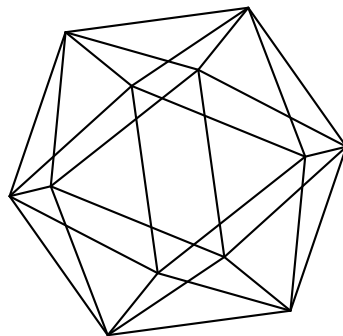


Max-Planck-Institut für Mathematik Bonn

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Preprint Series 2023 (20)

Date of submission: October 11, 2023

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SERRE ALGEBRA, MATRIX FACTORIZATION AND CATEGORICAL TORELLI THEOREM FOR HYPERSURFACES

XUN LIN AND SHIZHUO ZHANG

ABSTRACT. Let X be a smooth Fano variety. We attach a bi-graded associative algebra $\mathcal{A}_S = \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}(\text{Id}, S_{\mathcal{K}u(X)}^i[j])$ to the Kuznetsov component $\mathcal{K}u(X)$ whenever it is defined. Then we construct a natural sub-algebra of \mathcal{A}_S when X is a Fano hypersurface and establish its relation with Jacobian ring $J(X)$. As an application, we prove a categorical Torelli theorem for Fano hypersurface $X \subset \mathbb{P}^n$ ($n \geq 2$) of degree d if $\gcd(2(n+1), d) = 1$. In addition, we give a new proof of the main theorem [Pir22, Theorem 1.2] using a similar idea.

1. INTRODUCTION

Let X be a smooth complex projective variety. Reconstruction of X from its categorical invariant originates from Gabriel's thesis [Gab62], where the author proves X can be recovered from its category of coherent sheaves. Later on, this theorem is generalized to arbitrary quasi-separated scheme in [Ros98]. In the celebrated work [BO01], the authors prove smooth Fano variety X can be reconstructed from its bounded derived category $D^b(X)$ of coherent sheaves. Since the last decades, people are interested in reconstruction of Fano varieties from the non-trivial semi-orthogonal component $\mathcal{K}u(X)$, known as Kuznetsov component, of its bounded derived category of coherent sheaves, called *Categorical Torelli problem*. The first result in this direction is given in [BMMS12], where the authors prove categorical Torelli theorem for smooth cubic threefolds. Since then tremendous work have been carried out along this direction, see [PS23] for a review of known results. In this paper, we focus on Fano hypersurfaces $X \subset \mathbb{P}^n$ of degree $d \leq n$. Our work is inspired by the paper [HR16], where the authors relate a variant of Hochschild cohomology $\text{HH}(\mathcal{K}u(X), (1))$ of Kuznetsov component $\mathcal{K}u(X)$ to the Jacobian ring $J(X)$ of the hypersurface X , in particular they show the Hochschild cohomology ring is isomorphic to the Jacobian ring whenever $\mathcal{K}u(X)$ is a Calabi-Yau category, hence establish a categorical Torelli theorem for cubic fourfolds. In addition, they suggested using the category of graded matrix factorizations $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$ of a Fano hypersurface defined by a polynomial ω to reconstruct Jacobian ring via Hochschild cohomology $\text{HH}^\bullet(\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega))$, which motivates our approach to this problem. On the other hand, for any smooth DG category \mathcal{A} , there is a natural associative algebra \mathcal{A}_S attached to it, called *Serre algebra* (cf. Definition 3.3), which is a Morita invariant of \mathcal{A} . In [BO01], the authors construct a subring of \mathcal{A} , i.e. canonical ring to reconstruct smooth

2010 *Mathematics Subject Classification*. Primary 14F05; secondary 14J45, 14D20, 14D23.

Key words and phrases. Derived categories, Kuznetsov components, Categorical Torelli theorems, Fano hypersurfaces, Matrix factorization, Jacobian ring.

complex projective variety with canonical bundle ample or anti-ample. It is interesting to ask if certain sub-algebra of the Serre algebra $\mathcal{K}u(X)_S$ of a smooth Fano variety X can be used to determine the isomorphism class of it. In this article, we hope to answer this question.

1.1. Main results. Let $X, X' \subset \mathbb{P}^n (n \geq 2)$ be Fano hypersurfaces of degree $d \leq n$. Instead of making additional assumption that the equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ is compatible with degree shifting functor (1) and passing the equivalence $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \simeq \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega')$ to the equivalence $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)/(1) \simeq \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega')/(1)$, we only assume that there is an equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ and note that it commutes with Serre functors of $\mathcal{K}u(X), \mathcal{K}u(X')$ respectively. Then it is not hard to show that the associated *Serre algebra* (cf. Definition 3.3) of $\mathcal{K}u(X)$ and $\mathcal{K}u(X')$ are isomorphic. We construct a subalgebra of Serre algebra $\mathcal{K}u(X)_S$ and establish its relation with Jacobian ring.

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n$ defined by ω . Consider its affine LG model whose underlying stack is $\mathcal{Z} = [\mathbb{A}^{n+1}/\mathbb{Z}_d]$ and associate dg category of matrix factorization $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \simeq \mathcal{K}u(X)$.*

(1) *There is a sub-algebra of $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)_S$,*

$$\bigoplus_{i, j | (n+1)i+j \text{ even}} \text{Hom}(\Delta, \Delta(-ni - i + d(\frac{(n+1)i+j}{2}))).$$

(2) *Assume $\gcd(2(n+1), d) = 1$. Let $i = 2i', j = 2j'$, for appropriate choice of (i', j') , there is a sub-algebra $\bigoplus_{t \geq 1} \text{Hom}(\Delta, \Delta(t))$ of $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)_S$ such that*

$$\text{Hom}(\Delta, \Delta(t)) = \begin{cases} \text{Jac}(\omega)_t, & n = 2k, k \in \mathbb{Z} \\ \text{Jac}(\omega)_t, & n = 2k + 1, k \in \mathbb{Z}, t \neq \frac{(d-2)(n+1)}{2}, \end{cases}$$

In particular, if n is odd, the Serre algebra has a sub-graded algebra $\text{Jac}(\omega)$ and if n is even, the Serre algebra has a degree d graded subspace $\text{Jac}(\omega)_d$.

As an application, we establish *Categorical Torelli theorem* for smooth hypersurface $X \subset \mathbb{P}^n$ of degree d whenever $\gcd(2(n+1), d) = 1$. Namely, we show the Kuznetsov component $\mathcal{K}u(X)$ (cf. Proposition 2.2) determines its isomorphism class.

Theorem 1.2. *Let $X, X' \subset \mathbb{P}^n$ be degree d smooth hypersurfaces defined by ω and ω' respectively. Assume $\gcd(2(n+1), d) = 1$. If there is a Fourier-Mukai equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $X \cong X'$.*

Remark 1.3. Consider the smooth hypersurfaces of degree d in weighted projective space $\mathbb{P}(q_1, q_2, \dots, q_{n+1})$ with $\gcd(q_1, q_2, \dots, q_{n+1}) = 1$. One can prove the same statement if $\gcd(2 \cdot (\sum_{j=1}^{n+1} q_j), d) = 1$ with additional assumption $\frac{d(n+1)}{2} - \sum_{j=1}^{n+1} q_j \neq q_1, q_2, \dots, q_{n+1}$ when $n+1$ is even. In later version, we will provide more details.

Let us briefly explain the idea of the proof. We work on dg category of graded matrix factorization $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$. An equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ in $\text{Hqe}(\text{dg-cat})$ induces an equivalence $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \simeq \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega')$, commuting with Serre functors on both categories. Whenever n is even, we get graded ring isomorphism $\text{Jac}(\omega) \cong \text{Jac}(\omega')$ coming from a degree one map, then Mather-Yau reconstruction theorem [Don83, Proposition 1.1] gives $X \cong X'$. If n is odd, we get the isomorphism of degree d or $d - 1$ components of $\text{Jac}(\omega)$. Then similar arguments shows $X \cong X'$.

As a corollary, we give another proof of *Categorical Torelli theorem* for smooth cubic threefolds established in [BMMS12] and produce a new case: cubic surfaces.

Corollary 1.4. *Let X and X' be smooth cubic threefolds (or cubic surfaces). If there is a Fourier-Mukai equivalence $\mathcal{K}u(X) \cong \mathcal{K}u(X')$, then $X \cong X'$.*

In the paper [Pir22], the author shows in [Pir22, Theorem 1.2] that a class of Fano hypersurfaces X are determined by the Kuznetsov components $\mathcal{K}u(X)$ together with rotation functors $(1) : \mathcal{K}u(X) \simeq \mathcal{K}u(X)$, which generalizes a result in [HR16, Corollary 2.10]. Using the framework of matrix factorization, we give a simple proof for [Pir22, Theorem 1.2].

Theorem 1.5. *Let X and X' be smooth hypersurfaces of degree $d < n + 1$ in \mathbb{P}^n . If there is a Fourier-Mukai equivalence of pairs $(\mathcal{K}u(X), (1)) \cong (\mathcal{K}u(X'), (1)')$, then $X \cong X'$.*

1.2. Related Work. Our work is inspired by the paper [HR16], where the authors suggest using the framework of graded matrix factorization of a smooth hypersurface to relate the extended Hochschild cohomology $\text{HH}(\mathcal{K}u(X), (1))$ to the Jacobian ring $J(X)$, but they do all the work on derived category side. In [Pir22], the author generalized results [HR16, Corollary 2.10] to arbitrary Fano hypersurfaces. In an upcoming paper [Ren23], the author proves if d does not divide $n + 1$ and the pair (d, n) is not of the form $(4, 4k + 1)$, then the Kuznetsov component alone reconstructs X . In [Orl03], the author defined the Hochschild algebra HA by letting $\mathcal{K}u(X)$ be the whole bounded derived category $D^b(X)$.

Remark 1.6. While preparing the paper, we learned that the authors of the paper [BFK23] also define and study the Serre algebra under the name Hochschild-Serre cohomology in [BFK23, Definition A.1], where they give a formula for Hochschild cohomology of Hilbert scheme of points on a surface in terms of Hochschild-Serre cohomology (as a bi-graded vector space) on the surface.

1.3. Organization of the article. In Section 2 we introduce the terminology of graded matrix factorization associated with a hypersurface in projective space. Then we describe an important auto-equivalence on category of matrix factorization $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$. In Section 3, we introduce an associative algebra naturally attached to any smooth and proper differential graded (dg) category, called *Serre algebra* and show it is a Morita invariant. Then we give several examples of Serre algebra for various dg category, in particular, we construct interesting sub-algebra of

Serre algebra for category of matrix factorization $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$ corresponding to a Fano hypersurface in \mathbb{P}^n , proving Theorem 1.1. In Section 4.1 we prove Theorem 1.2 and Corollary 1.4. In Section 5 we prove Theorem 1.5.

1.4. Acknowledgement. We would like to thank Daniel Huybrechts, Ziqi Liu, Jørgen Rennemo, Ed Segal, Junwu Tu and Jieheng Zeng for useful conversation on related topics. SZ is supported by ANR project FanoHK, grant ANR-20-CE40-0023, Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy-EXC-2047/1-390685813, and partially supported by GSSCU2021092. Part of the work was finished when XL and SZ are visiting Max-Planck institute for mathematics and Hausdorff institute for mathematics. They are grateful for excellent working condition and hospitality.

2. DG CATEGORY OF GRADED MATRIX FACTORIZATIONS

In this section, we recall the terminology of dg -category of matrix factorization. We follow the context in [BFK14]. We refer the reader to [Kel06] for the basic of dg categories. Let $\text{Hqe}(\text{dg-cat})$ be the localizing of dg-cat with respect to the quasi-equivalences of dg categories. Let (X, G, L, ω) be a quadruple where X is a quasi-projective variety with G action, G is a reductive algebraic group, L is a G -equivariant line bundle and ω is a G -invariant section of L . Our main example is $(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$. The action of $\lambda \in \mathbb{C}^*$ on \mathbb{A}^{n+1} is given by $\lambda \cdot (x_0, x_1, \dots, x_n) = (\lambda \cdot x_0, \lambda \cdot x_1, \dots, \lambda \cdot x_n)$. ω is a \mathbb{C}^* -invariant section of $\mathcal{O}(d)$. Namely ω is a degree d polynomial. We always assume ω has only isolated singularity at $0 \in \mathbb{A}^{n+1}$.

We have dg category $\text{Fact}(X, G, L, \omega)$, whose objects are a quadruple $(\mathcal{E}_{-1}, \mathcal{E}_0, \Phi_{-1}, \Phi_0)$, where \mathcal{E}_{-1} and \mathcal{E}_0 are G -equivariant quasi-coherent sheaves, $\Phi_{-1} : \mathcal{E}_0 \rightarrow \mathcal{E}_{-1} \otimes L$ and $\Phi_0 : \mathcal{E}_{-1} \rightarrow \mathcal{E}_0$ are morphism of G -equivariant sheaves such that

$$\begin{aligned}\Phi_{-1} \circ \Phi_0 &= \omega. \\ (\Phi_0 \otimes L) \circ \Phi_{-1} &= \omega.\end{aligned}$$

The space of morphisms in $\text{Fact}(X, G, L, \omega)$ are the internal Hom of G -equivariant sheaves while extending the pairs of morphisms to certain \mathbb{Z} -graded complexes. We point out the reference [BFK14] for interested reader. Let $\text{Inj}(X, G, L, \omega) \subset \text{Fact}(X, G, L, \omega)$ be a dg sub-category whose components are G -equivariant injective quasi-coherent sheaves. There is a category $\text{Acycli}(\text{Fact}(X, G, L, \omega))$ which imitates acyclic complexes in category of complexes of sheaves. The absolute derived category $D^{\text{abs}}(\text{Fact}(X, G, L, \omega))$ is the homotopy category of dg quotient $\frac{\text{Fact}(X, G, L, \omega)}{\text{Acyclic}(\text{Fact}(X, G, L, \omega))}$ in $\text{Hqe}(\text{dg-cat})$.

Lemma 2.1. *The natural map $\text{Inj}(X, G, L, \omega) \rightarrow D^{\text{abs}}(\text{Fact}(X, G, L, \omega))$ induces isomorphism in homotopic categories.*

Let $\text{Inj}_{\text{coh}}(X, G, L, \omega) \subset \text{Inj}(X, G, L, \omega)$ be a dg sub-category whose objects are quasi-isomorphic to objects with coherent components in category $\text{Fact}(X, G, L, \omega)$.

Define shifting functor

$$[1] : (\mathcal{E}_{-1}, \mathcal{E}_0, \Phi_{-1}, \Phi_0) \mapsto (\mathcal{E}_0, \mathcal{E}_{-1} \otimes L, -\Phi_0, -\Phi_{-1} \otimes L).$$

With cone construction, the homotopic categories $[\text{Inj}_{\text{coh}}(X, G, L, \omega)]$ is a triangulated category which is isomorphic to graded matrix factorization in [Orl09] for $(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$.

Denote by

$$\{1\} : \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega) \rightarrow \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$$

the twisting functor which maps

$$\mathcal{E}_{-1} \xrightarrow{\Phi_0} \mathcal{E}_0 \xrightarrow{\Phi_{-1}} \mathcal{E}_{-1}(d)$$

to

$$\mathcal{E}_{-1}(1) \xrightarrow{\Phi_0(1)} \mathcal{E}_0(1) \xrightarrow{\Phi_{-1}(1)} \mathcal{E}_{-1}(d+1)$$

Clearly, we have equality of functors $\{d\} := \{1\}^d = [2]$.

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n$ defined by ω . Let

$$\mathcal{K}u(X) := \left\langle \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(n-d) \right\rangle^\perp.$$

Consider the natural enhancement $\text{Inj}_{\text{coh}}(X)$, and let $\mathcal{K}u_{dg}(X)$ be a dg subcategory that enhance $\mathcal{K}u(X)$. Write (1) as the quasi-endofunctor (Fourier-Mukai type) of $\mathcal{K}u_{dg}(X)$ that define degree shifting functor $(1) : \mathcal{K}u(X) \rightarrow \mathcal{K}u(X)$ in the sense of [HR16].

Theorem 2.2. [BFK14, Theorem 6.13] *There is an equivalence in $\text{Hqe}(\text{dg-cat})$,*

$$\Phi : \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \cong \mathcal{K}u_{dg}(X).$$

In particular, there is an isomorphism of quasi-functors

$$\Phi \circ \{1\} \cong (1) \circ \Phi.$$

Proof. Firstly, we have quasi-functor

$$\Phi : \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega) \rightarrow \mathcal{K}u_{dg}(X),$$

and quasi-functor

$$\Phi^\dagger : \text{Inj}_{\text{coh}}(X) \rightarrow \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$$

such that $\Phi^\dagger \circ \Phi \simeq \text{Id}$. Since Φ defines an equivalence of triangulated category $[\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)] \simeq \mathcal{K}u(X)$, and both $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \mathcal{O}(d), \omega)$ and $\mathcal{K}u_{dg}(X)$ are exact dg categories, therefore Φ is an isomorphism in $\text{Hqe}(\text{dg-cat})$. The equality $\Phi^\dagger \circ \Phi = \text{Id}$ implies Φ^\dagger is the inverse of Φ when restricting to $\mathcal{K}u_{dg}(X)$. Write $(1)_F : \text{Inj}_{\text{coh}}(X) \rightarrow \text{Inj}_{\text{coh}}(X)$ as the Fourier-Mukai functor that restricts to rotation functor (1) on $\mathcal{K}u_{dg}(X)$ in [BFK14, Theorem 6.13]. The isomorphisms of quasi-functors $\Phi^\dagger \circ (1)_F \circ \Phi \simeq \{1\}$ implies an isomorphism of quasi-functors $\Phi^{-1} \circ (1) \circ \Phi \simeq \{1\}$. \square

3. SERRE ALGEBRA

Theorem 3.1. [Toë07] Let \mathcal{A} be a dg category over the field k . In $\text{Hqe}(\text{dg-cat})$, we have isomorphism,

$$\mathcal{D}_{dg}(\mathcal{A}^{op} \otimes \mathcal{A}) \cong \mathcal{R}\text{Hom}_c(\mathcal{D}_{dg}(\mathcal{A}), \mathcal{D}_{dg}(\mathcal{A})),$$

where $\mathcal{R}\text{Hom}_c$ is the quasi-functor preserving coproduct.

If \mathcal{A} is a smooth proper dg category, then the bimodules for Serre functor and inverse of Serre functor are constructed explicitly in [Shk07], where the author defines Serre functor (inverse Serre functor) of the triangulated category $\text{Perf}(\mathcal{A})$ in the usual sense. From now on, we interpret those bimodules by quasi-functors by Theorem 3.1.

Definition 3.2. The Hochschild (co)homology of a smooth proper dg category \mathcal{A} are defined as,

$$\text{HH}^m(\mathcal{A}) = \text{Hom}(\text{Id}, \text{Id}[m]),$$

$$\text{HH}_m(\mathcal{A}) = \text{Hom}(\text{Id}, S[m]).$$

The Hochschild cohomology is an algebra, and the homology is a graded module over the Hochschild homology. We define an algebra which contains Hochschild cohomology and Hochschild homology, and encodes the algebra structure of Hochschild cohomology and the module structure of Hochschild homology over the Hochschild cohomology.

Definition 3.3. (Serre algebra) Let \mathcal{A} be a smooth proper dg category. Define bi-graded algebra

$$\mathcal{A}_S = \bigoplus_{m, n \in \mathbb{Z}} \text{Hom}(\text{Id}, S^m[n])$$

The multiplication map

$$\text{Hom}(\text{Id}, S^{m_1}[n_1]) \times \text{Hom}(\text{Id}, S^{m_2}[n_2]) \xrightarrow{\times} \text{Hom}(\text{Id}, S^{m_1+m_2}[n_1+n_2])$$

is defined as follows. For element $(a, b) \in \text{Hom}(\text{Id}, S^{m_1}[n_1]) \times \text{Hom}(\text{Id}, S^{m_2}[n_2])$, $a \times b$ is defined as the composition

$$\text{Id} \xrightarrow{b} \text{Id} \circ S^{m_2}[n_2] \xrightarrow{a \circ \text{Id}} S^{m_1}[n_1] \circ S^{m_2}[n_2] = S^{m_1+m_2}[n_1+n_2].$$

We check the associativity. Namely for elements $a \in \text{Hom}(\text{Id}, S^{m_1}[n_1])$, $b \in \text{Hom}(\text{Id}, S^{m_2}[n_2])$, and $c \in \text{Hom}(\text{Id}, S^{m_3}[n_3])$, we have $(ab)c = a(bc) = abc$. For example, $a(bc) = abc$ follows from the commutative diagram,

$$\begin{array}{ccccccc} \text{Id} & \xrightarrow{c} & S^{m_3}[n_3] & \xrightarrow{b} & S^{m_3}[n_3] \circ S^{m_2}[n_2] & \xrightarrow{a} & S^{m_3}[n_3] \circ S^{m_2}[n_2] \circ S^{m_1}[n_1] \\ & & & & \parallel & & \parallel \\ & & & \searrow^{bc} & S^{m_3+m_2}[n_3+n_2] & \xrightarrow{a} & S^{m_3+m_2+m_1}[n_3+n_2+n_1] \end{array}$$

Let $\text{Hmo}(\text{dg-cat})$ be the localization of dg-cat with respect to the Morita equivalences of dg categories. If \mathcal{A} and \mathcal{B} are smooth and proper, $\text{Hom}_{\text{Hmo}(\text{dg-cat})}(\mathcal{A}, \mathcal{B})$ is the isomorphism classes of $\text{Perf}(\mathcal{A}^{op} \otimes \mathcal{B})$ [Tab15, Corollary 1.44], and the composition corresponds to tensor product.

Theorem 3.4. *If $\mathcal{A} \cong \mathcal{B}$ in $\text{Hmo}(\text{dg-cat})$, then $\mathcal{A}_S \cong \mathcal{B}_S$.*

Proof. There is an isomorphism of Serre functor $S_{\mathcal{A}} \circ \Phi \cong \Phi \circ S_{\mathcal{B}}$. Hence, Φ induces isomorphism for any integer m, n ,

$$\text{Hom}(\text{Id}_{\mathcal{A}}, S_{\mathcal{A}}^m[n]) \cong \text{Hom}(\text{Id}_{\mathcal{B}}, S_{\mathcal{B}}^m[n]).$$

by the following commutative diagram,

$$\begin{array}{ccc} \text{Id}_{\mathcal{B}} & \longrightarrow & S_{\mathcal{B}}^m[n] \\ \downarrow \cong & & \downarrow \cong \\ \Phi^{-1} \circ \text{Id}_{\mathcal{A}} \circ \Phi & \longrightarrow & \Phi^{-1} \circ S_{\mathcal{A}}^m[n] \circ \Phi \end{array}$$

The isomorphism is an isomorphism of algebra since both algebra are defined by composition of functors. \square

3.1. Examples of Serre Algebras. In this section, we give examples of Serre algebra for various categories.

3.1.1. Orlov's algebra $\text{HA}(X)$. Let $\mathcal{A} = D^b(X)$ be the bounded derived category of coherent sheaves on a smooth projective variety X . In this case, $S_{\mathcal{A}} = - \otimes \omega_X[l]$, where $l = \dim X$. Thus the Serre algebra \mathcal{A}_S is given by

$$\begin{aligned} \mathcal{A}_S &:= \bigoplus_{m,n \in \mathbb{Z}} \text{Hom}(\text{Id}, S_{D^b(X)}^m[n]) \cong \bigoplus_{m,n \in \mathbb{Z}} \text{Hom}_{D^b(X \times X)}(\iota_* \mathcal{O}_X, \iota_* \omega_X^{\otimes m}[ml+n]) \\ &\cong \bigoplus_{m,n \in \mathbb{Z}} \text{Ext}_{X \times X}^{ml+n}(\iota_* \mathcal{O}_X, \iota_* \omega_X^{\otimes m}), \end{aligned}$$

where $\iota : X \hookrightarrow X \times X$ be the diagonal inclusion. It is clear that \mathcal{A}_S is isomorphic to the bi-graded algebra $\text{HA}(X)$ in [Orl03]. In particular, if $ml + n = 0$, then $\mathcal{A}_S = \bigoplus_{m \geq 0} \text{Hom}_{X \times X}(\iota_* \mathcal{O}_X, \iota_* \omega_X^{\otimes m}) \cong \bigoplus_{m \geq 0} \text{Hom}_X(\mathcal{O}_X, \omega_X^{\otimes m})$ is the canonical ring of X .

3.1.2. dg-category of matrix factorization on affine LG model. According to Orlov' sigma/LG correspondence, the Kuznetsov components of hypersurfaces $X \subset \mathbb{P}^n$ of degree d are affine LG model. First, the Serre functor of $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$ (Theorem 2.2) is $- \otimes \mathcal{O}_{\mathbb{A}^{n+1}}(-n-1)[n+1]$ [FK18, Theorem 1.2]. According to [BFK14], the natural functors can be reinterpreted as Fourier-Mukai transformation of kernels, and the natural transformation between these functors is morphism of kernels. We write $\Delta(m)$ as the kernel of functor $- \otimes \mathcal{O}_{\mathbb{A}^{n+1}}(m)$. Next we recall a key theorem in [BFK14, Theorem 1.2]. For $g \in \mathbb{C}^*$, we write W_g as the conormal sheaf of $(\mathbb{A}^{n+1})^g$, k_g the character of $\det(W_g)$. We write $H^\bullet(d\omega_g)$ as the Koszul cohomology of the Jacobian ideal of $\omega_g := \omega|_{(\mathbb{A}^{n+1})^g}$.

Theorem 3.5. [BFK14, Theorem 5.9] Assume ω has isolated singularity exactly at $0 \in \mathbb{A}^{n+1}$.

$$\begin{aligned} \mathrm{Hom}(\Delta, \Delta(m)[t]) \cong & \left(\bigoplus_{g \in \mu_d, l \geq 0, t - \mathrm{rk} \omega_g = 2u} H^{2l}(d\omega_g)(m - k_g + d(u - l)) \right. \\ & \left. \oplus \bigoplus_{g \in \mu_d, l \geq 0, t - \mathrm{rk} W_g = 2u+1} H^{2l+1}(d\omega_g)(m - k_g + d(u - l)) \right)^{\mathbb{C}^*} \end{aligned}$$

Furthermore, since $H^\bullet(d\omega_g)$ has only non-trivial cohomology at degree zero, namely $H^\bullet(d\omega_g) = H^0(d\omega_g) = \mathrm{Jac}(d\omega_g)$. Thus,

$$\mathrm{Hom}(\Delta, \Delta(m)[t]) \cong \left(\bigoplus_{g \in \mu_d, t - \mathrm{rk} W_g \text{ is even}} \mathrm{Jac}(d\omega_g)(m - k_g + d(\frac{t - \mathrm{rk} W_g}{2})) \right)^{\mathbb{C}^*}$$

Remark 3.6. As $(\mathbb{A}^{n+1})^g = (0, \dots, 0)$ if $g \neq 1$, $\mathrm{Jac}(\omega_g) = k(0)$. In the cases $g \neq 1$ has no contribution to $\mathrm{Hom}(\Delta, \Delta(t_1))$ and $\mathrm{Hom}(\Delta, \Delta(t_2))$, the multiplication

$$\mathrm{Hom}(\Delta, \Delta(t_1)) \times \mathrm{Hom}(\Delta, \Delta(t_2)) \rightarrow \mathrm{Hom}(\Delta, \Delta(t_1 + t_2))$$

is the composition of functions on \mathbb{A}^{n+1} (namely product of polynomials) while identifying with certain graded pieces of Jacobian algebra $\mathrm{Jac}(\omega)$.

Proposition 3.7. Consider the affine LG model $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$

(1) There is a sub-algebra of $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)_S$,

$$\bigoplus_{i, j | (n+1)i+j \text{ even}} \mathrm{Hom}(\Delta, \Delta(-ni - i + d(\frac{(n+1)i+j}{2}))).$$

(2) Assume $\mathrm{gcd}(2(n+1), d) = 1$. Let $i = 2i', j = 2j'$, for appropriate choice of (i', j') , there is a sub-algebra $\bigoplus_{t \geq 1} \mathrm{Hom}(\Delta, \Delta(t))$ of $\mathrm{Inj}_{\mathrm{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)_S$ such that

$$\mathrm{Hom}(\Delta, \Delta(t)) = \begin{cases} \mathrm{Jac}(\omega)_t, & n = 2k, k \in \mathbb{Z} \\ \mathrm{Jac}(\omega)_t, & n = 2k + 1, k \in \mathbb{Z}, t \neq \frac{(d-2)(n+1)}{2}, \end{cases}$$

Proof.

(1) Take all i, j such that $i(n+1) + j$ is even, the multiplication defined by composition is closed for the graded subspace

$$\bigoplus_{i, j | (n+1)i+j \text{ even}} \mathrm{Hom}(\Delta, \Delta(-ni - i + d(\frac{(n+1)i+j}{2}))).$$

Since $S = \Delta(-n-1)[n+1]$, we have $S^i[j] = \Delta(-i(n-1))[i(n+1) + j]$. Since $\Delta(d) \cong [2]$, if $i(n+1) + j$ is even, then $[i(n+1) + j] \cong \Delta(d(\frac{i(n+1)+j}{2}))$. Thus $S^i[j] \cong \Delta(-ni - i + d(\frac{i(n+1)+j}{2}))$, and the composition of Serre functor is compatible with the composition of functor $- \otimes \mathcal{O}(-n-1)[n+1]$.

(2) Let $i = 2i', j = 2j'$, then

$$-ni - i + d\left(\frac{(n+1)i + j}{2}\right) = (d-2)(n+1)i' + dj'.$$

Since $\gcd((d-2)(n+1), d) = \gcd(2(n+1), d) = 1$, we can choose integers i' , and j' such that $1 = (d-2)(n+1)i' + dj'$, and the same is true for any integer t . This implies by appropriate choice of (i, j) , we get sub-algebra $\bigoplus_{t \geq 1} \text{Hom}(\Delta, \Delta(t))$. According to Proposition 3.5,

$$\text{Hom}(\Delta, \Delta(t)) \cong \left(\bigoplus_{\substack{g \in \mu_d, \\ -\text{rk } W_g = 2u}} \text{Jac}(\omega_g)(t - k_g + du) \right)^{\mathbb{C}^*}.$$

In our case, if $g = 1$, then $\text{rk } W_g = 0$, and $k_g = 0$; if $g \neq 1$, then $\text{rk } W_g = n+1$, $k_g = -n-1$, and $\text{Jac}(\omega_g) = k(0)$. Therefore,

- If $n+1$ is odd, then

$$\text{Hom}(\Delta, \Delta(t)) \cong \text{Jac}(\omega)_t.$$

- If $n+1$ is even, then for $t \neq \frac{(d-2)(n+1)}{2}$,

$$\text{Hom}(\Delta, \Delta(t)) \cong \text{Jac}(\omega)_t.$$

□

Remark 3.8. In [BO01], the canonical ring as the sub-algebra of Orlov's algebra HA described in section 3.1.1 is used to reconstruct smooth projective varieties with canonical line bundles ample or anti-ample. So it is reasonable to expect the subring of Serre algebra $\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$ can be used to reconstruct the hypersurface defined by ω . We will prove this expectation in Section 4.

4. CATEGORICAL TORELLI THEOREM FOR FANO HYPERSURFACES

Theorem 4.1. *Let X and X' be degree $d \leq n$ smooth hypersurfaces in \mathbb{P}^n ($n \geq 2$) defined by ω and ω' respectively. Assume $\gcd(2(n+1), d) = 1$. If there is a Fourier-Mukai equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then we have isomorphism $X \cong X'$.*

Proof. First, if $n+1$ is odd, then the Serre algebra has a natural sub-algebra isomorphic to $\bigoplus_{t \geq 0} \text{Hom}(\Delta, \Delta(t))$ by Proposition 3.7(2). By the calculation in [BFK14, Theorem 5.39], the composition of $\bigoplus_{t \geq 0} \text{Hom}(\Delta, \Delta(t))$ is the composition of functions under isomorphism $\bigoplus_{t \geq 0} \text{Hom}(\Delta, \Delta(t)) \cong \text{Jac}(\omega)$, see also Remark 3.6. Therefore, $\mathcal{K}u(X) \cong \mathcal{K}u(X')$ implies an isomorphism of graded algebra $\text{Jac}(\omega) \cong \text{Jac}(\omega')$ by Theorem 3.4. Note that in degree one, it is a linear map A . Therefore, we have equality of ideal $\langle \partial_i(A\omega) \rangle = \langle \partial_i\omega' \rangle$. Then by Mather-Yau' reconstruction theorem [Don83, Proposition 1.1], $A\omega$ is projective equivalent to ω' . Thus, ω is projective equivalent to ω' , which implies $X \cong X'$.

Next, if $n+1$ is even, though we don't have $\text{Jac}(\omega)$ as a sub-algebra, but have natural graded piece $\text{Jac}(\omega)_d$ since $d \neq \frac{(d-2)(n+1)}{2}$ by Proposition 3.7(2), otherwise

$$d = \frac{(d-2)(n+1)}{2} > \frac{(d-2)d}{2}.$$

Then $d \leq 3$. If $d = 3$, then $n + 1 = 6$, contradicts that $\gcd(2(n + 1), d) = 1$. Thus $\mathcal{K}u(X) \cong \mathcal{K}u(X')$ implies isomorphism $\text{Jac}(\omega)_d \cong \text{Jac}(\omega')_d$ which is induced by a linear transformation of degree one polynomials. Hence similar argument above implies $X \cong X'$. \square

Corollary 4.2. *Let X and X' be cubic threefolds (or cubic surfaces). If there is a Fourier-Mukai equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $X \cong X'$.*

Proof.

- (1) If $d = 3$, and $n = 3$, then $\gcd(2(n + 1), d) = \gcd(8, 3) = 1$
- (2) If $d = 3$, and $n = 4$, then $\gcd(2(n + 1), d) = \gcd(10, 3) = 1$.

Then the statement follows from Theorem 4.1. \square

5. CATEGORICAL TORELLI WITH ROTATION FUNCTOR

In this section, we give a very simple proof of [Pir22, Theorem 1.2] via matrix factorizations.

Theorem 5.1. *Let X and X' be smooth hypersurfaces of degree $d < n + 1$ in \mathbb{P}^n . If there is a Fourier-Mukai equivalence of pairs $(\mathcal{K}u(X), (1)) \cong (\mathcal{K}u(X'), (1)')$, then $X \cong X'$.*

Proof. Let ω and ω' define X and X' respectively. According to Theorem 2.2, there are isomorphisms of pairs in $\text{Hqe}(\text{dg-cat})$

$$\begin{aligned} (\mathcal{K}u_{\text{dg}}(X), (1)) &\cong (\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega), \{1\}) \\ (\mathcal{K}u_{\text{dg}}(X'), (1)') &\cong (\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega'), \{1\}') \end{aligned}$$

Which induces an isomorphism of pairs by diagram chasing

$$(\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega), \{1\}) \cong (\text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega'), \{1\}').$$

If $n+1$ is odd, then we have isomorphism of graded algebra by Proposition 3.7(2) and Theorem 3.4,

$$\bigoplus_{t \geq 0}^{\binom{(n+1)(d-2)}{t}} \text{Hom}(\Delta, \Delta(t)) \cong \bigoplus_{t \geq 0}^{\binom{(n+1)(d-2)}{t}} \text{Hom}(\Delta', \Delta'(t)).$$

That is, we have isomorphism of graded algebra $\text{Jac}(\omega) \cong \text{Jac}(\omega')$. Thus, ω is projective equivalent to ω' .

If $n + 1$ is even, the case for $d = \frac{(d-2)(n+1)}{2}$ is $(d, n) = (3, 5)$. But then $d - 1 \neq \frac{(d-2)(n+1)}{2}$. So in this case we have isomorphism $\text{Jac}(\omega)_{d-1} \cong \text{Jac}(\omega')_{d-1}$ induced by linear map of degree one polynomials by Proposition 3.7 and Theorem 3.4. Thus $X \cong X'$. Similarly, the case for $d - 1 = \frac{(d-2)(n+1)}{2}$ is $(d, n) = (3, 3)$. So there is an isomorphism $\text{Jac}(\omega)_d \cong \text{Jac}(\omega')_d$ induced by linear map of degree one polynomials by Proposition 3.7 and Theorem 3.4, which implies $X \cong X'$. \square

Remark 5.2. Let $\{i\} : \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega) \simeq \text{Inj}_{\text{coh}}(\mathbb{A}^{n+1}, \mathbb{C}^*, \omega)$ be the degree shift functor with corresponding Fourier-Mukai kernel Q_i . Define $L_{\text{MF}}(X) := \bigoplus \text{Hom}(Q_0, Q_i)$. On the other hand, the degree shift auto-equivalence $(i) : \mathcal{K}u(X) \simeq \mathcal{K}u(X)$ is represented by Fourier-Mukai kernel P_i . Then we define another ring $L(X) := \bigoplus_i \text{Hom}(P_0, P_i)$. In [HR16], the authors conjecture that $L_{\text{MF}}(X) \cong L(X)$. Indeed, by [Pir22], $\text{Hom}(P_0, P_i) \cong \text{Hom}(\text{Id}, (1)^i)$. Now since we have equivalence of the pair $\langle \mathcal{K}u(X), (1) \rangle \simeq^\phi \langle \text{Inj}_{\text{coh}}(\mathcal{A}^{n+1}, G, \omega), \{1\} \rangle$, namely $(1) \cong \phi^{-1} \circ \{1\} \circ \phi$ from [BFK14]. We get

$$\text{Hom}(\text{Id}, (1)^i) \cong \text{Hom}(\text{Id}, \{1\}^i).$$

Then

$$L(X) := \bigoplus \text{Hom}(P_0, P_i) \cong \bigoplus \text{Hom}(Q_0, Q_i) \cong L_{\text{MF}}(X).$$

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