otimes \mathcal{O}(-1) \to P \otimes \mathcal{O} \to Q \to 0.$$

We want to prove that the rank 1 sheaf Q on $\mathbb{P}^1$ has no torsion. Since $\deg(Q) = 2$ and Q is generated by global sections, we only have to exclude the possibilities $Q \simeq \mathcal{O}_p \oplus \mathcal{O}(1)$ and $Q \simeq T \oplus \mathcal{O}$, where T is a torsion sheaf of length 2.

Assume first that $Q \simeq \mathcal{O}_p \oplus \mathcal{O}(1)$. Consider the composed surjection $f : P \otimes \mathcal{O} \to Q \to \mathcal{O}(1)$. It is induced by a surjection $P \to H^0(\mathcal{O}(1))$, which has 1-dimensional kernel $\langle v \rangle$. It follows that the inclusion of $D_p \otimes \mathcal{O}(-1)$ into $P \otimes \mathcal{O}$ factors as

$$D_p \otimes \mathcal{O}(-1) \to \langle v \rangle \otimes \mathcal{O} \oplus \mathcal{O}(-1) \to P \otimes \mathcal{O}.$$

It follows that D_p has a nontrivial intersection with $H^0(\mathcal{O}(1)) \otimes \langle v \rangle = L_p \otimes M/L_p \subset L_p \otimes V/L_p$, for some 3-dimensional $M \subset V$, containing L_p. This is a contradiction, as we proved that there could be no such M.

In the case $Q \simeq T \oplus \mathcal{O}$, we get that $D_p \otimes \mathcal{O}(-1)$ is contained in the kernel of a surjection $P \otimes \mathcal{O} \to \mathcal{O}$, i.e., $D_p \otimes \mathcal{O}(-1)$ is contained in $\mathcal{O}^2 \subset P \otimes \mathcal{O}$. But any embedding $\mathcal{O}(-1)^2 \to \mathcal{O}^2$

factors through some $\mathcal{O}(-1)\oplus\mathcal{O}\to\mathcal{O}^2$ (occurring as kernel of the surjection $\mathcal{O}^2\to\mathcal{O}_p$, for some point p in the support of the quotient). Hence, we can finish again as in the previous case. □

Remark 3.3.6. The rational map f from $G(2,V)$ to $\mathbb{P}^4$ has the following interpretation, which can be proved using projective duality. Start with a generic line $L\subset\mathbb{P}(V)$. Then the intersection $L\cap\mathcal{D}_E$ with the degeneration quintic of Π_E consists of 5 points. Taking the images of these points under the projection $\mathcal{D}_E\setminus S_E\to\widetilde{E}$ (see Cor. 3.2.5) we get a divisor D_L of degree 5 on $\widetilde{E}$. All these divisors will belong to a certain linear system $\mathbb{P}^4$ of degree 5, and the map $L\mapsto D_L$ is exactly our map f.

3.4. **Calculation of the Schouten bracket and proof of Theorem B.**

Lemma 3.4.1. *(i) Let $E\subset G(2,V)$ be the elliptic curve defined by $W\subset\bigwedge^2V$. Then for each point $p\in E$, the bivector Π_E vanishes on the projective line $\mathbb{P}L_p\subset\mathbb{P}V$, where $L_p\subset V$ is the 2-dimensional subspace corresponding to p. For a generic point v of L_p the Lie algebra $\mathfrak{g}=T^*_v\mathbb{P}V$ has a basis (h_1,h_2,e_1,e_2) such that*

$$[h_1,h_2]=[e_1,e_2]=0,$$

$$[h_i,e_i]=2e_i,\quad [h_j,e_i]=-e_i\quad \text{for } i\neq j.$$

Equivalently, the linearization of Π_E takes form

$$\Pi_E^{lin}=2e_1\partial_{h_1}\wedge\partial_{e_1}-e_1\partial_{h_2}\wedge\partial_{e_1}+2e_2\partial_{h_2}\wedge\partial_{e_2}-e_2\partial_{h_1}\wedge\partial_{e_2}.$$

Furthermore, the conormal subspace $N^\vee_{\mathbb{P}L_p,v}\subset\mathfrak{g}^*$ is spanned by e_1,e_2,h_1+h_2 (dually the tangent space to $T_{\mathbb{P}L_p}$ is spanned by $\partial_{h_1}-\partial_{h_2}$).
(ii) We have an identification

$$H^0(\mathbb{P}L_p,N_{\mathbb{P}L_p})\simeq H^0(\mathbb{P}L_p,V/L_p\otimes\mathcal{O}(1))\simeq L_p^*\otimes V/L_p\simeq T_pG(2,V).$$

Under this identification, the line $T_pE\subset T_pG(2,V)$ has the property that the corresponding global section of $N_{\mathbb{P}L_p}$ evaluated at generic $v\in\mathbb{P}L_p$ spans the line

$$\langle\partial_{h_1},\partial_{h_2}\rangle/\langle\partial_{h_1}-\partial_{h_2}\rangle\subset N_{\mathbb{P}L_p,v}\simeq V/L_p.$$

Equivalently, the tangent space at v to the surface $S_E\subset\mathbb{P}V$ is $\langle\partial_{h_1},\partial_{h_2}\rangle\subset T_v\mathbb{P}V$.
(iii) Let Π' be a Poisson bracket compatible with Π_E. Then for $p\in E$ and a generic $v\in L_p$, one has

$$\Pi'_v\in\langle(2\partial_{h_1}-\partial_{h_2})\wedge\partial_{e_1},(2\partial_{h_2}-\partial_{h_1})\wedge\partial_{e_2},\partial_{h_1}\wedge\partial_{h_2}\rangle. \tag{3.4}$$

Proof. (i) Extensions $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ by $\mathcal{O}$, corresponding to the line $\mathbb{P}L_p$, are exactly the extensions that split under $\mathcal{O}\to\mathcal{O}(p)$. We claim that for a generic point of $\mathbb{P}L_p$ we have $\widetilde{\mathcal{V}}\simeq\mathcal{O}(p)\oplus L_1\oplus L_2$, where L_1 and L_2 are nonisomorphic line bundles of degree 2. Indeed, by Lemma 3.2.1(ii), the only other possibility is $\widetilde{\mathcal{V}}\simeq\mathcal{O}(p)\oplus\mathcal{V}'$, where $\mathcal{V}'$ is a nontrivial extension of M by M, where $M^2\simeq\det(\mathcal{V})$. Since the corresponding extension splits over the unique embedding $M\to\mathcal{V}$, this gives one point on the line $\mathbb{P}L_p$ for each of the four possible line bundles M.

We can compute the Lie algebra $\mathfrak{g}$ for the point corresponding to $\widetilde{\mathcal{V}} \simeq \mathcal{O}(p) \oplus L_1 \oplus L_2$ using the isomorphism of Theorem 2.3.1,

$$\operatorname{End}(\widetilde{\mathcal{V}})/\langle \mathrm{id} \rangle \xrightarrow{\sim} \mathfrak{g} \subset H^0(\mathcal{V}). \tag{3.5}$$

We consider the following basis in $\operatorname{End}(\widetilde{\mathcal{V}})/\langle \mathrm{id} \rangle$:

$$h_i = \mathrm{id}_{L_i} - \mathrm{id}_{\mathcal{O}(p)},\ e_i \in \operatorname{Hom}(\mathcal{O}(p), L_i),\ i = 1, 2.$$

Then it is easy to check the claimed commutator relations between these elements.

The conormal subspace to $\mathbb{P}L_p$ is identified with $L_p^{\perp} = H^0(\mathcal{V}(-p))$. The image of the subspace $\operatorname{Hom}(\mathcal{O}(p), L_1 \oplus L_2)$ under the map (3.5) will consist of compositions

$$\mathcal{O} \to \mathcal{O}(p) \to L_1 \oplus L_2 \to \mathcal{V},$$

which vanish at p, so they are contained in $H^0(\mathcal{V}(-p))$. We have

$$h_1 + h_2 = \mathrm{id}_{L_1} \oplus \mathrm{id}_{L_2} - 2\,\mathrm{id}_{\mathcal{O}(p)} \equiv -3\,\mathrm{id}_{\mathcal{O}(p)} \bmod \langle \mathrm{id}_{\widetilde{\mathcal{V}}} \rangle,$$

and the element $\mathrm{id}_{\mathcal{O}(p)}$ is mapped under (3.5) to the composition

$$\mathcal{O} \to \mathcal{O}(p) \to \mathcal{V},$$

which also vanishes at p. This proves our claim about the conormal subspace.

(ii) To identify the direction corresponding to T_pE, we first recall that the map $E \to G(2, V)$ is associated with the subbundle $\mathcal{V}^{\vee} \hookrightarrow V \otimes \mathcal{O}$ over E. We have an exact sequence

$$0 \to T_p^*E \otimes \mathcal{V}|_p \to H^0(\mathcal{V}|_{2p}) \to \mathcal{V}|_p \to 0.$$

The dual of the natural map $V^* \to H^0(\mathcal{V}|_{2p})$ fits into a morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{V}^{\vee}|_p & \longrightarrow & H^0(\mathcal{V}|_{2p})^* & \longrightarrow & T_pE \otimes \mathcal{V}^{\vee}|_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sim} & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & L_p & \longrightarrow & V & \longrightarrow & V/L_p & \longrightarrow & 0
\end{array}$$

and the map β corresponds to a map $T_pE \to \operatorname{Hom}(\mathcal{V}^{\vee}|_p, V/L_p) = \operatorname{Hom}(L_p, V/L_p)$ which is the tangent map to $E \to G(2, V)$. Note that the dual to β is the natural linear map

$$(V/L_p)^* = \ker(H^0(\mathcal{V}) \to \mathcal{V}|_p) \to \ker(H^0(\mathcal{V}|_{2p}) \to \mathcal{V}|_p) \simeq T_p^*E \otimes \mathcal{V}|_p. \tag{3.6}$$

Now, given a functional $v : \mathcal{V}|_p \to k$, the image of T_pE under $\pi_v : L_p^* \otimes V/L_p \to V/L_p$ corresponds to the composition of (3.6) with v. In other words, it is given by the composition

$$L_p^{\perp} = H^0(\mathcal{V}(-p)) \to \mathcal{V}(-p)|_v \simeq \mathcal{V}|_p \xrightarrow{v} k$$

(here we use a trivialization of T_pE).

Let $\widetilde{\mathcal{V}} \to \mathcal{V}$ be the extension corresponding to v. As we have seen in (i), for a generic v, we have $\widetilde{V} \simeq \mathcal{O}(p) \oplus L_1 \oplus L_2$, where L_i are as above. As we have seen in (i), under the isomorphism (3.5), $L_p^{\perp} = H^0(\mathcal{V}(-p))$ is the image of the subspace $\langle h_1 + h_2, e_1, e_2 \rangle$.

Hence, it remains to check that under the composition

$$\langle e_1, e_1\rangle \to H^0(\mathcal{V}(-p)) \to \mathcal{V}(-p)|_p \simeq \mathcal{V}|_p \xrightarrow{v} k,$$

is zero (where the first arrow is induced by (3.5)). Let us consider the element e_1 (the case of e_2 is similar). It maps to the element of $H^0(\mathcal{V}(-p))$ given by the embedding

$$\mathcal{O} \to L_1(-p) \to \mathcal{V}(-p),$$

where we use the composed map $L_1 \to \widetilde{\mathcal{V}} \to \mathcal{V}$. Thus, we need to check that the composition $L_1 \to \mathcal{V} \xrightarrow{v} k$ is zero. But this follows from the fact that the extension $\widetilde{\mathcal{V}}$ is the pull-back of the standard extension $\mathcal{O}(p) \to \mathcal{O}_p$ via v, so that we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}(p) & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle v} & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \widetilde{\mathcal{V}} & \longrightarrow & \mathcal{V} & \longrightarrow & 0 \\
 & & & & \downarrow & & \uparrow & & \\
 & & & & L_1 \oplus L_2 & \xrightarrow{\mathrm{id}} & L_1 \oplus L_2 & &
\end{array}$$

(iii) This is obtained by a straightforward computation using the vanishing of $[\Pi_E, \Pi_{E'}]$ and the formula for Π_E^{lin} from part (i). $\square$

Lemma 3.4.2. *Let $E, E' \subset G(2, V)$ be a pair of elliptic curves obtained as linear sections, such that $[\Pi_E, \Pi_{E'}] = 0$. Then E is not contained in $\Sigma_{E'} \subset G(2, V)$.*

Proof. Assume $E \subset \Sigma_{E'}$. Then, by the description of $\Sigma_{E'}$ in Proposition 3.3.4, for every $p \in E$ there exists a line bundle L_2 of degree 2 on E' such that the image of $H^0(\mathcal{V}|_p)^* \to H^0(E, \mathcal{V})^* = V$ is contained in $H^0(E', L_2)^\perp \subset H^0(E', \mathcal{V}')^* = V$. In other words, each line $\mathbb{P}L_p \subset \mathbb{P}V$, for $p \in E$, is contained in the projective plane $\mathbb{P}H^0(E', L_2)^\perp \subset \mathbb{P}V$. This plane intersects the zero locus of $\Pi_{E'}$ in a smooth cubic (see Lemma 3.2.1(iii)), hence, for a generic point $v \in L_p$ the rank of $\Pi_{E'}|_v$ is 2.

Hence, $\Pi_{E'}|_v = w_1 \wedge w_2$, where $\langle w_1, w_2\rangle$ is the tangent plane to the leaf of $\Pi_{E'}$ (i.e., to the projective plane $\mathbb{P}H^0(E', L_2)^\perp$). Furthermore, the plane $\langle w_1, w_2\rangle$ contains the tangent line to $\mathbb{P}L_p$ at v. In the notation of Lemma 3.4.1(i), the latter tangent line is spanned by $\partial_{h_1} - \partial_{h_2}$. So, $\Pi_{E'}|_v = (\partial_{h_1} - \partial_{h_2}) \wedge w$ for some tangent vector w. But we also know by Lemma 3.4.1(iii) that $\Pi_{E'}|_v$ is a linear combination of $(2\partial_{h_1} - \partial_{h_2}) \wedge \partial_{e_1}$, $(2\partial_{h_2} - \partial_{h_1}) \wedge \partial_{e_2}$ and $\partial_{h_1} \wedge \partial_{h_2}$. This is possible only when $w \in \langle \partial_{h_1}, \partial_{h_2}\rangle$, which is the tangent plane to the surface S_E (see Lemma 3.4.1(ii)).

This implies that S_E is tangent to the corresponding projective plane $\mathbb{P}H^0(E', L_2)^\perp \subset \mathcal{D}_{E'}$. Assume first that $S_E \not\subset S_{E'}$. Then we get that the regular morphism

$$S_E \setminus S_{E'} \to \mathcal{D}_{E'} \setminus S_{E'} \to \mathrm{Pic}^2(E')$$

(see Corollary 3.2.5) has zero tangent map at every point. Hence, S_E is contained in a projective plane, which is a contradiction (since the map $\mathbb{P}(\mathcal{V}^\vee) \to \mathbb{P}H^0(\mathcal{V})^* = \mathbb{P}V$ induces an isomorphism on sections of $\mathcal{O}(1)$).

Finally, if $S_E \subset S_{E'}$ then $E = E' \subset G(2, V)$ and, we get a contradiction by Lemma 3.3.3. □

Proof of Theorem B. (i) We can assume that $E \neq E'$. We will check that for a generic point $p \in E$, one has

$$T_pE \subset D_{E',p} \subset T_pG(2, V). \tag{3.7}$$

By Lemma 3.4.2, for a generic $p \in E$, we have $p \notin \Sigma_E$, hence, the line $\mathbb{P}L_p$ is not contained in the degeneracy locus $\mathcal{D}_E$ of $\Pi_{E'}$. Let us pick a generic point v of L_p, so that the rank of $\Pi_{E',v}$ is 4. We want to study the normal projection

$$\Pi^{norm}_{E',v} \in \wedge^2(T_v\mathbb{P}V/T_v\mathbb{P}L_p) \simeq \wedge^2(V/L_p)$$

(see Lemma 3.3.1).

Recall that in the notation of Lemma 3.4.1, the tangent space to $\mathbb{P}L_p$ at v is spanned by $\partial_{h_1} - \partial_{h_2}$. Hence, the inclusion (3.4) implies that $\Pi^{norm}_{E',v}$ is proportional to a bivector of the form $\partial_{h_1} \wedge \xi$. By Lemma 3.4.1(ii), we can reformulate this as

$$\Pi^{norm}_{E',v} \in \pi_v(T_pE) \wedge V/L_p \subset \wedge^2(V/L_p).$$

By Lemma 3.3.1(ii), the subspace $\pi_v(D_{E',p}) \subset V/L_p$ consists of x such that $x \wedge \Pi^{norm}_{E',v} = 0$. Thus, we deduce the inclusion

$$\pi_v(T_pE) \subset \pi_v(D_{E',p}) \subset V/L_p$$

for generic $v \in L_p$.

In other words, the section s generating

$$T_pE \subset T_{L_p}G(2, V) \simeq \mathrm{Hom}(L_p, V/L_p) \simeq H^0(\mathbb{P}L_p, V/L_p \otimes \mathcal{O}(1))$$

has the property that for generic point $v \in \mathbb{P}L_p$ the evaluation $s(v)$ belongs to the image of the evaluation at v of the embedding $D_{E',p} \otimes \mathcal{O} \to V/L_p \otimes \mathcal{O}(1)$. Since by Lemma 3.3.5 the latter is an embedding of a subbundle, this implies that in fact $s \in D_{E',p}$ as claimed.

This proves the inclusion (3.7) for a generic $p \in E$. But this implies that the composed map

$$E \setminus E' \to G(2, V) \setminus E' \to \mathbb{P}^4$$

has zero derivative everywhere, so it is constant. Hence, E is contained in a linear section of $\mathbb{P}U \cap G(2, V)$, for some 6-dimensional subspace $U \subset \bigwedge^2 V$ containing W'. Hence, $\dim(W + W') \leq 6$.

Conversely, assume W and W' are such that $U = W + W'$ is 6-dimensional. Then we claim that $[\Pi_W, \Pi_{W'}] = 0$. Indeed, since the space of such pairs (W, W') is irreducible, it is enough to consider the case when the surface $S = \mathbb{P}U \cap G(2, V)$ is smooth. Then E_W

and $E_{W'}$ are anticanonical divisors on S, and we can apply [3, Thm. 4.4] to the bundle $\mathcal{V}_S := \mathcal{U}^\vee|_S$ on S. The fact that $(\mathcal{O}_S, \mathcal{V}_S)$ is an exceptional pair is easily checked using Koszul resolutions, as in Sec. 2.2.
(ii) It is well known that if a collection of k-dimensional subspaces in a vector space has the property that any two subspaces intersect in a $(k-1)$-dimensional space, then either all of them are contained in a fixed $(k+1)$-dimensional subspace, or they contain a fixed $(k-1)$-dimensional subspace. The statement immediately follows from (i) using this fact for $k=5$ and the collection (W_i). □

Proof of Corollary C. By Theorem B(ii), the brackets (Π_{W_i}) are pairwise compatible when either there exists a 6-dimensional subspace $U \subset \bigwedge^2 V$, containing all W_i, or there is a 4-dimensional subspace $K \subset \bigwedge^2 V$, contained in all W_i. In the former case the corresponding tensors $\bigwedge^2 W_i$ are all contained in the 6-dimensional subspace

$$\bigwedge^5 U \subset \bigwedge^5 (\bigwedge^2 V).$$

In the latter case all the tensors $\bigwedge^2 W_i$ are contained in the 6-dimensional subspace

$$\bigwedge^4 K \otimes (\bigwedge^2 V/K) \simeq (\bigwedge^4 K) \wedge (\bigwedge^2 V) \subset \bigwedge^5 (\bigwedge^2 V).$$

Conversely, by [3, Thm. 4.4], if we take a smooth linear section $S = \mathbb{P}U \cap G(2,V)$, where $\dim U = 6$, we claim that we will get a 6-dimensional subspace of compatible Poisson brackets coming from anticanonical divisors of S. We just need to show that the corresponding linear map from $H^0(S, \omega_S^{-1})$ to the space of Poisson bivectors on $\mathbb{P}(V)$ is injective. Suppose there exists an anticanonical divisor $E_0 \subset E$ such that the corresponding Poisson bivector is zero. Pick a generic anticanonical divisor E. Then all elliptic curves in the pencil $E + tE_0$ map to the same Poisson bivector. But this is impossible since we can recover $E \subset G(2,V)$ from the corresponding Poisson bracket Π_E on $\mathbb{P}(V)$, as the set of all lines lying in the zero locus S_E (see Sec. 3.2). □

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