Irreducible Modules of Quantized Enveloping Algebras at Roots of 1

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Let **A** be an associative algebra over a field. An interesting problem is to understand the structure of irreducible modules of **A** (of finite dimensions). More or less, this is equivalent to understand the structure of maximal left ideals of **A** (of finite codimensions). For the later, it would be helpful if we know the generators of the maximal left ideals.

In Lie theory, there are some infinite dimensional algebras associated to a semisimple Lie algebra $\mathfrak{g}$ over $\mathbf{C}$. We shall be only concerned with the following four of them.

(i). The universal enveloping algebra $\mathfrak{U}$ of $\mathfrak{g}$.

(ii). The hyperalgebra $\mathfrak{U}_{\mathfrak{k}} := \mathfrak{U}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathfrak{k}$, where $\mathfrak{U}_{\mathbf{Z}}$ is the Konstant $\mathbf{Z}$-form of $\mathfrak{U}$ and $\mathfrak{k}$ is an algebraic closed field of prime characteristic.

(iii). The quantized enveloping algebra U (over $\mathbf{Q}(v)$, v is an indeterminate) of $\mathfrak{g}$.

(iv). The quantized hyperalgebra $U_\xi := U_{\mathbf{Q}[v,v^{-1}]} \otimes_{\mathbf{Q}[v,v^{-1}]} \mathbf{Q}(\xi)$, where $\xi \in \mathbf{C}^*$ and $U_{\mathbf{Q}[v,v^{-1}]}$ is a $\mathbf{Q}[v, v^{-1}]$-form of U [L1, section 4.1, p.243], $\mathbf{Q}(\xi)$ is regarded as a $\mathbf{Q}[v, v^{-1}]$-algebra through the $\mathbf{Q}$-algebra homomorphism $\mathbf{Q}[v, v^{-1}] \to \mathbf{Q}(\xi),\ v \to \xi$.

We are mainly interested in finite dimensional irreducibles modules of these algebras, or equivalently, in maximal left ideals of the algebras of finite codimensions. The generators of maximal left ideals of $\mathfrak{U}$ of finite codimensions are known more than forty years ago [HC, Lemma 15, p.42]. Thanks to the works [L1, Theorem 4.12, p.247] and [APW, Corollary 7.7, p.40], a similar result holds for maximal left ideals of U and of U_ξ of finite codimensions provided that ξ is not a root of 1 or $\xi^2 = 1$. We will review these results in section 1.2.

The purpose of the paper is to find the counterparts of the above results for the hyperalgebra $\mathfrak{U}_{\mathfrak{k}}$ and for the quantized hyperalgebra U_ξ when ξ is a root of 1 of order ≥ 3. The main results might lead a way to compute the characters of finite dimensional irreducible modules of $\mathfrak{U}_{\mathfrak{k}}$ and of U_ξ.

The basic idea is simple. When ξ is a root of 1 of order ≥ 3, the algebra U_ξ has a Frobenius kernel $\mathbf{u}_\xi$ [L4, Theorem 8.3, p.107]. The Frobenius kernel $\mathbf{u}_\xi$ is a symmetric $\mathbf{Q}(\xi)$-algebra [X, Theorem 3.5] of finite dimension. Moreover, the algebra $\mathbf{u}_\xi$ has a triangular decomposition $\mathbf{u}_\xi = \mathbf{u}_\xi^- \mathbf{u}_\xi^0 \mathbf{u}_\xi^+$. Each Verma module of $\mathbf{u}_\xi$ has a unique irreducible submodule, and each irreducible $\mathbf{u}_\xi$-module L is an irreducible submodule of certain Verma module Z of $\mathbf{u}_\xi$. As a $\mathbf{u}_\xi^-$-module, Z is isomorphic to $\mathbf{u}_\xi^-$. Therefore there exists an element x in $\mathbf{u}_\xi^-$ such that L is isomorphic to $\mathbf{u}_\xi^- x$ as $\mathbf{u}_\xi^-$-module. It turns out that the element x is a monomial of the generators of U_ξ^- (the negative part of U_ξ). So the generators of the maximal left idea of $\mathbf{u}_\xi$ corresponding to L can be described explicitly (Theorem 5.3). But L is a restriction to $\mathbf{u}_\xi$ of certain irreducible U_ξ-module [L2, Prop. 7.1 (c), p.70]. Using tensor product theorem [L2, Therorem 7.4, p.73], we can give the generators of maximal left ideals of U_ξ of finite codimensions (Theorem 5.4). The same idea is valid to the hyperalgebra $\mathfrak{U}_\mathfrak{k}$.

The paper is organized as follows. In section 1 we recall some basic definitions and review some results in [APW, HC, L1-L4]. In section 2 we consider the Frobenius kernel $\mathbf{u}_\xi$. In section 3 we consider the category of finite dimensional U_ξ-modules of type $\mathbf{1}$. In section 4 we prove that certain monomials in U_ξ^- are actually in $\mathbf{u}_\xi$. In technique, this is the hardest part of the paper. In section 5 we give the main theorems of the paper. In section 6 we consider the hyperalgebra $\mathfrak{U}_\mathfrak{k}$. In section 7 we give some questions.

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1. Introduction

1.1. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{C}$ with rank n and let (a_{ij}) be the Cartan matrix associated to $\mathfrak{g}$. We can find integers d_i in $\{1,2,3\}$ such that (d_ia_{ij}) is a symmetric matrix. Assume the sum of all d_i is as small as possible.

Let U be the quantized enveloping algebra of $\mathfrak{g}$ over $\mathbf{Q}(v)$ with parameter v (v an indeterminate). By definition, U is an associative $\mathbf{Q}(v)$-algebra and has generators $E_i,\ F_i,\ K_i, K_i^{-1},\ i=1,2,...,n$ which satisfy certain relations (see for exemple, [L4, 1.1, p.90]). The algebra U is in fact a Hopf algebra, the coproduct Δ , antipode S, counit ϵ are defined as follows:

$$\Delta(E_i)=E_i\otimes 1+K_i\otimes E_i,\quad \Delta(F_i)=F_i\otimes K_i^{-1}+1\otimes F_i,\quad \Delta(K_i)=K_i\otimes K_i,$$

$$S(E_i)=-K_i^{-1}E_i,\quad S(F_i)=-F_iK_i,\quad S(K_i)=K_i^{-1},$$

$$\epsilon(E_i)=\epsilon(F_i)=0,\quad \epsilon(K_i)=1.$$

We need some notations to introduce quantized hyperalgerbas and for later uses. Given an integer a and positive integers b, d, set

$$[a]_d:=\frac{v^{ad}-v^{-ad}}{v^d-v^{-d}},\quad [b]_d^!:=\prod_{h=1}^{b}\frac{v^{hd}-v^{-hd}}{v^d-v^{-d}},\quad [0]_d^!:=1,\quad [-b]_d^!=(-1)^b[b]_d^!;$$

$$\begin{bmatrix} a \\ b \end{bmatrix}_d := \prod_{h=1}^{b} \frac{v^{(a-h+1)d} - v^{-(a-h+1)d}}{v^{hd} - v^{-hd}}, \quad \begin{bmatrix} a \\ 0 \end{bmatrix}_d := 1, \quad \begin{bmatrix} a \\ -b \end{bmatrix}_d := 0.$$

Note that $\begin{bmatrix} a \\ b \end{bmatrix}_d$ is in $\mathbf{Q}[v, v^{-1}]$, we shall denote $\begin{bmatrix} a \\ b \end{bmatrix}_{d,\xi}$ the evaluation of $\begin{bmatrix} a \\ b \end{bmatrix}_d$ at ξ for any ξ in $\mathbf{C}^* \cup \{v\}$. Of course, we have $\begin{bmatrix} a \\ b \end{bmatrix}_{d,v} = \begin{bmatrix} a \\ b \end{bmatrix}_d$.

The quantized hyperalgebra U_ξ ($\xi \in \mathbf{C}^*$) is defined as follows. Let $U_{\mathbf{Q}[v,v^{-1}]}$ be the $\mathbf{Q}[v, v^{-1}]$-subalgebra of U generated by all $E_i^{(a)} := E_i^a/[a]^!_{d_i}$, $F_i^{(a)} := F_i^a/[a]^!_{d_i}$, K_i, K_i^{-1}, $i = 1, 2, ..., n$, $a \geq 0$. Regard $\mathbf{Q}(\xi)$ as a $\mathbf{Q}[v, v^{-1}]$-algebra through the $\mathbf{Q}$-algebra homomorphism $\mathbf{Q}[v, v^{-1}] \to \mathbf{Q}(\xi)$, $v \to \xi$. Define $U_\xi := U_{\mathbf{Q}[v,v^{-1}]} \otimes_{\mathbf{Q}[v,v^{-1}]} \mathbf{Q}(\xi)$ and call U_ξ a quantized hyperalgebra (associated to (a_{ij}) with parameter ξ). For convenience, set $U_v := U$. The algebra U_ξ inherits a Hopf algebra structure from that of $U_{\mathbf{Q}[v,v^{-1}]}$, denote again by Δ the coproduct, by S the antipode and by ϵ the counit. The tensor product of two U_ξ-modules then has a natural U_ξ-module structure by means of the coproduct, and the antipode can be used to define the dual module of a U_ξ-module.

For an integer c and a positive integer a we set

$$\begin{bmatrix} K_i, c \\ a \end{bmatrix} := \prod_{h=1}^{a} \frac{K_i v^{(c-h+1)d_i} - K_i^{-1} v^{-(a-h+1)d_i}}{v^{hd_i} - v^{-hd_i}} \quad \text{and} \quad \begin{bmatrix} K_i, c \\ 0 \end{bmatrix} := 1.$$

We have $\begin{bmatrix} K_i, c \\ a \end{bmatrix} \in U_{\mathbf{Q}[v,v^{-1}]}$ [L1, Lemma 4.4, p.244]. For simplicity, the images in U_ξ of $E_i^{(a)}$, $F_i^{(a)}$, K_i, K_i^{-1}, $\begin{bmatrix} K_i, c \\ a \end{bmatrix}$, etc. will be denoted by the same notations.

The algebra U_ξ has a triangular decomposition. Let U_ξ^+ (resp. U_ξ^-; U_ξ^0) be the subalgerba of U_ξ generated by all $E_i^{(a)}$ (resp. $F_i^{(a)}$; K_i, K_i^{-1}, $\begin{bmatrix} K_i, c \\ a \end{bmatrix}$, $c \in \mathbf{Z}$), $i = 1, 2, ..., n$, $a \geq 0$. The multiplication in U_ξ defines a $\mathbf{Q}(\xi)$-space isomorphism between $U_\xi^- \otimes U_\xi^0 \otimes U_\xi^+$ and U_ξ.

1.2. Given $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{Z}_+^n$, $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in \{\pm 1\}^n$, let $I_{\lambda,\sigma}^+$ be the left ideal of U_ξ generated by all $E_i^{(a)}$, $K_i - \sigma_i \xi^{\lambda_i d_i}$, $\begin{bmatrix} K_i, c \\ a \end{bmatrix} - \sigma_i^a \begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{d_i,\xi}$, $i = 1, 2, ..., n$, $a \geq 1$, $c \in \mathbf{Z}$, and let $I_{\lambda,\sigma}^-$ be the left idea of U_ξ generated by all $F_i^{(a_i)}$, $i = 1, 2, ..., n$, $a_i \geq \lambda_i + 1$. Then let $I_{\lambda,\sigma}$ be the left ideal of U_ξ generated by all elements in $I_{\lambda,\sigma}^+ \cup I_{\lambda,\sigma}^-$. Then

(i). The U_ξ-module $V_\xi(\lambda, \sigma) := U_\xi / I_{\lambda,\sigma}$ is of finite $\mathbf{Q}(\xi)$-dimension and has a unique irreducible quotient module, denote by $L_\xi(\lambda, \sigma)$. The $\mathbf{Q}(\xi)$-dimension of $V_\xi(\lambda, \sigma)$ is given by the Weyl's character formula. [L1, Theorem 4.12, p.247]. We shall denote $v_{\lambda,\sigma}$ the image in $V_\xi(\lambda,\sigma)$ of the neutral element $1 \in U_\xi$, and denote $\bar{v}_{\lambda,\sigma}$ the image in $L_\xi(\lambda,\sigma)$ of $v_{\lambda,\sigma}$.

(ii). The map $(\lambda,\sigma) \to L_\xi(\lambda,\sigma)$ defines a bijection between the set $\mathbf{Z}_+^n \times \{\pm 1\}^n$ and the set of isomorphism classes of irreducible U_ξ-modules of finite dimensions. [L1, Prop. 2.6 and Prop. 3.2, p.241] and [L2, Prop. 6.4, p.69].

(iii). One has

$$V_\xi(\lambda,\sigma) \simeq V_\xi(\lambda,\mathbf{1}) \otimes \mathbf{Q}(\xi)_\sigma, \quad L_\xi(\lambda,\sigma) \simeq L_\xi(\lambda,\mathbf{1}) \otimes \mathbf{Q}(\xi)_\sigma,$$

where $\mathbf{1} = (1,1,...,1) \in \{\pm 1\}^n$ and $\mathbf{Q}(\xi)_\sigma$ is the one $\mathbf{Q}(\xi)$-dimensional U_ξ-module on which all $E_i^{(a)}$, $F_i^{(a)}$, $i = 1,2,...,n$, $a \geq 1$, act by scalar zero and K_i, $\begin{bmatrix} K_i, c \\ a \end{bmatrix}$ act by scalar σ_i, $\sigma_i^a \begin{bmatrix} c \\ a \end{bmatrix}$ respectively, $i = 1,2,...,n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$. [APW, 1.6, p.6-7].

(iv). Provided that ξ is not a root of 1 or $\xi^2 = 1$, then $V_\xi(\lambda,\sigma)$ is irreducible, i.e. $V_\xi(\lambda,\sigma) \simeq L_\xi(\lambda,\sigma)$. And every finite dimensional U_ξ-module is completely reducible. [L4, 7.2, p.105-106; APW, Corollary 7.7, p.40].

Therefore, the theory of finite dimensional U_ξ-module is well understood when ξ is not a root of 1 or $\xi^2 = 1$. When ξ is a root of 1 of order ≥ 3 we donot know much about the irreducible module $L_\xi(\lambda,\sigma)$. In section 5 we shall describe the generators of the maximal left ideal $J_{\lambda,\sigma}$ of U_ξ corresponding to $L_\xi(\lambda,\sigma)$. To have a look what the generators are we introduce some monomials of $F_i^{(a)}$, $i = 1,2,...,n$, $a \geq 0$. These monomials play a central role in the paper.

1.3. Set $\alpha_i = (a_{1i}, a_{2i}, ..., a_{ni}) \in \mathbf{Z}^n$. For every $\mu = (\mu_1,\mu_2,...,\mu_n) \in \mathbf{Z}^n$, we also write $\langle \mu, \alpha_i^\vee \rangle$ for μ_i. Define $s_i : \mathbf{Z}^n \to \mathbf{Z}^n$ by $s_\mu = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i$. The reflections $s_1, s_2, ..., s_n$ generate the Weyl group W of the Cartan matrix (a_{ij}).

Let $\lambda = (\lambda_1,\lambda_2,...,\lambda_n) \in \mathbf{Z}_+^n$. Assume that $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression of an element of w in W. Set $\lambda_{\mathbf{i},1} = \lambda_{i_1}$, $\lambda_{\mathbf{i},2} = \langle s_{i_1}\lambda, \alpha_{i_2}^\vee \rangle$, ..., $\lambda_{\mathbf{i},k} = \langle s_{i_{k-1}} \cdots s_{i_1}\lambda, \alpha_{i_k}^\vee \rangle$, where $\mathbf{i} = (i_1, i_2, ..., i_k)$. Define

$$x_{\lambda,w,\mathbf{i}} := F_{i_1}^{(\lambda_{\mathbf{i},1})} F_{i_2}^{(\lambda_{\mathbf{i},2})} \cdots F_{i_k}^{(\lambda_{\mathbf{i},k})}, \quad \text{and} \quad x'_{\lambda,w^{-1},\mathbf{i}} := F_{i_k}^{(\lambda_{\mathbf{i},k})} F_{i_{k-1}}^{(\lambda_{\mathbf{i},k-1})} \cdots F_{i_1}^{(\lambda_{\mathbf{i},1})}.$$

Note that in the universal enveloping algebra $\mathfrak{U}$ of $\mathfrak{g}$ similar elements are defined by Verma [V, Theorem 4, p.162].

Lemma 1.4. The elements $x_{\lambda,w,\mathbf{i}}$ and $x'_{\lambda,w^{-1},\mathbf{i}}$ are independent of the choice of the reduced expression of w, only depend on λ and w. We shall denote them $x_{\lambda,w}$ and $x'_{\lambda,w^{-1}}$ respectively. When w is the longest element of W, we simply write x_λ and x'_λ for $x_{\lambda,w}$ and $x'_{\lambda,w}$ respectively

Proof: Use the quantum Verma identity [L5, Prop. 39.3.7, p.313].

1.5. From now on ξ will be a root of 1 with oder $l \geq 3$. Let l_i be the order of ξ^{2d_i} and set $\kappa := (l_1 - 1, l_2 - 1, ..., l_n - 1)$. We say an element $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{Z}_+^n$ is l-restricted if $\lambda_1 \leq l_1 - 1, ..., \lambda_n \leq l_n - 1$. For each $\mu = (\mu_1, \mu_2, ..., \mu_n) \in \mathbf{Z}^n$ we set $\mathbf{l}\mu := (l_1\mu_1, l_2\mu_2, ..., l_n\mu_n)$.

Let $\lambda, \mu \in \mathbf{Z}_+^n$, $\sigma \in \{\pm 1\}^n$ and assume that λ is l-restricted. Let $J_{\mathbf{l}\mu+\lambda,\sigma}$ be the left ideal of U_ξ generated by elements in $I_{\mathbf{l}\mu+\lambda,\sigma}$ and by elements F in U_ξ^- such that $Fx_{\kappa-\lambda} = 0$, one main result of paper says that $U_\xi / J_{\mathbf{l}\mu+\lambda,\sigma} \simeq L_\xi(\mathbf{l}\mu + \lambda, \sigma)$ (Theorem 5.4). One key step to reach the result is the assertion that $x_{\kappa-\lambda}$ is in the Frobenius kernel (Theorem 4.2).

1.6. Remark: Some results in [L1-L4] are stated and proved in a full generality in [L5]. The other results in [L1-L4] can be stated and proved in a full generality along the same way in [L1-L4]. Therefore the author feels free to quote the results in [L1-L4] in full generality forms.

2. Frobenius Kernel

2.1. Recall that ξ is a root of 1 with order $l \geq 3$ and l_i is the order of ξ^{2d_i}. Let R^+ be the set of positive roots of the root system $R := W\{\alpha_1, \alpha_2, ...\alpha_n\} \subset \mathbf{Z}^n$. Set $l_\alpha := l_i$ if $\alpha = w(\alpha_i)$ for some w in W. For each positive root α in R^+, let E_α, F_α be the root vectors defined in [L4, Theorem 6.6 (iii), p.104].

Let $U_{\xi,l}$ be the subalgebra of U_ξ generated by all $E_i^{(al_i)}, F_i^{(al_i)}, K_i, K_i^{-1}, \begin{bmatrix} K_i, c \\ al_i \end{bmatrix}$, $i = 1, 2, ..., n$, $a \geq 0$. The positive part $U_{\xi,l}^+$, the negative part $U_{\xi,l}^-$ and the zero part $U_{\xi,l}^0$ are defined in an obvious way. Let $\mathbf{u}_\xi$ be the subalgebra of U_ξ generated by all $E_\alpha, F_\alpha, K_i, K_i^{-1}$, $\alpha \in R_l^+ := \{\alpha \in R^+ \mid l_\alpha \geq 2\}$, $i = 1, 2, ..., n$. The algebra is called the Frobenius kernel of U_ξ. The Frobenius kernel $\mathbf{u}_\xi$ is a Hopf algebra and $\dim_{\mathbf{Q}(\xi)} = 2^n \prod_{i=1}^n l_i \prod_{\alpha \in R^+} l_\alpha^2$ [L4, 8.11, p.111, and Theorem 8.3, p.107]. We define the positive part $\mathbf{u}_\xi^+$, the negative part $\mathbf{u}_\xi^-$ and the zero part $\mathbf{u}_\xi^0$ in an obvious manner.

2.2. The following are some properties concerned with the algebras $U_{\xi,l}$ and $\mathbf{u}_\xi$, which are due to Lusztig.

(i). There exists a unique $\mathbf{Q}(\xi)$-algebra isomorphism $U_{\xi,l} \to U_1^* \otimes_{\mathbf{Q}} \mathbf{Q}(\xi)$ such that $E_i^{(al_i)} \to E_i^{(a)}$, $F_i^{(al_i)} \to F_i^{(a)}$, $K_i^{\pm} \to K_i^{\pm}$, $\begin{bmatrix} K_i, c \\ al_i \end{bmatrix} \to \begin{bmatrix} K_i, c \\ a \end{bmatrix}$, for $i = 1, 2, ..., n$, $a \geq 0$, where (assume that (a_{ij}) is indecomposable) $U_1^* = U_1$ when $l_1 = l_2 = ... = l_n$ and U_1^* is the quantized hyperalgebra associated to the transpose matrix of (a_{ij}) when $l_k \neq l_m$ for some k, m [L5, Theorem 35.1.7-Theorem 35.1.9, p.270; L4, Theorem 8.10, p.110].

(ii). Let $\{x_a\}$ be a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi^-$ and $\{y_b\}$ be a $\mathbf{Q}(\xi)$-basis of $U_{\xi,l}^-$, then $\{x_a y_b\}$ is a basis of U_ξ^-, so is $\{y_b x_a\}$ [L4, Lemma 8.8, p.109].

(iii). The elements $\prod_{\alpha\in R_l^+} F_\alpha^{(a_\alpha)} \prod_{i=1}^n K_i^{b_i} \prod_{\alpha\in R_l^+} E_\alpha^{(a'_\alpha)}$, $0 \le a_\alpha, a'_\alpha \le l_\alpha - 1$, $0 \le b_i \le 2l_i - 1$, form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi$; the elements $\prod_{\alpha\in R_l^+} F_\alpha^{(a_\alpha)}$, $0 \le a_\alpha \le l_\alpha - 1$, form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi^-$; the elements $\prod_{i=1}^n K_i^{b_i}$, $0 \le b_i \le 2l_i - 1$, form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi^0$; the elements $\prod_{\alpha\in R_l^+} E_\alpha^{(a'_\alpha)}$, $0 \le a'_\alpha \le l_\alpha - 1$, form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi^+$. [L4, Theorem 8.3, p.107].

(iv). Let $\lambda, \nu \in \mathbf{Z}_+^n$ and $\sigma \in \{\pm 1\}^n$. Assume that λ is l-restricted, then [L2, Theorem 7.4., p.73]

$$L_\xi(\mathbf{l}\mu + \lambda, \sigma) \simeq L_\xi(\mathbf{l}\mu, \sigma) \otimes L_\xi(\lambda, \mathbf{1}) \simeq L_\xi(\mathbf{l}\mu, \mathbf{1}) \otimes L_\xi(\lambda, \sigma).$$

(v). The restriction $L_\xi(\mathbf{l}\mu, \sigma)$ to $U_{\xi,l}$ is an irreducible $U_{\xi,l}$-module, and $F_\alpha L_\xi(\mathbf{l}\mu, \sigma) = 0$ for all $\alpha \in R_l^+$. Moreover, through the isomorphism $U_{\xi,l} \simeq U_1^* \otimes_{\mathbf{Q}} \mathbf{Q}(\xi)$, the restriction becomes an irreducible module of $U_1^* \otimes_{\mathbf{Q}} \mathbf{Q}(\xi)$ corresponding to (μ, σ). [L2, Prop. 7.5 (b), p.74].

(vi). As a $\mathbf{u}_\xi$-module, $L_\xi(\lambda, \sigma)$ is irreducible if λ is l-restricted. The map $(\lambda, \sigma) \to L_\xi(\lambda, \sigma)$ defines a bijections between the set $\mathbf{Z}_{+,\mathbf{l}}^n \times \{\pm 1\}^n$ and the set of isomorphism classes of irreducible $\mathbf{u}_\xi$-modules, where $\mathbf{Z}_{+,\mathbf{l}}^n$ is the set of all l-restricted elements in $\mathbf{Z}_+^n$ [L3, Prop. 5.11, p.291].

According to (i-vi), the algebra $\mathbf{u}_\xi$ is a key to understand U_ξ. To be convenience, we consider the subalgebra $\tilde{\mathbf{u}}_\xi$ of U_ξ generated by all elements in $\mathbf{u}_\xi \cup U_\xi^0$. One has $\tilde{\mathbf{u}}_\xi = \mathbf{u}_\xi^- U_\xi^0 \mathbf{u}_\xi^+$. By (vi) we see

(vii). Assume $(\lambda, \sigma) \in \mathbf{Z}_{+,\mathbf{l}}^n \times \{\pm 1\}^n$, then the restriction to $\tilde{\mathbf{u}}_\xi$ of the irreducible U_ξ-module $L_\xi(\lambda, \sigma)$ is an irreducible $\tilde{\mathbf{u}}_\xi$-module, denote by $\tilde{L}_\xi(\lambda, \sigma)$.

2.3. To go further we need some notions. Let $\gamma \in \mathbf{Z}R$. An element x in U_ξ is said to have degree γ if $K_i x K_i^{-1} = \xi^{-\langle\gamma, \alpha_i^\vee\rangle d_i} x$ and $\begin{bmatrix} K_i, c \\ a \end{bmatrix} x = x \begin{bmatrix} K_i, c - \langle\gamma, \alpha_i^\vee\rangle \\ a \end{bmatrix}$ for $i = 1, 2, ..., n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$.

Let U'_ξ be a subalgebra of U_ξ containing U_ξ^0 and let M be a U'_ξ-module. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{Z}^n, \sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in \{\pm 1\}^n$. An element $m \in M$ is called to have weight (λ, σ) if

$$K_i m = \sigma_i \xi^{\lambda_i d_i} m, \quad \begin{bmatrix} K_i, c \\ a \end{bmatrix} m = \sigma_i^a \begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{d_i, \xi} m$$

for $i = 1, 2, ..., n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$. Denote $M_{\lambda,\sigma}$ the set of all elements in M of weight (λ, σ). We call (λ, σ) a weight of M if $M_{\lambda,\sigma}$ is not zero. If an element x in U'_ξ has degree γ, then obviously $x M_{\lambda,\sigma} \subseteq M_{\lambda-\gamma,\sigma}$.

As usual, for $(\lambda, \sigma), (\mu, \tau) \in \mathbf{Z}^n \times \{\pm 1\}^n$, we write $(\lambda, \sigma) \le (\mu, \tau)$ if $\mu - \lambda \in \mathbf{N}R^+$ and $\sigma = \tau$. This defines a partial order in $\mathbf{Z}^n \times \{\pm 1\}^n$.

2.4. Now we return to the algebra $\tilde{\mathbf{u}}_\xi$. Assume $\mu = (\mu_1, \mu_2, ..., \mu_n) \in \mathbf{Z}^n$ and $\tau = (\tau_1, \tau_2, ..., \tau_n) \in \{\pm 1\}^n$. Let $\tilde{I}^+_{\mu,\tau}$ be the left ideal of $\tilde{\mathbf{u}}_\xi$ generated by all $E_\alpha, \alpha \in R_l^+$, $K_i - \tau_i \xi^{\mu_i d_i}$, $\begin{bmatrix} K_i, c \\ a \end{bmatrix} - \tau_i^a \begin{bmatrix} \mu_i + c \\ a \end{bmatrix}_{d_i,\xi}$, $i = 1, 2, ..., n$, $c \in \mathbf{Z}, a \in \mathbf{N}$. Denote $\tilde{Z}_\xi(\mu, \tau)$ the Verma module $\tilde{\mathbf{u}}_\xi / \tilde{I}^+_{\mu,\tau}$ of $\tilde{\mathbf{u}}_\xi$. We shall write $\tilde{1}_{\mu,\tau}$ the image in $\tilde{Z}_\xi(\mu,\tau)$ of the neutral element $1 \in \tilde{\mathbf{u}}_\xi$. By 2.2 (iii), $\tilde{Z}_\xi(\mu,\tau)$ has $\mathbf{Q}(\xi)$-dimension $\prod_{\alpha \in R^+} l_\alpha$. We denote by $\mathbf{Q}(\xi)_{1\mu,\tau}$ the one $\mathbf{Q}(\xi)$-dimensional $\tilde{\mathbf{u}}_\xi$-module on which all $E_\alpha, F_\alpha, \alpha \in R_l^+$ act by scalar zero and K_i, $\begin{bmatrix} K_i, c \\ a \end{bmatrix}$ act by scalars $\tau_i, \tau_i^a \begin{bmatrix} l_i\mu_i + c \\ a \end{bmatrix}_{d_i,\xi}$, respectively, where $i = 1, 2, ..., n$, $c \in \mathbf{Z}, a \in \mathbf{N}$.

Let $(\lambda, \sigma), (\mu, \tau) \in \mathbf{Z}^n \times \{\pm 1\}^n$. Then (cf. [X, Prop. 2.4, Prop. 2.9])

(i). The Verma module $\tilde{Z}_\xi(\lambda,\sigma)$ has a unique irreducible quotient module, denote by $\tilde{L}_\xi(\lambda, \sigma)$. Moreover $\tilde{L}_\xi(\mathbf{l}\mu, \tau) \simeq \mathbf{Q}(\xi)_{\mathbf{l}\mu,\tau}$.

(ii). We have

$$\tilde{Z}_\xi(\lambda + \mathbf{l}\mu, \sigma\tau) \simeq \tilde{Z}_\xi(\lambda,\sigma) \otimes \tilde{L}_\xi(\mathbf{l}\mu, \tau) \simeq \tilde{Z}_\xi(\lambda, \tau) \otimes \tilde{L}_\xi(\mathbf{l}\mu, \sigma),$$

where the meaning of $\sigma\tau \in \{\pm 1\}^n$ is obvious.

(iii). We have

$$\tilde{L}_\xi(\lambda + \mathbf{l}\mu, \sigma\tau) \simeq \tilde{L}_\xi(\lambda,\sigma) \otimes \tilde{L}_\xi(\mathbf{l}\mu, \tau) \simeq \tilde{L}_\xi(\lambda, \tau) \otimes \tilde{L}_\xi(\mathbf{l}\mu, \sigma),$$

(iv). Let L be an irreducible $\tilde{\mathbf{u}}_\xi$-module such that L is the direct sum of its weight spaces, then L is isomorphic to certain $\tilde{L}_\xi(\lambda,\sigma)$. Two irreducible $\tilde{\mathbf{u}}_\xi$-modules $\tilde{L}_\xi(\lambda,\sigma)$ and $\tilde{L}_\xi(\mu,\tau)$ are isomorphic if and only if $(\lambda,\sigma) = (\mu,\tau)$.

Remark: It is easy to see there is a natural bijection between the set of isomorphism classes of irreducible $\tilde{\mathbf{u}}_\xi$-modules and the set of maximal ideals of U_ξ^0. Note that the subalgebra ${U'}^0_{\xi,l}$ of U_ξ generated by $\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix}$, $i = 1, 2, ..., n$, is isomorphic to a polynomial ring over $\mathbf{Q}(\xi)$ in n variables. And U_ξ^0 is generated by ${U'}^0_{\xi,l}$ and $\mathbf{u}_\xi^0$.

We need the following result to see that $\tilde{Z}_\xi(\lambda,\sigma)$ has a unique irreducible submodule.

Lemma 2.5. Given a nonzero element y in $\mathbf{u}_\xi^-$ we can find an element x in $\mathbf{u}_\xi^-$ such that $xy = F_\kappa$, where $F_\kappa = \prod_{\alpha \in R_l^+} F_\alpha^{(l_\alpha - 1)}$, the product takes the order opposite that in [L4, 4.3, p.93-94].

Proof: Set $r := |R^+|$. Let β_{r-q+1} be the q-th root in the total order on R^+ arranged in [L4, 4.3, p.93-94]. Then $\beta_1, \beta_2, ..., \beta_r$ give rise to a total order on R^+ opposite to that in [L4, 4.3, p.93-94]. By 2.2 (iii),

$$y = \sum_{\substack{0 \le a_m \le l_{\beta_m} - 1 \\ 1 \le m \le r}} A(a_1, a_2, ..., a_r) F_{\beta_1}^{(a_1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_r}^{(a_r)}, \quad A(a_1, a_2, ..., a_r) \in \mathbf{Q}(\xi).$$

Let $(b_1, b_2, ..., b_r)$ be the minimal element in $\{(a_1, a_2, ..., a_r) \in \mathbf{Z}_+^r \mid A(a_1, a_2, ..., a_r) \neq 0\}$. (Here we use the lexicographical order in $\mathbf{Z}_+^r$ such that $(0, 0, ..., 0, 1) < (0, 0, ..., 1, 0) < \cdots < (0, 1, ..., 0, 0) < (1, 0, ..., 0, 0)$.) Set $c_1 = l_{\beta_1} - 1 - b_1, ..., c_r = l_{\beta_r} - 1 - b_r$ and let $x' = F_{\beta_r}^{(c_r)} \cdots F_{\beta_2}^{(c_2)} F_{\beta_1}^{(c_1)}$. Using commutation relations in [L4, 5.3-4, p.95-97] and [L4, Theorem 6.6 (iii), p.104], we see $x'y = A(b_1, b_2, ..., b_r)x' F_{\beta_1}^{(b_1)} F_{\beta_2}^{(b_2)} \cdots F_{\beta_r}^{(b_r)} = \theta F_\kappa$ for some nonzero number θ in $\mathbf{Q}(\xi)$. Then the element $x := \theta^{-1} x'$ satisfys our requirements.

Proposition 2.6. let $(\lambda, \sigma) \in \mathbf{Z}^n \times \{\pm 1\}^n$, then

(i). The Verma module $\tilde{Z}_\xi(\lambda, \sigma)$ has a unique irreducible submodule.

(ii). Assume that λ is l-restricted, then the unique irreducible submodule of $\tilde{Z}_\xi(2\kappa + w_0\lambda)$ is isomorphic to $\tilde{L}_\xi(\lambda, \sigma)$, where w_0 is the longest element of W.

Proof: (i). By Lemma 2.5, each submodule of $\tilde{Z}_\xi(\lambda, \sigma)$ contains the element $F_\kappa \tilde{1}_{\lambda,\sigma}$. So $\tilde{Z}_\xi(\lambda, \sigma)$ has a unique irreducible submodule which is generated by $F_\kappa \tilde{1}_{\lambda,\sigma}$.

(ii). Since F_κ has degree 2κ, so $F_\kappa \tilde{1}_{2\kappa + w_0\lambda, \sigma}$ has weight $(w_0\lambda, \sigma)$. According to the symmetries [L5, Prop. 5.2.7, p. 45], the lowest weight of $L_\xi(\lambda, \sigma)$ is $(w_0\lambda, \sigma)$. According to 2.2 (vii), 2.4 (iii-iv) and the proof of (i) we see that the unique irreducible submodule of $\tilde{Z}_\xi(2\kappa + w_0\lambda)$ is isomorphic to $\tilde{L}_\xi(\lambda, \sigma)$.

Corollary 2.7. Assume that λ is l-restricted, then

(i). There exists a nonzero element y_λ (unique up to a scalar) in $\mathbf{u}_\xi^-$ such that $y_\lambda \tilde{1}_{2\kappa + w_0\lambda, \sigma}$ has weight (λ, σ) and $E_\alpha y_\lambda \tilde{1}_{2\kappa + w_0\lambda, \sigma} = 0$ for all $\alpha \in R_l^+$. Necessarily $y_\lambda \tilde{1}_{2\kappa + w_0\lambda, \sigma}$ generates the unique irreducible submodule of $\tilde{Z}_\xi(2\kappa + w_0\lambda)$.

(ii). There exists a nonzero element y'_λ (unique up to a scalar) in $\mathbf{u}_\xi^-$ such that $y'_\lambda \tilde{1}_{\kappa+\lambda,\sigma}$ has weight $(\kappa + w_0\lambda, \sigma)$ and $E_\alpha y'_\lambda \tilde{1}_{\kappa+\lambda,\sigma} = 0$ for all $\alpha \in R_l^+$. Necessarily $y'_\lambda \tilde{1}_{\kappa+\lambda,\sigma}$ generates the unique irreducible submodule of $\tilde{Z}_\xi(\kappa + \lambda)$.

We shall see that $y_\lambda = x_{\kappa-\lambda}$ and $y'_\lambda = x'_\lambda$ (see 1.5 for definitions of x_λ, x'_λ).

Proposition 2.8. Let $\sigma \in \{\pm 1\}^n$. Then

(i). The Verma module $\tilde{Z}_\xi(\kappa, \sigma)$ is an irreducible $\tilde{\mathbf{u}}_\xi$-module, i.e. $\tilde{Z}_\xi(\kappa, \sigma) \simeq \tilde{L}_\xi(\kappa, \sigma)$.

(ii). As a $\tilde{\mathbf{u}}_\xi$-module, $V_\xi(\kappa, \sigma)$ is isomprphic to $\tilde{Z}_\xi(\kappa, \sigma)$. In particular, $V_\xi(\kappa, \sigma)$ is an irreducible U_ξ-module.

(iii). For every $\mu \in \mathbf{Z}_+^n$, the module $V_\xi(\mathbf{l}\mu + \kappa, \sigma)$ is an irreducible U_ξ-module.

Proof: (i). Note that $w_0\kappa = -\kappa$. By Proposition 2.6 (ii), the unique irreducible submodule of $\tilde{Z}_\xi(\kappa, \sigma)$ is isomorphic to $\tilde{L}_\xi(\kappa, \sigma)$. But $\tilde{Z}_\xi(\kappa, \sigma)_{\kappa,\sigma}$ is one-dimensional $\mathbf{Q}(\xi)$-space, so the irreducible submodule of $\tilde{Z}_\xi(\kappa, \sigma)$ is generated by $\tilde{1}_{\kappa,\sigma}$. Hence $\tilde{Z}_\xi(\kappa, \sigma)$ is irreducible and isomorphic to $\tilde{L}_\xi(\kappa, \sigma)$.

(ii). By the definitions of $\tilde{Z}_\xi(\kappa, \sigma)$ and of $V_\xi(\kappa, \sigma)$, we have a natural $\tilde{\mathbf{u}}_\xi$-module homomorphism $\tilde{Z}_\xi(\kappa, \sigma) \to V_\xi(\kappa, \sigma)$. The homomorphism is surjective according to 2.2 (i) and the definition of $V_\xi(\kappa, \sigma)$. Weyl's character formula tells us that the $\mathbf{Q}(\xi)$-dimension of $V_\xi(\kappa, \sigma)$ is $\prod_{\alpha\in R^+} l_\alpha$. So the homomorphism is a $\tilde{\mathbf{u}}_\xi$-module isomorphism. This proves (ii).

(iii). By 1.2 (i), $L_\xi(\mathbf{l}\mu + \kappa, \sigma)$ is the unique irreducible quotient module of $V_\xi(\mathbf{l}\mu + \kappa, \sigma)$. Using (ii) and 2.2 (iv) we see $L_\xi(\mathbf{l}\mu + \kappa, \sigma)$ is isomorphic to $L_\xi(\mathbf{l}\mu, \sigma) \otimes V_\xi(\kappa, \mathbf{1})$. Combining 2.2 (v), 1.2 (i) and 1.2 (iv), we know that the $\mathbf{Q}(\xi)$-dimensions of $V_\xi(\mathbf{l}\mu + \kappa, \sigma)$ and $L_\xi(\mathbf{l}\mu, \sigma) \otimes V_\xi(\kappa, \mathbf{1})$ can be calculated by means of Weyl's character formula, they are equal. Hence $V_\xi(\mathbf{l}\mu + \kappa, \sigma)$ is an irreducible U_ξ-module.

The proposition is proved.

The following result will not be used in the sequel of the paper.

Theorem 2.9. (i). The algebra $\mathbf{u}_\xi$ is symmetric.

(ii). Let $\mathbf{k}$ be the two sided ideal of $\mathbf{u}_\xi$ generated by $K_1^{l_1} - 1, K_2^{l_2} - 1, ..., K_3^{l_n} - 1$. Then the algebra $\mathbf{u}'_\xi := \mathbf{u}_\xi/\mathbf{k}$ is symmetric.

Remark: The theorem was proved in [X, Theorem 3.5] with some restrictions on l. Since [X] is unpublished and Theorem 3.5 in [X] was quoted in some papers, it might be good to represent here a version without restrictions on l. The proof is the same as that in [X].

Proof: (i). We need to construct a bilinear form φ on $\mathbf{u}_\xi$ such that

(a). φ is associative, i.e. $\varphi(xy, z) = \varphi(x, yz)$ for any $x, y, z \in \mathbf{u}_\xi$;

(b). φ is non-degenerate, i.e. if $\varphi(x, x') = 0$ (resp. $\varphi(x', x) = 0$) for all $x' \in \mathbf{u}_\xi$, then $x = 0$;

(c). φ is symmetric, i.e. $\varphi(x, y) = \varphi(y, x)$.

Let $\beta_1, \beta_2, ..., \beta_r$ be as in the proof of Lemma 2.5. Set

$$\mathbf{Z}^r_{+,1} := \{(a_1, a_2, ..., a_r) \in \mathbf{Z}^r \mid 0 \le a_1 \le l_{\beta_1} - 1, ..., 0 \le a_r \le l_{\beta_r} - 1\},$$

$$\mathbf{Z}^n_{+,21} := \{(h_1, h_2, ..., h_n) \in \mathbf{Z}^n \mid 0 \le h_1 \le 2l_1 - 1, ..., 0 \le h_n \le 2l_n - 1\},$$

For $A = (a_1, a_2, ..., a_r) \in \mathbf{Z}^r_{+,1}$, $H = (h_1, h_2, ..., h_r) \in \mathbf{Z}^n_{+,21}$, we shall write F_A, E_A, K_H for $F_{\beta_1}^{(a_1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_r}^{(a_r)}$, $E_{\beta_r}^{(a_r)} \cdots E_{\beta_2}^{(a_2)} E_{\beta_1}^{(a_1)}$, $K_1^{h_1} K_2^{h_2} \cdots K_n^{h_n}$, respectively. Let φ_0 be the $\mathbf{Q}(\xi)$-linear function of $\mathbf{u}_\xi$ defined by

$$\varphi_0(F_A K_H E_{A'}) = \begin{cases} 1, & \text{if } F_A K_H E_{A'} = F_\iota E_\iota, \\ 0, & \text{otherwise,} \end{cases}$$

where $\iota = (l_{\beta_1} - 1, l_{\beta_2} - 1, ..., l_{\beta_r} - 1) \in \mathbf{Z}^r_{+,1}$. Set $\varphi(x, y) := \varphi_0(xy)$. Obviously φ defines an associative bilinear form on $\mathbf{u}_\xi$. We now show that φ is non-degenerate on $\mathbf{u}_\xi$.

Let

$$x = \sum_{\substack{A, A' \in \mathbf{Z}^r_{+,1} \\ H \in \mathbf{Z}^n_{+,21}}} \theta(A, H, A') F_A K_H E_{A'} \ne 0, \quad \theta(A, H, A') \in \mathbf{Q}(\xi).$$

Let $B=(b_1,b_2,...,b_r)$ be the minimal element in $\{A\in \mathbf{Z}^r_{+,1} \mid \theta(A,H,A')\neq 0 \text{ for some } H,A'\}$ and let $B'=(b_1,b_2,...,b_r)$ be the minimal element in $\{A'\in \mathbf{Z}^r_{+,1} \mid \theta(B,H,A')\neq 0 \text{ for some } H\}$. Here we use the lexicographical order on $\mathbf{Z}^r_+$ defined in the proof of Lemma 2.5. Set

$$y_1=F_{\beta_r}^{(c_r)}\cdots F_{\beta_2}^{(c_2)}F_{\beta_1}^{(c_1)},\quad y_2=E_{\beta_1}^{(c'_1)}E_{\beta_2}^{(c'_2)}\cdots E_{\beta_r}^{(c'_r)},$$

where $c_1=l_{\beta_1}-1-b_1,...,c_r=l_{\beta_r}-1-b_r$, and $c'_1=l_{\beta_1}-1-b'_1,...,c'_r=l_{\beta_r}-1-b'_r$. By the proof of Lemma 2.5 we have

$$y_2y_1x=\theta_1y_2\sum_{\substack{A'\in\mathbf{Z}^r_{+,1}\\ H\in\mathbf{Z}^n_{+,21}}}\theta(B,H,A')F_\iota K_HE_{A'}$$

for some nonzero number $\theta_1\in\mathbf{Q}(\xi)$.

By the commutation relations in [X, 3.3] we see that

$$y_2F_\iota=F_\iota y_2+\sum_{\substack{A,A'\in\mathbf{Z}^r_{+,1}\\ A\neq\iota\\ H\in\mathbf{Z}^n_{+,21}}}\eta(A,H,A')F_AK_HE_{A'},\quad \eta(A,H,A')\in\mathbf{Q}(\xi).$$

As in the proof of Lemma 2.5 we see $\theta(B,H,A')y_2E_{A'}\neq 0$ implies that $A'=B'$ and $y_2E_{B'}=\theta_2E_\iota$ for some nonzero number $\theta_2\in\mathbf{Q}(\xi)$. Thus

$$y_2y_1x=\theta_1\theta_2\sum_{H\in\mathbf{Z}^n_{+,21}}\theta(B,H,B')F_\iota K_HE_\iota+\sum_{\substack{A,A'\in\mathbf{Z}^r_{+,1}\\ A\neq\iota\\ H\in\mathbf{Z}^n_{+,21}}}\eta'(A,H,A')F_AK_HE_{A'},$$

where $\eta'(A,H,A')\in\mathbf{Q}(\xi)$. Let $I\in\mathbf{Z}^n_{+,21}$ be such that $\theta(B,I,B')\neq 0$, by the definition of φ we see $\varphi(K_I^{-1}y_2y_1,x)\neq 0$. We also have $\varphi(x,K_I^{-1}y_2y_1)\neq 0$ since φ is symmetric by the following argument.

Note that the elements $E_AK_HF_{A'}$, $A,A'\in\mathbf{Z}^r_{+,1}, H\in\mathbf{Z}^n_{+,21}$, also form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}_\xi$. Let $A=(a_k),B=(b_k),P=(p_k),Q=(q_k)$ be elements in $\mathbf{Z}^r_{+,1}$ and let $H=(h_i),H'=(h'_i)$ be elements in $\mathbf{Z}^n_{+,21}$. Using commutation relations in [L4, 5.3-5.4, p.95-97; X, 3.3] and [L4, Theorem 6.6, p.103-104] we see that $\varphi(F_AK_HE_P,E_QK_{H'}F_B)=\varphi(E_QK_{H'}F_B,F_AK_HE_P)=0$ if one of the following three cases happens: (a). $K_HK_{H'}\neq 1$, (b). $\sum_{k=1}^r(a_k+b_k)\beta_k\neq 2\kappa$, (c). $\sum_{k=1}^r(p_k+q_k)\beta_k\neq 2\kappa$. Using [L4, Theorem 6.6, p.103-104] and commutation relations in [L4, 5.3-5.4, p.95-97] and induction on P (resp. B) we know that (d). $E_PE_Q=E_QE_P$ (resp. $F_AF_B=F_BF_A$) if $\sum_{k=1}^r(a_k+b_k)\beta_k=2\kappa$ (resp. $\sum_{k=1}^r(p_k+q_k)\beta_k=2\kappa$). By this and the commutation relations in [X, 3.3], and noting that the coefficients of E_PE_Q,F_BF_A in $K_HE_PE_QK_H^{-1}$, $K_H^{-1}F_BF_AK_H$ are the same when $\sum_{k=1}^r(a_k+b_k)\beta_k=\sum_{k=1}^r(p_k+q_k)\beta_k=2\kappa$, we see that $\varphi(F_AK_HE_P,E_QK_{H'}F_B)=\varphi(E_QK_{H'}F_B,F_AK_HE_P)$ if $\sum_{k=1}^r(a_k+b_k)\beta_k=\sum_{k=1}^r(p_k+q_k)\beta_k=2\kappa$ and $K_HK_{H'}=1$. Therefore φ is symmetric. Part (i) is proved.

(ii). Since the images in $\mathbf{u}'_\xi$ of the elements $F_A K_H E_{A'}$, $A, A' \in \mathbf{Z}^r_{+,l}, H \in \mathbf{Z}^n_{+,l}$, form a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}'_\xi$, the proof of (i) is also valid to $\mathbf{u}'_\xi$.

The theorem is proved.

3. Category of finite dimensional U_ξ-module of type 1

3.1. Let M be a finite dimensional U_ξ-module. For each $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ in $\{\pm 1\}^n$, set $M_\sigma := \{m \in M \mid K_i^{l_i} m = \sigma_i m \text{ for } i = 1, 2, ..., n\}$. In U_ξ we have $K_i^{2l_i} = 1$ and $x K_i^{l_i} = K_i^{l_i} x$ for every $x \in U_\xi$ [L2, Lemma 4.4, p.64]. Therefore M_σ is a submodule of M and $M = \bigoplus_{\sigma \in \{\pm 1\}^n} M_\sigma$. We say that M has type σ if $M = M_\sigma$. All finite dimensional U_ξ-modules of type σ with usual U_ξ-module homomorphisms form a category of U_ξ-modules, denote by $\mathcal{C}_\sigma$. Clearly, the map $M \to M \otimes \mathbf{Q}(\xi)_\sigma$ gives rise to an isomorphism between the categories $\mathcal{C}_\mathbf{1}$ and $\mathcal{C}_\sigma$ [APW, 1.6, p.6-7]. What is more, the $\mathbf{Q}(\xi)$-algebra automorphism $U_\xi \to U_\xi$ defined by $E_i^{(a)} \to \sigma_i^a E_i^{(a)}$, $F_i^{(a)} \to F_i^{(a)}$, $K_i \to \sigma_i K_i$ $(i = 1, 2, ..., n,\ a \geq 0)$ interchanges the U_ξ-modules of type $\mathbf{1}$ to those of type σ [L2, 4.6, p.65].

Therefore, it suffices to work on the category $\mathcal{C}_\mathbf{1}$ of U_ξ-modules. Note that $V_\xi(\lambda, \mathbf{1})$, $L_\xi(\lambda, \mathbf{1}) \in \text{ob}\mathcal{C}_\mathbf{1}$ for each $\lambda \in \mathbf{Z}^n_+$. We shall drop the index $\mathbf{1}$ in all notations involved it. So $\mathcal{C}, V_\xi(\lambda), L_\xi(\lambda), v_\lambda$, etc. will stand for $\mathcal{C}_\mathbf{1}, V_\xi(\lambda, \mathbf{1}), L_\xi(\lambda, \mathbf{1}), v_{\lambda,\mathbf{1}}$, etc. respectively. One main result of the section is the following, which will be proved after establishing Lemma 3.4.

Theorem 3.2. Let $\mu \in \mathbf{Z}^n_+$.

(i). The module $V_\xi(\mathbf{l}\mu + \kappa)$ is injective as well as projective in the category $\mathcal{C}$.

(ii). The category $\mathcal{C}$ has enough injective objects and enough projective objects as well.

(iii). In $\mathcal{C}$ each injective object is also a projective object.

(iv). Every module M in ob$\mathcal{C}$ is integrable (i.e. $M = \bigoplus_{\lambda \in \mathbf{Z}^n} M_\lambda$ and $E_i^{(a)}, F_i^{(a)}$ are locally nilpotent on M for $i = 1, 2, ..., n$, $a \geq 1$).

(v). If M is a finite dimensional U_ξ-module, then $M = \bigoplus_{(\lambda,\sigma) \in \mathbf{Z}^n \times \{\pm 1\}^n} M_{\lambda,\sigma}$, i.e. M is integrable.

(vi). Let E be an injective object in $\mathcal{C}$, then E has a submodules filtration $0 = E_k \subset E_{k-1} \subset \cdots \subset E_2 \subset E_1 = E$ such that $E_a / E_{a+1} \simeq V_\xi(\nu_a)$ for some $\nu_a \in \mathbf{Z}^n_+$, $a = 1, ..., k-1$.

Remark: When l is a power of a prime number, the theorem is proved in [APW, 9.8, p.44; 9.12, p.45].

3.3. Let M be a U_ξ-module of type $\mathbf{1}$. An nonzero element m in M is called primitive if $m \in M_\lambda$ for some $\lambda \in \mathbf{Z}^n$ and $E_i^{(a)} m = 0$ for $i = 1, 2, ..., n$, $a \geq 1$. We have

(i). Let M be an integrable or finite dimensional U_ξ-module of type $\mathbf{1}$. Assume that m is a primitive element of weight λ, then $\lambda \in \mathbf{Z}^n_+$ and there is a unique U_ξ-module homomorphism $V_\xi(\lambda) \to M$ which carries v_λ to m. [L5, Prop. 3.5.8, p.33].

Given a finite dimensional U_ξ-module E of type $\mathbf{1}$, we define the dual modules E^*, $E^\star$ as in [APW, 1.18, p.9] by means of the antipode S of U_ξ and its inverse S^{-1} respectively. Then [APW, 1.18, p.9-10]

(ii). We have $(E^*)^\star \simeq E \simeq (E^\star)^*$.

(iii). For any U_ξ-modules M, N, one has

$$\mathrm{Hom}_{U_\xi}(M, N \otimes E) \simeq \mathrm{Hom}_{U_\xi}(M \otimes E^\star, N),$$

$$\mathrm{Hom}_{U_\xi}(E^* \otimes M, N) \simeq \mathrm{Hom}_{U_\xi}(M, E \otimes N).$$

Lemma 3.4. Let M be a finite dimensional U_ξ-module of type $\mathbf{1}$ and let $\mu \in \mathbf{Z}^n_+$.

(i). Assume that $V_\xi(\mathbf{l}\mu + \kappa)$ is a submodule of M, then $V_\xi(\mathbf{l}\mu + \kappa)$ is a direct summand of M, i.e. there exists a submodule M' of M such that M is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa) \oplus M'$.

(ii). Assume that $V_\xi(\mathbf{l}\mu+\kappa)$ is a quotient module of M, then $V_\xi(\mathbf{l}\mu+\kappa)$ is a direct summand of M.

Proof: The modules $V_\xi(\mathbf{l}\mu + \kappa)^*, V_\xi(\mathbf{l}\mu + \kappa), V_\xi(\mathbf{l}\mu + \kappa)^\star$ are isomorphic since $V_\xi(\mathbf{l}\mu + \kappa)$ is irreducible (Proposition 2.8 (iii)) and $w_0(\mathbf{l}\mu + \kappa) = -\mathbf{l}\mu - \kappa$. Now part (i) and part (ii) are equivalent by 3.3 (ii). We give a proof of part (i).

For $\nu = (\nu_1, \nu_2, ..., \nu_n) \in \mathbf{Z}^n, \lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{Z}^n_{+,\mathbf{l}}$, let $M_{(\mathbf{l}\nu+\lambda)}$ be the set of all elements m in M satisfying

$$K_i m = \xi^{d_i\lambda_i} m, \quad (\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix} - \nu_i)^a m = 0, \qquad \text{for } i = 1, 2, ..., n \text{ and some } a \in \mathbf{N}.$$

Then

(a). $M = \bigoplus_{\nu \in \mathbf{Z}^n} M_{(\nu)}$, and $E_i^{(a)} M_{(\nu)} \subseteq M_{(\nu+a\alpha_1)}$, $F_i^{(a)} M_{(\nu)} \subseteq M_{(\nu-a\alpha_1)}$. [L2, Prop. 5.1 and its proof, p.65-67].

(b). Obviously, $M_\nu \neq 0$ if and only if $M_{(\nu)} \neq 0$. So the set $P(M) := \{\nu \in \mathbf{Z}^n \mid M_{(\nu)} \neq 0\}$ is stable under the action of W [L5, Prop. 5.2.7, p. 45].

By induction on $\dim_{\mathbf{Q}(\xi)} M$ we may assume that $M/V_\xi(\mathbf{l}\mu + \kappa)$ is irreducible. One of the following three cases must happen.

(c). There is a maximal weight λ in $P(M)$ such that $\lambda \neq \mathbf{l}\mu + \kappa$.

(d). $\mathbf{l}\mu + \kappa$ is the unique maximal weight in $P(M)$ and $\dim_{\mathbf{Q}(\xi)} M_{(\mathbf{l}\mu+\kappa)} = 1$.

(e). $\mathbf{l}\mu + \kappa$ is the unique maximal weight in $P(M)$ and $\dim_{\mathbf{Q}(\xi)} M_{(\mathbf{l}\mu+\kappa)} = 2$.

Case (c). By (b), $M_\lambda \neq 0$. Choose an nonzero element m in M_λ, then m is a primitive element. Let M' be the submodule of M generated by m. We claim that $M' \cap V_\xi(\mathbf{l}\mu+\kappa) = 0$. Otherwise, $M' \cap V_\xi(\mathbf{l}\mu + \kappa) = V_\xi(\mathbf{l}\mu + \kappa)$. Then we can find an element y in U_ξ^- such that $v_{\mathbf{l}\mu+\kappa} = ym$. Note that $F_\kappa v_{\mathbf{l}\mu+\kappa} \neq 0$, using 2.2 (ii-iii) we see that $F_\kappa v_{\mathbf{l}\mu+\kappa} = F_\kappa y' m$

for some element y' in $U^-_{\xi,l}$. Therefore $\lambda = \mathbf{l}\nu + \mathbf{l}\mu + \kappa$ for certain nonzero element ν in $\mathbf{Z}^n$. By Proposition 2.8 (iii) and 3.2 (i), M' is irreducible. An contradiction to the assumption on $M' \cap V_\xi(\mathbf{l}\mu + \kappa) = V_\xi(\mathbf{l}\mu + \kappa)$. Hence $M' \cap V_\xi(\mathbf{l}\mu + \kappa) = 0$, in addition we have $M \simeq V_\xi(\mathbf{l}\mu + \kappa) \oplus M'$ and M' is irreducible.

Case (d). By (b), all the four spaces $M_{\mathbf{l}\mu+\kappa}, M_{(\mathbf{l}\mu+\kappa)}, M_{-\mathbf{l}\mu-\kappa}, M_{(-\mathbf{l}\mu-\kappa)}$ are of one $\mathbf{Q}(\xi)$-dimension. By (b) $\mathbf{l}\mu + \kappa$ is the unique maximal weight in $P(M^*)$. Let M_1 be the irreducible submodule of M^* such that M^*/M_1 is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa)^*$. By our assumptions on M we have $\mathbf{l}\mu + \kappa \notin P(M_1)$. Choose a nonzero element m in $M^*_{(\mathbf{l}\mu+\kappa)}$, then m is a primitve element and generates a submodule M_2 of M. By Proposition 2.8 (iii) and 3.2 (i), M_2 is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa)$. Hence M^* is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa) \oplus M_1$. Note that $V_\xi(\mathbf{l}\mu + \kappa)^\star \simeq V_\xi(\mathbf{l}\mu + \kappa)$, by 3.2 (ii) we see M is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa) \oplus M_1^\star$.

Case (e). Set $\mu_i := \langle \mu, \alpha_i^\vee \rangle$ for $i = 1, 2, ..., n$. By (b) we have $F_i^{(l_i\mu_i+l_i)} M_{(\mathbf{l}\mu+\kappa)} = 0$ for all i. Using (a) and our assumption on $\mathbf{l}\mu + \kappa$ we see

$$\begin{bmatrix} K_i, 0 \\ \mu_i l_i \end{bmatrix} M_{(\mathbf{l}\mu+\kappa)} = E_i^{(l_i\mu_i+l_i)} F_i^{(l_i\mu_i+l_i)} M_{(\mathbf{l}\mu+\kappa)} = 0.$$

But in U_ξ we have [L2, lemma 4.3, p.64]

$$\begin{bmatrix} K_i, 0 \\ \mu_i l_i \end{bmatrix} = \frac{1}{(\mu_i + 1)!} \prod_{j=0}^{\mu_i} (\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix} - j).$$

Now $\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix} - \mu_i$ is nilpotent on $M_{(\mathbf{l}\mu+\kappa)}$, so $(\begin{bmatrix} K_i, 0 \\ l_i \end{bmatrix} - \mu_i) M_{(\mathbf{l}\mu+\kappa)} = 0$. Therefore M is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa) \oplus V_\xi(\mathbf{l}\mu + \kappa)$.

The lemma is proved.

3.5. Now we prove Theorem 3.2. Part (i) is a trivial consequence of Lemma 3.4.

(ii). According to part (i) and 3.3 (iii), for any finite dimensional U_ξ-module M of type $\mathbf{1}$, the modules $V_\xi(\mathbf{l}\mu + \kappa) \otimes M$ and $M \otimes V_\xi(\mathbf{l}\mu + \kappa)$ are projevtive and injective as well in the category $\mathcal{C}$. For any λ in $\mathbf{Z}^n_+$, choose ν in $\mathbf{Z}^n_+$ such that $\mathbf{l}\nu + \kappa - \lambda \in \mathbf{Z}^n_+$. By 3.2 (i) we have a nonzero U_ξ-homomorphism $V_\xi(\mathbf{l}\nu + \kappa) \to L_\xi(\lambda) \otimes V_\xi(\mathbf{l}\nu + \kappa - \lambda)$. By 3.3 (iii), this gives rise to a nonzero U_ξ-homomorphism $V_\xi(\mathbf{l}\nu + \kappa) \otimes V_\xi(\mathbf{l}\nu + \kappa - \lambda)^\star \to L_\xi(\lambda)$, which is necessarily surjective. Further, this surjective gives rise to a nonzero U_ξ-homomorphism $L_\xi(-w_0\lambda) \simeq L_\xi(\lambda)^* \to (V_\xi(\mathbf{l}\nu + \kappa) \otimes V_\xi(\mathbf{l}\nu + \kappa - \lambda)^\star)^* \simeq V_\xi(\mathbf{l}\nu + \kappa) \otimes V_\xi(\mathbf{l}\nu + \kappa - \lambda)$. Therefore the category $\mathcal{C}$ has enough injective objects and enough projective objects as well. Part (ii) are proved.

(iii). Note that the modules $V_\xi(\mathbf{l}\mu + \kappa)^*, V_\xi(\mathbf{l}\mu + \kappa), V_\xi(\mathbf{l}\mu + \kappa)^\star$ are isomorphic. Since for each $M \in \text{ob}\mathcal{C}$, the modules $V_\xi(\mathbf{l}\mu + \kappa) \otimes M$ and $V_\xi(\mathbf{l}\mu + \kappa) \otimes M^*$ are projective and injective as well in the category $\mathcal{C}$, part (iii) follows.

(iv). We have seen that each indecomposable injective object is a direct summand of $V_\xi(\mathbf{l}\nu + \kappa) \otimes V_\xi(\delta)$ for some $\nu, \delta \in \mathbf{Z}^n_+$. So each injective object in ob$\mathcal{C}$ is an integrable

U_ξ-module. Let M be a finite dimensional U_ξ-module of type **1** and let M' be the maximal completely reducible submodule of M. By (ii), we can find an injective object E in ob$\mathcal{C}$ and an injective U_ξ-homomorphism $M' \hookrightarrow E$. Since E is injective in the category, the above injection can be extended to an injective U_ξ-homomorphism $M \hookrightarrow E$. Therefore M is integrable since E is integrable.

According to the statements in 3.1 we see (v) is an immediate consequence of (iv).

(vi). It is no harm to assume that E is indecomposable, then E is a direct summand of $V := V_\xi(\mathbf{l}\nu + \kappa) \otimes V_\xi(\delta)$ for some $\nu, \delta \in \mathbf{Z}_+^n$. By [L5, 27.3.3, p.221], V has a submodule filtration $0 = V_h \subset V_{h-1} \subset \cdots \subset V_2 \subset V_1 = V$ such that $V_a/V_{a+1} \simeq V_\xi(\delta_a)$ for some $\delta_a \in \mathbf{Z}_+^n$, $a = 1, ..., h-1$. Since E is a direct summand of V, the required filtration exists.

The theorem is proved.

Another main result of the section is the following.

Theorem 3.6. Let $\lambda \in \mathbf{Z}_{+,\mathbf{l}}^n$, $\mu \in \mathbf{Z}_+^n$. Then

(i). The module $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ contains a unique irreducible submodule.

(ii). The irreducible submodule $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ has highest weight $\mathbf{l}\mu + \kappa + w_0\lambda$ and is generated by $y'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$. (See Corollary 2.7 (ii) for the definition of y'_λ.)

Proof: (i). In the proof of Theorem 3.5 (ii) we have seen that $V_\xi(\mathbf{l}\mu + \kappa) \otimes V_\xi(\lambda)$ is an injective object in the category $\mathcal{C}$. According to [L5, 27.3.3, p.221], the submodule of $V_\xi(\mathbf{l}\mu + \kappa) \otimes V_\xi(\lambda)$ generated by $v_{\mathbf{l}\mu+\kappa} \otimes v_\lambda$ is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$. Let E be the indecomposable direct summand of $V_\xi(\mathbf{l}\mu + \kappa) \otimes V_\xi(\lambda)$ containing $v_{\mathbf{l}\mu+\kappa} \otimes v_\lambda$, then $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ is isomorphic to a submodule of E. The module E contains a unique irreducible submodule since E is an indecomposable injective object in the category $\mathcal{C}$. This forces that $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ contains a unique irreducible submodule.

(ii). We need to prove that

(a). The element $m := y'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$ is a primitive element in $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$.

(b). The element $y'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$ generates an irreducible submodule of $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$.

Note that in $V_\xi(\kappa + \lambda)$ we have $F_\kappa v_{\kappa+\lambda} \neq 0$ and $F_\alpha F_\kappa v_{\kappa+\lambda} = 0$ for all $\alpha \in R_l^+$. Since $F_i^{(l_i)} F_\kappa = F_\kappa F_i^{(l_i)}$ for $i = 1, 2, ..., n$ [L4, Lemma 8.5 (ii), p.108], so $F_i^{(l_i)} F_\kappa v_{\kappa+\lambda} = 0$ for $i = 1, 2, ..., n$. By the proof of Proposition 2.6 (i), $F_\kappa v_{\kappa+\lambda}$ generates an irreducible submodule M' of $V_\xi(\kappa + \lambda)$ whose highest weight is $\kappa + w_0\lambda$. By Corollary 2.7 (ii), the irreducible submodule M' is generated by $y'_\lambda v_{\kappa+\lambda}$. Thus $y'_\lambda v_{\kappa+\lambda}$ is a primitive element in $V_\xi(\kappa + \lambda)$ since it has weight $\kappa + w_0\lambda$. By [L4, Lemma 8.5 (i), p.108], for $i = 1, 2, ..., n$, we may write

$$E_i^{(l_i)} y'_\lambda = y'_\lambda E_i^{(l_i)} + \sum_{\substack{A, A' \in \mathbf{Z}_{+,\mathbf{l}}^r \\ K(A,A') \in \mathfrak{u}_\xi^0}} \theta(A, A') F_A K(A, A') E_{A'}, \quad \theta(A, A') \in \mathbf{Q}(\xi).$$

Since $F_A v_{\kappa+l}$, $A \in \mathbf{Z}^r_{+,l}$, are $\mathbf{Q}(\xi)$-linearly independent, so $K(A,A')v_{\kappa+\lambda} = 0$ when $E_{A'} = 1$ and $\theta(A,A') \neq 0$. But $K(A,A') \in \mathbf{u}^0_\xi$, so $K(A,A')v_{\kappa+\lambda} = 0$ is equivalent to $K(A,A')v_{l\mu+\kappa+\lambda} = 0$. Therefore $E_i^{(l_i)} y'_\lambda v_{l\mu+\kappa+\lambda} = 0$ for $i = 1,2,...,n$. By Corollary 2.7 (ii) and 2.4 (ii), $E_\alpha y'_\lambda v_{l\mu+\kappa+\lambda} = 0$ if $l_\alpha \geq 2$. Hence $y'_\lambda v_{l\mu+\kappa+\lambda}$ is a primitive element in $V_\xi(l\mu+\kappa+\lambda)$. This proves (a).

Let z_a, $a = 1,2,...,q,...,\prod_{\alpha\in R^+} l_\alpha$, be a $\mathbf{Q}(\xi)$-basis of $\mathbf{u}^-_\xi$ and let z'_b, $b = 1,2,...,q',...$, be a $\mathbf{Q}(\xi)$-basis of $U^-_{\xi,l}$ such that

(c). The elemnts $z_a \bar{v}_{\kappa+w_0\lambda}$, $a = 1,2,...,q$ is a basis of the irreducible module $L_\xi(\kappa+w_0\lambda)$, and $z_a \bar{v}_{\kappa+w_0\lambda} = 0$ for $a = q+1,...,\prod_{\alpha\in R^+} l_\alpha$, where $\bar{v}_{\kappa+w_0\lambda}$ is a nonzero element in $L_\xi(\kappa+w_0\lambda)$ of weight $\kappa+w_0\lambda$.

(d). The elemnts $z'_b \bar{v}_{l\mu}$, $b = 1,2,...,q'$ is a basis of the irreducible module $L_\xi(l\mu)$, $z'_b \bar{v}_{l\mu} = 0$ for $b = q'+1, q'+2...$, where $\bar{v}_{l\mu}$ is a nonzero element in $L_\xi(l\mu)$ of weight $l\mu$.

(e). For each positive integer b, the element z'_b has a degree $\gamma_b \in \mathbf{N}R^+$ (see 2.3 for the definition of degree).

We claim that the elements $z_a z'_b$, $a = 1,...,q$, $b = 1,...,q'$, span $M := U_\xi m$ as a $\mathbf{Q}(\xi)$-space. By 2.2 (i), 2.2 (v) and 3.2 (i) we see $z'_b m = 0$ whenever $b > q'$. Now assume that $1 \leq b \leq q'$ and $q < a \leq \prod_{\alpha\in R^+} l_\alpha$. We use induction on γ_b to prove that $z_a z'_b$ is a $\mathbf{Q}(\xi)$-linear combination of the elements $z_1 z'_1, ..., z_q z'_1, ..., z_1 z'_{q'}, ..., z_q z'_{q'}$. When $\gamma_b = 0$, this is obvious. If $1 \leq a \leq q$, nothing need to prove. Suppose tha $q < a \leq \prod_{\alpha\in R^+} l_\alpha$. Write $z_a z'_b = z'_b z_a + \sum A_{c,d} z_c z'_d$, $A_{c,d} \in \mathbf{Q}(\xi)$, then by [L4, Lemma 8.5 (ii), p.108], $\gamma_d < \gamma_b$ whenever $A_{c,d} \neq 0$. By induction hypothesis, $z_a z'_b m = \sum A_{c,d} z_c z'_d m$ is a $\mathbf{Q}(\xi)$-linear combination of $z_1 z'_1, ..., z_q z'_1, ..., z_1 z'_{q'}, ..., z_q z'_{q'}$. Therefore

$$\dim_{\mathbf{Q}(\xi)} M \leq qq' = \dim_{\mathbf{Q}(\xi)} L_\xi(\kappa+w_0\lambda) \cdot \dim_{\mathbf{Q}(\xi)} L_\xi(l\mu) = \dim_{\mathbf{Q}(\xi)} L_\xi(l\mu+\kappa+w_0\lambda).$$

This forces that $\dim_{\mathbf{Q}(\xi)} M = \dim_{\mathbf{Q}(\xi)} L_\xi(l\mu+\kappa+w_0\lambda)$ and M is an irreducible U_ξ-module of highest weight $l\mu+\kappa+w_0\lambda$.

4. The elements x'_λ

4.1. Recall that in 1.4 we have defined the element $x'_\lambda \in U^-_\xi$ and in Corollary 2.7 (ii) defined the element $y'_\lambda \in \mathbf{u}^-_\xi$ for every λ in $\mathbf{Z}^n_+$. The main result of this section is Theorem 4.2. We prove it after establishing several lemmas. It is a sorry that the author could not find a simple proof of Theorem 4.2 except for type A_n, B_2.

Theorem 4.2. Let $\lambda \in \mathbf{Z}^n_{+,l}, \mu \in \mathbf{Z}^n_+$. Then

(i). The elements $x'_\lambda v_{l\mu+\kappa+\lambda}$ is a primitive element in $V_\xi(l\mu+\kappa+\lambda)$.

(ii). We have $x'_\lambda = \theta y'_\lambda$ for some nonzero number $\theta \in \mathbf{Q}(\xi)$. In particular, x'_λ is in $\mathbf{u}^-_\xi$.

Lemma 4.3. Let M be an integrable U_ξ-module of type **1** and let $m \in M_\mu$ ($\mu \in \mathbf{Z}^n$). Let i, j be integers in $[1, n]$ and let a, b, c be non-negative integers.

(i). Assume that $E_i^{(h)} m = 0$ for $h \geq 1$, then $F_i^{(a)} F_j^{(b)} F_i^{(c)} m = 0$ if $a + \langle \alpha_j, \alpha_i^\vee \rangle b + c > \langle \mu, \alpha_i^\vee \rangle$.

(ii). Assume that $E_i^{(h)} m = 0, E_j^{(a)} m = 0$ for $h \geq 1$, then $F_i^{(a)} E_j^{(b)} F_j^{(c)} m = 0$ if $a + \langle \alpha_j, \alpha_i^\vee \rangle (c - b) > \langle \mu, \alpha_i^\vee \rangle$.

Proof: (i). By the commutation relations in [L4, 5.3-5.4, p.95-97], the element $F_i^{(a)} F_j^{(b)} F_i^{(c)}$ is in the left ideal of U_ξ^- generated by $F_i^{(h)}$, $h \geq a + \langle \alpha_j, \alpha_i^\vee \rangle b + c > \langle \mu, \alpha_i^\vee \rangle$. Now using 3.2 (i) to the subalgebra of U_ξ generated by all $E_i^{(h)}, F_i^{(h)}, K_i, K_i^{-1}$, $h \geq 0$, we see (i) is true.

(ii). If $b > c$, then

$$F_i^{(a)} E_j^{(b)} F_j^{(c)} m = F_i^{(a)} \sum_{0 \leq h \leq c} F_j^{(c-h)} \begin{bmatrix} K_j, 2h - c - b \\ h \end{bmatrix} E_j^{(b-h)} m = 0.$$

If $b \leq c$, using (i), then

$$F_i^{(a)} E_j^{(b)} F_j^{(c)} m = F_i^{(a)} F_j^{(c-b)} \begin{bmatrix} K_j, b - c \\ b \end{bmatrix} m = \begin{bmatrix} \langle \mu, \alpha_j^\vee \rangle + b - c \\ b \end{bmatrix} F_i^{(a)} F_j^{(c-b)} m = 0.$$

The lemma is proved.

Lemma 4.4. Let $\lambda \in \mathbf{Z}^n_{+,1}, \mu \in \mathbf{Z}^n_+, w \in W$. Then

(i). In $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$ we have $x'_{\lambda,w} v_{\mathbf{l}\mu+\kappa+\lambda} \neq 0$.

(ii). If $l_i \geq 2$, then $E_i x'_{\lambda,w} v_{\mathbf{l}\mu+\kappa+\lambda} = 0$.

(iii). If $l_\alpha \geq 2$, then $E_\alpha x'_{\lambda,w} v_{\mathbf{l}\mu+\kappa+\lambda} = 0$.

(iv). Assume that $x'_{\lambda,w} = F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)}$. For non negative integers $b_1, b_2, ..., b_k$, if $a_h - b_h \in l_{i_h} \mathbf{Z}$ for $h = 1, ..., k$, then $E_i F_{i_1}^{(b_1)} F_{i_2}^{(b_2)} \cdots F_{i_k}^{(b_k)} v_{\mathbf{l}\mu+\kappa+\lambda} = 0$ if $l_i \geq 2$.

Proof: Set $m := v_{\mathbf{l}\mu+\kappa+\lambda}$.

(i). According to [L5, Lemma 39.1.2, p.304], in $V_\xi(\lambda)$ we have $x'_{\lambda,w} v_\lambda \neq 0$. By 1.2 (i), this implies that $x'_{\lambda,w} m \neq 0$.

(ii). According to [L5, Lemma 39.1.4, p.305], there exists z in U_ξ^- such that

$$E_i x'_{\lambda,w} = x'_{\lambda,w} E_i + z \begin{bmatrix} K_i, 1 - \langle \lambda, \alpha_i^\vee \rangle \\ 1 \end{bmatrix}.$$

Since $l_i \geq 2$, $\begin{bmatrix} K_i, 1 - \langle \lambda, \alpha_i^\vee \rangle \\ 1 \end{bmatrix} m = \begin{bmatrix} l_i \langle \mu, \alpha_i^\vee \rangle + l_i \\ 1 \end{bmatrix} m = 0$. Therefore $E_i x'_{\lambda,w} m = 0$.

(iii). When all $l_i \geq 2$, this is a simple consequence of (ii) since $\mathbf{u}_\xi^-$ is genenerated by F_i, $i = 1, 2, ..., n$. When $l_i = 1$ for some i, we may check it directly.

Part (iv) is a simple consequence of (ii).

Lemma 4.5. Let $\lambda \in \mathbf{Z}^n_{+,1}$, $w \in W$. Assume that the Cartan matrix (a_{ij}) is symmetric. If $s_j w \geq w$, then $E_j^{(a)} x'_{\lambda,w} v_{\kappa+\lambda} = 0$ for all $a \geq 1$. (We also use "$\geq$" for the Bruhat order on W.)

Proof: Set $m := v_{\kappa+\lambda}$. Since all l_i, $i = 1, 2, ..., n$, are equal, we simply write l' for any one of them. Since U_ξ^+ is generated by $E_i, E_i^{(l')}$ for $i = 1, 2, ..., n$, [L2, Prop. 3.2 (b), p.62], by Lemma 4.4 (ii), it suffices to prove that $E_j^{(l')} x'_{\lambda,w} m = 0$. We use induction on the length $l(w)$ of w. Let $s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of w. We shall write a_h for $\langle s_{i_{h+1}} \cdots s_{i_k} \lambda, \alpha_{i_h}^\vee \rangle$ for $h = 1, ..., k$. When $k = 0, 1$, nohting need to prove. Now assume that $k \geq 2$. Set $i := i_1$ and let u be the shortest element of the coset $< s_i, s_j > w$. Since the Cartan matrix is symmetric, $k - 1 \geq l(u) \geq k - 2$.

If $l(u) = k - 1$, then $u = s_{i_2} \cdots s_{i_k}$ and $s_j u \geq u$. Note that $i \neq j$, using induction hypothesis, we see $E_j^{(l')} x'_{\lambda,w} m = F_i^{(a_1)} E_j^{(l')} x'_{\lambda,u} m = 0$.

If $l(u) = k - 2$, we may assume that $i_2 = j$ and $u = s_{i_3} \cdots s_{i_k}$. Then $s_i u \geq u$, $s_j u \geq u$ and $E_i^{(a)} x'_{\lambda,u} m = 0$, $E_j^{(a)} x'_{\lambda,u} m = 0$ for all $a \geq 1$. So $E_j^{(l')} x'_{\lambda,w} m = F_i^{(a_1)} E_j^{(l')} F_j^{a_2)} x'_{\lambda,u} m$. Note that $a_1 = \langle s_j u \lambda, \alpha_i^\vee \rangle = \langle u\lambda, \alpha_i^\vee + \alpha_j^\vee \rangle = \langle u\lambda, \alpha_i^\vee \rangle + a_2$ and $x'_{\lambda,u} m$ has weight $\kappa + u\lambda$, by Lemma 4.3 (ii) we see $E_j^{(l')} x'_{\lambda,w} m = 0$.

The lemma is proved

Lemma 4.6. Let $\lambda \in \mathbf{Z}^n_{+,1}$. Then in $V_\xi(\kappa + \lambda)$ the element $x'_\lambda v_{\kappa+\lambda}$ is primitive.

Proof: Set $m := v_{\kappa+\lambda}$. Since U_ξ^+ is generated by $E_i, E_i^{(l_i)}$ for $i = 1, 2, ..., n$, by Lemma 4.4 (ii), it suffices to prove that $E_i^{(l_i)} x'_{\lambda,w} m = 0$ for all i.

(a). Assume that (a_{ij}) is symmetric. Choose a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_r}$ of the longest element w_0 of W such that $i_1 = i$. Note that $a := \langle s_{i_2} \cdots s_{i_r} \lambda, \alpha_i^\vee \rangle < l_i$, so

$$E_i^{(l_i)} x'_\lambda m = \sum_{0 \leq h \leq a} F_i^{(a-h)} \begin{bmatrix} K_i, 2h - a - l_i \\ h \end{bmatrix} E_i^{(l_i - h)} x'_{\lambda,u} m,$$

where $u = s_{i_2} \cdots s_{i_r}$. By Lemma 4.5, $E_i^{(l_i - h)} x'_{\lambda,u} m = 0$ for $h = 0, 1, ..., a$. Therefore $E_i^{(l_i)} x'_\lambda m = 0$ for $i = 1, 2, ..., n$.

(b). Assume that (a_{ij}) is of type B_n. We number the simple roots in R^+ so that $\langle \alpha_2, \alpha_1^\vee \rangle = -2, \langle \alpha_1, \alpha_2^\vee \rangle = -1, \langle \alpha_2, \alpha_3^\vee \rangle = -1, ..., \langle \alpha_{n-1}, \alpha_n^\vee \rangle = -1$. We have $d_1 = 1, d_2 = ... = d_n = 2$, $l_2 = ... = l_n$, and $2l_j \geq l_1 \geq l_j$ for $j = 2, ..., n$. We use induction on n.

When $n = 2$, write $a := \langle \lambda, \alpha_1^\vee \rangle, b := \langle \lambda, \alpha_2^\vee \rangle$. Then

$$x'_\lambda = F_1^{(a)} F_2^{(a+b)} F_1^{(a+2b)} F_2^{(b)} = F_2^{(b)} F_1^{(a+2b)} F_2^{(a+b)} F_1^{(a)}.$$

Since $l_1 > a$, using Lemma 4.4 (ii) we see

$$E_1^{(l_1)} x'_\lambda m = F_1^{(a)} F_2^{(a+b)} E_1^{(l_1)} F_1^{(a+2b)} F_2^{(b)} m.$$

Note that $F_2^{(b)} m$ is a primitive element of weight $\kappa + \lambda - b\alpha_2$. Now

$$a + b - \langle \alpha_1, \alpha_2^\vee \rangle (l_1 - a - 2b) = l_1 - b > l_2 - 1 - b = \langle \kappa + \lambda - b\alpha_2, \alpha_2^\vee \rangle.$$

By Lemma 4.3 (ii) we have $E_1^{(l_1)} x'_\lambda m = 0$. Similarly we have

$$E_2^{(l_2)} x'_\lambda m = F_2^{(b)} F_1^{(a+2b)} E_2^{(l_2)} F_2^{(a+b)} F_1^{(a)} m = 0.$$

Now suppose the lemma is true for $n-1$. Let u be the longest element in $< s_1, s_2, ..., s_{n-1} >$. Then $w_0 = s_n s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} s_n u = u s_n s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} s_n$.∎ Set

$$a_h := \langle s_{h-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} s_n u \lambda, \alpha_h^\vee \rangle, \quad h = 2, ..., n,$$

and

$$b_h := \langle s_{h+1} \cdots s_{n-1} s_n u \lambda, \alpha_h^\vee \rangle, \quad h = 1, 2, ..., n.$$

Then $x'_\lambda = F_n^{(a_n)} \cdots F_2^{(a_2)} F_1^{(b_1)} F_2^{(b_2)} \cdots F_n^{(b_n)} x'_{\lambda,u}$. Note that $E_i^{(a)} x'_{\lambda,u} m = 0$ for all i, $a \geq 1$ and that $l_n > a_n$. Using Lemma 4.3 (ii) and Lemma 4.4 (ii) repeatedly we see $E_i^{(l_i)} x'_\lambda m = 0$ for $i = 2, ..., n$.

We need do a little more to see that $E_1^{(l_1)} x'_\lambda m = 0$. Let w be the longest element in $< s_2, ..., s_n >$. Then $E_i^{(a)} x'_{\lambda,w} m = 0$ for $i = 1, 2, ..., n$, $a \geq 1$. Note that $w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_n \cdots s_2 s_1 w$ and $l_1 > c_1 :=< \lambda, \alpha_1^\vee \rangle$. Using Lemma 4.3 (ii) and Lemma 4.4 (ii) repeatedly, one can check that

$$E_1^{(l_1)} x'_\lambda m = E_1^{(l_1)} F_1^{(c_1)} F_2^{(c_2)} F_1^{(c_3)} F_3^{(c_4)} \cdots F_n^{(c_k)} \cdots F_2^{(c_{h-1})} F_1^{(c_h)} x'_{\lambda,w} m = 0,$$

where $c_1, ..., c_h$ are defined according to the reduced expression $s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_n \cdots s_2 s_1 w$ and $k = 1 + n(n-1)/2, h = n(n+1)/2$. This completes the proof for type B_n.

(c). Similarly, we prove the Lemma for type C_n.

(d). (Sketch.) Similarly we prove the lemma for type F_4, G_2. Note that the longest element of the Weyl group of type F_4 is $s_4 s_3 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1 s_4 s_3 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_3$. Here we number the simple roots as usual. Moreover, if necessary, use the commutation relations in [L4 5.3-5.4, p.95-97] and note that $F_i^{(al_i+b)} F_i^{(cl_i+d)} = 0$ if $0 < b, d < l_i$ and $b + d \geq l_i$.

Lemma 4.7. Let $\delta_1, ..., \delta_k \in \mathbf{Z}_+^n$ and $\lambda \in \mathbf{Z}_{+,1}^n$. Then the submodule of $L_\xi(\mathbf{l}\delta_1) \otimes \cdots \otimes L_\xi(\mathbf{l}\delta_k) \otimes L_\xi(\kappa + \lambda)$ generated by $v_{\mathbf{l}\delta_1} \otimes \cdots \otimes v_{\mathbf{l}\delta_k} \otimes v_{\kappa+\lambda}$ is isomorphic to $V_\xi(\mathbf{l}\delta_1 + ... + \mathbf{l}\delta_k + \kappa + \lambda)$.

Proof: By 3.1 (i), we have a U_ξ-homomorphism

$$V_1 := V_\xi(\mathbf{l}\delta_1 + ... + \mathbf{l}\delta_k + \kappa + \lambda) \to V := L_\xi(\mathbf{l}\delta_1) \otimes \cdots \otimes L_\xi(\mathbf{l}\delta_k) \otimes L_\xi(\kappa + \lambda),$$

which carries $m_1 := v_{\mathbf{l}\delta_1+...+\mathbf{l}\delta_k+\kappa+\lambda}$ to $m := v_{\mathbf{l}\delta_1} \otimes \cdots \otimes v_{\mathbf{l}\delta_k} \otimes v_{\kappa+\lambda}$. By 2.2 (v), $y'_\lambda m = v_{\mathbf{l}\delta_1} \otimes \cdots \otimes v_{\mathbf{l}\delta_k} \otimes y'_\lambda v_{\kappa+\lambda} \neq 0$. But $y'_\lambda m_1$ generates the unique irreducible submodule of V_1 (Theorem 3.6 (ii)). Therefore, the submodule of V generated by m is isomorphic to V_1.

4.8. A sketch proof of Theroem 4.2. (i) For $i = 1, 2, ..., n$, denote $\delta_i \in \mathbf{Z}_+^n$ the unique element such that $\langle \delta_i, \alpha_j^\vee \rangle$ is $\langle \mu, \alpha_i^\vee \rangle$ if $i = j$, is 0 if $i \neq j$. By Lemma 4.7, the submodule M of $V := L_\xi(\mathbf{l}\delta_1) \otimes \cdots \otimes L_\xi(\mathbf{l}\delta_n) \otimes L_\xi(\kappa + \lambda)$ generated by $m := v_{\mathbf{l}\delta_1} \otimes \cdots \otimes v_{\mathbf{l}\delta_n} \otimes v_{\kappa+\lambda}$ is isomorphic to $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$. By Lemma 4.6, $m' := v_{\mathbf{l}\delta_1} \otimes \cdots \otimes v_{\mathbf{l}\delta_n} \otimes x'_\lambda v_{\kappa+\lambda}$ is a primitive element in V. But one can check that $x'_\lambda m = m'$. Therefore $x'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$ is a primitive element in $V_\xi(\mathbf{l}\mu + \kappa + \lambda)$.

(ii). Since $x'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda} \neq 0$ and has the same weight with $y'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$. By (i) and Theorem 3.6, we can find a nonzero number $\theta \in \mathbf{Q}(\xi)$ such that $x'_\lambda - \theta y'_\lambda \in I^-_{\mathbf{l}\mu+\kappa+\lambda}$. Choose $\mu \in \mathbf{Z}_+^n$ such that $\langle \mu, \alpha_i^\vee \rangle > l|R^+|$, then $x'_\lambda - \theta y'_\lambda \in I^-_{\mathbf{l}\mu+\kappa+\lambda}$ is equivalent to $x'_\lambda - \theta y'_\lambda = 0$.

The theorem is proved.

4.9. By Lemma 4.4 (iii), Theorem 4.2 is actually equivalent to that $x'_\lambda \in \mathbf{u}_\xi^-$ when λ is l-restricted. For type B_2, using the commutation relations in [L4, 5.3, p.96] we see easily that if λ is l-restricted then $x'_\lambda \in \mathbf{u}_\xi^-$. For type A_n there is a naive argument for the fact, which is based on the following Lemma 4.10. We need a notation. Given $i \in [1, n]$, let $\mathcal{H}_i$ be the $\mathbf{Q}(\xi)$-subspace of U_ξ^- spanned by all $F_{\beta_1}^{(a_1)} F_{\beta_2}^{(a_2)} \cdots F_{\beta_r}^{(a_r)}$, $a_1, ..., a_r \in \mathbf{N}$ and $a_h \leq l_{\beta_h} - 1$ whenever $\beta_h - \alpha_i \in \mathbf{N}R^+$, $h = 1, ..., r$. Obviously, $\bigcap_{i=1}^n \mathcal{H}_i = \mathbf{u}_\xi^-$.

Lemma 4.10. Let x be an element in U_ξ. Assume that x is expressed as a $\mathbf{Q}(\xi)$-linear combination of some monomials $z_1, ..., z_h$ of $F_\alpha^{(a)}, \alpha \in R^+, a \in \mathbf{N}$. Given $i \in [1, n]$. If $a \leq l_\alpha - 1$ whenever $F_\alpha^{(a)}$ appears in some monomial z_k and $\alpha - \alpha_i \in \mathbf{N}R^+$, then $x \in \mathcal{H}_i$.

Proof: Using commutation relations in [L4, 5.3-5.4, p.95-97] and [L4, Theorem 6.6, p.103-104].

4.11. Now we give a simple proof of Theorem 4.2 for type A_n by using Lemma 4.10. By Lemma 4.4 (ii), it suffices to prove that $x'_\lambda \in \mathbf{u}_\xi^-$ when λ is l-restricted. We use induction on n. Set $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle$, $i = 1, 2, ..., n$. When $1 \leq i < j \leq n$ we also write $\lambda_{i,j}$ for $\lambda_i + \lambda_{i+1} + \cdots + \lambda_j$. Then

$$x'_\lambda = F_1^{(\lambda_n)} F_2^{(\lambda_{n-1,n})} \cdots F_n^{(\lambda_{1,n})} F_1^{(\lambda_{n-1})} F_2^{(\lambda_{n-2,n-1})} \cdots F_{n-1}^{(\lambda_{1,n-1})} \cdots F_1^{(\lambda_2)} F_2^{(\lambda_{1,2})} F_1^{(\lambda_1)}.$$

Note that $l_1 = ... = l_n$, we see

(a). $x'_\lambda \in \mathcal{H}_1$. Symmetrically, we have $x'_\lambda \in \mathcal{H}_n$.

Let $w = s_1s_2s_1s_3s_2s_1\cdots s_{n-1}\cdots s_2s_1$. Set

$$y := F_1^{(\lambda_{n-1})}F_2^{(\lambda_{n-2,n-1})}F_1^{(\lambda_{n-2})}F_3^{(\lambda_{n-3,n-1})}F_2^{(\lambda_{n-3,n-2})}F_1^{(\lambda_{n-3})}\cdots F_{n-2}^{(\lambda_{2,n-1})}\cdots F_1^{(\lambda_2)},$$

$$y' := F_{n-1}^{(\lambda_{1,n-1})}\cdots F_2^{(\lambda_{1,2})}F_1^{(\lambda_1)}.$$

Then $x'_{\lambda,w} = yy'$. By induction hypothesis, $y, x'_{\lambda,w} \in \mathbf{u}_\xi^-$. By 2.2 (ii), then $x'_{\lambda,w} = yz$ for some $z \in \mathbf{u}_\xi^-$. Note that

$$x'_\lambda = F_1^{(\lambda_n)}F_2^{(\lambda_{n-1}+\lambda_n)}\cdots F_{n-1}^{(\lambda_2+\cdots+\lambda_n)}yF_n^{(\lambda_1+\cdots+\lambda_n)}z$$

and that

$$F_1^{(\lambda_n)}F_2^{(\lambda_{n-1}+\lambda_n)}\cdots F_{n-1}^{(\lambda_2+\cdots+\lambda_n)}y = x'_{\mu,w},$$

where $\mu := (\lambda_2, ..., \lambda_n, \lambda_1)$. According to induction hypothesis, $x'_{\mu,w} \in \mathbf{u}_\xi^-$. Therefore $x'_\lambda = x'_{\mu,w}F_n^{(\lambda_1+\cdots+\lambda_n)}z \in \bigcap_{i=1}^{n-1}\mathcal{H}_i$. Combine this and (a) we see $x'_\lambda \in \bigcap_{i=1}^{n}\mathcal{H}_i = \mathbf{u}_\xi^-$.

5. Main results

5.1. In this section we give the main results of the paper. Essentially, they reexpress some results in previous sections. Recall that in 1.4 we have defined the element $x_\lambda \in U_\xi^-$ for every λ in $\mathbf{Z}_+^n$.

Theorem 5.2. Assume that λ is l-restricted, then $x_\lambda \in \mathbf{u}_\xi^-$.

Proof: We have $x_\lambda = x'_{-w_0\lambda}$. Note that $-w_0\lambda$ is also l-restricted, by Theorem 4.2 (ii), $x_\lambda \in \mathbf{u}_\xi^-$.

Theorem 5.3. Assume that $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is l-restricted and $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in \{\pm 1\}^n$. Let $\mathbf{u}_\xi(\lambda,\sigma)$ be the left ideal of $\mathbf{u}_\xi$ generated by all $E_\alpha, K_i - \sigma_i\xi^{\lambda_i d_i}$, $\alpha \in R_l^+$, $i = 1, 2, ..., n$, and elements $F \in \mathbf{u}_\xi^-$ such that $Fx_{\kappa-\lambda} = 0$. Then

(i) $\mathbf{u}_\xi/\mathbf{u}_\xi(\lambda,\sigma)$ is an irreducible $\mathbf{u}_\xi$-module. Moreover, as a $\mathbf{u}_\xi$-module, $L_\xi(\lambda,\sigma)$ is isomorphic to $\mathbf{u}_\xi/\mathbf{u}_\xi(\lambda,\sigma)$.

(ii). For any $\gamma \in \mathbf{N}R^+$, denote $\mathbf{u}_{\xi,\gamma}^-$ the set of all elements in $\mathbf{u}_\xi^-$ of degree γ, and set $\mathbf{n}_\xi(\lambda,\gamma) := \{F \in \mathbf{u}_{\xi,\gamma}^- \mid Fx_{\kappa-\lambda} = 0\}$. Then $\dim_{\mathbf{Q}(\xi)}L_\xi(\lambda,\sigma)_{\lambda-\gamma,\sigma} = \dim_{\mathbf{Q}(\xi)}\mathbf{u}_{\xi,\gamma}^- - \mathbf{n}_\xi(\lambda,\gamma)$.

Proof: (i). Let $\tilde{J}_{\lambda,\sigma}$ be the left ideal of $\tilde{\mathbf{u}}_\xi$ generated by $E_\alpha, K_i - \sigma_i\xi^{\lambda_i d_i}, \begin{bmatrix} K_i, c \\ a \end{bmatrix} - \sigma_i^a\begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{d_i,\xi}$, $\alpha \in R_l^+$, $i = 1, 2, ..., n$, $c \in \mathbf{Z}, a \in \mathbf{N}$, elements $F \in \mathbf{u}_\xi^-$ such that $Fx_{\kappa-\lambda} = 0$. Since $x_{\kappa-\lambda} = x'_{w_0\lambda+\kappa}$, by Theorem 4.2 (ii) and Corollary 2.7 (ii) we see $\tilde{\mathbf{u}}_\xi/\tilde{J}_{\lambda,\sigma} \simeq \tilde{L}_\xi(\lambda,\sigma)$. But λ is l-restricted, so the restriction to $\mathbf{u}_\xi$ of $\tilde{L}_\xi(\lambda,\sigma)$ is an irreducible $\mathbf{u}_\xi$-module. Obviously, the restriction is isomorphic to $\mathbf{u}_\xi/\mathbf{u}_\xi(\lambda,\sigma)$. But $\tilde{L}_\xi(\lambda,\sigma)$

is the restriction to $\tilde{\mathbf{u}}_\xi$ of the irreducible U_ξ-module $L_\xi(\lambda,\sigma)$. So as a $\mathbf{u}_\xi$-module, $L_\xi(\lambda,\sigma)$ is isomorphic to $\mathbf{u}_\xi/\mathbf{u}_\xi(\lambda,\sigma)$.

Part (ii) is an immediate consequence of part (i).

The theorem is proved.

Theorem 5.4. Let $\lambda=(\lambda_1,\lambda_2,...,\lambda_n), \mu=(\mu_1,\mu_2,...,\mu_n)\in\mathbf{Z}_+^n$ and $\sigma=(\sigma_1,\sigma_2,...,\sigma_n)\in\{\pm1\}^n$. Assume that λ is l-restricted. Let $J_{\mathrm{l}\mu+\lambda,\sigma}$ be the left ideal of U_ξ generated by elements in $I_{\lambda,\sigma}$ and elements $F\in\mathbf{u}_\xi^-$ such that such that $Fx_{\kappa-\lambda}=0$. Then $U_\xi/J_{\mathrm{l}\mu+\lambda,\sigma}\simeq L_\xi(\mathrm{l}\mu+\lambda,\sigma)$.

Proof: Since $L_\xi(\mathrm{l}\mu+\lambda,\sigma)\simeq L_\xi(\mathrm{l}\mu,\mathbf{1})\otimes L_\xi(\lambda,\sigma))$, we have $J_{\mathrm{l}\mu+\lambda,\sigma}\bar{v}_{\mathrm{l}\mu+\lambda,\sigma}=0$. Note that

$$L_\xi(\mathrm{l}\mu,\mathbf{1})\otimes L_\xi(\kappa,\sigma)\simeq L_\xi(\mathrm{l}\mu,\mathbf{1})\otimes V_\xi(\kappa,\sigma)\simeq V_\xi(\mathrm{l}\mu+\kappa,\sigma).$$

Let $z_b',\ b=1,2,...,k,...,$ be a $\mathbf{Q}(\xi)$-basis of $U_{\xi,l}^-$ such that

(a). The elemnts $z_b'\bar{v}_{\mathrm{l}\mu},\ b=1,2,...,k$ is a basis of the irreducible module $L_\xi(\mathrm{l}\mu)$, $z_b'\bar{v}_{\mathrm{l}\mu}=0$ for $b=k+1,k+2...,$, where $\bar{v}_{\mathrm{l}\mu}$ is a non zero element in $L_\xi(\mathrm{l}\mu)$ of weight $\mathrm{l}\mu$. Let $\bar{I}$ be the $\mathbf{Q}(\xi)$-space spanned by $z_hF, 1\le h\le k, F\in\mathbf{u}_\xi^-$, then we have $\bar{I}+I^-_{\mathrm{l}\mu+\kappa,\sigma}=U_\xi^-$. Since $I^-_{\mathrm{l}\mu+\kappa,\sigma}\subseteq I_{\mathrm{l}\mu+\lambda,\sigma}$, as $\mathbf{Q}(\xi)$-spaces we have

$$U_\xi/J_{\mathrm{l}\mu+\lambda,\sigma}\simeq U_\xi^-/U_\xi^-\cap J_{\mathrm{l}\mu+\lambda,\sigma}\simeq\bar{I}/\bar{I}\cap J_{\mathrm{l}\mu+\lambda,\sigma}.$$

By Theorem 5.3, $\dim_{\mathbf{Q}(\xi)}\bar{I}\cap J_{\mathrm{l}\mu+\lambda,\sigma}\ge k(\dim_{\mathbf{Q}(\xi)}\mathbf{u}_\xi^- - \dim_{\mathbf{Q}(\xi)}L_\xi(\lambda,\sigma))$. Since $\dim_{\mathbf{Q}(\xi)}\bar{I}=k\dim_{\mathbf{Q}(\xi)}\mathbf{u}_\xi^-$, we have

$$\dim_{\mathbf{Q}(\xi)}U_\xi/J_{\mathrm{l}\mu+\lambda,\sigma}\le k\dim_{\mathbf{Q}(\xi)}L_\xi(\lambda,\sigma)=\dim_{\mathbf{Q}(\xi)}L_\xi(\mathrm{l}\mu+\lambda,\sigma).$$

This force that $U_\xi/J_{\mathrm{l}\mu+\lambda,\sigma}$ and $L_\xi(\mathrm{l}\mu+\lambda,\sigma)$ have the same $\mathbf{Q}(\xi)$-dimension and as U_ξ-modules, they are isomorphic.

The theorem is proved

From the above proof we get the following result.

Corollary 5.5. Keep the notations in Theorem 5.5. Then the left ideal $J_{\mathrm{l}\mu+\lambda,\sigma}\cap U_\xi^-$ of U_ξ^- is generated by $F_i^{(l_i\mu_i+l_i)}$, $i=1,2,...,n$, and elements $F\in\mathbf{u}_\xi^-$ such that such that $Fx_{\kappa-\lambda}=0$.

6. Hyperalgebra

6.1. Recall that $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbf{C}$. Let Let $e_\alpha,\ f_\alpha,\ h_i,\ \alpha\in R^+$, $i=1,2,...,n$ be a Chevalley basis of $\mathfrak{g}$. We also write e_i,f_i for $e_{\alpha_i},f_{\alpha_i}$, $i=1,2,...,n$. Let $\mathfrak{k}$ be an algebraic closed field of prime characteristic p. Recall that $\mathfrak{U}_\mathfrak{k}=\mathfrak{U}_\mathbf{Z}\otimes\mathfrak{k}$ is the hyperalgebra associated to $\mathfrak{g}$ and $\mathfrak{k}$. Let $\mathfrak{U}_{\mathfrak{k}}^+, \mathfrak{U}_{\mathfrak{k}}^-, \mathfrak{U}_{\mathfrak{k}}^0$ be the positive part, negative part, zero part of $\mathfrak{U}_{\mathfrak{k}}$ respectively. Given an positive integer a, let $\mathfrak{u}_a$ be the a-th Frobenius kernel of $\mathfrak{U}_{\mathfrak{k}}$, denote $\mathfrak{u}_a^+, \mathfrak{u}_a^-, \mathfrak{u}_a^0$ the positive part, negative part, zero part of $\mathfrak{u}_a$ respectively. Let $\tilde{\mathfrak{u}}_a$ be the subalgebra of $\mathfrak{U}_{\mathfrak{k}}$ generated by $\mathfrak{u}_a$ and $\mathfrak{U}_{\mathfrak{k}}^0$, then $\tilde{\mathfrak{u}}_a = \tilde{\mathfrak{u}}_a^- \mathfrak{U}_{\mathfrak{k}}^0 \tilde{\mathfrak{u}}_a^+$.

For any $\lambda \in \mathbf{Z}_+^n$, denote $V_{\mathfrak{k}}(\lambda)$ the Weyl module of $\mathfrak{U}_{\mathfrak{k}}$ of highest weight λ and denote $L_{\mathfrak{k}}(\lambda)$ the irreducible module of $\mathfrak{U}_{\mathfrak{k}}$ of highest weight λ. When λ is p^a-restricted (i.e. $0 \leq \langle \lambda, \alpha_i^\vee \rangle < p^a$ for $i = 1, 2, ..., n$), the restriction to $\mathfrak{u}_a$ (resp. $\tilde{\mathfrak{u}}_a$) of $L_{\mathfrak{k}}(\lambda)$ is an irreducible $\mathfrak{u}_a$-module (resp. $\tilde{\mathfrak{u}}_a$-module), denote the restriction by $L_{\mathfrak{k},a}(\lambda)$ (resp. $\tilde{L}_{\mathfrak{k},a}(\lambda)$).

For any $\mu \in \mathbf{Z}^n$, denote $\tilde{Z}_{\mathfrak{k},a}(\mu)$ the Verma mmodule of $\tilde{\mathfrak{u}}_a$ of highest weight μ, which contains a unique irreducible $\tilde{\mathfrak{u}}_a$-submodule. The following results (i) and (ii) are due to Jantzen [J, 6.2 (1), p.190; 6.3 Corollar, p.191], the assertion (iii) maybe is well known, all of them also can be proved along the proofs of Prop. 2.6, Theorem 3.6 and Theorem 3.2.

(i). Assume that λ is p^a-restricted, then the irreducible $\tilde{\mathfrak{u}}_a$-submodule of $\tilde{Z}_{\mathfrak{k},a}(2(p^a-1)\rho + w_0\lambda)$ is isomorphic to $\tilde{L}_{\mathfrak{k},a}(\lambda)$, where $\rho = (1, ..., 1) \in \mathbf{Z}_+^n$.

(ii). Assume that λ is p^a-restricted, then $V_{\mathfrak{k}}(2(p^a-1)\rho + w_0\lambda)$ contains a unique irreducible $\mathfrak{U}_{\mathfrak{k}}$-submodule, which is isomorphic to $L_{\mathfrak{k}}(\lambda)$.

(iii). The category $\mathfrak{C}$ of finite dimensional $\mathfrak{U}_{\mathfrak{k}}$-modules has enough injective objects and enough projective objects as well. And in $\mathfrak{C}$ each injective object is also a projective object. Each injective object in $\mathfrak{C}$ is a direct summand of the module $V_{\mathfrak{k}}((p^b-1)\rho) \otimes V_{\mathfrak{k}}(\delta)$ for some positive integer b and $\delta \in \mathbf{Z}_+^n$. Moreover, if E is an injective object in $\mathfrak{C}$ then E has a submodules filtration $0 = E_k \subset E_{k-1} \subset \cdots \subset E_2 \subset E_1 = E$ such that $E_a/E_{a+1} \simeq V_{\mathfrak{k}}(\nu_a)$ for some $\nu_a \in \mathbf{Z}_+^n$, $a = 1, ..., k-1$.

6.2. We shall fix the positive integer a. Assume that $\mathfrak{g}$ is ismple. If p is odd, and $p \geq 3$ when $\mathfrak{g}$ is of type G_2, choose a p^a-th primitive root ξ of 1. If $\mathfrak{g}$ is of type A, D, E and $p = 2$, choose a 2^{a+1}-th primitive root ξ of 1. Let U_ξ' be the $\mathbf{Z}[\xi]$-subalgebra of U_ξ generated by $E_i^{(k)}$ $F_i^{(k)}$, K_i, K_i^{-1}, $i = 1, 2, ..., n$, $k \geq 0$. Consider the $\mathfrak{k}$-algebra $\mathcal{U}_{\mathfrak{k}}' := U_\xi' \otimes_{\mathbf{Z}[\xi]} \mathfrak{k}$, where $\mathfrak{k}$ is regarded as a $\mathbf{Z}[\xi]$-algebra through the ring homomorphism $\mathbf{Z}[\xi] \to \mathfrak{k}$, $\xi \to 1$. For simplicity, the images in $\mathcal{U}_{\mathfrak{k}}'$ of $E_i^{(k)}$, $F_i^{(k)}$, K_i, K_i^{-1}, etc. will be denoted by the same notations.

Let $\mathcal{K}'$ be the two-sided ideal of $\mathcal{U}_{\mathfrak{k}}'$ generated by $K_1 - 1, ..., K_n - 1$. Set $\mathcal{U}_{\mathfrak{k}} := \mathcal{U}_{\mathfrak{k}}'/\mathcal{K}'$. Again for simplicity, the images in $\mathcal{U}_{\mathfrak{k}}'$ of $E_i^{(k)}$, $F_i^{(k)}$, K_i, K_i^{-1}, etc. will be denoted by the same notations. The following result is due to Lusztig [L3, 6.7 (d), p.295] (cf. 1.6).

(i). There is a unique $\mathfrak{k}$-algebra isomorphism $\mathcal{U}_{\mathfrak{k}} \to \mathfrak{U}_{\mathfrak{k}}$ such that $E_i^{(k)}$ maps to $e_i^{(k)} := e_i^k/k! \otimes 1$ and $F_i^{(k)}$ maps to $f_i^{(k)} := f_i^k/k! \otimes 1$ for $i = 1, 2, ..., n$. The image in $\mathfrak{U}_{\mathfrak{k}}$ of $\left[{K_i;0 \atop k}\right]$ will be denoted by $\binom{h_i}{k}$.

Given $\lambda \in \mathbf{Z}_+^n, w \in W$, define the monomials $\mathfrak{x}_{\lambda,w}, \mathfrak{x}'_{\lambda,w}, \mathfrak{x}_\lambda, \mathfrak{x}'_\lambda$ of $f_i^{(k)}$, $i = 1, 2, ..., n$, $k \geq 0$ as the same way in 1.4.

Theorem 6.3. Assume that $\lambda \in \mathbf{Z}_+^n$ is p^a-restricted. Then

(i). The elements $\mathfrak{x}_\lambda$ and $\mathfrak{x}'_\lambda$ are in $\mathfrak{u}_a^-$.

(ii). The element $\mathfrak{x}_{(p^a-1)\rho-\lambda}\tilde{\mathfrak{z}}$ generates the unique irredublе submodule of $\tilde{Z}_{\mathfrak{k},a}(2(p^a-1)\rho + w_0\lambda)$ (resp. $V_{\mathfrak{k}}(2(p^a-1)\rho+w_0\lambda)$), where $\tilde{\mathfrak{z}}$ is a nonzero element in $\tilde{Z}_{\mathfrak{k},a}(2(p^a-1)\rho+w_0\lambda)$ (resp. $V_{\mathfrak{k}}(2(p^a-1)\rho+w_0\lambda)$) of highest weight.

Proof: It is no harm to assume that $\mathfrak{g}$ is simple. When $\mathfrak{g}$ is of type A_n, D_n, E_n; or B_n, C_n, F_4 and p is odd; or type G_2 and $p \geq 5$, the theorem is a simple consequence of Theorem 4.2 and 6.2 (i). When $\mathfrak{g}$ is of type B_n, C_n, F_4 and $p=2$; or type G_2 and $p=2,3$, one may prove the theorem by direct calculations.

Theorem 6.4. Assume that $\lambda \in \mathbf{Z}_+^n$ is p^a-restricted.

(i). Let $\mathfrak{J}_\lambda$ be the left ideal of $\mathfrak{U}_{\mathfrak{k}}$ generated by all elements $e_i^{(k)}, \binom{h_i}{k} - \binom{\langle\lambda,\alpha_i^\vee\rangle}{k}, f_i^{(k_i)}$, $i=1,2,...,n$, $k \geq 1$, $k_i \geq p^a$, and all elements $f \in \mathfrak{u}_a^-$ such that $f\mathfrak{x}_{(p^a-1)\rho-\lambda} = 0$, then $\mathfrak{U}_{\mathfrak{k}}/\mathfrak{J}_\lambda \simeq L_{\mathfrak{k}}(\lambda)$.

(ii). Let $\mathfrak{I}_\lambda$ be the left ideal of $\mathfrak{u}_a$ generated by all elements $e_\alpha^{(k)}, \binom{h_i}{k} - \binom{\langle\lambda,\alpha_i^\vee\rangle}{k}$, $\alpha \in R^+$, $i=1,2,...,n$, $1 \leq k \leq p^a-1$, and all elements $f \in \mathfrak{u}_a^-$ such that $f\mathfrak{x}_{(p^a-1)\rho-\lambda} = 0$, then $\mathfrak{u}_a/\mathfrak{I}_\lambda \simeq L_{\mathfrak{k},a}(\lambda)$.

(iii) For any $\gamma \in \mathbf{N}R^+$, denote $\mathfrak{u}_{a,\gamma}^-$ the set of all elements in $\mathfrak{u}_a^-$ of degree γ and denote $\mathfrak{n}_a(\lambda,\gamma)$ the set $\{f \in \mathfrak{u}_{a,\gamma}^- \mid f\mathfrak{x}_{(p^a-1)\rho-\lambda} = 0\}$, then $\dim L_{\mathfrak{k}}(\lambda)_{\lambda-\gamma} = \dim \mathfrak{u}_{a,\gamma}^- - \dim \mathfrak{n}_a(\lambda,\gamma)$.

Proof: Similar to those in section 5.

7. Questions

7.1. Let ξ be root of 1 of order ≥ 3. For $i \in [1,n]$, $k \in \mathbf{N}$, denote $\Theta_{i,k}$ the $\mathbf{Q}(\xi)$-linear homomorphism $U_\xi \to U_\xi$, $x \to xF_i^{(k)}$. The kernel and the image of $\Theta_{i,k}$ are easily described by means of PBW Theorem. Assume that $\lambda \in \mathbf{Z}_+^n$ is l-restricted. Let $s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression of the longest element of W. Set $k_h := \langle s_{i_{h-1}}\cdots s_{i_1}(\kappa-\lambda), \alpha_{i_h}^\vee\rangle$, $\delta_h := k_1\alpha_{i_1} + \cdots + k_h\alpha_{i_h}$, $h=1,...,r$. Recall that for any $\gamma \in \mathbf{N}R^+$ we denote $\mathbf{u}_{\xi,\gamma}^-$ the set of all elments in $\mathbf{u}_\xi^-$ of degree γ. Given $\beta \in \mathbf{N}R^+$, set

$$D_{0,\beta} = \dim_{\mathbf{Q}(\xi)}\mathbf{u}_{\xi,\beta}^-,$$

$$D_{1,\beta} = \dim_{\mathbf{Q}(\xi)}\Theta_{i_1,k_1}(\mathbf{u}_{\xi,\beta}^-),$$

$$D_{2,\beta} = \min\{D_{1,\beta},\ \dim_{\mathbf{Q}(\xi)}\Theta_{i_2,k_2}(\mathbf{u}_{\xi,\beta+\delta_1}^-)\},$$

......

$$D_{h,\beta} = \min\{D_{h-1,\beta},\ \dim_{\mathbf{Q}(\xi)}\Theta_{i_h,k_h}(\mathbf{u}_{\xi,\beta+\delta_{h-1}}^-)\},$$

......

$$D_{r,\beta} = \min\{D_{r-1,\beta},\ \dim_{\mathbf{Q}(\xi)}\Theta_{i_r,k_r}(\mathbf{u}^-_{\xi,\beta+\delta_{r-1}})\}.$$

Conjecture A. The number $D_{r,\beta}$ is independent of the choice of the reduced expression w_0 and $\dim_{\mathbf{Q}(\xi)} L_\xi(\lambda)_{\lambda-\beta} = D_{r,\beta}$.

7.2. For $i \in [1,n]$, $k \in \mathbf{N}$, denote $\theta_{i,k}$ the $\mathfrak{k}$-linear homomorphism $\mathfrak{U}_\mathfrak{k} \to \mathfrak{U}_\mathfrak{k},\ x \to xf_i^{(k)}$. The kernel and the image of $\theta_{i,k}$ are easily described by means of PBW Theorem. Assume that $\lambda \in \mathbf{Z}^n_+$ is p^a-restricted. Let $s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression of the longest element of W. Set $k_h := \langle s_{i_{h-1}}\cdots s_{i_1}((p^a-1)\rho-\lambda), \alpha^\vee_{i_h}\rangle$, $\delta_h := k_1\alpha_{i_1}+\cdots+k_h\alpha_{i_h}$, $h = 1,\ldots,r$. Recall that for any $\gamma \in \mathbf{N}R^+$ we donte $\mathfrak{u}^-_{a,\gamma}$ the set of all elments in $\mathfrak{u}^-_a$ of degree γ. Given $\beta \in \mathbf{N}R^+$, set

$$\mathfrak{d}_{0,\beta} = \dim_{\mathbf{Q}(\xi)}\mathbf{u}^-_{\xi,\beta},$$

$$\mathfrak{d}_{1,\beta} = \dim_{\mathbf{Q}(\xi)}\theta_{i_1,k_1}(\mathbf{u}^-_{\xi,\beta}),$$

$$\mathfrak{d}_{2,\beta} = \min\{\mathfrak{d}_{1,\beta},\ \dim_{\mathbf{Q}(\xi)}\theta_{i_2,k_2}(\mathbf{u}^-_{\xi,\beta+\delta_1})\},$$

......

$$\mathfrak{d}_{h,\beta} = \min\{\mathfrak{d}_{h-1,\beta},\ \dim_{\mathbf{Q}(\xi)}\theta_{i_h,k_h}(\mathbf{u}^-_{\xi,\beta+\delta_{h-1}})\},$$

......

$$\mathfrak{d}_{r,\beta} = \min\{\mathfrak{d}_{r-1,\beta},\ \dim_{\mathbf{Q}(\xi)}\theta_{i_r,k_r}(\mathbf{u}^-_{\xi,\beta+\delta_{r-1}})\}.$$

Conjecture B. The number $\mathfrak{d}_{r,\beta}$ is independent of the choice of the reduced expression w_0 and $\dim_{\mathbf{Q}(\xi)} L_{\mathfrak{k},a}(\lambda)_{\lambda-\beta} = \mathfrak{d}_{r,\beta}$ provided that $p \geq$ the Coxeter number of the root system R associated to $\mathfrak{g}$.

7.3. Recall that $U_v = U$. We drop the index v and the index $\mathbf{1}$ in all notations involved them. So $V(\lambda)$ will stand for $V_v(\lambda)$. Let $\lambda,\mu \in \mathbf{Z}^n_+$. Assume that $\lambda \in \mathbf{Z}^n_+$ is $\mathbf{l}$-restricted. Given $w \in W$, set

$$H^w(\mathbf{l}\mu+\lambda) := \{yv_{\mathbf{l}\mu+\lambda} \mid y \in U,\ yx_{\kappa-\lambda,w} \in U_{\mathbf{Q}[v,v^{-1}]}\}.$$

Then $H^w(\lambda)$ is a free $\mathbf{Q}[v,v^{-1}]$-submodule of $V(\lambda)$.

Conjecture C. Essentially, $H^w(\mathbf{l}\mu+\lambda)$ is the free part of the cohomology group $H^{l(w)}(w(\mathbf{l}\mu+\lambda+\rho)-\rho)$ defined in [APW, section 3, p.22].

7.4. Keep the notations 7.3. Let $s_{i_k}\cdots s_{i_2}s_{i_1}$ be a reduced expression of w. Set $a_h := \langle s_{i_{h-1}}s_{i_{h-2}}\cdots s_{i_1}\lambda, \alpha^\vee_{i_h}\rangle$; $\nu_h := \langle \mathbf{l}\mu+\kappa, \alpha^\vee_{i_h}\rangle + a_h$; $d'_h = d_{i_h}$; $h = 1,\ldots,k$. Then set

$$a_{\lambda,w} := \begin{bmatrix}\nu_1\\ a_1\end{bmatrix}_{d'_1}\begin{bmatrix}\nu_2\\ a_2\end{bmatrix}_{d'_2}\cdots\begin{bmatrix}\nu_k\\ a_k\end{bmatrix}_{d'_k}.$$

Conjecture D. As $U_{\mathbf{Q}[v,v^{-1}]}$-modules, $U_{\mathbf{Q}[v,v^{-1}]}x'_{\lambda,w}v_{l\mu+\kappa+\lambda}/a_{\lambda,w}$ is isomorphic to $H^w(l\mu+\kappa+\lambda)$.

7.5. Keep the notations in 7.3. Let ϕ_l be the l-th cyclomatic polynomial (i.e. the minimal polynomial of ξ). For each integer $k \in \mathbf{N}$, set

$$M_k := \{yv_{l\mu+\lambda} \mid y \in U_{\mathbf{Q}[v,v^{-1}]},\ yx_{\kappa-\lambda} \in \phi_l^k U_{\mathbf{Q}[v,v^{-1}]}\}.$$

Conjecture E. (i). $M_{r+1} = 0$.

(ii).The filtration $0 = M_{r+1} \subseteq M_r \subseteq \cdots \subseteq M_1 \subseteq M_0 = U_{\mathbf{Q}[v,v^{-1}]}v_{l\mu+\lambda}$ is just the Jantzen filtration of $U_{\mathbf{Q}[v,v^{-1}]}v_{l\mu+\lambda}$.

7.6. Recall that in U^- a monomial of $F_i^{(k)}$, $i = 1, 2, ..., n$, $k \geq 0$, is called to be tight [L6, section 1] if the monomial is an element of the canonical basis of U^-.

Conjecture F. For each $\lambda \in \mathbf{Z}_+^n$ and $w \in W$, the monomials $x_{\lambda,w}, x'_{\lambda,w} \in U^-$ are tight.

Remark. It is enough to prove that x'_λ is tight.

7.7. Let ξ be a root of 1 of order ≥ 3. Assume that $\mathfrak{g}$ is simple. In $\mathbf{R}^n$, consider the hyperplanes

$$H_{\alpha,k} := \{e \in \mathbf{R}^n \mid \langle e+\rho, \alpha^\vee\rangle = kl_\alpha\}, \quad \alpha \in R^+, k \in \mathbf{Z}.$$

Denote $s_{\alpha,k}$ the corresponding reflections of $\mathbf{R}^n$, that is

$$s_{\alpha,k}(e) = e - (\langle e+\rho, \alpha^\vee\rangle - kl_\alpha)\alpha, \quad e \in \mathbf{R}^n.$$

These reflections generate an affine Weyl group W_l, which is the affine Weyl group associated to the Cartan matrix (a_{ij}) when $l_1 = ... = l_n$, the affine Weyl group associated to the transpose matrix of the Cartan matrix (a_{ij}) when $l_i \neq l_j$ for some i, j.

Conjecture G. The Conjecture 8.2 in [L2, p.75] is true in terms of W_l and U_ξ.

7.8. For Conjecture C - Conjecture E, one may states similar conjectures for $\mathfrak{U}_\mathbf{Z}$ and Weyl modules of $\mathfrak{U}$.

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