RECURRENT PROPERTY OF A CLASS OF MEROMORPHIC SELF-MAPS OF $\mathbb{P}^k$

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Abstract. We first introduce the class of quasi-algebraically stable meromorphic maps of $\mathbb{P}^k$. This class is strictly larger than that of algebraically stable meromorphic self-maps of $\mathbb{P}^k$. Then we prove that every quasi-algebraically stable meromorphic self-map enjoys a recurrent property. In particular, the first dynamical degree of such a map is always an algebraic integer.

1. Introduction

Let $f : \mathbb{P}^k \longrightarrow \mathbb{P}^k$ be a meromorphic self-map. It can be written $f := [F] := [F_0 : \ldots : F_k]$ in homogeneous coordinates where the F_j's are homogeneous polynomials in the $k+1$ variables $z_0, \ldots, z_k$ of the same degree d with no nontrivial common factor. The polynomial F will be called *a lifting* of f in $\mathbb{C}^{k+1}$. The number $\mathrm{d}(f) := d$ will be called *the algebraic degree* of f. Moreover f is said to be *dominating* if it is generically of maximal rank k, in other words, its jacobian determinant does not vanish identically (in any local chart). The *indeterminancy locus* $\mathcal{I}(f)$ of f is the set of all points of $\mathbb{P}^k$ where f is not holomorphic, or equivalently the common zero set of $k+1$ component polynomials $F_0, \ldots, F_k$. Observe that $\mathcal{I}(f)$ is a subvariety of codimension at least 2. From now on, we always consider dominating meromorphic self-maps f of $\mathbb{P}^k$ with $k \geq 2$. For such a map f, The *Julia set*, noted by $\mathcal{J}(f)$, is, by definition, the smallest closed set of $\mathbb{P}^k$ such that the family $\{f^n : n \geq 1\}$ is normal on $\mathbb{P}^k \setminus \mathcal{J}(f)$. A survey on recent development in the dynamical theory of meromorphic self-maps of $\mathbb{P}^k$ could be found in the expository article by Sibony [14].

For any map f with $\mathrm{d}(f) > 1$, consider the following limit in the sense of current

$$\lim_{n\to\infty} \frac{(f^*)^n \omega}{\mathrm{d}(f)^n}, \tag{1.1}$$

where ω denotes the Fubiny-Study Kähler form on $\mathbb{P}^k$ so normalized that $\int\limits_{\mathbb{P}^k} \omega^k = 1$.

It is well-known (see for example [14]) that this limit exists and defines the so-called *Green current* $T = T_f$ associated to f. Let us introduce the following

Definition 1.1. *A meromorphic self-map $f : \mathbb{P}^k \longrightarrow \mathbb{P}^k$ is said to be algebraically stable (or AS for short) if there is no hypersurface of $\mathbb{P}^k$ which would be sent, by some iterate f^N, to $\mathcal{I}(f)$.*

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The notion of Green current plays a peculiar role in complex dynamics. For example, if f is AS then the support of the Green current defined by formula (1.1) is contained in the Julia set $\mathcal{J}(f)$ (see Theorem 1.6.5 in [14]). In this case the Green current contains many important dynamical information of the corresponding map. We refer the reader to the recent works of Fornæss-Sibony [12, 13], Diller and Favre [4, 5, 6, 9], Dinh–Sibony [7, 8] for further explications.

However, the situation becomes much harder in the case of non AS maps. An explicit example where the support of the Green current defined by formula (1.1) is not contained in the Julia set is given by the birational $\mathcal{B}$-Fibonacci map

$$R\,[z:w:t] := \left[wt : wz : t^2\right].$$

Theorem 4.26 in the work of Bonifant–Fornæss [2] shows that $\mathcal{J}(R)$ contains the real hypersurface $\left\{[z:w:t]\in\mathbb{P}^2:\ |w|^{\frac{1+\sqrt{5}}{2}}|z| = |t|^{\frac{3+\sqrt{5}}{2}}\right\}$. On the other hand, as was proved in Theorem 2.4.6 of Favre's thesis [9] (see also [10]), the Green current (1.1) of a non AS map is supported in a countable union of complex hypersurfaces. Obviously, $\mathcal{J}(R)$ is not of this type. This phenomenon occurs because $\mathrm{d}(f)$ is strictly larger than its *first dynamical degree* $\lambda_1(f) := \lim\limits_{n\to\infty}\mathrm{d}(f^n)^{\frac{1}{n}}$. Therefore, the classical definition (1.1) of the Green current cannot be appropriate for the non AS case. This consideration motivates the study of the degree growth of non AS maps. One of the first works in this direction is the article of Bonifant–Fornæss [2] where some special non AS maps are thoroughly studied. In her thesis [1] Bonifant constructs an appropriate Green current for these maps and then write down the functional equation. Bedford and Kim [3] also study the degree growth of a class of birational mappings. Favre and Jonsson [11] give a complete study of the case of polynomial maps in $\mathbb{C}^2$. These works show in particular that the behavior of the algebraic degree of the n-iterate f^n for a non AS meromorphic self-map $f:\mathbb{P}^k\longrightarrow\mathbb{P}^k$ may be very complicated.

The purpose of this paper is to define a big class of self-maps of $\mathbb{P}^k$ which is called the class of quasi-algebraically stable (or QAS for short) meromorphic self-maps. This class contains strictly that of all AS self-maps. The QAS self-maps have a closed connection with the AS self-maps. More precisely, the QAS self-maps share a *recurrent property* with the AS ones. Let us explain this more explicitly. For an AS self-map f we may define a sequence of polynomial maps $(F_n)_{n=0}^{\infty}:\ \mathbb{C}^{k+1}\longrightarrow\mathbb{C}^{k+1}$ such that F_n is a lifting of f^n, $n\geq 0$, in the following recurrent way:

$$F_n := F_1\circ F_{n-1},\qquad n\geq 1,$$

where $F_1,\ F_0$ are arbitrarily fixed lifting of $f,\ f^0=\mathrm{Id}$. Following the same pattern, the *recurrent law* for a QAS self-map f which is not AS may be stated as follows:

$$F_n := \frac{F_1\circ F_{n-1}}{H_0\circ F_{n-n_0-1}}$$

for all $n>n_0$. Here $n_0\geq 1$ is an integer and H_0 is a homogeneous polynomial. The recurrent phenomenon happens when the orbits of the hypersurfaces which are sent to $\mathcal{I}(f)$ by some iterate f^N (see Definition 1.1 above) are, in some sense, not so complicated. That is the main point of our observation.

This paper is organized as follows.

We begin Section 2 by collecting some background and introducing some notation. This preparatory is necessary for us to state the results afterwards.

Section 3 starts with the definition of quasi-algebraically stable meromorphic self-maps. Then we provide some examples illustrating this definition.

The last section is devoted to the formulation and the proof of the main theorem. Some examples including some considered by Bonifant–Fornæss are also analyzed in the light of this theorem. Finally, we conclude the paper with some remarks.

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2. Background and notation

Let f be a dominating meromorphic self-map of $\mathbb{P}^k$, Γ the graph of f in $\mathbb{P}^k \times \mathbb{P}^k$ and π_1, π_2 the natural projections of $\mathbb{P}^k \times \mathbb{P}^k$ onto its factors. Let A be a (not necessarily irreducible) analytic subset of $\mathbb{P}^k$. We define the following analytic sets

$$f(A) := \overline{f|_{\mathbb{P}^k \setminus \mathcal{I}(f)}(A)} \qquad \text{and} \qquad f^{-1}(A) := \pi_1(\pi_2^{-1}(A)).$$

In the sequel, $\text{codim}(A)$ denotes the codimension of A. Moreover. we recall that a *hypersurface* is an analytic set of pure codimension 1 in $\mathbb{P}^k$. Let $\text{Crit}(f)$ denote the critical set of f (i.e. the hypersurface defined outside $\mathcal{I}(f)$ by the zero set of the jacobian of f in any local coordinates). The following result is very useful.

Proposition 2.1. *Let f be as above. Then, for every irreducible analytic set $A \subset \mathbb{P}^k$, $f(A)$ is also an irreducible analytic set.*

Proof. Suppose in order to get a contradiction that $f(A) = B_1 \cup B_2$, where B_1, B_2 are analytic sets in $\mathbb{P}^k$, distinct from $f(A)$. It follows that $(A \cap f^{-1}(B_1)) \cup (A \cap f^{-1}(B_2)) = A$ and the two analytic sets $A \cap f^{-1}(B_1)$, $A \cap f^{-1}(B_2)$ are distinct from A. We therefore get the desired contradiction. This finishes the proof. $\square$

We denote by $\mathcal{C}_1^+(\mathbb{P}^k)$ the set of positive closed currents of bidegree $(1,1)$. A current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$ can be written locally as $T = dd^c u$ for some plurisubharmonic function u (which is called a *local potential* of T). The mass of T is defined by $\|T\| := \int_{\mathbb{P}^k} T \wedge \omega^{k-1}$. Fix a point $z \in \mathbb{P}^k$ and local coordinates sending z to the origin in $\mathbb{C}^k$. Choose a local plurisubharmonic potential u for T defined around 0 in these coordinates. We can define the *Lelong number* of u at 0 as follows

$$\nu(u,0) := \max\{c \geq 0 : \ u(z) \leq c \log|z| + \mathcal{O}(1)\}$$

which is a finite nonnegative real number. We then set $\nu(T,z) := \nu(u,0)$, which does not depend on any choice we made.

For a current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$, we use local potentials to define the induced pull-back $f^*T \in \mathcal{C}_1^+(\mathbb{P}^k)$. More precisely, for any $z \in \mathbb{P}^k \setminus \mathcal{I}(f)$, T has a local potential u in a neghborhood of $f(z)$, and we define $f^*T := dd^c(u \circ f)$ in a neighborhood of z. This yields a well-defined, positive closed $(1,1)$-current on the set $\mathbb{P}^k \setminus \mathcal{I}(f)$. Since $\operatorname{codim}(\mathcal{I}(f)) > 1$, we can extend f^*T to $\mathbb{P}^k$ by assigning zero mass on the set $\mathcal{I}(f)$ to the coefficients measures of $(f^*T)|_{\mathbb{P}^k \setminus \mathcal{I}(f)}$.

Any hypersurface $\mathcal{H}$ of $\mathbb{P}^k$ defines a current of integration $[\mathcal{H}] \in \mathcal{C}_1^+(\mathbb{P}^k)$, and $\|[\mathcal{H}]\| = \deg(H)$, where $H :\ \mathbb{C}^{k+1} \longrightarrow \mathbb{C}$ is any homogeneous polynomial defining $\mathcal{H}$. Finally, for any current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$, it holds that

$$\|f^*T\| = \mathrm{d}(f) \cdot \|T\|. \tag{2.1}$$

For further information on this matter, the reader is invited to consult the surveys [12] and [14].

3. Quasi-algebraically stable meromorphic self-maps

In [13] Fornæss and Sibony establish the following definition.

Definition 3.1. *A hypersurface $\mathcal{H} \subset \mathbb{P}^k$ is said to be a degree lowering hypersurface of f if, for some (smallest) $n \geq 1$, $f^n(\mathcal{H}) \subset \mathcal{I}(f)$. The integer n is then called the height of $\mathcal{H}$.*

The following proposition gives us the structure of a non AS self-map.

Proposition 3.2. *Let f be a meromorphic self-map of $\mathbb{P}^k$. Then there are exactly one integer $M \geq 0$, M degree lowering hypersurfaces $\mathcal{H}_j$ with height n_j, $j = 1, \ldots, M$, satisfying the following properties:*
(i) all the numbers n_j , $j = 1, \ldots, M$, are pairwise different;
(ii) $\operatorname{codim}(f^m(\mathcal{H}_j)) > 1$ *for $m = 1, \ldots, n_j$, and $j = 1, \ldots, M$;*
(iii) for any degree lowering irreducible hypersurface $\mathcal{H}$ of f, there are integers $n \geq 0$ and $1 \leq j \leq M$ such that $f^n(\mathcal{H})$ is a hypersurface and $f^n(\mathcal{H}) \subset \mathcal{H}_j$

In particular, f is AS if and only if $M = 0$.

Proof. First, we give the construction of M and $\mathcal{H}_j$, n_j, $j = 1 \ldots, M$. To this end observe that every hypersurface $\mathcal{H}$ satisfying $\operatorname{codim}(f(\mathcal{H})) > 1$ should be contained in $\operatorname{Crit}(f)$. Therefore, one takes the family $\mathcal{F}$ of all degree lowering irreducible components $\mathcal{H}$ of $\operatorname{Crit}(f)$ such that $\operatorname{codim}(f^m(\mathcal{H})) > 1$ for $1 \leq m \leq n$, where n is the height of $\mathcal{H}$. Let $1 \leq n_1 < \cdots < n_M$ be all the heights of elements in $\mathcal{F}$ (it is easy to see that M is finite). Let $\mathcal{H}_j$ be the (finite) union of all elements in $\mathcal{F}$ with the same height n_j. Then properties (i) and (ii) are satisfied.

To prove (iii) let $\mathcal{H}$ be a degree lowering irreducible hypersurface with height h. In virtue of Proposition 2.1, let n be the greatest integer such that $0 \leq n < h$ and $f^n(\mathcal{H})$ is a hypersurface. The choice of n implies that $\operatorname{codim}(f^m(\mathcal{H})) > 1$, $m = n+1, \ldots, h$. Consequently, in virtue of the above construction, we deduce that $f^n(\mathcal{H}) \subset \mathcal{H}_j$ for some $1 \leq j \leq M$. This proves (iii).

Since the uniqueness of M and $\mathcal{H}_j$, n_j, $j = 1 \ldots, M$, is almost trivial, it is therefore left to the interested reader. This completes the proof. $\square$

Definition 3.3. *Under the hypothesis and the notation of Proposition 3.2, for every $j = 1, \ldots, M$, $\mathcal{H}_j$ is called the primitive degree lowering hypersurface of f with the height n_j.*

We are now able to define the class of quasi-algebraically stable self-maps.

Definition 3.4. *A meromorphic self-map f of $\mathbb{P}^k$ is said to be quasi-algebraically stable (or QAS for short) if either it is AS or it satisfies the following properties:*

- (i) *there is only one primitive degree lowering hypersurface (let $\mathcal{H}_0$ be this hypersurface and let n_0 be its height);*
- (ii) *for every irreducible component $\mathcal{H}$ of $\mathcal{H}_0$ and every $m = 1, \ldots, n_0$, $f^m(\mathcal{H}) \not\subset \mathcal{H}_0$;*
- (iii) *for every irreducible component $\mathcal{H}$ of $\mathcal{H}_0$, one of the following two conditions holds*
  $(iii)_1$ *$f^m(\mathcal{H}) \not\subset \mathcal{I}(f)$ for all $m \geq n_0 + 1$,*
  $(iii)_2$ *there is an $m_0 \geq n_0$ such that $f^{m_0+1}(\mathcal{H})$ is a hypersurface and $f^m(\mathcal{H}) \not\subset \mathcal{I}(f)$ for all m verifying $n_0 + 1 \leq m \leq m_0$.*

We conclude this section by studying some examples.

Example 3.5. Consider the following meromorphic self-map of $\mathbb{P}^2$:

$$f([z : w : t]) := \left[2tz - \left(z^2 + w^2\right) : 2tw - \left(z^2 + w^2\right) : 2t^2 - \left(z^2 + w^2\right)\right]. \tag{3.1}$$

It can be checked that $\mathcal{I}(f) = \{[1 : 1 : 1], [1 : i : 0], [1 : -i : 0]\}$, and $\operatorname{Crit}(f) = \{t(2t^2 + w^2 + z^2 - 2zt - 2wt) = 0\}$. Moreover we have

$$\begin{aligned} f(\{t = 0\}) &= [1 : 1 : 1] \in \mathcal{I}(f), \\ f\left(\left\{2t^2 + w^2 + z^2 - 2zt - 2wt = 0\right\}\right) &= \{t - z - w = 0\}. \end{aligned}$$

Therefore, $\{t = 0\}$ is the unique primitive degree lowering hypersurface and its height is 1. Since $[1 : 1 : 1] \notin \{t = 0\}$ and $f^2(\{t = 0\})$ is a hypersurface, f is QAS.

Example 3.6. For all integers $d \geq 2$ and $m \geq 1$, the following map is given by Bonifant-Fornæss in [2]

$$f([z : w : t]) := \left[zt^{d-1} : \left(wt^{d-1} + z^d\right)\cos\frac{\pi}{m} - t^d \sin\frac{\pi}{m} : \left(wt^{d-1} + z^d\right)\sin\frac{\pi}{m} + t^d\cos\frac{\pi}{m}\right].$$

It can be checked that $\mathcal{I}(f) = [0 : 1 : 0]$, $\operatorname{Crit}(f) = \{t = 0\}$, and $\{t = 0\}$ is the only primitive degree lowering hypersurface of f. Moreover, its height is m. Since $f^n(\{t = 0\}) = \left[0 : \cos\frac{n\pi}{m} : \sin\frac{n\pi}{m}\right] \in \{t = 0\}$ for $n = 1, \ldots, m$, f is not a QAS according to Definition 3.4 (ii). However, f satisfies conditions (i) and $(\mathrm{iii})_2$ of this definition. Similarly, it is not difficult to construct examples of meromorphic self-maps of $\mathbb{P}^2$ which satisfy (i)–(ii) but do not satisfy $(\mathrm{iii})_1$ (resp. but do not satisfy $(\mathrm{iii})_2$).

Example 3.7. Consider the following meromorphic self-map of $\mathbb{P}^2$:

$$\begin{aligned} f([z : w : t]) := \Big[&(z + w + t)^2(z^3 + w^3 + t^3)z^2 - 27z^3w^4 : \\ &(z + w + t)^2(z^3 + w^3 + t^3)w^2 - 27z^3w^4 : (z + w + t)^2(z^3 + w^3 + t^3)t^2 - 27z^3w^4\Big]. \end{aligned} \tag{3.2}$$

It can be checked that

$$\mathcal{I}(f) = [1:1:1] \cup \{[z:w:t],\ z+w+t = zw = 0\} \cup \Big\{[z:w:t],\ z^3+w^3+t^3 = zw = 0\Big\},$$

and

$$f\left(\{z+w+t=0\}\right) = f\left(\left\{z^3+w^3+t^3=0\right\}\right) = [1:1:1] \in \mathcal{I}(f).$$

Moreover, it is not difficult to see that there is no hypersurface $\mathcal{H}$ which is not contained in $\mathcal{H}_0 := \{(z+w+t)^2(z^3+w^3+t^3)=0\}$ and which satisfies $\operatorname{codim}(f(\mathcal{H})) > 1$. In other words, $\mathcal{H}_0$ is the unique primitive degree lowering hypersurface and its height is 1. Since $[1:1:1] \notin \mathcal{H}_0$, and $f^2(\{z+w+t=0\})$, $f^2(\{z^3+w^3+t^3=0\})$ are hypersurfaces, it follows that f is QAS.

4. The main result

Now we are ready to formulate the main result of this article.

The Main Theorem. *Let f be a QAS meromorphic self-map of $\mathbb{P}^k$ which is not AS. Let $\mathcal{H}_0$ be its unique primitive degree lowering hypersurface and let n_0 be its height. We define a sequence $\{F_n\ : n \geq 1\}$ of maps $\mathbb{C}^{k+1} \longrightarrow \mathbb{C}^{k+1}$ as follows :*

$F_1, \ldots, F_{n_0}, F_{n_0+1}$ are arbitrarily fixed liftings of $f^1 (\equiv f), \ldots, f^{n_0}, f^{n_0+1}$ respectively. Let H_0 be the unique homogeneous polynomial which verifies the equality

$$F_1 \circ F_{n_0} = H_0 \cdot F_{n_0+1}. \tag{4.1}$$

Next we define F_n for all $n > n_0+1$ as follows :

$$F_n := \frac{F_1 \circ F_{n-1}}{H_0 \circ F_{n-n_0-1}}. \tag{4.2}$$

Then $\mathcal{H}_0 = \{H_0(z) = 0\}$, and for any $n \geq 0$, F_n is a lifting of f^n. Moreover, for any current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$,

$$(f^n)^*T = \begin{cases} (f^{n-1})^*(f^*T), & n = 1, \ldots, n_0, \\ (f^{n-1})^*(f^*T) - \|T\| \cdot (f^{n-n_0-1})^*[\mathcal{H}_0], & n > n_0. \end{cases} \tag{4.3}$$

Prior to the proof of the theorem we need a preparatory result.

Lemma 4.1. *We keep the above hypothesis and notation. Let $m \in \mathbb{N}$, $m \geq 1$. Then, for any current $T := [l(z) = 0]$, where $\{l(z) = 0\}$ is a generic [1] complex hyperplane in $\mathbb{P}^k$, for any irreducible component $\mathcal{H}$ of the hypersurface $(f^m)^{-1}(\mathcal{H}_0)$, the following equalities hold*

$$\nu\Big((f^p)^*[\mathcal{H}_0], z\Big) = 0, \qquad p = \max\{0, m-n_0+1\}, \ldots, m-1; \tag{4.4}$$

$$\nu\Big((f^m)^*(f^{n_0+1})^*T, z\Big) = 0, \tag{4.5}$$

where z is a generic point of $\mathcal{H}$.

[1] A generic element of a family has a certain property means exactly that the set of elements in the family that do not have that property is contained in an analytic set of strictly smaller dimension.

Proof of Lemma 4.1. To prove (4.4), fix an arbitrary irreducible component $\mathcal{H}$ of the hypersurface $(f^m)^{-1}(\mathcal{H}_0)$, and an integer $p : \max\{0, m - n_0 + 1\} \le p \le m - 1$. Suppose in order to reach a contradiction that $\nu\Big((f^p)^*[\mathcal{H}_0], z\Big) > 0$ for any generic point $z \in \mathcal{H}$. Putting $\mathcal{G} := f^p(\mathcal{H})$, the latter inequality implies that

$$\mathcal{G} \subset \mathcal{H}_0. \tag{4.6}$$

In virtue of Proposition 2.1, there are two cases to consider.

Case 1: $\mathcal{G}$ is a hypersurface.

In this case it follows from the inclusion $\mathcal{H} \subset (f^m)^{-1}(\mathcal{H}_0)$ that $f^{m-p}(\mathcal{G}) = f^m(\mathcal{H}) \subset \mathcal{H}_0$. On the other hand, since $m - p \le n_0$, $\mathcal{G} \subset \mathcal{H}_0$ (by (4.6)), and f is QAS (see Definition 3.4(ii)), one gets $f^{m-p}(\mathcal{G}) \not\subset \mathcal{H}_0$. One therefore leads to a contradiction. This case cannot happen.

Case 2: $\mathcal{G}$ is an irreducible analytic set of codimension strictly greater than 1.

In virtue of Proposition 2.1, let q be the greatest integer such that $0 \le q < p$ and $f^q(\mathcal{H})$ is a hypersurface. Consider three subcases.

Case 2a: there is a smallest integer r such that $q < r < p$ and $f^r(\mathcal{H}) \subset \mathcal{I}(f)$.

In virtue of the choice of q, r, and of Definition 3.4, we see that $f^q(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$. In addition, by invoking Definition 3.4 (iii), we see that none of the following analytic sets $f^{r+1}(\mathcal{H}), \ldots, f^p(\mathcal{H})(= \mathcal{G})$ is a subset of $\mathcal{I}(f)$. On the other hand, using (4.6) and the hypothesis that f is QAS, we see that there is a smallest integer s such that $p < s \le p + n_0$, and $\operatorname{codim}(f^{p+1}(\mathcal{H})), \ldots, \operatorname{codim}(f^s(\mathcal{H})) > 1$, and none of the following sets $f^t(\mathcal{H})$ $(p + 1 \le t \le s - 1)$ is a subset of $\mathcal{I}(f)$, but $f^s(\mathcal{H}) \subset \mathcal{I}(f)$. We therefore obtain a contradiction with Definition 3.4 (iii).

Case 2b: $f^r(\mathcal{H}) \not\subset \mathcal{I}(f)$, $r = q + 1, \ldots, p - 1$, but $f^p(\mathcal{H}) \subset \mathcal{I}(f)$.

In virtue of the choice of q, and of Definition 3.4, we see that $f^q(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$, and $p = q + n_0$. However, by (4.6), $f^{n_0}(f^q(\mathcal{H})) = f^p(\mathcal{H}) \subset \mathcal{H}_0$. This is a contradiction with Definition 3.4 (ii).

Case 2c: $f^r(\mathcal{H}) \not\subset \mathcal{I}(f)$, $r = q + 1, \ldots, p$.

Using (4.6) and arguing as in Case 2a, we see that there is a smallest integer s such that $p < s \le p+n_0$, and none of the following sets $f^t(\mathcal{H})$ $(p+1 \le t \le s-1)$ is a subset of $\mathcal{I}(f)$, but $f^s(\mathcal{H}) \subset \mathcal{I}(f)$. This implies that $f^q(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$, and $s = q + n_0$. Since $f^p(\mathcal{H}) \subset \mathcal{H}_0$ (by (4.6)) we obtain a contradiction with Definition 3.4 (ii).

Hence, the proof of (4.4) is complete.

To prove (4.5) fix an arbitrary irreducible component $\mathcal{H}$ of the hypersurface $(f^m)^{-1}(\mathcal{H}_0)$, and a current $T := [l(z) = 0]$, where $\{l(z) = 0\}$ is a generic complex hyperplane of $\mathbb{P}^k$. Suppose in order to reach a contradiction that $\nu\Big((f^m)^*(f^{n_0+1})^*T, z\Big) > 0$ for any generic point $z \in \mathcal{H}$. Putting $\mathcal{G} := f^m(\mathcal{H})$, the latter inequality and the choice of $\mathcal{H}$ imply that

$$\mathcal{G} \subset \mathcal{H}_0 \cap \mathcal{I}(f^{n_0+1}). \tag{4.7}$$

Let q be the greatest integer such that $0 \le q \le m$ and $f^q(\mathcal{H})$ is a hypersurface. Clearly, $q < m$ because of (4.7): $\operatorname{codim}(\mathcal{I}(f^{n_0+1})) > 1$. In virtue of the choice of

q, and of Definition 3.4, and of the inclusion $\mathcal{G} = f^m(\mathcal{H}) \subset \mathcal{I}(f^{n_0+1})$ (see (4.7)), we conclude that $f^q(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$. Since $f^m(\mathcal{H}) \subset \mathcal{H}_0$, we obtain a contradiction with Definition 3.4 (ii)–(iii).

Hence, the proof of (4.5) is finished. This completes the proof of the lemma. □

Now we arrive at

The proof of the Main Theorem. The assertion $\mathcal{H}_0 = \{H_0(z) = 0\}$ follows immediately from (4.1) and the hypothesis on $\mathcal{H}_0$ and n_0. Moreover, the hypothesis of the theorem implies that F_n is a lifting of f^n and (4.3) is valid for $n = 1, \ldots, n_0$.

We will prove (4.3) and the fact that F_n is a lifting of f^n by induction on $n \geq n_0+1$. For $n = n_0 + 1$, these assertions are immediate consequences of (4.2). Suppose them true for $n - 1$, we like to show them for n.

To this end let G be the homogeneous polynomial given by

$$G \cdot F_n := F_1 \circ F_{n-1}, \tag{4.8}$$

and let $\mathcal{G}$ be the hypersurface $\{G(z) = 0\}$. We may rewrite (4.8) as

$$(f^{n-1})^*(f^*T) = (f^n)^*T + [\mathcal{G}], \tag{4.9}$$

for any current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$ of mass 1. In virtue of (4.2) and (4.9), we only need to show that

$$[\mathcal{G}] = (f^{n-n_0-1})^*[\mathcal{H}_0]. \tag{4.10}$$

One breaks the proof of this identity into two steps.
Step I: Proof of the inclusion $\mathcal{G} \subset (f^{n-n_0-1})^{-1}(\mathcal{H}_0)$.

Consider an arbitrary irreducible component $\mathcal{H}$ of $\mathcal{G}$. Then we deduce from (4.9) that

$$\nu\Big((f^{n-1})^*(f^*T), z\Big) > 0$$

for any current $T := [l(z) = 0]$, where $\{l(z) = 0\}$ is a generic complex hyperplane in $\mathbb{P}^k$, and for a generic point $z \in \mathcal{H}$. Since f^*T has bounded local potentials on $\mathbb{P}^k \setminus \mathcal{I}(f)$, it follows that $(f^{n-1})(\mathcal{H}) \subset \mathcal{I}(f)$.

Now let m be the greatest integer such that $0 \leq m < n - 1$ and $f^m(\mathcal{H})$ is a hypersurface. Put $\mathcal{F} := f^m(\mathcal{H})$. Therefore, $f^{m+1}(\mathcal{H}), \ldots, f^{n-1}(\mathcal{H})$ are analytic sets of codimension strictly greater than 1. Since we have shown that $f^{n-1}(\mathcal{H}) \subset \mathcal{I}(f)$, there is a smallest integer p such that $m + 1 \leq p \leq n - 1$ and $f^p(\mathcal{H}) \subset \mathcal{I}(f)$. Using the hypothesis that f is QAS, one concludes that $\mathcal{F} := f^m(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$. Moreover, one has $p = n_0 + m$.

Next, observe that $f^{n-1-m}(\mathcal{F}) = f^{n-1}(\mathcal{H}) \subset \mathcal{I}(f)$ with $n - 1 - m \geq p - m = n_0$. Invoking Definition 3.4(iii), it follows that $p = n - 1$, and hence $m = n - n_0 - 1$. In summary, we have shown that $f^{n-n_0-1}(\mathcal{H}) = \mathcal{F} \subset \mathcal{H}_0$. Since $\mathcal{H}$ is an arbitrary component of $\mathcal{G}$, we deduce that $\mathcal{G} \subset (f^{n-n_0-1})^{-1}(\mathcal{H}_0)$. This completes Step I.
Step II: Proof of identity (4.10).

In what follows T is a current in $\mathcal{C}_1^+(\mathbb{P}^k)$ of mass 1, and $d := \mathrm{d}(f)$. Moreover, we make the following convention $(f^m)^*[\mathcal{H}_0] := 0$ for all $m < 0$. Next, we apply the hypothesis of induction (i.e. identity (4.3)) for $n-1, \ldots, n-n_0$ repeatedly by taking into account the identity $\|(f^m)^*T\| = d^m$, $m = 1, \ldots, n_0$, (see (2.1)). Consequently, one gets

$$
\begin{aligned}
(f^{n-1})^*(f^*)T &= (f^{n-2})^*(f^*f^*)T - d(f^{n-n_0-2})^*[\mathcal{H}_0] \\
&= \cdots \\
&= (f^{n-n_0})^*(f^{n_0})^*T - d^{n_0-1}(f^{n-2n_0})^*[\mathcal{H}_0] - \cdots - d(f^{n-n_0-2})^*[\mathcal{H}_0] \\
&= (f^{n-n_0-1})^*f^*(f^{n_0})^*T - d^{n_0}(f^{n-2n_0+1})^*[\mathcal{H}_0] - \cdots - d(f^{n-n_0-2})^*[\mathcal{H}_0] \\
&= (f^{n-n_0-1})^*(f^{n_0+1})^*T + (f^{n-n_0-1})^*[\mathcal{H}_0] \\
&\quad - d^{n_0}(f^{n-2n_0+1})^*[\mathcal{H}_0] - \cdots - d(f^{n-n_0-2})^*[\mathcal{H}_0].
\end{aligned}
\tag{4.11}
$$

On the one hand, applying Lemma 4.1 to the right-hand side of (4.11), we deduce that

$$\nu\Big((f^{n-1})^*(f^*)T, z\Big) = \nu\Big((f^{n-n_0-1})^*[\mathcal{H}_0], z\Big)$$

for any current $T := [l(z) = 0]$, where $\{l(z) = 0\}$ is a generic complex hyperplane in $\mathbb{P}^k$, and for a generic point z in any irreducible component of $(f^{n-n_0-1})^{-1}(\mathcal{H}_0)$. On the other hand, under the same condition,

$$\nu\Big((f^n)^*T, z\Big) = 0.$$

We combine the latter two equalities with (4.9) and taking into account the result of Step I. Consequently, (4.10) follows. This completes Step II. The proof of the theorem is thereby finished. □

Corollary 4.2. *Let f be a QAS meromorphic self-map of $\mathbb{P}^k$. Then its first dynamical degree $\lambda_1(f)$ is an algebraic integer.*

Proof. If f is AS, then the corollary is trivial since $\lambda_1(f) = \mathrm{d}(f)$. Suppose now that f is non AS. Then in virtue of identities (4.1)–(4.2), we have that

$$\mathrm{d}(f^n) = \begin{cases} \mathrm{d}(f)^n, & n = 0, \ldots, n_0, \\ \mathrm{d}(f) \cdot \mathrm{d}(f^{n-1}) - \deg(H_0) \cdot \mathrm{d}(f^{n-n_0-1}), & n > n_0. \end{cases}$$

Consequently, the conclusion of the corollary follows. □

Applications. We apply the main theorem to the examples given in Section 3.

First consider Example 3.5. Put $H(z, w, t) := t$ and let $F : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be the lifting of f given by the right-hand side of (3.1). In the light of the Main Theorem, a sequence of liftings F_n of f^n may be defined as follows

$$F_0 := \mathrm{Id},\ F_1 := F, \qquad F_n := \frac{F \circ F_{n-1}}{H(F_{n-2})}, \qquad n \geq 2.$$

As a consequence, one obtains the equation $\mathrm{d}(f^n) - 2\,\mathrm{d}(f^{n-1}) + \mathrm{d}(f^{n-2}) = 0$. Therefore, a straightforward computation shows that $\mathrm{d}(f^n) = n + 1$ and $\lambda_1(f) = 1$.

Next consider Example 3.7. Put $H(z,w,t) := (z+w+t)^2(z^3+w^3+t^3)$ and let $F : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be the lifting of f given by the right-hand side of (3.2). Using the Main Theorem, a sequence of liftings F_n of f^n may be defined as follows

$$F_0 := \mathrm{Id},\ F_1 := F, \qquad F_n := \frac{F \circ F_{n-1}}{H(F_{n-2})}, \qquad n \geq 2.$$

As a consequence, one obtains the equation $\mathrm{d}(f^n) - 7\,\mathrm{d}(f^{n-1}) + 5\,\mathrm{d}(f^{n-2}) = 0$. Therefore, a straightforward computation shows that

$$\mathrm{d}(f^n) = \frac{1}{\sqrt{29}} \cdot \Big(\frac{7+\sqrt{29}}{2}\Big)^{n+1} - \frac{1}{\sqrt{29}} \cdot \Big(\frac{7-\sqrt{29}}{2}\Big)^{n+1},$$

and $\lambda_1(f) = \frac{7+\sqrt{29}}{2}$.

Concluding remarks. One may widen the class of QAS self-maps by weakening considerably the conditions in Definition 3.4. Of course the recurrent law would be then more complicating. One might also seek to

- for any given $k, d \geq 2$, find many families of QAS (but non AS) self-maps of $\mathbb{P}^k$ with the algebraic degree d;
- generalize the Main Theorem to meromorphic self-maps in compact Kähler manifolds;
- construct an *appropriate* Green current for every QAS self-map f with $\lambda_1(f) > 1$.

We hope to come back these issues in a future work.

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