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Abstract

In [25] the first author started with the development of harmonic analysis on infinite-dimensional groups. In this article, following these ideas, we construct an analogue of quasi-regular representations, when the group G acts on a $G-$space X equipped with a quasi-invariant measure. For the group G we take the inductive limit of the general linear groups $\mathrm{GL}_0(2\infty,\mathbb{R}) = \varinjlim_n \mathrm{GL}(2n-1,\mathbb{R})$, acting on the space X_m of m rows, infinite in both directions, with Gaussian measure. This measure is the infinite tensor product of one-dimensional arbitrary Gaussian non-centered measures. We prove an irreducibility criterion for $m=3$. In 2019, the first author [28] established a criterion for $m\leq 2$. Our proof is in the same spirit, but the details are far more involved.

Keywords: infinite-dimensional groups, irreducible unitary representation, Koopman representation, Ismagilov's conjecture, quasi-invariant, ergodic measure
2008 MSC: 22E65, (28C20, 43A80, 58D20)

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To all fearless Ukrainians defending not only their country, but the whole civilization against putin's rashism

Contents

1. Representations of the inductive limit of the general linear groups $\mathbf{GL_0(2\infty,\mathbb{R})}$

1.1. Finite-dimensional case

Consider the space

$$X_{m,n}=\Big\{x=\sum_{1\leq k\leq m}\ \sum_{-n\leq r\leq n}x_{kr}E_{kr},\ \ x_{kr}\in\mathbb{R}\Big\},$$

where $E_{kn},\ k,n\in\mathbb{Z}$ are infinite matrix unities, with the measure (see (1.5))

$$\mu^{m,n}_{(b,a)}(x)=\otimes_{k=1}^{m}\otimes_{r=-n}^{n}\mu_{(b_{kr},a_{kr})}(x_{kr}).$$

Two groups act on the space $X_{m,n}$, namely $\mathrm{GL}(m,\mathbb{R})$ from the left, and $\mathrm{GL}(2n+1,\mathbb{R})$ from the right, and their actions commute. Therefore, two von Neumann algebras $\mathfrak{A}_{1,n}$ and $\mathfrak{A}_{2,n}$ in the Hilbert space $L^2(X_{m,n},\mu^{m,n}_{(b,a)})$ generated respectively by the left and the right actions of the corresponding groups have the property that $\mathfrak{A}'_{1,n}\subseteq\mathfrak{A}_{2,n}$, where $\mathfrak{A}'$ is a commutant of a von Neuman algebra $\mathfrak{A}$. We study what happens as $n\to\infty$. In the limit we obtain some unitary representation $T^{R,\mu,m}$ (see (1.6)) of the group $G:=\mathrm{GL}_0(2\infty,\mathbb{R})=\varinjlim_{n,i^s}\mathrm{GL}(2n+1,\mathbb{R})$ acting from the right on X_m. In the generic case, the representation $T^{R,\mu,m}$ is reducible. Indeed, if there exists a non-trivial element $s\in\mathrm{GL}(m,\mathbb{R})$ such the left action is *admissible* for the measure $\mu^m_{(b,a)}$, i.e., $(\mu^m_{(b,a)})^{L_s}\sim\mu^m_{(b,a)}$ the operator $T^{L,\mu,m}_s$ naturally associated with the left action, is well defined and $[T^{R,\mu,m}_t,T^{L,\mu,m}_s]=0$ for all $t\in G$, $s\in\mathrm{GL}(m,\mathbb{R})$. We use notation $\mu^f(\Delta)=\mu\big(f^{-1}(\Delta)\big)$ for $f:X\to X$, where Δ is some measurable set in X.

The main result of the article is the following. The representation $T^{R,\mu,m}$ is irreducible, see Theorem 1.1 if and only if *no left actions are admissible*, i.e., when $(\mu^m_{(b,a)})^{L_s}\perp\mu^m_{(b,a)}$ for all $s\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$. This is again a manifestation of the *Ismagilov conjecture*, see [27].

Here, as in the case of the regular [18, 19] and quasiregular [21, 22] representations of the group $B_0^{\mathbb{N}}$, which is an inductive limit of upper-triangular real matrices, we obtain the remarkable result that the *irreducible representations* can be obtained as the *inductive limit of reducible representations*!

1.2. Infinite-dimensional case

Let us denote by $\mathrm{Mat}(2\infty,\mathbb{R})$ the space of all real matrices that are infinite in both directions:

$$\mathrm{Mat}(2\infty,\mathbb{R})=\Big\{x=\sum_{k,n\in\mathbb{Z}}x_{kn}E_{kn},\ x_{kn}\in\mathbb{R}\Big\}. \tag{1.1}$$

The group $\mathrm{GL}_0(2\infty,\mathbb{R})=\varinjlim_{n,i^s}\mathrm{GL}(2n+1,\mathbb{R})$ is defined as the inductive limit of the general linear groups $G_n=\mathrm{GL}(2n+1,\mathbb{R})$ with respect to the *symmetric embedding* i^s:

$$G_n\ni x\mapsto i^s_{n+1}(x)=x+E_{-(n+1),-(n+1)}+E_{n+1,n+1}\in G_{n+1}. \tag{1.2}$$

For a fixed natural number m, consider a G-space X_m as the following subspace of the space $\mathrm{Mat}(2\infty,\mathbb{R})$:

$$X_m=\Big\{x\in\mathrm{Mat}(2\infty,\mathbb{R})\mid x=\sum_{k=1}^{m}\sum_{n\in\mathbb{Z}}x_{kn}E_{kn}\Big\}. \tag{1.3}$$

The group $\mathrm{GL}_0(2\infty,\mathbb{R})$ acts from the right on the space X_m. Namely, the right action of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ is correctly defined on the space X_m by the formula $R_t(x)=xt^{-1},\ t\in G,\ x\in X_m$. We define a Gaussian non-centered product measure $\mu:=\mu^m:=\mu^m_{(b,a)}$ on the space X_m :

$$\mu^m_{(b,a)}(x)=\otimes_{k=1}^m\otimes_{n\in\mathbb{Z}}\mu_{(b_{kn},a_{kn})}(x_{kn}), \tag{1.4}$$

where

$$d\mu_{(b_{kn},a_{kn})}(x_{kn})=\sqrt{\frac{b_{kn}}{\pi}}e^{-b_{kn}(x_{kn}-a_{kn})^2}dx_{kn} \tag{1.5}$$

and $b=(b_{kn})_{k,n},\ b_{kn}>0,\ a=(a_{kn})_{k,n},\ a_{kn}\in\mathbb{R},\ 1\le k\le m,\ n\in\mathbb{Z}$. Define the unitary representation $T^{R,\mu,m}$ of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ on the space $L^2(X_m,\mu^m_{(b,a)})$ by the formula:

$$(T^{R,\mu,m}_t f)(x)=\big(d\mu^m_{(b,a)}(xt)/d\mu^m_{(b,a)}(x)\big)^{1/2}f(xt),\ f\in L^2(X_m,\mu^m_{(b,a)}). \tag{1.6}$$

Obviously, the *centralizer* $Z_{\mathrm{Aut}(X_m)}(R(G))\subset\mathrm{Aut}(X_m)$ contains the group $L(\mathrm{GL}(m,\mathbb{R}))$, i.e., the image of the group $\mathrm{GL}(m,\mathbb{R})$ with respect to the left action $L:\mathrm{GL}(m,\mathbb{R})\to\mathrm{Aut}(X_m)$, $L_s(x)=sx,\ s\in\mathrm{GL}(m,\mathbb{R}),\ x\in X_m$. We prove the following theorem.

Theorem 1.1. *The representation* $T^{R,\mu,m}:\mathrm{GL}_0(2\infty,\mathbb{R})\to U\Big(L^2(X_m,\mu^m_{(b,a)})\Big)$ *is irreducible, for* $m=3$*, if and only if*

$$\begin{array}{ll} (i) & (\mu^m_{(b,a)})^{L_s}\perp\mu^m_{(b,a)} \quad \text{for all} \quad s\in \mathrm{GL}(m,\mathbb{R})\backslash\{e\}; \\ (ii) & \text{the measure} \quad \mu^m_{(b,a)} \quad \text{is } G\text{-ergodic.} \end{array}$$

In [27, 28] this result was proved for $m\le 2$. Note that conditions (i) and (ii) are necessary conditions for irreducibility.

Remark 1.1. Any Gaussian product-measure $\mu^m_{(b,a)}$ on X_m is $\mathrm{GL}_0(2\infty,\mathbb{R})$-right-ergodic [34, §3, Corollary 1], see Definition 2.1. For non-product-measures this is not true in general.

In order to study the condition $(\mu^m_{(b,a)})^{L_t}\perp\mu^m_{(b,a)}$ for $t\in\mathrm{GL}(m,\mathbb{R})\setminus\{e\}$ set

$$t=(t_{rs})_{r,s=1}^m\in \mathrm{GL}(m,\mathbb{R}),\quad B_n=\mathrm{diag}(b_{1n},b_{2n},...,b_{mn}),\quad X_n(t)=B_n^{1/2}tB_n^{-1/2}. \tag{1.7}$$

Let $M^{i_1i_2\dots i_r}_{j_1j_2\dots j_r}(t)$ be the *minors* of the matrix t with $i_1,i_2,...,i_r$ rows and $j_1,j_2,...,j_r$ columns, $1\le r\le m$. Let δ_{rs} be the Kronecker symbols.

Lemma 1.2 ([27], Lemma 10.2.3; [28], Lemma 2.2)**.** *For the measures* $\mu^m_{(b,a)}$*, with* m *a natural number, the relation*

$$(\mu^m_{(b,a)})^{L_t}\perp\mu^m_{(b,a)} \ \text{ for all } \ t\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$$

holds if and only if

$$\prod_{n\in\mathbb{Z}}\frac{1}{2^m|\det\, t|}\det\left(I+X_n^*(t)X_n(t)\right)+\sum_{n\in\mathbb{Z}}\sum_{r=1}^m b_{rn}\Big(\sum_{s=1}^m(t_{rs}-\delta_{rs})a_{sn}\Big)^2=\infty,$$

where

$$\det\Big(I+X_n^*(t)X_n(t)\Big)=1+\sum_{r=1}^m\sum_{\substack{1\le i_1<i_2<\dots<i_r\le m\\ 1\le j_1<j_2<\dots<j_r\le m}}\left(M^{i_1i_2\dots i_r}_{j_1j_2\dots j_r}(X_n(t))\right)^2.$$

For the convenience of the reader this lemma is proved in Section 8.1.

Remark 1.2. (The idea of the proof of irreducibility.) Let us denote by $\mathfrak{A}^m$ the *von Neumann algebra* generated by the representation $T^{R,\mu,m}$, i.e.,

$\mathfrak{A}^m = (T_t^{R,\mu,m} \mid t \in G)''$. For $\alpha = (\alpha_k) \in \{0,1\}^m$ define the von Neumann algebra $L^\infty_\alpha(X_m, \mu^m)$ as follows:

$$L^\infty_\alpha(X_m, \mu^m) = \Big(\exp(itB^\alpha_{kn}) \mid 1 \leq k \leq m,\ t \in \mathbb{R},\ n \in \mathbb{Z} \Big)'',$$

where $B^\alpha_{kn} = \begin{cases} x_{kn}, & \text{if} \quad \alpha_k = 0 \\ i^{-1} D_{kn}, & \text{if} \quad \alpha_k = 1 \end{cases}$ and $D_{kn} = \partial/\partial x_{kn} - b_{kn}(x_{kn} - a_{kn})$.

The proof of the irreducibility is based on four facts:

1) we can approximate by generators $A_{kn} = A^{R,m}_{kn} = \frac{d}{dt} T^{R,\mu,m}_{I+tE_{kn}}|_{t=0}$ the set of operators $(B^\alpha_{kn})^m_{k=1}$, $n \in \mathbb{Z}$ *for some* $\alpha \in \{0,1\}^m$ depending on the measure μ^m using the orthogonality condition $(\mu^m)^{L_s} \perp \mu^m$ for all $s \in \mathrm{GL}(m,\mathbb{R}) \setminus \{e\}$,

2) it is sufficient to verify the approximation only on the cyclic vector $\mathbf{1}(x) \equiv 1$, since the representation $T^{R,\mu,m}$ is *cyclic*,

3) the subalgebra $L^\infty_\alpha(X_m, \mu^m)$ is a *maximal abelian subalgebra* in $\mathfrak{A}^m$,

4) the measure μ^m is G-ergodic.

Here the *generators* A_{kn} are given by the formulas:

$$A_{kn} = \sum_{r=1}^m x_{rk} D_{rn}, \quad k, n \in \mathbb{Z}, \quad \text{where} \quad D_{kn} = \partial/\partial x_{kn} - b_{kn}(x_{kn} - a_{kn}).$$

Remark 1.3. *Scheme of the proof.* We prove the irreducibility as follows

$$\big(\mu^{L_s} \perp \mu \ \text{ for all } \ s \in \mathrm{GL}(3,\mathbb{R}) \setminus \{e\}\big) \Leftrightarrow \begin{pmatrix} \text{criteria} \\ \text{of} \\ \text{orthogonality} \end{pmatrix} \& \tag{1.8}$$

$$\begin{pmatrix} \text{Lemma 8.15} \\ \text{about} \\ \text{three vectors } f,g,h \notin l_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \text{some of} & \Delta^{(1)},\ \Delta_1 \\ \text{the expressions} & \Delta^{(2)},\ \Delta_2 \\ \text{are divergent:} & \Delta^{(3)},\ \Delta_3 \end{pmatrix} \Rightarrow \text{irreducibility},$$

$$\text{where} \quad \Delta^{(i)} := \Delta(Y^{(i)}_i, Y^{(i)}_j, Y^{(i)}_k), \quad \Delta_i := \Delta(Y_i, Y_j, Y_k), \tag{1.9}$$

$\Delta(f,g,h)$ is defined by (8.15), and $\{i,j,k\}$ is a cyclic permutation of $\{1,2,3\}$, see for details Lemmas 5.4–5.6, Lemmas 5.1–5.3 and Lemmas 5.14-5.15.

Remark 1.4. The fact that the conditions $(\mu^3)^{L_t} \perp \mu^3$ for all $t \in \mathrm{GL}(3,\mathbb{R}) \setminus \{e\}$ imply the possibility of the approximation of x_{kn} and D_{kn} by combinations of generators is based on some completely i*ndependent statement* about three infinite vectors $f, g, h \notin l_2$ such that

$$C_1 f + C_2 g + C_3 h \notin l_2 \quad \text{for arbitrary} \quad (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}, \tag{1.10}$$

see Lemma 8.11 for $m = 2$ [28] and Lemma 8.15 for $m = 3$. A similar result for general m is studied in [29]. These lemmas are the *key ingredients* of the proof of the irreducibility of the representation.

Remark 1.5. Note that in the case of the "nilpotent group" $B_0^{\mathbb{N}}$ and the infinite product of *arbitrary* Gaussian measures on $\mathbb{R}^m$ (see [2]) the proof of the *irreducibility* is also based on another completely *independent statement* namely, *the Hadamard – Fischer inequality*, see Lemma 1.3.

Lemma 1.3 (Hadamard – Fischer inequality [11, 12]). *For any positive definite matrix* $C \in \mathrm{Mat}(m,\mathbb{R})$, $m \in \mathbb{N}$ *and any two subsets* α *and* β *with* $\emptyset \subseteq \alpha,\ \beta \subseteq \{1,...,m\}$ *the following inequality holds:*

$$\begin{vmatrix} M(\alpha) & M(\alpha\bigcap\beta) \\ M(\alpha\bigcup\beta) & M(\beta) \end{vmatrix} = \begin{vmatrix} A(\hat{\alpha}) & A(\hat{\alpha}\bigcup\hat{\beta}) \\ A(\hat{\alpha}\bigcap\hat{\beta}) & A(\hat{\beta}) \end{vmatrix} \geq 0, \tag{1.11}$$

where $M(\alpha) = M_\alpha^\alpha(C),\ A(\alpha) = A_\alpha^\alpha(C)$ *and* $\hat{\alpha} = \{1,...,m\} \setminus \alpha$.

For the details see [11, p.573] and [12, Chapter 2.5, problem 36]. In [2] the conditions of orthogonality $\mu^{L_t} \perp \mu$ with respect to the left action of the group $B(m,\mathbb{R})$ on X^m were expressed as the divergence of some series, $S_{kn}^L(\mu) = \infty,\ 1 \leq k < n \leq m$. Conditions on the measure μ for the variables x_{kn} to be approximated by combinations of generators A_{pq} were expressed in terms of the divergence of another series Σ_{kn}. The proof of the fact that conditions $S_{kn}^L(\mu) = \infty,\ 1 \leq k < n \leq m$ imply the conditions $\Sigma_{kn} = \infty,\ 1 \leq k < n \leq m$ was based on the Hadamard – Fischer inequality.

2. Some orthogonality problem in measure theory

2.1. General setting

Our aim now is to find the minimal generating set of conditions of the orthogonality $(\mu_{(b,a)}^m)^{L_t} \perp \mu_{(b,a)}^m$ for all $t \in \mathrm{GL}(m,\mathbb{R}) \setminus \{e\}$. To be more precise, consider the following more general situation. Let $\alpha : G \to \mathrm{Aut}(X)$ be a *measurable action* of a group G on a measurable space (X,μ) with the following property: $\mu^{\alpha_t} \perp \mu$ for all $t \in G \setminus \{e\}$. Define a *generating subset* $G^\perp(\mu)$ in the group G as follows:

$$\text{if} \quad \mu^{\alpha_t} \perp \mu \ \text{ for all } \ t \in G^\perp(\mu), \quad \text{then} \quad \mu^{\alpha_t} \perp \mu \ \text{ for all } \ t \in G \setminus \{e\}. \tag{2.1}$$

Problem 2.1. *Find a* minimal generating subset $G_0^\perp(\mu)$ *satisfying* (2.1).

Definition 2.1. Recall that the probability measure μ on some G-space X is called *ergodic* if any function $f \in L^1(X,\mu)$ with property $f(\alpha_t(x)) = f(x) \bmod \mu$ is constant.

2.2. Orthogonality criteria $\mu^{L_t} \perp \mu$ *for* $t \in \mathrm{GL}(2,\mathbb{R}) \setminus \{e\}$

Remark 2.1. By Lemma 4.1 proved in [28] or Lemma 10.4.1 in [27] for $m = 2$ we conclude that the minimal generating set $G_0^{\perp}(\mu) = \mathrm{GL}(2,\mathbb{R})_0^{\perp}(\mu)$ (see Problem (2.1)) is reduced to the following subgroups, families and elements:

$$\exp(tE_{12}) = I + tE_{12} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \ \exp(tE_{21}) = I + tE_{21} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad (2.2)$$

$$\exp(tE_{12})P_1 = \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}, \quad \exp(tE_{21})P_2 = \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}, \quad (2.3)$$

$$\tau_-(\phi, s) = \begin{pmatrix} \cos\phi & s^2\sin\phi \\ s^{-2}\sin\phi & -\cos\phi \end{pmatrix} = D_2(s) \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} D_2^{-1}(s)P_2. \quad (2.4)$$

The families (2.2), (2.3) are one-parameter, the family (2.4) is two-parameter. All elements are of order 2 except the elements in subgroups given in (2.2)! It suffices to verify the conditions (2.2) only for some $t \in \mathbb{R} \setminus \{0\}$. The family $\tau_-(\phi, s)$, actually, coincides with $D_2(s)\mathrm{O}(2)D_2^{-1}(s)P_2$, where $D_2(s) = \mathrm{diag}(s, s^{-1})$. All points t in (2.3) and all points (ϕ, s) in (2.4) are essential, i.e., we can not remove any single point.

Remark 2.2. We note [16, Chapter V, §8 Problems, 2, p. 147] that every element of $\mathrm{SL}(2,\mathbb{R})$ is conjugate to at least one matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \ a \neq 0, \ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix}, \ \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

Remark 2.3. Some elements $a = gP_r$ in the set $G_0^{\perp}(\mu) = \mathrm{GL}(2,\mathbb{R})_0^{\perp}(\mu)$ are of order 2 (see Remark 2.1):

$$a^2 = (gP_r)^2 = 1. \quad (2.5)$$

This follows from the relation

$$P_r g P_r = g^{-1}. \quad (2.6)$$

To see this we note that if (2.6) holds, then we get (2.5):

$$a^2 = (gP_r)^2 = gP_r gP_r = gg^{-1} = 1.$$

For example, for $g=\exp(tE_{12})$ we get $P_1gP_1=g^{-1}$, for $g=\exp(tE_{21})$ we get $P_2gP_2=g^{-1}$ and for $g=\tau(\phi,s)$ we get $P_2gP_2=g^{-1}$. See (2.3) and (2.4) for details, where

$$\tau_-(\phi,s)=\left(\begin{array}{cc}\cos\phi & s^2\sin\phi\\ s^{-2}\sin\phi & -\cos\phi\end{array}\right),\quad \tau(\phi,s)=\left(\begin{array}{cc}\cos\phi & -s^2\sin\phi\\ s^{-2}\sin\phi & \cos\phi\end{array}\right).$$

We recall some useful lemmas from [28].

Lemma 2.2. *For* $t=\left(\begin{smallmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{smallmatrix}\right)\in\mathrm{GL}(2,\mathbb{R})\setminus\{e\}$ *we have, if* $\det t>0$,

$$(\mu^2_{(b,0)})^{L_t}\perp\mu^2_{(b,0)}\quad\Leftrightarrow$$

$$\sum_{n\in\mathbb{Z}}\left[(1-\mid\det t\mid)^2+(t_{11}-t_{22})^2+\left(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}+t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2\right]=\infty.\qquad(2.7)$$

If $\det t<0$ *we have*

$$(\mu^2_{(b,0)})^{L_t}\perp\mu^2_{(b,0)}\quad\Leftrightarrow$$

$$\sum_{n\in\mathbb{Z}}\left[(1-\mid\det t\mid)^2+(t_{11}+t_{22})^2+\left(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}-t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2\right]=\infty.\qquad(2.8)$$

Lemma 2.3. *For* $t\in\mathrm{GL}(2,\mathbb{R})\setminus\{e\}$ *we have*

$$(\mu^2_{(b,a)})^{L_t}\perp\mu^2_{(b,a)}\quad\text{if}\quad|\det t|\neq 1.$$

If $\det t=1$, *we have*

$$(\mu^2_{(b,a)})^{L_t}\perp\mu^2_{(b,a)}\quad\Leftrightarrow\quad\Sigma^+(t):=\Sigma^+_1(t)+\Sigma_2(t)=\infty.$$

If $\det t=-1$, *we have*

$$(\mu^2_{(b,a)})^{L_t}\perp\mu^2_{(b,a)}\quad\Leftrightarrow\quad\Sigma^-(t):=\Sigma^-_1(t)+\Sigma_2(t)=\infty,$$

where

$$\Sigma^+_1(t)=\sum_{n\in\mathbb{Z}}\left[(t_{11}-t_{22})^2+\left(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}+t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2\right],$$

$$\Sigma^-_1(t)=\sum_{n\in\mathbb{Z}}\left[(t_{11}+t_{22})^2+\left(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}-t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2\right],$$

$$\Sigma_2(t^{-1})=\sum_{n\in\mathbb{Z}}\left(b_{1n}\big[(t_{11}-1)a_{1n}+t_{12}a_{2n}\big]^2+b_{2n}\big[t_{21}a_{1n}+(t_{22}-1)a_{2n}\big]^2\right).\quad(2.9)$$

Remark 2.4. By Lemma 2.3 we have

$$(\mu^2_{(b,a)})^{L_t} \perp \mu^2_{(b,a)} \quad \text{for} \quad t \in \mathrm{GL}(2,\mathbb{R}) \setminus \{e\}$$

if and only if this holds for two subsets of the group $\pm\mathrm{SL}(2,\mathbb{R})$ defined as follows:

$$G_2^+ = \{t \in \mathrm{SL}(2,\mathbb{R}) \mid t_{11} = A_1^1(t)\}, \tag{2.10}$$

$$G_2^- = \{t \in -\mathrm{SL}(2,\mathbb{R}) \mid t_{11} = -A_1^1(t)\}. \tag{2.11}$$

The set G_2^+ is reduced to two families of one-parameter subgroups (2.2). The set G_2^- is reduced to the one-parameter family (2.3), the reflections of (2.2) by P_2, and two parameter family (2.4) of elements from $D_2(s)\mathrm{O}(2)D_2^{-1}(s)P_2$.

Lemma 2.4. *If $t \in G_2^+$ we have*

$$(\mu^2_{(b,a)})^{L_t} \perp \mu^2_{(b,a)} \quad \Leftrightarrow \quad \Sigma^+(t) := \Sigma_1^+(t) + \Sigma_2(t) = \infty.$$

If $t \in G_2^-$ we have

$$(\mu^2_{(b,a)})^{L_t} \perp \mu^2_{(b,a)} \quad \Leftrightarrow \quad \Sigma^-(t) := \Sigma_1^-(t) + \Sigma_2(t) = \infty,$$

where $\Sigma_2(t^{-1})$ is defined by (2.9) and

$$\Sigma_1^+(t) = \sum_{n\in\mathbb{Z}} \left(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}} + t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2, \tag{2.12}$$

$$\Sigma_1^-(t) = \sum_{n\in\mathbb{Z}} \left(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}} - t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2. \tag{2.13}$$

The conditions of orthogonality with respect to elements defined by (2.2)–(2.4) are transformed in the divergence of the following series:

$$S^L_{12}(\mu) = \sum_{n\in\mathbb{Z}} \frac{b_{1n}}{2}\left(\frac{1}{2b_{2n}} + a_{2n}^2\right), \quad S^L_{21}(\mu) = \sum_{n\in\mathbb{Z}} \frac{b_{2n}}{2}\left(\frac{1}{2b_{1n}} + a_{1n}^2\right), \tag{2.14}$$

$$S^{L,-}_{kn}(\mu,t) = \frac{t^2}{4}\sum_{m\in\mathbb{Z}} \frac{b_{km}}{b_{nm}} + \sum_{m\in\mathbb{Z}} \frac{b_{km}}{2}(-2a_{km} + t a_{nm})^2, \ t \in \mathbb{R}, \tag{2.15}$$

$$\Sigma^-_{12}\big(\tau_-(\phi,s)\big) = \sin^2\phi\, \Sigma_1(s) + \Sigma_2^-\big(\tau_-(\phi,s)\big), \ \phi \in [0,2\pi), \ s > 0, \tag{2.16}$$

$$\text{where}\quad \Sigma_1(s):=\sum_{n\in\mathbb{Z}}\left(s^2\sqrt{\frac{b_{1n}}{b_{2n}}}-s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2, \tag{2.17}$$

$$\Sigma_2^-\big(\tau_-(\phi,s)\big):=\sum_{n\in\mathbb{Z}}\left(4b_{1n}\sin^2\frac{\phi}{2}+4s^{-4}b_{2n}\cos^2\frac{\phi}{2}\right)\left(a_{1n}\sin\frac{\phi}{2}-s^2a_{2n}\cos\frac{\phi}{2}\right)^2. \tag{2.18}$$

Recall Remark 4.3 from [28].

Remark 2.5. The following three conditions are equivalent:

$$\begin{array}{lll} (i) & \mu^{L_{\tau_-(\phi,s)}}\perp\mu, & \phi\in[0,2\pi),\ s>0,\\ (ii) & \Sigma_{12}^-\big(\tau_-(\phi,s)\big)=\sin^2\phi\,\Sigma_1(s)+\Sigma_2^-\big(\tau_-(\phi,s)\big)=\infty, & \phi\in[0,2\pi),\ s>0,\\ (iii) & \Sigma_1(s)+\Sigma_2(C_1,C_2)=\infty, & s>0,\ (C_1,C_2)\in\mathbb{R}^2\setminus\{0\}, \end{array}$$

where $\Sigma_1(s)$ is defined by (2.17) and

$$\Sigma_2(C_1,C_2):=\sum_{n\in\mathbb{Z}}(C_1^2b_{1n}+C_2^2b_{2n})(C_1a_{1n}+C_2a_{2n})^2. \tag{2.19}$$

2.3. Equivalent series and equivalent sequences

There is an extensive theory of convergent and divergent series. In our context we are only interested in when a series with positive coefficients is divergent or convergent.

Definition 2.2. We say that two series $\sum_{n\in\mathbb{N}}a_n$ and $\sum_{n\in\mathbb{N}}b_n$ with positive a_n, b_n are *equivalent* if they are divergent or convergent simultaneously. We will write $\sum_{n\in\mathbb{N}}a_n\sim\sum_{n\in\mathbb{N}}b_n$. We say that two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are *equivalent* if for some $C_1,C_2>0$ we have $C_1b_n\le a_n\le C_2b_n$ for all $n\in\mathbb{N}$. We will use the same notation $a_n\sim b_n$.

Lemma 2.5. *Let $1+c_n>0$ for all $n\in\mathbb{Z}$. Then two series are equivalent:*

$$\Sigma_1:=\sum_{n\in\mathbb{Z}}\frac{c_n^2}{1+c_n},\quad \Sigma_2:=\sum_{n\in\mathbb{Z}}c_n^2. \tag{2.20}$$

Proof. Fix some $\varepsilon\in(0,1)$ and a big N. We have three cases:
(a) $1+c_n\in(\varepsilon,N)$,
(b) for infinite subset $\mathbb{Z}_1$ we have $\lim_{n\in\mathbb{Z}_1}c_n=\infty$,
(c) for infinite subset $\mathbb{Z}_1$ we have $\lim_{n\in\mathbb{Z}_1}(1+c_n)=0$.

In the case (a) we have

$$\frac{1}{N}\sum_{n\in\mathbb{Z}} c_n^2 < \sum_{n\in\mathbb{Z}} \frac{c_n^2}{1+c_n} < \frac{1}{\varepsilon}\sum_{n\in\mathbb{Z}} c_n^2, \tag{2.21}$$

In the case (b) and (c) both series are divergent. □

We will make systematic use of the following statement.

Remark 2.6 ([27]). Let $a_n, b_n > 0$ for all $n \in \mathbb{N}$. The following two series are equivalent

$$\sum_{n\in\mathbb{N}} \frac{a_n}{a_n+b_n} \sim \sum_{n\in\mathbb{N}} \frac{a_n}{b_n}. \tag{2.22}$$

2.4. Orthogonality criteria $\mu^{L_t} \perp \mu$ *for* $t \in \mathrm{GL}(3,\mathbb{R}) \setminus \{e\}$

For $m = 2$ and $\det t > 0$ we have, here $H_{m,n}(t)$ is defined by (8.8)

$$2^2 \mid \det t \mid \left(H_{2,n}^{-2}(t) - 1\right) = \left[(1- \mid \det t \mid)^2 + (t_{11}-t_{22})^2 + \left(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}} + t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2\right] =$$
$$\left[\left(M_\emptyset^\emptyset(X(t)) - A_\emptyset^\emptyset(X(t))\right)^2 + \left(M_1^1(X(t)) - A_1^1(X(t))\right)^2 + \left(M_2^1(X(t)) - A_2^1(X(t))\right)^2\right].$$

For $m = 3$ using (1.7) we have $X(t) = B^{1/2}tB^{-1/2}$, hence

$$X(t) = \begin{pmatrix} b_{1n} & 0 & 0 \\ 0 & b_{2n} & 0 \\ 0 & 0 & b_{3n} \end{pmatrix}^{1/2} \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} b_{1n} & 0 & 0 \\ 0 & b_{2n} & 0 \\ 0 & 0 & b_{3n} \end{pmatrix}^{-1/2} =$$

$$\begin{pmatrix} t_{11} & \sqrt{\frac{b_{1n}}{b_{2n}}}t_{12} & \sqrt{\frac{b_{1n}}{b_{3n}}}t_{13} \\ \sqrt{\frac{b_{2n}}{b_{1n}}}t_{21} & t_{22} & \sqrt{\frac{b_{2n}}{b_{3n}}}t_{23} \\ \sqrt{\frac{b_{3n}}{b_{1n}}}t_{31} & \sqrt{\frac{b_{3n}}{b_{2n}}}t_{32} & t_{33} \end{pmatrix}.$$

Therefore, using (7.8) and the fact that $X = X^*(t)X(t)$ we obtain

$$2^3 \mid \det t \mid H_{3,n}^{-2}(t) = \Big(1 + \mid \det t \mid^2 + t_{11}^2 + \frac{b_{1n}}{b_{2n}}t_{12}^2 + \frac{b_{1n}}{b_{3n}}t_{13}^2 + \frac{b_{2n}}{b_{1n}}t_{21}^2$$
$$+ t_{22}^2 + \frac{b_{2n}}{b_{3n}}t_{23}^2 + \frac{b_{3n}}{b_{1n}}t_{31}^2 + \frac{b_{3n}}{b_{2n}}t_{32}^2 + t_{33}^2 + (M_{12}^{12}(t))^2 + \frac{b_{2n}}{b_{3n}}(M_{13}^{12}(t))^2 +$$

$$
\frac{b_{1n}}{b_{3n}}(M^{12}_{23}(t))^2+\frac{b_{3n}}{b_{2n}}(M^{13}_{12}(t))^2+(M^{13}_{13}(t))^2+\frac{b_{1n}}{b_{2n}}(M^{13}_{23}(t))^2
+\frac{b_{3n}}{b_{1n}}(M^{23}_{12}(t))^2+\frac{b_{2n}}{b_{2n}}(M^{23}_{13}(t))^2+(M^{23}_{23}(t))^2\Big)
$$

$$
=1+\mid\det t\mid^2+\sum_{1\le i\le j\le 3}\Bigg[\left(t^i_j\sqrt{\frac{b_{in}}{b_{jn}}}\right)^2+\left(A^i_j\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2\Bigg]
$$

$$
=1+\mid\det t\mid^2+\sum_{1\le i\le j\le 3}\left(|M^i_j(X(t))|^2+|A^i_j(X(t))|^2\right).
$$

Using the notation $t^i_j=t_{ij}$ and the fact

$$
\det t=t^k_1A^k_1+t^k_2A^k_2+t^k_3A^k_3,\quad k=1,2,3,
$$

we get

$$
2^3\mid\det t\mid\left(H^{-2}_{3,n}(t)-1\right)=(1-\mid\det t\mid)^2+\sum_{1\le i,j\le 3}\Big(M^i_j(X(t))-A^i_j(X(t))\Big)^2
$$

$$
=(1-\mid\det t\mid)^2+\sum_{1\le i\le j\le 3}\left(t^i_j\sqrt{\frac{b_{in}}{b_{jn}}}-A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2. \tag{2.23}
$$

Similar to [28, Lemmas 2.22] in the case $m=2$, or [27, Lemma 10.4.30] we get the following lemma, for $m=3$.

Lemma 2.6. *For $t\in\mathrm{GL}(3,\mathbb{R})\setminus\{e\}$ we have, if $\det t>0$,*

$$
(\mu^3_{(b,0)})^{L_t}\perp\mu^3_{(b,0)}\quad\Leftrightarrow
$$

$$
\sum_{n\in\mathbb{Z}}\Bigg[(1-\mid\det t\mid)^2+\sum_{1\le i\le 3}\Big(t^i_i-A^i_i(t)\Big)^2+\sum_{1\le i<j\le 3}\left(t^i_j\sqrt{\frac{b_{in}}{b_{jn}}}-A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2\Bigg]=\infty. \tag{2.24}
$$

If $\det t<0$ we have

$$
(\mu^3_{(b,0)})^{L_t}\perp\mu^3_{(b,0)}\quad\Leftrightarrow
$$

$$
\sum_{n\in\mathbb{Z}}\Bigg[(1-\mid\det t\mid)^2+\sum_{1\le i\le 3}\Big(t^i_i+A^i_i(t)\Big)^2+\sum_{1\le i<j\le 3}\left(t^i_j\sqrt{\frac{b_{in}}{b_{jn}}}+A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2\Bigg]=\infty. \tag{2.25}
$$

By Lemma 8.6 and (8.9) the following lemma holds true.

Lemma 2.7. *For $t\in \mathrm{GL}(3,\mathbb{R})\setminus\{e\}$ we have*

$$(\mu^3_{(b,a)})^{L_t}\perp \mu^3_{(b,a)}\quad \textit{if}\quad \mid \det t\mid\neq 1.$$

If $\det t=1$, *we have*

$$(\mu^3_{(b,a)})^{L_t}\perp \mu^3_{(b,a)}\quad\Leftrightarrow\quad \Sigma^+(t):=\Sigma^+_1(t)+\Sigma_2(t)=\infty.$$

If $\det t=-1$, *we have*

$$(\mu^3_{(b,a)})^{L_t}\perp \mu^3_{(b,a)}\quad\Leftrightarrow\quad \Sigma^-(t):=\Sigma^-_1(t)+\Sigma_2(t)=\infty,$$

where

$$\Sigma^+_1(t)=\sum_{n\in\mathbb{Z}}\Big[\sum_{k=1}^3(t_{kk}-A^k_k(t))^2+\sum_{1\leq i<j\leq 3}\Big(t^i_j\sqrt{\frac{b_{in}}{b_{jn}}}-A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\Big)^2\Big],\tag{2.26}$$

$$\Sigma^-_1(t)=\sum_{n\in\mathbb{Z}}\Big[\sum_{k=1}^3(t_{kk}+A^k_k(t))^2+\sum_{1\leq i<j\leq 3}\Big(t^i_j\sqrt{\frac{b_{in}}{b_{jn}}}+A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\Big)^2\Big],\tag{2.27}$$

$$\Sigma_2(t^{-1})=\sum_{n\in\mathbb{Z}}\Big[b_{1n}\big((t_{11}-1)a_{1n}+t_{12}a_{2n}+t_{13}a_{3n}\big)^2+\tag{2.28}$$

$$b_{2n}\big(t_{21}a_{1n}+(t_{22}-1)a_{2n}+t_{23}a_{3n}\big)^2+b_{3n}\big(t_{31}a_{1n}+t_{32}a_{2n}+(t_{33}-1)a_{3n}\big)^2\Big].$$

Remark 2.7. By Lemma 2.7, it suffices to verify, the condition of orthogonality

$$(\mu^3_{(b,a)})^{L_t}\perp \mu^3_{(b,a)}\quad\text{for}\quad t\in \mathrm{GL}(3,\mathbb{R})\setminus\{e\}$$

for the following two subsets of the group $\pm\mathrm{SL}(3,\mathbb{R})$:

$$G^+_3:=\{t\in \mathrm{SL}(3,\mathbb{R})\mid t_{kk}=A^k_k(t),\ 1\leq k\leq 3\},\tag{2.29}$$
$$G^-_3:=\{t\in -\mathrm{SL}(3,\mathbb{R})\mid t_{kk}=-A^k_k(t),\ 1\leq k\leq 3\}.\tag{2.30}$$

Lemma 2.8. *If $t \in G_3^{\pm}$, we have respectively*

$$(\mu_{(b,a)}^3)^{L_t} \perp \mu_{(b,a)}^3 \quad \Leftrightarrow \quad \Sigma^{\pm}(t) = \Sigma_1^{\pm}(t) + \Sigma_2(t) = \infty,$$

$$\Sigma_1^{+}(t) = \sum_{1\leq i<j\leq 3} \sum_{n\in\mathbb{Z}} \left(t_j^i \sqrt{\frac{b_{in}}{b_{jn}}} - A_j^i(t)\sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 = \sum_{1\leq i<j\leq 3} \Sigma_{ij}^{+}(t), \tag{2.31}$$

$$\Sigma_1^{-}(t) = \sum_{1\leq i<j\leq 3} \sum_{n\in\mathbb{Z}} \left(t_j^i \sqrt{\frac{b_{in}}{b_{jn}}} + A_j^i(t)\sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 = \sum_{1\leq i<j\leq 3} \Sigma_{ij}^{-}(t), \tag{2.32}$$

$$\Sigma_{ij}^{\pm}(t) = \sum_{n\in\mathbb{Z}} \left(t_j^i \sqrt{\frac{b_{in}}{b_{jn}}} \mp A_j^i(t)\sqrt{\frac{b_{jn}}{b_{in}}} \right)^2, \tag{2.33}$$

where $\Sigma_2(t)$ is defined by (2.28).

Remark 2.8. We note that the *Iwasawa decomposition* holds for $\mathrm{SL}(3,\mathbb{R})$, i.e., $\mathrm{SL}(3,\mathbb{R}) = KAN$, where $K = \mathrm{O}(3)$,

$$A = \left\{ D_3(s) = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}, \ \det D_3(s) = 1 \right\}, \ N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}. \tag{2.34}$$

Next we will show that the set G_3^{+} can be reduced to the six family of one-parameter subgroups $\exp(tE_{kr})$, $1 \leq k \neq r \leq 3$, see (2.36), or three families of two-parameter subgroups, see (2.37). The set G_3^{-} can be reduced to the three two-parameter family (2.38) reflections of (2.37) by P_r. The remaining part is reduced to the sets $D_3(s)\mathrm{O}(3)D_3^{-1}(s)P_r$ or five parameter family of elements $\tau_r(t,s) = D_3(s)tD_3^{-1}(s)P_r$, see (2.42).

Lemma 2.9. *In case $m = 3$ the* minimal generating set $\mathrm{GL}(3,\mathbb{R})_0^{\perp}(\mu)$ *is defined as follows (compare with Remark 2.1) :*

$$\mathrm{GL}(3,\mathbb{R})_0^{\perp}(\mu) = \{e_r(t,s), e_r(t,s)P_r, \mid 1 \leq r \leq 3, \ (t,s) \in \mathbb{R}^2\}$$
$$\cup \{\mathrm{O}_r^A(3), \ 1 \leq r \leq 3\}, \quad \textit{where} \tag{2.35}$$

$$e_{kn}(t) := \exp(tE_{kn}) = I + tE_{kn}, \ 1 \leq k \neq n \leq 3, \ t \in \mathbb{R}, \tag{2.36}$$

$$e_1(t,s) = \begin{pmatrix} 1 & t & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ e_2(t,s) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \ e_3(t,s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & s & 1 \end{pmatrix}, \tag{2.37}$$

$$e_1(t,s)P_1=\begin{pmatrix} -1 & t & s\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ e_2(t,s)P_2=\begin{pmatrix} 1 & 0 & 0\\ t & -1 & s\\ 0 & 0 & 1\end{pmatrix},\ e_3(t,s)P_3=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ t & s & -1\end{pmatrix}, \quad (2.38)$$

$$P_1=\begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad P_2=\begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad P_3=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad (2.39)$$

$$\mathrm{O}^A(3) := \{D_3(s)\mathrm{O}(3)D_3^{-1}(s) \mid D_3(s)\in A\}, \quad (2.40)$$

$$\mathrm{O}^A_r(3) := \{D_3(s)\mathrm{O}(3)D_3^{-1}(s)P_r \mid D_3(s)\in A\},\ 1\le r\le 3, \quad (2.41)$$

$$\tau_r(t,s) := D_3(s)tD_3^{-1}(s)P_r, \quad t\in O(3),\ D_3(s)=\mathrm{diag}(s_1,s_2,s_3)\in A, \quad (2.42)$$

and A is defined by (2.34).

The families (2.36) give us respectively the divergence of the following series:

$$S^L_{kr}(\mu)=\sum_{n\in\mathbb{Z}}\frac{b_{kn}}{2}\Big(\frac{1}{2b_{rn}}+a^2_{rn}\Big), \quad 1\le k,r\le 3,\ k\ne r. \quad (2.43)$$

The families (2.37) give us, respectively, the divergence of the following series:

$$S^L_{1,23}(\mu,t,s)=\sum_{n\in\mathbb{Z}}\left[\frac{t^2}{4}\frac{b_{1n}}{b_{2n}}+\frac{s^2}{4}\frac{b_{1n}}{b_{3n}}+\frac{b_{1n}}{2}\big(-2a_{1n}+ta_{2n}+sa_{3n}\big)^2\right], \quad (2.44)$$

$$S^L_{2,13}(\mu,t,s)=\sum_{n\in\mathbb{Z}}\left[\frac{t^2}{4}\frac{b_{2n}}{b_{1n}}+\frac{s^2}{4}\frac{b_{2n}}{b_{3n}}+\frac{b_{2n}}{2}\big(ta_{1n}-2a_{2n}+sa_{3n}\big)^2\right], \quad (2.45)$$

$$S^L_{3,12}(\mu,t,s)=\sum_{n\in\mathbb{Z}}\left[\frac{t^2}{4}\frac{b_{3n}}{b_{1n}}+\frac{s^2}{4}\frac{b_{3n}}{b_{2n}}+\frac{b_{3n}}{2}\big(ta_{1n}+sa_{2n}-2a_{3n}\big)^2\right]. \quad (2.46)$$

The families (2.42) give us the conditions (2.50), see Lemma 2.10 below.

Proof. Consider the subset $\mathrm{GL}(3,\mathbb{R})_0^{\perp}(\mu)$ of $\mathrm{GL}(3,\mathbb{R})$ described by (2.35). The fact that this set is *minimal generating* will follow from Lemma 4.1, more precisely, from the following implications:

$$\Big(\mu^{L_t}\perp\mu\ \text{ for all }\ t\in\mathrm{GL}(3,\mathbb{R})_0^{\perp}(\mu)\Big)\Rightarrow\Big(\text{irreducibility}\Big) \quad (2.47)$$
$$\Rightarrow\Big(\mu^{L_t}\perp\mu\ \text{ for all }\ t\in\mathrm{GL}(3,\mathbb{R})\setminus\{e\}\Big).$$

The first implication is just Lemma 4.1. The second implication follows from irreducibility. Indeed, suppose that $\mathrm{GL}(3,\mathbb{R})_0^{\perp}(\mu)$ is not a minimal generating set. Then we can find an $s\in\mathrm{GL}(3,\mathbb{R})\setminus\{e\}$ having the property

$$\big(\mu^3_{(b,a)}\big)^{L_s}\sim\mu^3_{(b,a)}.$$

Hence the non-trivial operator $T_s^{L,\mu,3}$ can be defined by

$$(T_s^{L,\mu,3}f)(x)=\big(d\mu^3_{(b,a)}(s^{-1}x)/d\mu^3_{(b,a)}(x)\big)^{1/2}f(s^{-1}x),\ \ f\in L^2(X_3,\mu^3_{(b,a)}). \tag{2.48}$$

This operator commutes with the representations $T^{R,\mu,3}$:

$$[T_t^{R,\mu,3},T_s^{L,\mu,3}]=0\quad \text{for all}\quad t\in G,$$

contradicting the irreducibility.

The relations (2.43)–(2.46) follows from (2.26)–(2.28). The relation (2.45), for example, follows from (2.27) and (2.28). The relation (2.50) we obtain from (2.26) for $\tau_r(t,s),\ t\in\mathrm{O}(3),\ s\in\big(\mathbb{R}^*\big)^3$ defined by

$$\tau_r(t,s)=D_3(s)tD_3^{-1}(s)P_r,\quad \text{where}\quad D_3(s)=\mathrm{diag}(s_1,s_2,s_3). \tag{2.49}$$

□

Lemma 2.10. *Set*

$$\tau(s,t):=D_3(s)tD_3^{-1}(s),\quad \tau_r(s,t):=\tau(s,t)P_r$$

for $t\in\pm\mathrm{O}(3),\ D_3(s)=\mathrm{diag}(s_1,s_2,s_3),\ s=(s_1,s_2,s_3)\in(\mathbb{R}^*)^3$ *and* $1\le r\le 3$. *Then*

$$\big(\mu^3_{(b,a)}\big)^{L_{\tau_r(s,t)}}\perp\mu^3_{(b,a)}\Leftrightarrow\Sigma_1^{\pm}\big(\tau_r(s,t)\big)+\Sigma_2\big(\tau_r(s,t)\big)=\infty, \tag{2.50}$$

where $\Sigma_1^{\pm}(t)$ *are defined by* (2.31), (2.32) *and* $\Sigma_2(t)$ *is defined by* (2.28).

In particular, if we denote $s_{ij}=s_is_j^{-1}$ *we get*

$$\Sigma_1^{+}\big(\tau(t,s)\big)=\Sigma_1^{+}(t,s)=t_{12}^2\Sigma_{12}(s_{12}^{1/2})+t_{13}^2\Sigma_{13}(s_{13}^{1/2})+t_{23}^2\Sigma_{23}(s_{23}^{1/2}). \tag{2.51}$$

Proof. For $T:=\tau(s,t)$ and $T(3):=\tau_3(s,t)$ we have respectively:

$$T=D_3(s)tD_3^{-1}(s)=\begin{pmatrix}t_{11}&\frac{s_1}{s_2}t_{12}&\frac{s_1}{s_3}t_{13}\\ \frac{s_2}{s_1}t_{21}&t_{22}&\frac{s_2}{s_3}t_{23}\\ \frac{s_3}{s_1}t_{31}&\frac{s_3}{s_2}t_{32}&t_{33}\end{pmatrix}, \tag{2.52}$$

$$\begin{pmatrix}t_{11}&\frac{s_1}{s_2}t_{12}&-\frac{s_1}{s_3}t_{13}\\ \frac{s_2}{s_1}t_{21}&t_{22}&-\frac{s_2}{s_3}t_{23}\\ \frac{s_3}{s_1}t_{31}&\frac{s_3}{s_2}t_{32}&-t_{33}\end{pmatrix}=D_3(s)tD_3^{-1}(s)P_3=:T(3). \tag{2.53}$$

By Lemma 2.11 we have for $t\in\mathrm{O}(3)$

$$t_{kr}=A_r^k(t),\ 1\le k,r\le 3. \tag{2.54}$$

Therefore, for T and $T(3)$ we have for $1\leq k,r\leq 3$:

$$M^k_r(T)=T_{kr}=\frac{s_k}{s_r}t_{kr},\ \ A^k_r(T)=\frac{s_r}{s_k}A^k_r(t)\overset{(2.57)}{=}\frac{s_r}{s_k}t_{kr},\quad M^k_r(T(3)) \tag{2.55}$$

$$=(-1)^{\delta_{3,r}}\frac{s_k}{s_r}t_{kr},\ \ A^k_r(T(3))=(-1)^{\delta_{3,r}}\frac{s_r}{s_k}A^k_r(t)=(-1)^{\delta_{3,r}}\frac{s_r}{s_k}t_{kr}. \tag{2.56}$$

Finally, we get

$$\Sigma^+_1(T)=\Sigma^+_1\big(\tau(s,t)\big)=\sum_{n\in\mathbb{Z}}\Big[\sum_{1\leq i<j\leq 3}\Big(M^i_j(T)\sqrt{\frac{b_{in}}{b_{jn}}}-A^i_j(T)\sqrt{\frac{b_{jn}}{b_{in}}}\Big)^2\Big]$$

$$=\sum_{n\in\mathbb{Z}}\Big[t^2_{12}\Big(s_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}-s_{12}^{-1}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2+t^2_{13}\Big(s_{13}\sqrt{\frac{b_{1n}}{b_{3n}}}-s_{13}^{-1}\sqrt{\frac{b_{3n}}{b_{1n}}}\Big)^2+$$

$$t^2_{23}\Big(s_{23}\sqrt{\frac{b_{2n}}{b_{3n}}}-s_{23}^{-1}\sqrt{\frac{b_{3n}}{b_{2n}}}\Big)^2\Big]=t^2_{12}\Sigma_{12}(s_{12}^{1/2})+t^2_{13}\Sigma_{13}(s_{13}^{1/2})+t^2_{23}\Sigma_{23}(s_{23}^{1/2}).$$

$$\Sigma^-_1(T(3))=t^2_{12}\Sigma_{12}(s_{12}^{1/2})+t^2_{13}\Sigma_{13}(s_{13}^{1/2})+t^2_{23}\Sigma_{23}(s_{23}^{1/2})$$ □

Lemma 2.11. *For an arbitrary orthogonal matrix $t\in\pm\mathrm{O}(3)$ we have*

$$t_{kn}=\pm A^k_n(t),\ \ 1\leq k,n\leq 3,\quad \textit{where}\quad t=\begin{pmatrix} t_{11} & t_{12} & t_{13}\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{pmatrix}. \tag{2.57}$$

Proof. Denote the three rows of the matrix t by, respectively, $t_1,t_2,t_3\in\mathbb{R}^3$. Since $t\in\pm\mathrm{O}(3)$ we get

$$\|t_1\|^2=\|t_2\|^2=\|t_3\|^2=1\quad\text{and}\quad t_l\perp t_r,\ \ l\neq r. \tag{2.58}$$

Moreover, since t_1 is orthogonal to the hyperplane V_{23} generated by the vectors t_2 and t_3 and $t\in\pm\mathrm{O}(3)$ we get respectively $t_l=\pm[t_r,t_s]$, where $[x,y]$ is the *vector product* or *cross product* of two vectors $x,y\in\mathbb{R}^3$ and the triple $\{l,r,s\}$ denotes any cyclic permutations of $\{1,2,3\}$. For $t\in\mathrm{O}(3)$ and $l=1$ we get

$$t_1=[t_2,t_3]=\begin{vmatrix} i & j & k\\ t_{21} & t_{22} & t_{23}\\ t_{31} & t_{32} & t_{33}\end{vmatrix}=i\begin{vmatrix} t_{22} & t_{23}\\ t_{32} & t_{33}\end{vmatrix}-j\begin{vmatrix} t_{21} & t_{23}\\ t_{31} & t_{33}\end{vmatrix}+k\begin{vmatrix} t_{21} & t_{22}\\ t_{31} & t_{32}\end{vmatrix}, \tag{2.59}$$

where i,j,k is the standard orthonormal basis in $\mathbb{R}^3$, i.e.,

$$i=(1,0,0),\ \ j=(0,1,0),\ \ k=(0,0,1).$$

Define X *formally* as the matrix:

$$X = \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Then

$$t_1 = (t_{11}, t_{12}, t_{13}) = \left(A_1^1(X), A_2^1(X), A_3^1(X)\right),$$

thus proving (2.57) for $k = 1$. For other rows the proof is similar. □

Remark 2.9. For $t \in \pm\mathrm{O}(n)$ we can prove a similar statement.

3. Irreducibility, the cases $m = 1$ and $m = 2$

For convenience of the reader, we recall the previous results (see details in [28]).

3.1. Case $m = 1$

Let us denote by $\langle f_n \mid n \in \mathbb{N}\rangle$ the *closure of the linear space* generated by the set of vectors $(f_n)_{n\in\mathbb{N}}$ in a Hilbert space H. Consider the measure $\mu_{(b,a)}$ on the space $X_1 \sim \mathbb{R}^{\mathbb{Z}} = \otimes_{n\in\mathbb{Z}}\mathbb{R}$, the infinite product of the real lines:

$$d\mu_{(b,a)}(x) = \otimes_{n\in\mathbb{Z}} \sqrt{\frac{b_{1n}}{\pi}} e^{-b_{1n}(x_{1n}-a_{1n})^2} dx_{1n},$$

for $b = (b_{1n})_{n\in\mathbb{Z}}$ and $a = (a_{1n})_{n\in\mathbb{Z}}$ with $b_{1n} > 0,\ a_{1n} \in \mathbb{R}$ where $x = (x_{1n})_{n\in\mathbb{Z}}$. In the case $m = 1$ the generators $A_{kn} := A_{kn}^{R,1}$ have the form

$$A_{kn} = x_{1k} D_{1n}, \quad k, n \in \mathbb{Z},$$

where $D_{kn} = \frac{\partial}{\partial x_{kn}} - b_{kn}(x_{kn} - a_{kn})$. The following lemmas are proved in [1]

Lemma 3.1. *The following three conditions are equivalent:*
(i) $(\mu_{(b,a)})^{L_t} \perp \mu_{(b,a)}$ *for all* $t \in \mathrm{GL}(1, \mathbb{R}) \setminus \{e\}$,
(ii) $(\mu_{(b,a)})^{L_{-E_{11}}} \perp \mu_{(b,a)}$,
(iii) $S_{11}^L(\mu) = 4\sum_{n\in\mathbb{Z}} b_{1n} a_{1n}^2 = \infty$.

Lemma 3.2. *For* $k, m \in \mathbb{Z}$ *we have*

$$x_{1k}x_{1m}\mathbf{1} \in \langle A_{kn}A_{mn}\mathbf{1} = x_{1k}x_{1m}D_{1n}^2\mathbf{1} \mid n \in \mathbb{Z}\rangle.$$

Lemma 3.3. *For any $k \in \mathbb{Z}$ we have*

$$x_{1k}\mathbf{1} \in \langle x_{1k}x_{1n}\mathbf{1} \mid n \in \mathbb{Z}\rangle \Leftrightarrow S_{11}^{L}(\mu) = \infty.$$

So, the operators $x_{1k},\ k \in \mathbb{Z}$ are *affiliated* with the von Neumann algebra $\mathfrak{A}^1$ generated by the representation, which completes the proof of the irreducibility for $m = 1$.

Definition 3.1. Recall (see [8]) that, a not necessarily bounded self-adjoint operator A in a Hilbert space H, is said to be *affiliated* with a von Neumann algebra M of operators in this Hilbert space H if $e^{itA} \in M$ for all $t \in \mathbb{R}$. One writes $A\ \eta\ M$.

3.2. Case $m = 2$, approximation of x_{kn} and D_{kn}

In this case the generators $A_{kn} := A_{kn}^{R,2} := \frac{d}{dt}T_{I+tE_{kn}}^{R,\mu,2}\mid_{t=0}$ have the form:

$$A_{kn} = x_{1k}D_{1n} + x_{2k}D_{2n}, \quad k, n \in \mathbb{Z}.$$

We will formulate several useful lemmas for approximation of the operators of multiplication by the independent variables x_{kn} and operators D_{kn} by combinations of the generators A_{kn}. In what follows we use the following notation for $f, g \in \mathbb{R}^m$ (see Remark 2.6 for notations $\sim$)

$$\Delta(f,g) := \frac{\Gamma(f) + \Gamma(f,g)}{\Gamma(g) + 1} \sim \frac{I + \Gamma(f) + \Gamma(g) + \Gamma(f,g)}{I + \Gamma(g)} = \frac{\det(I + \gamma(f,g))}{\det(I + \gamma(g))}, \quad (3.1)$$

where $\Gamma(x_1, \ldots, x_n)$ is the Gram determinant of vectors $x_1, x_2, ..., x_n \in \mathbb{R}^m$ defined by (6.5). To make the notations of the article [28] compatible with the notations in the case $m = 3$ (see (4.5)), we denote

$$\|Y_1^{(1)}\|^2 := \|f^1\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{1k}^2}{b_{1k}^2 + 2b_{1k}b_{2k}}, \quad \|Y_2^{(1)}\|^2 := \|g^1\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{2k}^2}{b_{1k}^2 + 2b_{1k}b_{2k}},$$
$$\|Y_1^{(2)}\|^2 := \|g^2\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{1k}^2}{b_{2k}^2 + 2b_{1k}b_{2k}}, \quad \|Y_2^{(2)}\|^2 := \|f^2\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{2k}^2}{b_{2k}^2 + 2b_{1k}b_{2k}},$$
$$\|Y_1\|^2 := \|f\|^2 = \sum_{k\in\mathbb{Z}} \frac{a_{1k}^2}{\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}}, \quad \|Y_2\|^2 := \|g\|^2 = \sum_{k\in\mathbb{Z}} \frac{a_{2k}^2}{\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}}. \quad (3.2)$$

Lemma 3.4. *For any $k, t \in \mathbb{Z}$ one has*

$$x_{1k}x_{1t} \in \langle A_{kn}A_{tn}\mathbf{1} \mid n \in \mathbb{Z}\rangle \ \Leftrightarrow\ \Delta(Y_1^{(1)}, Y_2^{(1)}).$$

Lemma 3.5. *For any $k, t \in \mathbb{Z}$ we have*
$$x_{2k}x_{2t} \in \langle A_{kn}A_{tn}\mathbf{1} \mid n \in \mathbb{Z}\rangle \;\Leftrightarrow\; \Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty.)$$
Lemma 3.6. *Set $\Sigma^{rs} = \sum_{n\in\mathbb{Z}} \frac{b_{rn}}{b_{sn}}$, $1 \le r, s \le 2$. For any $k \in \mathbb{Z}$ we get*
$$x_{1k}\mathbf{1} \in \langle D_{1n}A_{kn}\mathbf{1} \mid n \in \mathbb{Z}\rangle \quad\Leftrightarrow\quad \Sigma^{12} = \infty.$$
Lemma 3.7. *For any $k \in \mathbb{Z}$ we have*
$$x_{2k}\mathbf{1} \in \langle D_{2n}A_{kn}\mathbf{1} \mid n \in \mathbb{Z}\rangle \quad\Leftrightarrow\quad \Sigma^{21} = \infty.$$
Lemma 3.8. *For any $n \in \mathbb{Z}$ we have*
$$D_{1n}\mathbf{1} \in \langle A_{kn}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad\Leftrightarrow\quad \Delta(Y_1, Y_2) = \infty.$$
Lemma 3.9. *For any $n \in \mathbb{Z}$ we have*
$$D_{2n}\mathbf{1} \in \langle A_{kn}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad\Leftrightarrow\quad \Delta(Y_2, Y_1) = \infty.$$

3.2.1. Technical part of the proof of irreducibility

Lemma 3.10. *If $\mu^{L_t} \perp \mu$ for all $t \in \mathrm{GL}(2,\mathbb{R}) \setminus \{e\}$, we can approximate by combinations of generators at least one of the following pairs of operators: (x_{1n}, x_{2n}), (x_{1n}, D_{2n}), (D_{1n}, x_{2n}) or (D_{1n}, D_{2n}).*

Proof. Recall the orthogonality conditions for the case $m = 2$
$$S_{kr}^{L}(\mu) = \sum_{n\in\mathbb{Z}} \frac{b_{kn}}{2}\left(\frac{1}{2b_{rn}} + a_{rn}^2\right), \quad 1 \le k, r \le 2,\; k \ne r,$$
$$S_{kr}^{L,-}(\mu, t) = \frac{t^2}{4}\sum_{n\in\mathbb{Z}} \frac{b_{kn}}{b_{rn}} + \sum_{n\in\mathbb{Z}} \frac{b_{kn}}{2}(-2a_{kn} + t a_{rn})^2, \quad 1 \le k \ne r \le 2,$$
$$\Sigma_{12}^{-}\big(\tau_-(\phi, s)\big) = \sin^2\phi\, \Sigma_1(s) + \Sigma_2^{-}\big(\tau_-(\phi, s)\big), \quad \text{where}$$
$$\Sigma_1(s) := \sum_{n\in\mathbb{Z}} \left(s^2\sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2,$$
$$\Sigma_2^{-}\big(\tau_-(\phi, s)\big) = \sum_{n\in\mathbb{Z}} \left(4b_{1n}\sin^2\frac{\phi}{2} + 4s^{-4}b_{2n}\cos^2\frac{\phi}{2}\right)\left(a_{1n}\sin\frac{\phi}{2} - s^2 a_{2n}\cos\frac{\phi}{2}\right)^2.$$
Let $\mathfrak{A}^2$ be the von Neumann algebra generated by our representation. In order to approximate operators x_{kn} or D_{kn} by the corresponding generators, by Lemmas 3.4–3.5, Lemmas 3.2– 3.2 and Lemmas 3.8– 3.9 we have:
$$x_{1n}x_{1t}\;\eta\;\mathfrak{A}^2 \Leftrightarrow \Delta(Y_1^{(1)}, Y_2^{(1)}) = \infty, \quad x_{2n}x_{2t}\;\eta\;\mathfrak{A}^2 \Leftrightarrow \Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty,$$
$$D_{1n}\;\eta\;\mathfrak{A}^2 \Leftrightarrow \Delta(Y_1, Y_2) = \infty, \quad D_{2n}\;\eta\;\mathfrak{A}^2 \Leftrightarrow \Delta(Y_2, Y_1) = \infty,$$
where $Y_r^{(s)}$ and Y_r for $1 \le r, s \le 2$ are defined by (3.2). □

3.2.2. Scheme of the proof for two lines

There are two different cases:

I. Approximation of $x_{rk}x_{rt}$ for $1 \leq r \leq 2$ by $A_{kn}A_{tn}$,

II. Approximation of D_{rk} for $1 \leq r \leq 2$ by A_{kn}.

In the case $m = 2$ the analysis of the divergence

$$\Delta(Y_1^{(1)}, Y_2^{(1)}) = \infty \quad \text{and} \quad \Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty$$

is governed by the convergence or divergence of Σ^{12} and Σ^{21}, since

$$\|Y_1^{(1)}\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{1k}^2}{b_{1k}^2 + 2b_{1k}b_{2k}} \sim \frac{1}{2}\sum_{k\in\mathbb{Z}} \frac{b_{1k}^2}{b_{1k}b_{2k}} = \frac{1}{2}\Sigma^{12},$$

$$\|Y_2^{(2)}\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{2k}^2}{b_{2k}^2 + 2b_{1k}b_{2k}} \sim \frac{1}{2}\sum_{k\in\mathbb{Z}} \frac{b_{2k}^2}{b_{1k}b_{2k}} = \frac{1}{2}\Sigma^{21}.$$

Remark 3.1. To guess the right generalisation for the case $m = 3$ we note that $\Sigma^{12} = S_1(2)$ and $\Sigma^{21} = S_2(2)$, where we denote

$$S_1(2) := \sum_{k\in\mathbb{Z}} \frac{b_{1k}^2}{b_{1k}b_{2k}}, \;\; S_2(2) := \sum_{k\in\mathbb{Z}} \frac{b_{2k}^2}{b_{1k}b_{2k}}, \;\; \Sigma^{12} = \sum_{k\in\mathbb{Z}} \frac{b_{1k}}{b_{2k}}, \;\; \Sigma^{21} = \sum_{k\in\mathbb{Z}} \frac{b_{2k}}{b_{1k}}.$$

We also observe that

$$\|Y_1^{(1)}\|^2 \sim S_1(2), \quad \|Y_2^{(2)}\|^2 \sim S_2(2). \tag{3.3}$$

Hence, in the case $m = 3$ it is natural to replace $\Sigma^{12} = S_1(2)$ and $\Sigma^{21} = S_2(2)$ by $S_r(3)$, which is defined as follows:

$$S_r(3) = \sum_{n\in\mathbb{Z}} \frac{b_{rn}^2}{b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}}, \quad 1 \leq r \leq 3. \tag{3.4}$$

3.2.3. I, Approximation of x_{1n} and x_{2n}

Lemma 3.11. *We have*

$$S_1(2) + S_2(2) = \infty, \tag{3.5}$$

$$\|Y_r^{(r)}\|^2 \sim S_r(2) \quad \textit{for all} \quad 1 \leq r \leq 2, \tag{3.6}$$

$$\|Y_1^{(2)}\|^2 < \frac{1}{2}S_1(2), \quad \|Y_2^{(1)}\|^2 < \frac{1}{2}S_2(2), \tag{3.7}$$

$$\|Y_1^{(i)}\|^2 + \|Y_2^{(j)}\|^2 = \infty, \quad i, j \in \{1, 2\}. \tag{3.8}$$

Proof. Since $a^2+b^2\geq 2ab$ we get

$$S_1(2)+S_2(2)=\sum_{k\in\mathbb{Z}}\frac{b_{1k}^2+b_{2k}^2}{b_{1k}b_{2k}}\geq\sum_{k\in\mathbb{Z}}2=\infty.$$

Further, for $1\leq r\leq 2$

$$\|Y_r^{(r)}\|^2=\sum_{k\in\mathbb{Z}}\frac{b_{rk}^2}{b_{rk}^2+2b_{1k}b_{2k}}\sim\sum_{k\in\mathbb{Z}}\frac{b_{rk}^2}{2b_{1k}b_{2k}}=\frac{1}{2}S_r(2),$$
$$\|Y_1^{(2)}\|^2=\sum_{k\in\mathbb{Z}}\frac{b_{1k}^2}{b_{2k}^2+2b_{1k}b_{2k}}<\sum_{k\in\mathbb{Z}}\frac{b_{1k}^2}{2b_{1k}b_{2k}}=\frac{1}{2}S_1(2),$$
$$\|Y_2^{(1)}\|^2=\sum_{k\in\mathbb{Z}}\frac{b_{2k}^2}{b_{1k}^2+2b_{1k}b_{2k}}<\sum_{k\in\mathbb{Z}}\frac{b_{2k}^2}{2b_{1k}b_{2k}}=\frac{1}{2}S_2(2).$$

In addition, we have

$$\|Y_1^{(i)}\|^2+\|Y_2^{(j)}\|^2=\sum_{k\in\mathbb{Z}}\frac{b_{1k}^2}{b_{ik}^2+2b_{1k}b_{2k}}+\sum_{k\in\mathbb{Z}}\frac{b_{2k}^2}{b_{jk}^2+2b_{1k}b_{2k}}$$
$$>\sum_{k\in\mathbb{Z}}\frac{b_{1k}^2+b_{2k}^2}{(b_{1k}+b_{2k})^2}=\infty.$$ □

Remark 3.2. In what follows if some expression $<\infty$ (resp. $=\infty$) we denote this case by 0 (respectively, by 1).

Set $S=(S_1(2),S_2(2))$, since $S_1(2)+S_2(2)=\infty$ we have two cases:
I(1) $S=(0,1)$, if $S=(1,0)$ we interchange $(b_{1n},a_{1n})_n$ with $(b_{2n},a_{2n})_n$,
I(2) $S=(1,1)$, i.e., $S_1(2)=\infty,\ S_2(2)=\infty$.

3.2.4. Case $S=(0,1)$

Lemma 3.12. *In the case $S=(0,1)$ the representation is irreducible, moreover we can approximate:*

(1) *$x_{2k}x_{2t}$ by $A_{kn}A_{tn}$, since $\Delta(Y_2^{(2)},Y_1^{(2)})=\infty$,*
(2) *D_{1n}, D_{2n} by A_{kn}, since $\Delta(Y_1,Y_2)=\infty$ and $\Delta(Y_2,Y_1)=\infty$.*

Proof. (1) Set (compare with (4.12) in the case $m=3$)

$$y^{(k)}=(y_1^{(k)},y_2^{(k)})=(\|Y_1^{(k)}\|^2,\|Y_2^{(k)}\|^2),\ 1\leq k\leq 2, \tag{3.9}$$
$$\text{or } y=\begin{pmatrix}y^{(1)}\\ y^{(2)}\end{pmatrix}=\begin{pmatrix}y_1^{(1)} & y_2^{(1)}\\ y_1^{(2)} & y_2^{(2)}\end{pmatrix}. \tag{3.10}$$

In the case $S=(0,1)$ we have by Lemma 3.11 (see Remark 3.2)

$$y^{(1)}=(0,1),\ y^{(2)}=(0,1)\text{ or }\left(\begin{array}{c}y^{(1)}\\ y^{(2)}\end{array}\right)=\left(\begin{array}{cc}0&1\\ 0&1\end{array}\right). \tag{3.11}$$

Indeed, $\|Y_1^{(1)}\|^2\sim S_1(2)<\infty$, hence $\|Y_2^{(1)}\|^2=\infty$ by (3.8). Further, $\Gamma(Y_2^{(2)})=\|Y_2^{(2)}\|^2\sim S_2(2)=\infty$, and $\Gamma(Y_1^{(2)})<\infty$, since $S_1(2)<\infty$. Therefore,

$$\Delta(Y_2^{(2)},Y_1^{(2)})=\frac{\Gamma(Y_2^{(2)})+\Gamma(Y_2^{(2)},Y_1^{(2)})}{1+\Gamma(Y_1^{(2)})}>\frac{\Gamma(Y_2^{(2)})}{1+\Gamma(Y_1^{(2)})}=\infty,$$

and we conclude that $x_{2n}\ \eta\ \mathfrak{A}^2$.

(2) Further, since

$$\|Y_1\|^2=\|Y_2\|^2=\|Y_1-sY_2\|^2=\infty \tag{3.12}$$

by Lemma 8.11, we conclude that $\Delta(Y_1,Y_2)=\Delta(Y_2,Y_1)=\infty$, so $D_{1n},\ D_{2n}\ \eta\ \mathfrak{A}^2$ by Lemmas 3.8 and 3.9. Finally, $x_{2n},\ D_{1n},\ D_{2n}\,\eta\ \mathfrak{A}^2$. Now we get

$$A_{kn}-x_{2k}D_{2n}=x_{1k}D_{1n},\quad k,n\in\mathbb{Z},$$

and the proof is complete since we are in the case $m=1$.

Relations (3.12) follows from $S_1(2)=\Sigma^{12}=\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}}<\infty$. Indeed, we have

$$\|Y_1\|^2=\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}=\sum_{k\in\mathbb{Z}}\frac{b_{1k}a_{1k}^2}{\frac{1}{2}+\frac{b_{1k}}{2b_{2k}}}\sim 2\sum_{k\in\mathbb{Z}}b_{1k}a_{1k}^2=S_{11}^L(\mu)=\infty,$$

$$\|Y_2\|^2=\sum_{k\in\mathbb{Z}}\frac{b_{1k}a_{2k}^2}{\frac{1}{2}+\frac{b_{1k}}{2b_{2k}}}\sim\sum_{k\in\mathbb{Z}}b_{1k}a_{2k}^2\sim\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{2}\Big(\frac{1}{2b_{2k}}+a_{2k}^2\Big)=S_{12}^L(\mu)=\infty,$$

$$\|Y_1-sY_2\|^2=\sum_{k\in\mathbb{Z}}\frac{b_{1k}(a_{1k}-sa_{2k})^2}{\frac{1}{2}+\frac{b_{1k}}{2b_{2k}}}\sim\sum_{k\in\mathbb{Z}}b_{1k}(a_{1k}-sa_{2k})^2=\frac{1}{4}\sum_{k\in\mathbb{Z}}b_{1k}(-2a_{1k}+2sa_{2k})^2$$

$$\sim\frac{1}{2}\Big[\frac{(2s)^2}{4}\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}}+\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{2}(-2a_{1k}+2sa_{2k})^2\Big]=\frac{1}{2}S_{12}^{L,-}(\mu,t)=\infty,$$

for $t=2s$. $\square$

3.2.5. Case $S = (1,1)$

In the case $S = (1,1)$ we have three possibilities (see Remark 3.2)

$$\mathrm{I(2a)}\ y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ \mathrm{I(2b)}\ y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \mathrm{I(2c)}\ y = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{3.13}$$

i.e., I(2a) $y_2^{(1)} < \infty$ and $y_1^{(2)} = \infty$, I(2b) $y_2^{(1)} = \infty$ and $y_1^{(2)} < \infty$, I(2c) $y_2^{(1)} = y_1^{(2)} = \infty$, since $\|Y_1^{(1)}\|^2 \sim S_1(2) = \infty$ and $\|Y_2^{(2)}\|^2 \sim S_2(2) = \infty$ and $y_2^{(1)} + y_1^{(2)} = \infty$ by (3.8).

In the first case I(2a) or in the second case I(2b), i.e., if $\|Y_2^{(1)}\|^2 < \infty$ or $\|Y_1^{(2)}\|^2 < \infty$, we conclude respectively that $\Delta(Y_1^{(1)}, Y_2^{(1)}) = \infty$ or $\Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty$ hence, $x_{1n}x_{1t}\ \eta\ \mathfrak{A}^2$ or $x_{2n}x_{2t}\ \eta\ \mathfrak{A}^2$. So, we will get respectively x_{1n} or x_{2n}. We shall come back to these cases later.

It remains to consider the case I(2c): $\|Y_2^{(1)}\|^2 = \|Y_1^{(2)}\|^2 = \infty$. In the case $S = (1,1)$ set $c_n = \frac{b_{2n}}{b_{1n}}$, $n \in \mathbb{Z}$. Then $\sum_{k\in\mathbb{Z}} \frac{b_{1k}}{b_{2k}} = \sum_{k\in\mathbb{Z}} \frac{b_{2k}}{b_{1k}} = \infty$. We get

$$\|Y_1^{(1)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{1}{1+2c_n} \sim \sum_{n\in\mathbb{Z}} \frac{1}{c_n}, \quad \|Y_2^{(1)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{c_n^2}{1+2c_n}, \tag{3.14}$$

$$\|Y_1^{(2)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{1}{c_n^2+2c_n}, \quad \|Y_2^{(2)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{c_n^2}{c_n^2+2c_n} \sim \sum_{n\in\mathbb{Z}} c_n. \tag{3.15}$$

3.2.6. II, Approximation of D_{1n} and D_{2n}

Set

$$y_{12} = (y_1, y_2) = (\|Y_1\|^2, \|Y_2\|^2), \tag{3.16}$$

$$\text{where} \quad \|Y_1\|^2 = \sum_{n\in\mathbb{Z}} \frac{a_{1n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}}, \quad \|Y_2\|^2 = \sum_{n\in\mathbb{Z}} \frac{a_{2n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}}. \tag{3.17}$$

The case II splits into four subcases

$$(1)\,(y_1,y_2)=(1,0),\ (2)\,(y_1,y_2)=(0,1),\ (3)\,(y_1,y_2)=(1,1),\ (4)\,(y_1,y_2)=(0,0). \tag{3.18}$$

We have $4 = 2^2$ possibilities for $y = (y_1, y_2) \in \{0,1\}^2$:

	(1)	(2)	$(3a)$	$(3b)$	$(3c)$	(4)
y_1	1	0	1	1	1	0
y_2	0	1	1	1	1	0
α			1	0	$C_1 \le \alpha_m \le C_2$	

where $\alpha=\lim_{m\to\infty}\alpha_m$, with $\alpha_m=\frac{\|Y_1(m)\|^2}{\|Y_2(m)\|^2}$, and

$$\|Y_1(m)\|^2=\sum_{n=-m}^{m}\frac{a_{1n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}},\quad \|Y_2(m)\|^2=\sum_{n=-m}^{m}\frac{a_{2n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}}.$$

All the different cases are presented in the following table:

Table II	(1)	(2)	(3a)	(3b)	(3c)	(4)
$\|Y_1\|^2$	∞	$<\infty$	∞	∞	∞	$<\infty$
$\|Y_2\|^2$	$<\infty$	∞	∞	∞	∞	$<\infty$
$\alpha_m=\frac{\|Y_1(m)\|^2}{\|Y_2(m)\|^2}$			$\to\infty$	$\to 0$	$C_1\leq\alpha_m\leq C_2$	
Lemma	3.8 3.6	3.9 3.7	3.8 3.6	3.9 3.7	3.8 , 3.9 3.14, 8.11	
	$D_{1n},\ x_{1n}$	$D_{2n},\ x_{2n}$	$D_{1n},\ x_{1n}$	$D_{2n},\ x_{2n}$	$D_{1n},\ D_{2n}$	

Remark 3.3. We show that if $\|Y_2\|^2<\infty$ and $S_{12}^L(\mu)=\infty$, then $\sum_n\frac{b_{1n}}{b_{2n}}=\infty$. Indeed, let us suppose that $\sum_n\frac{b_{1n}}{b_{2n}}<\infty$, then

$$\|Y_2\|^2=\sum_{n\in\mathbb{Z}}\frac{a_{2n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}}\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{2n}^2\sim\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{2}\Big(\frac{1}{2b_{2n}}+a_{2n}^2\Big)=S_{12}^L(\mu)=\infty. \tag{3.19}$$

We explain Table II in detail. The first two case (1) and (2) are independent of the case I(2), i.e., $S=(1,1)$.

(1) If $\|Y_2\|^2<\infty$ and $\|Y_1\|^2=\infty$, then $D_{1k}\ \eta\ \mathfrak{A}^2$ by Lemma 3.8. The condition $\|Y_2\|^2<\infty$ implies $\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}}=\infty$, by Remark 3.3 therefore, $x_{1k}\ \eta\ \mathfrak{A}^2$, by Lemma 3.6. Further, $A_{kn}-x_{1k}D_{1n}=x_{2k}D_{2n},\ k,n\in\mathbb{Z}$ and the proof is complete since we are reduced to the case $m=1$.

(2) If $\|Y_2\|^2=\infty$ and $\|Y_1\|^2<\infty$, then $D_{2k}\ \eta\ \mathfrak{A}^2$ by Lemma 3.9. Reasoning as in Remark 3.3, we conclude that $\sum_{k\in\mathbb{Z}}\frac{b_{2k}}{b_{1k}}=\infty$ and therefore, $x_{2k}\ \eta\ \mathfrak{A}^2$ by Lemma 3.7 and $A_{kn}-x_{2k}D_{2n}=x_{1k}D_{1n},\ k,n\in\mathbb{Z}$, case $m=1$.

(3) Consider now the case I(2). Suppose that both series are divergent: $\|Y_2\|^2=\infty$ and $\|Y_1\|^2=\infty$. We show that in the case (B) (see (3.25)) holds

$$\|Y_1+sY_2\|^2=\infty\quad\text{for all}\quad s\in\mathbb{R}$$

by Lemma 3.14 therefore, by Lemma 8.11, we can approximate D_{1n} and D_{2n}. To be more precise consider three possibilities:

(3a) Let $\frac{\|Y_1(m)\|^2}{\|Y_2(m)\|^2} \to \infty$, then $D_{1k}\ \eta\ \mathfrak{A}^2$. Since $\sum_n \frac{b_{1n}}{b_{2n}} = \infty$, case I(2), we have $x_{1n}\ \eta\ \mathfrak{A}^2$ by Lemma 3.6 and finally, $x_{1n},\ D_{1n}\ \eta\ \mathfrak{A}^2,\ n \in \mathbb{Z}$. We are reduced to the case $m = 1$.

(3b) let $\frac{\|Y_1(m)\|^2}{\|Y_2(m)\|^2} \to 0$, then $D_{2k}\ \eta\ \mathfrak{A}^2$. Since $\sum_n \frac{b_{2n}}{b_{1n}} = \infty$, case I(2), we get $x_{2n}\ \eta\ \mathfrak{A}$, by Lemma 3.7 and finally, $x_{2n},\ D_{2n}\ \eta\ \mathfrak{A}^2,\ n \in \mathbb{Z}$. We are reduced to the case $m = 1$.

(3c) The case when $\|Y_1\|^2 = \|Y_2\|^2 = \infty$ and $C_1 \leq \frac{\|Y_1(m)\|^2}{\|Y_2(m)\|^2} \leq C_2,\ \ m \in \mathbb{N}$.

(4) The case when $\|Y_1\|^2 + \|Y_2\|^2 < \infty$.

To complete the proof of the lemma it remains to consider I(2), i.e., $S = (1,1)$ and the last two cases in the table II, i.e., II(3c) and II(4), where:

$$\text{I}(2) \qquad \sum_{k\in\mathbb{Z}} \frac{b_{1k}}{b_{2k}} = \sum_{k\in\mathbb{Z}} \frac{b_{2k}}{b_{1k}} = \infty, \tag{3.20}$$

$$\text{II}(3c) \qquad \sum_{k\in\mathbb{Z}} a_{1k}^2\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\Big)^{-1} = \sum_{k\in\mathbb{Z}} a_{2k}^2\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\Big)^{-1} = \infty, \tag{3.21}$$

$$\text{II}(4) \qquad \sum_{k\in\mathbb{Z}} \Big(a_{1k}^2 + a_{2k}^2\Big)\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\Big)^{-1} < \infty. \tag{3.22}$$

We come back to the condition $\mu^{L_t} \perp \mu$. By Remark 2.5 we have

$$\mu^{L_{\tau_-(\phi,s)}} \perp \mu,\ \ \phi \in [0, 2\pi),\ \ s > 0 \Leftrightarrow \Sigma_1(s) + \Sigma_2(C_1, C_2) = \infty,\ \ s > 0,$$

for $(C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}$. To make the notation consistent for the case $m = 3$ we replace everywhere $\Sigma_1(s)$ (defined by (2.17)) with $\Sigma_{12}(s)$ and $\Sigma_2(C_1, C_2)$ defined by (2.18) with $\Sigma_{12}(C_1, C_2)$ for $(C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}$:

$$\Sigma_{12}(s) = \sum_{n\in\mathbb{Z}} \Big(s^2\sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2, \quad s \in \mathbb{R}\setminus\{0\}, \tag{3.23}$$

$$\Sigma_{12}(C_1, C_2) = \sum_{n\in\mathbb{Z}} (C_1^2 b_{1n} + C_2^2 b_{2n})(C_1 a_{1n} + C_2 a_{2n})^2. \tag{3.24}$$

The condition $\Sigma_{12}(s) + \Sigma_{12}(C_1, C_2) = \infty$, splits into two cases:

$$\begin{array}{ll} (A) & \Sigma_{12}(s) = \infty, \\ (B) & \Sigma_{12}(s) < \infty \quad \text{and} \quad \Sigma_{12}(C_1, C_2) = \infty. \end{array} \tag{3.25}$$

Finally, we need to consider the following 12 possibilities:

A	I(3a)	I(3b)	I(3c)
II(3c)			
II(4)			

.

B	I(3a)	I(3b)	I(3c)
II(3c)			
II(4)			

.

Briefly:

(**A**)&**I**(**2**). In this case independently of the conditions II(3c) and II(4) we can approximate x_{1n} and x_{2n} using Lemma 3.4 and 3.5.

(**B**)&**II**(**3c**) In this case we can approximate D_{1n} and D_{2n} using Lemmas 3.8 and 3.9 respectively. More precisely, to be able to use Lemma 8.11 we show that conditions (8.17) are satisfied for the two vectors Y_1 and Y_2 defined by (3.2) (see Lemma 3.14).

(**B**)&**II**(**4**) Since $\Sigma_{12}(C_1, C_2) = \infty$, this case (see (3.22)) cannot be realized

More details:

Case (A)&I(2). Using Lemma 8.11 we conclude that

$$\Delta(Y_1^{(1)}, Y_2^{(1)}) = \infty \quad \text{and} \quad \Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty. \tag{3.26}$$

To use Lemma 8.11, it is sufficient to show that in the case (A) relations (8.17) hold for $Y_1^{(1)}, Y_2^{(1)}$ and $Y_2^{(2)}, Y_1^{(2)}$, i.e., for all $s \in \mathbb{R} \setminus \{0\}$ we have (see Lemma 3.13)

$$\begin{aligned} \|Y_1^{(1)}\|^2 = \|Y_2^{(1)}\|^2 = \|Y_1^{(1)} + sY_2^{(1)}\|^2 = \infty, \\ \|Y_2^{(2)}\|^2 = \|Y_1^{(2)}\|^2 = \|Y_2^{(2)} + sY_1^{(2)}\|^2 = \infty. \end{aligned} \tag{3.27}$$

Consider the following three possibilities in the case I(2):

I(2a) If $\|Y_2^{(1)}\| < \infty$, then $\|Y_1^{(1)}\| = \infty$ by (3.8) therefore, $\Delta(Y_1^{(1)}, Y_2^{(1)}) = \infty$ so, $x_{1n} \;\eta\; \mathfrak{A}^2$ by Lemma 8.10 (a). In the case (A) by Lemma 3.13 holds

$$\|Y_2^{(2)}\|^2 = \|Y_1^{(2)}\|^2 = \|Y_2^{(2)} + sY_1^{(2)}\|^2 = \infty,$$

therefore, $x_{2n} \;\eta\; \mathfrak{A}^2$ by Lemma 8.11.

I(2b) If $\|Y_1^{(2)}\| < \infty$, then $\|Y_2^{(2)}\| = \infty$ by (3.8) therefore, $\Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty$ so, $x_{2n} \;\eta\; \mathfrak{A}^2$ by Lemma 8.10 (a). In the case (A) by Lemma 3.13 we have

$$\|Y_2^{(1)}\|^2 = \|Y_1^{(1)}\|^2 = \|Y_1^{(1)} + sY_2^{(2)}\|^2 = \infty,$$

and therefore, $x_{1n} \;\eta\; \mathfrak{A}^2$ by Lemma 8.11.

I(2c) If $\|Y_2^{(1)}\| = \|Y_1^{(2)}\| = \infty$, then by Lemma 3.13 all relations (3.27) hold in the case (A) and therefore, $x_{1n},\; x_{2n} \;\eta\; \mathfrak{A}^2$. To prove (3.27) we need Lemma 4.8.

Lemma 3.13. *If $\Sigma_{12}(s)=\infty$ for any $s>0$, then*

$$\|Y_1^{(1)}-CY_2^{(1)}\|^2=\infty \quad \textit{and} \quad \|Y_2^{(2)}-CY_1^{(2)}\|^2=\infty, \quad \textit{for any} \quad C\in\mathbb{R}\setminus\{0\}.$$

So, in the case (A)&I(2) we can approximate x_{1n} and x_{2n}.

Case (B)&II(3c).

Lemma 3.14. *When $\Sigma_{12}(s)<\infty$ and $\Sigma_{12}(C_1,C_2)=\infty$, we get*

$$\sigma(C_1,C_2):=\|C_1Y_1+C_2Y_2\|^2=\sum_{n\in\mathbb{Z}}\frac{(C_1a_{1n}+C_2a_{2n})^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}}=\infty,\ (C_1,C_2)\in\mathbb{R}^2\setminus\{0\}. \tag{3.28}$$

Finally, we can approximate D_{1n} and D_{2n} in the case (B)&II(3c).

Case (B)&II(4). The last case (B)&II(4) (see (3.22)) can not be realized. Indeed, in this case $\Sigma_{12}(s)<\infty$ and $\Sigma_{12}(C_1,C_2)=\infty$. Therefore, by Lemma 4.8 we have $s^4\lim_{n\to\infty}\frac{b_{1n}}{b_{2n}}=1$ and hence,

$$\sigma(C_1,C_2)\sim\Sigma_{12}(C_1,C_2)=\infty$$

(see also the proof of Lemma 3.14 in [25]). This contradicts (3.22):

$$\sigma(1,1)=\sum_{k\in\mathbb{Z}}\left(a_{1k}^2+a_{2k}^2\right)\left(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\right)^{-1}<\infty.$$

Thus the proof of Lemma 3.10 for $m=2$ is completed.

The proof of the irreducibility for $m=2$ follows from Remark 1.2. Depending on the measure, we can approximate four different families of commuting operators $B^\alpha=(B^\alpha_{1n},B^\alpha_{2n})_{n\in\mathbb{Z}}$ for $\alpha\in\{0,1\}^2$:

$$B^{(0,0)}{=}(x_{1n},x_{2n})_n,\ B^{(0,1)}{=}(x_{1n},D_{2n})_n,\ B^{(1,0)}{=}(D_{1n},x_{2n})_n,\ B^{(0,0)}{=}(D_{1n},D_{2n})_n.$$

The von Neumann algebra $L^\infty_\alpha(X_2,\mu^2)$ consists of all essentially bounded functions $f(B^\alpha)$ in the commuting family of operators B^α (see, e.g., [5]) as, in particular, $L^\infty_{(0,0)}(X_2,\mu^2)=L^\infty(X_2,\mu^2)$. Since the von Neumann algebras $L^\infty_\alpha(X_2,\mu^2)$ are maximal abelian, the commutant $\left(\mathfrak{A}^2\right)'$ of the von Neumann algebra $\mathfrak{A}^2$ generated by the representation is contained in $L^\infty_\alpha(X_2,\mu^2)$. Hence, the bounded operator $A\in\left(\mathfrak{A}^2\right)'$ will be some function $A=a(B^\alpha)\in L^\infty_\alpha(X_2,\mu^2)$. The commutation relation $[A,T^{R,\mu,2}_t]=0$ gives us the following relations: $a(B^\alpha)^{R_t}=a(B^\alpha)$ for all $t\in\mathrm{GL}_0(2\infty,\mathbb{R})$. Set $B^\alpha_r=$

$(B^{\alpha}_{rn})_n$, $x_r = (x_{rn})_n$, $D_r = (x_{rn})_n$, $r = 1, 2$, $n \in \mathbb{Z}$ and set as before, $E_{kn}(t) := I + tE_{kn}$, $t \in \mathbb{R}$, $k, n \in \mathbb{Z}$, $k \neq n$. Then the action $(B^{\alpha})^{R_t}$ is defined as follows:

$$(B^{\alpha}_1, B^{\alpha}_2)^{R_t} = ((B^{\alpha}_1)^{R_t}, (B^{\alpha}_2)^{R_t}), \quad (x_r)^{R_t} = x_r t, \quad (D_r)^{R_t} = D_r t^T,$$
$$a(\dots, x_{rk}, \dots, x_{rn}, \dots)^{R_{E_{kn}(t)}} := a(\dots, x_{rk}, \dots, x_{rn} + t x_{rk}, \dots),$$
$$a(\dots, D_{rk}, \dots, D_{rn}, \dots)^{R_{E_{kn}(t)}} := a(\dots, D_{rk} + tD_{rn}, \dots, D_{rn}, \dots), \ \ t \in \mathbb{R}.$$

In all the cases, by ergodicity of the measure μ^2, we conclude that a is constant.

4. Irreducibility, the case $m = 3$

4.1. Technical part of the proof of irreducibility

Lemma 4.1. *If $\mu^{L_t} \perp \mu$ for all $t \in \mathrm{GL}(3, \mathbb{R}) \setminus \{e\}$, we can approximate at least one of the following eight triplets of operators:*

$$(x_{1n}, x_{2n}, x_{3n}),\ (x_{1n}, x_{2n}, D_{3n}),\ (x_{1n}, D_{2n}, x_{3n}),\ (D_{1n}, x_{2n}, x_{3n}),$$
$$(x_{1n}, D_{2n}, D_{3n}), (D_{1n}, x_{2n}, D_{3n}),\ (D_{1n}, D_{2n}, x_{3n}),\ (D_{1n}, D_{2n}, D_{3n}).$$

Proof. By Lemma 2.8, the condition of orthogonality $(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)}$ for $t \in \pm\mathrm{SL}(3, \mathbb{R}) \setminus \{e\}$ are,

$$\Sigma^{\pm}(t) = \Sigma^{\pm}_1(t) + \Sigma_2(t) = \infty, \tag{4.1}$$

where $\Sigma_2(t)$ is defined by (2.28) and $\Sigma^+_1(t)$, $\Sigma^-_1(t)$ are defined by (2.31), (2.32). Let $\mathfrak{A}^3$ be the von Neumann algebra generated by the representation. We write compactly:

$$x_{kn}\ \eta\ \mathfrak{A}^3 \Leftrightarrow \Delta^{(k)} = \infty, \quad D_{kn}\ \eta\ \mathfrak{A}^3 \Leftrightarrow \Delta_k = \infty, \tag{4.2}$$
$$\text{where } \Delta^{(k)} := \Delta(Y^{(k)}_k, Y^{(k)}_r, Y^{(k)}_s), \quad \Delta_k := \Delta(Y_k, Y_r, Y_s), \tag{4.3}$$

and $\{k, r, s\}$ is a cyclic permutation of $\{1, 2, 3\}$.

Case I. *Approximation of $x_{rk}x_{rt}$ for $1 \le r \le 3$ by $A_{kn}A_{tn}$.*
Set $B_{3k} = b_{1k} + b_{2k} + b_{3k}$. To approximate the operators x_{kn} by the corresponding operators, by Lemmas 5.1–5.3 we get:

$$x_{1n}x_{1t}\ \eta\ \mathfrak{A}^3 \Leftrightarrow\ \Delta^{(1)} = \infty,\ x_{2n}x_{2t}\ \eta\ \mathfrak{A}^3 \Leftrightarrow \Delta^{(2)} = \infty,\ x_{3n}x_{3t}\ \eta\ \mathfrak{A}^3 \Leftrightarrow \Delta^{(3)} = \infty, \tag{4.4}$$

where

$$\|Y_s^{(r)}\|^2=\sum_{k\in\mathbb{Z}}\frac{b_{rk}^2}{B_{3k}^2-(b_{1k}^2+b_{2k}^2+b_{3k}^2-b_{sk}^2)},\quad 1\leq r,s\leq 3. \tag{4.5}$$

Case II. *Approximation of D_{rn} by A_{kn}.*

By Lemmas 5.4–5.5 we have for $1\leq r\leq 3$ (see (4.3)):

$$D_{rn}\ \eta\ \mathfrak{A}^3\Leftrightarrow\Delta_r=\infty,\quad\text{where}\quad \|Y_r\|^2=\sum_{k\in\mathbb{Z}}\frac{a_{rk}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}}. \tag{4.6}$$

Case III. *Approximation of x_{rk} by $D_{rn}A_{kn}$.*

$$\begin{aligned}D_{1n}A_{kn}&=x_{1k}D_{1n}^2+x_{2k}D_{1n}D_{2n}+x_{3k}D_{1n}D_{3n},\\D_{2n}A_{kn}&=x_{1k}D_{1n}D_{2n}+x_{2k}D_{2n}^2+x_{3k}D_{2n}D_{3n},\\D_{3n}A_{kn}&=x_{1k}D_{1n}D_{3n}+x_{2k}D_{2n}D_{3n}+x_{3k}D_{3n}^2.\end{aligned}$$

By Lemmas 5.7–5.9 we have

$$\begin{aligned}x_{1n}\mathbf{1}\in\langle D_{1n}A_{kn}\mathbf{1}\mid k\in\mathbb{Z}\rangle&\quad\Leftrightarrow\quad\Sigma_1=\infty,\\x_{2n}\mathbf{1}\in\langle D_{2n}A_{kn}\mathbf{1}\mid k\in\mathbb{Z}\rangle&\quad\Leftrightarrow\quad\Sigma_2=\infty,\\x_{3n}\mathbf{1}\in\langle D_{3n}A_{kn}\mathbf{1}\mid k\in\mathbb{Z}\rangle&\quad\Leftrightarrow\quad\Sigma_3=\infty,\end{aligned}$$

where $\Sigma_r=\sum_{k\in\mathbb{Z}}\frac{b_{rk}}{b_{1k}+b_{2k}+b_{3k}}$.

Case IV. *Approximation of D_{rn} by $x_{rk}A_{kn}$.*

$$\begin{aligned}x_{1k}A_{kn}&=x_{1k}^2D_{1n}+x_{1k}x_{2k}D_{2n}+x_{1k}x_{3k}D_{3n},\\x_{2k}A_{kn}&=x_{1k}x_{2k}D_{1n}+x_{2k}^2D_{2n}+x_{2k}x_{3k}D_{3n},\\x_{3k}A_{kn}&=x_{1k}x_{3k}D_{1n}+x_{2k}x_{3k}D_{2n}+x_{3k}^2D_{3n}.\end{aligned}$$

By Lemmas 5.10–5.12 we have

$$\begin{aligned}D_{1n}\mathbf{1}\in\langle x_{1k}A_{kn}\mathbf{1}\mid k\in\mathbb{Z}\rangle&\quad\Leftrightarrow\quad\Delta(Y_{11},Y_{12},Y_{13})=\infty,\\D_{2n}\mathbf{1}\in\langle x_{2k}A_{kn}\mathbf{1}\mid k\in\mathbb{Z}\rangle&\quad\Leftrightarrow\quad\Delta(Y_{22},Y_{23},Y_{21})=\infty,\\D_{3n}\mathbf{1}\in\langle x_{3k}A_{kn}\mathbf{1}\mid k\in\mathbb{Z}\rangle&\quad\Leftrightarrow\quad\Delta(Y_{33},Y_{31},Y_{32})=\infty,\end{aligned}$$

where Y_{kr} for $1\leq k,r\leq 3$ are defined by (5.13)–(5.20).

Case V. By Lemma 2.8 we have two conditions

$$\begin{aligned}&(A)\quad\Sigma_1^+(t)=\infty,\ \det t=1\quad\text{or}\quad\Sigma_1^-(t)=\infty,\ \det t=-1,\\&(B)\quad\Sigma_1^+(t)<\infty\quad\text{or}\quad\Sigma_1^-(t)<\infty\quad\text{but}\quad\Sigma_2(t)=\infty,\end{aligned}\tag{4.7}$$

where $\Sigma_1^+(t)$, $\Sigma_1^-(t)$, $\Sigma_2(t)$ are defined respectively by (2.31), (2.32) and (2.28). The rest of this section will be devoted to the proof of Lemma 4.1. □

4.1.1. Notations and the change of the variables

In what follows we will systematically use the following notations:

$$S_r(3) = \sum_{n\in\mathbb{Z}} \frac{b_{rn}^2}{b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}}, \quad 1 \le r \le 3, \tag{4.8}$$

$$\Sigma_r := \sum_{n\in\mathbb{Z}} \frac{b_{rn}}{b_{1n} + b_{2n} + b_{3n}}, \quad 1 \le r \le 3, \tag{4.9}$$

$$\Sigma^{rs} := \sum_{k\in\mathbb{Z}} \frac{b_{rk}}{b_{sk}}, \quad 1 \le r \ne s \le 3, \quad C_k = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}, \tag{4.10}$$

$$y_{123} = (y_1, y_2, y_3), \quad \text{where} \quad y_r := \|Y_r\|^2, \tag{4.11}$$

$$y^{(k)} = (y_1^{(k)}, y_2^{(k)}, y_3^{(k)}) = (\|Y_1^{(k)}\|^2, \|Y_2^{(k)}\|^2, \|Y_3^{(k)}\|^2),\ 1 \le k \le 3, \tag{4.12}$$

$$y = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} y_1^{(1)} & y_2^{(1)} & y_3^{(1)} \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{pmatrix}, \ \text{where } y_s^{(r)} := \|Y_s^{(r)}\|^2, \tag{4.13}$$

$$\Sigma_{123}(s) = (\Sigma_{12}(s_{12}), \Sigma_{23}(s_{23}), \Sigma_{13}(s_{13})), \quad s = (s_{12}, s_{23}, s_{13}). \tag{4.14}$$

The expressions $S_r(3)$ in the case $m = 3$ can be generalized for an arbitrary $m \in \mathbb{N}$ as follows:

$$S_k(m) = \sum_{n\in\mathbb{Z}} \frac{b_{kn}^2}{\sum_{1\le r<s\le m} b_{rn}b_{sn}}, \quad 1 \le k \le m. \tag{4.15}$$

Lemma 4.2. *We have*

$$S_1(3) + S_2(3) + S_3(3) = \infty, \tag{4.16}$$

$$\|Y_r^{(r)}\|^2 \sim S_r(3) \quad for\ all \quad 1 \le r \le 3, \tag{4.17}$$

$$\|Y_r^{(s)}\|^2 < \frac{1}{2} S_r(3) \quad for\ all \quad 1 \le r \ne s \le 3, \tag{4.18}$$

$$\|Y_1^{(i_1)}\|^2 + \|Y_2^{(i_2)}\|^2 + \|Y_3^{(i_3)}\|^2 = \infty, \quad i_1, i_2, i_3 \in \{1, 2, 3\}. \tag{4.19}$$

Proof. Since $3(a^2 + b^2 + c^2) \ge 2(ab + ac + bc)$ we get

$$S_1(3) + S_2(3) + S_3(3) = \sum_{n\in\mathbb{Z}} \frac{b_{1n}^2 + b_{2n}^2 + b_{3n}^2}{b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}} \ge \sum_{k\in\mathbb{Z}} 2/3 = \infty.$$

Further by (4.5)

$$\|Y_r^{(r)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{b_{rn}^2}{b_{rn}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})} \sim S_r(3),$$
$$\|Y_r^{(s)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{b_{rn}^2}{b_{sn}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})} < \frac{1}{2}S_r(3), \ \ s \neq r.$$

To prove (4.19) we get by (4.5)

$$\|Y_1^{(i_1)}\|^2 + \|Y_2^{(i_2)}\|^2 + \|Y_3^{(i_3)}\|^2 =$$
$$\sum_{r=1}^{3}\sum_{n\in\mathbb{Z}} \frac{b_{rn}^2}{b_{i_r n}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})} > \sum_{n\in\mathbb{Z}} \frac{\sum_{r=1}^3 b_{rn}^2}{(\sum_{r=1}^3 b_{rn})^2} = \infty. \qquad \square$$

We make the following change of the variables:

$$\begin{pmatrix} b_{1n} & b_{2n} & b_{3n} \\ a_{1n} & a_{2n} & a_{3n} \end{pmatrix} \to \begin{pmatrix} b'_{1n} & b'_{2n} & b'_{3n} \\ a'_{1n} & a'_{2n} & a'_{3n} \end{pmatrix} = \begin{pmatrix} 1 & d_{2n} := \frac{b_{2n}}{b_{1n}} & d_{3n} := \frac{b_{3n}}{b_{1n}} \\ a_{1n}\sqrt{b_{1n}} & a_{2n}\sqrt{b_{1n}} & a_{3n}\sqrt{b_{1n}} \end{pmatrix}, \qquad (4.20)$$

motivated by the following formulas:

$$d\mu_{(b,a)}(x) = \sqrt{\frac{b}{\pi}}\exp(-b(x-a)^2)dx = \sqrt{\frac{1}{\pi}}\exp(-(x'-a')^2)dx' = d\mu_{(b',a')}(x'),$$
$$d\mu_{(b_2,a_2)}(x) = \sqrt{\frac{b_2}{\pi}}\exp(-b_2(x-a_2)^2)dx = \sqrt{\frac{b_2}{b_1\pi}}\exp\Big(-\frac{b_2}{b_1}(x'-a'_2)^2\Big)dx'$$
$$= d\mu_{(b'_2,a'_2)}(x'), \quad (b',a') = (1, a\sqrt{b}), \quad (b'_2,a'_2) = \Big(\frac{b_2}{b_1}, a_2\sqrt{b_1}\Big).$$

Remark 4.1. All the expressions, given in the list (2.26) (2.27), (2.28) and (4.1) are invariant under the transformations (4.20)

$$S_{kr}^L(\mu) = \sum_{n\in\mathbb{Z}} \frac{b_{kn}}{2}\left(\frac{1}{2b_{rn}} + a_{rn}^2\right), \quad Y_r = \left(a_{rk}\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}\Big)^{-1/2}\right)_{k\in\mathbb{Z}},$$

etc., and $S_r(3)$ that are defined by (3.4).

4.2. Approximation scheme

Case I. *Approximation of x_{kt} by $A_{kn}A_{tn}$.* Recall that we will write 1 if some values $=\infty$ and 0 in the case $<\infty$ (see Remark 3.2). We use the following notation $S:=(S_1(3),S_2(3),S_3(3))$. By Lemma 4.2 we get

$$\sum_{r=1}^{3} S_r(3)=\infty.$$

Therefore, without loss of generality, it suffices to consider the following three cases:

$$(1)\ S=(0,0,1),\quad (2)\ S=(0,1,1),\quad (3)\ S=(1,1,1). \tag{4.21}$$

By Lemma 2.8, the condition of orthogonality $(\mu^3_{(b,a)})^{L_t}\perp\mu^3_{(b,a)}$ for $t\in \pm\mathrm{SL}(3,\mathbb{R})\setminus\{e\}$, i.e., $\Sigma^{\pm}(t)=\Sigma^{\pm}_1(t)+\Sigma_2(t)=\infty$, splits into two cases:

$$\begin{array}{lll} (A)\quad \Sigma^{\pm}_1(t)=\infty, & & \Sigma^{\pm}_1(t)=\sum_{1\leq i<j\leq 3}\Sigma^{\pm}_{ij}(t),\\ (B)\quad \Sigma^{\pm}_1(t)<\infty & \text{but} & \Sigma_2(t)=\infty, \end{array} \tag{4.22}$$

where $\Sigma^{\pm}_1(t)$, $\Sigma^{\pm}_{ij}(t)$ and $\Sigma_2(t)$ are defined by (2.31), (2.32) , (2.33) and (2.28).

4.3. Case $S=(0,0,1)$

Lemma 4.3. *The case $S=(0,0,1)$ is equivalent with*

$$\Sigma^{13}+\Sigma^{23}<\infty,\quad S_3(3)\sim\sum_n\frac{b_{3n}^2}{b_{1n}b_{2n}}=\infty. \tag{4.23}$$

Proof. To prove the first part of (4.23) we set $c_n=\frac{b_{3n}}{b_{1n}+b_{2n}}$ and note that

$$\infty>S_1(3)+S_2(3)=\sum_{n\in\mathbb{Z}}\frac{b_{1n}^2+b_{2n}^2}{b_{1n}b_{2n}+b_{1n}b_{3n}+b_{2n}b_{3n}}\overset{(2.22)}{\sim}$$
$$\sum_{n\in\mathbb{Z}}\frac{b_{1n}^2+b_{2n}^2}{(b_{1n}+b_{2n}+b_{3n})^2-b_{3n}^2}\sim\sum_{n\in\mathbb{Z}}\frac{(b_{1n}+b_{2n})^2}{(b_{1n}+b_{2n}+b_{3n})^2-b_{3n}^2}=$$
$$\sum_{n\in\mathbb{Z}}\frac{1}{(1+c_n)^2-c_n^2}=\sum_{n\in\mathbb{Z}}\frac{1}{1+2c_n}\overset{(2.22)}{\sim}\sum_{n\in\mathbb{Z}}\frac{1}{c_n}=\sum_{n\in\mathbb{Z}}\frac{b_{1n}+b_{2n}}{b_{3n}}=\Sigma^{13}+\Sigma^{23}.$$

To prove the second part of (4.23) we have by the first part of (4.23)

$$S_3(3)=\sum_{n\in\mathbb{Z}}\frac{b_{3n}^2}{b_{1n}b_{2n}+b_{1n}b_{3n}+b_{2n}b_{3n}}=\sum_{n\in\mathbb{Z}}\frac{1}{\frac{b_{1n}b_{2n}}{b_{3n}^2}+\frac{b_{1n}}{b_{3n}}+\frac{b_{2n}}{b_{3n}}}\sim\sum_{n\in\mathbb{Z}}\frac{b_{3n}^2}{b_{1n}b_{2n}}.\ \square$$

In the case $S = (0,0,1)$ we have

$$\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) \sim \Delta(Y_3^{(3)}) \sim \|Y_3^{(3)}\|^2 = \infty,$$

so we can approximate $x_{3n}x_{3t}$ using Lemma 5.3 and after that we can approximate x_{3n} by an analogue of Lemma 3.3. *From now on we will say that we can approximate x_{3n} using Lemma 5.3, without mentioning Lemma 3.3.*

We can not approximate x_{1n} and x_{2n} using Lemma 5.1-5.2, since we have

$$\Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}) + \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}) < \infty.$$

We can try to approximate some of D_{rn} for $1 \le r \le 3$ using Lemmas 5.4–5.6, see Section 4.4.4 for details. We have for $1 \le k \le 3$ (see (4.3)):

$$D_{kn}\ \eta\ \mathfrak{A}^3 \Leftrightarrow \Delta_k = \infty, \quad \text{where} \quad \Delta_k := \Delta(Y_k, Y_r, Y_s),$$

and $\{k, r, s\}$ is a cyclic permutation of $\{1,2,3\}$. Recall that by $\Sigma^{12}+\Sigma^{13} < \infty$ we get (see (4.6) for the expressions of $\|Y_r\|^2,\ 1 \le r \le 3$)

$$\|Y_1\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{1n}^2, \quad \|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{2n}^2, \quad \|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{3n}^2. \tag{4.24}$$

By (4.23) we have $\Sigma^{13} + \Sigma^{23} < \infty$. We distinguish two cases:
(1) $\Sigma^{12} < \infty$,
(2) $\Sigma^{12} = \infty$.

In the case (1), since $\Sigma^{12} + \Sigma^{13} < \infty$ we have

$$S_{1,23}^L(\mu, t, s) \stackrel{(2.44)}{=} \sum_{n\in\mathbb{Z}} \left[\frac{t^2}{4}\frac{b_{1n}}{b_{2n}} + \frac{s^2}{4}\frac{b_{1n}}{b_{3n}} + \frac{b_{1n}}{2}\big(-2a_{1n}+ta_{2n}+sa_{3n}\big)^2\right] \sim$$
$$\sum_{n\in\mathbb{Z}} \frac{b_{1n}}{2}\big(-2a_{1n}+ta_{2n}+sa_{3n}\big)^2 \stackrel{(4.24)}{\sim} \|C_1Y_1 + C_2Y_2 + C_3Y_3\|^2.$$

Finally, in the case (1) we can approximate all D_{rn}, $1 \le r \le 3$ using Lemmas 5.4–5.6 and Lemma 8.15 and the proof is finished.

The case (2) can be divided into three cases (if necessary, we can chose an appropriate subsequence of $\left(\frac{b_{1n}}{b_{2n}}\right)_n$):

$$\lim_n \frac{b_{1n}}{b_{2n}} = \begin{cases} (a) & 0 \\ (b) & b > 0 \\ (c) & \infty \end{cases}. \tag{4.25}$$

The case (c) is reduced to the case (a) by exchanging (b_{2n}, a_{2n}) with (b_{1n}, a_{1n}).

This transfomation does not change the first condition in (4.23). In the case (2.a-b), by (4.6) we obtain the following expressions for $\|Y_r\|^2,\ 1 \le r \le 3$:

$$\|Y_1\|^2 = \sum_{n\in\mathbb{Z}} \frac{a_{1n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}} + \frac{1}{2b_{3n}}} = \sum_{k\in\mathbb{Z}} \frac{2b_{1n}a_{1n}^2}{1 + \frac{b_{1n}}{b_{2n}} + \frac{b_{1n}}{b_{3n}}} \stackrel{\Sigma^{13}<\infty}{\sim} \sum_{n\in\mathbb{Z}} 2b_{1n}a_{1n}^2,$$
$$\|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} 2b_{1n}a_{2n}^2, \quad \|Y_3\|^2 = \sum_{n\in\mathbb{Z}} b_{1n}a_{3n}^2.$$

Since
$$\|Y_1\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{1n}^2 \sim S_{11}^L(\mu) = \infty,$$

we have four possibilities for $y_{23} := (y_2, y_3) \in \{0,1\}^2$ as in (4.54), see Section 4.4.4:

	(1.0)	(1.1)	(1.2)	(1.3)
y_1	1	1	1	1
y_2	0	1	0	1
y_3	0	0	1	1

We just follow the instructions given in Remark 4.4. We note that the cases (1.0) and (1.1) can not occur since the following conditions are contradictory:

$$S_{13}^L(\mu) \stackrel{(2.43)}{=} \sum_{n\in\mathbb{Z}} \frac{b_{1n}}{2}\Big(\frac{1}{2b_{3n}} + a_{3n}^2\Big) = \infty, \ \ \|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{3n}^2 < \infty, \ \ \Sigma^{13} \stackrel{(4.23)}{<} \infty.$$

We have two cases (1.2.1) and (1.3.1) according to whether respectively the expressions in (4.58) or (4.60) are divergent. We can approximate in these cases respectively D_{1n} and D_{3n} in (4.56) and all $D_{1n},\ D_{2n},\ D_{3n}$ in (4.57). The proof of irreducibilty is finished in both cases because we have $x_{3n},\ D_{3n}\ \eta\ \mathfrak{A}^3$ and the problem is reduced to the case $m = 2$ [28], since $A_{kn} = \sum_{r=1}^3 x_{rk}D_{rn} - x_{3k}D_{3n} = \sum_{r=1}^2 x_{rk}D_{rn}$.

If the opposite holds, we have two different cases (1.2.0) and (1.3.0). We try to approximate D_{3n} using Lemma 5.15. If one of the expressions $\Sigma_3(D,s)$ or $\Sigma_3^\vee(D,s)$ is divergent for some sequence $s = (s_k)_{k\in\mathbb{Z}}$, we can approximate D_{3k} and the proof is finished, since we have $x_{3n},\ D_{3n}\ \eta\ \mathfrak{A}^3$ and the problem is reduced to the case $m = 2$. Let us suppose, as in Remark 4.6, that for every sequence $s = (s_k)_{k\in\mathbb{Z}}$ we have

$$\Sigma_3(D,s) + \Sigma_3^\vee(D,s) < \infty.$$

Then, in particular, we have for $s^{(3)} = (s_k)_{k\in\mathbb{Z}}$ with $\frac{s_k^2}{b_{3k}} \equiv 1$

$$\infty > \Sigma_3(D, s^{(3)}) + \Sigma_3^\vee(D, s^{(3)}) \sim \Sigma_3(D) + \Sigma_3^\vee(D) =$$
$$\sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2} \overset{(2.22)}{\sim} \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{\frac{1}{2b_{1k}} + a_{1k}^2 + \frac{1}{2b_{2k}} + a_{2k}^2} =$$
$$\sum_k \frac{\frac{b_{1k}}{b_{3k}} + 2b_{1k}a_{3k}^2}{1 + 2b_{1k}a_{1k}^2 + \frac{b_{1k}}{b_{2k}} + 2b_{1k}a_{2k}^2} \overset{(4.23)}{\sim} \sum_k \frac{2b_{1k}a_{3k}^2}{1 + 2b_{1k}a_{1k}^2 + 2b_{1k}a_{2k}^2} =: \Sigma_3^+(D).$$

Remark 4.2. Finally, we have $\Sigma_3^+(D) \sim \sum_k \frac{2a_{3k}^2}{1+2a_{1k}^2+2a_{2k}^2}$, we take $b_{1n} \equiv 1$ by (4.20). In the case (1.2.0) we have $\|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{2n}^2 < \infty$, and therefore $\Sigma_3^+(D) \sim \sum_k \frac{2a_{3k}^2}{1+2a_{1k}^2}$, and hence $\Sigma_3^+(D) = \infty$ by Lemma 4.10. In the case (1.3.0) we have $a_3 = \pm a_1 \pm a_2 + h$ or $a_3 - h = \pm a_1 \pm a_2$, see the proof of Lemma 4.11. Therefore,

$$\infty > \Sigma_3^+(D) \sim \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} \geq \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + 2|a_{1k}||a_{2k}| + a_{2k}^2} =$$
$$\sum_k \frac{a_{3k}^2}{1 + \big(|a_{1k}| + |a_{2k}|\big)^2}, \quad \infty > \Sigma_3^+(D) \sim \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} \geq \tag{4.26}$$
$$\sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2 + (|a_{1k}| - |a_{2k}|)^2} \sim \sum_k \frac{a_{3k}^2}{1 + 2a_{1k}^2 - 2|a_{1k}||a_{2k}| + 2a_{2k}^2}$$
$$\sim \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 - 2|a_{1k}||a_{2k}| + a_{2k}^2} \sim \sum_k \frac{a_{3k}^2}{1 + \big(|a_{1k}| - |a_{2k}|\big)^2}. \tag{4.27}$$

Hence, we have by (4.26) and (4.27)

$$\infty > \Sigma_3^+(D) \geq \sum_k \frac{a_{3k}^2}{1 + \big(\pm a_{1k} \pm a_{2k}\big)^2} = \sum_k \frac{a_{3k}^2}{1 + \big(a_{3k} - h_k\big)^2} = \infty \tag{4.28}$$

by Lemma 4.10, contradiction. Therefore, in both cases we can approximate D_{3n} and the proof is finished.

4.4. Case $S = (0, 1, 1)$

Lemma 4.4. *In the case $S = (0, 1, 1)$ we have*

$$\lim_n d_{2n} = \lim_n d_{3n} = \infty. \tag{4.29}$$

PROOF. Setting as before $d_{rn}=\frac{b_{rn}}{b_{1n}}$, we obtain by (3.4) and (2.22)

$$S_1(3) = \sum_{n\in\mathbb{Z}} \frac{1}{d_{2n}+d_{3n}+d_{2n}d_{3n}} \sim \sum_{n\in\mathbb{Z}} \frac{1}{(1+d_{2n})(1+d_{3n})} < \infty, \tag{4.30}$$

$$S_2(3)=\sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{d_{2n}+d_{3n}+d_{2n}d_{3n}} \overset{(2.22)}{\sim} \sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{(1+d_{2n})(d_{2n}+d_{3n})} = \infty, \tag{4.31}$$

$$S_3(3)= \sum_{n\in\mathbb{Z}} \frac{d_{3n}^2}{d_{2n}+d_{3n}+d_{2n}d_{3n}} \overset{(2.22)}{\sim} \sum_{n\in\mathbb{Z}} \frac{d_{3n}^2}{(1+d_{3n})(d_{2n}+d_{3n})} = \infty. \tag{4.32}$$

Suppose that $d_{2n} \leq C$ for all $n \in \mathbb{Z}$. Then by (4.30) and (4.31) we conclude

$$S_1(3) \sim \sum_{n\in\mathbb{Z}} \frac{1}{(1+d_{2n})(1+d_{3n})} \sim \sum_{n\in\mathbb{Z}} \frac{1}{1+d_{3n}} \sim \sum_{n\in\mathbb{Z}} \frac{1}{d_{3n}} < \infty, \quad \infty = S_2(3)$$

$$\sim \sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{(1+d_{2n})(d_{2n}+d_{3n})} \sim \sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{d_{2n}+d_{3n}} \leq \sum_{n\in\mathbb{Z}} \frac{C^2}{C+d_{3n}} \overset{(2.22)}{\sim} \sum_{n\in\mathbb{Z}} \frac{1}{d_{3n}} < \infty,$$

a contradiction. We use the fact that for any fixed $D > 0$ the function

$$f_D(x) = \frac{x^2}{x + D}$$

is strictly increasing when $x > 0$. Similarly, if we suppose that $d_{3n} \leq C$ for all $n \in \mathbb{Z}$ we will obtain a contradiction too. □

Lemma 4.5. *The case* $S = (0, 1, 1)$ *is equivalent with*

$$S_1(3) \sim \sum_n \frac{b_{1n}^2}{b_{2n}b_{3n}} < \infty, \;\; S_2(3) \sim \sum_n \frac{1}{d_n} = \infty, \;\; S_3(3) \sim \sum_n d_n = \infty. \tag{4.33}$$

PROOF. Recall that $d_n = \frac{d_{3n}}{d_{2n}}$. Denote $D_n := 1 + d_{2n}^{-1} + d_{3n}^{-1}$. By Lemma 4.4 we have

$$1 \leq D_n = 1 + d_{2n}^{-1} + d_{3n}^{-1} \leq C, \quad \text{for all} \quad n \in \mathbb{Z}. \tag{4.34}$$

Therefore, we get

$$S_1(3)=\sum_{n\in\mathbb{Z}} \frac{1}{d_{2n} + d_{3n}+d_{2n}d_{3n}} = \sum_{n\in\mathbb{Z}} \frac{1}{D_n d_{2n}d_{3n}} \sim \sum_{n\in\mathbb{Z}} \frac{1}{d_{2n}d_{3n}} = \sum_n \frac{b_{1n}^2}{b_{2n}b_{3n}},$$

$$S_2(3)=\sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{d_{2n}+d_{3n}+d_{2n}d_{3n}} = \sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{D_n d_{2n}d_{3n}} \sim \sum_{n\in\mathbb{Z}} \frac{1}{d_n},$$

$$S_3(3) = \sum_{n\in\mathbb{Z}} \frac{d_{3n}^2}{d_{2n}+d_{3n}+d_{2n}d_{3n}} = \sum_{n\in\mathbb{Z}} \frac{d_{3n}^2}{D_n d_{2n}d_{3n}} \sim \sum_{n\in\mathbb{Z}} d_n. \qquad □$$

By Lemma 4.2, (4.18) we get $\|Y_1^{(r)}\|^2<\infty,\ 1\le r\le 3$ therefore, we get

Lemma 4.6. *In the case $S=(0,1,1)$ we have*

$$\begin{aligned}&\Delta(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)})<\infty,\quad \Delta(Y_2^{(2)},Y_3^{(2)},Y_1^{(2)})\sim\Delta(Y_2^{(2)},Y_3^{(2)}),\\ &\Delta(Y_3^{(3)},Y_1^{(3)},Y_2^{(3)})\sim\Delta(Y_3^{(3)},Y_2^{(3)}).\end{aligned}\tag{4.35}$$

PROOF. Set $(f_1,f_2,f_3)=(Y_3^{(3)},Y_1^{(3)},Y_2^{(3)})$. Then

$$\Delta(f_1,f_2,f_3)\overset{(8.15)}{=}\frac{\Gamma(f_1)+\Gamma(f_1,f_2)+\Gamma(f_1,f_3)+\Gamma(f_1,f_2,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}>$$
$$\frac{\Gamma(f_1)+\Gamma(f_1,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}\overset{(4.36)}{\ge}\frac{\Gamma(f_1)+\Gamma(f_1,f_3)}{(1+\Gamma(f_2))(1+\Gamma(f_3))}\sim\Delta(f_1,f_3),$$

since $f_2\in l_2(\mathbb{Z})$. Indeed, for $f,g\in l_2(\mathbb{Z})$ and $f\in l_2(\mathbb{Z}),g\notin l_2(\mathbb{Z})$ we have respectively

$$\begin{aligned}&\Gamma(f,g)\le\Gamma(f)\Gamma(g)<\infty,\ \ \Gamma(f,g)\le\Gamma(f)\Gamma(g),\ \ \text{where}\ \ \Gamma(f,g),\\ &\Gamma(g)\quad\text{are defined by}\quad \Gamma(f,g):=\lim_n\Gamma(f_{(n)},g_{(n)})\quad \Gamma(g):=\lim_n\Gamma(g_{(n)}),\end{aligned}\tag{4.36}$$

and $g_{(n)}:=(g_k)_{k=-n}^n\in\mathbb{R}^{2n+1}$. Similarly, set $(f_1,f_2,f_3)=(Y_2^{(2)},Y_3^{(2)},Y_1^{(2)})$, then

$$\Delta(f_1,f_2,f_3)\overset{(8.15)}{=}\frac{\Gamma(f_1)+\Gamma(f_1,f_2)+\Gamma(f_1,f_3)+\Gamma(f_1,f_2,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}>$$
$$\frac{\Gamma(f_1)+\Gamma(f_1,f_2)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}\overset{(4.36)}{\ge}\frac{\Gamma(f_1)+\Gamma(f_1,f_2)}{(1+\Gamma(f_2))(1+\Gamma(f_3))}\sim\Delta(f_1,f_2),$$

since $f_3\in l_2(\mathbb{Z})$. Finally, we derive both equivalences in (4.35). To prove that
$\Delta(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)})<\infty$ we set $(f_1,f_2,f_3)=(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)})$, and note that

$$\Delta(f_1,f_2,f_3)\overset{(8.15)}{=}\frac{\Gamma(f_1)+\Gamma(f_1,f_2)+\Gamma(f_1,f_3)+\Gamma(f_1,f_2,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}\le$$
$$\frac{\Gamma(f_1)\Big(1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)\Big)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}=\Gamma(f_1)<\infty.\qquad\square$$

In order to approximate x_{2n} or x_{3n}, it remains to study when

$$\Delta(Y_2^{(2)},Y_3^{(2)})=\infty,\quad \Delta(Y_3^{(3)},Y_2^{(3)})=\infty,\tag{4.37}$$

where $\Delta(f_1, f_2) = \frac{\Gamma(f_1)+\Gamma(f_1,f_2)}{1+\Gamma(f_2)}$. For $2 \le r \le 3$, denote

$$\rho_r(C_2, C_3) := \|C_2 Y_2^{(r)} + C_3 Y_3^{(r)}\|^2, \quad (C_2, C_3) \in \mathbb{R}^2, \tag{4.38}$$

$$\nu(C_1, C_2, C_3) := \|C_1 Y_1 + C_2 Y_2 + C_3 Y_3\|^2,\ (C_1, C_2, C_3) \in \mathbb{R}^3. \tag{4.39}$$

Lemma 4.7. *In the case $S = (0, 1, 1)$ we have*

$$\rho_2(C_2, C_3) \sim \sum_{n\in\mathbb{Z}} \frac{\big(C_2 + C_3 d_n\big)^2}{1 + 2d_n},\ \rho_3(C_2, C_3) \sim \sum_{n\in\mathbb{Z}} \frac{\big(C_2 + C_3 d_n\big)^2}{d_n^2 + 2d_n} \tag{4.40}$$

$$= \sum_{n\in\mathbb{Z}} \frac{\big(C_2 l_n + C_3\big)^2}{1 + 2l_n}, \quad \nu(C_1, C_2, C_3) \sim \sum_{n\in\mathbb{Z}} b_{1n}\Big(\sum_{r=1}^{3} C_r a_{rn}\Big)^2. \tag{4.41}$$

Proof. Set as before $d_n = \frac{d_{3n}}{d_{2n}}$. By (4.5) and (4.6) we get

$$\|Y_2^{(2)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{d_{2n}^2 + 2(d_{2n} + d_{3n} + d_{2n}d_{3n})} = \sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{d_{2n}^2 + 2D_n d_{2n} d_{3n}} \sim \sum_{n\in\mathbb{Z}} \frac{1}{D_n d_n},$$

$$\|Y_3^{(2)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{d_{3n}^2}{d_{2n}^2 + 2(d_{2n} + d_{3n} + d_{2n}d_{3n})} = \sum_{n\in\mathbb{Z}} \frac{d_{3n}^2}{d_{2n}^2 + 2D_n d_{2n} d_{3n}} =$$

$$\sum_{n\in\mathbb{Z}} \frac{d_n^2}{1 + 2D_n d_n}, \quad \|Y_2^{(3)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{d_{3n}^2 + 2(d_{2n} + d_{3n} + d_{2n}d_{3n})} =$$

$$\sum_{n\in\mathbb{Z}} \frac{d_{2n}^2}{d_{3n}^2 + 2D_n d_{2n} d_{3n}} = \sum_{n\in\mathbb{Z}} \frac{1}{d_n^2 + 2D_n d_n},$$

$$\|Y_3^{(3)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{d_{3n}^2}{d_{3n}^2 + 2(d_{2n} + d_{3n} + d_{2n}d_{3n})} = \sum_{n\in\mathbb{Z}} \frac{d_{3n}^2}{d_{3n}^2 + 2D_n d_{2n} d_{3n}} \sim \sum_{n\in\mathbb{Z}} \frac{d_n}{D_n},$$

$$\|Y_1\|^2 = \sum_{n\in\mathbb{Z}} \frac{a_{1n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}} + \frac{1}{2b_{3n}}} = \sum_{k\in\mathbb{Z}} \frac{2b_{1n} a_{1n}^2}{1 + d_{2n}^{-1} + d_{3n}^{-1}} = \sum_{n\in\mathbb{Z}} \frac{2b_{1n} a_{1n}^2}{D_n},$$

$$\|Y_2\|^2 = \sum_{n\in\mathbb{Z}} \frac{2b_{1n} a_{2n}^2}{D_n}, \quad \|Y_3\|^2 = \sum_{n\in\mathbb{Z}} \frac{b_{1n} a_{3n}^2}{D_n}. \tag{4.42}$$

Recall that $d_{rn} = \frac{b_{rn}}{b_{1n}}$. By (4.34), we obtain

$$\|Y_2^{(2)}\|^2 \sim \sum_{n\in\mathbb{Z}} \frac{1}{1 + 2d_n} \sim \sum_{n\in\mathbb{Z}} \frac{1}{d_n}, \quad \|Y_3^{(2)}\|^2 \sim \sum_{n\in\mathbb{Z}} \frac{d_n^2}{1 + 2d_n}, \tag{4.43}$$

$$\|Y_2^{(3)}\|^2 \sim \sum_{n\in\mathbb{Z}} \frac{1}{d_n^2 + 2d_n}, \quad \|Y_3^{(3)}\|^2 \sim \sum_{n\in\mathbb{Z}} \frac{d_n^2}{d_n^2 + 2d_n} \sim \sum_{n\in\mathbb{Z}} d_n,$$

$$\|Y_1\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{1n}^2,\ \ \|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{2n}^2,\ \ \|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{3n}^2,$$
$$\|C_1Y_1 + C_2Y_2 + C_3Y_3\|^2 \overset{(4.34)}{\sim} \sum_{n\in\mathbb{Z}} b_{1n}\big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\big)^2. \quad (4.44)$$

By (4.43) and (4.44) the proof is finished. □

4.4.1. Approximation of x_{2n}, x_{3n}

To approximate x_{2n}, x_{3n}, we need several lemmas. Denote $l_n = d_n^{-1}$.

Lemma 4.8. *The following five series are equivalent:*

$$(i-ii) \quad \sum_{n\in\mathbb{Z}} \frac{\big(C_2-C_3d_n\big)^2}{1+2d_n} \sim \sum_{n\in\mathbb{Z}} c_n^2, \quad (4.45)$$

$$(iii-iv) \quad \sum_{n\in\mathbb{Z}} \frac{\big(C_2l_n-C_3\big)^2}{1+2l_n} \sim \sum_{n\in\mathbb{Z}} e_n^2, \quad (4.46)$$

$$(v)\ \Sigma_{23}(s)=\sum_{n\in\mathbb{Z}} \left(s^2\sqrt{\frac{b_{2n}}{b_{3n}}}-s^{-2}\sqrt{\frac{b_{3n}}{b_{2n}}}\right)^2=\sum_{n\in\mathbb{Z}}\left(\frac{s^2}{\sqrt{d_n}}-\frac{\sqrt{d_n}}{s^2}\right)^2, \quad (4.47)$$

where

$$d_n=C_2C_3^{-1}(1+c_n),\ \ l_n=C_3C_2^{-1}(1+e_n),\ \ s^4=C_2C_3^{-1}>0, \quad l_n=d_n^{-1}. \quad (4.48)$$

Proof. To prove (4.45) and (4.46) we get by Lemma 2.5 using (4.48)

$$\sum_{n\in\mathbb{Z}} \frac{\big(C_2-C_3d_n\big)^2}{1+2d_n} = \sum_{n\in\mathbb{Z}} \frac{C_2^2c_n^2}{1+2C_2C_3^{-1}(1+c_n)} \sim \sum_{n\in\mathbb{Z}} c_n^2,$$
$$\sum_{n\in\mathbb{Z}} \frac{\big(C_2l_n-C_3\big)^2}{1+2l_n} = \sum_{n\in\mathbb{Z}} \frac{C_3^2e_n^2}{1+2C_3C_2^{-1}(1+e_n)} \sim \sum_{n\in\mathbb{Z}} e_n^2.$$

To finish the proof we make use of the following lemma □

Lemma 4.9. *Let $(c_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers with $1+c_n>0$ and $(1+c_n)(1+e_n)=1$. Then the following three series are equivalent:*

$$\sum_{n\in\mathbb{Z}} \left((1+e_n)^{1/2}-(1+e_n)^{-1/2}\right)^2, \quad \sum_{n\in\mathbb{Z}} c_n^2 \quad and \quad \sum_{n\in\mathbb{Z}} e_n^2.$$

Proof. Set $s^4 = C_2 C_3^{-1}$, replacing $1+c_n$ by $(1+e_n)^{-1}$ in Lemma 8.2 gives

$$\Sigma_{23}(s) = \sum_{n\in\mathbb{Z}} \left((1+c_n)^{-1/2} - (1+c_n)^{1/2}\right)^2 = \sum_{n\in\mathbb{Z}} \left((1+e_n)^{1/2} - (1+e_n)^{-1/2}\right)^2.$$

Therefore, $\sum_{n\in\mathbb{Z}} \frac{c_n^2}{1+c_n} = \sum_{n\in\mathbb{Z}} \frac{e_n^2}{1+e_n}$ and hence, by Lemma 2.5, the two series are equivalent: $\sum_{n\in\mathbb{Z}} c_n^2 \sim \sum_{n\in\mathbb{Z}} e_n^2$. □

4.4.2. Two remaining possibilities

By Lemma 4.8 there are only two cases:
(1) when $\rho_2(C_2,C_3) = \rho_3(C_2,C_3) = \infty$ for all $(C_2,C_3) \in \mathbb{R}^2 \setminus \{0\}$,
(2) when both $\rho_2(C_2,C_3)$ and $\rho_3(C_2,C_3)$ are finite and hence, $\Sigma_{23}(s) < \infty$.
To illustrate this we start with the following example

Example 4.1. Set $d_n = n^\alpha$ for $n \in \mathbb{N}$ with $\alpha \in \mathbb{R}$. We have

$$\lim_n d_n = \begin{cases} \infty & \text{if} \quad \alpha > 0 \\ 1 & \text{if} \quad \alpha = 0 \\ 0 & \text{if} \quad \alpha < 0 \end{cases}. \tag{4.49}$$

For the general sequence $(d_n)_{n\in\mathbb{Z}}$ we have four cases (if necessary, we can chose an appropriate subsequence):

$$\lim_n d_n = \begin{cases} (a) & \infty \\ (b) & d > 0 \quad \text{with} \quad \sum_n c_n^2 = \infty \\ (c) & d > 0 \quad \text{with} \quad \sum_n c_n^2 < \infty \\ (d) & d = 0 \end{cases}, \tag{4.50}$$

where $d_n = d(1+c_n)$ and $\lim_n c_n = 0$.

4.4.3. Cases (a), (b), (d)

Remark 4.3. In the cases (a) we see by (4.40) that

$$\rho_2(C_2,C_3) = \rho_3(C_2,C_3) = \infty \quad \text{for all} \quad (C_2,C_3) \in \mathbb{R}^2 \setminus \{0\}.$$

The case (d) is reduced to the case (a) by exchanging (b_{2n}, a_{2n}) with (b_{3n}, a_{3n}). In the cases (b) by Lemma 4.8 and (4.50) we conclude that

$$\rho_2(C_2,-C_3) = \rho_3(C_2,-C_3) = \infty \quad \text{for} \quad C_2 C_3^{-1} > 0$$

hence, $\rho_2(C_2,C_3) = \rho_3(C_2,C_3) = \infty$ for all $(C_2,C_3) \in \mathbb{R}^2 \setminus \{0\}$. Therefore, in cases (a), (b) and (d) we get $x_{2n},\, x_{3n}\, \eta\, \mathfrak{A}^3$.

To finish the proof in these cases, it is sufficient to approximate one of operators D_{rn}, $1 \le r \le 3$ by opertors $(A_{kn})_{k\in\mathbb{Z}}$ using Lemmas 5.4–5.6, see Section 4.4.4. Alternatively we can try to approximate D_{3n}, D_{2n} using Lemma 5.13 and its analogue, see Section 4.4.5, or to approximate D_{3n}, D_{2n} using Lemma 5.14 and its analogue, see Section 4.4.6.

Note that by Lemma 4.4 we have $\lim_n b_{2n} = \lim_n b_{3n} = \infty$. In the cases (a) and (b) the conditions (4.33) are expressed by (4.50) as follows:

$$b = (1, b_{2n}, d_n b_{2n}), \ \sum_n \frac{1}{b_{2n}^2 d_n} < \infty, \ \sum_n \frac{1}{d_n} = \infty, \ \lim_n d_n = \infty, \tag{4.51}$$

$$b = \big(1, b_{2n}, d b_{2n}(1 + c_n)\big), \quad \sum_n \frac{1}{b_{2n}^2} < \infty, \quad \sum_n c_n^2 = \infty. \tag{4.52}$$

Indeed, to get (4.51) we observe that (4.33) are expressed as follows:

$$S_1(3) \sim \sum_n \frac{1}{b_{2n} b_{3n}} = \sum_n \frac{1}{b_{2n}^2 d_n} < \infty, \quad S_2(3) \sim \sum_n \frac{1}{d_n} = \infty.$$

Condition $S_3(3) \sim \sum_n d_n = \infty$ holds by $\lim_n d_n = \infty$.

In order to get (4.52), we express the conditions (4.33) as follows:

$$S_1(3) \sim \sum_n \frac{1}{b_{2n} d b_{2n}(1 + c_n)} \sim \sum_n \frac{1}{b_{2n}^2} < \infty,$$
$$S_2(3) \sim \sum_n \frac{1}{d_n} = \sum_n \frac{1}{1 + c_n} = \infty, \quad S_3(3) \sim \sum_n d_n = \sum_n (1 + c_n) = \infty.$$

The conditions $S_2(3) = \infty$ holds by $\lim_n c_n = 0$.

4.4.4. Approximation of D_{rn}, $1 \le r \le 3$, 1

By Lemmas 5.4–5.6 we have for $1 \le k \le 3$ (see (4.3)):

$$D_{kn} \ \eta \ \mathfrak{A}^3 \Leftrightarrow \Delta_k = \infty, \quad \text{where} \quad \Delta_k := \Delta(Y_k, Y_r, Y_s),$$

and $\{k, r, s\}$ is a cyclic permutation of $\{1, 2, 3\}$.

Recall that by (4.42) we have

$$\|Y_1\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n} a_{1n}^2, \quad \|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n} a_{2n}^2, \quad \|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n} a_{3n}^2. \tag{4.53}$$

Since $\|Y_1\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}a_{1n}^2 \sim S_{11}^L(\mu) = \infty$, we have four possibilities for $y_{23} := (y_2, y_3) \in \{0,1\}^2$:

$$\begin{array}{ccccc} & (1.0) & (1.1) & (1.2) & (1.3) \\ y_1 & 1 & 1 & 1 & 1 \\ y_2 & 0 & 1 & 0 & 1 \\ y_3 & 0 & 0 & 1 & 1 \end{array} \tag{4.54}$$

In the case (1.0) we have $\Delta(Y_1, Y_2, Y_3) \sim \|Y_1\|^2 = \infty$, so we can approximate D_{1n} using Lemma 5.10 and the proof is finished. We should consider the three following cases:

$$(1.1) \quad (1.2) \quad (1.3)$$

In the cases (1.1), (1.2) and (1.3) we have respectively (see the proof of Lemma 4.6)

$$\Delta(Y_1, Y_2, Y_3) \sim \Delta(Y_1, Y_2), \quad \Delta(Y_2, Y_3, Y_1) \sim \Delta(Y_2, Y_1), \tag{4.55}$$

$$\Delta(Y_1, Y_2, Y_3) \sim \Delta(Y_1, Y_3), \quad \Delta(Y_3, Y_1, Y_2) \sim \Delta(Y_3, Y_1), \tag{4.56}$$

$$\Delta(Y_1, Y_2, Y_3), \quad \Delta(Y_2, Y_3, Y_1), \quad \Delta(Y_3, Y_1, Y_2). \tag{4.57}$$

By (4.42) and Lemma 4.4 we have respectively in the cases (1.1)–(1.3):

$$\nu_{12}(C_1, C_2) := \|C_1Y_1 + C_2Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}\Big(C_1a_{1n} + C_2a_{2n}\Big)^2, \tag{4.58}$$

$$\nu_{13}(C_1, C_3) := \|C_1Y_1 + C_3Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}\Big(C_1a_{1n} + C_3a_{3n}\Big)^2, \tag{4.59}$$

$$\nu(C_1, C_2, C_3) = \|C_1Y_1 + C_2Y_2 + C_3Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n}\Big(C_1a_{1n} + C_2a_{2n} + C_3a_{3n}\Big)^2. \tag{4.60}$$

Remark 4.4. We have three cases (1.1.1), (1.2.1) and (1.3.1) according to whether respectively the expressions in (4.58), (4.59) or (4.60) are divergent. We can approximate in these cases respectively D_{1n} and D_{2n} in (4.55), D_{1n} and D_{3n} in (4.56) all D_{1n}, D_{2n}, D_{3n} in (4.57). The proof of irreducibilty is finished in these cases because we have $D_{rn},\ x_{2n},\ x_{3n}\,\eta\,\mathfrak{A}^3$ for some $1 \leq r \leq 3$.

If the opposite holds, we have three different cases:

(1.1.0) $\|C_1Y_1 + C_2Y_2\| < \infty$ for some $(C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}$,

(1.2.0) $\|C_1Y_1 + C_3Y_3\| < \infty$ for some $(C_1, C_3) \in \mathbb{R}^2 \setminus \{0\}$,

(1.3.0) $\nu(C_1, C_2, C_3) < \infty$ for some $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$.

Recall that by (2.44) we have

$$S_{1,23}^{L}(\mu,t,s)=\sum_{n\in\mathbb{Z}}\left[\frac{t^2}{4}\frac{b_{1n}}{b_{2n}}+\frac{s^2}{4}\frac{b_{1n}}{b_{3n}}+\frac{b_{1n}}{2}\Big(-2a_{1n}+ta_{2n}+sa_{3n}\Big)^2\right].$$

Remark 4.5. In the case (1.1.0) we have $\Sigma^{12}=\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{b_{2n}}=\infty$, since

$$S_{1,23}^{L}(\mu,t,0)=\infty,\qquad \text{but}\quad \nu_{12}(C_1,C_2)<\infty,$$

and $\Sigma^{13}=\infty$, since $S_{13}^{L}(\mu)=\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{2}\Big(\frac{1}{2b_{3n}}+a_{3n}^2\Big)=\infty$, but $\|Y_3\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{3n}^2<\infty$, see (2.43) for definition of $S_{kr}^{L}(\mu)$.

In the case (1.2.0) we conclude that $\Sigma^{13}=\infty$, since $S_{1,23}^{L}(\mu,0,s)=\infty$, but $\nu_{13}(C_1,C_3)<\infty$, and $\Sigma^{12}=\infty$, since $S_{12}^{L}(\mu)=\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{2}\Big(\frac{1}{2b_{2n}}+a_{2n}^2\Big)=\infty$, but $\|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{2n}^2<\infty$.

In the case (1.3.0) we have $\Sigma^{12}=\Sigma^{13}=\infty$, since $S_{1,23}^{L}(\mu,t,s)=\infty$, but $\nu(C_1,C_2,C_3)\sim\sum_{n\in\mathbb{Z}}b_{1n}\Big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\Big)^2<\infty$.

So, it remains to consider only the three following cases, when $\Sigma^{12}=\Sigma^{13}=\infty$:

$$(1.1.0)\quad (1.2.0)\quad (1.3.0)$$

4.4.5. Approximation of D_{2n} and D_{3n}, 2

Recall that by Lemma 5.13 we have $D_{3n}\,\eta\,\mathfrak{A}^3\Leftrightarrow\Sigma_3(\mu)=\infty$ where $\Sigma_3(\mu)$ is defined by (5.21)

$$\Sigma_3(\mu):=\sum_{k\in\mathbb{Z}}\frac{\frac{1}{2b_{3k}}}{C_k+a_{1k}^2+a_{2k}^2+a_{3k}^2},\qquad C_k=\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}.$$

Similarly, by analogue of Lemma 5.13 we can prove that $D_{2n}\,\eta\,\mathfrak{A}^3\Leftrightarrow\Sigma_2(\mu)=\infty$, where $\Sigma_2(\mu)$ is defined as follows

$$\Sigma_2(\mu):=\sum_{k\in\mathbb{Z}}\frac{\frac{1}{2b_{2k}}}{C_k+a_{1k}^2+a_{2k}^2+a_{3k}^2}.$$

If one of $\Sigma_2(\mu)$ or $\Sigma_3(\mu)$ is infinite, we can approximate D_{2n} or D_{3n} and the proof is finished. If $\Sigma_2(\mu)+\Sigma_3(\mu)<\infty$, we conclude that

$$\Sigma_{23}(\mu):=\sum_{k\in\mathbb{Z}}\frac{\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}}{C_k+a_{1k}^2+a_{2k}^2+a_{3k}^2}<\infty. \tag{4.61}$$

4.4.6. Approximation of D_{2n} and D_{3n}, 3

By Lemmas 5.11–5.12 we have

$$D_{2l}\mathbf{1} \in \langle x_{2k}A_{kl}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_{22}, Y_{23}, Y_{21}) = \infty,$$
$$D_{3l}\mathbf{1} \in \langle x_{3k}A_{kl}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_{33}, Y_{31}, Y_{32}) = \infty,$$

where vectors Y_{rs} for $2 \leq r \leq 3,\ 1 \leq s \leq 3$ are defined by (5.17)–(5.20). We can not prove that $\Delta(Y_{22}, Y_{23}, Y_{21}) = \infty$ or $\Delta(Y_{33}, Y_{31}, Y_{32}) = \infty$. We can try to approximate D_{3n} using Lemma 5.14 or to approximate D_{2n} using an analogue of Lemma 5.14, but it does not work. Therefore, to approximate D_{3n} we are forced to prove Lemma 5.15 the refinement of Lemma 5.14 and its analogue for D_{2n}, see Remark 4.6 below.

4.4.7. Two technical lemmas

Lemma 4.10. *Let $a_1, a_2 \notin l_2(\mathbb{Z})$ and $C_1a_1 + C_2a_2 \in l_2$ for some $(C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}$, $C_2 \neq 0$, where $a_r = (a_{rk})_{k\in\mathbb{Z}}$, $1 \leq r \leq 2$. Then we have*

$$\sum_{k\in\mathbb{Z}} \frac{a_{1k}^2}{1 + a_{2k}^2} = \infty. \tag{4.62}$$

Proof. We set $Y_r = a_r$, in the case (1.1.0) when $C_1Y_1 + C_2Y_2 = h \in l_2(\mathbb{Z})$ with $C_1C_2 > 0$ (we have $C_1C_2 \neq 0$) we should take $a_2 = -a_1 + h$, in the case when $C_1C_2 < 0$ we take $a_2 = a_1 + h$. The series $\sum_{k\in\mathbb{Z}} \frac{a_{1k}^2}{1+a_{2k}^2}$ will remain equivalent with the initial one, if we replace (C_1, C_2) with $(\pm 1, 1)$ in the expression for h. Fix a small $\varepsilon > 0$ and a large $N \in \mathbb{N}$. Since $|\pm a + b| \leq |a| + |b|$, we get

$$\sum_{k\in\mathbb{Z}} \frac{a_{1k}^2}{1 + a_{2k}^2} = \sum_{k\in\mathbb{Z}} \frac{a_{1k}^2}{1 + \big(\pm a_{1k} + h_k\big)^2} \geq \sum_{k\in\mathbb{Z}} \frac{a_{1k}^2}{1 + a_{1k}^2 + 2|a_{1k}||h_k| + h_k^2} \overset{(2.22)}{\sim}$$
$$\sum_{k\in\mathbb{Z}} \frac{a_{1k}^2}{1 + 2|a_{1k}||h_k| + h_k^2} \overset{(*)}{>} \sum_{k\in\mathbb{Z}_N} \frac{a_{1k}^2}{1 + 2|a_{1k}|\varepsilon + \varepsilon^2} \overset{(2.20)}{\sim} \sum_{k\in\mathbb{Z}_N} a_{1k}^2 = \infty,$$

where $\mathbb{Z}_N := \{n \in \mathbb{Z} \mid |n| > N\}$. The inequality (*) holds, since $h \in l_2(\mathbb{Z})$ and we have $\sum_{k\in\mathbb{Z}_N} h_k^2 < \varepsilon^2$ for sufficiently large $N \in \mathbb{N}$. □

Lemma 4.11. *Let $a_1, a_2, a_3 \notin l_2(\mathbb{Z})$ and $C_1a_1 + C_2a_2 + C_3a_3 \in l_2(\mathbb{Z})$ for some $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$, $C_3 \neq 0$, where $a_r = (a_{rk})_{k\in\mathbb{Z}}$ for $1 \leq r \leq 3$. Then we have*

$$\sum_{k\in\mathbb{Z}} \frac{a_{1k}^2 + a_{2k}^2}{1 + a_{3k}^2} = \infty. \tag{4.63}$$

Proof. We set $Y_r = a_r$, in the case (1.3.0), we have $C_1a_1 + C_2a_2 + C_3a_3 = h \in l_2(\mathbb{Z})$ for some $(C_1, C_2, C_3) \in \mathbb{R}^3$, see Remark 4.4. We can take $C_3 = 1$, then $a_3 = -C_1a_1 - C_2a_2 + h$. When $C_1 = 0$ or $C_2 = 0$ lemma is reduced to Lemma 4.10. Suppose $C_1C_2 \neq 0$. The series $\sum_{k\in\mathbb{Z}} \frac{a_{1k}^2+a_{2k}^2}{1+a_{3k}^2}$ will remain equivalent with the initial one, if we replace (C_1, C_2, C_3) with $(\pm1, \pm1, 1)$ in the expression for h. Fix a small $\varepsilon > 0$ and a large $N \in \mathbb{N}$. Suppose the opposite, i.e.,

$$\infty > \sum_{k\in\mathbb{Z}} \frac{a_{1k}^2 + a_{2k}^2}{1 + \big(\pm a_{1k} \pm a_{2k} + h_k\big)^2}, \text{ then } \infty > \sum_{k\in\mathbb{Z}} \frac{(|a_{1k}| + |a_{2k}|)^2}{1 + \big(\pm a_{1k} \pm a_{2k} + h_k\big)^2}$$
$$\geq \sum_{k\in\mathbb{Z}} \frac{(|a_{1k}| + |a_{2k}|)^2}{1 + a_{1k}^2 + a_{2k}^2 + 2|a_{1k}||a_{2k}| + 2|a_{1k}||h_k| + 2|a_{2k}||h_k| + h_k^2} \overset{(2.22)}{\sim}$$
$$\sum_{k\in\mathbb{Z}} \frac{(|a_{1k}| + |a_{2k}|)^2}{1 + 2|a_{1k}||h_k| + 2|a_{2k}||h_k| + h_k^2} \overset{(*)}{>} \sum_{k\in\mathbb{Z}_N} \frac{(|a_{1k}| + |a_{2k}|)^2}{1 + 2\big(|a_{1k}| + |a_{2k}|\big)\varepsilon + \varepsilon^2} \overset{(2.20)}{\sim}$$
$$\sum_{k\in\mathbb{Z}_N} (|a_{1k}| + |a_{2k}|)^2 = \infty,$$

where $\mathbb{Z}_N := \{n \in \mathbb{Z} \mid |n| > N\}$, contradiction. The inequality (*) holds, since $h \in l_2(\mathbb{Z})$ and we have $\sum_{k\in\mathbb{Z}_N} h_k^2 < \varepsilon^2$ for sufficiently large $N \in \mathbb{N}$. □

Remark 4.6. It is possible to prove an analogue of Lemma 5.15 to approximate D_{2n} with corresponding expressions $\Sigma_2(D, s)$, $\Sigma_2^\vee(D, s)$ and $\Sigma_3(D)$, $\Sigma_3^\vee(D)$. If one of the expressions $\Sigma_2(D, s)$, $\Sigma_2^\vee(D, s)$, $\Sigma_3(D, s)$ or $\Sigma_3^\vee(D, s)$ is divergent for some sequence $s = (s_k)_{k\in\mathbb{Z}}$, we can approximate D_{2k} or D_{3k} and the proof is finished when $S = (0.1.1)$ in the cases (a) and (b). Suppose that for all sequence $s = (s_k)_{k\in\mathbb{Z}}$ we have

$$\Sigma_2(D, s) + \Sigma_2^\vee(D, s) + \Sigma_3(D, s) + \Sigma_3^\vee(D, s) < \infty.$$

Then, in particularly, we have for $s^{(r)} = (s_{rk})_{k\in\mathbb{Z}},\ 2 \leq r \leq 3$ with $\frac{s_{rk}^2}{b_{rk}} \equiv 1$

$$\infty > \Sigma_2(D, s^{(2)}) + \Sigma_2^\vee(D, s^{(2)}) + \Sigma_3(D, s^{(3)}) + \Sigma_3^\vee(D, s^{(3)}) \sim$$
$$\Sigma_2(D) + \Sigma_2^\vee(D) + \Sigma_3(D) + \Sigma_3^\vee(D) = \tag{4.64}$$
$$\sum_k \frac{\frac{1}{2b_{2k}} + a_{2k}^2 + \frac{1}{2b_{3k}} + a_{3k}^2}{C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2} \overset{(2.22)}{\sim} \sum_k \frac{\frac{1}{2b_{2k}} + a_{2k}^2 + \frac{1}{2b_{3k}} + a_{3k}^2}{\frac{1}{2b_{1k}} + a_{1k}^2} =: \Sigma_{23}^\vee(D)$$
$$\sim \sum_k \frac{\frac{b_{1k}}{b_{2k}} + 2b_{1k}a_{2k}^2 + \frac{b_{1k}}{b_{3k}} + 2b_{1k}a_{3k}^2}{1 + 2b_{1k}a_{1k}^2} \overset{(4.34)}{\sim} \sum_k \frac{a_{2k}^2 + a_{3k}^2}{1 + a_{1k}^2} =: \Sigma_{23}^a(D).$$

Remark 4.7. In the case (1.1.0) (resp. the case (1.2.0)) we have $\|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} a_{3n}^2 < \infty$ (resp. $\|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} a_{2n}^2 < \infty$) and therefore,

$$\Sigma_{23}^a(D) \sim \sum_k \frac{a_{2k}^2}{1+a_{1k}^2} = \infty, \quad \text{resp.} \quad \Sigma_{23}^a(D) \sim \sum_k \frac{a_{3k}^2}{1+a_{1k}^2} = \infty,$$

by Lemma 4.10, contradicting (4.64). In the case (1.3.0) we have four cases:
(0) when $C_1C_2C_3 \neq 0$, $C_1a_1 + C_2a_2 + C_3a_3 = h \in l_2(\mathbb{Z})$,
(1) when $C_1 = 0$ hence, $C_2C_3 \neq 0$, $C_2a_2 + C_3a_3 = h \in l_2(\mathbb{Z})$,
(2) when $C_2 = 0$ hence, $C_1C_3 \neq 0$, $C_1a_1 + C_3a_3 = h \in l_2(\mathbb{Z})$,
(3) when $C_3 = 0$ hence, $C_1C_2 \neq 0$ $C_1a_1 + C_2a_2 = h \in l_2(\mathbb{Z})$.

In the case (0) we have $\Sigma_{23}^a(D) = \infty$ by Lemma 4.11, contradicting (4.64). In the cases (2) and (3) we get $\Sigma_{23}^a(D) = \infty$ by Lemma 4.10, contradicting (4.64). Therefore, one of the expressions $\Sigma_2(D,s)$, $\Sigma_2^\vee(D,s)$, $\Sigma_3(D,s)$ or $\Sigma_3^\vee(D,s)$ is convergent hence, we can approximate D_{2n} or D_{3n} and the proof is finished. To study the case (1) we need the following statement.

Lemma 4.12. *Let $C_2Y_2 + C_3Y_3 = h_{23} \in l_2$ for some $(C_2, C_3) \in \big(\mathbb{R}\setminus\{0\}\big)^2$ and $C_1Y_1 + C_2Y_2 \notin l_2$ or $C_1Y_1 + C_3Y_3 \notin l_2$ for all $(C_1, C_r) \in \big(\mathbb{R}\setminus\{0\}\big)^2$, then*

$$\Delta(Y_1, Y_2, Y_3) = \infty. \tag{4.65}$$

Proof. To prove (4.65) we have by (8.15)

$$\Delta(Y_1,Y_2,Y_3) = \frac{\Gamma(Y_1)+\Gamma(Y_1,Y_2)+\Gamma(Y_1,Y_3)+\Gamma(Y_1,Y_2,Y_3)}{1+\Gamma(Y_2)+\Gamma(Y_3)+\Gamma(Y_2,Y_3)} \stackrel{(*)}{>}$$
$$\frac{\Gamma(Y_1,Y_2)+\Gamma(Y_1,Y_3)}{1+(1+c_2)\Gamma(Y_2)+\Gamma(Y_3)} \sim \frac{\Gamma(Y_1,Y_2)+\Gamma(Y_1,Y_3)}{\Gamma(Y_2)+\Gamma(Y_3)} \stackrel{(4.67)}{\sim}$$
$$\frac{\Gamma(Y_1,Y_2)+\Gamma(Y_1,Y_3)}{2\Gamma(Y_2)} \stackrel{(4.67)}{\sim} \frac{\Gamma(Y_1,Y_2)}{\Gamma(Y_2)} + \frac{\Gamma(Y_1,Y_3)}{\Gamma(Y_3)} = \infty, \tag{4.66}$$
$$\Gamma(Y_2) \sim \Gamma(Y_3), \quad \text{since} \quad C_2Y_2 + C_3Y_3 = h \in l_2. \tag{4.67}$$

The relation (*) holds by the inequality $\Gamma(Y_2,Y_3) \leq c_2\Gamma(Y_2)$, since $C_2Y_2 + C_3Y_3 \in l_2$ for some $(C_1, C_3) \in \big(\mathbb{R}\setminus\{0\}\big)^2$, the relation (4.66) holds by Lemma 8.11. To prove (4.67) we get since $Y_2 \notin l_2$ and $h \in l_2$,

$$\frac{\Gamma(Y_3)}{\Gamma(Y_2)} = \frac{\|Y_3\|^2}{\|Y_2\|^2} = \frac{\|Y_2+h\|^2}{\|Y_2\|^2} \leq \Big(\frac{\|Y_2\|+\|h\|}{\|Y_2\|}\Big)^2 = 1,$$

If $C_1Y_1+C_2Y_2 \notin l_2$ for all $(C_1,C_2) \in \big(\mathbb{R}\setminus\{0\}\big)^2$, or $C_1Y_1+C_3Y_3 \notin l_2$ for all $(C_1,C_3) \in \big(\mathbb{R}\setminus\{0\}\big)^2$, by Lemma 4.12 we get $\Delta(Y_1,Y_2,Y_3)=\infty$ hence, we can approximate D_{1n} using Lemma 5.10 and the proof is finished. If $C_1Y_1+C_2Y_2 = h_{12} \in l_2$ for some for $(C_1,C_2) \in \big(\mathbb{R}\setminus\{0\}\big)^2$ or $C_1Y_1+C_3Y_3 = h_{13} \in l_2$ for some $(C_1,C_3) \in \big(\mathbb{R}\setminus\{0\}\big)^2$, then we have $h_{12}+\alpha h_{23} = C_1Y_1+C_2Y_2+C_3Y_3 \in l_2$ or $h_{12}+\beta h_{13} = C_1Y_1+C_2Y_2+C_3Y_3 \in l_2$ with $C_1C_2C_3 \neq 0$ for an appropriate $\alpha\beta \neq 0$, and we are in the case (0). □

4.4.8. Case (c)

In this case both $\rho_2(C_2,-C_3)$ and $\rho_3(C_2,-C_3)$ are finite, i.e., we are in the case (2) therefore, we can not approximate $x_{2n}x_{2t}$, $x_{3n}x_{3t}$ by Lemmas 5.2–5.3. By Lemma 4.8 $\Sigma_{23}(s)<\infty$ and hence, $\Sigma_{23}(C_2,C_3)=\infty$. Indeed, reasoning as in Remark 2.5 we see that

$$\mu^{L_{\tau_{23}(\phi,s)}} \perp \mu,\ \ \phi \in [0,2\pi),\ \ s>0 \Leftrightarrow \Sigma_{23}(s)+\Sigma_{23}(C_2,C_3)=\infty,\ \ s>0, \quad (4.68)$$

for $(C_2,C_3) \in \mathbb{R}^2\setminus\{0\}$. where $\tau_{23}(\phi,s)$, $\Sigma_{23}(s)$ and $\Sigma_{23}(C_2,C_3)$ are defined as follows:

$$\tau_{23}(\phi,s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & s^2\sin\phi \\ 0 & s^{-2}\sin\phi & -\cos\phi \end{pmatrix}, \quad (4.69)$$

$$\Sigma_{ij}(s) = \sum_{n\in\mathbb{Z}} \left(s^2\sqrt{\frac{b_{in}}{b_{jn}}} - s^{-2}\sqrt{\frac{b_{jn}}{b_{in}}} \right)^2, \quad s \in \mathbb{R}\setminus\{0\}, \quad (4.70)$$

$$\Sigma_{ij}(C_i,C_j) = \sum_{n\in\mathbb{Z}} (C_i^2 b_{in} + C_j^2 b_{jn})(C_i a_{in} + C_j a_{jn})^2. \quad (4.71)$$

In this case there are four possibilities for the pair $(\Sigma^{12},\Sigma^{13})$:
(2.1) $(\Sigma^{12},\Sigma^{13})=(0,0)$, i.e., $\Sigma^{12}<\infty$ and $\Sigma^{13}<\infty$,
(2.2) $(\Sigma^{12},\Sigma^{13})=(0,1)$, i.e., $\Sigma^{12}<\infty$, but $\Sigma^{13}=\infty$,
(2.3) $(\Sigma^{12},\Sigma^{13})=(1,0)$, i.e., $\Sigma^{12}=\infty$, but $\Sigma^{13}<\infty$,
(2.4) $(\Sigma^{12},\Sigma^{13})=(1,1)$, i.e., $\Sigma^{12}=\infty$ and $\Sigma^{13}=\infty$.

Lemma 4.13. *In the case* (2.1)*, i.e., when* $(\Sigma^{12},\Sigma^{13})=(0,0)$*, we can approximate* D_{rn} *for* $1\leq r\leq 3$*, hence the representation is irreducible.*

Proof. Let $\Sigma^{12} < \infty$ and $\Sigma^{13} < \infty$ we have by (4.41)

$$\nu(C_1, C_2, C_3) \sim \sum_{k\in\mathbb{Z}} b_{1k}(C_1 a_{1k} + C_2 a_{2k} + C_3 a_{3k})^2$$
$$\overset{(2.1)}{\sim} \sum_{k\in\mathbb{Z}} \left[\frac{t^2}{4}\frac{b_{1k}}{b_{2k}} + \frac{s^2}{4}\frac{b_{1k}}{b_{3k}} + \frac{b_{1k}}{2}(-2a_{1k} + ta_{2k} + sa_{3k})^2\right] \overset{(2.44)}{=} S^L_{1,23}(\mu, t, s) = \infty.$$

Hence, D_{1n}, D_{2n}, D_{3n} η $\mathfrak{A}^3$ and the proof is finished. □

Remark 4.8. The cases (2.2) and (2.3) do not occur.

Indeed, by Lemma 4.8 the three series $\Sigma_{23}(s)$ (defined by (4.70)), $\sum_{n\in\mathbb{Z}} c_n^2$ and $\sum_{n\in\mathbb{Z}} e_n^2$ are equivalent where $\frac{s^4 b_{2n}}{b_{3n}} = (1 + c_n)$, see Lemma 4.9. In the case (c) we have $\sum_{n\in\mathbb{Z}} c_n^2 < \infty$, therefore, $\lim_n c_n = 0$ and hence, $\lim_n d_n^{-1} = \lim_n \frac{b_{2n}}{b_{3n}} = s^{-4} > 0$. Recall that $d_n = \frac{d_{2n}}{d_{3n}} = \frac{b_{2n}}{b_{3n}}$. But this contradicts $(\Sigma^{12}, \Sigma^{13}) = (0, 1)$, or $(\Sigma^{12}, \Sigma^{13}) = (1, 0)$, since the two series

$$\Sigma^{12} = \sum_n d_{2n}^{-1} \quad \text{and} \quad \Sigma^{13} = \sum_n d_{3n}^{-1}$$

should be equivalent by $\lim_n \frac{d_{2n}}{d_{3n}} = s^{-4} > 0$.

In the **case (2.4)** we have

$$\Sigma_{23}(s) < \infty, \quad \Sigma_{23}(C_2, C_3) = \infty, \quad \Sigma^{12} = \Sigma^{13} = \infty. \tag{4.72}$$

To approximate D_{rn} we need to estimate $\nu(C_1, C_2, C_3)$ defined by (4.39). By (4.41) we have

$$\nu(C_1, C_2, C_3) \sim \sum_{n\in\mathbb{Z}} b_{1n}\Big(\sum_{r=1}^{3} C_r a_{rn}\Big)^2.$$

Since $\|Y_1\|^2 = \sum_{n\in\mathbb{Z}} b_{1n} a_{1n}^2 \sim S^L_{11}(\mu) = \infty$, in the case (2.4) we have four possibilities for $y_{23} := (y_2, y_3) \in \{0, 1\}^2$:

	(2.4.1)	(2.4.2)	(2.4.3)	(2.4.4)
y_1	1	1	1	1
y_2	0	1	0	1
y_3	0	0	1	1

Remark 4.9. The cases (2.4.1)–(2.4.3) are not compatible with the condition $\Sigma_{23}(C_2, C_3) = \infty$ for all $(C_2, C_3) \in \mathbb{R}^2 \setminus \{0\}$.

So it suffices to consider only the **case (2.4.4)** when $y_{123} = (1, 1, 1)$.

The case (2.4.4) splits into two subcases:
(2.4.4.1) when $\Sigma_{12}(s_{12}) < \infty$ (resp. $\Sigma_{13}(s_{13}) < \infty$) for some s_{12}, $s_{13} > 0$,
(2.4.4.2) when both $\Sigma_{12}(s_{12}) = \Sigma_{13}(s_{13}) = \infty$ for all s_{12}, $s_{13} > 0$.

The case (2.4.4.1) does not occur. Indeed, we have in this case $\Sigma_{13}(s_{12}s_{23}) < \infty$ (resp. $\Sigma_{12}(s_{13}s_{23}^{-1}) < \infty$) since

$$\Sigma_{12}(s_{12}) < \infty \Leftrightarrow \mu_{(s_{12}^4 b_1,0)} \sim \mu_{(b_2,0)}, \quad \Sigma_{23}(s_{23}) < \infty \Leftrightarrow \mu_{(s_{23}^4 b_2,0)} \sim \mu_{(b_3,0)}$$

where $\mu_{(b_r,0)} = \otimes_{n\in\mathbb{Z}} \mu_{(b_{rk},0)}$ for $1 \le r \le 3$. Therefore,

$$\mu_{((s_{12}s_{23})^4 b_1,0)} \sim \mu_{(b_3,0)} \quad \Leftrightarrow \quad \Sigma_{13}(s_{12}s_{23}) < \infty.$$

Similarly, if $\Sigma_{13}(s_{13}) < \infty$ and $\Sigma_{23}(s_{23}) < \infty$ we have

$$\mu_{(s_{13}^4 b_1,0)} \sim \mu_{(b_3,0)}, \quad \mu_{(s_{23}^4 b_2,0)} \sim \mu_{(b_3,0)} \Rightarrow \mu_{\left((s_{13}s_{23}^{-1})^4 b_1,0\right)} \sim \mu_{(b_2,0)}$$

hence, $\Sigma_{12}(s_{13}s_{23}^{-1}) < \infty$. But condition $\Sigma_{13}(s_{12}s_{23}) + \Sigma_{12}(s_{13}s_{23}^{-1}) < \infty$ contradicts the first condition of (4.33). Indeed, we have by Lemma 4.9

$$\Sigma_{12}(s) = \sum_{n\in\mathbb{Z}} \left(s^2 \sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2} \sqrt{\frac{b_{2n}}{b_{1n}}} \right)^2 \sim \sum_{n\in\mathbb{Z}} c_n^2 < \infty, \quad s^2 \sqrt{\frac{b_{1n}}{b_{2n}}} = 1 + c_n,$$
$$\Sigma_{13}(s) = \sum_{n\in\mathbb{Z}} \left(s^2 \sqrt{\frac{b_{1n}}{b_{3n}}} - s^{-2} \sqrt{\frac{b_{3n}}{b_{1n}}} \right)^2 \sim \sum_{n\in\mathbb{Z}} f_n^2 < \infty, \quad s^2 \sqrt{\frac{b_{1n}}{b_{3n}}} = 1 + f_n,$$

and $\lim_n c_n = \lim_n f_n = 0$. This contradicts $S_1(3) \sim \sum_n \frac{b_{1n}^2}{b_{2n}b_{3n}} < \infty$. Indeed,

$$\lim_{n\to\infty} \frac{b_{1n}^2}{b_{2n}b_{3n}} = s^{-4} \lim_{n\to\infty} (1 + c_n)^2 (1 + f_n)^2 = s^{-4} > 0.$$

Finally, to finish the case $S = (0, 1, 1)$, we need to consider only the case (2.4.4.2) when $\Sigma_{12}(s_{12}) = \Sigma_{13}(s_{13}) = \infty$ for all s_{12}, $s_{13} > 0$.

By (4.33) and all the previous considerations we have the conditions:

$$S_1(3)\sim\sum_n\frac{b_{1n}^2}{b_{2n}b_{3n}}<\infty,\quad S_2(3)\sim\sum_n\frac{b_{2n}}{b_{3n}}=\infty,\quad S_3(3)\sim\sum_n\frac{b_{3n}}{b_{2n}}=\infty,$$

$$\Sigma_{23}(C_2,C_3)=\infty,\quad \Sigma^{12}=\sum_n\frac{b_{1n}}{b_{2n}}=\infty,\quad \Sigma^{13}=\sum_n\frac{b_{1n}}{b_{3n}}=\infty, \tag{4.73}$$

$$\Sigma_{12}(s_{12})=\Sigma_{13}(s_{13})=\infty \text{ for all } s_{12},\ s_{13}>0,\ \Sigma_{23}(s_{23})<\infty \text{ for some } s_{23}>0.$$

Remark 4.10. By (4.20) without loss of generality we can suppose that (b_{1n}, b_{2n}, b_{3n}) is replaced with $(1, d_{2n}, d_{3n})$. Since $\Sigma_{23}(s)<\infty$, using notations (4.47) and (4.48) of Lemma 4.8

$$\Sigma_{23}(s)=\sum_{n\in\mathbb{Z}}\Big(\frac{s^2}{\sqrt{d_n}}-\frac{\sqrt{d_n}}{s^2}\Big)^2=\sum_{n\in\mathbb{Z}}\Big(s^2\sqrt{\frac{d_{2n}}{d_{3n}}}-s^{-2}\sqrt{\frac{d_{3n}}{d_{2n}}}\Big)^2,$$

and taking into consideration (4.73), we can chose d_{2n} and d_{3n} as follows:

$$d_n=\frac{d_{3n}}{d_{2n}}=s^4(1+c_n),\ \sum_n c_n^2<\infty,\ \sum_n\frac{1}{d_{2n}^2}<\infty,\ \ \sum_n\frac{1}{d_n}=\sum_n d_n=\infty. \tag{4.74}$$

Since $\sum_n c_n^2<\infty$ we have $\sum_n\frac{b_{1n}^2}{b_{2n}b_{3n}}\sim\sum_n\frac{1}{d_{2n}^2}$ and the measures $\mu_{(d_3^{c,s},0)}$ and $\mu_{(d_3^s,0)}$ are equivalent, where

$$\mu_{(d_3^{c,s},0)}=\otimes_n\mu_{(s^4d_{2n}(1+c_n),0)},\qquad \mu_{(d_3^s,0)}=\otimes_n\mu_{(s^4d_{2n},0)},$$

hence, we can choose $c_n\equiv 0$ and $s=1$. So, to finish the case $S=(0,1,1)$ we should prove the irreducibility for $b=(1,d_{2n},d_{2n})_{n\in\mathbb{Z}}$ with the only condition:

$$\sum_n d_{2n}^{-2}<\infty.\quad \text{Since}\quad d_n\equiv 1,\quad \text{we have}\quad \sum_n d_n^{-1}=\sum_n d_n=\infty. \tag{4.75}$$

As usual, $a=(a_{1n},a_{2n},a_{3n})_{n\in\mathbb{Z}}$ should satisfy the orthogonality condition:

$$(\mu^3_{(b,a)})^{L_t}\perp\mu^3_{(b,a)}\quad \text{for all}\quad t\in \mathrm{GL}(3,\mathbb{R})\setminus\{e\}.$$

Example 4.2. The pairwise conditions

$$\|C_rY_r+C_sY_s\|^2=\infty\ \text{ for } 1\le r<s\le 3\quad \text{do not imply}\quad \|\sum_{r=1}^3 C_rY_r\|^2=\infty.$$

Let $a_{r,n} = a_{r,-n}$ for $n \in \mathbb{N}$ and $a_{1,0} = 1,\ a_{2,0} = 2,\ a_{3,0} = 3$. We define $a_{r,n}$ for $n \in \mathbb{N}$ as follows

$$a_{1n} = \begin{cases} 2 & n = 2k+1 \\ 1 & n = 2k \end{cases}, \quad a_{2n} = \begin{cases} 1 & n = 2k+1 \\ 2 & n = 2k \end{cases}, \quad a_{3n} \equiv 3. \tag{4.76}$$

Then we have clearly for arbitrary $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$

$$\|C_1a_1+C_2a_2\|^2=\infty, \quad \|C_1a_1+C_3a_3\|^2=\infty, \quad \|C_2a_2+C_3a_3\|^2=\infty, \tag{4.77}$$

$$\text{but} \quad a_1 + a_2 - a_3 = 0 \quad \text{hence,} \quad \|a_1 + a_2 - a_3\|^2 = 0. \tag{4.78}$$

Example 4.3. Let us consider the measure $\mu^3_{(b,a)}$ with $a = \left(a_{rn}\right)_{r,n}$ from Example 4.2 and $b = (b_{1n}, b_{2n}, b_{3n})$ defined as follows:

$$b_{1n} \equiv 1, \quad d_{2n} = d_{3n} = |n| \quad \text{for} \quad n \in \mathbb{Z} \setminus \{0\}, \quad d_{20} = d_{30} = 1. \tag{4.79}$$

Lemma 4.14. *In Example 4.2 we have (we consider only $n \in \mathbb{N}$)*

$$\Delta(a_1, a_2, a_3) = 2, \quad \Delta(a_2, a_3, a_1) = 2, \quad \Delta(a_3, a_1, a_2) = 2, \tag{4.80}$$

where $a_r = (a_{rn})_{n\in\mathbb{N}},\ 1 \le r \le 3$.

Proof. Set $a_r(n) = (a_{rl})_{l=1}^n$ for $1 \le r \le 3$ and $n \in \mathbb{N}$, then for $1 \le k < r \le 3$

$$\Gamma(a_k(n)) \sim \Gamma(a_1(n)+a_2(n)) \sim n, \quad \Gamma(a_k(n), a_r(n)) \sim \frac{n(n-1)}{2}, \quad \Gamma(a_1, a_2, a_3)=0.$$

We observe that $\Gamma(a_k, a_k + a_r) = \Gamma(a_k, a_r)$ for $k \ne r$. Since $a_3 = a_1 + a_2$ we get

$$\Delta(a_1, a_2, a_3) = \frac{\Gamma(a_1) + \Gamma(a_1, a_2) + \Gamma(a_1, a_3) + \Gamma(a_1, a_2, a_3)}{1 + \Gamma(a_2) + \Gamma(a_3) + \Gamma(a_2, a_3)} =$$

$$\frac{\Gamma(a_1) + \Gamma(a_1, a_2) + \Gamma(a_1, a_1 + a_2) + \Gamma(a_1, a_2, a_1 + a_2)}{1 + \Gamma(a_2) + \Gamma(a_1 + a_2) + \Gamma(a_2, a_1 + a_2)} =$$

$$\frac{\Gamma(a_1) + 2\Gamma(a_1, a_2)}{1 + \Gamma(a_2) + \Gamma(a_1 + a_2) + \Gamma(a_1, a_2)} = 2,$$

$$\Delta(a_2, a_3, a_1) = \frac{\Gamma(a_2) + \Gamma(a_2, a_3) + \Gamma(a_2, a_1) + \Gamma(a_2, a_3, a_1)}{1 + \Gamma(a_3) + \Gamma(a_1) + \Gamma(a_3, a_1)} =$$

$$\frac{\Gamma(a_2) + \Gamma(a_2, a_1 + a_2) + \Gamma(a_2, a_1) + \Gamma(a_2, a_1 + a_2, a_1)}{1 + \Gamma(a_1 + a_2) + \Gamma(a_1) + \Gamma(a_1 + a_2, a_1)} =$$

$$\frac{\Gamma(a_2) + 2\Gamma(a_2, a_1)}{1 + \Gamma(a_1 + a_2) + \Gamma(a_1) + \Gamma(a_2, a_1)} = 2,$$

$$\Delta(a_3,a_1,a_2)=\frac{\Gamma(a_3)+\Gamma(a_3,a_1)+\Gamma(a_3,a_2)+\Gamma(a_3,a_1,a_2)}{1+\Gamma(a_1)+\Gamma(a_2)+\Gamma(a_1,a_2)}=$$
$$\frac{\Gamma(a_1+a_2)+\Gamma(a_1+a_2,a_1)+\Gamma(a_1+a_2,a_2)+\Gamma(a_1+a_2,a_1,a_2)}{1+\Gamma(a_1)+\Gamma(a_2)+\Gamma(a_1,a_2)}=$$
$$\frac{\Gamma(a_1+a_2)+2\Gamma(a_1,a_2)}{1+\Gamma(a_1)+\Gamma(a_2)+\Gamma(a_1,a_2)}=2.$$

We use two facts for $1\leq r\leq 2$:

$$\frac{\Gamma(a_1,a_2)}{\Gamma(a_r)}=\infty \quad \text{and} \quad \Gamma(a_1+a_2)\leq\Gamma(a_1)+\Gamma(a_2)+2\sqrt{\Gamma(a_1)\Gamma(a_2)}.$$

The first relation follows from Lemma 8.11 since $\|C_1a_1+C_2a_2\|^2=\infty$. We get

$$\frac{\Gamma(a_1,a_2)}{\Gamma(a_r)}=\lim_{n\to\infty}\frac{\Gamma(a_1(n),a_2(n))}{\Gamma(a_r(n))}=\infty.$$

Recall that $\Gamma(a)=\|a\|^2$. The inequality follows from $\|a_1+a_2\|\leq\|a_1\|+\|a_2\|$, i.e., $\sqrt{\Gamma(a_1+a_2)}\leq\sqrt{\Gamma(a_1)}+\sqrt{\Gamma(a_2)}$. □

By Lemma‘2.7 we have

$$(\mu^3_{(b,a)})^{L_t}\perp\mu^3_{(b,a)} \quad\Leftrightarrow\quad \Sigma^{\pm}(t):=\Sigma^{\pm}_1(t)+\Sigma_2(t)=\infty,$$

where

$$\Sigma^+_1(t)=\sum_{n\in\mathbb{Z}}\sum_{1\leq i<j\leq 3}\left(t^i_j\sqrt{\frac{b_{in}}{b_{jn}}}-A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2,$$

$$\Sigma^-_1(t)=\sum_{n\in\mathbb{Z}}\sum_{1\leq i<j\leq 3}\left(t^i_j\sqrt{\frac{b_{in}}{b_{jn}}}+A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2,$$

$$\Sigma_2(t^{-1})=\sum_{n\in\mathbb{Z}}\Big[b_{1n}\big((t_{11}-1)a_{1n}+t_{12}a_{2n}+t_{13}a_{3n}\big)^2+$$
$$b_{2n}\big(t_{21}a_{1n}+(t_{22}-1)a_{2n}+t_{23}a_{3n}\big)^2+b_{3n}\big(t_{31}a_{1n}+t_{32}a_{2n}+(t_{33}-1)a_{3n}\big)^2\Big].$$

In Example 4.3 we can not approximate x_{2n}, x_{3n} since in this case we have

$$\Delta(Y^{(2)}_2,Y^{(2)}_3)=1,\quad \Delta(Y^{(3)}_3,Y^{(3)}_2)=1. \tag{4.81}$$

Indeed, by (4.37) we have

$$\Delta(Y_2^{(2)},Y_3^{(2)}) = \frac{\Gamma(Y_2^{(2)})+\Gamma(Y_2^{(2)},Y_3^{(2)})}{1+\Gamma(Y_3^{(2)})},\ \ \Delta(Y_3^{(3)},Y_2^{(3)}) = \frac{\Gamma(Y_3^{(3)})+\Gamma(Y_3^{(3)},Y_2^{(3)})}{1+\Gamma(Y_2^{(3)})}. \tag{4.82}$$

In Example 4.3 we have $d_n = \frac{d_{3n}}{d_{2n}} \equiv 1$ and hence, by (4.43) we have

$$\|Y_2^{(2)}\|^2 \sim \sum_{n\in\mathbb{Z}} \frac{1}{1+2d_n} = \sum_{n\in\mathbb{Z}} \frac{1}{3}, \quad \|Y_3^{(2)}\|^2 \sim \sum_{n\in\mathbb{Z}} \frac{d_n^2}{1+2d_n} = \sum_{n\in\mathbb{Z}} \frac{1}{3},$$
$$\|Y_2^{(3)}\|^2 \sim \sum_{n\in\mathbb{Z}} \frac{1}{d_n^2+2d_n} = \sum_{n\in\mathbb{Z}} \frac{1}{3}, \quad \|Y_3^{(3)}\|^2 \sim \sum_{n\in\mathbb{Z}} \frac{d_n^2}{d_n^2+2d_n} = \sum_{n\in\mathbb{Z}} \frac{1}{3}.$$

Therefore, $\Gamma(Y_2^{(2)},Y_3^{(2)}) = \Gamma(Y_3^{(3)},Y_2^{(3)}) = 0$, and

$$\Delta(Y_2^{(2)},Y_3^{(2)}) = \frac{\Gamma(Y_2^{(2)})}{1+\Gamma(Y_3^{(2)})} = 1, \quad \Delta(Y_3^{(3)},Y_2^{(3)}) = \frac{\Gamma(Y_3^{(3)})}{1+\Gamma(Y_2^{(3)})} = 1.$$

Since $b_{1n} \equiv 1$, by (4.44) we get

$$\|Y_1\|^2 \sim \sum_{n\in\mathbb{Z}} a_{1n}^2, \quad \|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} a_{2n}^2, \quad \|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} a_{3n}^2,$$

so by (4.41) we have

$$\nu(C_1,C_2,C_3) \sim \sum_{n\in\mathbb{Z}} b_{1n}\Big(\sum_{r=1}^{3} C_r a_{rn}\Big)^2 = \sum_{n\in\mathbb{Z}} \Big(\sum_{r=1}^{3} C_r a_{rn}\Big)^2.$$

But in Example 4.2 there does not exist $t \in \pm\mathrm{SL}(3,\mathbb{R}) \setminus \{e\}$ such that $\nu(C_1,C_2,C_3)=\infty$ for all $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$ to approximate some D_{rn}.

4.4.9. Approximations of $x_{2k}x_{2r} + x_{3k}x_{3r}$ in the case (c)

Since we can not approximate $x_{2n}x_{2t}$, $x_{3n}x_{3t}$ using Lemmas 5.2–5.3 in the case (c), we shall try to approximate $x_{2k}x_{2r} + s^4 x_{3k}x_{3r}$ by an appropriate combinations of $A_{kn}A_{rn}$ for $n \in \mathbb{Z}$. Let $s = 1$, the general case is similar.

Lemma 4.15. *For any $k, r \in \mathbb{Z}$ one has*

$$(x_{2k}x_{2r}+x_{3k}x_{3r})\mathbf{1} \in \langle A_{kn}A_{rn}\mathbf{1} \mid n\in\mathbb{Z}\rangle \ \Leftrightarrow\ \Delta(Y^{(2)},Y^{(1)})=\infty, \tag{4.83}$$

where $Y^{(r)} = \Big(\frac{b_{rn}}{\sqrt{\lambda_n}}\Big)_{n\in\mathbb{Z}}$, $1\le r\le 2$, $\lambda_n = (b_{1n}+b_{2n}+b_{3n})^2 - b_{1n}^2$.

Proof. The proof of Lemma 4.15 is based on Lemma 7.2. We study when $(x_{2k}x_{2r}+x_{3k}x_{3r})\mathbf{1}\in\langle A_{kn}A_{rn}\mathbf{1}\mid n\in\mathbb{Z}\rangle$. Since

$$\begin{aligned}A_{kn}A_{rn}\quad &= (x_{1k}D_{1n}+x_{2k}D_{2n}+x_{3k}D_{3n})(x_{1r}D_{1n}+x_{2r}D_{2n}+x_{3r}D_{3n})\\ &= x_{1k}x_{1r}D_{1n}^2+x_{2k}x_{2r}D_{2n}^2+x_{3k}x_{3r}D_{3n}^2+(x_{1k}x_{2r}+x_{2k}x_{1r})D_{1n}D_{2n}\\ &+(x_{1k}x_{3r}+x_{3k}x_{1r})D_{1n}D_{3n}+(x_{2k}x_{3r}+x_{3k}x_{2r})D_{2n}D_{3n},\end{aligned}$$

and $MD_{rn}^2\mathbf{1}=-\frac{b_{rn}}{2}$, for $2\le r\le 3$ we take $t=(t_n)_{n=-m}^m$ as follows:

$$(t,b_2)=(t,b_3)=1,\ \text{ where } t=(t_n)_{k=-m}^m,\ \ b_2=-(b_{2n}/2)_{n=-m}^m,\ \ b_3=-(b_{3n}/2)_{n=-m}^m.$$

We have

$$\|\big[\sum_{n=-m}^m t_nA_{kn}A_{rn}-\big(x_{2k}x_{2r}+x_{3k}x_{3r}\big)\big]\mathbf{1}\|^2=$$

$$\|\sum_{n=-m}^m t_n\big[x_{1k}x_{1r}D_{1n}^2+x_{2k}x_{2r}\big(D_{2n}^2+\frac{b_{2n}}{2}\big)+x_{3k}x_{3r}\big(D_{3n}^2+\frac{b_{3n}}{2}\big)+(x_{1k}x_{2r}+$$

$$x_{2k}x_{1r})D_{1n}D_{2n}+(x_{1k}x_{3r}+x_{3k}x_{1r})D_{1n}D_{3n}+(x_{2k}x_{3r}+x_{3k}x_{2r})D_{2n}D_{3n}\Big]\mathbf{1}\|^2$$

$$=\quad\sum_{-m\le n,l\le m}(f_n,f_l)t_nt_l=:(A_{2m+1}t,t),$$

where $A_{2m+1}=(f_n,f_l)_{n,l=-m}^m$ and

$$f_n=\sum_{i=1}^3 f_n^i+\sum_{1\le i<j\le 3}f_n^{ij},\quad\text{with}\tag{4.84}$$

$$f_n^i=x_{ik}x_{ir}\Big(D_{in}^2+\frac{b_{in}}{2}(1-\delta_{i1})\Big)\mathbf{1},\quad f_n^{ij}=(x_{ik}x_{jr}+x_{jk}x_{ir})D_{in}D_{jn}\mathbf{1}$$

for $1\le i\le 3,\ 1\le i<j\le 3$. Since $f_n^{i'}\perp f_n^{ij},\quad f_n^{ij}\perp f_n^{i'j'}$ for different $(ij),\ (i'j')$, writing $c_{kn}=\|x_{kn}\|^2=\frac{1}{2b_{kn}}+a_{kn}^2$, we get

$$(f_n,f_n)=\sum_{i=1}^3\|f_n^i\|^2+\sum_{1\le i<j\le 3}\|f_n^{ij}\|^2=$$

$$c_{1k}c_{1r}3\Big(\frac{b_{1n}}{2}\Big)^2+c_{2k}c_{2r}2\Big(\frac{b_{2n}}{2}\Big)^2+c_{3k}c_{3r}2\Big(\frac{b_{3n}}{2}\Big)^2+$$

$$\big(c_{1k}c_{2r}+c_{2k}c_{1r}+2a_{1k}a_{2r}a_{2k}a_{1r}\big)\frac{b_{1n}}{2}\frac{b_{2n}}{2}+\big(c_{1k}c_{3r}+c_{3k}c_{1r}+2a_{1k}a_{3r}a_{3k}a_{1r}\big)$$
$$\times\frac{b_{1n}}{2}\frac{b_{3n}}{2}+\big(c_{2k}c_{3r}+c_{3k}c_{2r}+2a_{2k}a_{3r}a_{3k}a_{2r}\big)\frac{b_{2n}}{2}\frac{b_{3n}}{2}\sim(b_{1n}+b_{2n}+b_{3n})^2,$$
$$(f_n,f_l)=(f_n^1,f_l^1)=c_{1k}c_{1r}\frac{b_{1n}}{2}\frac{b_{1l}}{2}\sim b_{1n}b_{1l}.$$

Finally, we get

$$(f_n,f_n)\sim(b_{1n}+b_{2n}+b_{3n})^2,\quad (f_n,f_l)\sim b_{1n}b_{1l},\quad n\neq l. \tag{4.85}$$

Set

$$\lambda_n=(b_{1n}+b_{2n}+b_{3n})^2-b_{1n}^2,\quad g_n=(b_{1n}), \tag{4.86}$$

then

$$(f_n,f_n)\sim\lambda_n+(g_n,g_n),\quad (f_n,f_l)\sim(g_n,g_l). \tag{4.87}$$

For $A_{2m+1}=\big((f_n,f_l)\big)_{n,l=-m}^m$ and $b_2=b_3=-(b_{2n}/2)_{n=-m}^m\in\mathbb{R}^{2m+1}$ we have

$$A_{2m+1}=\sum_{n=-m}^m\lambda_nE_{nn}+\gamma(g_{-m},\dots,g_0,\dots,g_m).$$

To finish the proof, it suffices to use Lemma 7.2. □

Remark 4.11. In the case (c) we can approximate $x_{2k}x_{2r}+x_{3k}x_{3r}$ since $\Delta(Y^{(2)},Y^{(1)})=\infty$.

Indeed, by (4.83) we have

$$\Delta(Y^{(2)},Y^{(1)})=\frac{\Gamma(Y^{(2)})+\Gamma(Y^{(2)},Y^{(1)})}{1+\Gamma(Y^{(1)})}>\frac{\Gamma(Y^{(2)})}{1+\Gamma(Y^{(1)})}=$$
$$\frac{\sum_{n\in\mathbb{Z}}\frac{b_{2n}^2}{\lambda_n}}{1+\sum_{n\in\mathbb{Z}}\frac{b_{1n}^2}{\lambda_n}}=\frac{\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{(1+2d_{2n})^2-1}}{1+\sum_{n\in\mathbb{Z}}\frac{1}{(1+2d_{2n})^2-1}}\sim\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{d_n+d_{2n}^2}=\infty$$

since by (4.74) we have $\sum_n\frac{1}{d_{2n}^2}<\infty$. Therefore,

$$\Gamma(Y^{(1)})=\sum_{n\in\mathbb{Z}}\frac{1}{(1+2d_{2n})^2-1}\sim\sum_{n\in\mathbb{Z}}\frac{1}{d_n+d_{2n}^2}<\infty.$$

Lemma 4.16. *We have for all $k \in \mathbb{Z}$*

$$x_{2k}\mathbf{1} \in \langle (x_{2k}x_{2n} + x_{3k}x_{3n})\mathbf{1} \mid n \in \mathbb{Z}\rangle \;\Leftrightarrow\; \sigma_2(\mu) = \sum_{n\in\mathbb{Z}} \frac{a_{2n}^2}{\frac{1}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{3n}^2} = \infty. \tag{4.88}$$

Proof. Recall the notation $c_{rn} = \frac{1}{2b_{rn}} + a_{rn}^2$. Since $Mx_{2n}\mathbf{1} = a_{2n}$ we take $t = (t_n)_{n=-m}^m$ as follows: $(t, a_2) = 1$, where $a_2 = (a_{2n})_{n=-m}^m$. We have

$$\|\Big[\sum_{n=-m}^{m} t_n\big(x_{2k}x_{2n} + x_{3k}x_{3n}\big) - x_{2k}x_{2n}\Big]\mathbf{1}\|^2 =$$

$$\|\Big[\sum_{n=-m}^{m} t_n\big(x_{2k}(x_{2n} - a_{2n}) + x_{3k}x_{3n}\big)\Big]\mathbf{1}\|^2 = \|x_{2k}\mathbf{1}\|^2\|\sum_{n=-m}^{m} t_n(x_{2n} - a_{2n})\mathbf{1}\|^2$$

$$+\|x_{3k}\mathbf{1}\|^2\|\sum_{n=-m}^{m} t_n x_{3n}\mathbf{1}\|^2 = c_{2k}\sum_{n=-m}^{m} t_n^2\frac{1}{2b_{2n}} + c_{3k}\sum_{n=-m}^{m} t_n^2\Big(\frac{1}{2b_{3n}} + a_{3n}^2\Big)$$

$$\sim \sum_{n=-m}^{m} t_n^2\Big(\frac{1}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{3n}^2\Big).$$

By (6.3) we get (4.88). □

Similarly, we prove the following lemma.

Lemma 4.17. *We have for all $k \in \mathbb{Z}$*

$$x_{3k}\mathbf{1} \in \langle (x_{2k}x_{2n} + x_{3k}x_{3n})\mathbf{1} \mid n \in \mathbb{Z}\rangle \;\Leftrightarrow\; \sigma_3(\mu) = \sum_{\in\mathbb{Z}} \frac{a_{3n}^2}{\frac{1}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{2n}^2} = \infty. \tag{4.89}$$

Remark 4.12. Suppose that $\sigma_2(\mu)+\sigma_3(\mu) < \infty$, this contradicts $\Sigma_{23}(C_2, C_3) = \infty$ for $(C_2, C_3) \in \mathbb{R}^2 \setminus \{0\}$, where $\Sigma_{23}(C_2, C_3)$ is defined by (4.71):

$$\Sigma_{23}(C_2, C_3) = \sum_{n\in\mathbb{Z}}(C_2^2 b_{2n} + C_3^2 b_{3n})(C_2 a_{2n} + C_3 a_{3n})^2.$$

Proof. Indeed, we have

$$\infty > \sigma_2(\mu) + \sigma_3(\mu) = \sum_{\in\mathbb{Z}} \frac{a_{2n}^2}{\frac{1}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{3n}^2} + \sum_{\in\mathbb{Z}} \frac{a_{3n}^2}{\frac{1}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{2n}^2} \sim$$

$$\sum_{\in\mathbb{Z}} \frac{a_{2n}^2 + a_{3n}^2}{\frac{1}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{2n}^2 + a_{3n}^2} \sim \sum_{\in\mathbb{Z}} \frac{a_{2n}^2 + a_{3n}^2}{\frac{1}{2b_{2n}} + \frac{1}{2b_{3n}}} \stackrel{(4.74)}{=} \frac{2}{1+s^{-4}}\sum_{n\in\mathbb{Z}} b_{2n}(a_{2n}^2 + a_{3n}^2).$$

This contradicts $\Sigma_{23}(C_2,C_3)=\infty$. Indeed, by $b_{3n}=s^4 b_{2n}$ (see (4.74)) we have

$$\Sigma_{23}(C_2,C_3)=\sum_{n\in\mathbb{Z}}\big(C_2^2+C_3^2s^4\big)b_{2n}\big(C_2a_{2n}+C_3a_{3n}\big)^2<\infty. \qquad \square$$

Finally, we have $\sigma_2(\mu)+\sigma_3(\mu)=\infty$, and therefore we have $x_{rn}\,\eta\,\mathfrak{A}^3$ for some $2\le r\le 3$. Let $x_{3n}\,\eta\,\mathfrak{A}^3$, then we can approximate x_{2n} by combinations of $x_{2n}x_{2k}$, $k\in\mathbb{Z}$ using an analogue of Lemma 3.3. To approximate D_{rn}, $1\le r\le 3$ we again follows Section 4.4.4. As in (4.53) we get

$$\|Y_1\|^2\sim\sum_{n\in\mathbb{Z}}a_{1n}^2,\quad \|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}a_{2n}^2,\quad \|Y_3\|^2\sim\sum_{n\in\mathbb{Z}}a_{3n}^2.$$

Indeed, for example, by (4.6) we get

$$\|Y_1\|^2=\sum_{n\in\mathbb{Z}}\frac{a_{1n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}}=\sum_{n\in\mathbb{Z}}\frac{a_{1n}^2}{\frac{1}{2}+\frac{1}{d_{2n}}}\overset{(4.75)}{\sim}\sum_{n\in\mathbb{Z}}a_{1n}^2.$$

Again, as in (4.54) we have four possibilities: (1.0), (1.1), (1.2) and (1.3). The corresponding expressions in (4.58), (4.59), (4.60) becomes as follows:

$$\nu_{12}(C_1,C_2):=\|C_1Y_1+C_2Y_2\|^2\sim\sum_{n\in\mathbb{Z}}\Big(C_1a_{1n}+C_2a_{2n}\Big)^2,$$

$$\nu_{13}(C_1,C_3):=\|C_1Y_1+C_3Y_3\|^2\sim\sum_{n\in\mathbb{Z}}\Big(C_1a_{1n}+C_3a_{3n}\Big)^2,$$

$$\nu(C_1,C_2,C_3)=\sum_{n\in\mathbb{Z}}\Big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\Big)^2.$$

To study the cases (1.1.1)–(1.3.1) we should use Remark 4.4. We can approximate in these cases respectively D_{1n} and D_{2n} in (4.55), D_{1n} and D_{3n} in (4.56) all D_{1n}, D_{2n}, D_{3n} in (4.57). The proof of irreducibilty is finished in these cases because we have D_{rn}, x_{2n}, $x_{3n}\,\eta\,\mathfrak{A}^3$ for some $1\le r\le 3$. Following Remark 4.6 we can use Lemma 5.15 and its analogue to approximate D_{2n} and D_{3n} with corresponding expressions $\Sigma_2(D,s)$, $\Sigma_2^\vee(D,s)$ and $\Sigma_3(D)$, $\Sigma_3^\vee(D)$. If one of the expressions $\Sigma_2(D,s)$, $\Sigma_2^\vee(D,s)$, $\Sigma_3(D,s)$ or $\Sigma_3^\vee(D,s)$ is divergent for some sequence $s=(s_k)_{k\in\mathbb{Z}}$, we can approximate D_{2k} or D_{3k} and the proof is finished. Suppose that for all sequence $s=(s_k)_{k\in\mathbb{Z}}$ we have

$$\Sigma_2(D,s)+\Sigma_2^\vee(D,s)+\Sigma_3(D,s)+\Sigma_3^\vee(D,s)<\infty.$$

Then, by (4.64) we have

$$\infty > \Sigma_{23}^{\vee}(D) = \sum_k \frac{\frac{1}{2b_{2k}} + a_{2k}^2 + \frac{1}{2b_{3k}} + a_{3k}^2}{\frac{1}{2b_{1k}} + a_{1k}^2} = \sum_k \frac{\frac{1}{d_{2k}} + a_{2k}^2 + a_{3k}^2}{\frac{1}{2} + a_{1k}^2} \overset{(4.75)}{\sim}$$
$$\sum_k \frac{a_{2k}^2 + a_{3k}^2}{1 + a_{1k}^2} =: \Sigma_{23}^a(D).$$

To study the cases (1.1.0)–(1.3.0) we should follow Remark 4.7.

4.5. Case $S = (1,1,1)$

Denote by

$$\Sigma_{123}(s) = (\Sigma_{12}(s_1), \Sigma_{23}(s_2), \Sigma_{13}(s_3)), \tag{4.90}$$

where $s = (s_1, s_2, s_3)$ and $\Sigma_{ij}(s)$ are defined by (4.70) for $1 \le i < j \le 3$. In terms of Remark 3.2, we have 2^3 possibilities for $\Sigma_{123}(s) \in \{0,1\}^3$:

	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$\Sigma_{12}(s_1)$	0	0	0	0	1	1	1	1
$\Sigma_{23}(s_2)$	0	0	1	1	0	0	1	1
$\Sigma_{13}(s_3)$	0	1	0	1	0	1	0	1

The cases (1), (2) and (4) and respectively the cases (3), (5) and (6) result from cyclic permutations of three measures $\mu^{(1)}, \mu^{(2)}, \mu^{(3)}$ defined as follows:

$$\mu^{(r)} = \otimes_{n\in\mathbb{Z}} \mu_{(b_{rn}, a_{rn})}, \ 1 \le r \le 3, \quad \mu_0^{(r)} = \otimes_{n\in\mathbb{Z}} \mu_{(b_{rn}, 0)}, \ 1 \le r \le 3. \tag{4.91}$$

The case (1), (2) and (4) can not be realized. We prove this only in the case (1). By Lemma 8.1 we have $\Sigma_{12}(s_1) < \infty \ \Leftrightarrow \ \mu_0^{(1)} \sim \mu_0^{(2)}$ and $\Sigma_{23}(s_2) < \infty \Leftrightarrow \mu_0^{(2)} \sim \mu_0^{(3)}$ hence, $\mu_0^{(1)} \sim \mu_0^{(3)}$, that contradicts $\Sigma_{13}(s_2) = \infty \Leftrightarrow \mu_0^{(1)} \perp \mu_0^{(3)}$. Finally, we are left with the three cases (0), (3) and (7):
the **case (0)**, i.e., $\Sigma_{123}(s) = (0,0,0)$,
the **case (3)**, i.e., $\Sigma_{123}(s) = (0,1,1)$,
the **case (7)**, i.e., $\Sigma_{123}(s) = (1,1,1)$.

4.5.1. Case $\Sigma_{123}(s) = (0,0,0)$

In the case (0), we have for some $s = (s_1, s_2, s_3) \in (\mathbb{R}_+)^3$

$$\Sigma_{12}(s_1) < \infty, \quad \Sigma_{23}(s_2) < \infty, \quad \Sigma_{13}(s_3) < \infty.$$

In this case we get $\mu_0^{(1)} \sim \mu_0^{(2)} \sim \mu_0^{(3)}$. By (4.20) we can make the following change of the variables:

$$\left(\begin{smallmatrix} b_{1n} & b_{2n} & b_{3n} \\ a_{1n} & a_{2n} & a_{3n} \end{smallmatrix}\right) \to \left(\begin{smallmatrix} b'_{1n} & b'_{2n} & b'_{3n} \\ a'_{1n} & a'_{2n} & a'_{3n} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & \frac{b_{2n}}{b_{1n}} & \frac{b_{3n}}{b_{1n}} \\ a_{1n}\sqrt{b_{1n}} & a_{2n}\sqrt{b_{1n}} & a_{3n}\sqrt{b_{1n}} \end{smallmatrix}\right).$$

Remark 4.13. By Lemma 8.2, we can suppose that

$$b=(b_{1n}, b_{2n}, b_{3n})_{n\in\mathbb{Z}}=(1, 1+c_n, 1+e_n)_{n\in\mathbb{Z}}, \quad \sum_n c_n^2 < \infty, \quad \sum_n e_n^2 < \infty. \quad (4.92)$$

But the two measures $\mu_{(b,a)}$ and $\mu_{(\mathbb{I},a)}$ are equivalent, where b is defined by (4.92) and

$$\mathbb{I} := (1, 1, 1)_{n\in\mathbb{Z}}. \quad (4.93)$$

Finally, it is sufficient to consider the measure $\mu_{(\mathbf{1},a)}$.

Example 4.4. Let $b_{1n} = b_{2n} = b_{3n} \equiv 1,\ n \in \mathbb{Z}$.
(a) Take $a_n = (a_{1n}, a_{2n}, a_{3n}),\ n \in \mathbb{Z}$ as it was defined in Example 4.2:

$$a_{1n} = \begin{cases} 2 & n = 2k+1 \\ 1 & n = 2k \end{cases}, \quad a_{2n} = \begin{cases} 1 & n = 2k+1 \\ 2 & n = 2k \end{cases}, \quad a_{3n} \equiv 3.$$

Then $a_1 + a_2 - a_3 = 0$, where $a_r = (a_{rn})_{n\in\mathbb{Z}}$.
(b) Take any $a_r = (a_{rn})_{n\in\mathbb{Z}}$ such that $a_1, a_2, a_3 \notin l_2$, but $C_1a_1 + C_2a_2 + C_3a_3 \in l_2(\mathbb{Z})$ for some $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$.

Example 4.5. Let $b_{1n} = b_{2n} = b_{3n} \equiv 1,\ n \in \mathbb{Z}$ and $a = (a_{1n}, a_{2n}, a_{3n})_{n\in\mathbb{Z}}$ such that $a_1, a_2, a_3 \notin l_2$, but the measure $\mu^3_{(b,a)}$ satisfies the orthogonality conditions. The case $\Sigma_{123}(s) = (0, 0, 0)$ is reduced to this example.

Remark 4.14. Since the measure $\mu^3_{(b,0)}$ is *standard* in Example 4.4 and 4.5, i.e., it is invariant under rotations $\pm\mathrm{O}(3)$, we have

$$(\mu^3_{(b,0)})^{L_t} = \mu^3_{(b,0)} \quad \text{for all} \quad t \in \pm\mathrm{O}(3). \quad (4.94)$$

By Lemma 2.8, the orthogonality condition $(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)}$ for $t \in \pm\mathrm{O}(3) \setminus \{e\}$, is equivalent to

$$\Sigma_1^{\pm}(t) + \Sigma_2(t) = \infty,$$

where $\Sigma_1^+(t)$, $\Sigma_1^-(t)$ are defined by (2.31), (2.32) and $\Sigma_2(t)$ is defined by (2.28). By (4.94) we get $\Sigma_1^{\pm}(t) < \infty$ in Example 4.4 and 4.5 hence the orthogonality condition $(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)}$ for $t \in \pm\mathrm{O}(3) \setminus \{e\}$ is equivalent to $\Sigma_2(t) = \infty$. Further, to prove the irreducibility in Example 4.4 and 4.5 we should show that $\Sigma_2(t) = \infty$ for all $t \in \pm\mathrm{O}(3) \setminus \{e\}$ implies

$$\|C_1 Y_1 + C_2 Y_2 + C_3 Y_3\|^2 = \infty \quad \text{for all} \quad (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}.$$

Lemma 4.18. *(1) The representations corresponding to the measures in Example 4.4 (a) and (b) are reducible.*
(2) The representations corresponding to the measures in Example 4.5 are irreducible.

Proof. To prove the part (1) of theorem, by Remark 4.14 and (4.94), we should find for the measure in Example 4.4 an element $t \in \pm\mathrm{O}(3) \setminus \{e\}$ such that $\Sigma_2(t) < \infty$. This will imply $(\mu^3_{(b,a)})^{L_t} \sim \mu^3_{(b,a)}$ hence, the *reducibility*.

Finally, it is sufficient to find $t \in \pm\mathrm{O}(3) \setminus \{e\}$ such that

$$t - 1 = \begin{pmatrix} \lambda_1 C_1 & \lambda_1 C_2 & \lambda_1 C_3 \\ \lambda_2 C_1 & \lambda_2 C_2 & \lambda_2 C_3 \\ \lambda_3 C_1 & \lambda_3 C_2 & \lambda_3 C_3 \end{pmatrix}, \tag{4.95}$$

where $(C_1, C_2, C_3) = (1, 1, -1)$, in part (a), or for an arbitrary $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$ in the part (b). Such an element exists by Lemma 4.19 below. For such an element t we get respectively in the cases (a), (b) and Example 4.5 (see (2.28)):

$$\Sigma_2(t^{-1}) = \sum_{n\in\mathbb{Z}} (b_{1n}\lambda_1^2 + b_{2n}\lambda_2^2 + b_{3n}\lambda_3^2)\big(a_{1n} + a_{2n} - a_{3n}\big)^2 = 0,$$

$$\Sigma_2(t^{-1}) = \sum_{n\in\mathbb{Z}} (b_{1n}\lambda_1^2 + b_{2n}\lambda_2^2 + b_{3n}\lambda_3^2)\big(C_1 a_{1n} + C_2 a_{2n} + C_3 a_{3n}\big)^2 < \infty,$$

$$\Sigma_2(t^{-1}) = \sum_{n\in\mathbb{Z}} (b_{1n}\lambda_1^2 + b_{2n}\lambda_2^2 + b_{3n}\lambda_3^2)\big(C_1 a_{1n} + C_2 a_{2n} + C_3 a_{3n}\big)^2 = \infty. \tag{4.96}$$

Note that the measure in Example 4.4 does not satisfy the orthogonality conditions.

(2) *Irreducibility.* In Example 4.5 we can not approximate x_{rn} by Lemmas 5.1–5.3, since all the expressions

$$\Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}), \quad \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}), \quad \Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)})$$

are bounded. To approximate D_{rn} using Lemmas 5.4–5.6, we should estimate the following expressions:

$$\Delta(Y_1,Y_2,Y_3),\quad \Delta(Y_2,Y_3,Y_1),\quad \Delta(Y_3,Y_1,Y_2).$$

By Lemma 8.15, all these expressions are infinite, if for all $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$ holds

$$\nu(C_1,C_2,C_3) := \|C_1Y_1+C_2Y_2+C_3Y_3\|^2 = \sum_{n\in\mathbb{Z}}\frac{\Big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\Big)^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}} = \infty.$$

In Examples 4.5 we have

$$\nu(C_1,C_2,C_3) = \|C_1Y_1+C_2Y_2+C_3Y_3\|^2 \sim \sum_{k\in\mathbb{Z}} b_{1n}(C_1a_{1k}+C_2a_{2k}+C_3a_{3k})^2$$
$$\sim \sum_{n\in\mathbb{Z}}(b_{1n}\lambda_1^2+b_{2n}\lambda_2^2+b_{3n}\lambda_3^2)\big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\big)^2 = \Sigma_2(t^{-1}) = \infty. \quad \square$$

Lemma 4.19. *For an arbitrary $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$, and an arbitrary $D_3(s)=\mathrm{diag}(s_1,s_2,s_3)$ with $(s_1,s_2,s_3)\in(\mathbb{R}_+)^3$, there exists a unique element $t\in\pm\mathrm{O}(3)\setminus\{e\}$ and $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3\setminus\{0\}$ such that*

$$D_3(s)tD_3^{-1}(s)-\mathrm{I} = \begin{pmatrix}\lambda_1C_1 & \lambda_1C_2 & \lambda_1C_3\\ \lambda_2C_1 & \lambda_2C_2 & \lambda_2C_3\\ \lambda_3C_1 & \lambda_3C_2 & \lambda_3C_3\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}C_1 & 0 & 0\\ 0 & C_2 & 0\\ 0 & 0 & C_3\end{pmatrix}. \quad (4.97)$$

Proof. By (4.97) we get

$$\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix} := t = \begin{pmatrix}t_{11}&t_{12}&t_{13}\\t_{21}&t_{22}&t_{23}\\t_{31}&t_{32}&t_{33}\end{pmatrix} = \begin{pmatrix}C_1\lambda_1+1 & \frac{s_2}{s_1}C_2\lambda_1 & \frac{s_3}{s_1}C_3\lambda_1\\ \frac{s_1}{s_2}C_1\lambda_2 & C_2\lambda_2+1 & \frac{s_3}{s_2}C_3\lambda_2\\ \frac{s_1}{s_3}C_1\lambda_3 & \frac{s_2}{s_3}C_2\lambda_3 & C_3\lambda_3+1\end{pmatrix}, \quad (4.98)$$

$$\text{where}\quad \|e_k\|^2=1 \quad\text{and}\quad e_k\perp e_r,\quad 1\le k<r\le 3. \quad (4.99)$$

By (4.98) and the first relations in (4.99) we get

$$\lambda_k = -\frac{2s_k^2C_k}{s_1^2C_1^2+s_2^2C_2^2+s_3^2C_3^2},\quad 1\le k\le 3. \quad (4.100)$$

Then the matrix elements $t = (t_{kr})_{k,r=1}^3$ are defined by (4.98). To verify $e_k \perp e_r$ we need to show that

$$(e_1, e_2) = \frac{(s_1^2C_1^2 + s_2^2C_2^2 + s_3^2C_3^2)\lambda_1\lambda_2}{s_1s_2} + \frac{s_1^2C_1\lambda_2 + s_2^2C_2\lambda_1}{s_1s_2} = 0,$$
$$(e_1, e_3) = \frac{(s_1^2C_1^2 + s_2^2C_2^2 + s_3^2C_3^2)\lambda_1\lambda_3}{s_1s_3} + \frac{s_1^2C_1\lambda_3 + s_3^2C_3\lambda_1}{s_1s_3} = 0,$$
$$(e_2, e_3) = \frac{(s_1^2C_1^2 + s_2^2C_2^2 + s_3^2C_3^2)\lambda_2\lambda_3}{s_2s_3} + \frac{s_2^2C_2\lambda_3 + s_3^2C_2\lambda_3}{s_2s_3} = 0.$$

Indeed, for example, for (e_1, e_2) we have

$$(e_1, e_2) = \frac{(s_1^2C_1^2 + s_2^2C_2^2 + s_3^2C_3^2)\lambda_1\lambda_2}{s_1s_2} + \frac{s_1^2C_1\lambda_2 + s_2^2C_2\lambda_1}{s_1s_2} =$$
$$\frac{1}{s_1s_2(s_1^2C_1^2 + s_2^2C_2^2 + s_3^2C_3^2)}\Big(4s_1^2s_2^2 - (2s_1^2s_2^2 + 2s_1^2s_2^2)\Big)C_1C_2 = 0.$$

The proofs of $e_1 \perp e_3$ and $e_2 \perp e_3$ are similar. □

Similarly, for any $m \geq 2$ we can prove the following lemma:

Lemma 4.20. *For an arbitrary* $(C_k)_{k=1}^m \in \mathbb{R}^m \backslash \{0\}$, *and* $D_m(s) = \mathrm{diag}(s_k)_{k=1}^m$ *with* $s_k \in \mathbb{R}_+,\ 1 \leq k \leq m$ *there exists a unique element* $t \in \pm\mathrm{O}(m) \setminus \{e\}$ *and* $(\lambda_k)_{k=1}^m \in \mathbb{R}^m \setminus \{0\}$ *such that*

$$D_m(s)tD_m^{-1}(s) - \mathrm{I} = \begin{pmatrix} \lambda_1C_1 & \lambda_1C_2 & \dots & \lambda_1C_m \\ \lambda_2C_1 & \lambda_2C_2 & \dots & \lambda_2C_m \\ & & \dots & \\ \lambda_mC_1 & \lambda_mC_2 & \dots & \lambda_mC_m \end{pmatrix}. \tag{4.101}$$

The formulas for the corresponding λ_k *are as follows:*

$$\lambda_k = -\frac{2s_k^2C_k}{\sum_{r=1}^m s_r^2C_r^2}, \quad 1 \leq k \leq m. \tag{4.102}$$

Lemma 4.21. *For an arbitrary* $(C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}$ *and arbitrary* $D_2(s) = \mathrm{diag}(s_1, s_2)$ *with* $s_1, s_2 \neq 0$ *there exists a unique element* $t \in \pm\mathrm{O}(2) \setminus \{e\}$ *and* $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{0\}$ *such that*

$$D_2(s)tD_2^{-1}(s) - \mathrm{I} = \begin{pmatrix} \lambda_1C_1 & \lambda_1C_2 \\ \lambda_2C_1 & \lambda_2C_2 \end{pmatrix}. \tag{4.103}$$

The formulas for the corresponding λ_k *are as follows:*

$$\lambda_1 = -\frac{2s_1^2C_1}{s_1^2C_1^2 + s_2^2C_2^2}, \quad \lambda_2 = -\frac{2s_2C_2}{s_1^2C_1^2 + s_2^2C_2^2}. \tag{4.104}$$

In particular, we have

$$\begin{pmatrix} t_{11} & \frac{s_1}{s_2}t_{12} \\ \frac{s_2}{s_1}t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} C_1\lambda_1+1 & C_2\lambda_1 \\ C_1\lambda_2 & C_2\lambda_2+1 \end{pmatrix} = \begin{pmatrix} \frac{s_2^2C_2^2-s^2C_1^2}{s_1^2C_1^2+s_2^2C_2^2} & -\frac{2s_1^2C_1C_2}{s_1^2C_1^2+s_2^2C_2^2} \\ -\frac{2s_2^2C_1C_2}{s_1^2C_1^2+s_2^2C_2^2} & \frac{s_1^2C_1^2-s_2^2C_2^2}{s_1^2C_1^2+s_2^2C_2^2} \end{pmatrix} = \tau_{12}(\phi, s_1, s_2).$$

We can verify that

$$\tau_{12}(\phi, s, s^{-1}) = \tau_-(\phi, s)$$

where $\tau_-(\phi, s)$ is defined by (2.4). We just set

$$\cos\phi = \frac{s^{-2}C_2^2 - s^2C_1^2}{s^2C_1^2 + s^{-2}C_2^2} \quad \textit{and} \quad \sin\phi = -\frac{2C_1C_2}{s^2C_1^2 + s^{-2}C_2^2}.$$

In addition $\det t = -1$.

4.5.2. Case $\Sigma_{123}(s) = (0, 1, 1)$

We have for some $s_1 \in \mathbb{R}_+$ and all $(s_2, s_3) \in \left(\mathbb{R}_+\right)^2$

$$, \quad \Sigma_{23}(s_2) = \infty, \quad \Sigma_{13}(s_3) = \infty.$$

Remark 4.15. Since $\Sigma_{12}(s_1) < \infty$, by (4.20) and Lemma 8.2, we can suppose that

$$b = (b_{1n}, b_{2n}, b_{3n})_{n\in\mathbb{Z}} = (1, s_1^4(1+c_n), b_{3n})_{n\in\mathbb{Z}}, \quad \sum_n c_n^2 < \infty,$$

therefore, we can take $\quad b = (1, 1, b_{3n})_{n\in\mathbb{Z}},\ s = 1,\ c_n \equiv 0.$

Since $\Sigma_{13}(s) = \sum_{n\in\mathbb{Z}}\left(\frac{s^2}{\sqrt{b_{3n}}} - \frac{\sqrt{b_{3n}}}{s^2}\right)^2 = \infty$, we have as in (4.50) three cases:

$$\lim_n b_{3n} = \begin{cases} (a) & \infty \\ (b) & b > 0 \quad \text{with} \quad \sum_n b_n^2 = \infty, \\ (c) & 0 \end{cases} \tag{4.105}$$

where $b_{3n} = b(1 + b_n)$ with $\lim_n b_n = 0$ in the case (b). Note that condition $S_3(3) = \infty$, implies $\sum_n b_{3n}^2 = \infty$. Indeed, by (4.8) we have for $1 \le r \le 3$

$$S_r(3) = \sum_{n\in\mathbb{Z}} \frac{b_{rn}^2}{b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}}, \quad S_1(3) = \sum_{n\in\mathbb{Z}} \frac{1}{1 + 2b_{3n}} = \infty,$$

$$S_2(3) = \sum_{n\in\mathbb{Z}} \frac{1}{1+2b_{3n}} = \infty, \quad \infty = S_3(3) = \sum_{n\in\mathbb{Z}} \frac{b_{3n}^2}{1+2b_{3n}} \overset{(2.20)}{\sim} \sum_{n\in\mathbb{Z}} b_{3n}^2. \tag{4.106}$$

By (4.5) we have

$$\|Y_r^{(r)}\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{rk}^2}{b_{rk}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})},$$
$$\|Y_r^{(s)}\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{rk}^2}{b_{sk}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})}, \ s \neq r.$$

Let us denote

$$\begin{pmatrix} Y_{1n}^{(1)} & Y_{2n}^{(1)} & Y_{3n}^{(1)} \\ Y_{1n}^{(2)} & Y_{2n}^{(2)} & Y_{3n}^{(2)} \\ Y_{1n}^{(3)} & Y_{2n}^{(3)} & Y_{3n}^{(3)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3+4b_{3n}}} & \frac{1}{\sqrt{3+4b_{3n}}} & \frac{b_{3n}}{\sqrt{3+4b_{3n}}} \\ \frac{1}{\sqrt{3+4b_{3n}}} & \frac{1}{\sqrt{3+4b_{3n}}} & \frac{b_{3n}}{\sqrt{3+4b_{3n}}} \\ \frac{1}{\sqrt{b_{3n}^2+4b_{3n}+2}} & \frac{1}{\sqrt{b_{3n}^2+4b_{3n}+2}} & \frac{b_{3n}}{\sqrt{b_{3n}^2+4b_{3n}+2}} \end{pmatrix} \tag{4.107}$$

We have $\Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}) = \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}) < \infty$, indeed, since $Y_1^{(2)} = Y_2^{(2)}$ we get for example

$$\Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}) = \frac{\Gamma(Y_2^{(2)}) + \Gamma(Y_2^{(2)}, Y_3^{(2)})}{1 + \Gamma(Y_3^{(2)}) + \Gamma(Y_1^{(2)}) + \Gamma(Y_3^{(2)}, Y_1^{(2)})} < 1 < \infty.$$

Lemma 4.22. *In the cases (a), (b) and (c) given by (4.105) we have*

$$\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = \infty. \tag{4.108}$$

Proof. In all these cases we have $Y_1^{(3)} = Y_2^{(3)}$ hence, $\Gamma(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = 0$ and $\Gamma(Y_1^{(3)}, Y_2^{(3)}) = 0$. Therefore, by (8.15)

$$\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = \frac{\Gamma(Y_3^{(3)}) + \Gamma(Y_3^{(3)}, Y_1^{(3)}) + \Gamma(Y_3^{(3)}, Y_2^{(3)})}{1 + \Gamma(Y_1^{(3)}) + \Gamma(Y_2^{(3)})}$$
$$= \frac{\Gamma(Y_3^{(3)}) + 2\Gamma(Y_3^{(3)}, Y_1^{(3)})}{1 + 2\Gamma(Y_1^{(3)})} \sim \Delta(Y_3^{(3)}, Y_1^{(3)}). \tag{4.109}$$

We have two cases:
(a.1) when $\|Y_1^{(3)}\| < \infty$, and (a.2) when $\|Y_1^{(3)}\| = \infty$.
In the case (a.1) we have $\Delta(Y_3^{(3)}, Y_1^{(3)}) \sim \Gamma(Y_3^{(3)}) = \infty$. Therefore, (4.108) holds. In the case (a.2) we should verify that

$$\|C_1 Y_1^{(3)} + C_3 Y_3^{(3)}\|^2 = \infty \quad \text{for all} \quad (C_1, C_3) \in \mathbb{R}^2 \setminus \{0\}. \tag{4.110}$$

Then this will imply (4.108). We have

$$\|C_1Y_1^{(3)}+C_3Y_3^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{\big(C_1+C_3b_{3n}\big)^2}{b_{3n}^2+4b_{3n}+2}=:\sum_{n\in\mathbb{Z}}g_n.$$

If $C_1=0$ or $C_3=0$ the later expression is divergent since $Y_1^{(3)}=Y_3^{(3)}=\infty$. Let $C_1C_3\neq 0$. In this case $\lim_n g_n=C_3^2>0$ since $\lim_n b_{3n}=\infty$, case (a). Therefore, $\sum_{n\in\mathbb{Z}}g_n=\infty$. By Lemma 8.11 this implies $\Delta(Y_3^{(3)},Y_1^{(3)})=\infty$ therefore, (4.108). In the case (b) we have by (4.109)

$$\Delta(Y_3^{(3)},Y_1^{(3)},Y_2^{(3)})=\Delta(Y_3^{(3)},Y_1^{(3)}).$$

To prove that $\Delta(Y_3^{(3)},Y_1^{(3)})=\infty$ using Lemma 8.11 we should verify (4.110). We have $\|Y_3^{(3)}\|^2=\infty$ since $S=(0,1,1)$. By (4.107)

$$\|Y_1^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{1}{b_{3n}^2+4b_{3n}+2}\sim\sum_{n\in\mathbb{Z}}\frac{1}{b^2+4b+2}=\infty.$$

The expression $\|C_1Y_1^{(3)}+C_3Y_3^{(3)}\|^2$ can be finite only for $(C_1,C_3)=\lambda(b,-1)$. Take $\lambda=1$, we get in the case (b)

$$\|C_1Y_1^{(3)}+C_3Y_3^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{(b-b_{3n})^2}{b_{3n}^2+4b_{3n}+2}=\sum_{n\in\mathbb{Z}}\frac{b^2b_n^2}{b^2(1+b_n)^2+4b(1+b_n)+2}$$
$$\overset{(2.22)}{\sim}\sum_{n\in\mathbb{Z}}\frac{b_n^2}{(4b+2b^2)b_n+b^2+4b+2}\overset{(2.20)}{\sim}\sum_{n\in\mathbb{Z}}b_n^2=\infty.$$

In the case (c), we have by (4.109)

$$\Delta(Y_3^{(3)},Y_1^{(3)},Y_2^{(3)})\sim\Delta(Y_3^{(3)},Y_1^{(3)}).$$

To prove that $\Delta(Y_3^{(3)},Y_1^{(3)})=\infty$ using Lemma 8.11 we should verify (4.110). Again, we have $\|Y_3^{(3)}\|^2=\infty$ since $S=(0,1,1)$. Because of $\lim_n b_{3n}=0$, we have by (4.107)

$$\|Y_1^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{1}{b_{3n}^2+4b_{3n}+2}\sim\sum_{n\in\mathbb{Z}}\frac{1}{2}=\infty,$$

Let $C_1C_3\neq 0$, then since $\lim_n b_{3n}=0$ we get

$$\|C_1Y_1^{(3)}+C_3Y_3^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{\big(C_1+C_3b_{3n}\big)^2}{b_{3n}^2+4b_{3n}+2}=\sum_{n\in\mathbb{Z}}\frac{C_3^2\big(b_{3n}+C_1C_3^{-1}\big)^2}{b_{3n}^2+4b_{3n}+2}=\infty.\quad\square$$

By Lemma 4.22 we can approximate x_{3n}. By (4.6) we have

$$\|Y_1\|^2 = \sum_{n\in\mathbb{Z}} \frac{a_{1n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}} + \frac{1}{2b_{3n}}} = \sum_{k\in\mathbb{Z}} \frac{a_{1n}^2}{1+\frac{1}{2b_{3n}}} = \sum_{k\in\mathbb{Z}} \frac{2b_{3n}a_{1n}^2}{1+2b_{3n}},$$

$$\|Y_2\|^2 = \sum_{n\in\mathbb{Z}} \frac{a_{2n}^2}{1+\frac{1}{2b_{3n}}} = \sum_{k\in\mathbb{Z}} \frac{2b_{3n}a_{2n}^2}{1+2b_{3n}}, \quad \|Y_3\|^2 = \sum_{n\in\mathbb{Z}} \frac{a_{3n}^2}{1+\frac{1}{2b_{3n}}} = \sum_{k\in\mathbb{Z}} \frac{2b_{3n}a_{3n}^2}{1+2b_{3n}}.$$

Therefore, in the case (a) and (b) we have

$$\|Y_1\|^2 \sim \sum_{k\in\mathbb{Z}} a_{1n}^2, \quad \|Y_2\|^2 \sim \sum_{k\in\mathbb{Z}} a_{2n}^2, \quad \|Y_3\|^2 \sim \sum_{k\in\mathbb{Z}} a_{3n}^2,$$

In the case (c) we get

$$\|Y_1\|^2 \sim \sum_{k\in\mathbb{Z}} b_{3n}a_{1n}^2, \quad \|Y_2\|^2 \sim \sum_{k\in\mathbb{Z}} b_{3n}a_{2n}^2, \quad \|Y_3\|^2 \sim \sum_{k\in\mathbb{Z}} b_{3n}a_{3n}^2.$$

Since in the case (a), (b)

$$\|Y_1\|^2 \sim \sum_{n\in\mathbb{Z}} a_{1n}^2 = \sum_{n\in\mathbb{Z}} b_{1n}a_{1n}^2 \sim S_{11}^L(\mu) = \infty,$$
$$\|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} a_{2n}^2 = \sum_{n\in\mathbb{Z}} b_{2n}a_{2n}^2 = S_{22}^L(\mu) = \infty,$$

we have two possibilities for $y_{23} := (y_2, y_3) \in \{0,1\}^2$, see Section 4.4.4:

$$\begin{array}{ccc} & (1.1) & (1.3) \\ y_1 & 1 & 1 \\ y_2 & 1 & 1 \\ y_3 & 0 & 1 \end{array} \tag{4.111}$$

In the case (c) we have

$$\|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} b_{3n}a_{3n}^2 \sim S_{33}^L(\mu) = \infty.$$

Therefore, we have four possibilities for $y_{12} := (y_1, y_2) \in \{0,1\}^2$, see (4.54),

$$\begin{array}{ccccc} & (1.0) & (1.1) & (1.2) & (1.3) \\ y_1 & 0 & 1 & 0 & 1 \\ y_2 & 0 & 0 & 1 & 1 \\ y_3 & 1 & 1 & 1 & 1 \end{array} \tag{4.112}$$

Further, in the case (a), (b) we have four possibilities: (1.1.1), (1.3.1) and (1.1.0), (1.3.0), see Remark 4.4. In the case (1.1.1) we can approximate D_{1n}, D_{2n}, in the case (1.3.1) we can approximate all D_{rn}, $1 \leq r \leq 3$. In these cases the proof is finished, since we get respectively $D_{1n}, D_{2n}, x_{3n}\, \eta\, \mathfrak{A}^3$. The cases (a),(b) subcases (1.1.0), (1.3.0) we consider below.

In the case (c) subcase (1.0) we can approximate D_{3n} using Lemma 5.6, since $\Delta(Y_3, Y_2, Y_1) \sim \|Y_3\|^2 = \infty$, so we have $D_{3n}, x_{3n}\, \eta\, \mathfrak{A}^3$, and the proof is finished.

Further, in the case (c) we have six cases (1.1.1), (1.2.1), (1.3.1) and (1.1.0), (1.2.0), (1.3.0), according to whether corresponding expressions are divergent (see analogue in Remark 4.4). We can approximate in the three first cases by respectively D_{1n} and D_{3n} in the case (1.1.1), D_{2n} and D_{3n} in the case (1.1.2) and all D_{1n}, D_{2n}, D_{3n} in (1.1.3). The proof of irreducibilty is finished in these cases because we have respectively D_{1n}, D_{3n}, $x_{3n}\, \eta\, \mathfrak{A}^3$, D_{2n}, D_{3n}, $x_{3n}\, \eta\, \mathfrak{A}^3$, or D_{1n}, D_{2n}, D_{3n}, $x_{3n}\, \eta\, \mathfrak{A}^3$.

If the opposite holds, in the cases (a), (b) or (c), i.e., we are in the cases (1.1.0), (1.2.0) and (1.3.0) respectively, we try to approximate D_{3n} using Lemma 5.15. If one of the expressions $\Sigma_3(D, s)$ or $\Sigma_3^{\vee}(D, s)$ is divergent, we can approximate D_{3k} and the proof is finished, since we have x_{3n}, $D_{3n}\ \eta\ \mathfrak{A}^3$. Let us suppose, as in Remark 4.2, that for every sequence $s = (s_k)_{k\in\mathbb{Z}}$ holds

$$\Sigma_3(D, s) + \Sigma_3^{\vee}(D, s) < \infty.$$

Then, in particularly, we have for $s^{(3)} = (s_k)_{k\in\mathbb{Z}}$ with $\frac{s_k^2}{b_{3k}} \equiv 1$

$$\infty > \Sigma_3(D, s^{(3)}) + \Sigma_3^{\vee}(D, s^{(3)}) \sim \Sigma_3(D) + \Sigma_3^{\vee}(D) = \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2}$$

$$\overset{(2.22)}{\sim} \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{\frac{1}{2b_{1k}} + a_{1k}^2 + \frac{1}{2b_{2k}} + a_{2k}^2} = \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} =: \Sigma_3^{\vee,+}(D). \quad (4.113)$$

In the case (a), (b) and (c) we have respectively

$$\Sigma_3^{\vee,+}(D) \sim \Sigma_3^{+}(D) = \sum_k \frac{2a_{3k}^2}{1 + 2a_{1k}^2 + 2a_{2k}^2}, \quad \Sigma_3^{\vee,+}(D) = \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2}.$$

In particular, in the case (c) we have by (4.113)

$$\infty > \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} > \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} \sim \Sigma_3^{+}(D). \quad (4.114)$$

The cases (a), subcase (1.1.0), where $\|Y_3\|^2 < \infty$ can not occur, because conditions $\Sigma_{12}(s_1) < \infty$ and $\nu_{12}(C_1, C_2) < \infty$ defined by (4.58), contradict the orthogonality condition for the matrix $\tau_{12}(\phi, s)$:

$$\tau_{12}(\phi, s) = \begin{pmatrix} \cos\phi & s^2 \sin\phi & 0 \\ s^{-2}\sin\phi & -\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (4.115)$$

Indeed, recall Remark 2.5 (instead of $\mu^2_{(b,a)}$ we can write $\mu^3_{(b,a)}$)

$$\left(\mu^3_{(b,a)}\right)^{L_{\tau_{12}(\phi,s)}} \perp \mu^3_{(b,a)} \Leftrightarrow \Sigma_{12}(s) + \Sigma_{12}(C_1, C_2) = \infty,$$

where $\Sigma_{12}(s) = \Sigma_1(s)$ is defined by (2.17) and $\Sigma_{12}(C_1, C_2) = \Sigma_2(C_1, C_2)$ is defined by (2.19)

$$\Sigma_{12}(C_1, C_2) := \sum_{n\in\mathbb{Z}} (C_1^2 b_{1n} + C_2^2 b_{2n})(C_1 a_{1n} + C_2 a_{2n})^2 \sim \nu_{12}(C_1, C_2).$$

We get contradiction:

$$\infty > \Sigma_{12}(s) + \nu_{12}(C_1, C_2) \sim \Sigma_{12}(s) + \Sigma_{12}(C_1, C_2) = \infty.$$

In the case (a), (b), subcase (1.3.0) we get $\Sigma_3^+(D) = \infty$ by Lemma 4.11, contradiction with (4.113) hence, $D_{3n}\,\eta\,\mathfrak{A}^3$.

In the case (c), subcases (1.1.0) and (1.2.0) we have respectively $\|Y_2\|^2 < \infty$ and $\|Y_1\|^2 < \infty$ hence,

$$\Sigma_3^+(D) \sim \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2} = \infty, \quad \Sigma_3^+(D) \sim \sum_k \frac{a_{3k}^2}{1 + a_{2k}^2} = \infty.$$

by Lemma 4.10, contradiction with (4.113) hence, $D_{3n}\,\eta\,\mathfrak{A}^3$. In the case (c), subcase (1.3.0) we get

$$\Sigma_3^+(D) = \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} = \infty$$

by Lemma 4.11, contradiction with (4.113) hence, $D_{3n}\,\eta\,\mathfrak{A}^3$.

4.5.3. Case $\Sigma_{123}(s)=(1,1,1)$

We have for all $s=(s_{12},s_{23},s_{13})\in\mathbb{R}^3_+\setminus\{0\}$

$$\Sigma_{12}(s_{12})=\infty,\quad \Sigma_{23}(s_{23})=\infty,\quad \Sigma_{13}(s_{13})=\infty, \tag{4.116}$$
$$b=(b_{1n},b_{2n},b_{3n})_{n\in\mathbb{Z}}\overset{(4.20)}{=}(1,d_{2n},d_{3n})_{n\in\mathbb{Z}}.$$

Recall (4.34), that we denote $D_n:=d_{2n}^{-1}+d_{3n}^{-1}+1$ and $d_n=\frac{d_{3n}}{d_{3n}}$. Set

$$\begin{pmatrix} Y^{(1)}_{1n} & Y^{(1)}_{2n} & Y^{(1)}_{3n}\\ Y^{(2)}_{1n} & Y^{(2)}_{2n} & Y^{(2)}_{3n}\\ Y^{(3)}_{1n} & Y^{(3)}_{2n} & Y^{(3)}_{3n}\end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{1+2D_nd_{2n}d_{3n}}} & \frac{d_{2n}}{\sqrt{1+2D_nd_{2n}d_{3n}}} & \frac{d_{3n}}{\sqrt{1+2D_nd_{2n}d_{3n}}}\\ \frac{1}{\sqrt{d_{2n}^2+2D_nd_{2n}d_{3n}}} & \frac{d_{2n}}{\sqrt{d_{2n}^2+2D_nd_{2n}d_{3n}}} & \frac{d_{3n}}{\sqrt{d_{2n}^2+2D_nd_{2n}d_{3n}}}\\ \frac{1}{\sqrt{d_{3n}^2+2D_nd_{2n}d_{3n}}} & \frac{d_{2n}}{\sqrt{d_{3n}^2+2D_nd_{2n}d_{3n}}} & \frac{d_{3n}}{\sqrt{d_{3n}^2+2D_nd_{2n}d_{3n}}}\end{pmatrix}. \tag{4.117}$$

Remark 4.16. For (r,s) such that $1\le r<s\le 3$ the following equivalence hold

$$\Sigma_{rs}(s_{rs})<\infty\ \Leftrightarrow\ \sum_{n\in\mathbb{Z}}c^2_{rs,n}<\infty\ \Leftrightarrow\ \sum_{n\in\mathbb{Z}}c^2_{sr,n}<\infty,\ \text{where} \tag{4.118}$$
$$\frac{b_{rn}}{b_{sn}}=:s_{rs}^{-4}(1+c_{rs,n}),\ \ \frac{b_{sn}}{b_{rn}}=s_{rs}^4(1+c_{sr,n}),\ \lim_n\frac{b_{rn}}{b_{sn}}\in(0,\infty). \tag{4.119}$$

Proof. By Lemma 8.2 we have

$$\Sigma_{rs}(s_{rs})=\sum_{n\in\mathbb{Z}}\left(s_{rs}^2\sqrt{\frac{b_{rn}}{b_{sn}}}-s_{rs}^{-2}\sqrt{\frac{b_{sn}}{b_{rn}}}\right)^2=\sum_{n\in\mathbb{Z}}\frac{c^2_{rs,n}}{1+c_{rs,n}}\sim\sum_{n\in\mathbb{Z}}c^2_{rs,n},$$
$$\Sigma_{sr}(s_{rs}^{-1})=\sum_{n\in\mathbb{Z}}\left(s_{rs}^{-2}\sqrt{\frac{b_{sn}}{b_{rn}}}-s_{rs}^{2}\sqrt{\frac{b_{rn}}{b_{sn}}}\right)^2=\sum_{n\in\mathbb{Z}}\frac{c^2_{sr,n}}{1+c_{sr,n}}\sim\sum_{n\in\mathbb{Z}}c^2_{sr,n},$$
$$\text{note also that}\quad 1=\frac{b_{rn}}{b_{sn}}\frac{b_{sn}}{b_{rn}}=(1+c_{rs,n})(1+c_{sr,n}). \tag{4.120}$$

□

By Remark 4.16, the condition $\Sigma_{rs}(s_{rs})=\infty$ means the following:

$$l_{sr}:=\lim_n\frac{b_{sn}}{b_{rn}}=\begin{cases}(a) & \infty & & \\ (b) & s_{rs}^4>0 & \text{with} & \sum_{n\in\mathbb{Z}}c^2_{sr,n}=\infty.\\ (c) & 0 & & \\ (d) & \lim & \text{does not exist} & \end{cases} \tag{4.121}$$

Remark 4.17. In the case (d) we can use the fact that some *subsequence* of $\left(\frac{b_{sn}}{b_{rn}}\right)_{n\in\mathbb{Z}}$ has property (a), (b) or (c). We can avoid the case (c). Namely, if $l_{sr}=0$ for some pair (r,s) with $1\leq r<s\leq 3$, we can exchange the two lines (b_{sn},a_{sn}) and (b_{rn},a_{rn}) to obtain $l_{sr}=\infty$.

Formally, we have $3^3=\#(A)^{\#(B)}$ possibilities where $A=\{(21),(32),(31)\}$ and $B=\{(a),(b),(d)\}$. Since $l_{32}l_{21}=l_{31}$ we get only the following cases:

$e\setminus(rs)$	(21)	(32)	(31)
(1)	b	b	b
(2)	a	a	a
(3)	a	b	a
(4)	b	a	b

To be able to approximate x_{rn} for $1\leq r\leq 3$ we should study when the following expressions are infinite:

$$\rho_r(C_1,C_2,C_3)=\|C_1Y_1^{(r)}+C_2Y_2^{(r)}+C_2Y_3^{(r)}\|^2. \tag{4.122}$$

By (4.117) we have

$$\rho_r(C_1,C_2,C_3)=:\sum_n\frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{C_{rn}}, \tag{4.123}$$

where $C_{1n}=1+2D_nd_{2n}d_{3n},\ C_{2n}=d_{2n}^2+2D_nd_{2n}d_{3n},\ C_{3n}=d_{3n}^2+2D_nd_{2n}d_{3n}$.

Consider the **case (1)=(bbb)**. We prove the analogue of Lemma 3.13 for the case $m=3$.

Lemma 4.23. *Let for all* $s=(s_{12},s_{23},s_{13})\in\left(\mathbb{R}_+\right)^3$ *holds* (4.116). *Then*

$$\Delta(Y_3^{(3)},Y_1^{(3)},Y_2^{(3)})=\Delta(Y_2^{(2)},Y_3^{(2)},Y_1^{(2)})=\Delta(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)})=\infty. \tag{4.124}$$

Proof. For $1\leq r<s\leq 3$ set

$$\frac{b_{sn}}{b_{rn}}=s_{rs}^4(1+c_{sr,n})\quad\text{with}\quad\sum_{n\in\mathbb{Z}}c_{sr,n}^2=\infty,\quad\lim_{n\to\infty}c_{sr,n}=0.$$

For $b_{1n}\equiv 1$ we have

$$b_{2n}=s_{12}^4(1+c_{21,n}),\ b_{3n}=s_{13}^4(1+c_{31,n}),$$

$$\frac{b_{3n}}{b_{2n}}=\frac{s_{13}^4}{s_{12}^4}\frac{1+c_{31,n}}{1+c_{21,n}}=s_{23}^4(1+c_{32,n}),\quad c_{32,n}=\frac{1+c_{31,n}}{1+c_{21,n}}-1,\ s_{23}=\frac{s_{13}}{s_{12}},$$

$$\sum_n c_{32,n}^2=\sum_n\left(\frac{1+c_{31,n}}{1+c_{21,n}}-1\right)^2=\sum_n\left(\frac{c_{31,n}-c_{21,n}}{1+c_{21,n}}\right)^2\sim\sum_n(c_{21,n}-c_{31,n})^2=\infty.$$

Finally, we get

$$\sum_n c_{21,n}^2 = \infty, \quad \sum_n c_{31,n}^2 = \infty, \quad \sum_n (c_{21,n} - c_{31,n})^2 = \infty. \tag{4.125}$$

By (4.122) and (4.123) we get

$$\rho_r(C_1, C_2, C_3) = \|C_1 Y_1^{(r)} + C_2 Y_2^{(r)} + C_2 Y_3^{(r)}\|^2 = \sum_n \frac{|C_1 + C_2 d_{2n} + C_3 d_{3n}|^2}{C_{rn}}$$
$$= \sum_n \frac{|C_1 + C_2 s_{12}^4 (1 + c_{21,n}) + C_3 s_{13}^4 (1 + c_{31,n})|^2}{C_{rn}}.$$

The latter expression is divergent if $C_1 + C_2 s_{12}^4 + C_3 s_{13}^4 \neq 0$ since $\lim_{n\to\infty} c_{21,n} = \lim_{n\to\infty} c_{31,n} = 0$ and $A_1 \leq C_{rn} \leq A_2$.

In the case when $C_1 + C_2 s_{12}^4 + C_3 s_{13}^4 = 0$ we get

$$\rho_r(C_1, C_2, C_3) = \sum_{n\in\mathbb{Z}} \frac{|C_2 s_{12}^4 c_{21,n} + C_3 s_{13}^4 c_{31,n}|^2}{C_{rn}} =: \rho_r(C_2, C_3). \tag{4.126}$$

The latter expression is divergent by the first two relations in (4.125) when 1) $C_2 C_3 > 0$, 2) $C_2 = 0$ and $C_3 \neq 0$, 3) $C_2 \neq 0$ and $C_3 = 0$.

If $C_2 C_3 < 0$ we have by the last relation in (4.125)

$$\sum_{n\in\mathbb{Z}} \frac{|C_2 s_{12}^4 c_{21,n} - C_3 s_{13}^4 c_{31,n}|^2}{C_{rn}} \sim \sum_{n\in\mathbb{Z}} |C_2 s_{12}^4 c_{21,n} - C_3 s_{13}^4 c_{31,n}|^2 = \infty,$$

since $(s_{12}, s_{13}) = \frac{1}{s_1}(s_2, s_3) \in (\mathbb{R}^*)^2$ are arbitrary.

Consider the **case (2)=(aaa)**. Now, see (4.121), we have

$$l_{21} = \lim_n \frac{b_{2n}}{b_{1n}} = \infty, \quad l_{32} = \lim_n \frac{b_{3n}}{b_{2n}} = \infty, \quad \text{therefore,} \quad l_{31} = \lim_n \frac{b_{3n}}{b_{1n}} = \infty. \tag{4.127}$$

Since $b_{1n} \equiv 1$ we conclude that

$$l_{21} = \lim_n d_{2n} = \infty \quad \text{and} \quad l_{31} = \lim_n d_{3n} = \infty. \tag{4.128}$$

Therefore, we get for some $C > 0$ and all $n \in \mathbb{Z}$

$$1 \leq D_n = 1 + \frac{1}{d_{2n}} + \frac{1}{d_{3n}} \leq C. \tag{4.129}$$

By (4.122) and (4.123) we obtain

$$\rho_r(C_1,C_2,C_3)=\|C_1Y_1^{(r)}+C_2Y_2^{(r)}+C_2Y_3^{(r)}\|^2=\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{C_{rn}}$$
$$\sim \sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{C'_{rn}} =: \rho'_r(C_1,C_2,C_3),$$

where $\quad C'_{rn}=1+2d_{2n}d_{3n}, \quad C'_{rn}=d_{2n}^2+2d_{2n}d_{3n}, \quad C'_{rn}=d_{3n}^2+2d_{2n}d_{3n}.$

We should study when $\rho'_r(C_1,C_2,C_3)=\infty$ for some $1\le r\le 3$:

$$\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{1+2d_{2n}d_{3n}},\ \sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{d_{2n}^2+2d_{2n}d_{3n}},\ \sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{d_{3n}^2+2d_{2n}d_{3n}}.$$

Denoting as before $d_n=\frac{d_{3n}}{d_{2n}}=: l_n^{-1}$, we get

$$\rho'_1(C_1,C_2,C_3)=\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{1+2d_{2n}d_{3n}}=\sum_n \frac{|\frac{C_1}{d_{2n}}+C_2+C_3d_n|^2}{\frac{1}{d_{2n}^2}+2d_n}$$
$$\sim \sum_n \frac{|C_2+C_3d_n|^2}{2d_n},\quad \rho'_2(C_1,C_2,C_3)=\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{d_{2n}^2+2d_{2n}d_{3n}}=$$
$$\sum_n \frac{|\frac{C_1}{d_{2n}}+C_2+C_3d_n|^2}{1+2d_n}\sim \sum_n \frac{|C_2+C_3d_n|^2}{1+2d_n},$$
$$\rho'_3(C_1,C_2,C_3)=\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{d_{3n}^2+2d_{2n}d_{3n}}=$$
$$\sum_n \frac{|\frac{C_1}{d_{2n}}+C_2+C_3d_n|^2}{d_n^2+2d_n}\sim \sum_n \frac{|C_2+C_3d_n|^2}{d_n^2+2d_n}=\sum_n \frac{|C_2l_n+C_3|^2}{1+2l_n}.$$

By Lemma 4.8 we get when $C_2C_3^{-1}>0$

$$\rho'_1(C_1,C_2,-C_3)\sim \sum_n \frac{|C_2-C_3d_n|^2}{2d_n}>\sum_n \frac{|C_2-C_3d_n|^2}{1+2d_n}\sim \sum_n c_n^2\sim \Sigma_{23}(s),$$
$$\rho'_2(C_1,C_2,-C_3)\sim \sum_n \frac{|C_2-C_3d_n|^2}{1+2d_n}\sim \sum_n c_n^2\sim \Sigma_{23}(s), \tag{4.130}$$
$$\rho'_3(C_1,C_2,-C_3)\sim \sum_n \frac{|C_2l_n-C_3|^2}{1+2l_n}\sim \sum_n e_n^2\sim \Sigma_{23}(s),$$

where $\quad d_n=C_2C_3^{-1}(1+c_n),\ \ l_n=C_3C_2^{-1}(1+e_n), \quad s^4=C_2C_3^{-1}>0.$

But $\Sigma_{23}(s)=\infty$ for all $s>0$ therefore, for $C_2C_3^{-1}>0$ we have

$$\rho_r(C_1,C_2,-C_3)\sim\Sigma_{23}(s)=\infty. \tag{4.131}$$

If $C_2C_3^{-1}>0$, by (4.130) we get

$$\sum_n\frac{|C_2{+}C_3d_n|^2}{1+2d_n}>\sum_n\frac{|C_2{-}C_3d_n|^2}{1+2d_n}\sim\sum_n c_n^2\sim\Sigma_{23}(s)=\infty,$$
$$\sum_n\frac{|C_2{+}C_3d_n|^2}{1+2d_n}\sum_n\frac{|C_2{-}C_3d_n|^2}{1+2d_n}\sim\sum_n c_n^2\sim\Sigma_{23}(s)=\infty,$$
$$\sum_n\frac{|C_2l_n{+}C_3|^2}{1+2l_n}>\sum_n\frac{|C_2l_n{-}C_3|^2}{1+2l_n}\sim\sum_n e_n^2\sim\Sigma_{23}(s)=\infty.$$

Therefore, $\rho_r(C_1,C_2,C_3)=\infty$ for every $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$.

Consider the **case (3)=(aba)**. In this case, see (4.121), we have

$$l_{21}=\lim_n\frac{b_{2n}}{b_{1n}}=\infty,\quad l_{32}=\lim_n\frac{b_{3n}}{b_{2n}}<\infty,\quad \text{therefore,}\quad l_{31}=\lim_n\frac{b_{3n}}{b_{1n}}=\infty.$$

So, we have again, see (4.128)

$$l_{21}=\lim_n d_{2n}=\infty,\quad \text{and}\quad l_{31}=\lim_n d_{3n}=\infty.$$

We are reduced to the case (2).

Consider the **case (4)=(baa).** Now, see (4.121), we have

$$l_{21}=\lim_n d_{2n}<\infty,\quad \text{and}\quad l_{31}=\lim_n d_{3n}=\infty.$$

Hence (4.129) holds too and we can use all estimations of the case (1). □

Remark 4.18. In the cases (1)–(4) by Lemma 4.23 and Lemmas 5.1– 5.3 we can approximate x_{rn} for all $1\le r\le 3$ and $n\in\mathbb{Z}$ and the irreducibility is proved.

5. Approximation of D_{kn} and x_{kn}

5.1. Approximation of x_{kn} by $A_{nk}A_{tk}$

For $m=3$, consider three rows as follows

$$\begin{pmatrix} \dots & b_{11} & b_{12} & \dots & b_{1n} & \dots\\ \dots & b_{21} & b_{22} & \dots & b_{2n} & \dots\\ \dots & b_{31} & b_{32} & \dots & b_{3n} & \dots \end{pmatrix}.$$

Set

$$\lambda_k^{(r)} = (b_{1k} + b_{2k} + b_{3k})^2 - (b_{1k}^2 + b_{2k}^2 + b_{3k}^2 - b_{rk}^2),\ r = 1, 2, 3,\ k \in \mathbb{Z}. \quad (5.1)$$

Denote by $Y_r^{(s)}$ the following vectors:

$$x_{rk}^{(s)} = b_{rk}/\sqrt{\lambda_k^{(s)}},\ k \in \mathbb{Z}, \quad Y_r^{(s)} = (x_{rk}^{(s)})_{k\in\mathbb{Z}}. \quad (5.2)$$

Lemma 5.1. *For any $n, t \in \mathbb{Z}$ one has*

$$x_{1n}x_{1t}\mathbf{1} \in \langle A_{nk}A_{tk}\mathbf{1} \mid k \in \mathbb{Z}\rangle \ \Leftrightarrow\ \Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}) = \infty.$$

Similarly, using the cyclic permutation of vectors, and changing $\lambda_k^{(r)}$ we arrive at the following lemma.

Lemma 5.2. *For any $n, t \in \mathbb{Z}$ one has*

$$x_{2n}x_{2t}\mathbf{1} \in \langle A_{nk}A_{tk}\mathbf{1} \mid k \in \mathbb{Z}\rangle \ \Leftrightarrow\ \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)})) = \infty.$$

Lemma 5.3. *For any $n, t \in \mathbb{Z}$ one has*

$$x_{3n}x_{3t}\mathbf{1} \in \langle A_{nk}A_{tk}\mathbf{1} \mid k \in \mathbb{Z}\rangle \ \Leftrightarrow\ \Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = \infty.$$

Proof. The proof of Lemma 5.1 is also based on Lemma 7.3. We study when $x_{1n}x_{1t}\mathbf{1} \in \langle A_{nk}A_{tk}\mathbf{1} \mid k \in \mathbb{Z}\rangle$. Since

$$\begin{aligned}A_{nk}A_{tk} &= (x_{1n}D_{1k} + x_{2n}D_{2k} + x_{3n}D_{3k})(x_{1t}D_{1k} + x_{2t}D_{2k} + x_{3t}D_{3k})\\ &= x_{1n}x_{1t}D_{1k}^2 + x_{2n}x_{2t}D_{2k}^2 + x_{3n}x_{3t}D_{3k}^2 + (x_{1n}x_{2t} + x_{2n}x_{1t})D_{1k}D_{2k} +\\ &(x_{1n}x_{3t} + x_{3n}x_{1t})D_{1k}D_{3k} + (x_{2n}x_{3t} + x_{3n}x_{2t})D_{2k}D_{3k}\end{aligned}$$

and $MD_{1k}^2\mathbf{1} = -\frac{b_{1k}}{2}$, we take $t = (t_k)$ as follows: $-\sum_{k=-m}^{m} t_k\frac{b_{1k}}{2} = (t, b') = 1$, where $t = (t_k)_{k=-m}^{m}$ and $b' = -(\frac{b_{1k}}{2})_{k=-m}^{m} \sim b = (b_{1k})_{k=-m}^{m}$. We have

$$\begin{aligned}&\|\Big[\sum_{k=-m}^{m} t_kA_{nk}A_{tk} - x_{1n}x_{1t}\Big]\mathbf{1}\|^2 =\\ &\|\sum_{k=-m}^{m} t_k\Big[x_{1n}x_{1t}\Big(D_{1k}^2+\frac{b_{1k}}{2}\Big)+x_{2n}x_{2t}D_{2k}^2+x_{3n}x_{3t}D_{3k}^2+(x_{1n}x_{2t} + x_{2n}x_{1t}) \times\\ &D_{1k}D_{2k} + (x_{1n}x_{3t} + x_{3n}x_{1t})D_{1k}D_{3k} + (x_{2n}x_{3t} + x_{3n}x_{2t})D_{2k}D_{3k}\Big]\mathbf{1}\|^2\\ = &\sum_{-m\le k,r\le m}(f_k, f_r)t_kt_r =: (A_{2m+1}t, t),\end{aligned}$$

where $A_{2m+1} = \big((f_k, f_r)\big)_{k,r=-m}^{m}$ and $f_k = \sum_{r=1}^{3} f_k^r + \sum_{1\le i<j\le 3} f_k^{ij}$,

$$f_k^r = x_{rn}x_{rt}\Big(D_{rk}^2 + \frac{b_{rk}}{2}\delta_{1r}\Big)\mathbf{1}, \quad f_k^{ij} = (x_{in}x_{jt} + x_{jn}x_{it})D_{ik}D_{jk}\mathbf{1} \qquad (5.3)$$

for $1 \le r \le 3,\ 1 \le i < j \le 3$. Since

$$f_k^r \perp f_k^{ij}, \quad f_k^{ij} \perp f_k^{i'j'}$$

for different $(ij),\ (i'j')$, writing $c_{kn} = \|x_{kn}\|^2 = \frac{1}{2b_{kn}} + a_{kn}^2$, we get

$$(f_k, f_k) = \sum_{r=1}^{3} \|f_k^r\|^2 + \sum_{1\le i<j\le 3} \|f_k^{ij}\|^2 =$$
$$c_{1n}c_{1t}2\Big(\frac{b_{1k}}{2}\Big)^2 + c_{2n}c_{2t}3\Big(\frac{b_{2k}}{2}\Big)^2 + c_{3n}c_{3t}3\Big(\frac{b_{3k}}{2}\Big)^2 +$$
$$\big(c_{1n}c_{2t} + c_{1t}c_{2n} + 2a_{1n}a_{2t}a_{1t}a_{2n}\big)\frac{b_{1k}}{2}\frac{b_{2k}}{2} + \big(c_{1n}c_{3t} + c_{3t}c_{1n} + 2a_{1n}a_{3t}a_{3t}a_{1n}\big) \times$$
$$\frac{b_{1k}}{2}\frac{b_{3k}}{2} + \big(c_{2n}c_{3t} + c_{3t}c_{2n} + 2a_{2n}a_{3t}a_{3t}a_{2n}\big)\frac{b_{2k}}{2}\frac{b_{3k}}{2} \sim (b_{1k} + b_{2k} + b_{3k})^2,$$
$$(f_k, f_r) = (f_k^2, f_r^2) + (f_k^3, f_r^3) = c_{2n}c_{2t}\frac{b_{2k}}{2}\frac{b_{2r}}{2} + c_{3n}c_{3t}\frac{b_{3k}}{2}\frac{b_{3r}}{2} \sim b_{2k}b_{2r} + b_{3k}b_{3r}.$$

Finally, we have

$$(f_k, f_k) \sim (b_{1k} + b_{2k} + b_{3k})^2, \quad (f_k, f_r) \sim b_{2k}b_{2r} + b_{3k}b_{3r}, \quad k \ne r. \qquad (5.4)$$

Set

$$\lambda_k = (b_{1k} + b_{2k} + b_{3k})^2 - (b_{2k}^2 + b_{3k}^2), \quad g_k = (b_{2k}, b_{3k}), \qquad (5.5)$$

then

$$(f_k, f_k) \sim \lambda_k + (g_k, g_k), \quad (f_k, f_r) \sim (g_k, r_r). \qquad (5.6)$$

For $A_{2m+1} = \big((f_k, f_r)\big)_{k,r=-m}^{m}$, and $b = -(b_{1k}/2)_{k=-m}^{m} \in \mathbb{R}^{2m+1}$ we have

$$A_{2m+1} = \sum_{k=-m}^{m} \lambda_k E_{kk} + \gamma(g_{-m}, \dots, g_0, \dots, g_m).$$

To finish the proof, it suffices to invoke Lemma 7.3. $\square$

5.2. Approximation of D_{rn} by A_{kn}

We will formulate several lemmas, which will be useful for approximation of the independent variables x_{kn} and operators D_{kn} by combinations of the generators A_{kn}. The generators A_{kn} have the following form:

$$A_{kn} = x_{1k}D_{1n} + x_{2k}D_{2n} + x_{3k}D_{3n},\ \ k, n \in \mathbb{Z}.$$

For $m = 3$, consider three rows as follows

$$\begin{pmatrix} \dots & a_{11} & a_{12} & \dots & a_{1n} & \dots \\ \dots & a_{21} & a_{22} & \dots & a_{2n} & \dots \\ \dots & a_{31} & a_{32} & \dots & a_{2n} & \dots \end{pmatrix}.$$

Set

$$\lambda_k = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}. \tag{5.7}$$

Denote by Y_1, Y_2 and Y_3 the three following vectors:

$$x_{rk} = \frac{a_{rk}}{\sqrt{\lambda_k}},\ \ k \in \mathbb{Z}, \quad Y_r = (x_{rk})_{k\in\mathbb{Z}}. \tag{5.8}$$

The proofs of Lemmas 5.4–5.6 and 5.1–5.3 are based on Lemma 7.3.

Lemma 5.4. *For any $l \in \mathbb{Z}$ we have*

$$D_{1l}\mathbf{1} \in \langle A_{kl}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_1, Y_2, Y_3) = \infty.$$

Similarly, by cyclic permutation of the vectors, we obtain the following two lemmas.

Lemma 5.5. *For any $l \in \mathbb{Z}$ we have*

$$D_{2l}\mathbf{1} \in \langle A_{kl}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_2, Y_3, Y_1) = \infty.$$

Lemma 5.6. *For any $l \in \mathbb{Z}$ we have*

$$D_{3l}\mathbf{1} \in \langle A_{kl}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_3, Y_1, Y_2) = \infty.$$

Proof. We determine when the inclusion

$$D_{1n}\mathbf{1} \in \langle A_{kn}\mathbf{1} = (x_{1k}D_{1n} + x_{2k}D_{2n} + x_{3k}D_{3n})\mathbf{1} \mid k \in \mathbb{Z}\rangle$$

holds. Fix $m \in \mathbb{N}$, since $Mx_{1k} = a_{1k}$, we put $\sum_{k=-m}^{m} t_k a_{1k} = (t, b) = 1$, where $t = (t_k)_{k=-m}^{m}$ and $b = (a_{1k})_{k=-m}^{m}$. We have

$$\begin{aligned}
&\| \Big[\sum_{k=-m}^{m} t_k (x_{1k} D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}) - D_{1n} \Big] \mathbf{1} \|^2 \\
= \ & \| \sum_{k=-m}^{m} t_k [(x_{1k} - a_{1k}) D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}] \mathbf{1} \|^2 \\
= \ & \sum_{-m \leq k, r \leq m} (f_k, f_r) t_k t_r =: (A_{2m+1} t, t), \ \text{where } A_{2m+1} = \big((f_k, f_r) \big)_{k,r=-m}^{m}, \\
& \quad \text{and} \quad f_k = [(x_{1k} - a_{1k}) D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}] \mathbf{1}.
\end{aligned}$$

We get

$$\begin{aligned}
&(f_k, f_k) = \| \, [(x_{1k} - a_{1k}) D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}] \, \mathbf{1} \|^2 \\
&= \frac{1}{2b_{1k}} \frac{b_{1n}}{2} + \Big(\frac{1}{2b_{2k}} + a_{2k}^2 \Big) \frac{b_{2n}}{2} + \Big(\frac{1}{2b_{3k}} + a_{3k}^2 \Big) \frac{b_{3n}}{2} \\
&\sim \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2, \\
&(f_k, f_r) = \Big(\big[(x_{1k} - a_{1k}) D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n} \big] \mathbf{1}, \\
&\big[(x_{1r} - a_{1r}) D_{1n} + x_{2r} D_{2n} + x_{3r} D_{3n} \big] \mathbf{1} \Big) \\
&= (x_{2k} \mathbf{1}, x_{2r} \mathbf{1})(D_{2n} \mathbf{1}, D_{2n} \mathbf{1}) + (x_{3k} \mathbf{1}, x_{3r} \mathbf{1})(D_{3n} \mathbf{1}, D_{3n} \mathbf{1}) \\
&= a_{2k} a_{2r} \frac{b_{2n}}{2} + a_{3k} a_{3r} \frac{b_{3n}}{2} \sim a_{2k} a_{2r} + a_{3k} a_{3r}.
\end{aligned}$$

Finally, we have

$$(f_k, f_k) \sim \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2, \quad (f_k, f_r) \sim a_{2k} a_{2r} + a_{3k} a_{3r}, \ k \neq r. \tag{5.9}$$

If we denote

$$\lambda_k = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}, \quad g_k = (a_{2k}, a_{3k}), \tag{5.10}$$

then we have

$$(f_k, f_k) \sim \lambda_k + (g_k, g_k), \quad (f_k, f_r) \sim (g_k, g_r). \tag{5.11}$$

For $A_{2m+1}=\big((f_k,f_r)\big)_{k,r=-m}^m$, and $b=(a_{1k})_{k=-m}^m\in\mathbb{R}^{2m+1}$ we have

$$A_{2m+1}=\sum_{k=-m}^m\lambda_kE_{kk}+\gamma(g_{-m},\dots,g_0,\dots,g_m).$$

To finish the proof, it suffices to apply Lemma 7.3. □

The proofs of Lemmas 5.5 and 5.6 are exactly the same.

5.3. Approximation of x_{rk} by $D_{rn}A_{kn}$

Lemma 5.7. *For any $k\in\mathbb{Z}$ we get*

$$x_{1k}\mathbf{1}\in\langle D_{1n}A_{kn}\mathbf{1}\mid n\in\mathbb{Z}\rangle\quad\Leftrightarrow\quad\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{b_{1n}+b_{2n}+b_{3n}}=\infty.$$

PROOF. Since

$$D_{1n}A_{kn}=x_{1k}D_{1n}^2+x_{2k}D_{1n}D_{2n}+x_{3k}D_{1n}D_{3n}$$

and $MD_{1k}^2\mathbf{1}=-\frac{b_{1k}}{2}$, we take $t=(t_k)_{k=-m}^m$ as follows: $(t,b')=1$, where $t=(t_k)_{k=-m}^m$ and $b'=-(\frac{b_{1k}}{2})_{k=-m}^m\sim b=-(b_{1k})_{k=-m}^m$. We have

$$\begin{aligned}&\|\big[\sum_{n=-m}^m t_nD_{1n}A_{kn}-x_{1k}\big]\mathbf{1}\|^2\\ =\ &\|\sum_{n=-m}^m t_n\left[x_{1k}\Big(D_{1n}^2+\frac{b_{1n}}{2}\Big)+x_{2k}D_{1n}D_{2n}+x_{3k}D_{1n}D_{3n}\right]\mathbf{1}\|^2\\ =\ &\sum_{-m\le n,r\le m}(f_n,f_r)t_kt_r=:(A_{2m+1}t,t),\end{aligned}$$

where $A_{2m+1}=\big((f_n,f_r)\big)_{n,r=-m}^m$ and

$$f_n=\left[x_{1k}\Big(D_{1n}^2+\frac{b_{1n}}{2}\Big)+x_{2k}D_{1n}D_{2n}+x_{3k}D_{1n}D_{3n}\right]\mathbf{1}.$$

We have

$$(f_n,f_n)\sim b_{1k}(b_{1n}+b_{2n}+b_{3n})\quad\text{and}\quad(f_n,f_k)=0\quad\text{for}\quad n\neq k.$$

Therefore, by (6.3)

$$\min_{t\in\mathbb{R}^{2m+1}}\|\left[\sum_{n=-m}^m t_nD_{1n}A_{kn}-x_{1k}\right]\mathbf{1}\|^2=\left(\sum_{n=-m}^m\frac{b_{1k}}{b_{1n}+b_{2n}+b_{3n}}\right)^{-1}.$$ □

Lemma 5.8. *For any $k \in \mathbb{Z}$ we have*

$$x_{2k}\mathbf{1} \in \langle D_{2n}A_{kn}\mathbf{1} \mid n \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \sum_{n\in\mathbb{Z}} \frac{b_{2n}}{b_{1n}+b_{2n}+b_{3n}} = \infty.$$

Lemma 5.9. *For any $k \in \mathbb{Z}$ we have*

$$x_{3k}\mathbf{1} \in \langle D_{3n}A_{kn}\mathbf{1} \mid n \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \sum_{n\in\mathbb{Z}} \frac{b_{3n}}{b_{1n}+b_{2n}+b_{3n}} = \infty.$$

5.4. Approximation of D_{kn} by $x_{rk}A_{kn}$

Set

$$\lambda_k^{(1)} = \Big(\frac{1}{2b_{1k}}+a_{1k}^2\Big)\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}+a_{2k}^2+a_{3k}^2\Big)-a_{1k}^2(a_{2k}^2+a_{3k}^2) \quad (5.12)$$

$$Y_{11} = \left(\frac{\frac{1}{2b_{1k}}+a_{1k}^2}{\sqrt{\lambda_k^{(1)}}}\right)_{k\in\mathbb{Z}}, Y_{12} = \left(\frac{a_{1k}a_{2k}}{\sqrt{\lambda_k^{(1)}}}\right)_{k\in\mathbb{Z}}, Y_{13} = \left(\frac{a_{1k}a_{3k}}{\sqrt{\lambda_k^{(1)}}}\right)_{k\in\mathbb{Z}} \quad (5.13)$$

Lemma 5.10. *For any $n \in \mathbb{Z}$ we have*

$$D_{1n}\mathbf{1} \in \langle x_{1k}A_{kn}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_{11}, Y_{12}, Y_{13}) = \infty.$$

PROOF. We determine when the inclusion

$$D_{1n}\mathbf{1} \in \langle x_{1k}A_{kn}\mathbf{1} = (x_{1k}^2 D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n})\mathbf{1} \mid k \in \mathbb{Z}\rangle$$

holds. Fix $m \in \mathbb{N}$, since $Mx_{1k}^2 = \frac{1}{2b_{1k}} + a_{1k}^2$, we put

$$\sum_{k=-m}^{m} t_k\big(\frac{1}{2b_{1k}} + a_{1k}^2\big) = (t, b) = 1,$$

where $t = (t_k)_{k=-m}^m$ and $b = (\frac{1}{2b_{1k}} + a_{1k}^2)_{k=-m}^m$. We have

$$\begin{aligned}
&\|\big[\sum_{k=-m}^{m} t_k(x_{1k}^2 D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n}) - D_{1n}\big]\mathbf{1}\|^2 \\
= \;& \|\sum_{k=-m}^{m} t_k\Big[\Big(x_{1k}^2 - \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\Big)D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n}\Big]\mathbf{1}\|^2 \\
= \;& \sum_{-m\le k,r\le m} (f_k, f_r)t_k t_r =: (A_{2m+1}t, t), \text{ where } A_{2m+1} = \big((f_k, f_r)\big)_{k,r=-m}^m, \\
& \text{and} \quad f_k = \Big[\Big(x_{1k}^2 - \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\Big)D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n}\Big]\mathbf{1}.
\end{aligned}$$

Since $M|\psi - M|\psi||^2 = M\psi^2 - |M\psi|^2$ we have

$$M\left|x_{1k}^2 - \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\right|^2 = Mx_{1k}^4 - \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)^2$$
$$= \frac{3}{(2b_{1k})^2} + 6\frac{1}{2b_{1k}}a_{1k}^2 + a_{1k}^4 - \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)^2 = \frac{1}{2b_{1k}}\Big(\frac{2}{2b_{1k}} + 4a_{1k}^2\Big),$$

we get

$$(f_k, f_k) = \|\Big[\Big(x_{1k}^2 - \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\Big)D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n}\Big]\mathbf{1}\|^2 =$$
$$\frac{1}{2b_{1k}}\Big(\frac{2}{2b_{1k}} + 4a_{1k}^2\Big)\frac{b_{1n}}{2} + \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\Big(\frac{1}{2b_{2k}} + a_{2k}^2\Big)\frac{b_{2n}}{2} + \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\frac{b_{3n}}{2}$$
$$\times\Big(\frac{1}{2b_{3k}} + a_{3k}^2\Big) \sim \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2\Big)$$
$$(f_k, f_r) = \Big(\Big[\Big(x_{1k}^2 - \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\Big)D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n}\Big]\mathbf{1},$$
$$\Big[\Big(x_{1r}^2 - \Big(\frac{1}{2b_{1r}} + a_{1r}^2\Big)\Big)D_{1n} + x_{1r}x_{2r}D_{2n} + x_{1r}x_{3r}D_{3n}\Big]\mathbf{1}\Big) =$$
$$(x_{1k}\mathbf{1}, x_{1r}\mathbf{1})(x_{2k}\mathbf{1}, x_{2r}\mathbf{1})(D_{2n}\mathbf{1}, D_{2n}\mathbf{1}) + (x_{1k}\mathbf{1}, x_{1r}\mathbf{1})(x_{3k}\mathbf{1}, x_{3r}\mathbf{1})(D_{3n}\mathbf{1}, D_{3n}\mathbf{1})$$
$$= a_{1k}a_{1r}a_{2k}a_{2r}\frac{b_{2n}}{2} + a_{1k}a_{1r}a_{3k}a_{3r}\frac{b_{3n}}{2} \simeq a_{1k}a_{1r}(a_{2k}a_{2r} + a_{3k}a_{3r}).$$

Finally, we have

$$(f_k, f_k) \sim \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2\Big), \quad (5.14)$$
$$(f_k, f_r) \sim a_{1k}a_{1r}(a_{2k}a_{2r} + a_{3k}a_{3r}),\ \ k \neq r.$$

Set

$$\lambda_k^{(1)} = \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2\Big) - a_{1k}^2(a_{2k}^2 + a_{3k}^2),$$
$$g_k = a_{1k}(a_{2k}, a_{3k}), \quad (5.15)$$

then

$$(f_k, f_k) = \lambda_k^{(1)} + (g_k, g_k) \quad (f_k, f_r) \sim (g_k, g_r),\ \ k \neq r. \quad (5.16)$$

For $A_{2m+1} = \big((f_k, f_r)\big)_{k,r=-m}^m$, and $b = (a_{1k})_{k=-m}^m \in \mathbb{R}^{2m+1}$ we have

$$A_{2m+1} = \sum_{k=-m}^{m} \lambda_k E_{kk} + \gamma(g_{-m}, \ldots, g_0, \ldots, g_m).$$

To finish the proof, it suffices to apply Lemma 7.3. □

Similarly, by cyclic permutation of the indices, we obtain the following lemmas.

Lemma 5.11. *For any $l \in \mathbb{Z}$ we have*

$$D_{2l}\mathbf{1} \in \langle x_{2k}A_{kl}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_{22}, Y_{23}, Y_{21}) = \infty.$$

Lemma 5.12. *For any $l \in \mathbb{Z}$ we have*

$$D_{3l}\mathbf{1} \in \langle x_{3k}A_{kl}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_{33}, Y_{31}, Y_{32}) = \infty.$$

Here we set

$$\lambda_k^{(2)} = \Big(\frac{1}{2b_{2k}} + a_{2k}^2\Big)\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{3k}^2\Big) - a_{2k}^2(a_{1k}^2 + a_{3k}^2), \quad (5.17)$$

$$Y_{21} = \left(\frac{a_{1k}a_{2k}}{\sqrt{\lambda_k^{(2)}}}\right)_{k\in\mathbb{Z}}, Y_{22} = \left(\frac{\frac{1}{2b_{2k}} + a_{2k}^2}{\sqrt{\lambda_k^{(2)}}}\right)_{k\in\mathbb{Z}}, Y_{23} = \left(\frac{a_{2k}a_{3k}}{\sqrt{\lambda_k^{(2)}}}\right)_{k\in\mathbb{Z}}, \quad (5.18)$$

$$\lambda_k^{(3)} = \Big(\frac{1}{2b_{3k}} + a_{3k}^2\Big)\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2\Big) - a_{3k}^2(a_{1k}^2 + a_{2k}^2), \quad (5.19)$$

$$Y_{31} = \left(\frac{a_{1k}a_{3k}}{\sqrt{\lambda_k^{(3)}}}\right)_{k\in\mathbb{Z}}, Y_{32} = \left(\frac{a_{2k}a_{3k}}{\sqrt{\lambda_k^{(3)}}}\right)_{k\in\mathbb{Z}}, Y_{33} = \left(\frac{\frac{1}{2b_{3k}} + a_{3k}^2}{\sqrt{\lambda_k^{(3)}}}\right)_{k\in\mathbb{Z}} \quad (5.20)$$

5.5. Approximation of D_{rn} by $(x_{3k} - a_{3k})A_{kn}$ and $\exp(is_kx_{rk})A_{kn}$

Lemma 5.13. *For any $n \in \mathbb{Z}$ we have*

$$D_{3n}\mathbf{1} \in \langle (x_{3k} - a_{3k})A_{kn}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow$$
$$\Sigma_3(\mu) := \sum_{k\in\mathbb{Z}} \frac{1}{2b_{3k}}\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2 + a_{3k}^2\Big)^{-1} = \infty. \quad (5.21)$$

Proof. We determine when the following inclusion holds

$$D_{3n}\mathbf{1} \in \langle (x_{3k} - a_{3k})A_{kn}\mathbf{1}$$
$$= \big(x_{1k}(x_{3k} - a_{3k})D_{1n} + x_{2k}(x_{3k} - a_{3k})D_{2n} + x_{3k}(x_{3k} - a_{3k})D_{3n}\big)\mathbf{1} \mid k \in \mathbb{Z}\rangle.$$

Set $\xi_{3k} = x_{3k}(x_{3k} - a_{3k})$. Fix $m \in \mathbb{N}$. We have

$$M\xi_{3k}\mathbf{1} = M(x_{3k} - a_{3k})^2\mathbf{1} = \frac{1}{2b_{3k}}, \quad \text{chose} \quad (t_k) \text{ as } \sum_{k=-m}^{m} t_k\frac{1}{2b_{3k}} = (t, b) = 1,$$

where we denote $t=(t_k)_{k=-m}^m$ and $b=(\frac{1}{2b_{3k}})_{k=-m}^m$. We have

$$\begin{aligned}
&\|\Big[\sum_{k=-m}^m t_k\big(x_{1k}(x_{3k}-a_{3k})D_{1n}+x_{2k}(x_{3k}-a_{3k})D_{2n}+\xi_{3k}D_{3n}\big)-D_{3n}\Big]\mathbf{1}\|^2\\
=\ &\|\sum_{k=-m}^m t_k\Big[x_{1k}(x_{3k}-a_{3k})D_{1n}+x_{2k}(x_{3k}-a_{3k})D_{2n}+\big(\xi_{3k}-M\xi_{3k}\big)D_{3n}\Big]\mathbf{1}\|^2\\
=\ &\sum_{-m\le k,r\le m}(f_k,f_r)t_kt_r=\sum_{-m\le k\le m}(f_k,f_k)t_k^2,\ \text{ since }\quad f_k\perp f_r,\quad \text{where}\quad (5.22)\\
&f_k=\Big[x_{1k}(x_{3k}-a_{3k})D_{1n}+x_{2k}(x_{3k}-a_{3k})D_{2n}+\big(\xi_{3k}-M\xi_{3k}\big)D_{3n}\Big]\mathbf{1}.
\end{aligned}$$

To calculate $M|\xi_{3k}-M\xi_{3k}|^2$ set

$$d\mu_{(b,a)}(x)=\sqrt{\frac{b}{\pi}}\exp(-b(x-a)^2)dx\quad\text{and}\quad d\mu_{(b,0)}(x)=\sqrt{\frac{b}{\pi}}\exp(-bx^2)dx, \tag{5.23}$$

then we get $M\xi_{3k}^2=\frac{3}{(2b_{3k})^2}+\frac{a_{3k}^2}{2b_{3k}}$. Indeed,

$$\begin{aligned}
&Mx^2(x-a)^2=\int x^2(x-a)^2d\mu_{(b,a)}(x)=\int x^2(x+a)^2d\mu_{(b,0)}(x)\\
&=\int(x^4+2ax^3+a^2x^2)d\mu_{(b,0)}(x)=\frac{3}{(2b)^2}+\frac{a^2}{2b}.
\end{aligned}$$

Since $M|\xi-M|\xi||^2=M\xi^2-|M\xi|^2$ we get

$$M\xi_{3k}^2-M|\xi_{3k}|^2=\frac{3}{(2b_{3k})^2}+\frac{a_{3k}^2}{2b_{3k}}-\frac{1}{(2b_{3k})^2}=\frac{1}{2b_{3k}}\Big(\frac{2}{2b_{3k}}+a_{3k}^2\Big).$$

Set $f_{sk}=x_{sk}(x_{3k}-a_{3k})D_{1n}\mathbf{1},\ 1\le s\le 2$ and $f_{3k}=\big(\xi_{3k}-M\xi_{3k}\big)D_{3n}\mathbf{1}$, then

$$\begin{aligned}
&(f_k,f_k)=\|\Big[x_{1k}(x_{3k}-a_{3k})D_{1n}+x_{2k}(x_{3k}-a_{3k})D_{2n}+\big(\xi_{3k}-M\xi_{3k}\big)D_{3n}\Big]\mathbf{1}\|^2\\
&=\Big(\frac{1}{2b_{1k}}+a_{1k}^2\Big)\frac{1}{2b_{3k}}\frac{b_{1n}}{2}+\Big(\frac{1}{2b_{2k}}+a_{2k}^2\Big)\frac{1}{2b_{3k}}\frac{b_{2n}}{2}+\Big(\frac{2}{2b_{3k}}+a_{3k}^2\Big)\frac{1}{2b_{3k}}\frac{b_{3n}}{2}\\
&\sim\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}+a_{1k}^2+a_{2k}^2+a_{3k}^2\Big)\frac{1}{2b_{3k}},\ \text{since } f_{lk}\perp f_{sk},\ l\neq s,\\
&(f_k,f_r)=\Big(\Big[x_{1k}(x_{3k}-a_{3k})D_{1n}+x_{2k}(x_{3k}-a_{3k})D_{2n}+\big(\xi_{3k}-M\xi_{3k}\big)D_{3n}\Big]\mathbf{1},\\
&\Big[x_{1r}(x_{3r}-a_{3r})D_{1n}+x_{2r}(x_{3r}-a_{3r})D_{2n}+\big(\xi_{3r}-M\xi_{3r}\big)D_{3n}\Big]\mathbf{1}\Big)=0.
\end{aligned}$$

The previous equality holds since $f_{lr} \perp f_{sk}$ for $1 \leq l, s \leq 3$ and $r \neq k$. For $l \neq s$ this follows from $(D_{ln}\mathbf{1}, D_{sn}\mathbf{1}) = 0$. For $l = s$ it follows from the equalities:

$$(x_{sk}\mathbf{1}, x_{sr}\mathbf{1}) = a_{sk}a_{sr},\ 1 \leq s \leq 2, \quad \Big((\xi_{3k} - M\xi_{3k})\mathbf{1}, (\xi_{3r} - M\xi_{3r})\mathbf{1}\Big) = 0.$$

and $\big((x_{3k} - a_{3k})\mathbf{1}, (x_{3r} - a_{3r})\mathbf{1}\big) = 0$. Finally, we have

$$(f_k, f_k) \sim \Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2 + a_{3k}^2\Big)\frac{1}{2b_{3k}},\ (f_k, f_r) = 0,\ k \neq r.$$

Set $a_k = (f_k, f_k)$ and $b_k = M\xi_k$, then by Lemma 6.1, (6.3) and (5.22) the proof is completed since

$$\sum_k \frac{b_k^2}{a_k} = \sum_{k\in\mathbb{Z}} \frac{1}{2b_{3k}}\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}+a_{1k}^2+a_{2k}^2+a_{3k}^2\Big)^{-1}. \qquad \square$$

Now we would like to approximate D_{3n} by combinations of $\exp\big(is_k(x_{3k}-a_{3k})iA_{kn}$. Set $s = (s_k)_{k\in\mathbb{Z}}$

$$\lambda_k^{(3)}(s_k) = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{3k}^2 - \Big(\frac{s^2}{4b_{3k}^2} + a_{3k}^2\Big)\exp\big(-\frac{s^2}{2b_{3k}}\big), \tag{5.24}$$

$$Y_{31}(s) = \left(\frac{a_{1k}}{\sqrt{\lambda_k^{(3)}(s_k)}}\right)_{k\in\mathbb{Z}}, \quad Y_{32}(s) = \left(\frac{a_{2k}}{\sqrt{\lambda_k^{(3)}(s_k)}}\right)_{k\in\mathbb{Z}}, \tag{5.25}$$

$$Y_{32}(s) = \left(\frac{\Big(-\frac{s_k}{2b_{3k}} + ia_{3k}\Big)\exp\big(-\frac{s_k^2}{4b_{3k}}\big)}{\sqrt{\lambda_k^{(3)}(s_k)}}\right)_{k\in\mathbb{Z}}.$$

In particular for $s_k = \sqrt{2b_{3k}}$ we get

$$\lambda_k^{(3)}(s_k) = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \Big(\frac{1}{2b_{3k}} + a_{3k}^2\Big)\big(1 - \frac{1}{e}\big). \tag{5.26}$$

Lemma 5.14. *For any $n \in \mathbb{Z}$ we have*

$$D_{3n}\mathbf{1} \in \langle \exp\big(is_k(x_{3k}-a_{3k})iA_{kn}\mathbf{1} \mid k \in \mathbb{Z}\rangle \ \Leftrightarrow\ \Delta(Y_{33}(s), Y_{31}(s), Y_{32}(s)) = \infty.$$

Proof. We determine when the inclusion

$$D_{3n}\mathbf{1} \in \langle \exp\big(is_k(x_{3k} - a_{3k})iA_{kn}\mathbf{1} = \Big(ix_{1k}\exp\big(is_k(x_{3k} - a_{3k})\big)D_{1n}$$
$$+ix_{2k}\exp\big(is_k(x_{3k} - a_{3k})\big)D_{2n} + ix_{3k}\exp\big(is_k(x_{3k} - a_{3k})\big)D_{3n}\Big)\mathbf{1} \mid k \in \mathbb{Z}\rangle$$

holds. Set $\xi_{rk}(s_k) = ix_{rk}\exp\big(is_k(x_{3k}-a_{3k})\big)$ for $1 \le r \le 3$ and

$$\begin{aligned} f_k(s_k) &= \Big(\xi_{1k}(s_k)D_{1n} + \xi_{2k}(s_k)D_{2n} + \big[\xi_{3k}(s_k) - M\xi_{3k}(s_k)\big]D_{3n}\Big)\mathbf{1} \quad (5.27)\\ &= \Big(ix_{1k}\exp\big(is_k(x_{3k}-a_{3k}\big)D_{1n} + ix_{2k}\exp\big(is_k(x_{3k}-a_{3k}\big)D_{2n}\\ &+ \big[ix_{3k}\exp\big(is_k(x_{3k}-a_{3k}\big) - M\xi_{3k}(s_k)\big]D_{3n}\Big)\mathbf{1}. \end{aligned}$$

We show that

$$M\xi_{3k}(s) = \Big(-\frac{s}{2b_{3k}} + ia_{3k}\Big)\exp\Big(-\frac{s^2}{4b_{3k}}\Big), \tag{5.28}$$

$$(f_k, f_k) \sim \lambda_k^{(3)}(s_k) + (g_k, g_k), \tag{5.29}$$

$$(f_k, f_r) \sim a_{1k}a_{1r} + a_{2k}a_{2r} = (g_k, g_r), \tag{5.30}$$

where $g_k = (a_{1k}, a_{2k}) \in \mathbb{R}^2$. Indeed, set $F_b(s) = \int_{\mathbb{R}}\exp\big(is(x-a)\big)d\mu_{(b,a)}(x)$, then

$$F_b(s) = \int_{\mathbb{R}}\exp\big(isx\big)d\mu_{(b,0)}(x) = \exp\Big(-\frac{s^2}{4b}\Big), \tag{5.31}$$

where $d\mu_{(b,a)}(x)$ and $d\mu_{(b,0)}(x)$ are defined by (5.23). Therefore,

$$H_{a,b}(s) = \int_{\mathbb{R}} ix\exp\big(is(x-a)\big)d\mu_{(b,a)}(x) = \int_{\mathbb{R}} i(x+a)\exp(isx)d\mu_{(b,0)}(x) \tag{5.32}$$

$$= \frac{dF_b(s)}{ds} + iaF_b(s) = \Big(-\frac{s}{2b} + ia\Big)\exp\Big(-\frac{s^2}{4b}\Big). \tag{5.33}$$

This implies (5.28). Further, to obtain (5.29) and (5.30) we write

$$(f_k, f_r) = \sum_{1\le t,l\le 2}(x_{tk}, x_{lr})(D_{tn}\mathbf{1}, D_{ln}\mathbf{1}) = \sum_{1\le t\le 2}(x_{tk}, x_{tr})(D_{tn}\mathbf{1}, D_{tn}\mathbf{1})$$

$$= a_{1k}a_{1r}\frac{b_{1n}}{2} + a_{2k}a_{2r}\frac{b_{2n}}{2} \sim a_{1k}a_{1r} + a_{2k}a_{2r} = (g_k, g_r),$$

$$(f_k, f_k) = \sum_{1\le t\le 2}\|x_{tk}\|^2\|D_{tn}\mathbf{1}\|^2 + \Big(M|\xi_{3k}(s_k)|^2 - |M\xi_{3k}(s_k)|^2\Big)\|D_{3n}\mathbf{1}\|^2$$

$$= \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)\frac{b_{1n}}{2} + \Big(\frac{1}{2b_{2k}} + a_{2k}^2\Big)\frac{b_{2n}}{2} + \Big(M|\xi_{3k}(s_k)|^2 - |M\xi_{3k}(s_k)|^2\Big)\frac{b_{3n}}{2}$$

$$\sim \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big) + \Big(\frac{1}{2b_{2k}} + a_{2k}^2\Big) + \Big(M|\xi_{3k}(s_k)|^2 - |M\xi_{3k}(s_k)|^2\Big)$$

$$= \lambda_k^{(3)}(s_k) + (g_k, g_k).$$

Set for brevity $\xi(s) = ix\exp\big(is(x-a)\big)$, then

$$f(s) := M|\xi(s)|^2 - |M\xi(s)|^2 = \Big(\frac{1}{2b}+a^2\Big) - \Big(\frac{s^2}{4b^2}+a^2\Big)\exp\Big(-\frac{s^2}{2b}\Big),$$
$$\min_s f(s) = \begin{cases} \frac{1-e^{-(1-2ba^2)}}{2b}+a^2, & \text{if } 1-2ba^2 \geq 0 \\ \frac{1}{2b}, & \text{if } 1-2ba^2 < 0 \end{cases}. \tag{5.34}$$

Indeed, consider the function $g_{a,b}(x) = \Big(\frac{x^2}{2b}+a^2\Big)\exp(-x^2)$ and set $x_0 = \sqrt{1-2ba^2}$. We have

$$\max_{x\in\mathbb{R}} g_{a,b}(x) = \begin{cases} g_{a,b}(x_0) = \frac{\exp(-(1-2ba^2))}{2b}, & \text{if } 1-2ba^2 \geq 0, \\ g_{a,b}(0) = a^2, & \text{if } 1-2ba^2 < 0. \end{cases} \tag{5.35}$$

To calculate $\lambda_k^{(3)}(s_k)$, we get finally

$$\lambda_k^{(3)}(s_k) = \frac{1}{2b_{1k}} + a_{1k}^2 + \frac{1}{2b_{2k}} + a_{2k}^2 + \Big(M|\xi_{3k}(s_k)|^2 - |M\xi_{3k}(s_k)|^2\Big) - (g_k, g_k)$$
$$= \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2 + a_{3k}^2 - \Big(\frac{s_k^2}{4b_{3k}^2} + a_{3k}^2\Big)\exp\big(-\frac{s_k^2}{2b_{3k}}\big)$$
$$-(g_k, g_k) = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{3k}^2 - \Big(\frac{s_k^2}{4b_{3k}^2} + a_{3k}^2\Big)\exp\big(-\frac{s_k^2}{2b_{3k}}\big) =$$
$$\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \Big(\frac{1}{2b_{3k}} + a_{3k}^2\Big)\big(1-e^{-1}\big), \quad \text{for} \quad \frac{s_k^2}{2b_{3k}} = 1.$$

Therefore, we get (5.24) and (5.26)

$$\lambda_k^{(3)}(s_k) = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \Big(\frac{1}{2b_{3k}} + a_{3k}^2\Big)\big(1-\frac{1}{e}\big).$$

For $A_{2m+1} = \big((f_k(s_k), f_r(s_r))\big)_{k,r=-m}^m$, and $b = (M\xi_{3k}(s_k))_{k=-m}^m \in \mathbb{R}^{2m+1}$ we have

$$A_{2m+1} = \sum_{k=-m}^m \lambda_k E_{kk} + \gamma(g_{-m}, \ldots, g_0, \ldots, g_m).$$

The proof is now finished on invoking Lemma 7.3. $\square$

Lemma 5.15. *We have*

$$D_{3k}\mathbf{1} \in \langle \sin\big(s_k(x_{3k}-a_{3k})\big)A_{kn}\mathbf{1} \mid k\in\mathbb{Z}\rangle \ \Leftrightarrow\ \Sigma_3(D,s)=\infty, \tag{5.36}$$

$$D_{3k}\mathbf{1} \in \langle \cos\big(s_k(x_{3k}-a_{3k})\big)A_{kn}\mathbf{1} \mid k\in\mathbb{Z}\rangle \ \Leftrightarrow\ \Sigma_3^{\vee}(D,s)=\infty, \tag{5.37}$$

$$\textit{where } \Sigma_3(D,s)=\sum_{k\in\mathbb{Z}}\frac{|M\eta_{3k}(s_k)|^2}{\|g_k(s_k)\|^2},\ \ \Sigma_3^{\vee}(D,s)=\sum_{k\in\mathbb{Z}}\frac{|M\eta_{3k}^{\vee}(s_k)|^2}{\|g_k^{\vee}(s_k)\|^2}, \tag{5.38}$$

$$\textit{moreover,}\quad \Sigma_3(D,s^{(3)})\sim\Sigma_3(D):=\sum_k\frac{\frac{1}{2b_{3k}}}{C_k+a_{1k}^2+a_{2k}^2+a_{3k}^2}, \tag{5.39}$$

$$\textit{and}\quad \Sigma_3^{\vee}(D,s^{(3)})\sim\Sigma_3^{\vee}(D):=\sum_k\frac{a_{3k}^2}{C_k+a_{1k}^2+a_{2k}^2+a_{3k}^2}, \tag{5.40}$$

where $s^{(3)}=(s_{3k})_k$ *with* $\frac{s_{3k}^2}{b_{3k}}\equiv 1,\ k\in\mathbb{Z}$.

Proof. We shall try to obtain separately the real part and imaginary part of $M\xi_{3k}(s)$, where $\xi_{3k}(s_k)=ix_{3k}\exp\big(is_k(x_{3k}-a_{3k})\big)$. Using Lemma 5.14 formulas (5.28) and (5.33) we get

$$H_{a,b}(s)=\int_{\mathbb{R}} ix\exp\big(is(x-a)\big)d\mu_{(b,a)}(x)=\int_{\mathbb{R}} i(x+a)\exp(isx)d\mu_{(b,0)}(x)$$
$$=\frac{dF_b(s)}{ds}+iaF_b(s)=\Big(-\frac{s}{2b}+ia\Big)\exp\Big(-\frac{s^2}{4b}\Big)=:M\xi_{3k}(s),$$

Recall the Euler formulas

$$e^{it}=\cos t+i\sin t,\quad e^{-it}=\cos t-i\sin t, \tag{5.41}$$
$$\cos t=\frac{e^{it}+e^{-it}}{2},\quad \sin t=\frac{e^{it}-e^{-it}}{2i}.$$

More precisely, we denote for $1\le r\le 3$

$$\eta_{rk}(s)=x_{rk}\cos\big(s_k(x_{3k}-a_{3k})\big),\quad \eta_{rk}^{\vee}(s)=x_{rk}\cos\big(s_k(x_{3k}-a_{3k})\big). \tag{5.42}$$

We determine when the inclusion holds:

$$D_{3k}\mathbf{1}\in\langle\sin\big(s_k(x_{3k}-a_{3k})\big)A_{kn}\mathbf{1}=\Big(x_{1k}\sin\big(s_k(x_{3k}-a_{3k})\big)D_{1n}$$
$$+x_{2k}\sin\big(s_k(x_{3k}-a_{3k})\big)D_{2n}+x_{3k}\sin\big(s_k(x_{3k}-a_{3k})\big)D_{3n}\Big)\mathbf{1}\mid k\in\mathbb{Z}\rangle,$$
$$D_{3k}\mathbf{1}\in\langle\cos\big(s_k(x_{3k}-a_{3k})\big)A_{kn}\mathbf{1}=\Big(x_{1k}\cos\big(s_k(x_{3k}-a_{3k})\big)D_{1n}$$
$$+x_{2k}\cos\big(s_k(x_{3k}-a_{3k})\big)D_{2n}+x_{3k}\cos\big(s_k(x_{3k}-a_{3k})\big)D_{3n}\Big)\mathbf{1}\mid k\in\mathbb{Z}\rangle.$$

Set

$$g_k(s_k)=\Big(\eta_{1k}(s_k)D_{1n}+\eta_{2k}(s_k)D_{2n}+\big[\eta_{3k}(s_k)-M\eta_{3k}(s_k)\big]D_{3n}\Big)\mathbf{1}, \quad (5.43)$$

$$g_k^\vee(s_k)=\Big(\eta_{1k}^\vee(s_k)D_{1n}+\eta_{2k}^\vee(s_k)D_{2n}+\big[\eta_{3k}^\vee(s_k)-M\eta_{3k}^\vee(s_k)\big]D_{3n}\Big)\mathbf{1}, \quad (5.44)$$

We show that (compare with (5.28))

$$M\eta_{3k}(s)=-\frac{1}{2}\Big(H_{a,b}(s)+\overline{H_{a,b}(s)}\Big)=\frac{s}{2b_{3k}}\exp\Big(-\frac{s^2}{4b_{3k}}\Big), \quad (5.45)$$

$$M\eta_{3k}^\vee(s)=\frac{1}{2i}\Big(H_{a,b}(s)-\overline{H_{a,b}(s)}\Big)=a_{3k}\exp\Big(-\frac{s^2}{4b_{3k}}\Big). \quad (5.46)$$

Recall the definition of the function $F_b(s)$ defined by (5.31):

$$F_b(s)=\int_{\mathbb{R}}\exp\big(is(x-a)\big)d\mu_{(b,a)}(x)=\int_{\mathbb{R}}\exp\big(isx\big)d\mu_{(b,0)}(x)=\exp\Big(-\frac{s^2}{4b}\Big). \quad (5.47)$$

We have

$$M\eta(s)=\int_{\mathbb{R}}x\sin\big(s(x-a)\big)d\mu_{(b,a)}(x)=\int_{\mathbb{R}}(x+a)\sin(sx)d\mu_{(b,0)}(x)=$$
$$\int_{\mathbb{R}}(x+a)\frac{e^{isx}-e^{-isx}}{2i}d\mu_{(b,0)}(x)=-\frac{1}{2}\int_{\mathbb{R}}i(x+a)\big(e^{isx}-e^{-isx}\big)d\mu_{(b,0)}(x)=$$
$$-\frac{1}{2}\Big(H_{a,b}(s)+\overline{H_{a,b}(s)}\Big)=\frac{s}{2b}\exp\Big(-\frac{s^2}{4b}\Big),$$

this implies (5.45). Similarly we get

$$M\eta^\vee(s)=\int_{\mathbb{R}}x\cos\big(s(x-a)\big)d\mu_{(b,a)}(x)=\int_{\mathbb{R}}(x+a)\cos(sx)d\mu_{(b,0)}(x)=$$
$$\frac{1}{2i}\int_{\mathbb{R}}i(x+a)\Big(e^{isx}+e^{-isx}\Big)d\mu_{(b,0)}(x)=\frac{1}{2i}\Big(H_{a,b}(s)-\overline{H_{a,b}(s)}\Big)$$
$$=a\exp\Big(-\frac{s^2}{4b}\Big)$$

this implies (5.45). Fix $m\in\mathbb{N}$, we put $\sum_{k=-m}^m t_kM\eta_{3k}(s_k)=(t,b)=1$, where $t=(t_k)_{k=-m}^m$ and $b=(M\eta_{3k}(s_k))_{k=-m}^m$. We have

$$\begin{aligned}
&\|\big[\sum_{k=-m}^{m} t_k \sin\big(s_k(x_{3k}-a_{3k})\big)A_{kn}-D_{3n}\big]\mathbf{1}\|^2\\
=\ &\|\sum_{k=-m}^{m} t_k\Big(\eta_{1k}(s_k)D_{1n}+\eta_{2k}(s_k)D_{2n}+\big[\eta_{3k}(s_k)-M\eta_{3k}(s_k)\big]D_{3n}\Big)\mathbf{1}\|^2\\
=\ &\sum_{k=-m}^{m} t_k^2\|g_k(s_k)\|^2,\ \text{ since } \Big(D_{rn}\mathbf{1},D_{ln}\mathbf{1}\Big)=0,\ \ 1\le r<l\le 3, \qquad (5.48)
\end{aligned}$$

where the $g_k(s_k)$ are defined by (5.43). To calculate $\|g_k(s_k)\|^2$ we have

$$\begin{aligned}
&\|g_k(s_k)\|^2=(g_k(s_k),g_k(s_k))=\\
&\Big(\big(\eta_{1k}(s_k)D_{1n}+\eta_{2k}(s_k)D_{2n}+\big[\eta_{3k}(s_k)-M\eta_{3k}(s_k)\big]D_{3n}\big)\mathbf{1},\\
&\big(\eta_{1k}(s_k)D_{1n}+\eta_{2k}(s_k)D_{2n}+\big[\eta_{3k}(s_k)-M\eta_{3k}(s_k)\big]D_{3n}\big)\mathbf{1}\Big)=\\
&\|x_{1k}\mathbf{1}\|^2\|\sin\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{1k}\mathbf{1}\|^2+\\
&\|x_{2k}\mathbf{1}\|^2\|\sin\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{2k}\mathbf{1}\|^2+\\
&\Big(M|\eta_{kn}(s_k)|^2-|M\eta_{kn}(s_k)|^2\Big)\|D_{3k}\mathbf{1}\|^2=\Big(\frac{1}{2b_{1k}}+a_{1k}^2\Big)I_3\frac{b_{1n}}{2}+\\
&\Big(\frac{1}{2b_{2k}}+a_{2k}^2\Big)I_3\frac{b_{2n}}{2}+\Big(M|\eta_{kn}(s_k)|^2-|M\eta_{kn}(s_k)|^2\Big)\frac{b_{3n}}{2}. \qquad (5.49)
\end{aligned}$$

We need to calculate $I_3=\|\sin\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2$, $M|\eta_{kn}(s_k)|^2$ and $|M\eta_{kn}(s_k)|^2$. If we set $a:=a_{3k},\ b:=b_{3k}$, we get

$$\begin{aligned}
&I_3=\|\sin\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2=\int_{\mathbb{R}}\frac{e^{isx}-e^{-isx}}{2i}\frac{e^{-isx}-e^{isx}}{-2i}d\mu_{(b,0)}(x)=\\
&\frac{1}{2}\int_{\mathbb{R}}\Big(1-\frac{e^{2isx}+e^{-2isx}}{2}\Big)d\mu_{(b,0)}(x)\overset{(5.47)}{=}\frac{1-e^{-\frac{s^2}{b}}}{2}, \qquad (5.50)\\
&|M\eta_{kn}(s_k)|^2=\frac{s_k^2}{4b_{3k}^2}\exp\Big(-\frac{s_k^2}{2b_{3k}}\Big), \qquad (5.51)\\
&M|\eta_{kn}(s_k)|^2=\int_{\mathbb{R}}(x^2+2xa+a^2)\frac{e^{isx}-e^{-isx}}{2i}\frac{e^{-isx}-e^{isx}}{-2i}d\mu_{(b,0)}(x)=\\
&\frac{1}{2}\int_{\mathbb{R}}(x^2+2xa+a^2)\Big(1-\frac{e^{2isx}+e^{-2isx}}{2}\Big)d\mu_{(b,0)}(x)=
\end{aligned}$$

$$\frac{1}{2}\Big[\int_{\mathbb{R}}(x^2+a^2)d\mu_{(b,0)}(x)-\int_{\mathbb{R}}(x^2+a^2)\frac{e^{2isx}+e^{-2isx}}{2}d\mu_{(b,0)}(x)\Big]=$$
$$\frac{1}{2}\Big[\frac{1}{2b}+a^2-\frac{d^2F_b(2s)}{ds^2}-a^2F_b(2s)\Big]\stackrel{(5.47)}{=}\frac{1}{2}\Big[\frac{1}{2b}+a^2-\frac{1}{(2i)^2}\Big[\Big(\frac{2s}{b}\Big)^2-\frac{2}{b}\Big]\times$$
$$e^{-\frac{s^2}{b}}-a^2e^{-\frac{s^2}{b}}\Big]=\frac{1}{2}\Big[\Big(\frac{1}{2b}+a^2\Big)(1-e^{-\frac{s^2}{b}})+\frac{s^2}{b^2}e^{-\frac{s^2}{b}}\Big].\qquad(5.52)$$

Finally, we get

$$M|\eta_{kn}(s_k)|^2-|M\eta_{kn}(s_k)|^2=\frac{1}{2}\Big[\Big(\frac{1}{2b}+a^2\Big)(1-e^{-\frac{s^2}{b}})+\frac{s^2}{b^2}e^{-\frac{s^2}{b}}\Big]-\frac{s^2}{4b^2}e^{-\frac{s^2}{2b}}.\qquad(5.53)$$

By (5.58),(5.49), (5.50), (5.53) and (6.3) we prove (5.36), where

$$\Sigma_3(D,s)=\sum_{k\in\mathbb{Z}}\frac{|M\eta_{kn}(s_k)|^2}{\|g_k(s_k)\|^2}=$$
$$\sum_{k\in\mathbb{Z}}\frac{\frac{s_k^2}{4b_{3k}^2}e^{-\frac{s_k^2}{2b_{3k}}}}{\Big(\frac{1}{2b_{1k}}+a_{1k}^2\Big)I_3\frac{b_{1n}}{2}+\Big(\frac{1}{2b_{2k}}+a_{2k}^2\Big)I_3\frac{b_{2n}}{2}+\Big(M|\eta_{kn}(s_k)|^2-|M\eta_{kn}(s_k)|^2\Big)\frac{b_{3n}}{2}}\sim$$
$$\sum_{k\in\mathbb{Z}}\frac{\frac{s_k^2}{4b_{3k}^2}e^{-\frac{s_k^2}{2b_{3k}}}}{\frac{1-e^{-\frac{s_k^2}{b_{3k}}}}{2}\Big(c_{1k}+c_{2k}\Big)+\frac{1}{2}\Big[c_{3k}(1-e^{-\frac{s_k^2}{b_{3k}}})+\frac{s_k^2}{b_{3k}^2}e^{-\frac{s_k^2}{b_{3k}}}\Big]-\frac{s_k^2}{4b_{3k}^2}e^{-\frac{s_k^2}{2b_{3k}}}}=$$
$$\sum_{k\in\mathbb{Z}}\frac{\frac{x_k^2}{4b_{3k}}e^{-\frac{x_k^2}{2}}}{\frac{1-e^{-x_k^2}}{2}\Big(c_{1k}+c_{2k}\Big)+\frac{1}{2}\Big[c_{3k}(1-e^{-x_k^2})+\frac{x_k^2}{b_{3k}}e^{-x_k^2}\Big]-\frac{x_k^2}{4b_{3k}}e^{-\frac{x_k^2}{2}}}=:\Sigma_3(D,x),\qquad(5.54)$$

where $x_k^2=\frac{s_k^2}{b_{3k}}$ and $c_{rk}=\frac{1}{2b_{rk}}+a_{rk}^2$. For $x^{(3)}=(x_k)_k$ with $x_k\equiv 1$ we get

$$\Sigma_3(D,x^{(3)})=\sum_{k\in\mathbb{Z}}\frac{\frac{1}{4b_{3k}}e^{-\frac{1}{2}}}{\frac{1-e^{-1}}{2}\Big(c_{1k}+c_{2k}\Big)+\frac{1}{2}\Big[c_{3k}(1-e^{-1})+\frac{1}{b_{3k}}e^{-1}\Big]-\frac{1}{4b_{3k}}e^{-\frac{1}{2}}}=$$
$$\sum_{k\in\mathbb{Z}}\frac{\frac{1}{4b_{3k}}e^{-\frac{1}{2}}}{\frac{1-e^{-1}}{2}\Big(c_{1k}+c_{2k}+c_{3k}\Big)+\frac{1}{2b_{3k}}\Big(e^{-1}-\frac{1}{2}e^{-\frac{1}{2}}\Big)}\stackrel{(2.6)}{\sim}$$
$$\sum_{k\in\mathbb{Z}}\frac{\frac{1}{2b_{3k}}}{c_{1k}+c_{2k}+c_{3k}}=\sum_{k\in\mathbb{Z}}\frac{\frac{1}{2b_{3k}}}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}+a_{1k}^2+a_{2k}^2+a_{3k}^2}=\Sigma_3(D).\qquad(5.55)$$

So, we have proved (5.39) for $x = (x_k)_k$ with $x_k \equiv 1$. To approximate D_{3n} in terms of functions involving the cosine, fix $m \in \mathbb{N}$, and put $\sum_{k=-m}^{m} t_k M\eta_{3k}^{\vee}(s_k) = (t,b) = 1$, where $t = (t_k)_{k=-m}^{m}$ and $b = (M\eta_{3k}^{\vee}(s_k))_{k=-m}^{m}$. We have

$$\begin{aligned}
&\|\Big[\sum_{k=-m}^{m} t_k \cos\big(s_k(x_{3k}-a_{3k})\big)A_{kn} - D_{3n}\Big]\mathbf{1}\|^2 \\
= &\ \|\sum_{k=-m}^{m} t_k\Big(\eta_{1k}^{\vee}(s_k)D_{1n} + \eta_{2k}^{\vee}(s_k)D_{2n} + \big[\eta_{3k}^{\vee}(s_k) - M\eta_{3k}^{\vee}(s_k)\big]D_{3n}\Big)\mathbf{1}\|^2 \\
= &\sum_{k=-m}^{m} t_k^2\|g_k^{\vee}(s_k)\|^2, \ \text{ since } \Big(D_{rn}\mathbf{1}, D_{ln}\mathbf{1}\Big)=0, \ \ 1 \le r < l \le 3, \qquad (5.56)
\end{aligned}$$

where the $g_k^{\vee}(s_k)$ are defined by (5.43). To calculate $\|g_k^{\vee}(s_k)\|^2$ we have

$$\begin{aligned}
&\|g_k^{\vee}(s_k)\|^2 = (g_k^{\vee}(s_k), g_k^{\vee}(s_k)) = \\
&\Big(\big(\eta_{1k}^{\vee}(s_k)D_{1n} + \eta_{2k}^{\vee}(s_k)D_{2n} + \big[\eta_{3k}^{\vee}(s_k) - M\eta_{3k}^{\vee}(s_k)\big]D_{3n}\big)\mathbf{1}, \\
&\big(\eta_{1k}^{\vee}(s_k)D_{1n} + \eta_{2k}^{\vee}(s_k)D_{2n} + \big[\eta_{3k}^{\vee}(s_k) - M\eta_{3k}^{\vee}(s_k)\big]D_{3n}\big)\mathbf{1}\Big) = \\
&\|x_{1k}\mathbf{1}\|^2\|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{1k}\mathbf{1}\|^2 + \\
&\|x_{2k}\mathbf{1}\|^2\|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{2k}\mathbf{1}\|^2 +
\end{aligned}$$

$$\begin{aligned}
&\Big(M|\eta_{kn}^{\vee}(s_k)|^2 - |M\eta_{kn}^{\vee}(s_k)|^2\Big)\|D_{3k}\mathbf{1}\|^2 = \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)I_3^{\vee}\frac{b_{1n}}{2} + \\
&\Big(\frac{1}{2b_{2k}} + a_{2k}^2\Big)I_3^{\vee}\frac{b_{2n}}{2} + \Big(M|\eta_{kn}^{\vee}(s_k)|^2 - |M\eta_{kn}^{\vee}(s_k)|^2\Big)\frac{b_{3n}}{2}. \qquad (5.57)
\end{aligned}$$

We need to calculate $I_3^{\vee} = \|\cos\big(is_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2$, $M|\eta_{kn}^{\vee}(s_k)|^2$ and $|M\eta_{kn}^{\vee}(s_k)|^2$. Finally, we get (we set $b := b_{3k}$) To approximate D_{3n} in terms of functions involving cos, fix $m \in \mathbb{N}$, and put $\sum_{k=-m}^{m} t_k M\eta_{3k}^{\vee}(s_k) = (t,b) = 1$, where $t = (t_k)_{k=-m}^{m}$ and $b = (M\eta_{3k}^{\vee}(s_k))_{k=-m}^{m}$. We have

$$\begin{aligned}
&\|\Big[\sum_{k=-m}^{m} t_k \cos\big(s_k(x_{3k}-a_{3k})\big)A_{kn} - D_{3n}\Big]\mathbf{1}\|^2 \\
= &\ \|\sum_{k=-m}^{m} t_k\Big(\eta_{1k}^{\vee}(s_k)D_{1n} + \eta_{2k}^{\vee}(s_k)D_{2n} + \big[\eta_{3k}^{\vee}(s_k) - M\eta_{3k}^{\vee}(s_k)\big]D_{3n}\Big)\mathbf{1}\|^2 \\
= &\sum_{k=-m}^{m} t_k^2\|g_k^{\vee}(s_k)\|^2, \ \text{ since } \Big(D_{rn}\mathbf{1}, D_{ln}\mathbf{1}\Big)=0, \ \ 1 \le r < l \le 3, \qquad (5.58)
\end{aligned}$$

where the $g_k^\vee(s_k)$ are defined by (5.43). To calculate $\|g_k^\vee(s_k)\|^2$ we have

$$\|g_k^\vee(s_k)\|^2 = (g_k^\vee(s_k), g_k^\vee(s_k)) =$$
$$\Big(\big(\eta_{1k}^\vee(s_k)D_{1n} + \eta_{2k}^\vee(s_k)D_{2n} + \big[\eta_{3k}^\vee(s_k) - M\eta_{3k}^\vee(s_k)\big]D_{3n}\big)\mathbf{1},$$
$$\big(\eta_{1k}^\vee(s_k)D_{1n} + \eta_{2k}^\vee(s_k)D_{2n} + \big[\eta_{3k}^\vee(s_k) - M\eta_{3k}^\vee(s_k)\big]D_{3n}\big)\mathbf{1}\Big) =$$
$$\|x_{1k}\mathbf{1}\|^2\|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{1k}\mathbf{1}\|^2 +$$
$$\|x_{2k}\mathbf{1}\|^2\|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{2k}\mathbf{1}\|^2 +$$
$$\Big(M|\eta_{kn}^\vee(s_k)|^2 - |M\eta_{kn}^\vee(s_k)|^2\Big)\|D_{3k}\mathbf{1}\|^2 = \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)I_3^\vee\frac{b_{1n}}{2} +$$
$$\Big(\frac{1}{2b_{2k}} + a_{2k}^2\Big)I_3^\vee\frac{b_{2n}}{2} + \ \Big(M|\eta_{kn}^\vee(s_k)|^2 - |M\eta_{kn}^\vee(s_k)|^2\Big)\frac{b_{3n}}{2}. \qquad (5.59)$$

We need to calculate $I_3^\vee = \|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2$, $M|\eta_{kn}^\vee(s_k)|^2$ and $|M\eta_{kn}^\vee(s_k)|^2$. If we set $b := b_{3k}$, we get

$$I_3^\vee = \|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2 = \int_{\mathbb{R}} \frac{e^{isx}+e^{-isx}}{2}\frac{e^{-isx}+e^{isx}}{2}d\mu_{(b,0)}(x) =$$
$$\frac{1}{2}\int_{\mathbb{R}}\Big(1+\frac{e^{2isx}+e^{-2isx}}{2}\Big)d\mu_{(b,0)}(x) \overset{(5.47)}{=} \frac{1+e^{-\frac{s^2}{b}}}{2}, \qquad (5.60)$$
$$|M\eta_{kn}^\vee(s_k)|^2 = a_{3k}^2\exp\Big(-\frac{s_k^2}{2b_{3k}}\Big), \qquad (5.61)$$
$$M|\eta_{kn}^\vee(s_k)|^2 = \int_{\mathbb{R}}(x^2+2xa+a^2)\frac{e^{isx}+e^{-isx}}{2}\frac{e^{-isx}+e^{isx}}{2}d\mu_{(b,0)}(x) =$$

$$\frac{1}{2}\int_{\mathbb{R}}(x^2+2xa+a^2)\Big(1+\frac{e^{2isx}+e^{-2isx}}{2}\Big)d\mu_{(b,0)}(x) =$$
$$\frac{1}{2}\Big[\int_{\mathbb{R}}(x^2+a^2)d\mu_{(b,0)}(x) + \int_{\mathbb{R}}(x^2+a^2)\frac{e^{2isx}+e^{-2isx}}{2}d\mu_{(b,0)}(x)\Big] =$$
$$\frac{1}{2}\Big[\frac{1}{2b}+a^2+\frac{d^2F_b(2s)}{ds^2}+a^2F_b(2s)\Big] \overset{(5.47)}{=} \frac{1}{2}\Big[\frac{1}{2b}+a^2+\frac{1}{(2i)^2}\Big[\Big(\frac{2s}{b}\Big)^2-\frac{2}{b}\Big]\times$$
$$e^{-\frac{s^2}{b}}+a^2e^{-\frac{s^2}{b}}\Big] = \frac{1}{2}\Big[\Big(\frac{1}{2b}+a^2\Big)(1+e^{-\frac{s^2}{b}}) - \frac{s^2}{b^2}e^{-\frac{s^2}{b}}\Big]. \qquad (5.62)$$

Finally, we get

$$M|\eta_{kn}^\vee(s_k)|^2 - |M\eta_{kn}^\vee(s_k)|^2 = \frac{1}{2}\Big[\Big(\frac{1}{2b}+a^2\Big)(1+e^{-\frac{s^2}{b}}) - \frac{s^2}{b^2}e^{-\frac{s^2}{b}}\Big] - a^2e^{-\frac{s^2}{2b}}. \quad (5.63)$$

By (5.56), (5.57), (5.60), (5.63) and (6.3) we prove (5.37), where

$$\Sigma_3^\vee(D,s) = \sum_{k\in\mathbb{Z}} \frac{|M\eta_{kn}^\vee(s_k)|^2}{\|g_k^\vee(s_k)\|^2} =$$

$$\sum_{k\in\mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{s_k^2}{2b_{3k}}}}{\left(\frac{1}{2b_{1k}}+a_{1k}^2\right)I_3^\vee\frac{b_{1n}}{2}+\left(\frac{1}{2b_{2k}}+a_{2k}^2\right)I_3^\vee\frac{b_{2n}}{2}+\left(M|\eta_{kn}^\vee(s_k)|^2-|M\eta_{kn}^\vee(s_k)|^2\right)\frac{b_{3n}}{2}}$$

$$\sim \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{s_k^2}{2b_{3k}}}}{\frac{1+e^{-\frac{s_k^2}{b_{3k}}}}{2}\left(c_{1k}+c_{2k}\right)+\frac{1}{2}\left[c_{3k}(1+e^{-\frac{s_k^2}{b_{3k}}})-\frac{s_k^2}{b_{3k}^2}e^{-\frac{s_k^2}{b_{3k}}}\right]-a_{3k}^2 e^{-\frac{s_k^2}{2b_{3k}}}} =$$

$$\sum_{k\in\mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{x_k^2}{2}}}{\frac{1-e^{-x_k^2}}{2}\left(c_{1k}+c_{2k}\right)+\frac{1}{2}\left[c_{3k}(1+e^{-x_k^2})-\frac{x_k^2}{b_{3k}}e^{-x_k^2}\right]-a_{3k}^2 e^{-\frac{x_k^2}{2}}} = \Sigma_3(D,x), \tag{5.64}$$

where $x_k^2=\frac{s_k^2}{b_{3k}}$ and $c_{rk}=\frac{1}{2b_{rk}}+a_{rk}^2$. For $x^{(3)}=(x_k)_k$ with $x_k\equiv 1$ we get

$$\Sigma_3^\vee(D,x^{(3)}) = \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{1}{2}}}{\frac{1+e^{-1}}{2}\left(c_{1k}+c_{2k}\right)+\frac{1}{2}\left[c_{3k}(1+e^{-1})-\frac{1}{b_{3k}}e^{-1}\right]-a_{3k}^2 e^{-\frac{1}{2}}} =$$

$$\sum_{k\in\mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{1}{2}}}{\frac{1+e^{-1}}{2}\left(c_{1k}+c_{2k}+c_{3k}\right)-\left(\frac{1}{2b_{3k}}e^{-1}+a_{3k}^2 e^{-\frac{1}{2}}\right)} \overset{(2.6)}{\sim}$$

$$\sum_{k\in\mathbb{Z}} \frac{a_{3k}^2}{c_{1k}+c_{2k}+c_{3k}} = \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2}{C_k+a_{1k}^2+a_{2k}^2+a_{3k}^2} = \Sigma_3^\vee(D). \tag{5.65}$$

So, we have proved (5.40) for $x=(x_k)_k$ with $x_k\equiv 1$. □

6. How far is a vector from a hyperplane?

6.1. Some estimates

We recall some material from [27], Section 1.4.1, pp. 24–25.

Lemma 6.1 ([24]). *For a strictly positive operator A (i.e., $(Af,f)>0$ for $f\neq 0$) acting in $\mathbb{R}^n$ and a vector $b\in\mathbb{R}^n\setminus\{0\}$, we have*

$$\min_{x\in\mathbb{R}^n}\Big((Ax,x)\mid (x,b)=1\Big) = \frac{1}{(A^{-1}b,b)}. \tag{6.1}$$

The minimum is assumed for $x = \frac{A^{-1}b}{(A^{-1}b,b)}$.

Lemma 6.1 is a direct generalization of the well known result (see, for example, [4], Chap. I, §52), stating that for $a_k > 0,\ 1 \le k \le n$ we have

$$\min_{x\in\mathbb{R}^n}\Big(\sum_{k=1}^n a_k x_k^2 \mid \sum_{k=1}^n x_k = 1\Big) = \Big(\sum_{k=1}^n \frac{1}{a_k}\Big)^{-1}. \tag{6.2}$$

We will also use the same result in a slightly different form:

$$\min_{x\in\mathbb{R}^n}\Big(\sum_{k=1}^n a_k x_k^2 \mid \sum_{k=1}^n x_k b_k = 1\Big) = \Big(\sum_{k=1}^n \frac{b_k^2}{a_k}\Big)^{-1}, \tag{6.3}$$

with the minimum being assumed for $x_k = \frac{b_k}{a_k}\Big(\sum_{k=1}^n \frac{b_k^2}{a_k}\Big)^{-1}$.

6.2. The distance of a vector from a hyperplane

We follow closely the exposition [30]. We start with a classical result, see, e.g. [9]. Consider the hyperplane V_n generated by n arbitrary vectors $f_1, \dots, f_n$ in some Hilbert space H.

Lemma 6.2. *The square of the distance* $d(f_0, V_n)$ *of a vector* f_0 *from the hyperplane* V_n *is given by the ratio of two Gram determinants:*

$$d^2(f_0, V_n) = \frac{\Gamma(f_0, f_1, f_2, \dots, f_n)}{\Gamma(f_1, f_2, \dots, f_n)}. \tag{6.4}$$

6.3. Gram determinants and Gram matrices

Definition 6.1. Let us recall the definition of the Gram determinant and the Gram matrix (see [9], Chap IX, §5). Given the vectors $x_1, x_2, ..., x_m$ in some Hilbert space H the *Gram matrix* $\gamma(x_1, x_2, ..., x_m)$ is defined by the formula

$$\gamma(x_1, x_2, ..., x_m) = \big((x_k, x_n)\big)_{k,n=1}^m.$$

The determinant of this matrix is called the *Gram determinant* for the vectors $x_1, x_2, ..., x_m$ and is denoted by $\Gamma(x_1, x_2, ..., x_m)$. Thus,

$$\Gamma(x_1, x_2, \dots, x_m) := \det \gamma(x_1, x_2, \dots, x_m). \tag{6.5}$$

Notations. For a matrix $C \in \mathrm{Mat}(n,\mathbb{R})$ and $1 \le i_1 < i_2 < \dots i_r \le n$, $1 \le j_1 < j_2 < \dots j_r \le n,\ r \le n$ denote by

$$M^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(C) \quad \text{and} \quad A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(C)$$

the corresponding *minors* and *cofactors* of the matrix C.

Definition 6.2. ([27, Ch.1.4.3]) For the matrix $C \in \mathrm{Mat}(m,\mathbb{C})$ and $\lambda = (\lambda_1,\dots,\lambda_m) \in \mathbb{C}^m$ define the *generalization of the characteristic polynomial*, $p_C(t) = \det(tI - C)$, $t \in \mathbb{C}$ as follows:

$$P_C(\lambda) = \det C(\lambda), \quad \text{where} \quad C(\lambda) = \mathrm{diag}(\lambda_1,\dots,\lambda_m) + C. \tag{6.6}$$

Lemma 6.3. ([27, Ch.1.4.3]) *For the generalized characteristic polynomial $P_C(\lambda)$ of $C \in \mathrm{Mat}(m,\mathbb{C})$ and $\lambda = (\lambda_1,\lambda_2,...,\lambda_m) \in \mathbb{C}^m$ we have*

$$P_C(\lambda) = \det C + \sum_{r=1}^{m} \sum_{1 \le i_1 < i_2 < \dots < i_r \le m} \lambda_{i_1}\lambda_{i_2}...\lambda_{i_r} A^{i_1 i_2 \dots i_r}_{i_1 i_2 \dots i_r}(C). \tag{6.7}$$

Remark 6.1. If we set $\lambda_\alpha = \lambda_{i_1}\lambda_{i_2}...\lambda_{i_r}$, where $\alpha = (i_1,i_2,...,i_r)$ and $A^\alpha_\alpha(C) = A^{i_1 i_2 \dots i_r}_{i_1 i_2 \dots i_r}(C)$, $M^\alpha_\alpha(C) = M^{i_1 i_2 \dots i_r}_{i_1 i_2 \dots i_r}(C)$, $\lambda_\emptyset = 1$, $A^\emptyset_\emptyset(C) = \det C$ we may write (6.7) as follows:

$$P_C(\lambda) = \det C(\lambda) = \sum_{\emptyset \subseteq \alpha \subseteq \{1,2,\dots,m\}} \lambda_\alpha A^\alpha_\alpha(C), \tag{6.8}$$

$$P_C(\lambda) = \det C(\lambda) = \left(\prod_{k=1}^{n} \lambda_k\right) \sum_{\emptyset \subseteq \alpha \subseteq \{1,2,\dots,m\}} \frac{M^\alpha_\alpha(C)}{\lambda_\alpha}, \tag{6.9}$$

Let

$$X = X_{mn} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}. \tag{6.10}$$

Setting

$$x_k = (x_{1k}, x_{2k}, ..., x_{mk}) \in \mathbb{R}^m, \quad y_r = (x_{r1}, x_{r2}, ..., x_{rn}) \in \mathbb{R}^n, \tag{6.11}$$

we get

$$X^*X = \begin{pmatrix} (x_1,x_1) & (x_1,x_2) & ... & (x_1,x_n) \\ (x_2,x_1) & (x_2,x_2) & ... & (x_2,x_n) \\ ... & ... & ... & ... \\ (x_n,x_1) & (x_n,x_2) & ... & (x_n,x_n) \end{pmatrix} = \gamma(x_1,x_2,...,x_n), \tag{6.12}$$

$$XX^* = \begin{pmatrix} (y_1,y_1) & (y_1,y_2) & ... & (y_1,y_m) \\ (y_2,y_1) & (y_2,y_2) & ... & (y_2,y_m) \\ ... & ... & ... & ... \\ (y_m,y_1) & (y_m,y_2) & ... & (y_m,y_m) \end{pmatrix} = \gamma(y_1,y_2,...,y_m), \tag{6.13}$$

therefore, we obtain

$$\Gamma(x_1,x_2,...,x_n) = \det(X^*X) = \det(XX^*) = \Gamma(y_1,y_2,...,y_m). \tag{6.14}$$

7. Explicit expressions for $C^{-1}(\lambda)$ and $(C^{-1}(\lambda)a, a)$

In this section we follow [30]. Fix $C \in \mathrm{Mat}(n,\mathbb{R})$, $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}^n$. Our aim is to find the explicit formulas for $C^{-1}(\lambda)$ and $(C^{-1}(\lambda)a, a)$, where $C(\lambda)$ is defined by (6.6). Set $M(i_1i_2\dots i_r)(C) = M^{i_1i_2\dots i_r}_{i_1i_2\dots i_r}(C)$ and $a_{i_1i_2\dots i_r} = (a_{i_1}, a_{i_2}, \dots, a_{i_r})$. Let also $C_{i_1i_2\dots i_r}$ be the corresponding *submatrix* of the matrix C. The elements of this matrix are on the intersection of $i_1, i_2, \dots, i_r$ rows and column of the matrix C. Denote by $A(C_{i_1i_2\dots i_r})$ the matrix of the cofactors of the first order of the matrix $C_{i_1i_2\dots i_r}$, i.e.

$$A(C_{i_1i_2\dots i_r}) = (A^i_j(C_{i_1i_2\dots i_r}))_{1\le i,j\le r} \tag{7.1}$$

Let $n = 3$, then $A(C_{123}) = A(C)$ is the following matrix:

$$A(C) = A(C_{123}) = \begin{pmatrix} A^1_1 & A^1_2 & A^1_3 \\ A^2_1 & A^2_2 & A^2_3 \\ A^3_1 & A^3_2 & A^3_3 \end{pmatrix} = \begin{pmatrix} M^{23}_{23} & -M^{23}_{13} & M^{23}_{12} \\ -M^{13}_{23} & M^{13}_{13} & -M^{13}_{12} \\ M^{12}_{23} & -M^{12}_{13} & M^{12}_{12} \end{pmatrix}, \tag{7.2}$$

where we write M^{ij}_{rs} instead of $M^{ij}_{rs}(C)$ and A^i_j instead of $A^i_j(C)$.

Remark 7.1. Let A^T be the transposed matrix of A. Then

$$A^T(C_{i_1i_2\dots i_r}) = \det C^{-1}_{i_1i_2\dots i_r}\Big(C^{-1}_{i_1i_2\dots i_r}\Big), \tag{7.3}$$

In what follows we will consider the submatrix $C_{i_1 i_2 \ldots i_r}$ of the matrix $C \in \mathrm{Mat}(n, \mathbb{R})$ as an appropriate element of $\mathrm{Mat}(n, \mathbb{R})$.

Theorem 7.1. *For the matrix $C(\lambda)$ defined by (6.6) $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}^n$ we have*

$$P_C(\lambda) = \Big(\prod_{k=1}^{n} \lambda_k\Big) \sum_{r=1}^{n} \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \frac{M(i_1 i_2 \ldots i_r)}{\lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_r}}, \tag{7.4}$$

$$C^{-1}(\lambda) = \frac{1}{P_C(\lambda)} \Big(\prod_{k=1}^{n} \lambda_k\Big) \sum_{r=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots i_r \le n} \frac{A(C_{i_1 i_2 \ldots i_r})}{\lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_r}}, \tag{7.5}$$

$$(C^{-1}(\lambda)a, a) = \frac{1}{P_C(\lambda)} \Big(\prod_{k=1}^{n} \lambda_k\Big) \sum_{r=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots i_r \le n} \frac{(A(C_{i_1 i_2 \ldots i_r}) a_{i_1 i_2 \ldots i_r}, a_{i_1 i_2 \ldots i_r})}{\lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_r}}. \tag{7.6}$$

7.1. The case where C is the Gram matrix

Fix the matrix X_{mn} defined by (6.10). Denote by C the Gram matrix $\gamma(x_1, x_2, ..., x_n)$, i.e.,

$$C = \gamma(x_1, x_2, ..., x_n), \tag{7.7}$$

where $(x_1, x_2, ..., x_n)$ are defined by (6.11) and $\gamma(x_1, x_2, ..., x_n)$ by (6.12). In what follows we consider the operator $C(\lambda)$ defined by (6.6).

Remark 7.2. In this case we have

$$\begin{aligned} P_C(\lambda) &= \det\Big(\sum_{k=1}^{n} \lambda_k E_{kk} + \gamma(x_1, x_2, ..., x_n)\Big) \qquad (7.8) \\ &= \prod_{k=1}^{n} \lambda_k \Big(1 + \sum_{r=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le m} \Big(\lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_r}\Big)^{-1} \Gamma(x_{i_1}, x_{i_2}, ..., x_{i_r})\Big) \\ &= \prod_{k=1}^{n} \lambda_k \Big(1 + \sum_{r=1}^{n} \sum_{\substack{1 \le i_1 < i_2 < \ldots < i_r \le n; \\ 1 \le j_1 < j_2 < \ldots < j_r \le n}} \Big(\lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_r}\Big)^{-1} \Big(M^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r}(X)\Big)^2\Big), \end{aligned}$$

where we have used the following formula (see [9], Chap IX, §5 formula (25)):

$$\Gamma(x_{i_1}, x_{i_2}, ..., x_{i_r}) = \sum_{1 \le j_1 < j_2 < \ldots < j_r \le m} \big(M^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r}(X)\big)^2. \tag{7.9}$$

7.2. Case $m=2$

Fix two natural numbers $n, m \in \mathbb{N}$ with $m \le n$, two matrices A_{mn} and X_{mn}, vectors $g_k \in \mathbb{R}^{m-1}$, $1 \le k \le n$ and $a \in \mathbb{R}^n$ as follows

$$A_{mn} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad g_k = \begin{pmatrix} a_{2k} \\ a_{3k} \\ \dots \\ a_{mk} \end{pmatrix} \in \mathbb{R}^{m-1}, \quad a = (a_{1k})_{k=1}^n \in \mathbb{R}^n. \tag{7.10}$$

Set $C = \gamma(g_1, g_2, \dots, g_n)$. We calculate $\Delta_n(\lambda, C)$ and $(C^{-1}(\lambda)a, a)$ for an arbitrary n. Consider the matrix (6.10)

$$X_{mn} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ & & \dots & \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}, \quad \text{where} \quad x_{kr} = \frac{a_{1k}}{\sqrt{\lambda_k}}, \quad y_r = (x_{rk})_{k=1}^n \in \mathbb{R}^n. \tag{7.11}$$

Lemma 7.2 ([30], Lemma 2.2)**.** *For $m=2$ we have*

$$(C^{-1}(\lambda)a, a) = \Delta(y_1, y_2) = \frac{\Gamma(y_1) + \Gamma(y_1, y_2)}{1 + \Gamma(y_2)}, \tag{7.12}$$

where y_1 and y_2 are defined as follows

$$y_1 = \left(\frac{a_{1k}}{\sqrt{\lambda_k}} \right)_{k=1}^n \quad y_2 = \left(\frac{a_{2k}}{\sqrt{\lambda_k}} \right)_{k=1}^n. \tag{7.13}$$

7.3. Case $m=3$

By (7.8) we get

$$\Delta(y_1, y_2, y_3) = \frac{\Gamma(y_1) + \Gamma(y_1, y_2) + \Gamma(y_1, y_3) + \Gamma(y_1, y_2, y_3)}{1 + \Gamma(y_2) + \Gamma(y_3) + \Gamma(y_2, y_3)}.$$

Lemma 7.3. *For $m=3$ we have*

$$(C^{-1}(\lambda)a, a) = \Delta(y_1, y_2, y_3), \tag{7.14}$$

where the y_r are defined as follows:

$$y_r = \left(\frac{a_{rk}}{\sqrt{\lambda_k}} \right)_{k=1}^n \in \mathbb{R}^n, \quad 1 \le r \le 3. \tag{7.15}$$

8. Appendix

8.1. Comparison of two Gaussian measures

For two centered Gaussian measures $\mu_{(b,0)}$ and $\mu_{(b',0)}$ on the real line $\mathbb{R}$ defined by (1.5) it is well known that

$$H(\mu_{(b,0)}, \mu_{(b',0)}) = \left(\frac{4bb'}{(b+b')^2}\right)^{1/4}. \tag{8.1}$$

By Kakutani's criterion for product measures on $\mathbb{R}^{\mathbb{N}}$ [13], and (8.1) we see that the following lemma holds true.

Lemma 8.1. *Two Gaussian measures $\mu_{(b,0)} = \otimes_{n\in\mathbb{Z}}\mu_{(b,0)}$ and $\mu_{(b',0)} = \otimes_{n\in\mathbb{Z}}\mu_{(b',0)}$ are equivalent if and only if the product*

$$\prod_{n\in\mathbb{Z}} \frac{4b_n b'_n}{(b_n + b'_n)^2} \tag{8.2}$$

does not converge to 0*. The equivalent condition is*

$$\sum_{n\in\mathbb{Z}} \left(\sqrt{\frac{b_n}{b'_n}} - \sqrt{\frac{b'_n}{b_n}}\right)^2 < \infty. \tag{8.3}$$

Consider two measures: $\mu_{(\mathbb{I},0)} = \otimes_{n\in\mathbb{Z}}\mu_{(1,0)}$ and $\mu_{(\mathbb{I}+c,0)} = \otimes_{n\in\mathbb{Z}}\mu_{(1+c_n,0)}$ on the space X_1, where the measure $\mu_{(b,a)}$ on the real line $\mathbb{R}$ is defined by (1.5).

Lemma 8.2. *Two measures $\mu_{(\mathbb{I},0)}$ and $\mu_{(\mathbb{I}+c,0)}$ are equivalent if and only if*

$$\sum_{n\in\mathbb{Z}} c_n^2 < \infty \tag{8.4}$$

Proof. By Lemma 8.1 and (8.3), the measures $\mu_{(\mathbb{I},0)}$ and $\mu_{(\mathbb{I}+c,0)}$ are equivalent if and only if

$$\sum_{n\in\mathbb{Z}} \left(\frac{1}{\sqrt{1+c_n}} - \sqrt{1+c_n}\right)^2 = \sum_{n\in\mathbb{Z}} \frac{c_n^2}{1+c_n} < \infty.$$

By Lemma 2.5, two series $\sum_{n\in\mathbb{Z}} \frac{c_n^2}{1+c_n}$ and $\sum_{n\in\mathbb{Z}} c_n^2$ are equivalent. $\square$

The next lemma is also a consequence of Kakutani's criterion [13].

Lemma 8.3. *Two Gaussian measures* $\mu^m_{(b,0)}$ *and* $\mu^m_{(b',0)}$ *are equivalent if and only if the product*

$$\prod_{r=1}^{m}\prod_{n\in\mathbb{Z}}\frac{4b_{rn}b'_{rn}}{(b_{rn}+b'_{rn})^2} \tag{8.5}$$

does not converge to 0*. The equivalent condition is*

$$\sum_{r=1}^{m}\sum_{n\in\mathbb{Z}}\left(\sqrt{\frac{b_{rn}}{b'_{rn}}}-\sqrt{\frac{b'_{rn}}{b_{rn}}}\right)^2<\infty. \tag{8.6}$$

Lemma 1.2 follows from Lemmas 8.4 – 8.7.

Lemma 8.4. *For* $t\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$ *we have* $(\mu^m_{(b,a)})^{L_t}\perp\mu^m_{(b,a)}$ *if and only if*

$$(\mu^m_{(b,0)})^{L_t}\perp\mu^m_{(b,0)}\quad \textit{or}\quad \mu^m_{(b,L_ta)}\perp\mu^m_{(b,a)}. \tag{8.7}$$

Let us define the following measures on the spaces $\mathbb{R}^m$ and X_m:

$$\mu_m^{(B_n,0)}=\otimes_{k=1}^m\mu_{(b_{kn},0)},\quad \mu_m^{(B_n,a_n)}=\otimes_{k=1}^m\mu_{(b_{kn},a_{kn})},$$

where $a_n=(a_{1n},...,a_{mn})\in\mathbb{R}^m$ and $B_n=\mathrm{diag}(b_{1n},...,b_{mn})\in\mathrm{Mat}(m,\mathbb{R})$. Since

$$\mu^m_{(b,a)}=\otimes_{n\in\mathbb{Z}}\mu_m^{(B_n,a_n)},\quad \mu^m_{(b,0)}=\otimes_{n\in\mathbb{Z}}\mu_m^{(B_n,0)},$$

$$\left(\mu^m_{(b,a)}\right)^{L_t}=\otimes_{n\in\mathbb{Z}}\left(\mu_m^{(B_n,a_n)}\right)^{L_t},\quad \left(\mu^m_{(b,0)}\right)^{L_t}=\otimes_{n\in\mathbb{Z}}\left(\mu_m^{(B_n,0)}\right)^{L_t},$$

and

$$\mu^m_{(b,L_ta)}=\otimes_{n\in\mathbb{Z}}\mu_m^{(B_n,L_ta_n)},$$

by the Kakutani criterion [13], we derive the following two lemmas:

Lemma 8.5. *For the measures* $\mu^m_{(b,0)}$, $m\in\mathbb{N}$ *and* $t\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$, *we obtain*

$$(\mu^m_{(b,0)})^{L_t}\perp\mu^m_{(b,0)}\ \Leftrightarrow\prod_{n\in\mathbb{Z}}H\left(\left(\mu_m^{(B_n,0)}\right)^{L_t},\mu_m^{(B_n,0)}\right)=0.$$

Lemma 8.6. *For the measures* $\mu^m_{(b,0)}$, $m\in\mathbb{N}$ *and* $t\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$, *we get*

$$\mu^m_{(b,L_ta)}\perp\mu^m_{(b,a)}\Leftrightarrow\prod_{n\in\mathbb{Z}}H\left(\mu_m^{(B_n,L_ta_n)},\mu_m^{(B_n,a_n)}\right)=0.$$

To prove Lemma 1.2 it is sufficient to show, by Lemma 8.4, that

$$H_{m,n}(t)=H\Big(\Big(\mu_m^{(B_n,0)}\Big)^{L_t},\mu_m^{(B_n,0)}\Big)=\Big(\frac{1}{2^m|\det\ t|}\det\left(I+X_n^*(t)X_n(t)\right)\Big)^{-1/2}, \tag{8.8}$$

to prove the equivalence

$$\prod_{n\in\mathbb{Z}}H\left(\mu_m^{(B_n,L_ta_n)},\mu_m^{(B_n,a_n)}\right)=0\Leftrightarrow\sum_{n\in\mathbb{Z}}\sum_{r=1}^m b_{rn}\Big(\sum_{s=1}^m(t_{rs}-\delta_{rs})a_{sn}\Big)^2=\infty, \tag{8.9}$$

and to apply the following lemma.

Lemma 8.7. *For $X\in\mathrm{Mat}(m,\mathbb{R})$ we have*

$$\det\left(I+X^*X\right)=1+\sum_{r=1}^m\sum_{1\leq i_1<i_2<\ldots<i_r\leq m;1\leq j_1<j_2<\ldots<j_r\leq m}\left(M_{j_1j_2\ldots j_r}^{i_1i_2\ldots i_r}(X)\right)^2. \tag{8.10}$$

The proof of the equivalence (8.9) is based on the following theorem that one can find, e.g., in [35, Ch. III, §16, Theorem 2].

Theorem 8.8. *Two Gaussian measures $\mu_{B,a}$ and $\mu_{B,b}$ in a Hilbert space H are equivalent if and only of $B^{-1/2}(a-b)\in H$.*

Indeed, we have

$$\|C^{-1/2}(ta-a)\|_H^2=\sum_{n\in\mathbb{Z}}\|C_n^{-1/2}(t-I)a_n\|_{H_n}^2=2\sum_{n\in\mathbb{Z}}\sum_{r=1}^m\frac{b_{kn}}{d_{kn}}\Big(\sum_{s=1}^m(t_{rs}-\delta_{rs})a_{sn}\Big)^2d_{kn}.$$

To explain the latter equality let us describe H and C. To find an operator C we present the measure $\mu_{(b,a)}^m$ in the canonical form $\mu_{C,a}$ defined by its Fourier transform:

$$\int_H\exp i(y,x)d\mu_{C,a}(x)=\exp\left(i(a,y)-\frac{1}{2}(Cy,y)\right),\ y\in H, \tag{8.11}$$

where C is a positive *nuclear operator* (called the *covariance operator*) on the Hilbert space H, and $a\in H$ is the *mathematical expectation* or *mean.*

Recall the *Kolmogorov zero-one law.* Let us consider in the space $\mathbb{R}^\infty=\mathbb{R}\times\mathbb{R}\times\cdots$ the infinite tensor product $\mu_b=\otimes_{n\in\mathbb{N}}\mu_{b_k}$ of one-dimensional Gaussian measures μ_{b_k} on $\mathbb{R}$ defined as follows:

$$d\mu_b(x)=\sqrt{\frac{b}{\pi}}\exp(-bx^2)dx. \tag{8.12}$$

Consider a Hilbert space $l_2(a)$ defined by

$$l_2(a) = \{x \in \mathbb{R}^\infty \, : \, \|x\|^2_{l_2(a)} = \sum_{k\in\mathbb{N}} x_k^2 a_k < \infty\},$$

where $a = (a_k)_{k\in\mathbb{N}}$ is an infinite sequence of positive numbers.

Theorem 8.9 (Kolmogorov's zero-one law, [34]). *We have*

$$\mu_b(l_2(a)) = \begin{cases} 0, & \text{if} \quad \sum_{k\in\mathbb{N}} \frac{a_k}{b_k} = \infty; \\ 1, & \text{otherwise.} \end{cases}$$

8.2. Properties of two vectors $f, g \notin l_2$

In what follows we will use systematically the following notation. For k vectors $f_1, f_2, \dots, f_k \in \mathbb{R}^n$ with $k \le n$ we set

$$\Delta(f_1, f_2, \dots, f_k) = \frac{\det(I + \gamma(f_1, f_2, \dots, f_k))}{\det(I + \gamma(f_2, \dots, f_k))} - 1. \tag{8.13}$$

For $k = 2$ and $k = 3$ we get respectively:

$$\Delta(f_1, f_2) = \frac{\det(I + \gamma(f_1, f_2))}{\det(I + \gamma(f_2))} - 1 = \frac{I + \Gamma(f_1) + \Gamma(f_2) + \Gamma(f_1, f_2)}{I + \Gamma(f_2)}. \tag{8.14}$$

$$\Delta(f_1, f_2, f_3) = \frac{\det(I + \gamma(f_1, f_2, f_3))}{\det(I + \gamma(f_2, f_3))} - 1 = \tag{8.15}$$
$$\frac{\Gamma(f_1) + \Gamma(f_1, f_2) + \Gamma(f_1, f_3) + \Gamma(f_1, f_2, f_3)}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)}.$$

Lemma 8.10 ([28], and [27], Ch.10). *Let $f = (f_k)_{k\in\mathbb{N}}$ and $g = (g_k)_{k\in\mathbb{N}}$ be two real vectors such that $\|f\|^2 = \infty$, where $\|f\|^2 = \sum_k f_k^2$. Denote by $f_{(n)}$, $g_{(n)} \in \mathbb{R}^n$ their projections to the subspace $\mathbb{R}^n$, i.e., $f_{(n)} = (f_k)_{k=1}^n$, $g_{(n)} = (g_k)_{k=1}^n$. Then*

$$\Delta(f, g) := \lim_{n\to\infty} \Delta(f_{(n)}, g_{(n)}) = \infty, \tag{8.16}$$

where $\Delta(f_{(n)}, g_{(n)})$ is defined by (8.14), in the following cases:

(a) $\|g\|^2 < \infty$,

(b) $\|g\|^2 = \infty$, *and* $\lim_{n\to\infty} \frac{\|f_{(n)}\|}{\|g_{(n)}\|} = \infty$,

(c) $\|f\|^2 = \|g\|^2 = \|f + sg\|^2 = \infty$, *for all* $s \in \mathbb{R} \setminus \{0\}$.

PROOF. Obviously $\lim_{n\to\infty}\Delta(f_{(n)},g_{(n)})=\infty$ if conditions (a) or (b) hold. The implication $(c)\Rightarrow$ (8.16) is based on the following lemma. □

Lemma 8.11. *Let $f=(f_k)_{k\in\mathbb{N}}$ and $g=(g_k)_{k\in\mathbb{N}}$ be two real vectors such that*

$$\|f\|^2=\|g\|^2=\|C_1f+C_2g\|^2=\infty \quad \text{for all} \quad (C_1,C_2)\in\mathbb{R}^2\setminus\{0\}, \tag{8.17}$$

$$\text{then} \quad \lim_{n\to\infty}\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(g_{(n)})}=\infty \quad \text{and} \quad \lim_{n\to\infty}\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(f_{(n)})}=\infty. \tag{8.18}$$

Lemma 8.12. *Let $f_1,f_2\notin l_2$ and $\Delta(f_1,f_2)<\infty$, then for some $(C_1,C_2)\in\mathbb{R}^2\setminus\{0\}$ we have $C_1f_1+C_2f_2\in l_2$.*

PROOF. Let assume the opposite, i.e., $C_1f_1+C_2f_2\notin l_2$ for all $(C_1,C_2)\in\mathbb{R}^2\setminus\{0\}$. Then by Lemma 8.11

$$\Delta(f_1,f_2)=\frac{\Gamma(f_1)+\Gamma(f_1,f_2)}{1+\Gamma(f_2)}>\frac{\Gamma(f_1,f_2)}{1+\Gamma(f_2)}\sim\frac{\Gamma(f_1,f_2)}{\Gamma(f_2)}=\infty. \qquad \square$$

Lemma 8.13. *Let $f_1,f_2,f_3\notin l_2$ and $\Delta(f_1,f_2,f_3)<\infty$, then for some $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$ we have $C_1f_1+C_2f_2+C_3f_3\in l_2$.*

PROOF. Let assume the opposite, i.e., that $C_1f_1+C_2f_2+C_3f_3\notin l_2$ for all $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$. Then by Lemmas 8.15–8.11 and (8.15) we get

$$\Delta(f_1,f_2,f_3)=\frac{\Gamma(f_1)+\Gamma(f_1,f_2)+\Gamma(f_1,f_3)+\Gamma(f_1,f_2,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}>$$
$$\frac{\Gamma(f_1,f_2,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}\sim\frac{\Gamma(f_1,f_2,f_3)}{\Gamma(f_2,f_3)}=\infty. \qquad \square$$

8.3. Properties of three vectors $f,g,h\notin l_2$

Lemma 8.14. *Let $f=(f_k)_{k\in\mathbb{N}}$, $g=(g_k)_{k\in\mathbb{N}}$ and $h=(h_k)_{k\in\mathbb{N}}$ be three real vectors such that $\|f\|^2=\infty$ where $\|f\|^2=\sum_k f_k^2$. Denote by $f_{(n)},\ g_{(n)},\ h_{(n)}\in\mathbb{R}^n$ their projections to the subspace $\mathbb{R}^n$, i.e., $f_{(n)}=(f_k)_{k=1}^n$, $g_{(n)}=(g_k)_{k=1}^n$, $h_{(n)}=(h_k)_{k=1}^n$. Then*

$$\Delta(f,g,h):=\lim_{n\to\infty}\Delta(f_{(n)},g_{(n)},h_{(n)})=\infty, \tag{8.19}$$

where $\Delta(f,g,h)$ is defined by (8.15), in the following cases:

(a) $\|g\|^2<\infty$ *and* $\|h\|^2<\infty$,

(b) $\|g\|^2<\infty$ *and* $\|h\|^2=\infty$, *or* $\|g\|^2=\infty$ *and* $\|h\|^2<\infty$,

(c) $\|C_1f+C_2g+C_3h\|^2=\infty$, *for all* $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$.

PROOF. (a) In this case $\|g_{(n)}\|^2 \leq C$, $\|h_{(n)}\|^2 \leq C$ and $\Gamma(g_{(n)}, h_{(n)}) \leq C$ and therefore,

$$\lim_{n\to\infty} \Delta(f_{(n)}, g_{(n)}, h_{(n)}) \geq \lim_{n\to\infty} \frac{\|f_{(n)}\|^2}{1+3C} = \infty.$$

(b) Let $\|g\|^2 < \infty$, $\|h\|^2 = \infty$. In this case we have

$$\Gamma(g_{(n)}, h_{(n)}) \leq \|g_{(n)}\|^2 \|h_{(n)}\|^2 \sin^2(\alpha_n) \leq C_1 \Gamma(h_{(n)}),$$

where α_n is the angle betwen two vectors $g_{(n)}$ and $h_{(n)}$. Therefore,

$$1 + \Gamma(g_{(n)}) + \Gamma(h_{(n)}) + \Gamma(g_{(n)}, h_{(n)}) \leq (1+C)(1+\Gamma(h_{(n)}),$$

$$\Delta(f_{(n)}, g_{(n)}, h_{(n)}) \geq \frac{\Gamma(f_{(n)}) + \Gamma(f_{(n)}, h_{(n)})}{(1+C)(1+\Gamma(h_{(n)})} \sim \Delta(f_{(n)}, h_{(n)}).$$

So, this case is reduced to the case $m = 2$, see Lemma 8.11.

The implication $(c) \Rightarrow$ (8.19) is based on the following lemma (the proof of which will appear in [29]. Compare with Lemma 8.10. □

Lemma 8.15 (see [29])**.** *Let f_0, f_1, f_2 be three infinite real vectors $f_r = (f_{rk})_{k\in\mathbb{N}}$, $0 \leq r \leq 2$ such that for all $(C_0, C_1, C_2) \in \mathbb{R}^3 \setminus \{0\}$ holds*

$$\sum_{r=0}^{2} C_r f_r \notin l_2, \quad i.e., \quad \sum_{k\in\mathbb{N}} |C_0 f_{0k} + C_1 f_{1k} + C_2 f_{2k}|^2 = \infty. \tag{8.20}$$

Denote by $f_r(n) = (f_{rk})_{k=1}^n \in \mathbb{R}^n$ the projections of the vectors f_r on the subspace $\mathbb{R}^n$. Then

$$\frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)} := \lim_{n\to\infty} \frac{\Gamma(f_0(n), f_1(n), f_2(n))}{\Gamma(f_1(n), f_2(n))} = \infty. \tag{8.21}$$

The idea of the proof (for details see [29]). We assume there exists a real number C such that

$$\frac{\Gamma(f_0(n), f_1(n), f_2(n))}{\Gamma(f_1(n), f_2(n))} \leq C, \tag{8.22}$$

for every integer n and will show that this leads to a contradiction.

For $t \in \mathbb{R}^2$ and $f_0, f_1, f_2 \in H$ we define the function

$$\begin{aligned} F_{20}(t) &= \| \sum_{k=1}^{2} t_k f_k - f_0 \|^2 = \sum_{k,r=1}^{2} t_k t_r (f_k, f_r) - 2 \sum_{k=1}^{2} t_k (f_k, f_0) + (f_0, f_0) \\ &= (At, t) - 2(t, b) + (f_0, f_0), \end{aligned}$$

where $b = (f_k, f_0)_{k=1}^2 \in \mathbb{R}^2$ and A is the Gram matrix

$$A = \gamma(f_1, f_2) = \big((f_k, f_r)\big)_{k,r=1}^2.$$

Suppose that $At_0 = b$, then we have

$$(At, t) - 2(t, b) + (f_0, f_0) = (A(t - t_0), (t - t_0)) + \frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)},$$

and therefore

$$\begin{aligned} F_{20}(t) &= \big(A(t - t_0), (t - t_0)\big) + \frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)}, \\ F_{20}(t_0) &= \min_{(t_1, t_2) \in \mathbb{R}^2} \| \sum_{k=1}^{2} t_k f_k - f_0 \|^2 = \frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)}, \end{aligned} \tag{8.23}$$

Consider the matrix

$$X_{3n} = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ f_{01} & f_{02} & \dots & f_{0n} \end{pmatrix} \tag{8.24}$$

and its minors

$$M_{krs}^{123} = \begin{vmatrix} f_{1k} & f_{1r} & f_{1s} \\ f_{2k} & f_{2r} & f_{2s} \\ f_{0k} & f_{0r} & f_{0s} \end{vmatrix}, \quad M_{kr}^{12} = \begin{vmatrix} f_{1k} & f_{1r} \\ f_{2k} & f_{2r} \end{vmatrix}.$$

Then by [9] we have

$$\begin{aligned} \Gamma(f_1(n), f_2(n)) &= \sum_{1 \le r < s \le n} |M_{rs}^{12}|^2, \\ \Gamma(f_0(n), f_1(n), f_2(n)) &= \sum_{1 \le k < r < s \le n} |M_{krs}^{123}|^2. \end{aligned}$$

Therefore, the inequality (8.22) will have the following form

$$\frac{\Gamma(f_0(n), f_1(n), f_2(n))}{\Gamma(f_1(n), f_2(n))} = \frac{\sum_{1\leq k<r<s\leq n} |M_{krs}^{123}|^2}{\sum_{1\leq r<s\leq n} |M_{rs}^{12}|^2} \leq C \tag{8.25}$$

for all $n \in \mathbb{N}$. Set now

$$a_{2n} = \gamma(f_1(n), f_2(n), f_0(n)),$$

$$A_{2n} = \gamma(f_1(n), f_2(n)), \quad A_{2n}t_0^{(n)} = b_{2n}, \tag{8.26}$$

$$b_{2n} = (f_k(n), f_0(n))_{k=1}^2 \in \mathbb{R}^2, \quad t_0^{(n)} = (t_{0r}^{(n)})_{r=1}^2. \tag{8.27}$$

More explicitly

$$a_{2n} = \begin{pmatrix} (f_1(n), f_1(n)) & (f_1(n), f_2(n)) & (f_1(n), f_0(n)) \\ (f_2(n), f_1(n)) & (f_2(n), f_2(n)) & (f_2(n), f_0(n)) \\ (f_0(n), f_1(n)) & (f_0(n), f_2(n)) & (f_0(n), f_0(n)) \end{pmatrix}, \tag{8.28}$$

$$A_{2n} = \begin{pmatrix} (f_1(n), f_1(n)) & (f_1(n), f_2(n)) \\ (f_2(n), f_1(n)) & (f_2(n), f_2(n)) \end{pmatrix}. \tag{8.29}$$

If we replace the vectors f_0, f_1, f_2 with $f_0(n), f_1(n), f_2(n)$, the equality (8.23) then becomes

$$F_{20}^{(n)}(t_0^{(n)}) = \min_{(t_1,t_2)\in\mathbb{R}^2} \| \sum_{k=1}^{2} t_k f_k(n) - f_0(n)\|^2 = \frac{\Gamma(f_0(n), f_1(n), f_2(n))}{\Gamma(f_1(n), f_2(n))}. \tag{8.30}$$

For $t \in \mathbb{R}^2$ and $f_0(n), f_1(n), f_2(n) \in \mathbb{R}^n$ and $0 \leq s \leq 2$, define the functions

$$F_{2s}^{(n)}(t) = \| \sum_{0\leq r\leq 2, r\neq s} t_r f_r(n) - f_s(n)\|^2. \tag{8.31}$$

The minimum of the corresponding expressions $F_{2s}^{(n)}(t)$ for $0 \leq s \leq 2$ is attained respectively at $t_0^{(n)}$, $t_1^{(n)}$, $t_2^{(n)}$. The proof of the fact that one of the sequences $t_0^{(n)}$, $t_1^{(n)}$, $t_2^{(n)}$ is bounded, is based on the positive definiteness of the matrices $\gamma(f_1(n), f_2(n), f_0(n))$, for the details see [29]. Let for example, the sequence $t_0^{(n)} \in \mathbb{R}^2$ is bounded. Therefore, there exists a subsequence $(t^{(n_k)})_{k\in\mathbb{N}}$ that converges to some $t \in \mathbb{R}^2$. This contradics (8.22). Indeed

$$\lim_{n\to\infty} F_{20}^{(n)}(t) = \infty, \quad F_{20}^{(n)}(t_0^{(n_k)}) \leq C, \quad \lim_{k\to\infty} t_0^{(n_k)} = t. \tag{8.32}$$

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