EVOLUTION OPERATOR FOR A QUANTUM PENDULUM

S. SERGEEV

To the memory of A. N. Tyurin

Abstract. One remarkable relation for the infinite series in the simple Weyl algebra is conjectured. This relation expresses an evolution operator for a quantum pendulum via its Hamiltonian.

Talking about infinite series over the simple Weyl algebra

$$\mathbf{u}\,\mathbf{v} \;=\; q\,\mathbf{v}\,\mathbf{u} \tag{1}$$

with invertible elements $\mathbf{u}$, $\mathbf{v}$, people basically keep in mind the quantum dilogarithm. The compact q-dilogarithm (synonym q-exponent) by the definition [1] is

$$(x;q)_\infty \;=\; \prod_{n=0}^{\infty}(1\,-\,q^n\,x) \;=\; \sum_{n=0}^{\infty}\frac{(-)^n\,q^{n(n-1)/2}}{(q;q)_n}\,x^n \tag{2}$$

where it is supposed $|q| \;<\; 1$ and $(x;q)_n \;=\; (1-x)(1-qx)...(1-q^{n-1}x)$. When the arguments of q-dilogarithms belongs to the simple Weyl algebra (1), the may be used for the construction of the group of rational canonical transformations of (1) and thus are related to the quantum mechanics. For example, the operator-valued function $(\mathbf{u};q)_\infty$ makes

$$(\mathbf{u};q)_\infty\;\mathbf{v}\;(\mathbf{u};q)_\infty^{-1} \;=\; \mathbf{v}\,(1\,-\,\mathbf{u})^{-1}\,. \tag{3}$$

Several remarkable relations for the q-series over the simple Weyl algebra (1) are to be mentioned. The addition theorem and the pentagon identity are the most known ones [1]:

$$(\mathbf{u};q)_\infty\,(\mathbf{v};q)_\infty \;=\; (\mathbf{u}+\mathbf{v};q)_\infty\,,\quad (\mathbf{v};q)_\infty\,(\mathbf{u};q)_\infty \;=\; (\mathbf{u}-\mathbf{v}\mathbf{u}+\mathbf{v};q)_\infty\,. \tag{4}$$

1991 *Mathematics Subject Classification.* 82B23.

Key words and phrases. Quantum dilogarithm.

This work was supported by the grants INTAS OPEN 00-00055, RFBR 01-01-00201, CGP CRDF RM1-2334-MO-02.

Here we should like to present one more complicated result concerning the following "five-term function":

$$\mathbf{s} \;=\; (a_1\mathbf{v}^{-1};q)_\infty\,(b_0\mathbf{u};q)_\infty\,(-c_0\mathbf{v}\mathbf{u};q)_\infty\,(a_0\mathbf{v};q)_\infty\,(b_1\mathbf{u}^{-1};q)_\infty\,, \tag{5}$$

where $\mathbf{u},\mathbf{v}$ are Weyl's generators and $a_0, a_1, b_0, b_1, c_0 \in \mathbb{C}$. Let

$$\mathbf{h} \;=\; H(\mathbf{u},\mathbf{v}) \;=\; a_0\mathbf{v} + b_0\mathbf{u} + a_1\mathbf{v}^{-1} + b_1\mathbf{u}^{-1} - c_0\mathbf{v}\mathbf{u} - c_1\mathbf{v}^{-1}\mathbf{u}^{-1}\,, \tag{6}$$

where $c_1 \;=\; a_1\,b_1$. Polynomial $\mathbf{h}$ is the Hamiltonian of $\mathbf{s}$ in the following sense. Operator $\mathbf{s}$ defines the rational transformations

$$\mathbf{u} \mapsto \mathbf{u}' \;=\; \mathbf{s}\,\mathbf{u}\,\mathbf{s}^{-1}\,,\quad \mathbf{v} \mapsto \mathbf{v}' \;=\; \mathbf{s}\,\mathbf{v}\,\mathbf{s}^{-1}\,, \tag{7}$$

where as the result of repeated use of (3), $\mathbf{u}'$ for example is

$$\mathbf{u}' \;=\; \widetilde{\mathbf{u}} - \mathbf{v}\,(a_0 - c_0\widetilde{\mathbf{u}})\,\widetilde{\mathbf{u}}\,(1 - b_0\widetilde{\mathbf{u}})^{-1}\,,\quad \widetilde{\mathbf{u}} \;=\; \mathbf{u}\,(1 - a_1\mathbf{v}^{-1})^{-1}\,. \tag{8}$$

One may verify "by hands" that $H(\mathbf{u}',\mathbf{v}') = H(\mathbf{u},\mathbf{v})$, i.e. $\mathbf{h}$ is indeed the invariant of $\mathbf{s}$,

$$\mathbf{s}\,\mathbf{h} \;=\; \mathbf{h}\,\mathbf{s}\,. \tag{9}$$

Polynomial (6) is called sometimes the Hamiltonian for a kind of the quantum pendulum, although its form ad the form of the evolution mapping is a bit unusual.

Operator $\mathbf{s}$ and its invariant $\mathbf{h}$ appeared in [2] as the evolution operator and the Hamiltonian of a simplest case of some hierarchy of $2+1$ dimensional integrable systems. In [2] there was formulated the following

Conjecture 1. *The operator-valued function* $\mathbf{s}$*, defined by (5) as the series with respect to* $\mathbf{u},\mathbf{v}$

$$\mathbf{s} \;=\; \sum_{n,m\in\mathbb{Z}} f_{n,m}(a_0,a_1,b_0,b_1,c_0)\,\mathbf{v}^n\,\mathbf{u}^m \tag{10}$$

may be re-written identically as the $\mathbf{v}^n\mathbf{u}^m$*-expansion of s series*

$$\mathbf{s} \;=\; \sum_{k=0}^{\infty} g_k(a_0a_1, b_0b_1, c_0a_1b_1)\,\mathbf{h}^k \tag{11}$$

for generic a_0, a_1, b_0, b_1, c_0*.*

To get (10) from (11), one has to make an *infinite* re-summation: let

$$\mathbf{h}^k \;=\; \sum_{n,m=-k}^{n,m=k} \chi_{k,n,m}\, \mathbf{v}^n\, \mathbf{u}^m \,, \tag{12}$$

then the statement of the conjecture 1 is that

$$f_{n,m} \;=\; \sum_{k=0}^{\infty} g_k\, \chi_{k,n,m} \tag{13}$$

are convergent seria. Note that the identities (4), being the particular cases of the representation of $\mathbf{s}$ as the function of corresponding $\mathbf{h}$, contain the *finite* re-summations like (13).

Next, we may guess the form of the function (11). Denote for the shortness

$$a \;=\; a_0 a_1\,, \quad b \;=\; b_0 b_1\,, \quad c \;=\; c_0 a_1 b_1\,. \tag{14}$$

For arbitrary complex h, a, b, c let

$$S(x) \;=\; \begin{pmatrix} 1 - hx + bx^2 & 1 \\ & \\ q^{-1}x(1-x)(a - q^{-1}cx) & 0 \end{pmatrix} \tag{15}$$

and

$$S_n(x) \;=\; S(x)\cdot S(qx)\cdot S(q^2x)\,\cdots\, S(q^{n-1}x)\,. \tag{16}$$

For $|q| < 1$ one may easily convinced

$$S_\infty(x) \;=\; \begin{pmatrix} s(x) & s(x) \\ & \\ \widetilde{s}(x) & \widetilde{s}(x) \end{pmatrix}, \quad \widetilde{s}(x) \;=\; q^{-1}x(1-x)(a - q^{-1}cx)\, s(qx)\,, \tag{17}$$

Due to the definition $s(x)$ is also the function of a, b, c and h:

$$s(x) \;=\; s(a,b,c;h;x) \;=\; s(b,a,c;h;x)\,. \tag{18}$$

The second equality is the remarkable symmetry hidden well in the definition (17) of s. In an equivalent formulation, if $F(x)$ is the solution of the functional equation

$$F(x) \;=\; 1 \,-\, h\,x \,+\, b\,x^2 \,+\, \frac{x(1-qx)(a-cx)}{F(qx)} \tag{19}$$

with $F(0)=1$, then $s(x) \;=\; \prod_{n=0}^{\infty} F(q^n x)$.

Conjecture 2. *Operator* $\mathbf{s}$ *as function of* a, b, c *and* $\mathbf{h}$ *is*

$$\mathbf{s} \;=\; s(a,b,c;\mathbf{h};1)\;. \tag{20}$$

For $\mathbf{s}$ given by (20) we have checked the conjectured identities (13) analytically as the series on q up to q^{21}.

The nature of the conjectured formula (20) for $\mathbf{s}$ may be described as follows.

Classical limit

Consider at the first the situation when $q = 1$ so that $\mathbf{u},\mathbf{v},\mathbf{h} = u,v,h \in \mathbb{C}$ where $h = H(u,v)$ is given by (6) with commutative u, v. As it is well known [3], instead of the Weyl algebra relation (1) one has at the classical level the Poisson algebra $\{u,v\} = uv$.

The equations of motion (e.g. for u) are then [1]

$$\frac{du}{dt} \;\equiv\; \dot{u} \;=\; \{h,u\}\;, \tag{21}$$

that may be rewritten excluding v as

$$\dot{u}^2 \;=\; u^2\left(h - b_0 u - b_1 u^{-1}\right)^2 \;-\; 4u^2(a_0 - c_0 u)(a_1 - c_1 u^{-1})\;. \tag{22}$$

Of course, the right hand side of (22) is the discriminant of $H(u,v) = h$ with respect to v.

$$v \;=\; \frac{h - b_0 u - b_1 u^{-1} - \dot{u}u^{-1}}{2(a_0 - c_0 u)}\;. \tag{23}$$

[1]Solution of the equations of motion is following. Define at the first $\sigma(x) = \dfrac{\theta_1(x)}{\theta_1(x+\eta)}$ Then in general one has

$$\begin{array}{llll} v \;=\; v(t) \;=\; \dfrac{\sigma(t-\alpha_2)}{\sigma(t-\alpha_3)}\,, & & u \;=\; u(t) \;=\; \dfrac{\sigma(t-\alpha_1)}{\sigma(t-\alpha_2)}\,, & \\ a_0 \;=\; -\dfrac{\sigma(\alpha_2-\alpha_3)\,\sigma(\alpha_1-\alpha_2)}{\sigma(\alpha_3-\alpha_2)\,\sigma(\alpha_1-\alpha_3)}\,, & & a_1 \;=\; \dfrac{\sigma(\alpha_1-\alpha_2)}{\sigma(\alpha_1-\alpha_3)}\,, & \\ b_0 \;=\; -\dfrac{\sigma(\alpha_2-\alpha_3)\,\sigma(\alpha_1-\alpha_2)}{\sigma(\alpha_2-\alpha_1)\,\sigma(\alpha_1-\alpha_3)}\,, & & b_1 \;=\; \dfrac{\sigma(\alpha_2-\alpha_3)}{\sigma(\alpha_1-\alpha_3)}\,, & \\ c_0 \;=\; -\dfrac{\sigma(\alpha_2-\alpha_3)\,\sigma(\alpha_1-\alpha_2)}{\sigma(\alpha_3-\alpha_1)\,\sigma(\alpha_1-\alpha_3)}\,, & & c_1 \;=\; \dfrac{\sigma(\alpha_2-\alpha_3)\,\sigma(\alpha_1-\alpha_2)}{\sigma(\alpha_1-\alpha_3)^2}\,. & \end{array}$$

The variable t here is proportional to used before in (22). The periodicity of this solution explains of the term "pendulum". The time shift τ is proportional to $\alpha_1 - \alpha_3 + \eta$.

The transformation $u, v \mapsto u', v'$ (7) is equivalent to a shift $t \mapsto t + \tau$, where

$$\tau \;=\; \int_u^{u'} \frac{dt}{\dot u} \tag{24}$$

The limits of the integration are actually two points on the elliptic curve $H(u, v) = h$. Of course, τ does not depend on the initial point on the curve. There exists one remarkable simple choose of the initial point:

$$u = 0, \quad uv = \text{finite}\;, \quad \text{then} \quad \dot u \;=\; u' \;=\; b_1\;. \tag{25}$$

It gives after the re-scaling $u = b_1 x$

$$\tau(h) \;=\; \int_0^1 \frac{dx}{\sqrt{(1 - hx + bx^2)^2 + 4x(1 - x)(a - cx)}}\;. \tag{26}$$

Define "the action"

$$\mathcal{A}(h) \;=\; \int^h d\chi\, \tau(\chi)\;. \tag{27}$$

Evidently, defining this like the action, we regard $\int du/\dot u$ as the angle. The lower limit in (27) is inessential. Integration over χ may be done, the answer is

$$\begin{aligned} \mathcal{A} \;&=\; \mathcal{A}(a, b, c; h) \;=\; \mathcal{A}(b, a, c; h) \;= \\ &-\int_0^1 \frac{dx}{x}\, \log\left(\frac{1 - hx + bx^2 \;+\; \sqrt{(1 - hx + bx^2)^2 + 4x(1 - x)(a - cx)}}{2}\right)\;. \end{aligned} \tag{28}$$

To establish the symmetry $a \leftrightarrow b$, it is convenient to re-parameterize the curve $H(u, v) = h$ as follows: let

$$u \;=\; b_1\, x\;, \quad v \;=\; -\,\frac{(1 - x)}{x(a_0 - c_0 x)}\, y^{-1}\;, \tag{29}$$

then

$$h \;=\; x^{-1} \,+\, y^{-1} \,-\, x^{-1} y^{-1} \,+\, bx \,+\, ay \,-\, cxy\;. \tag{30}$$

Formula (28) may be rewritten in the obviously symmetrical form

$$\mathcal{A}(h) \;=\; \frac{\pi^2}{6} \,+\, \int_{\text{in}}^{\text{out}} \log y\; d\log x\;, \tag{31}$$

where

$$\text{in} = (x = 0, y = 1)\;, \quad \text{out} = (x = 1, y = 0) \tag{32}$$

and x and y are related by (30). The extra constant $\dfrac{\pi^2}{6} = -\displaystyle\int_0^1 \frac{dx}{x}\log(1-x)$. Note, the integrals of the type (32) were appeared in [4].

Quantization

Turn now to the quantum case of the Weyl algebra (1). To get the formula for $\mathbf{s}$, one has to consider the quantum analogue of (25). Choose then the following formal representation of (1):

$$\mathbf{u}\,|x\rangle \;=\; |x\rangle\, x\, b_1\;, \quad \mathbf{v}\,|x\rangle \;=\; |qx\rangle\, \frac{-1}{x\,(a_0 - c_0 b_1 x)}\;. \tag{33}$$

In this definition one may check

$$\mathbf{u}\,|0\rangle \;=\; 0\;, \quad \mathbf{vu}\,|0\rangle \;=\; |0\rangle \left(-\frac{b_1}{a_0}\right) \tag{34}$$

as well as using the explicit form (8)

$$\mathbf{u}'\,|0\rangle \;=\; |0\rangle\, b_1\;, \tag{35}$$

sf. (25). Since $\mathbf{u}' = \mathbf{s}\mathbf{u}\mathbf{s}^{-1}$, equation (35) means

$$\mathbf{s}^{-1}\,|0\rangle \;=\; |1\rangle\;. \tag{36}$$

Let $\langle\psi_h|$ is defined as $\langle\psi_h|\mathbf{h} = h\langle\psi_h|$. According to the conjecture 1, then $\langle\psi_h|\,\mathbf{s} = s(a,b,c;h)\,\langle\psi_h|$ and

$$s(a,b,c;h) \;=\; \frac{\langle\psi_h|0\rangle}{\langle\psi_h|1\rangle}\;. \tag{37}$$

The "Schrödinger equation" $\langle\psi_h|\mathbf{h}|x\rangle = h\psi_h(x)$ for $\langle\psi_h|x\rangle \equiv \psi_h(x)$

$$\psi_h(x)(1-hx+bx^2) - \psi_h(qx) + \psi_h(q^{-1}x)q^{-1}x(1-x)(a-cq^{-1}x) = 0 \tag{38}$$

may be rewritten as the matrix one-step recursion with the help of the matrix (15), so as (17) produces

$$\psi_h(0) \;=\; \psi_h(x)\,s(x) \;+\; \psi_h(q^{-1}x)\,q^{-1}x(1-x)(a-cq^{-1}x)\,s(qx)\;. \tag{39}$$

Substituting here $x=1$, one gets *unambiguously* from (37) the conjectured formula (20).

From the other hand side, (20) is the appropriate discretisation (i.e. the quantization) of the continuous integration (28). In particular, the following assymptotics

$$s(a,b,c;h;1) \;=\; \exp\left\{-\epsilon^{-1}\mathcal{A}(a,b,c;h) \,+\, O(\epsilon^0)\right\} \tag{40}$$

for $q = \exp(-\epsilon)$, $\epsilon \,\mapsto\, 0$, is thus established.

Our way do "derive" $\mathbf{s} = s(\mathbf{h})$ in the terms of the wave functions $\psi_h(x)$ with $\psi_h(0) \neq 0$ is of course rather formal. But with a small improvement our considerations become valuable when one considers the pair of the Schrödinger equations in the framework of modular dualisation [5]. The defining equations for a physical wave function would be the modular pair of relations (39) with $\psi_h(0) = 0$ (note, used condition $\psi_h(0) \neq 0$ à priori means that this $\psi_h \neq L_2$, i.e. it is unphysical). The dualised evolution operator $\mathbf{s}$, given as the product of five non-compact dilogarithms in the same way as (5), equals to the non-compact version of the function $s(a,b,c;h)$ which is the ratio of two s-functions (18) calculated for $q = \mathrm{e}^{2i\pi\tau}$ and its modular partner $\overline{q} = \mathrm{e}^{-2i\pi\tau^{-1}}$ in the same way as the usual non-compact quantum dilogarithm is defined.

References

1. L.D. Faddeev and R.M. Kashaev 1994 Quantum dilogarithm *Mod. Phys. Lett.* **A9** 427-434
2. S. Sergeev 1999 Quantum 2+1 evolution model *J. Phys. A: Math. Gen.* **32** 5693-5714
3. V.V. Bazhanov and N.Yu. Reshetikhin 1995 Remarks on the quantum dilogarithm *J. Phys. A: Math. Gen.* **28** 2217-2226
4. S. Sergeev 2002 An evidence for a phase transition in three dimensional lattice models *Max-Plank-Institut für Mathematik Preprint series 2002 (139)*
5. L.D. Faddeev 1995 Discrete Heisenberg-Weyl group and modular group *Lett. Math. Phys.* **34** 249-254
   L.D. Faddeev 1999 Modular double of quantum group *Conférence Moshé Flato 1999 Vol. I (Dijon)* 149–156 Math. Phys. Stud. 21 *Kluwer Acad. Publ. Dordrecht* 2000

Sergei Sergeev, Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Moscow reg., Russia

E-mail address: `sergeev@thsun1.jinr.ru`

Max Plank Institute fur Mathematic, Vivatsgasse 7, D-53111, Bonn, Germany

E-mail address: `sergeev@mpim-bonn.mpg.de`