Irreducibility of the Koopman representations for the group $\text{GL}_0(2^{\infty}, \mathbb{R})$ acting on three infinite rows

by

Alexandre V. Kosyak
Pieter Moree
Irreducibility of the Koopman representations for the group $\text{GL}_0(2^\infty, \mathbb{R})$ acting on three infinite rows

by

Alexandre V. Kosyak
Pieter Moree

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institute of Mathematics
Ukrainian National Academy of Sciences
3 Tereshchenkivs'ka Str.
Kyiv 01601
Ukraine

London Institute for Mathematical Sciences
21 Albemarle St.
London W1S 4BS
UK

MPIM 23-15
Irreducibility of the Koopman representations for the group $\text{GL}_0(2\infty, \mathbb{R})$ acting on three infinite rows

A.V. Kosyak*

Institute of Mathematics, Ukrainian National Academy of Sciences,
3 Tereshchenkivs’ka Str., Kyiv, 01601, Ukraine

London Institute for Mathematical Sciences,
21 Albemarle St, London W1S 4BS, UK

P. Moree

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany

Abstract

In [25] the first author started with the development of harmonic analysis on infinite-dimensional groups. In this article, following these ideas, we construct an analogue of quasi-regular representations, when the group $G$ acts on a $G$–space $X$ equipped with a quasi-invariant measure. For the group $G$ we take the inductive limit of the general linear groups $\text{GL}_0(2\infty, \mathbb{R}) = \lim \rightarrow GL(2n−1, \mathbb{R})$, acting on the space $X_m$ of $m$ rows, infinite in both directions, with Gaussian measure. This measure is the infinite tensor product of one-dimensional arbitrary Gaussian non-centered measures. We prove an irreducibility criterion for $m = 3$. In 2019, the first author [28] established a criterion for $m \leq 2$. Our proof is in the same spirit, but the details are far more involved.

Keywords: infinite-dimensional groups, irreducible unitary representation, Koopman representation, Ismagilov’s conjecture, quasi-invariant, ergodic measure

2008 MSC: 22E65, (28C20, 43A80, 58D20)

*Corresponding author

Email address: kosyak02@gmail.com (A.V. Kosyak)

To all fearless Ukrainians defending not only their country, but the whole civilization against putin’s rashism

Preprint submitted to Journal of Functional Analysis August 18, 2023
Contents

1 Representations of the inductive limit of the general linear groups $GL_0(2\infty, \mathbb{R})$ 4
   1.1 Finite-dimensional case 4
   1.2 Infinite-dimensional case 5

2 Some orthogonality problem in measure theory 8
   2.1 General setting 8
   2.2 Orthogonality criteria $\mu^t \perp \mu$ for $t \in GL(2, \mathbb{R}) \setminus \{e\}$ 9
   2.3 Equivalent series and equivalent sequences 12
   2.4 Orthogonality criteria $\mu^t \perp \mu$ for $t \in GL(3, \mathbb{R}) \setminus \{e\}$ 13

3 Irreducibility, the cases $m = 1$ and $m = 2$ 20
   3.1 Case $m = 1$ 20
   3.2 Case $m = 2$, approximation of $x_{kn}$ and $D_{kn}$ 21
      3.2.1 Technical part of the proof of irreducibility 22
      3.2.2 Scheme of the proof for two lines 23
      3.2.3 I, Approximation of $x_{1n}$ and $x_{2n}$ 23
      3.2.4 Case $S = (0, 1)$ 24
      3.2.5 Case $S = (1, 1)$ 26
      3.2.6 II, Approximation of $D_{1n}$ and $D_{2n}$ 26

4 Irreducibility, the case $m = 3$ 31
   4.1 Technical part of the proof of irreducibility 31
      4.1.1 Notations and the change of the variables 33
   4.2 Approximation scheme 35
   4.3 Case $S = (0, 0, 1)$ 35
   4.4 Case $S = (0, 1, 1)$ 38
      4.4.1 Approximation of $x_{2n}$, $x_{3n}$ 42
      4.4.2 Two remaining possibilities 43
      4.4.3 Cases (a), (b), (d) 43
      4.4.4 Approximation of $D_{rn}$, $1 \leq r \leq 3$, 1 44
      4.4.5 Approximation of $D_{2n}$ and $D_{3n}$, 2 46
      4.4.6 Approximation of $D_{2n}$ and $D_{3n}$, 3 47
      4.4.7 Two technical lemmas 47
      4.4.8 Case (c) 50
1. Representations of the inductive limit of the general linear groups \( \text{GL}_0(2\infty, \mathbb{R}) \)

1.1. Finite-dimensional case

Consider the space 

\[
X_{m,n} = \left\{ x = \sum_{1 \leq k \leq m} \sum_{-n \leq r \leq n} x_{kr}E_{kr}, \ x_{kr} \in \mathbb{R} \right\},
\]

where \( E_{kn}, k, n \in \mathbb{Z} \) are infinite matrix unities, with the measure (see (1.5))

\[
\mu_{(b,a)}^{m,n}(x) = \bigotimes_{k=1}^{m} \bigotimes_{r=-n}^{n} \mu_{(b_{kr},a_{kr})}(x_{kr}).
\]

Two groups act on the space \( X_{m,n} \), namely \( \text{GL}(m, \mathbb{R}) \) from the left, and \( \text{GL}(2n+1, \mathbb{R}) \) from the right, and their actions commute. Therefore, two von Neumann algebras \( \mathcal{A}_{1,n} \) and \( \mathcal{A}_{2,n} \) in the Hilbert space \( L^2(X_{m,n}, \mu_{(b,a)}^{m,n}) \) generated respectively by the left and the right actions of the corresponding groups have the property that \( \mathcal{A}_{1,n}' \subseteq \mathcal{A}_{2,n} \), where \( \mathcal{A}' \) is a commutant of a von Neuman algebra \( \mathcal{A} \). We study what happens as \( n \to \infty \). In the limit we obtain some unitary representation \( T_{R,\mu,m} \) (see (1.6)) of the group \( G := \text{GL}_0(2\infty, \mathbb{R}) = \lim_{n\to \infty} \text{GL}(2n+1, \mathbb{R}) \) acting from the right on \( X_m \). In the generic case, the representation \( T_{R,\mu,m} \) is reducible. Indeed, if there exists a non-trivial element \( s \in \text{GL}(m, \mathbb{R}) \) such the left action is admissible for the measure \( \mu_{(b,a)}^{m,n} \), i.e., \((\mu_{(b,a)}^{m,n})^{Ls} \sim \mu_{(b,a)}^{m,n}\) the operator \( T_{s}^{L,\mu,m} \) naturally associated with the left action, is well defined and \([T_{t}^{R,\mu,m}, T_{s}^{L,\mu,m}] = 0\) for all \( t \in G, s \in \text{GL}(m, \mathbb{R}) \). We use notation \( \mu^{f}(\Delta) = \mu(f^{-1}(\Delta)) \) for \( f : X \to X \), where \( \Delta \) is some measurable set in \( X \).

The main result of the article is the following. The representation \( T_{R,\mu,m} \) is irreducible, see Theorem 1.1 if and only if no left actions are admissible, i.e., when \((\mu_{(b,a)}^{m,n})^{Ls} \perp \mu_{(b,a)}^{m,n}\) for all \( s \in \text{GL}(m, \mathbb{R}) \setminus \{e\} \). This is again a manifestation of the Ismagilov conjecture, see [27].

Here, as in the case of the regular [18, 19] and quasiregular [21, 22] representations of the group \( B_{0}^n \), which is an inductive limit of upper-triangular real matrices, we obtain the remarkable result that the irreducible representations can be obtained as the inductive limit of reducible representations!
1.2. Infinite-dimensional case

Let us denote by \( \text{Mat}(2\infty, \mathbb{R}) \) the space of all real matrices that are infinite in both directions:
\[
\text{Mat}(2\infty, \mathbb{R}) = \left\{ x = \sum_{k,n \in \mathbb{Z}} x_{kn}E_{kn}, \; x_{kn} \in \mathbb{R} \right\}.
\] (1.1)

The group \( \text{GL}_0(2\infty, \mathbb{R}) = \lim_{\to n,i} \text{GL}(2n+1, \mathbb{R}) \) is defined as the inductive limit of the general linear groups \( \text{GL}(2n+1, \mathbb{R}) \) with respect to the symmetric embedding \( i^*: \)
\[
G_n \ni x \mapsto i^*_n(x) = x + E_{-(n+1)} - (n+1) + E_{n+1,n+1} \in G_{n+1}.
\] (1.2)

For a fixed natural number \( m \), consider a \( G \)-space \( X_m \) as the following subspace of the space \( \text{Mat}(2\infty, \mathbb{R}) \):
\[
X_m = \left\{ x \in \text{Mat}(2\infty, \mathbb{R}) \mid x = \sum_{k=1}^{m} \sum_{n \in \mathbb{Z}} x_{kn}E_{kn} \right\}.
\] (1.3)

The group \( \text{GL}_0(2\infty, \mathbb{R}) \) acts from the right on the space \( X_m \). Namely, the right action of the group \( \text{GL}_0(2\infty, \mathbb{R}) \) is correctly defined on the space \( X_m \) by the formula
\[
R_t(x) = xt^{-1}, \; t \in G, \; x \in X_m.
\]

We define a Gaussian non-centered product measure \( \mu := \mu^m := \mu^m_{(b,a)} \) on the space \( X_m \):
\[
\mu^m_{(b,a)}(x) = \otimes_{k=1}^{m} \otimes_{n \in \mathbb{Z}} \mu_{(b_{kn},a_{kn})}(x_{kn}),
\] (1.4)

where
\[
d\mu_{(b_{kn},a_{kn})}(x_{kn}) = \sqrt{\frac{b_{kn}}{\pi}} e^{-b_{kn}(x_{kn} - a_{kn})^2} dx_{kn}
\] (1.5)

and \( b = (b_{kn})_{k,n}, \; b_{kn} > 0, \; a = (a_{kn})_{k,n}, \; a_{kn} \in \mathbb{R}, \; 1 \leq k \leq m, \; n \in \mathbb{Z} \). Define the unitary representation \( T^{R,\mu^m} \) of the group \( \text{GL}_0(2\infty, \mathbb{R}) \) on the space \( L^2(X_m, \mu^m_{(b,a)}) \) by the formula:
\[
(T^R_t \mu^m f)(x) = \left( \frac{d\mu^m_{(b,a)}(xt)}{d\mu^m_{(b,a)}(x)} \right)^{1/2} f(x), \; f \in L^2(X_m, \mu^m_{(b,a)}).
\] (1.6)

Obviously, the centralizer \( Z_{\text{Aut}(X_m)}(R(G)) \subset \text{Aut}(X_m) \) contains the group \( L(\text{GL}(m, \mathbb{R})), \) i.e., the image of the group \( \text{GL}(m, \mathbb{R}) \) with respect to the left action \( L : \text{GL}(m, \mathbb{R}) \to \text{Aut}(X_m), \; L_s(x) = sx, \; s \in \text{GL}(m, \mathbb{R}), \; x \in X_m \). We prove the following theorem.
**Theorem 1.1.** The representation $T_{R,\mu,m}: \text{GL}_0(2\infty, \mathbb{R}) \to U\left(L^2(X_m, \mu^m_{(b,a)})\right)$ is irreducible, for $m = 3$, if and only if

1. $(\mu^m_{(b,a)})^{L_t} \perp \mu^m_{(b,a)}$ for all $s \in \text{GL}(m, \mathbb{R}) \setminus \{e\}$;
2. the measure $\mu^m_{(b,a)}$ is $G$-ergodic.

In [27, 28] this result was proved for $m \leq 2$. Note that conditions (i) and (ii) are necessary conditions for irreducibility.

**Remark 1.1.** Any Gaussian product-measure $\mu^m_{(b,a)}$ on $X_m$ is $\text{GL}_0(2\infty, \mathbb{R})$-right-ergodic [34, §3, Corollary 1], see Definition 2.1. For non-product-measures this is not true in general.

In order to study the condition $(\mu^m_{(b,a)})^{L_t} \perp \mu^m_{(b,a)}$ for all $t \in \text{GL}(m, \mathbb{R}) \setminus \{e\}$ set

$$t = (t_{rs})_{r,s=1}^m \in \text{GL}(m, \mathbb{R}), \quad B_n = \text{diag}(b_{1n}, b_{2n}, \ldots, b_{mn}), \quad X_n(t) = B_n^{1/2}tB_n^{-1/2}.$$  

(1.7)

Let $M_{i_1j_1 \ldots i_r}^{i_2j_2 \ldots j_r}(t)$ be the minors of the matrix $t$ with $i_1, i_2, \ldots, i_r$ rows and $j_1, j_2, \ldots, j_r$ columns, $1 \leq r \leq m$. Let $\delta_{rs}$ be the Kronecker symbols.

**Lemma 1.2 ([27], Lemma 10.2.3; [28], Lemma 2.2).** For the measures $\mu^m_{(b,a)}$, with $m$ a natural number, the relation

$$(\mu^m_{(b,a)})^{L_t} \perp \mu^m_{(b,a)} \text{ for all } t \in \text{GL}(m, \mathbb{R}) \setminus \{e\}$$

holds if and only if

$$\prod_{n \in \mathbb{Z}} \frac{1}{2^m |\det t|} \det(I + X_n^*(t)X_n(t)) + \sum_{n \in \mathbb{Z}} \sum_{r=1}^m b_{rn} \left(\sum_{s=1}^m (t_{rs} - \delta_{rs})a_{sn}\right)^2 = \infty,$$

where

$$\det(I + X_n^*(t)X_n(t)) = 1 + \sum_{r=1}^m \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq m}^{1 \leq j_1 < j_2 < \ldots < j_r \leq m} (M_{i_1j_1 \ldots i_r}^{i_2j_2 \ldots j_r}(X_n(t)))^2.$$  

For the convenience of the reader this lemma is proved in Section 8.1.

**Remark 1.2.** (The idea of the proof of irreducibility.) Let us denote by $\mathfrak{A}^m$ the von Neumann algebra generated by the representation $T_{R,\mu,m}$, i.e.,
\( \mathfrak{A}^m = \{ T_{t}^{R,\mu,m} \mid t \in G \}'' \). For \( \alpha = (\alpha_k) \in \{0,1\}^m \) define the von Neumann algebra \( L^\infty_\alpha(X_m, \mu^m) \) as follows:
\[
L^\infty_\alpha(X_m, \mu^m) = \left\{ \exp(itB^\alpha_{kn}) \mid 1 \leq k \leq m, \, t \in \mathbb{R}, \, n \in \mathbb{Z} \right\}''.
\]
where \( B^\alpha_{kn} = \begin{cases} x_{kn}, & \text{if } \alpha_k = 0 \\ i^{-1}D_{kn}, & \text{if } \alpha_k = 1 \end{cases} \), and \( D_{kn} = \partial/\partial x_{kn} - b_{kn}(x_{kn} - a_{kn}) \).

The proof of the irreducibility is based on four facts:
1) we can approximate by generators \( A_{kn} = \frac{d}{dt}T_{t}^{R,\mu,m} \bigg|_{t=0} \) the set \( (B^\alpha_{kn})_{k=1}^m, n \in \mathbb{Z} \) for some \( \alpha \in \{0,1\}^m \) depending on the measure \( \mu^m \) using the orthogonality condition \((\mu^m)^{Ls} \perp \mu^m \) for all \( s \in \text{GL}(m, \mathbb{R}) \) \( \setminus \{e\} \),
2) it is sufficient to verify the approximation only on the cyclic vector \( 1(x) \equiv 1 \), since the representation \( T_{t}^{R,\mu,m} \) is cyclic,
3) the subalgebra \( L^\infty_\alpha(X_m, \mu^m) \) is a maximal abelian subalgebra in \( \mathfrak{A}^m \),
4) the measure \( \mu^m \) is \( G \)-ergodic.

Here the generators \( A_{kn} \) are given by the formulas:
\[
A_{kn} = \sum_{r=1}^{m} x_{rk}D_{rn}, \quad k, n \in \mathbb{Z}, \quad \text{where } D_{kn} = \partial/\partial x_{kn} - b_{kn}(x_{kn} - a_{kn}).
\]

**Remark 1.3.** Scheme of the proof. We prove the irreducibility as follows
\[
(\mu^{Ls} \perp \mu \text{ for all } s \in \text{GL}(3, \mathbb{R}) \setminus \{e\}) \iff \left( \begin{array}{c} \text{criteria of orthogonality} \\ \text{Lemma 8.15 about three vectors } f, g, h \notin l_2 \end{array} \right) \Rightarrow \text{irreducibility},
\]
where \( \Delta^{(i)} := \Delta(Y^{(i)}_i, Y^{(i)}_j, Y^{(i)}_k), \quad \Delta := \Delta(Y_i, Y_j, Y_k) \).

**Remark 1.4.** The fact that the conditions \((\mu^3)^{Lt} \perp \mu^3 \) for all \( t \in \text{GL}(3, \mathbb{R}) \) \( \setminus \{e\} \) imply the possibility of the approximation of \( x_{kn} \) and \( D_{kn} \) by combinations of generators is based on some completely independent statement about three infinite vectors \( f, g, h \notin l_2 \) such that
\[
C_1f + C_2g + C_3h \notin l_2 \quad \text{for arbitrary } (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\},
\]
see Lemma 8.11 for \( m = 2 \) [28] and Lemma 8.15 for \( m = 3 \). A similar result for general \( m \) is studied in [29]. These lemmas are the key ingredients of the proof of the irreducibility of the representation.
Remark 1.5. Note that in the case of the “nilpotent group” $B_0^N$ and the infinite product of arbitrary Gaussian measures on $\mathbb{R}^m$ (see [2]) the proof of the irreducibility is also based on another completely independent statement namely, the Hadamard – Fischer inequality, see Lemma 1.3.

Lemma 1.3 (Hadamard – Fischer inequality [11, 12]). For any positive definite matrix $C \in \text{Mat}(m, \mathbb{R})$, $m \in \mathbb{N}$ and any two subsets $\alpha$ and $\beta$ with $\emptyset \subseteq \alpha, \beta \subseteq \{1, \ldots, m\}$ the following inequality holds:

$$\begin{vmatrix} M(\alpha) & M(\alpha \cap \beta) \\ M(\alpha \cup \beta) & M(\beta) \end{vmatrix} = \begin{vmatrix} A(\hat{\alpha}) & A(\hat{\alpha} \cup \hat{\beta}) \\ A(\hat{\alpha} \cap \hat{\beta}) & A(\beta) \end{vmatrix} \geq 0,$$  \hspace{1cm} (1.11)

where $M(\alpha) = M^\alpha(C)$, $A(\alpha) = A^\alpha(C)$ and $\hat{\alpha} = \{1, \ldots, m\} \setminus \alpha$.

For the details see [11, p.573] and [12, Chapter 2.5, problem 36]. In [2] the conditions of orthogonality $\mu^{L_t} \perp \mu$ with respect to the left action of the group $B(m, \mathbb{R})$ on $X^m$ were expressed as the divergence of some series, $S_{kn}^L(\mu) = \infty$, $1 \leq k < n \leq m$. Conditions on the measure $\mu$ for the variables $x_{kn}$ to be approximated by combinations of generators $A_{pq}$ were expressed in terms of the divergence of another series $\Sigma_{kn}$. The proof of the fact that conditions $S_{kn}^L(\mu) = \infty$, $1 \leq k < n \leq m$ imply the conditions $\Sigma_{kn} = \infty$, $1 \leq k < n \leq m$ was based on the Hadamard – Fischer inequality.

2. Some orthogonality problem in measure theory

2.1. General setting

Our aim now is to find the minimal generating set of conditions of the orthogonality $(\mu_{b,a}^m)^{L_t} \perp \mu_{b,a}^m$ for all $t \in \text{GL}(m, \mathbb{R}) \setminus \{e\}$. To be more precise, consider the following more general situation. Let $\alpha : G \to \text{Aut}(X)$ be a measurable action of a group $G$ on a measurable space $(X, \mu)$ with the following property: $\mu^\alpha_t \perp \mu$ for all $t \in G \setminus \{e\}$. Define a generating subset $G^\perp(\mu)$ in the group $G$ as follows:

if $\mu^\alpha_t \perp \mu$ for all $t \in G^\perp(\mu)$, then $\mu^\alpha_t \perp \mu$ for all $t \in G \setminus \{e\}$.  \hspace{1cm} (2.1)

Problem 2.1. Find a minimal generating subset $G^\perp(\mu)$ satisfying (2.1).

Definition 2.1. Recall that the probability measure $\mu$ on some $G$-space $X$ is called ergodic if any function $f \in L^1(X, \mu)$ with property $f(\alpha_t(x)) = f(x) \text{ mod } \mu$ is constant.
2.2. Orthogonality criteria \( \mu^t \perp \mu \) for \( t \in \text{GL}(2, \mathbb{R}) \setminus \{e\} \)

**Remark 2.1.** By Lemma 4.1 proved in [28] or Lemma 10.4.1 in [27] for \( m = 2 \) we conclude that the minimal generating set \( G_0^+(\mu) = \text{GL}(2, \mathbb{R})_0^+(\mu) \) (see Problem (2.1)) is reduced to the following subgroups, families and elements:

\[
\exp(tE_{12}) = I + tE_{12} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \exp(tE_{21}) = I + tE_{21} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},
\]

(2.2)

\[
\exp(tE_{12})P_1 = \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}, \quad \exp(tE_{21})P_2 = \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix},
\]

(2.3)

\[
\tau_-(\phi, s) = \begin{pmatrix} \cos \phi & s^2 \sin \phi \\ s^{-2} \sin \phi & -\cos \phi \end{pmatrix} = D_2(s) \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} D_2^{-1}(s)P_2.
\]

(2.4)

The families (2.2), (2.3) are one-parameter, the family (2.4) is two-parameter.

All elements are of order 2 except the elements in subgroups given in (2.2)!

It suffices to verify the conditions (2.2) only for some \( t \in \mathbb{R} \setminus \{0\} \).

The family \( \tau_-(\phi, s) \), actually, coincides with \( D_2(s)O(2)D_2^{-1}(s)P_2 \), where \( D_2(s) = \text{diag}(s, s^{-1}) \). All points \( t \) in (2.3) and all points \( (\phi, s) \) in (2.4) are essential, i.e., we can not remove any single point.

**Remark 2.2.** We note [16, Chapter V, § 8 Problems, 2, p. 147] that every element of \( \text{SL}(2, \mathbb{R}) \) is conjugate to at least one matrix of the form

\[
\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a \neq 0, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.
\]

**Remark 2.3.** Some elements \( a = gP_r \) in the set \( G_0^+(\mu) = \text{GL}(2, \mathbb{R})_0^+(\mu) \) are of order 2 (see Remark 2.1):

\[
a^2 = (gP_r)^2 = 1.
\]

(2.5)

This follows from the relation

\[
P_r gP_r = g^{-1}.
\]

(2.6)

To see this we note that if (2.6) holds, then we get (2.5):

\[
a^2 = (gP_r)^2 = gP_r gP_r = gg^{-1} = 1.
\]
For example, for \( g = \exp(tE_{12}) \) we get \( P_1 g P_1 = g^{-1} \), for \( g = \exp(tE_{21}) \) we get \( P_2 g P_2 = g^{-1} \) and for \( g = \tau(\phi, s) \) we get \( P_2 g P_2 = g^{-1} \). See (2.3) and (2.4) for details, where
\[
\tau_-(\phi, s) = \begin{pmatrix} \cos \phi & s^2 \sin \phi \\ s^{-2} \sin \phi & -\cos \phi \end{pmatrix}, \quad \tau(\phi, s) = \begin{pmatrix} \cos \phi & -s^2 \sin \phi \\ s^{-2} \sin \phi & \cos \phi \end{pmatrix}.
\]

We recall some useful lemmas from [28].

**Lemma 2.2.** For \( t = \left(t_{11}, t_{21}, t_{22}\right) \in \text{GL}(2, \mathbb{R}) \setminus \{e\} \) we have, if \( \det t > 0 \),
\[
(\mu^2_{(b,0)})^{L_t} \perp \mu^2_{(b,0)} \iff \sum_{n \in \mathbb{Z}} \left[(1 - |\det t|)^2 + (t_{11} - t_{22})^2 + \left(t_{12} \sqrt{\frac{b_{1n}}{b_{2n}}} + t_{21} \sqrt{\frac{b_{2n}}{b_{1n}}} \right)^2 \right] = \infty. \tag{2.7}
\]
If \( \det t < 0 \) we have
\[
(\mu^2_{(b,0)})^{L_t} \perp \mu^2_{(b,0)} \iff \sum_{n \in \mathbb{Z}} \left[(1 - |\det t|)^2 + (t_{11} + t_{22})^2 + \left(t_{12} \sqrt{\frac{b_{1n}}{b_{2n}}} - t_{21} \sqrt{\frac{b_{2n}}{b_{1n}}} \right)^2 \right] = \infty. \tag{2.8}
\]

**Lemma 2.3.** For \( t \in \text{GL}(2, \mathbb{R}) \setminus \{e\} \) we have
\[
(\mu^2_{(b,a)})^{L_t} \perp \mu^2_{(b,a)} \text{ if } |\det t| \neq 1.
\]
If \( \det t = 1 \), we have
\[
(\mu^2_{(b,a)})^{L_t} \perp \mu^2_{(b,a)} \iff \Sigma^+(t) := \Sigma^+_{1}(t) + \Sigma^+_{2}(t) = \infty.
\]
If \( \det t = -1 \), we have
\[
(\mu^2_{(b,a)})^{L_t} \perp \mu^2_{(b,a)} \iff \Sigma^-(t) := \Sigma^-_{1}(t) + \Sigma^-_{2}(t) = \infty,
\]
where
\[
\Sigma^+_{1}(t) = \sum_{n \in \mathbb{Z}} \left[(t_{11} - t_{22})^2 + \left(t_{12} \sqrt{\frac{b_{1n}}{b_{2n}}} + t_{21} \sqrt{\frac{b_{2n}}{b_{1n}}} \right)^2 \right],
\]
\[
\Sigma^-_{1}(t) = \sum_{n \in \mathbb{Z}} \left[(t_{11} + t_{22})^2 + \left(t_{12} \sqrt{\frac{b_{1n}}{b_{2n}}} - t_{21} \sqrt{\frac{b_{2n}}{b_{1n}}} \right)^2 \right],
\]
\[
\Sigma^+_2(t^{-1}) = \sum_{n \in \mathbb{Z}} \left[b_{1n} [(t_{11} - 1)a_{1n} + t_{12}a_{2n}]^2 + b_{2n} [t_{21}a_{1n} + (t_{22} - 1)a_{2n}]^2 \right]. \tag{2.9}
\]
**Remark 2.4.** By Lemma 2.3 we have

\[(\mu_{(b,a)}^2)^{L_t} \perp \mu_{(b,a)}^2 \quad \text{for} \quad t \in \text{GL}(2, \mathbb{R}) \setminus \{e\}\]

if and only if this holds for two subsets of the group \(\pm \text{SL}(2, \mathbb{R})\) defined as follows:

\[G^+_2 = \{ t \in \text{SL}(2, \mathbb{R}) \mid t_{11} = A^1_1(t) \}, \quad (2.10)\]

\[G^-_2 = \{ t \in -\text{SL}(2, \mathbb{R}) \mid t_{11} = -A^1_1(t) \}. \quad (2.11)\]

The set \(G^+_2\) is reduced to two families of one-parameter subgroups (2.2). The set \(G^-_2\) is reduced to the one-parameter family (2.3), the reflections of (2.2) by \(P_2\), and two parameter family (2.4) of elements from \(D_2\text{O}(2)D_2^{-1}(s)P_2\).

**Lemma 2.4.** If \(t \in G^+_2\) we have

\[(\mu_{(b,a)}^2)^{L_t} \perp \mu_{(b,a)}^2 \iff \Sigma^+(t) := \Sigma_1^+(t) + \Sigma_2(t) = \infty.\]

If \(t \in G^-_2\) we have

\[(\mu_{(b,a)}^2)^{L_t} \perp \mu_{(b,a)}^2 \iff \Sigma^-(t) := \Sigma_1^-(t) + \Sigma_2(t) = \infty,\]

where \(\Sigma_2(t^{-1})\) is defined by (2.9) and

\[\Sigma^+_1(t) = \sum_{n \in \mathbb{Z}} \left( t_{12} \sqrt{\frac{b_{1n}^2}{b_{2n}^2} + t_{21} \sqrt{\frac{b_{2n}^2}{b_{1n}^2}}} \right)^2, \quad (2.12)\]

\[\Sigma^-_1(t) = \sum_{n \in \mathbb{Z}} \left( t_{12} \sqrt{\frac{b_{1n}^2}{b_{2n}^2} - t_{21} \sqrt{\frac{b_{2n}^2}{b_{1n}^2}}} \right)^2. \quad (2.13)\]

The conditions of orthogonality with respect to elements defined by (2.2)--(2.4) are transformed in the divergence of the following series:

\[S_{12}^L(\mu) = \sum_{n \in \mathbb{Z}} \frac{b_{1n}^2}{2} \left( \frac{1}{2b_{2n}} + a_{2n}^2 \right), \quad S_{21}^L(\mu) = \sum_{n \in \mathbb{Z}} \frac{b_{2n}^2}{2} \left( \frac{1}{2b_{1n}} + a_{1n}^2 \right), \quad (2.14)\]

\[S_{kn}^L(\mu, t) = \frac{t^2}{4} \sum_{m \in \mathbb{Z}} \frac{b_{km}}{b_{nm}} + \sum_{m \in \mathbb{Z}} \frac{b_{km}}{2} (-2a_{km} + t a_{nm})^2, \quad t \in \mathbb{R}, \quad (2.15)\]

\[\Sigma_{12}(\tau_-(\phi, s)) = \sin^2 \phi \Sigma_1(s) + \Sigma_{2}^{-} (\tau_-(\phi, s)), \quad \phi \in [0, 2\pi), \quad s > 0, \quad (2.16)\]
where \( \Sigma_1(s) := \sum_{n \in \mathbb{Z}} \left( s^2 \sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2} \sqrt{\frac{b_{2n}}{b_{1n}}} \right)^2, \) \hspace{1cm} (2.17)

\[ \Sigma_2^-(\tau_-(\phi, s)) := \sum_{n \in \mathbb{Z}} \left( 4b_{1n} \sin^2 \frac{\phi}{2} + 4s^{-4}b_{2n} \cos^2 \frac{\phi}{2} \right) \left( a_{1n} \sin \frac{\phi}{2} - s^2a_{2n} \cos \frac{\phi}{2} \right)^2. \] \hspace{1cm} (2.18)

Recall Remark 4.3 from [28].

**Remark 2.5.** The following three conditions are equivalent:

1. (i) \( \mu_{L_{\tau_-(\phi, s)}} \perp \mu, \quad \phi \in [0, 2\pi), \quad s > 0, \)
2. (ii) \( \Sigma_1(\tau_-(\phi, s)) = \sin^2 \phi \Sigma_1(s) + \Sigma_2^-(\tau_-(\phi, s)) = \infty, \quad \phi \in [0, 2\pi), \quad s > 0, \)
3. (iii) \( \Sigma_1(s) + \Sigma_2(C_1, C_2) = \infty, \quad s > 0, \quad (C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}, \)

where \( \Sigma_1(s) \) is defined by (2.17) and

\[ \Sigma_2(C_1, C_2) := \sum_{n \in \mathbb{Z}} (C_1^2b_{1n} + C_2^2b_{2n})(C_1a_{1n} + C_2a_{2n})^2. \] \hspace{1cm} (2.19)

### 2.3. Equivalent series and equivalent sequences

There is an extensive theory of convergent and divergent series. In our context we are only interested in when a series with positive coefficients is divergent or convergent.

**Definition 2.2.** We say that two series \( \sum_{n \in \mathbb{N}} a_n \) and \( \sum_{n \in \mathbb{N}} b_n \) with positive \( a_n, b_n \) are equivalent if they are divergent or convergent simultaneously. We will write \( \sum_{n \in \mathbb{N}} a_n \sim \sum_{n \in \mathbb{N}} b_n \). We say that two sequences \( (a_n)_{n \in \mathbb{N}} \) and \( (b_n)_{n \in \mathbb{N}} \) are equivalent if for some \( C_1, C_2 > 0 \) we have \( C_1b_n \leq a_n \leq C_2b_n \) for all \( n \in \mathbb{N} \). We will use the same notation \( a_n \sim b_n \).

**Lemma 2.5.** Let \( 1 + c_n > 0 \) for all \( n \in \mathbb{Z} \). Then two series are equivalent:

\[ \Sigma_1 := \sum_{n \in \mathbb{Z}} \frac{c_n^2}{1 + c_n}, \quad \Sigma_2 := \sum_{n \in \mathbb{Z}} c_n^2. \] \hspace{1cm} (2.20)

**Proof.** Fix some \( \varepsilon \in (0, 1) \) and a big \( N \). We have three cases:

(a) \( 1 + c_n \in (\varepsilon, N), \)
(b) for infinite subset \( Z_1 \) we have \( \lim_{n \in Z_1} c_n = \infty, \)
(c) for infinite subset \( Z_1 \) we have \( \lim_{n \in Z_1} (1 + c_n) = 0. \)
In the case (a) we have

\[ \frac{1}{N} \sum_{n \in \mathbb{Z}} c_n^2 < \sum_{n \in \mathbb{Z}} \frac{c_n^2}{1 + c_n} < \frac{1}{\varepsilon} \sum_{n \in \mathbb{Z}} c_n^2, \]  

(2.21)

In the case (b) and (c) both series are divergent.

\[ \square \]

We will make systematic use of the following statement.

**Remark 2.6 ([27]).** Let \( a_n, b_n > 0 \) for all \( n \in \mathbb{N} \). The following two series are equivalent

\[ \sum_{n \in \mathbb{N}} \frac{a_n}{a_n + b_n} \sim \sum_{n \in \mathbb{N}} \frac{a_n}{b_n}. \]

(2.22)

### 2.4. Orthogonality criteria \( \mu^{\mu t} \perp \mu \) for \( t \in \text{GL}(3, \mathbb{R}) \setminus \{e\} \)

For \( m = 2 \) and \( \det t > 0 \) we have, here \( H_{m,n}(t) \) is defined by (8.8)

\[ 2^2 | \det t | (H_{2,n}^{-2}(t) - 1) = \left[(1 - | \det t |)^2 + (t_{11} - t_{22})^2 + \left(t_{12} \sqrt{\frac{b_{11}}{b_{22}}} + t_{21} \sqrt{\frac{b_{21}}{b_{11}}} \right)^2 \right] \]

\[ \left[ (M_0^0(X(t)) - A_0^0(X(t)))^2 + (M_1^1(X(t)) - A_1^1(X(t)))^2 + (M_2^2(X(t)) - A_2^2(X(t)))^2 \right]. \]

For \( m = 3 \) using (1.7) we have \( X(t) = B^{1/2} t B^{-1/2} \), hence

\[ X(t) = \begin{pmatrix} b_{1n} & 0 & 0 \\ 0 & b_{2n} & 0 \\ 0 & 0 & b_{3n} & \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} & \end{pmatrix} \begin{pmatrix} b_{1n} & 0 & 0 \\ 0 & b_{2n} & 0 \\ 0 & 0 & b_{3n} & \end{pmatrix}^{-1/2} = \]

\[ \begin{pmatrix} t_{11} & \sqrt{\frac{b_{1n}}{b_{2n}}} t_{12} & \sqrt{\frac{b_{1n}}{b_{3n}}} t_{13} \\ \sqrt{\frac{b_{2n}}{b_{1n}}} t_{21} & t_{22} & \sqrt{\frac{b_{2n}}{b_{3n}}} t_{23} \\ \sqrt{\frac{b_{3n}}{b_{1n}}} t_{31} & \sqrt{\frac{b_{3n}}{b_{2n}}} t_{32} & t_{33} & \end{pmatrix}. \]

Therefore, using (7.8) and the fact that \( X = X^*(t) \) we obtain

\[ 2^3 | \det t | H_{3,n}^{-2}(t) = \left(1 + | \det t |^2 + t_{11}^2 + b_{1n} t_{11}^2 + b_{1n} t_{13}^2 + b_{2n} t_{21}^2ight. \]

\[ + t_{22}^2 + b_{2n} t_{23}^2 + b_{3n} t_{31}^2 + b_{3n} t_{32}^2 + t_{33}^2 + (M_{12}^2(t))^2 + b_{2n} (M_{13}^2(t))^2 + \]

13
\[ \frac{b_{1n}}{b_{3n}} (M_{23}^{12}(t))^2 + \frac{b_{3n}}{b_{2n}} (M_{12}^{13}(t))^2 + (M_{13}^{13}(t))^2 + \frac{b_{1n}}{b_{2n}} (M_{23}^{13}(t))^2 \\
+ \frac{b_{3n}}{b_{1n}} (M_{12}^{23}(t))^2 + \frac{b_{2n}}{b_{2n}} (M_{13}^{23}(t))^2 + (M_{23}^{23}(t))^2 \]
\[ = 1 + |\det t|^2 + \sum_{1 \leq i \leq j \leq 3} \left[ \left( t^i_j \sqrt{\frac{b_{in}}{b_{jn}}} \right)^2 + \left( A^i_j \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 \right] \]
\[ = 1 + |\det t|^2 + \sum_{1 \leq i \leq j \leq 3} \left( |M_j^i(X(t))|^2 + |A_j^i(X(t))|^2 \right). \]

Using the notation \( t^i_j = t_{ij} \) and the fact
\[ \det t = t_1^k A_1^k + t_2^k A_2^k + t_3^k A_3^k, \quad k = 1, 2, 3, \]
we get
\[ 2^3 |\det t| (H_{3,n}^{-2}(t) - 1) = (1 - |\det t|)^2 + \sum_{1 \leq i \leq j \leq 3} \left( M_j^i(X(t)) - A_j^i(X(t)) \right)^2 \]
\[ = (1 - |\det t|)^2 + \sum_{1 \leq i \leq j \leq 3} \left( t^i_j \sqrt{\frac{b_{in}}{b_{jn}}} - A_j^i(t) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2. \quad (2.23) \]

Similar to [28, Lemmas 2.22] in the case \( m = 2 \), or [27, Lemma 10.4.30] we get the following lemma, for \( m = 3 \).

**Lemma 2.6.** For \( t \in \text{GL}(3, \mathbb{R}) \setminus \{e\} \) we have, if \( |\det t| > 0 \),
\[ (\mu_3^{(b,0)})^{L_t} \perp \mu_3^{(b,0)} \iff \]
\[ \sum_{n \in \mathbb{Z}} \left[ (1 - |\det t|)^2 + \sum_{1 \leq i \leq 3} \left( t^i_i - A^i_i(t) \right)^2 + \sum_{1 \leq i < j \leq 3} \left( t^i_j \sqrt{\frac{b_{in}}{b_{jn}}} - A_j^i(t) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 \right] = \infty. \quad (2.24) \]

If \( |\det t| < 0 \) we have
\[ (\mu_3^{(b,0)})^{L_t} \perp \mu_3^{(b,0)} \iff \]
\[ \sum_{n \in \mathbb{Z}} \left[ (1 - |\det t|)^2 + \sum_{1 \leq i \leq 3} \left( t^i_i + A^i_i(t) \right)^2 + \sum_{1 \leq i < j \leq 3} \left( t^i_j \sqrt{\frac{b_{in}}{b_{jn}}} + A_j^i(t) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 \right] = \infty. \quad (2.25) \]
By Lemma 8.6 and (8.9) the following lemma holds true.

**Lemma 2.7.** For \( t \in \text{GL}(3, \mathbb{R}) \setminus \{e\} \) we have

\[
(\mu_{(b,a)}^3)^L \perp \mu_{(b,a)}^3 \quad \text{if} \quad | \det t | \neq 1.
\]

If \( \det t = 1 \), we have

\[
(\mu_{(b,a)}^3)^L \perp \mu_{(b,a)}^3 \quad \iff \quad \Sigma^+(t) := \Sigma_1^+(t) + \Sigma_2(t) = \infty.
\]

If \( \det t = -1 \), we have

\[
(\mu_{(b,a)}^3)^L \perp \mu_{(b,a)}^3 \quad \iff \quad \Sigma^-(t) := \Sigma_1^-(t) + \Sigma_2(t) = \infty,
\]

where

\[
\Sigma_1^+(t) = \sum_{n \in \mathbb{Z}} \left[ \sum_{k=1}^{3} (t_{kk} - A_k^k(t))^2 + \sum_{1 \leq i < j \leq 3} \left( t_{ij}^i \sqrt{\frac{b_{in}}{b_{jn}}} - A_{ij}^i(t) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 \right], \tag{2.26}
\]

\[
\Sigma_1^-(t) = \sum_{n \in \mathbb{Z}} \left[ \sum_{k=1}^{3} (t_{kk} + A_k^k(t))^2 + \sum_{1 \leq i < j \leq 3} \left( t_{ij}^i \sqrt{\frac{b_{in}}{b_{jn}}} + A_{ij}^i(t) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 \right], \tag{2.27}
\]

\[
\Sigma_2(t^{-1}) = \sum_{n \in \mathbb{Z}} \left[ b_{1n} \left( (t_{11} - 1)a_{1n} + t_{12}a_{2n} + t_{13}a_{3n} \right)^2 \right. + \left. b_{2n} \left( t_{21}a_{1n} + (t_{22} - 1)a_{2n} + t_{23}a_{3n} \right)^2 + b_{3n} \left( t_{31}a_{1n} + t_{32}a_{2n} + (t_{33} - 1)a_{3n} \right)^2 \right]. \tag{2.28}
\]

**Remark 2.7.** By Lemma 2.7, it suffices to verify, the condition of orthogonality

\[
(\mu_{(b,a)}^3)^L \perp \mu_{(b,a)}^3 \quad \text{for} \quad t \in \text{GL}(3, \mathbb{R}) \setminus \{e\}
\]

for the following two subsets of the group \( \pm \text{SL}(3, \mathbb{R}) \):

\[
G_3^+ := \{ t \in \text{SL}(3, \mathbb{R}) \mid t_{kk} = A_k^k(t), \ 1 \leq k \leq 3 \}, \tag{2.29}
\]

\[
G_3^- := \{ t \in -\text{SL}(3, \mathbb{R}) \mid t_{kk} = -A_k^k(t), \ 1 \leq k \leq 3 \}. \tag{2.30}
\]
Lemma 2.8. If $t \in G_3^\pm$, we have respectively

$$(\mu_{(b,a)}^3)^L \perp \mu_{(b,a)}^3 \iff \sum^\pm(t) = \sum^+_1(t) + \sum^+_2(t) = \infty,$$

$$\sum^+_1(t) = \sum_{1 \leq i < j \leq 3} \sum_{n \in \mathbb{Z}} \left( t_i \sqrt{\frac{b_{in}}{b_{jn}}} - A_j(t) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 = \sum_{1 \leq i < j \leq 3} \sum^+_ij(t),$$

(2.31)

$$\sum^-_1(t) = \sum_{1 \leq i < j \leq 3} \sum_{n \in \mathbb{Z}} \left( t_i \sqrt{\frac{b_{in}}{b_{jn}}} + A_j(t) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 = \sum_{1 \leq i < j \leq 3} \sum^-_ij(t),$$

(2.32)

$$\sum^\pm_1(t) = \sum_{n \in \mathbb{Z}} \left( t_j \sqrt{\frac{b_{jn}}{b_{jn}}} \mp A_j(t) \sqrt{\frac{b_{jn}}{b_{jn}}} \right)^2,$$

(2.33)

where $\sum_2(t)$ is defined by (2.28).

Remark 2.8. We note that the Iwasawa decomposition holds for $SL(3, \mathbb{R})$, i.e., $SL(3, \mathbb{R}) = KAN$, where $K = O(3)$,

$$A = \left\{ D_3(s) = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}, \det D_3(s) = 1 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} | x, y, z \in \mathbb{R} \right\}.$$  

(2.34)

Next we will show that the set $G_3^+$ can be reduced to the six family of one-parameter subgroups $\exp(tE_{kn})$, $1 \leq k \neq r \leq 3$, see (2.36), or three families of two-parameter subgroups, see (2.37). The set $G_3^-$ can be reduced to the three two-parameter family (2.38) reflections of (2.37) by $P_r$. The remaining part is reduced to the sets $D_3(s)O(3)D_3^{-1}(s)P_r$ or five parameter family of elements $\tau_r, s = D_3(s)tD_3^{-1}(s)P_r$, see (2.42).

Lemma 2.9. In case $m = 3$ the minimal generating set $GL(3, \mathbb{R})^\perp_0(\mu)$ is defined as follows (compare with Remark 2.1):

$$GL(3, \mathbb{R})^\perp_0(\mu) = \{ e_r(t, s) = e_r(t, s)P_r, 1 \leq r \leq 3, (t, s) \in \mathbb{R}^2 \} \cup \{ O^A_r(3), 1 \leq r \leq 3 \},$$

where

(2.35)

$$e_{kn}(t) := \exp(tE_{kn}) = I + tE_{kn}, 1 \leq k \neq n \leq 3, t \in \mathbb{R},$$

(2.36)

$$e_1(t, s) = \begin{pmatrix} 1 & t & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2(t, s) = \begin{pmatrix} 1 & 0 & s \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_3(t, s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & s & 1 \end{pmatrix},$$

(2.37)
The families (2.36) give us, respectively, the divergence of the following series:

\[ e_1(t, s)P_1 = \begin{pmatrix} -1 & t & s \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2(t, s)P_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{t} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_3(t, s)P_3 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{t} & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \] (2.38)

\[ P_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \] (2.39)

\[ O^A(3) := \{ D_3(s)O(3)D_3^{-1}(s) \mid D_3(s) \in A \}, \] (2.40)

\[ O_r^A(3) := \{ D_3(s)O(3)D_3^{-1}(s)P_r \mid D_3(s) \in A \}, \quad 1 \leq r \leq 3, \] (2.41)

\[ \tau_r(t, s) := D_3(s)tD_3^{-1}(s)P_r, \quad t \in O(3), \quad D_3(s) = \text{diag}(s_1, s_2, s_3) \in A, \] (2.42)

and \( A \) is defined by (2.34).

The families (2.36) give us respectively the divergence of the following series:

\[ S_{kr}^L(\mu) = \sum_{n \in \mathbb{Z}} \frac{b_{kn}}{2} \left( \frac{1}{2b_{rn}} + a_{rn}^2 \right), \quad 1 \leq k, r \leq 3, \quad k \neq r. \] (2.43)

The families (2.37) give us, respectively, the divergence of the following series:

\[ S_{1,23}^L(\mu, t, s) = \sum_{n \in \mathbb{Z}} \left[ \frac{t^2 b_{1n}}{4 b_{2n}} + \frac{s^2 b_{1n}}{4 b_{3n}} + \frac{b_{1n}}{2} \left( -2a_{1n} + ta_{2n} + sa_{3n} \right)^2 \right], \] (2.44)

\[ S_{2,13}^L(\mu, t, s) = \sum_{n \in \mathbb{Z}} \left[ \frac{t^2 b_{2n}}{4 b_{1n}} + \frac{s^2 b_{2n}}{4 b_{3n}} + \frac{b_{2n}}{2} \left( ta_{1n} - 2a_{2n} + sa_{3n} \right)^2 \right], \] (2.45)

\[ S_{3,12}^L(\mu, t, s) = \sum_{n \in \mathbb{Z}} \left[ \frac{t^2 b_{3n}}{4 b_{1n}} + \frac{s^2 b_{3n}}{4 b_{2n}} + \frac{b_{3n}}{2} \left( ta_{1n} + sa_{2n} - 2a_{3n} \right)^2 \right]. \] (2.46)

The families (2.42) give us the conditions (2.50), see Lemma 2.10 below.

**Proof.** Consider the subset \( \text{GL}(3, \mathbb{R})^0_0(\mu) \) of \( \text{GL}(3, \mathbb{R}) \) described by (2.35). The fact that this set is minimal generating will follow from Lemma 4.1, more precisely, from the following implications:

\[ \left( \mu^L_t \perp \mu \text{ for all } t \in \text{GL}(3, \mathbb{R})^+_0(\mu) \right) \Rightarrow \text{irreducibility} \] (2.47)

\[ \Rightarrow \left( \mu^L_t \perp \mu \text{ for all } t \in \text{GL}(3, \mathbb{R}) \setminus \{e\} \right). \]

The first implication is just Lemma 4.1. The second implication follows from irreducibility. Indeed, suppose that \( \text{GL}(3, \mathbb{R})^+_0(\mu) \) is not a minimal generating set. Then we can find an \( s \in \text{GL}(3, \mathbb{R}) \setminus \{e\} \) having the property

\[ (\mu_{(b,a)}^3)^{L_s} \sim \mu_{(b,a)}^3. \]
Hence the non-trivial operator $T_{s}^{L,\mu:3}$ can be defined by
\[
(T_{s}^{L,\mu:3} f)(x) = (d\mu_{(b,a)}(s^{-1}x)/d\mu_{(b,a)}(x))^{1/2} f(s^{-1}x), \quad f \in L^{2}(X_{3}, \mu_{(b,a)}^{3}).
\]  
This operator commutes with the representations $T^{R,\mu:3}$.

\[
[T_{t}^{R,\mu:3}, T_{s}^{L,\mu:3}] = 0 \quad \text{for all} \quad t \in G,
\]
contradicting the irreducibility.

The relations (2.43)–(2.46) follows from (2.26)–(2.28). The relation (2.45), for example, follows from (2.27) and (2.28). The relation (2.50) we obtain from (2.26) for $\tau_{r}(t, s), \quad t \in O(3), \quad s \in (\mathbb{R}^{*})^{3}$ defined by
\[ \tau_{r}(t, s) = D_{3}(t)D_{3}^{-1}(s)P_{r}, \quad \text{where} \quad D_{3}(s) = \text{diag}(s_{1}, s_{2}, s_{3}). \] (2.49)

\[ \square \]

**Lemma 2.10.** Set
\[ \tau(s, t) := D_{3}(s)D_{3}^{-1}(s), \quad \tau_{r}(s, t) := \tau(s, t)P_{r} \]
for $t \in \pm O(3), \quad D_{3}(s) = \text{diag}(s_{1}, s_{2}, s_{3}), \quad s = (s_{1}, s_{2}, s_{3}) \in (\mathbb{R}^{*})^{3}$ and $1 \leq r \leq 3$. Then
\[ (\mu_{(b,a)})^{L_{\tau}(s,t)} \perp \mu_{(b,a)} \Leftrightarrow \Sigma_{1}^{+}(\tau_{r}(s, t)) + \Sigma_{2}(\tau_{r}(s, t)) = \infty, \] (2.50)
where $\Sigma_{1}^{+}(t)$ are defined by (2.31), (2.32) and $\Sigma_{2}(t)$ is defined by (2.28).

In particular, if we denote $s_{ij} = s_{i}s_{j}^{-1}$ we get
\[ \Sigma_{1}^{+}(\tau(t, s)) = \Sigma_{1}^{+}(t, s) = t_{1}^{2}\Sigma_{12}(s_{12}^{1/2}) + t_{3}^{2}\Sigma_{13}(s_{13}^{1/2}) + t_{23}^{2}\Sigma_{23}(s_{23}^{1/2}). \] (2.51)

**Proof.** For $T := \tau(s, t)$ and $T(3) := \tau_{3}(s, t)$ we have respectively:
\[ T = D_{3}(s)D_{3}^{-1}(s), \quad \left( \begin{array}{ccc} t_{11} & \frac{t_{12}}{t_{2}} & \frac{t_{13}}{t_{3}} \\ \frac{t_{21}}{t_{1}} & t_{22} & \frac{t_{23}}{t_{3}} \\ \frac{t_{31}}{t_{1}} & \frac{t_{32}}{t_{2}} & t_{33} \end{array} \right) \] (2.52),
\[ \left( \begin{array}{ccc} t_{11} & \frac{t_{12}}{t_{2}} & \frac{t_{13}}{t_{3}} \\ \frac{t_{21}}{t_{1}} & t_{22} & \frac{t_{23}}{t_{3}} \\ \frac{t_{31}}{t_{1}} & \frac{t_{32}}{t_{2}} & t_{33} \end{array} \right) = D_{3}(s)D_{3}^{-1}(s)P_{3} =: T(3). \] (2.53)
By Lemma 2.11 we have for $t \in O(3)$
\[ t_{kr} = A_{r}^{k}(t), \quad 1 \leq k, r \leq 3. \] (2.54)
Therefore, for $T$ and $T(3)$ we have for $1 \leq k, r \leq 3$:

$$M_r^k(T) = T_{kr} = \frac{s_k}{s_r} t_{kr}, \quad A_r^k(T) = \frac{s_r}{s_k} A_r^k(t) \quad (2.57) \quad M_r^k(T(3)) = \frac{s_r}{s_k} t_{kr}, \quad M_r^k(T(3)) \quad (2.55)$$

Finally, we get

$$\Sigma_1^+(T) = \Sigma_1^+(\tau(s, t)) = \sum_{n \in \mathbb{Z}} \left[ \sum_{1 \leq i < j \leq 3} \left( M_{ij}^j(T) \sqrt{\frac{b_{in}}{b_{jn}}} - A_{ij}^j(T) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2 \right]$$

$$= \sum_{n \in \mathbb{Z}} \left[ t_{12}^2 \left( s_{12} \sqrt{\frac{b_{1n}}{b_{2n}}} - s_{12}^{-1} \sqrt{\frac{b_{2n}}{b_{1n}}} \right)^2 + t_{13}^2 \left( s_{13} \sqrt{\frac{b_{1n}}{b_{3n}}} - s_{13}^{-1} \sqrt{\frac{b_{3n}}{b_{1n}}} \right)^2 + t_{23}^2 \left( s_{23} \sqrt{\frac{b_{2n}}{b_{3n}}} - s_{23}^{-1} \sqrt{\frac{b_{3n}}{b_{2n}}} \right)^2 \right] = t_{12}^2 \Sigma_{12}(s_{12}^{1/2}) + t_{13}^2 \Sigma_{13}(s_{13}^{1/2}) + t_{23}^2 \Sigma_{23}(s_{23}^{1/2}).$$

$$\Sigma_1^-(T(3)) = t_{12}^2 \Sigma_{12}(s_{12}^{1/2}) + t_{13}^2 \Sigma_{13}(s_{13}^{1/2}) + t_{23}^2 \Sigma_{23}(s_{23}^{1/2}) \quad \square$$

**Lemma 2.11.** For an arbitrary orthogonal matrix $t \in \pm O(3)$ we have

$$t_{kn} = \pm A_n^k(t), \quad 1 \leq k, n \leq 3, \quad \text{where} \quad t = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}. \quad (2.57)$$

**Proof.** Denote the three rows of the matrix $t$ by, respectively, $t_1, t_2, t_3 \in \mathbb{R}^3$. Since $t \in \pm O(3)$ we get

$$\|t_1\|^2 = \|t_2\|^2 = \|t_3\|^2 = 1 \quad \text{and} \quad t_l \perp t_r, \quad l \neq r. \quad (2.58)$$

Moreover, since $t_1$ is orthogonal to the hyperplane $V_{23}$ generated by the vectors $t_2$ and $t_3$ and $t \in \pm O(3)$ we get respectively $t_i = \pm [t_r, t_s]$, where $[x, y]$ is the vector product or cross product of two vectors $x, y \in \mathbb{R}^3$ and the triple $\{l, r, s\}$ denotes any cyclic permutations of $\{1, 2, 3\}$. For $t \in O(3)$ and $l = 1$ we get

$$t_1 = [t_2, t_3] = \begin{vmatrix} i & j & k \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix} = i \left| t_{22} t_{33} - t_{23} t_{32} \right| - j \left| t_{21} t_{33} - t_{23} t_{31} \right| + k \left| t_{21} t_{32} - t_{22} t_{31} \right|, \quad (2.59)$$

where $i, j, k$ is the standard orthonormal basis in $\mathbb{R}^3$, i.e.,

$$i = (1, 0, 0), \quad j = (0, 1, 0), \quad k = (0, 0, 1).$$

19
Define $X$ formally as the matrix:

$$X = \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$ 

Then

$$t_1 = (t_{11}, t_{12}, t_{13}) = (A_{11}^1(X), A_{11}^2(X), A_{11}^3(X)),$$

thus proving (2.57) for $k = 1$. For other rows the proof is similar. \qed

**Remark 2.9.** For $t \in \pm O(n)$ we can prove a similar statement.

3. **Irreducibility, the cases $m = 1$ and $m = 2$**

For convenience of the reader, we recall the previous results (see details in [28]).

3.1. **Case $m = 1$**

Let us denote by $\langle f_n \mid n \in \mathbb{N} \rangle$ the closure of the linear space generated by the set of vectors $(f_n)_{n \in \mathbb{N}}$ in a Hilbert space $H$. Consider the measure $\mu_{(b,a)}$ on the space $X_1 \sim \mathbb{R}^{\mathbb{Z}} = \otimes_{n \in \mathbb{Z}} \mathbb{R}$, the infinite product of the real lines:

$$d\mu_{(b,a)}(x) = \otimes_{n \in \mathbb{Z}} \sqrt{\frac{b_{1n}}{\pi}} e^{-b_{1n}(x_{1n} - a_{1n})^2} dx_{1n},$$

for $b = (b_{1n})_{n \in \mathbb{Z}}$ and $a = (a_{1n})_{n \in \mathbb{Z}}$ with $b_{1n} > 0$, $a_{1n} \in \mathbb{R}$ where $x = (x_{1n})_{n \in \mathbb{Z}}$.

In the case $m = 1$ the generators $A_{kn} := A_{R}^{1,1}$ have the form

$$A_{kn} = x_{1k} D_{1n}, \quad k, n \in \mathbb{Z},$$

where $D_{kn} = \frac{\partial}{\partial x_{kn}} - b_{kn}(x_{kn} - a_{kn})$. The following lemmas are proved in [1]

**Lemma 3.1.** The following three conditions are equivalent:

(i) $(\mu_{(b,a)})^L \perp \mu_{(b,a)}$ for all $t \in \text{GL}(1, \mathbb{R}) \setminus \{e\}$,

(ii) $(\mu_{(b,a)})^{L - E_{11}} \perp \mu_{(b,a)}$,

(iii) $S_{11}^L(\mu) = 4 \sum_{n \in \mathbb{Z}} b_{1n} a_{1n}^2 = \infty$.

**Lemma 3.2.** For $k, m \in \mathbb{Z}$ we have

$$x_{1k} x_{1m} 1 \in \langle A_{kn} A_{mn} 1 = x_{1k} x_{1m} D_{1n}^2 1 \mid n \in \mathbb{Z} \rangle.$$
Lemma 3.3. For any $k \in \mathbb{Z}$ we have

$$x_{1k}1 \in \langle x_{1k}x_{1n}1 \mid n \in \mathbb{Z} \rangle \iff S^L_{11}(\mu) = \infty.$$ 

So, the operators $x_{1k}$, $k \in \mathbb{Z}$ are affiliated with the von Neumann algebra $\mathfrak{A}^1$ generated by the representation, which completes the proof of the irreducibility for $m = 1$.

Definition 3.1. Recall (see [8]) that, a not necessarily bounded self-adjoint operator $A$ in a Hilbert space $H$, is said to be affiliated with a von Neumann algebra $M$ of operators in this Hilbert space $H$ if $e^{itA} \in M$ for all $t \in \mathbb{R}$. One writes $A \in M$.

3.2. Case $m = 2$, approximation of $x_{kn}$ and $D_{kn}$

In this case the generators $A_{kn} := A^R_{kn} := \frac{d}{dt} e^{R \mu_2 t} |_{t=0}$ have the form:

$$A_{kn} = x_{1k}D_{1n} + x_{2k}D_{2n}, \quad k, n \in \mathbb{Z}.$$ 

We will formulate several useful lemmas for approximation of the operators of multiplication by the independent variables $x_{kn}$ and operators $D_{kn}$ by combinations of the generators $A_{kn}$. In what follows we use the following notation for $f, g \in \mathbb{R}^m$ (see Remark 2.6 for notations $\sim$)

$$\Delta(f, g) := \frac{\Gamma(f) + \Gamma(f, g) + \Gamma(g)}{\Gamma(g) + 1} = \frac{\det(I + \gamma(f, g))}{\det(I + \gamma(g))}, \quad (3.1)$$

where $\Gamma(x_1, \ldots, x_n)$ is the Gram determinant of vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^m$. To make the notations of the article [28] compatible with the notations in the case $m = 3$ (see (4.5)), we denote

$$\|Y^{(1)}_1\|^2 := \|f_1\|^2 := \sum_{k \in \mathbb{Z}} \frac{b_{1k}^2}{b_{1k}^2 + 2b_{1k}b_{2k}}, \quad \|Y^{(1)}_2\|^2 := \|g_1\|^2 := \sum_{k \in \mathbb{Z}} \frac{b_{2k}^2}{b_{1k}^2 + 2b_{1k}b_{2k}},$$

$$\|Y^{(2)}_1\|^2 := \|f_2\|^2 := \sum_{k \in \mathbb{Z}} \frac{b_{2k}^2}{b_{2k}^2 + 2b_{1k}b_{2k}}, \quad \|Y^{(2)}_2\|^2 := \|g_2\|^2 := \sum_{k \in \mathbb{Z}} \frac{b_{1k}^2}{b_{2k}^2 + 2b_{1k}b_{2k}},$$

$$\|Y_1\|^2 := \|f\|^2 := \sum_{k \in \mathbb{Z}} \frac{a_{1k}^2}{2b_{1k}^2 + 2b_{2k}^2}, \quad \|Y_2\|^2 := \|g\|^2 := \sum_{k \in \mathbb{Z}} \frac{a_{2k}^2}{2b_{1k}^2 + 2b_{2k}^2}. \quad (3.2)$$

Lemma 3.4. For any $k, t \in \mathbb{Z}$ one has

$$x_{1k}x_{1t} \in \langle A_{kn}A_{tn}1 \mid n \in \mathbb{Z} \rangle \iff \Delta(Y^{(1)}_1, Y^{(1)}_2).$$

21
Lemma 3.5. For any \( k, t \in \mathbb{Z} \) we have
\[
x_{2k} x_{2t} \in \langle A_{kn} A_{tn} 1 \mid n \in \mathbb{Z} \rangle \iff \Delta(Y_{2}^{(2)}, Y_{1}^{(2)}) = \infty.
\]

Lemma 3.6. Set \( \Sigma^r_s = \sum_{n \in \mathbb{Z}} \frac{b_{kn}}{b_{rn}}, \ 1 \leq r, s \leq 2 \). For any \( k \in \mathbb{Z} \) we get
\[
x_{1k} 1 \in \langle D_{1n} A_{kn} 1 \mid n \in \mathbb{Z} \rangle \iff \Sigma_{12}^1 = \infty.
\]

Lemma 3.7. For any \( k \in \mathbb{Z} \) we have
\[
x_{2k} 1 \in \langle D_{2n} A_{kn} 1 \mid n \in \mathbb{Z} \rangle \iff \Sigma_{21}^1 = \infty.
\]

Lemma 3.8. For any \( n \in \mathbb{Z} \) we have
\[
D_{1n} 1 \in \langle A_{kn} 1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_{1}, Y_{2}) = \infty.
\]

Lemma 3.9. For any \( n \in \mathbb{Z} \) we have
\[
D_{2n} 1 \in \langle A_{kn} 1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_{2}, Y_{1}) = \infty.
\]

3.2.1. Technical part of the proof of irreducibility

Lemma 3.10. If \( \mu^{L_{t}} \perp \mu \) for all \( t \in \text{GL}(2, \mathbb{R}) \setminus \{e\} \), we can approximate by combinations of generators at least one of the following pairs of operators: \((x_{1n}, x_{2n}), (x_{1n}, D_{2n}), (D_{1n}, x_{2n}) \) or \((D_{1n}, D_{2n})\).

PROOF. Recall the orthogonality conditions for the case \( m = 2 \)
\[
S_{kr}^L (\mu) = \sum_{n \in \mathbb{Z}} \frac{b_{kn}}{2} \left( \frac{1}{2b_{rn}} + a_{rn}^2 \right), \quad 1 \leq k, r \leq 2, \ k \neq r,
\]
\[
S_{kr}^L (\mu, t) = \frac{t^2}{4} \sum_{n \in \mathbb{Z}} \frac{b_{kn}}{b_{rn}} + \sum_{n \in \mathbb{Z}} \frac{b_{kn}}{2} (-2a_{kn} + ta_{rn})^2, \quad 1 \leq k \neq r \leq 2,
\]
\[
\Sigma_{12}^r (\tau_{-}(\phi, s)) = \sin^2 \phi \Sigma_{1}(s) + \Sigma_{2}^{r} (\tau_{-}(\phi, s)), \quad \text{where}
\]
\[
\Sigma_{1}(s) := \sum_{n \in \mathbb{Z}} \left( s^2 \sqrt{\frac{b_{kn}}{b_{2n}}} - s^2 \sqrt{\frac{b_{kn}}{b_{1n}}} \right)^2,
\]
\[
\Sigma_{2}^{r} (\tau_{-}(\phi, s)) = \sum_{n \in \mathbb{Z}} \left( 4b_{1n} \sin^2 \frac{\phi}{2} + 4s^{-2} b_{2n} \cos^2 \frac{\phi}{2} \right) \left( a_{1n} \sin \frac{\phi}{2} - s^2 a_{2n} \cos \frac{\phi}{2} \right)^2.
\]

Let \( \mathfrak{A}^2 \) be the von Neumann algebra generated by our representation. In order to approximate operators \( x_{kn} \) or \( D_{kn} \) by the corresponding generators, by Lemmas 3.4–3.5, Lemmas 3.2–3.2 and Lemmas 3.8–3.9 we have:
\[
x_{1n} x_{1t} \eta \mathfrak{A}^2 \iff \Delta(Y_{1}^{(1)}, Y_{2}^{(1)}) = \infty, \quad x_{2n} x_{2t} \eta \mathfrak{A}^2 \iff \Delta(Y_{2}^{(2)}, Y_{1}^{(2)}) = \infty,
\]
\[
D_{1n} \eta \mathfrak{A}^2 \iff \Delta(Y_{1}, Y_{2}) = \infty, \quad D_{2n} \eta \mathfrak{A}^2 \iff \Delta(Y_{2}, Y_{1}) = \infty,
\]
where \( Y_{r}^{(s)} \) and \( Y_{r} \) for \( 1 \leq r, s \leq 2 \) are defined by (3.2). \( \square \)
3.2.2. Scheme of the proof for two lines

There are two different cases:

I. Approximation of $x_{rk}x_{rt}$ for $1 \leq r \leq 2$ by $A_{kn}A_{tn}$,

II. Approximation of $D_{rk}$ for $1 \leq r \leq 2$ by $A_{kn}$.

In the case $m = 2$ the analysis of the divergence

$$\Delta(Y_{1(1)}, Y_{2(1)}) = \infty \quad \text{and} \quad \Delta(Y_{2(2)}, Y_{1(2)}) = \infty$$

is governed by the convergence or divergence of $\Sigma_{12}$ and $\Sigma_{21}$, since

$$\|Y_{1(1)}\|^2 = \sum_{k \in Z} \frac{b_{1k}^2}{b_{1k}^2 + 2b_{1k}b_{2k}} \sim \frac{1}{2} \sum_{k \in Z} \frac{b_{1k}^2}{b_{1k}b_{2k}} = \frac{1}{2} \Sigma_{12},$$

$$\|Y_{2(2)}\|^2 = \sum_{k \in Z} \frac{b_{2k}^2}{b_{2k}^2 + 2b_{1k}b_{2k}} \sim \frac{1}{2} \sum_{k \in Z} \frac{b_{2k}^2}{b_{1k}b_{2k}} = \frac{1}{2} \Sigma_{21}.$$ 

**Remark 3.1.** To guess the right generalisation for the case $m = 3$ we note that $\Sigma_{12} = S_1(2)$ and $\Sigma_{21} = S_2(2)$, where we denote

$$S_1(2) := \sum_{k \in Z} \frac{b_{1k}^2}{b_{1k}b_{2k}}, \quad S_2(2) := \sum_{k \in Z} \frac{b_{2k}^2}{b_{1k}b_{2k}}, \quad \Sigma_{12} := \sum_{k \in Z} \frac{b_{1k}}{b_{2k}}, \quad \Sigma_{21} := \sum_{k \in Z} \frac{b_{2k}}{b_{1k}}.$$ 

We also observe that

$$\|Y_{1(1)}\|^2 \sim S_1(2), \quad \|Y_{2(2)}\|^2 \sim S_2(2). \quad (3.3)$$

Hence, in the case $m = 3$ it is natural to replace $\Sigma_{12} = S_1(2)$ and $\Sigma_{21} = S_2(2)$ by $S_r(3)$, which is defined as follows:

$$S_r(3) = \sum_{n \in Z} \frac{b_{rn}^2}{b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}}, \quad 1 \leq r \leq 3. \quad (3.4)$$

3.2.3. I, Approximation of $x_{1n}$ and $x_{2n}$

**Lemma 3.11.** We have

$$S_1(2) + S_2(2) = \infty, \quad (3.5)$$

$$\|Y_r^{(r)}\|^2 \sim S_r(2) \quad \text{for all} \quad 1 \leq r \leq 2, \quad (3.6)$$

$$\|Y_{1(2)}\|^2 < \frac{1}{2} S_1(2), \quad \|Y_{2(1)}\|^2 < \frac{1}{2} S_2(2), \quad (3.7)$$

$$\|Y_{1(i)}\|^2 + \|Y_{2(j)}\|^2 = \infty, \quad i, j \in \{1, 2\}. \quad (3.8)$$
Proof. Since \(a^2 + b^2 \geq 2ab\) we get

\[ S_1(2) + S_2(2) = \sum_{k \in \mathbb{Z}} \frac{b_{1k}^2 + b_{2k}^2}{b_{1k}b_{2k}} \geq \sum_{k \in \mathbb{Z}} 2 = \infty. \]

Further, for \(1 \leq r \leq 2\)

\[ \left\| Y_r^{(r)} \right\|^2 = \sum_{k \in \mathbb{Z}} \frac{b_{rk}^2}{b_{rk}^2 + 2b_{1k}b_{2k}} \sim \sum_{k \in \mathbb{Z}} \frac{b_{rk}^2}{2b_{1k}b_{2k}} = \frac{1}{2} S_r(2), \]

\[ \left\| Y_1^{(2)} \right\|^2 = \sum_{k \in \mathbb{Z}} \frac{b_{1k}^2}{b_{1k}^2 + 2b_{1k}b_{2k}} < \sum_{k \in \mathbb{Z}} \frac{b_{1k}^2}{2b_{1k}b_{2k}} = \frac{1}{2} S_1(2), \]

\[ \left\| Y_2^{(1)} \right\|^2 = \sum_{k \in \mathbb{Z}} \frac{b_{2k}^2}{b_{2k}^2 + 2b_{1k}b_{2k}} < \sum_{k \in \mathbb{Z}} \frac{b_{2k}^2}{2b_{1k}b_{2k}} = \frac{1}{2} S_2(2). \]

In addition, we have

\[ \left\| Y_1^{(i)} \right\|^2 + \left\| Y_2^{(j)} \right\|^2 = \sum_{k \in \mathbb{Z}} \frac{b_{1k}^2}{b_{1k}^2 + 2b_{1k}b_{2k}} + \sum_{k \in \mathbb{Z}} \frac{b_{2k}^2}{b_{2k}^2 + 2b_{1k}b_{2k}} \]

\[ > \sum_{k \in \mathbb{Z}} \frac{b_{1k}^2 + b_{2k}^2}{(b_{1k} + b_{2k})^2} = \infty. \]

Remark 3.2. In what follows if some expression \(< \infty\) (resp. \(= \infty\)) we denote this case by 0 (respectively, by 1).

Set \(S = (S_1(2), S_2(2))\), since \(S_1(2) + S_2(2) = \infty\) we have two cases:

I(1) \(S = (0, 1)\), if \(S = (1, 0)\) we interchange \((b_{1n}, a_{1n})_n\) with \((b_{2n}, a_{2n})_n\),

I(2) \(S = (1, 1)\), i.e., \(S_1(2) = \infty, S_2(2) = \infty\).

3.2.4. Case \(S = (0, 1)\)

Lemma 3.12. In the case \(S = (0, 1)\) the representation is irreducible, moreover we can approximate:

1. \(x_{2k}x_{2t}\) by \(A_{kn}A_{tn}\), since \(\Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty\),
2. \(D_{1n}, D_{2n}\) by \(A_{kn}\), since \(\Delta(Y_1, Y_2) = \infty\) and \(\Delta(Y_2, Y_1) = \infty\).

Proof. (1) Set (compare with (4.12) in the case \(m = 3\))

\[ y^{(k)} = (y_1^{(k)}, y_2^{(k)}) = (||Y_1^{(k)}||^2, ||Y_2^{(k)}||^2), 1 \leq k \leq 2, \quad (3.9) \]

or \(y = (y_1^{(1)} y_2^{(1)} y_1^{(2)} y_2^{(2)})\). (3.10)

24
In the case $S = (0, 1)$ we have by Lemma 3.11 (see Remark 3.2)

$$y^{(1)} = (0, 1), \ y^{(2)} = (0, 1) \ or \ \begin{pmatrix} y^{(1)} \\ y^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \quad (3.11)$$

Indeed, $\|Y_1^{(1)}\|^2 \sim S_1(2) < \infty$, hence $\|Y_2^{(1)}\|^2 = \infty$ by (3.8). Further, $\Gamma(Y_2^{(2)}) = \|Y_2^{(2)}\|^2 \sim S_2(2) = \infty$, and $\Gamma(Y_1^{(2)}) < \infty$, since $S_1(2) < \infty$. Therefore,

$$\Delta(Y_2^{(2)}, Y_1^{(2)}) = \frac{\Gamma(Y_2^{(2)}) + \Gamma(Y_2^{(2)}, Y_1^{(2)})}{1 + \Gamma(Y_1^{(2)})} > \frac{\Gamma(Y_2^{(2)})}{1 + \Gamma(Y_1^{(2)})} = \infty,$$

and we conclude that $x_{2n} \not\in \mathfrak{A}^2$.

(2) Further, since

$$\|Y_1\|^2 = \|Y_2\|^2 = \|Y_1 - sY_2\|^2 = \infty \quad (3.12)$$

by Lemma 8.11, we conclude that $\Delta(Y_1, Y_2) = \Delta(Y_2, Y_1) = \infty$, so $D_{1n}, D_{2n} \not\in \mathfrak{A}^2$ by Lemmas 3.8 and 3.9. Finally, $x_{2n}, D_{1n}, D_{2n} \not\in \mathfrak{A}^2$. Now we get

$$A_{kn} - x_{2k}D_{2n} = x_{1k}D_{1n}, \quad k, n \in \mathbb{Z},$$

and the proof is complete since we are in the case $m = 1$.

Relations (3.12) follows from $S_1(2) = S_{12} = \sum_{k \in \mathbb{Z}} \frac{b_{1k}}{b_{2k}} < \infty$. Indeed, we have

$$\|Y_1\|^2 = \sum_{k \in \mathbb{Z}} \frac{a_{1k}^2}{2b_{1k}} + \frac{1}{2b_{2k}} = \sum_{k \in \mathbb{Z}} \frac{1}{2} + \frac{b_{1k}}{2b_{2k}} \sim 2 \sum_{k \in \mathbb{Z}} b_{1k}a_{1k}^2 = S_{11}^L(\mu) = \infty,$$

$$\|Y_2\|^2 = \sum_{k \in \mathbb{Z}} \frac{b_{1k}a_{2k}^2}{1 + \frac{b_{1k}}{2b_{2k}}} \sim \sum_{k \in \mathbb{Z}} b_{1k}a_{2k}^2 \sim \sum_{k \in \mathbb{Z}} \frac{b_{1k}}{2} \left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) = S_{12}^L(\mu) = \infty,$$

$$\|Y_1 - sY_2\|^2 = \sum_{k \in \mathbb{Z}} \frac{b_{1k}(a_{1k} - sa_{2k})^2}{1 + \frac{b_{1k}}{2b_{2k}}} \sim \sum_{k \in \mathbb{Z}} b_{1k}(a_{1k} - sa_{2k})^2 = \frac{1}{4} \sum_{k \in \mathbb{Z}} b_{1k}(-2a_{1k} + 2s a_{2k})^2$$

$$\sim \frac{1}{2} \left[ \frac{(2s)^2}{4} \sum_{k \in \mathbb{Z}} \frac{b_{1k}}{b_{2k}} + \sum_{k \in \mathbb{Z}} \frac{b_{1k}}{2}(-2a_{1k} + 2s a_{2k})^2 \right] = \frac{1}{2} S_{12}^L(\mu, t) = \infty,$$

for $t = 2s$. \hfill \Box
3.2.5. Case $S = (1, 1)$

In the case $S = (1, 1)$ we have three possibilities (see Remark 3.2)

$$I(2a)\ y = \begin{pmatrix} 1 & 0 \\
1 & 1 \end{pmatrix},\ I(2b)\ y = \begin{pmatrix} 1 & 1 \\
0 & 1 \end{pmatrix},\ I(2c)\ y = \begin{pmatrix} 1 & 1 \\
1 & 1 \end{pmatrix},$$  \hspace{1cm} (3.13)

i.e., $I(2a)$ $y_1^{(1)} < \infty$ and $y_1^{(2)} = \infty$, $I(2b)$ $y_2^{(1)} = \infty$ and $y_1^{(2)} < \infty$, $I(2c)$ $y_2^{(1)} = y_1^{(2)} = \infty$, since $\|Y_1^{(1)}\|^2 \sim S_1(2) = \infty$ and $\|Y_2^{(2)}\|^2 \sim S_2(2) = \infty$ and $y_2^{(1)} + y_1^{(2)} = \infty$ by (3.8).

In the first case I(2a) or in the second case I(2b), i.e., if $\|Y_2^{(1)}\|^2 < \infty$ or $\|Y_1^{(2)}\|^2 < \infty$, we conclude respectively that $\Delta(Y_1^{(1)}, Y_2^{(1)}) = \infty$ or $\Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty$ hence, $x_{1n}x_{1t} \in \mathfrak{A}^2$ or $x_{2n}x_{2t} \in \mathfrak{A}^2$. So, we will get respectively $x_{1n}$ or $x_{2n}$. We shall come back to these cases later.

It remains to consider the case I(2c): $\|Y_2^{(1)}\|^2 = \|Y_2^{(2)}\|^2 = \infty$. In the case $S = (1, 1)$ set $c_n = \frac{b_{2n}}{b_{1n}},\ n \in \mathbb{Z}$. Then $\sum_{k \in \mathbb{Z}} \frac{b_{1k}}{b_{2k}} = \sum_{k \in \mathbb{Z}} \frac{b_{2k}}{b_{1k}} = \infty$. We get

$$\|Y_1^{(1)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{1}{1 + 2c_n} \sim \sum_{n \in \mathbb{Z}} \frac{1}{c_n},\ \|Y_2^{(1)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{c_n^2}{1 + 2c_n},$$  \hspace{1cm} (3.14)

$$\|Y_1^{(2)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{1}{c_n^2 + 2c_n},\ \|Y_2^{(2)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{c_n^2}{c_n^2 + 2c_n} \sim \sum_{n \in \mathbb{Z}} c_n.$$  \hspace{1cm} (3.15)

3.2.6. II, Approximation of $D_{1n}$ and $D_{2n}$

Set

$$y_{12} = (y_1, y_2) = (\|Y_1\|^2, \|Y_2\|^2),$$  \hspace{1cm} (3.16)

where $\|Y_1\|^2 = \sum_{n \in \mathbb{Z}} \frac{d_{1n}^2}{1 + \frac{1}{2b_{1n}}},\ \|Y_2\|^2 = \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{1 + \frac{1}{2b_{2n}}}.$  \hspace{1cm} (3.17)

The case II splits into four subcases

(1) $(y_1, y_2) = (1, 0),\ (2) (y_1, y_2) = (0, 1),\ (3) (y_1, y_2) = (1, 1),\ (4) (y_1, y_2) = (0, 0).$ \hspace{1cm} (3.18)

We have $4 = 2^2$ possibilities for $y = (y_1, y_2) \in \{0, 1\}^2$:

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3a)</th>
<th>(3b)</th>
<th>(3c)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>$C_1 \leq \alpha_m \leq C_2$</td>
</tr>
</tbody>
</table>

26
where \( \alpha = \lim_{m \to \infty} \alpha_m \), with \( \alpha_m = \frac{\|Y_1(m)\|^2}{\|Y_2(m)\|^2} \), and

\[
\|Y_1(m)\|^2 = \sum_{n=-m}^{m} \frac{a_{1n}^2}{2b_{1n} + \frac{1}{2b_{2n}}} \quad \text{and} \quad \|Y_2(m)\|^2 = \sum_{n=-m}^{m} \frac{a_{2n}^2}{2b_{1n} + \frac{1}{2b_{2n}}} .
\]

All the different cases are presented in the following table:

<table>
<thead>
<tr>
<th>( |Y_1|^2 )</th>
<th>(1)</th>
<th>(2)</th>
<th>(3a)</th>
<th>(3b)</th>
<th>(3c)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( |Y_2|^2 )</td>
<td>( \infty )</td>
<td>( &lt; \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( &lt; \infty )</td>
</tr>
<tr>
<td>( \alpha_m = \frac{|Y_1(m)|^2}{|Y_2(m)|^2} )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( \rightarrow \infty )</td>
<td>( \rightarrow 0 )</td>
<td>( C_1 \leq \alpha_m \leq C_2 )</td>
<td></td>
</tr>
<tr>
<td>Lemma</td>
<td>3.8</td>
<td>3.9</td>
<td>3.8</td>
<td>3.9</td>
<td>3.8, 3.9</td>
<td></td>
</tr>
<tr>
<td>( D_{1n}, x_{1n}, D_{2n}, x_{2n} )</td>
<td>3.6</td>
<td>3.7</td>
<td>3.6</td>
<td>3.7</td>
<td>3.14, 8.11</td>
<td></td>
</tr>
<tr>
<td>( D_{1n}, x_{1n}, D_{2n}, x_{2n} )</td>
<td>3.8</td>
<td>3.9</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td></td>
</tr>
</tbody>
</table>

**Remark 3.3.** We show that if \( \|Y_2\|^2 < \infty \) and \( S_{12}^L(\mu) = \infty \), then \( \sum_n \frac{b_{1n}}{b_{2n}} = \infty \). Indeed, let us suppose that \( \sum_n \frac{b_{1n}}{b_{2n}} < \infty \), then

\[
\|Y_2\|^2 = \sum_{n \in \mathbb{Z}} \frac{a_{2n}^2}{2b_{1n} + \frac{1}{2b_{2n}}} \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{2n}^2 \sim \sum_{n \in \mathbb{Z}} \frac{b_{1n}}{2} \left( \frac{1}{2b_{2n}} + a_{2n}^2 \right) = S_{12}^L(\mu) = \infty .
\]

(3.19)

We explain Table II in detail. The first two case (1) and (2) are independent of the case I(2), i.e., \( S = (1, 1) \).

(1) If \( \|Y_2\|^2 < \infty \) and \( \|Y_1\|^2 = \infty \), then \( D_{1k} \eta \mathcal{A}_2 \) by Lemma 3.8. The condition \( \|Y_2\|^2 < \infty \) implies \( \sum_{k \in \mathbb{Z}} \frac{b_{1k}}{b_{2k}} = \infty \), by Remark 3.3 therefore, \( x_{1k} \eta \mathcal{A}_2 \), by Lemma 3.6. Further, \( A_{kn} - x_{1k}D_{1n} = x_{2k}D_{2n}, k, n \in \mathbb{Z} \) and the proof is complete since we are reduced to the case \( m = 1 \).

(2) If \( \|Y_2\|^2 = \infty \) and \( \|Y_1\|^2 < \infty \), then \( D_{2k} \eta \mathcal{A}_2 \) by Lemma 3.9. Reasoning as in Remark 3.3, we conclude that \( \sum_{k \in \mathbb{Z}} \frac{b_{1k}}{b_{2k}} = \infty \) and therefore, \( x_{2k} \eta \mathcal{A}_2 \) by Lemma 3.7 and \( A_{kn} - x_{2k}D_{2n} = x_{1k}D_{1n}, k, n \in \mathbb{Z} \), case \( m = 1 \).

(3) Consider now the case I(2). Suppose that both series are divergent: \( \|Y_2\|^2 = \infty \) and \( \|Y_1\|^2 = \infty \). We show that in the case \( (B) \) (see (3.25)) holds

\[
\|Y_1 + sY_2\|^2 = \infty \quad \text{for all} \quad s \in \mathbb{R}
\]

by Lemma 3.14 therefore, by Lemma 8.11, we can approximate \( D_{1n} \) and \( D_{2n} \).

To be more precise consider three possibilities:
(3a) Let \( \frac{\|Y_1(m)\|^2}{\|Y_2(m)\|^2} \to \infty \), then \( D_{1k} \parallel \mathfrak{A}^2 \). Since \( \sum_n \frac{b_{1n}}{b_{2n}} = \infty \), case I(2), we have \( x_{1n} \parallel \mathfrak{A}^2 \) by Lemma 3.6 and finally, \( x_{1n}, D_{1n} \parallel \mathfrak{A}^2, \ n \in \mathbb{Z} \). We are reduced to the case \( m = 1 \).

(3b) let \( \frac{\|Y_1(m)\|^2}{\|Y_2(m)\|^2} \to 0 \), then \( D_{2k} \parallel \mathfrak{A}^2 \). Since \( \sum_n \frac{b_{2n}}{b_{1n}} = \infty \), case I(2), we get \( x_{2n} \parallel \mathfrak{A} \), by Lemma 3.7 and finally, \( x_{2n}, D_{2n} \parallel \mathfrak{A}^2, \ n \in \mathbb{Z} \). We are reduced to the case \( m = 1 \).

(3c) The case when \( \|Y_1\|^2 = \|Y_2\|^2 = \infty \) and \( C_1 \leq \frac{\|Y_1(m)\|^2}{\|Y_2(m)\|^2} \leq C_2, \ m \in \mathbb{N} \).

(4) The case when \( \|Y_1\|^2 + \|Y_2\|^2 < \infty \).

To complete the proof of the lemma it remains to consider I(2), i.e., \( S = (1, 1) \) and the last two cases in the table II, i.e., II(3c) and II(4), where:

\[
\begin{align*}
I(2) & \quad \sum_{k \in \mathbb{Z}} \frac{b_{1k}}{b_{2k}} = \sum_{k \in \mathbb{Z}} \frac{b_{2k}}{b_{1k}} = \infty, \\
II(3c) & \quad \sum_{k \in \mathbb{Z}} a_{1k}^2 \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} \right)^{-1} = \sum_{k \in \mathbb{Z}} a_{2k}^2 \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} \right)^{-1} = \infty, \\
II(4) & \quad \sum_{k \in \mathbb{Z}} \left( a_{1k}^2 + a_{2k}^2 \right) \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} \right)^{-1} < \infty.
\end{align*}
\]

We come back to the condition \( \mu^{L_t} \perp \mu \). By Remark 2.5 we have

\[
\mu^{L_{r-(0,s)}} \perp \mu, \ \phi \in [0, 2\pi), \ s > 0 \Leftrightarrow \Sigma_1(s) + \Sigma_2(C_1, C_2) = \infty, \ s > 0,
\]

for \((C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}\). To make the notation consistent for the case \( m = 3 \) we replace everywhere \( \Sigma_1(s) \) (defined by (2.17)) with \( \Sigma_{12}(s) \) and \( \Sigma_2(C_1, C_2) \) defined by (2.18) with \( \Sigma_{12}(C_1, C_2) \) for \((C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}\):

\[
\begin{align*}
\Sigma_{12}(s) & = \sum_{n \in \mathbb{Z}} \left( s^2 \sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2} \sqrt{\frac{b_{2n}}{b_{1n}}} \right)^2, \ s \in \mathbb{R} \setminus \{0\}, \\
\Sigma_{12}(C_1, C_2) & = \sum_{n \in \mathbb{Z}} (C_1 b_{1n} + C_2 b_{2n})(C_1 a_{1n} + C_2 a_{2n})^2.
\end{align*}
\]

The condition \( \Sigma_{12}(s) + \Sigma_{12}(C_1, C_2) = \infty \), splits into two cases:

\[
\begin{align*}
(A) & \quad \Sigma_{12}(s) = \infty, \\
(B) & \quad \Sigma_{12}(s) < \infty \quad \text{and} \quad \Sigma_{12}(C_1, C_2) = \infty.
\end{align*}
\]

Finally, we need to consider the following 12 possibilities:
(A)&I(2). In this case independently of the conditions II(3c) and II(4) we can approximate \( x_{1n} \) and \( x_{2n} \) using Lemma 3.4 and 3.5.

(B)&II(3c) In this case we can approximate \( D_{1n} \) and \( D_{2n} \) using Lemmas 3.8 and 3.9 respectively. More precisely, to be able to use Lemma 8.11 we show that conditions (8.17) are satisfied for the two vectors \( Y_1 \) and \( Y_2 \) defined by (3.2) (see Lemma 3.14).

(B)&II(4) Since \( \Sigma_{12}(C_1, C_2) = \infty \), this case (see (3.22)) cannot be realized.

More details:
Case (A)&I(2). Using Lemma 8.11 we conclude that
\[
\Delta(Y_1^{(1)}, Y_2^{(1)}) = \infty \quad \text{and} \quad \Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty.
\]

To use Lemma 8.11, it is sufficient to show that in the case (A) relations (8.17) hold for \( Y_1^{(1)}, Y_2^{(1)}, Y_2^{(2)}, Y_1^{(2)} \), i.e., for all \( s \in \mathbb{R} \setminus \{0\} \) we have (see Lemma 3.13)
\[
\|Y_1^{(1)}\|^2 = \|Y_2^{(1)}\|^2 = \|Y_1^{(1)} + sY_2^{(1)}\|^2 = \infty,
\]
\[
\|Y_2^{(2)}\|^2 = \|Y_1^{(2)}\|^2 = \|Y_2^{(2)} + sY_1^{(2)}\|^2 = \infty.
\]

Consider the following three possibilities in the case I(2):

I(2a) If \( \|Y_2^{(1)}\| < \infty \), then \( \|Y_1^{(1)}\| = \infty \) by (3.8) therefore, \( \Delta(Y_1^{(1)}, Y_2^{(1)}) = \infty \) so, \( x_{1n} \notin \mathfrak{A}^2 \) by Lemma 8.10 (a). In the case (A) by Lemma 3.13 holds
\[
\|Y_2^{(2)}\|^2 = \|Y_1^{(2)}\|^2 = \|Y_2^{(2)} + sY_1^{(2)}\|^2 = \infty,
\]
therefore, \( x_{2n} \notin \mathfrak{A}^2 \) by Lemma 8.11.

I(2b) If \( \|Y_2^{(2)}\| < \infty \), then \( \|Y_2^{(2)}\| = \infty \) by (3.8) therefore, \( \Delta(Y_2^{(2)}, Y_1^{(2)}) = \infty \) so, \( x_{2n} \notin \mathfrak{A}^2 \) by Lemma 8.10 (a). In the case (A) by Lemma 3.13 we have
\[
\|Y_2^{(1)}\|^2 = \|Y_1^{(1)}\|^2 = \|Y_2^{(1)} + sY_1^{(1)}\|^2 = \infty,
\]
and therefore, \( x_{1n} \notin \mathfrak{A}^2 \) by Lemma 8.11.

I(2c) If \( \|Y_2^{(1)}\| = \|Y_1^{(2)}\| = \infty \), then by Lemma 3.13 all relations (3.27) hold in the case (A) and therefore, \( x_{1n}, x_{2n} \notin \mathfrak{A}^2 \). To prove (3.27) we need Lemma 4.8.
Lemma 3.13. If $\Sigma_{12}(s) = \infty$ for any $s > 0$, then
\[
\|Y^{(1)}_1 - CY^{(1)}_2\|^2 = \infty \text{ and } \|Y^{(2)}_2 - CY^{(2)}_1\|^2 = \infty, \text{ for any } C \in \mathbb{R} \setminus \{0\}.
\]

So, in the case (A)&I(2) we can approximate $x_{1n}$ and $x_{2n}$.

Case (B)&II(3c).

Lemma 3.14. When $\Sigma_{12}(s) < \infty$ and $\Sigma_{12}(C_1, C_2) = \infty$, we get
\[
\sigma(C_1, C_2) := \|C_1 Y_1 + C_2 Y_2\|^2 = \sum_{n \in \mathbb{Z}} \left(\frac{C_1 a_{1n} + C_2 a_{2n}}{2b_{1n} + 1} + \frac{1}{2b_{2n}}\right)^2 = \infty, \text{ (}C_1, C_2\text{) } \in \mathbb{R}^2 \setminus \{0\}.
\]

Finally, we can approximate $D_{1n}$ and $D_{2n}$ in the case (B)&II(3c).

Case (B)&II(4). The last case (B)&II(4) (see (3.22)) cannot be realized. Indeed, in this case $\Sigma_{12}(s) < \infty$ and $\Sigma_{12}(C_1, C_2) = \infty$. Therefore, by Lemma 4.8 we have $s^4 \lim_{n \to \infty} \frac{b_{1n}}{b_{2n}} = 1$ and hence,
\[
\sigma(C_1, C_2) \sim \Sigma_{12}(C_1, C_2) = \infty
\]
(see also the proof of Lemma 3.14 in [25]). This contradicts (3.22):
\[
\sigma(1, 1) = \sum_{k \in \mathbb{Z}} \left(a_{1k}^2 + a_{2k}^2\right) \left(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\right)^{-1} < \infty.
\]
Thus the proof of Lemma 3.10 for $m = 2$ is completed.

The proof of the irreducibility for $m = 2$ follows from Remark 1.2. Depending on the measure, we can approximate four different families of commuting operators $B^\alpha = (B^\alpha_{1n}, B^\alpha_{2n})_{n \in \mathbb{Z}}$ for $\alpha \in \{0, 1\}^2$:
\[
B^{(0,0)} = (x_{1n}, x_{2n})_n, \quad B^{(0,1)} = (x_{1n}, D_{2n})_n, \quad B^{(1,0)} = (D_{1n}, x_{2n})_n, \quad B^{(0,0)} = (D_{1n}, D_{2n})_n.
\]

The von Neumann algebra $L_\infty^\infty(X_2, \mu^2)$ consists of all essentially bounded functions $f(B^\alpha)$ in the commuting family of operators $B^\alpha$ (see, e.g., [5]) as, in particular, $L^\infty_{(0,0)}(X_2, \mu^2) = L^\infty(X_2, \mu^2)$. Since the von Neumann algebras $L_\infty^\infty(X_2, \mu^2)$ are maximal abelian, the commutant $(\mathcal{A}^2)'$ of the von Neumann algebra $\mathcal{A}^2$ generated by the representation is contained in $L_\infty^\infty(X_2, \mu^2)$. Hence, the bounded operator $A \in (\mathcal{A}^2)'$ will be some function $A = a(B^\alpha) \in L_\infty^\infty(X_2, \mu^2)$. The commutation relation $[A, T_t^{R, \mu, 2}] = 0$ gives us the following relations: $a(B^\alpha)^{R_t} = a(B^\alpha)$ for all $t \in \text{GL}_0(2\infty, \mathbb{R})$. Set $B_t^\alpha =
\((B_{rn}^{a})_{n}, \ x_{r} = (x_{rn})_{n}, \ D_{r} = (x_{rn})_{n}, \ r = 1, 2, n \in \mathbb{Z}\) and set as before, 
\(E_{kn}(t) := I + tE_{kn}, \ t \in \mathbb{R}, \ k, n \in \mathbb{Z}, \ k \neq n.\) Then the action \((B^{a})_{rt}\) is defined as follows:
\[
(B_{1}^{a}, B_{2}^{a})_{rt} = ((B_{1}^{a})_{rt}, (B_{2}^{a})_{rt}), \ (x_{r})_{rt} = x_{r}t, \ (D_{r})_{rt} = D_{r}t^{T},
\]
\[
a(\ldots, x_{rk}, \ldots, x_{rn}, \ldots)_{rt} := a(\ldots, x_{rk}, \ldots, x_{rn} + tx_{rk}, \ldots),
\]
\[
a(\ldots, D_{rk}, \ldots, D_{rn}, \ldots)_{rt} := a(\ldots, D_{rk} + tD_{rn}, \ldots, D_{rn}, \ldots), \ t \in \mathbb{R}.
\]

In all the cases, by ergodicity of the measure \(\mu^{2}\), we conclude that \(a\) is constant.

4. Irreducibility, the case \(m = 3\)

4.1. Technical part of the proof of irreducibility

Lemma 4.1. If \(\mu^{lt} \perp \mu\) for all \(t \in \text{GL}(3, \mathbb{R}) \setminus \{e\}\), we can approximate at least one of the following eight triplets of operators:
\[
(x_{1n}, x_{2n}, x_{3n}), \ (x_{1n}, x_{2n}, D_{3n}), \ (x_{1n}, D_{2n}, x_{3n}), \ (D_{1n}, x_{2n}, x_{3n}),
\]
\[
(x_{1n}, D_{2n}, D_{3n}), \ (D_{1n}, x_{2n}, D_{3n}), \ (D_{1n}, D_{2n}, x_{3n}), \ (D_{1n}, D_{2n}, D_{3n}).
\]

PROOF. By Lemma 2.8, the condition of orthogonality \((\mu_{(b,a)}^{3})_{lt} \perp \mu_{(b,a)}^{3}\) for \(t \in \pm \text{SL}(3, \mathbb{R}) \setminus \{e\}\) are,
\[
\Sigma^{\pm}(t) = \Sigma_{1}^{\pm}(t) + \Sigma_{2}(t) = \infty, \tag{4.1}
\]
where \(\Sigma_{2}(t)\) is defined by (2.28) and \(\Sigma_{1}^{+}(t), \ \Sigma_{1}^{-}(t)\) are defined by (2.31), (2.32). Let \(\mathfrak{A}^{3}\) be the von Neumann algebra generated by the representation.

We write compactly:
\[
x_{kn} \eta \mathfrak{A}^{3} \Leftrightarrow \Delta^{(k)} = \infty, \quad D_{kn} \eta \mathfrak{A}^{3} \Leftrightarrow \Delta_{k} = \infty, \tag{4.2}
\]
where \(\Delta^{(k)} := \Delta(Y_{k}^{(k)}, Y_{r}^{(k)}, Y_{s}^{(k)}), \quad \Delta_{k} := \Delta(Y_{k}, Y_{r}, Y_{s}). \tag{4.3}\)

and \(\{k, r, s\}\) is a cyclic permutation of \(\{1, 2, 3\}\).

Case I. Approximation of \(x_{rk}\) for \(1 \leq r \leq 3\) by \(A_{kn}A_{tn}\).

Set \(B_{3k} = b_{1k} + b_{2k} + b_{3k}\). To approximate the operators \(x_{kn}\) by the corresponding operators, by Lemmas 5.1–5.3 we get:
\[
x_{1n}x_{1t} \eta \mathfrak{A}^{3} \Leftrightarrow \Delta^{(1)} = \infty, \quad x_{2n}x_{2t} \eta \mathfrak{A}^{3} \Leftrightarrow \Delta^{(2)} = \infty, \quad x_{3n}x_{3t} \eta \mathfrak{A}^{3} \Leftrightarrow \Delta^{(3)} = \infty, \tag{4.4}
\]
where
\[ \|Y_s^{(r)}\|^2 = \sum_{k \in \mathbb{Z}} \frac{b_{rk}^2}{B_{3k}^2 - (b_{1k}^2 + b_{2k}^2 + b_{3k}^2 - b_{sk}^2)}, \quad 1 \leq r, s \leq 3. \]  

(4.5)

Case II. Approximation of \( D_{rn} \) by \( A_{kn} \).

By Lemmas 5.4–5.5 we have for \( 1 \leq r \leq 3 \) (see (4.3)):
\[ D_{rn} \eta \mathfrak{A}^3 \Leftrightarrow \Delta_r = \infty, \quad \text{where} \quad \|Y_r\|^2 = \sum_{k \in \mathbb{Z}} \frac{a_{rk}^2}{2b_{1k} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}} . \]

(4.6)

Case III. Approximation of \( x_{rk} \) by \( D_{rn}A_{kn} \).

\[
\begin{align*}
D_{1n}A_{kn} &= x_{1k}D_{1n}^2 + x_{2k}D_{1n}D_{2n} + x_{3k}D_{1n}D_{3n}, \\
D_{2n}A_{kn} &= x_{1k}D_{1n}D_{2n} + x_{2k}D_{2n}^2 + x_{3k}D_{2n}D_{3n}, \\
D_{3n}A_{kn} &= x_{1k}D_{1n}D_{3n} + x_{2k}D_{2n}D_{3n} + x_{3k}D_{3n}^2.
\end{align*}
\]

By Lemmas 5.7–5.9 we have
\[
\begin{align*}
x_{1n}1 \in \langle D_{1n}A_{kn}1 \mid k \in \mathbb{Z} \rangle & \Leftrightarrow \Sigma_1 = \infty, \\
x_{2n}1 \in \langle D_{2n}A_{kn}1 \mid k \in \mathbb{Z} \rangle & \Leftrightarrow \Sigma_2 = \infty, \\
x_{3n}1 \in \langle D_{3n}A_{kn}1 \mid k \in \mathbb{Z} \rangle & \Leftrightarrow \Sigma_3 = \infty,
\end{align*}
\]

where \( \Sigma_r = \sum_{k \in \mathbb{Z}} \frac{b_{rk}}{b_{1k} + b_{2k} + b_{3k}} \).

Case IV. Approximation of \( D_{rn} \) by \( x_{rk}A_{kn} \).

\[
\begin{align*}
x_{1k}A_{kn} &= x_{1k}D_{1n}^2 + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n}, \\
x_{2k}A_{kn} &= x_{1k}x_{2k}D_{1n} + x_{2k}D_{2n}^2 + x_{2k}x_{3k}D_{3n}, \\
x_{3k}A_{kn} &= x_{1k}x_{3k}D_{1n} + x_{2k}x_{3k}D_{2n} + x_{3k}D_{3n}.
\end{align*}
\]

By Lemmas 5.10–5.12 we have
\[
\begin{align*}
x_{1k}A_{kn}1 \in \langle x_{1k}A_{kn}1 \mid k \in \mathbb{Z} \rangle & \Leftrightarrow \Delta(Y_{11}, Y_{12}, Y_{13}) = \infty, \\
x_{2k}A_{kn}1 \in \langle x_{2k}A_{kn}1 \mid k \in \mathbb{Z} \rangle & \Leftrightarrow \Delta(Y_{22}, Y_{23}, Y_{21}) = \infty, \\
x_{3k}A_{kn}1 \in \langle x_{3k}A_{kn}1 \mid k \in \mathbb{Z} \rangle & \Leftrightarrow \Delta(Y_{33}, Y_{31}, Y_{32}) = \infty,
\end{align*}
\]

where \( Y_{kr} \) for \( 1 \leq k, r \leq 3 \) are defined by (5.13)–(5.20).

Case V. By Lemma 2.8 we have two conditions
\[
\begin{align*}
(A) & \quad \Sigma_1^+(t) = \infty, \quad \det t = 1 \quad \text{or} \quad \Sigma_1^-(t) = \infty, \quad \det t = -1, \\
(B) & \quad \Sigma_2^+(t) < \infty \quad \text{or} \quad \Sigma_2^-(t) < \infty \quad \text{but} \quad \Sigma_2(t) = \infty,
\end{align*}
\]

(4.7)

where \( \Sigma_1^+(t), \Sigma_1^-(t), \Sigma_2(t) \) are defined respectively by (2.31), (2.32) and (2.28). The rest of this section will be devoted to the proof of Lemma 4.1.
4.1.1. Notations and the change of the variables

In what follows we will systematically use the following notations:

\[ S_r(3) = \sum_{n \in \mathbb{Z}} \frac{b_r n}{b_{1n} b_{2n} + b_{1n} b_{3n} + b_{2n} b_{3n}}, \quad 1 \leq r \leq 3, \quad (4.8) \]

\[ \Sigma_r := \sum_{n \in \mathbb{Z}} \frac{b_r n}{b_{1n} + b_{2n} + b_{3n}}, \quad 1 \leq r \leq 3, \quad (4.9) \]

\[ \Sigma_{rs} := \sum_{k \in \mathbb{Z}} \frac{b_{rk}}{b_{sk}}, \quad 1 \leq r \neq s \leq 3, \quad C_k = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}, \quad (4.10) \]

\[ y_{123} = (y_1, y_2, y_3), \quad \text{where} \quad y_r := \|Y_r\|^2, \quad (4.11) \]

\[ y^{(k)} = (y_1^{(k)}, y_2^{(k)}, y_3^{(k)}) = (\|Y_1^{(k)}\|^2, \|Y_2^{(k)}\|^2, \|Y_3^{(k)}\|^2), \quad 1 \leq k \leq 3, \quad (4.12) \]

\[ y = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} y_1^{(1)} & y_2^{(1)} & y_3^{(1)} \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{pmatrix}, \quad \text{where} \quad y^{(r)} := \|Y^{(r)}\|^2, \quad (4.13) \]

\[ \Sigma_{123}(s) = (\Sigma_{12}(s_{12}), \Sigma_{23}(s_{23}), \Sigma_{13}(s_{13})), \quad s = (s_{12}, s_{23}, s_{13}). \quad (4.14) \]

The expressions \( S_r(3) \) in the case \( m = 3 \) can be generalized for an arbitrary \( m \in \mathbb{N} \) as follows:

\[ S_k(m) = \sum_{n \in \mathbb{Z}} \sum_{1 \leq r < s \leq m} \frac{b_{kn}^2}{b_{rn} b_{sn}}, \quad 1 \leq k \leq m. \quad (4.15) \]

**Lemma 4.2.** We have

\[ S_1(3) + S_2(3) + S_3(3) = \infty, \quad (4.16) \]

\[ \|Y_r^{(r)}\|^2 \sim S_r(3) \quad \text{for all} \quad 1 \leq r \leq 3, \quad (4.17) \]

\[ \|Y_r^{(s)}\|^2 < \frac{1}{2} S_r(3) \quad \text{for all} \quad 1 \leq r \neq s \leq 3, \quad (4.18) \]

\[ \|Y_1^{(i_1)}\|^2 + \|Y_2^{(i_2)}\|^2 + \|Y_3^{(i_3)}\|^2 = \infty, \quad i_1, i_2, i_3 \in \{1, 2, 3\}. \quad (4.19) \]

**Proof.** Since \( 3(a^2 + b^2 + c^2) \geq 2(ab + ac + bc) \) we get

\[ S_1(3) + S_2(3) + S_3(3) = \sum_{n \in \mathbb{Z}} \frac{b_{1n}^2 + b_{2n}^2 + b_{3n}^2}{b_{1n} b_{2n} + b_{1n} b_{3n} + b_{2n} b_{3n}} \geq \sum_{k \in \mathbb{Z}} 2/3 = \infty. \]
Further by (4.5)

\[ \| Y_r^{(r)} \|^2 = \sum_{n \in \mathbb{Z}} \frac{b_{rn}^2}{b_{rn}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})} \sim S_r(3), \]

\[ \| Y_r^{(s)} \|^2 = \sum_{n \in \mathbb{Z}} \frac{b_{sn}^2}{b_{sn}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})} < \frac{1}{2} S_r(3), \quad s \neq r. \]

To prove (4.19) we get by (4.5)

\[ \| Y_{i1}^{(i1)} \|^2 + \| Y_{i2}^{(i2)} \|^2 + \| Y_{i3}^{(i3)} \|^2 = \sum_{r=1}^{3} \sum_{n \in \mathbb{Z}} \frac{b_{rn}^2}{b_{rn}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})} > \sum_{n \in \mathbb{Z}} \left( \sum_{r=1}^{3} \frac{b_{rn}^2}{(b_{rn})^2} \right) = \infty. \]

We make the following change of the variables:

\[ \left( \begin{array}{c} b_{1n} \\ b_{2n} \\ b_{3n} \\ a_{1n} \\ a_{2n} \\ a_{3n} \end{array} \right) \rightarrow \left( \begin{array}{c} b_{1n}' \\ b_{2n}' \\ b_{3n}' \\ a_{1n}' \\ a_{2n}' \\ a_{3n}' \end{array} \right) = \left( \begin{array}{c} \frac{1}{a_{1n}} \\ \frac{d_{1n}}{b_{1n}} \\ \frac{d_{2n}}{b_{1n}} \\ \frac{1}{a_{2n}} \sqrt{b_{1n}} \\ \frac{a_{3n}}{b_{1n}} \sqrt{b_{1n}} \end{array} \right), \quad \text{(4.20)} \]

motivated by the following formulas:

\[
\begin{align*}
\mu_{(b,a)}(x) &= \sqrt{\frac{b}{\pi}} \exp(-b(x-a)^2)dx = \sqrt{\frac{1}{\pi}} \exp(-(x'-a')^2)dx' = \mu_{(b',a')}(x'), \\
\mu_{(b_2,a_2)}(x) &= \sqrt{\frac{b_2}{\pi}} \exp(-b_2(x-a_2)^2)dx = \sqrt{\frac{b_2}{b_1 \pi}} \exp\left(-\frac{b_2}{b_1}(x'-a'_2)^2\right)dx' = \mu_{(b'_2,a'_2)}(x'), \\
&= \mu_{(b'_2,a'_2)}(x'), \quad (b', a') = (1, a \sqrt{b}), \quad (b'_2, a'_2) = \left( \frac{b_2}{b_1}, a_2 \sqrt{b_1} \right).
\end{align*}
\]

**Remark 4.1.** All the expressions, given in the list (2.26) (2.27), (2.28) and (4.1) are invariant under the transformations (4.20)

\[ S_{kr}^L(\mu) = \sum_{n \in \mathbb{Z}} \frac{b_{kn}}{2} \left( \frac{1}{2b_{rn}} + a_{rn}^2 \right), \quad Y_r = \left( a_{rk} \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} \right)^{-1/2} \right)_{k \in \mathbb{Z}}, \]

etc., and \( S_r(3) \) that are defined by (3.4).
4.2. Approximation scheme

**Case I.** Approximation of $x_{kt}$ by $A_{kn}A_{tn}$. Recall that we will write 1 if some values = $\infty$ and 0 in the case < $\infty$ (see Remark 3.2). We use the following notation $S := (S_1(3), S_2(3), S_3(3))$. By Lemma 4.2 we get

$$\sum_{r=1}^{3} S_r(3) = \infty.$$ 

Therefore, without loss of generality, it suffices to consider the following three cases:

(1) $S = (0, 0, 1)$, (2) $S = (0, 1, 1)$, (3) $S = (1, 1, 1)$. \hfill (4.21)

By Lemma 2.8, the condition of orthogonality $(\mu_{3(n)}^{(b,a)})^L \perp \mu_{3(n)}^{(b,a)}$ for $t \in \pm SL(3, \mathbb{R}) \setminus \{e\}$, i.e., $\Sigma^\pm(t) = \Sigma_1^\pm(t) + \Sigma_2(t) = \infty$, splits into two cases:

(A) $\Sigma_1^\pm(t) = \infty$, \hspace{1cm} $\Sigma_2^\pm(t) = \sum_{1 \leq i < j \leq 3} \Sigma_{ij}^\pm(t)$, \hfill (4.22)

(B) $\Sigma_1^\pm(t) < \infty$ but $\Sigma_2(t) = \infty$,

where $\Sigma_1^\pm(t)$, $\Sigma_{ij}^\pm(t)$ and $\Sigma_2(t)$ are defined by (2.31), (2.32), (2.33) and (2.28).

4.3. Case $S = (0, 0, 1)$

**Lemma 4.3.** The case $S = (0, 0, 1)$ is equivalent with

$$\Sigma_{13} + \Sigma_{23} < \infty, \hspace{0.5cm} S_3(3) \sim \sum_n \frac{b_{3n}^2}{b_1b_2} = \infty.$$ \hfill (4.23)

**Proof.** To prove the first part of (4.23) we set $c_n = \frac{b_{3n}}{b_1b_2 + b_{2n}}$ and note that

$$\infty > S_1(3) + S_2(3) = \sum_{n \in \mathbb{Z}} (\frac{b_{1n}^2 + b_{2n}^2}{b_1b_2 + b_{1n}b_{2n} + b_{2n}b_{3n}}) \sim \sum_{n \in \mathbb{Z}} (\frac{b_{1n} + b_{2n}}{b_1b_2 + b_{1n}b_{2n} + b_{2n}b_{3n}})^2 \sim \sum_{n \in \mathbb{Z}} (\frac{1}{1 + c_n})^2 = \sum_{n \in \mathbb{Z}} \frac{1}{1 + 2c_n} \sim \sum_{n \in \mathbb{Z}} \frac{1}{c_n} \sum_{n \in \mathbb{Z}} \frac{b_{1n} + b_{2n}}{b_{3n}} = \Sigma_{13} + \Sigma_{23}.$$

To prove the second part of (4.23) we have by the first part of (4.23)

$$S_3(3) = \sum_{n \in \mathbb{Z}} \frac{b_{3n}^2}{b_1b_2 + b_{1n}b_{3n} + b_{2n}b_{3n}} = \sum_{n \in \mathbb{Z}} \frac{1}{b_1b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}} \sim \sum_{n \in \mathbb{Z}} \frac{b_{2n}^2}{b_1b_2} \Box$$

35
In the case \( S = (0, 0, 1) \) we have
\[
\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) \sim \Delta(Y_3^{(3)}) \sim \|Y_3^{(3)}\|^2 = \infty,
\]
so we can approximate \( x_{3n}x_{3t} \) using Lemma 5.3 and after that we can approximate \( x_{3n} \) by an analogue of Lemma 3.3. From now on we will say that we can approximate \( x_{3n} \) using Lemma 5.3, without mentioning Lemma 3.3.

We can not approximate \( x_{1n} \) and \( x_{2n} \) using Lemma 5.1-5.2, since we have
\[
\Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}) + \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}) < \infty.
\]

We can try to approximate some of \( D_{rn} \) for \( 1 \leq r \leq 3 \) using Lemmas 5.4–5.6, see Section 4.4.4 for details. We have for \( 1 \leq k \leq 3 \) (see (4.3)):
\[
\|Y_1\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{1n}^2, \quad \|Y_2\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{2n}^2, \quad \|Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{3n}^2. \quad (4.24)
\]

By (4.23) we have \( \Sigma^{13} + \Sigma^{23} < \infty \). We distinguish two cases:

(1) \( \Sigma^{12} < \infty \),

(2) \( \Sigma^{12} = \infty \).

In the case (1), since \( \Sigma^{12} + \Sigma^{13} < \infty \) we have
\[
S_{1,23}^t(\mu, t, s) \overset{(2.44)}{=} \sum_{n \in \mathbb{Z}} \left[ \frac{t^2 b_{1n}}{4} b_{2n} + \frac{s^2 b_{1n}}{4} b_{3n} + \frac{b_{1n}}{2} (-2a_{1n} + ta_{2n} + sa_{3n})^2 \right] \sim \sum_{n \in \mathbb{Z}} \frac{b_{1n}}{2} (-2a_{1n} + ta_{2n} + sa_{3n})^2 \overset{(4.24)}{=} \|C_1Y_1 + C_2Y_2 + C_3Y_3\|^2.
\]

Finally, in the case (1) we can approximate all \( D_{rn} \), \( 1 \leq r \leq 3 \) using Lemmas 5.4–5.6 and Lemma 8.15 and the proof is finished.

The case (2) can be divided into three cases (if necessary, we can chose an appropriate subsequence of \( \left( \frac{b_{1n}}{b_{2n}} \right)_n \)):
\[
\lim_{n} \frac{b_{1n}}{b_{2n}} = \begin{cases} 
(a) & 0 \\
(b) & b > 0 \\
(c) & \infty
\end{cases} \quad . \quad (4.25)
\]
The case (c) is reduced to the case (a) by exchanging \((b_{2n}, a_{2n})\) with \((b_{1n}, a_{1n})\).

This transformation does not change the first condition in (4.23). In the case (2.a-b), by (4.6) we obtain the following expressions for \(\|Y_r\|^2, \ 1 \leq r \leq 3:\)

\[
\begin{align*}
\|Y_1\|^2 &= \sum_{n \in \mathbb{Z}} \frac{a_{1n}^2}{2b_{1n}} + \frac{1}{2b_{2n}} + \frac{1}{2b_{3n}} = \sum_{k \in \mathbb{Z}} \frac{2b_{1n}a_{1n}^2}{1 + b_{1n}b_{2n} + b_{1n}b_{3n}} \sim \sum_{n \in \mathbb{Z}} 2b_{1n}a_{1n}^2, \\
\|Y_2\|^2 &\sim \sum_{n \in \mathbb{Z}} 2b_{1n}a_{2n}^2, \quad \|Y_3\|^2 = \sum_{n \in \mathbb{Z}} b_{1n}a_{3n}^2.
\end{align*}
\]

Since

\[
\|Y_1\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{1n}^2 \sim S^{L}_{11}(\mu) = \infty,
\]

we have four possibilities for \(y_{23} := (y_2, y_3) \in \{0, 1\}^2\) as in (4.54), see Section 4.4.4:

<table>
<thead>
<tr>
<th></th>
<th>(1.0)</th>
<th>(1.1)</th>
<th>(1.2)</th>
<th>(1.3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y_1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(y_2)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(y_3)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We just follow the instructions given in Remark 4.4. We note that the cases (1.0) and (1.1) can not occur since the following conditions are contradictory:

\[
\Sigma_{13}^{L}(\mu) \overset{2.43}{=} \sum_{n \in \mathbb{Z}} b_{1n}^2 + a_{3n}^2 = \infty, \quad \|Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{3n}^2 < \infty, \quad \Sigma_{13}^{L} < \infty.
\]

We have two cases (1.2.1) and (1.3.1) according to whether respectively the expressions in (4.58) or (4.60) are divergent. We can approximate in these cases respectively \(D_{1n}\) and \(D_{3n}\) in (4.58) and all \(D_{1n}, D_{2n}, D_{3n}\) in (4.57). The proof of irreducibility is finished in both cases because we have \(x_{3n}, D_{3n} \eta \mathbb{A}^3\) and the problem is reduced to the case \(m = 2\) [28], since \(A_{kn} = \sum_{r=1}^{3} x_{rk}D_{rn} - x_{3k}D_{3n} = \sum_{r=1}^{2} x_{rk}D_{rn}\).

If the opposite holds, we have two different cases (1.2.0) and (1.3.0). We try to approximate \(D_{3n}\) using Lemma 5.15. If one of the expressions \(\Sigma_3(D, s)\) or \(\Sigma_{3}^{\prime}(D, s)\) is divergent for some sequence \(s = (s_k)_{k \in \mathbb{Z}}\), we can approximate \(D_{3k}\) and the proof is finished, since we have \(x_{3n}, D_{3n} \eta \mathbb{A}^3\) and the problem is reduced to the case \(m = 2\). Let us suppose, as in Remark 4.6, that for every sequence \(s = (s_k)_{k \in \mathbb{Z}}\) we have

\[
\Sigma_3(D, s) + \Sigma_{3}^{\prime}(D, s) < \infty.
\]
Then, in particular, we have for $s^{(3)} = (s_k)_{k \in \mathbb{Z}}$ with $\frac{s_k^2}{b_{1k}} \equiv 1$

$$\infty > \Sigma_3(D, s^{(3)}) + \Sigma_3^+(D, s^{(3)}) - \Sigma_3(D) + \Sigma_3^+(D) = \sum_k \frac{C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2}{2b_{1k} + a_{1k}^2 + a_{2k}^2 + a_{3k}^2} \sim \sum_k \frac{2b_{1k}a_{3k}^2}{1 + 2b_{1k}a_{1k}^2 + 2b_{1k}a_{2k}^2} =: \Sigma_3^+(D).$$

Remark 4.2. Finally, we have $\Sigma_3^+(D) \sim \sum_k \frac{2a_{3k}^2}{1 + 2a_{1k}^2 + 2a_{2k}^2}$, we take $b_{1n} \equiv 1$ by (4.20). In the case (1.2.0) we have $||Y_2||^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{2n}^2 < \infty$, and therefore $\Sigma_3^+(D) \sim \sum_k \frac{2a_{3k}^2}{1 + 2a_{1k}^2 + 2a_{2k}^2}$, and hence $\Sigma_3^+(D) = \infty$ by Lemma 4.10. In the case (1.3.0) we have $a_3 = \pm a_1 \pm a_2 + h$ or $a_3 - h = \pm a_1 \pm a_2$, see the proof of Lemma 4.11. Therefore,

$$\infty > \Sigma_3^+(D) \sim \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} \geq \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + 2|a_{1k}||a_{2k}| + a_{2k}^2} = \sum_k \frac{a_{3k}^2}{1 + (|a_{1k}| + |a_{2k}|)^2}, \quad \infty > \Sigma_3^+(D) \sim \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} \geq \sum_k \frac{a_{2k}^2}{1 + a_{1k}^2 + a_{2k}^2 + (|a_{1k}| - |a_{2k}|)^2} \sim \sum_k \frac{a_{2k}^2}{1 + 2a_{1k}^2 - 2|a_{1k}||a_{2k}| + 2a_{2k}^2} \sim \sum_k \frac{a_{2k}^2}{1 + (|a_{1k}| - |a_{2k}|)^2}.$$

Hence, we have by (4.26) and (4.27)

$$\infty > \Sigma_3^+(D) \geq \sum_k \frac{a_{3k}^2}{1 + (\pm a_{1k} \pm a_{2k})^2} = \sum_k \frac{a_{3k}^2}{1 + (a_{3k} - h_k)^2} = \infty$$

by Lemma 4.10, contradiction. Therefore, in both cases we can approximate $D_{3n}$ and the proof is finished.

4.4. Case $S = (0,1,1)$

Lemma 4.4. In the case $S = (0,1,1)$ we have

$$\lim_n d_{2n} = \lim_n d_{3n} = \infty.$$  

(4.29)
PROOF. Setting as before $d_{rn} = \frac{b_{rn}}{b_{rn}}$, we obtain by (3.4) and (2.22)

$$S_1(3) = \sum_{n \in \mathbb{Z}} \frac{1}{d_{2n} + d_{3n} + d_{2n}d_{3n}} \sim \sum_{n \in \mathbb{Z}} \frac{1}{(1+d_{2n})(1+d_{3n})} < \infty, \quad (4.30)$$

$$S_2(3) = \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{d_{2n} + d_{3n} + d_{2n}d_{3n}} \sim \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{(1+d_{2n})(d_{2n} + d_{3n})} = \infty, \quad (4.31)$$

$$S_3(3) = \sum_{n \in \mathbb{Z}} \frac{d_{3n}^2}{d_{2n} + d_{3n} + d_{2n}d_{3n}} \sim \sum_{n \in \mathbb{Z}} \frac{d_{3n}^2}{(1+d_{3n})(d_{2n} + d_{3n})} = \infty. \quad (4.32)$$

Suppose that $d_{2n} \leq C$ for all $n \in \mathbb{Z}$. Then by (4.30) and (4.31) we conclude

$$S_1(3) \sim \sum_{n \in \mathbb{Z}} \frac{1}{(1+d_{2n})(1+d_{3n})} \sim \sum_{n \in \mathbb{Z}} \frac{1}{1+d_{3n}} \sim \sum_{n \in \mathbb{Z}} \frac{1}{d_{3n}} < \infty, \quad \infty = S_2(3)$$

$$\sim \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{(1+d_{2n})(d_{2n} + d_{3n})} \sim \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{d_{2n} + d_{3n}} \leq \sum_{n \in \mathbb{Z}} \frac{C^2}{C + d_{3n}} \sim \sum_{n \in \mathbb{Z}} \frac{1}{d_{3n}} < \infty,$$

a contradiction. We use the fact that for any fixed $D > 0$ the function

$$f_D(x) = \frac{x^2}{x + D}$$

is strictly increasing when $x > 0$. Similarly, if we suppose that $d_{3n} \leq C$ for all $n \in \mathbb{Z}$ we will obtain a contradiction too. \hfill \Box

**Lemma 4.5.** The case $S = (0, 1, 1)$ is equivalent with

$$S_1(3) \sim \sum_n \frac{b_{2n}^2}{b_{2n}^3} < \infty, \quad S_2(3) \sim \sum_n \frac{1}{d_n} = \infty, \quad S_3(3) \sim \sum_n d_n = \infty. \quad (4.33)$$

**Proof.** Recall that $d_n = \frac{d_{3n}}{d_{2n}}$. Denote $D_n := 1 + d_{2n}^{-1} + d_{3n}^{-1}$. By Lemma 4.4 we have

$$1 \leq D_n = 1 + d_{2n}^{-1} + d_{3n}^{-1} \leq C, \quad \text{for all } n \in \mathbb{Z}. \quad (4.34)$$

Therefore, we get

$$S_1(3) = \sum_{n \in \mathbb{Z}} \frac{1}{d_{2n} + d_{3n} + d_{2n}d_{3n}} = \sum_{n \in \mathbb{Z}} \frac{1}{D_n d_{2n}d_{3n}} \sim \sum_{n \in \mathbb{Z}} \frac{1}{d_{2n}d_{3n}} = \sum_{n} \frac{b_{1n}^2}{b_{2n}b_{3n}},$$

$$S_2(3) = \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{d_{2n} + d_{3n} + d_{2n}d_{3n}} = \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{D_n d_{2n}d_{3n}} \sim \sum_{n} \frac{1}{d_n},$$

$$S_3(3) = \sum_{n \in \mathbb{Z}} \frac{d_{3n}^2}{d_{2n} + d_{3n} + d_{2n}d_{3n}} = \sum_{n \in \mathbb{Z}} \frac{d_{3n}^2}{D_n d_{2n}d_{3n}} \sim \sum_{n} d_n. \quad \Box$$
By Lemma 4.2, (4.18) we get \( \|Y^{(r)}\|^2 < \infty, \ 1 \leq r \leq 3 \) therefore, we get

**Lemma 4.6.** In the case \( S = (0, 1, 1) \) we have

\[
\Delta(Y^{(1)}, Y^{(1)}, Y^{(1)}) < \infty, \quad \Delta(Y^{(2)}, Y^{(2)}, Y^{(1)}) \sim \Delta(Y^{(2)}, Y^{(2)}), \\
\Delta(Y^{(3)}, Y^{(3)}, Y^{(3)}) \sim \Delta(Y^{(3)}, Y^{(3)}). \quad (4.35)
\]

**Proof.** Set \((f_1, f_2, f_3) = (Y^{(3)}, Y^{(1)}, Y^{(3)}). \) Then

\[
\Delta(f_1, f_2, f_3) \overset{(8.15)}{=} \frac{\Gamma(f_1) + \Gamma(f_1, f_2) + \Gamma(f_1, f_3) + \Gamma(f_1, f_2, f_3)}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)} > \Gamma(f_1, f_2, f_3) \overset{(4.36)}{=} \frac{\Gamma(f_1) + \Gamma(f_1, f_3)}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)} \sim \Delta(f_1, f_3),
\]

since \( f_2 \in l_2(\mathbb{Z}). \) Indeed, for \( f, g \in l_2(\mathbb{Z}) \) and \( f \in l_2(\mathbb{Z}), g \notin l_2(\mathbb{Z}) \) we have respectively

\[
\Gamma(f, g) \leq \Gamma(f) \Gamma(g) < \infty, \quad \Gamma(f, g) \leq \Gamma(f) \Gamma(g), \quad \text{where} \quad \Gamma(f, g), \quad (4.36)
\]

\( \Gamma(g) \) are defined by \( \Gamma(f, g) := \lim_n \Gamma(f(n), g(n)) \) \( \Gamma(g) := \lim_n \Gamma(g(n)), \)

and \( g(n) := (g_k)_{k=-n}^{n} \in \mathbb{R}^{2n+1}. \) Similarly, set \((f_1, f_2, f_3) = (Y^{(2)}, Y^{(2)}, Y^{(1)}), \) then

\[
\Delta(f_1, f_2, f_3) \overset{(8.15)}{=} \frac{\Gamma(f_1) + \Gamma(f_1, f_2) + \Gamma(f_1, f_3) + \Gamma(f_1, f_2, f_3)}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)} > \frac{\Gamma(f_1) + \Gamma(f_1, f_2)}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)} \overset{(4.36)}{=} \frac{\Gamma(f_1) + \Gamma(f_1, f_2)}{(1 + \Gamma(f_2))(1 + \Gamma(f_3))} \sim \Delta(f_1, f_2),
\]

since \( f_3 \in l_2(\mathbb{Z}). \) Finally, we derive both equivalences in (4.35). To prove that

\[
\Delta(Y^{(1)}, Y^{(1)}, Y^{(1)}) < \infty \text{ we set } (f_1, f_2, f_3) = (Y^{(1)}, Y^{(1)}, Y^{(1)}), \text{ and note that}
\]

\[
\Delta(f_1, f_2, f_3) \overset{(8.15)}{=} \frac{\Gamma(f_1) + \Gamma(f_1, f_2) + \Gamma(f_1, f_3) + \Gamma(f_1, f_2, f_3)}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)} \leq \frac{\Gamma(f_1)(1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3))}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)} = \Gamma(f_1) < \infty. \quad \square
\]

In order to approximate \( x_{2n} \) or \( x_{3n}, \) it remains to study when

\[
\Delta(Y^{(2)}, Y^{(2)}) = \infty, \quad \Delta(Y^{(3)}, Y^{(3)}) = \infty, \quad (4.37)
\]
where $\Delta(f_1, f_2) = \frac{\Gamma(f_1) + \Gamma(f_2)}{1 + \Gamma(f_2)}$. For $2 \leq r \leq 3$, denote

$$
\rho_r(C_2, C_3) := \|C_2 Y_2^{(r)} + C_3 Y_3^{(r)}\|_2, \quad (C_2, C_3) \in \mathbb{R}^2, \quad (4.38)
$$

$$
\nu(C_1, C_2, C_3) := \|C_1 Y_1 + C_2 Y_2 + C_3 Y_3\|_2, \quad (C_1, C_2, C_3) \in \mathbb{R}^3. \quad (4.39)
$$

**Lemma 4.7.** In the case $S = (0, 1, 1)$ we have

$$
\rho_2(C_2, C_3) \sim \sum_{n \in \mathbb{Z}} \frac{(C_2 + C_3 d_n)^2}{1 + 2d_n}, \quad \rho_3(C_2, C_3) \sim \sum_{n \in \mathbb{Z}} \frac{(C_2 + C_3 d_n)^2}{d_n^2 + 2d_n} \quad (4.40)
$$

$$
= \sum_{n \in \mathbb{Z}} \frac{(C_2 d_n + C_3)^2}{1 + 2d_n}, \quad \nu(C_1, C_2, C_3) \sim \sum_{n \in \mathbb{Z}} b_1 \left( \sum_{r=1}^3 C_r a_{rn} \right)^2. \quad (4.41)
$$

**PROOF.** Set as before $d_n = \frac{d_1}{d_2}$. By (4.5) and (4.6) we get

$$
\|Y_2^{(2)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{d_{2n}^2 + 2(d_{2n} + d_{3n} + d_{2n} d_{3n})} = \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{d_{2n}^2 + 2D_n d_{2n} d_{3n}} \sim \sum_{n \in \mathbb{Z}} \frac{1}{D_n d_n},
$$

$$
\|Y_3^{(2)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{d_{3n}^2}{d_{2n}^2 + 2(d_{2n} + d_{3n} + d_{2n} d_{3n})} = \sum_{n \in \mathbb{Z}} \frac{d_{3n}^2}{d_{2n}^2 + 2D_n d_{2n} d_{3n}} = \sum_{n \in \mathbb{Z}} \frac{d_{3n}^2}{1 + 2D_n d_n}, \quad \|Y_3^{(2)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{d_{2n}^2 + 2D_n d_{2n} d_{3n}} = \sum_{n \in \mathbb{Z}} \frac{1}{d_{2n}^2 + 2D_n d_n},
$$

$$
\|Y_4^{(3)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{d_{3n}^2}{d_{3n}^2 + 2(d_{2n} + d_{3n} + d_{2n} d_{3n})} = \sum_{n \in \mathbb{Z}} \frac{d_{3n}^2}{d_{3n}^2 + 2D_n d_{2n} d_{3n}} \sim \sum_{n \in \mathbb{Z}} \frac{d_n}{D_n},
$$

$$
\|Y_1\|^2 = \sum_{n \in \mathbb{Z}} \frac{d_{1n}^2}{1 + d_{1n}^2 + 1 + d_{2n}^2 + d_{3n}^2} = \sum_{n \in \mathbb{Z}} \frac{2b_{1n} d_{1n}^2}{1 + d_{2n}^2 + d_{3n}^2} = \sum_{n \in \mathbb{Z}} \frac{2b_{1n} d_{1n}^2}{D_n},
$$

$$
\|Y_2\|^2 = \sum_{n \in \mathbb{Z}} \frac{2b_{1n} d_{2n}^2}{D_n}, \quad \|Y_3\|^2 = \sum_{n \in \mathbb{Z}} \frac{b_{1n} d_{3n}^2}{D_n}. \quad (4.42)
$$

Recall that $d_n = \frac{b_{1n}}{b_{1n}}$. By (4.34), we obtain

$$
\|Y_2^{(2)}\|^2 \sim \sum_{n \in \mathbb{Z}} \frac{1}{1 + 2d_n} \sim \sum_{n \in \mathbb{Z}} \frac{1}{d_n}, \quad \|Y_3^{(2)}\|^2 \sim \sum_{n \in \mathbb{Z}} \frac{d_n^2}{1 + 2d_n}, \quad (4.43)
$$

$$
\|Y_2^{(3)}\|^2 \sim \sum_{n \in \mathbb{Z}} \frac{1}{d_n^2 + 2d_n}, \quad \|Y_3^{(3)}\|^2 \sim \sum_{n \in \mathbb{Z}} \frac{d_n^2}{d_n^2 + 2d_n} \sim \sum_{n \in \mathbb{Z}} d_n.
$$
\[ \|Y_1\|^2 \sim \sum_{n \in \mathbb{Z}} b_1 n a_1^2, \quad \|Y_2\|^2 \sim \sum_{n \in \mathbb{Z}} b_1 n a_2^2, \quad \|Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} b_1 n a_3^2. \]

\[ \|C_1 Y_1 + C_2 Y_2 + C_3 Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} b_1 (C_1 a_1 n + C_2 a_2 n + C_3 a_3 n)^2. \quad (4.44) \]

By (4.43) and (4.44) the proof is finished. \qed

4.4.1. Approximation of \( x_{2n}, x_{3n} \)

To approximate \( x_{2n}, x_{3n} \), we need several lemmas. Denote \( l_n = d_n^{-1}. \)

Lemma 4.8. The following five series are equivalent:

(i - ii) \[ \sum_{n \in \mathbb{Z}} \frac{(C_2 - C_3 d_n)^2}{1 + 2d_n} \sim \sum_{n \in \mathbb{Z}} c_n^2, \quad (4.45) \]

(iii - iv) \[ \sum_{n \in \mathbb{Z}} \frac{(C_2 l_n - C_3)^2}{1 + 2l_n} \sim \sum_{n \in \mathbb{Z}} e_n^2, \quad (4.46) \]

(v) \[ \sum_{n \in \mathbb{Z}} \left( s^2 \sqrt{\frac{b_{2n}}{b_{3n}}} - s^2 \sqrt{\frac{b_{3n}}{b_{2n}}} \right) = \sum_{n \in \mathbb{Z}} \left( \frac{s^2}{\sqrt{d_n}} - \frac{\sqrt{d_n}}{s^2} \right)^2, \quad (4.47) \]

where

\[ d_n = C_2 C_3^{-1} (1 + c_n), \quad l_n = C_3 C_2^{-1} (1 + e_n), \quad s^4 = C_2 C_3^{-1} > 0, \quad l_n = d_n^{-1}. \quad (4.48) \]

PROOF. To prove (4.45) and (4.46) we get by Lemma 2.5 using (4.48)

\[ \sum_{n \in \mathbb{Z}} \frac{(C_2 - C_3 d_n)^2}{1 + 2d_n} = \sum_{n \in \mathbb{Z}} \frac{C_2^2 c_n^2}{1 + 2C_2 C_3^{-1} (1 + c_n)} \sim \sum_{n \in \mathbb{Z}} c_n^2, \]

\[ \sum_{n \in \mathbb{Z}} \frac{(C_2 l_n - C_3)^2}{1 + 2l_n} = \sum_{n \in \mathbb{Z}} \frac{C_3^2 e_n^2}{1 + 2C_3 C_2^{-1} (1 + e_n)} \sim \sum_{n \in \mathbb{Z}} e_n^2. \]

To finish the proof we make use of the following lemma \qed

Lemma 4.9. Let \((c_n)_{n \in \mathbb{Z}} \) be a sequence of real numbers with \( 1 + c_n > 0 \) and \( (1 + c_n)(1 + e_n) = 1 \). Then the following three series are equivalent:

\[ \sum_{n \in \mathbb{Z}} \left( (1 + e_n)^{1/2} - (1 + e_n)^{-1/2} \right)^2, \quad \sum_{n \in \mathbb{Z}} c_n^2 \quad \text{and} \quad \sum_{n \in \mathbb{Z}} e_n^2. \]
Proof. Set \( s^4 = C_2 C_3^{-1} \), replacing \( 1 + c_n \) by \( (1 + e_n)^{-1} \) in Lemma 8.2 gives

\[
\sum_{n \in \mathbb{Z}} \left( (1 + c_n)^{-1/2} - (1 + c_n)^{1/2} \right)^2 = \sum_{n \in \mathbb{Z}} \left( (1 + e_n)^{1/2} - (1 + e_n)^{-1/2} \right)^2.
\]

Therefore, \( \sum_{n \in \mathbb{Z}} \frac{c_n}{1 + c_n} = \sum_{n \in \mathbb{Z}} \frac{e_n}{1 + e_n} \) and hence, by Lemma 2.5, the two series are equivalent: \( \sum_{n \in \mathbb{Z}} c_n^2 \sim \sum_{n \in \mathbb{Z}} e_n^2 \).

4.4.2. Two remaining possibilities

By Lemma 4.8 there are only two cases:

1. when \( \rho_2(C_2, C_3) = \rho_3(C_2, C_3) = \infty \) for all \( (C_2, C_3) \in \mathbb{R}^2 \setminus \{0\} \),
2. when both \( \rho_2(C_2, C_3) \) and \( \rho_3(C_2, C_3) \) are finite and hence, \( \Sigma_{23}(s) < \infty \).

To illustrate this we start with the following example

Example 4.1. Set \( d_n = n^\alpha \) for \( n \in \mathbb{N} \) with \( \alpha \in \mathbb{R} \). We have

\[
\lim_{n} d_n = \begin{cases} 
\infty & \text{if } \alpha > 0 \\
1 & \text{if } \alpha = 0 \\
0 & \text{if } \alpha < 0
\end{cases}.
\]

For the general sequence \( (d_n)_{n \in \mathbb{Z}} \) we have four cases (if necessary, we can choose an appropriate subsequence):

\[
\lim_{n} d_n = \begin{cases} 
(a) & \infty \\
(b) & d > 0 \text{ with } \sum_n c_n^2 = \infty \\
(c) & d > 0 \text{ with } \sum_n c_n^2 < \infty \\
(d) & d = 0
\end{cases},
\]

where \( d_n = d(1 + c_n) \) and \( \lim_n c_n = 0 \).

4.4.3. Cases (a), (b), (d)

Remark 4.3. In the cases (a) we see by (4.40) that

\[
\rho_2(C_2, C_3) = \rho_3(C_2, C_3) = \infty \quad \text{for all } (C_2, C_3) \in \mathbb{R}^2 \setminus \{0\}.
\]

The case (d) is reduced to the case (a) by exchanging \( (b_2n, a_{2n}) \) with \( (b_{3n}, a_{3n}) \).

In the cases (b) by Lemma 4.8 and (4.50) we conclude that

\[
\rho_2(C_2, -C_3) = \rho_3(C_2, -C_3) = \infty \quad \text{for } C_2 C_3^{-1} > 0
\]

hence, \( \rho_2(C_2, C_3) = \rho_3(C_2, C_3) = \infty \) for all \( (C_2, C_3) \in \mathbb{R}^2 \setminus \{0\} \). Therefore, in cases (a), (b) and (d) we get \( x_{2n}, x_{3n} \notin \mathbb{A}^3 \).
To finish the proof in these cases, it is sufficient to approximate one of operators $D_{rn}$, $1 \leq r \leq 3$ by operators $(A_{kn})_{k \in \mathbb{Z}}$ using Lemmas 5.4–5.6, see Section 4.4.4. Alternatively we can try to approximate $D_{3n}$, $D_{2n}$ using Lemma 5.13 and its analogue, see Section 4.4.5, or to approximate $D_{3n}$, $D_{2n}$ using Lemma 5.14 and its analogue, see Section 4.4.6.

Note that by Lemma 4.4 we have $\lim_{n} b_{2n} = \lim_{n} b_{3n} = \infty$. In the cases (a) and (b) the conditions (4.33) are expressed by (4.50) as follows:

$$b = (1, b_{2n}, d_{n}b_{2n}), \quad \sum_{n} \frac{1}{b_{2n}d_{n}} < \infty, \quad \lim_{n} d_{n} = \infty, \quad (4.51)$$

$$b = (1, b_{2n}, db_{2n}(1 + c_{n})), \quad \sum_{n} \frac{1}{b_{2n}^{2}} < \infty, \quad \sum_{n} c_{n}^{2} = \infty. \quad (4.52)$$

Indeed, to get (4.51) we observe that (4.33) are expressed as follows:

$$S_{1}(3) \sim \sum_{n} \frac{1}{b_{2n}b_{3n}} = \sum_{n} \frac{1}{b_{2n}^{2}d_{n}} < \infty, \quad S_{2}(3) \sim \sum_{n} \frac{1}{d_{n}} = \infty.$$ 

Condition $S_{3}(3) \sim \sum_{n} d_{n} = \infty$ holds by $\lim_{n} d_{n} = \infty$.

In order to get (4.52), we express the conditions (4.33) as follows:

$$S_{1}(3) \sim \sum_{n} \frac{1}{b_{2n}db_{2n}(1 + c_{n})} \sim \sum_{n} \frac{1}{b_{2n}^{2}} < \infty,$$

$$S_{2}(3) \sim \sum_{n} \frac{1}{d_{n}} = \sum_{n} \frac{1}{1 + c_{n}} = \infty, \quad S_{3}(3) \sim \sum_{n} d_{n} = \sum_{n} (1 + c_{n}) = \infty.$$ 

The conditions $S_{2}(3) = \infty$ holds by $\lim_{n} c_{n} = 0$.

4.4.4. Approximation of $D_{rn}$, $1 \leq r \leq 3$, 1

By Lemmas 5.4–5.6 we have for $1 \leq k \leq 3$ (see (4.3)):

$$D_{kn} \eta \mathfrak{A}_{3} \Leftrightarrow \Delta_{k} = \infty, \quad \text{where} \quad \Delta_{k} := \Delta(Y_{k}, Y_{r}, Y_{s}),$$

and $\{k, r, s\}$ is a cyclic permutation of $\{1, 2, 3\}$.

Recall that by (4.42) we have

$$\|Y_{1}\|^{2} \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{1n}^{2}, \quad \|Y_{2}\|^{2} \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{2n}^{2}, \quad \|Y_{3}\|^{2} \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{3n}^{2}. \quad (4.53)$$
Since \( \|Y_1\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{1n}^2 \sim S_{11}^2(\mu) = \infty \), we have four possibilities for \( y_{23} := (y_2, y_3) \in \{0, 1\}^2 \):

\[
\begin{array}{cccc}
(1.0) & (1.1) & (1.2) & (1.3) \\
y_1 & 1 & 1 & 1 \\
y_2 & 0 & 1 & 0 & 1 \\
y_3 & 0 & 0 & 1 & 1
\end{array}
\]  

(4.54)

In the case (1.0) we have \( \Delta(Y_1, Y_2, Y_3) \sim \|Y_1\|^2 = \infty \), so we can approximate \( D_{1n} \) using Lemma 5.10 and the proof is finished. We should consider the three following cases:

\[
\begin{array}{ccc}
(1.1) & (1.2) & (1.3)
\end{array}
\]

In the cases (1.1), (1.2) and (1.3) we have respectively (see the proof of Lemma 4.6)

\[
\begin{align*}
\Delta(Y_1, Y_2, Y_3) &\sim \Delta(Y_1, Y_2), & \Delta(Y_2, Y_3, Y_1) &\sim \Delta(Y_2, Y_1), \\
\Delta(Y_1, Y_2, Y_3) &\sim \Delta(Y_1, Y_3), & \Delta(Y_3, Y_1, Y_2) &\sim \Delta(Y_3, Y_1), \\
\Delta(Y_1, Y_2, Y_3) &\sim \Delta(Y_1, Y_3), & \Delta(Y_3, Y_1, Y_2) &\sim \Delta(Y_3, Y_1).
\end{align*}
\]

(4.55) (4.56) (4.57)

By (4.42) and Lemma 4.4 we have respectively in the cases (1.1)–(1.3):

\[
\begin{align*}
\nu_{12}(C_1, C_2) &:= \|C_1Y_1 + C_2Y_2\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}\left(C_1a_{1n} + C_2a_{2n}\right)^2, \\
\nu_{13}(C_1, C_3) &:= \|C_1Y_1 + C_3Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}\left(C_1a_{1n} + C_3a_{3n}\right)^2, \\
\nu(C_1, C_2, C_3) &:= \|C_1Y_1 + C_2Y_2 + C_3Y_3\|^2 \sim \\
\sum_{n \in \mathbb{Z}} b_{1n}\left(C_1a_{1n} + C_2a_{2n} + C_3a_{3n}\right)^2.
\end{align*}
\]

(4.58) (4.59) (4.60)

**Remark 4.4.** We have three cases (1.1.1), (1.2.1) and (1.3.1) according to whether respectively the expressions in (4.58), (4.59) or (4.60) are divergent. We can approximate in these cases respectively \( D_{1n} \) and \( D_{2n} \) in (4.55), \( D_{1n} \) and \( D_{3n} \) in (4.56) all \( D_{1n}, D_{2n}, D_{3n} \) in (4.57). The proof of irreducibility is finished in these cases because we have \( D_{rn}, x_{2n}, x_{3n} \in \mathfrak{A}^3 \) for some \( 1 \leq r \leq 3 \).

If the opposite holds, we have three different cases:

\[
\begin{align*}
(1.1.0) & \quad \|C_1Y_1 + C_2Y_2\| < \infty \quad \text{for some} \quad (C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}, \\
(1.2.0) & \quad \|C_1Y_1 + C_3Y_3\| < \infty \quad \text{for some} \quad (C_1, C_3) \in \mathbb{R}^2 \setminus \{0\}, \\
(1.3.0) & \quad \nu(C_1, C_2, C_3) < \infty \quad \text{for some} \quad (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}.
\end{align*}
\]

45
Recall that by (2.44) we have
\[ S_{1,23}^L(\mu, t, s) = \sum_{n \in \mathbb{Z}} \left( \frac{t^2b_{1n}}{4b_{2n}} + \frac{s^2b_{1n}}{4b_{3n}} + \frac{b_{1n}}{2}(-2a_{1n} + ta_{2n} + sa_{3n})^2 \right). \]

**Remark 4.5.** In the case (1.1.0) we have \( \Sigma^{12} = \sum_{n \in \mathbb{Z}} \frac{b_{1n}}{b_{2n}} = \infty \), since \( S_{1,23}^L(\mu, t, 0) = \infty \), but \( \nu_{12}(C_1, C_2) < \infty \), and \( \Sigma^{13} = \infty \), since \( S_{1,23}^L(\mu) = \sum_{n \in \mathbb{Z}} \frac{b_{1n}}{2}(\frac{1}{2b_{3n}} + a_{3n}^2) = \infty \), but \( \|Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{3n}^2 < \infty \), see (2.43) for definition of \( S_{kr}^L(\mu) \).

In the case (1.2.0) we conclude that \( \Sigma^{13} = \infty \), since \( S_{1,23}^L(\mu, 0, s) = \infty \), but \( \nu_{13}(C_1, C_3) < \infty \), and \( \Sigma^{12} = \infty \), since \( S_{1,23}^{L}(\mu) = \sum_{n \in \mathbb{Z}} \frac{b_{1n}}{2}(\frac{1}{2b_{2n}} + a_{2n}^2) = \infty \), but \( \|Y_2\|^2 \sim \sum_{n \in \mathbb{Z}} b_{1n}a_{2n}^2 < \infty \).

In the case (1.3.0) we have \( \Sigma^{12} = \Sigma^{13} = \infty \), since \( S_{1,23}^L(\mu, t, s) = \infty \), but \( \nu(C_1, C_2, C_3) \sim \sum_{n \in \mathbb{Z}} b_{1n}(C_1a_{1n} + C_2a_{2n} + C_3a_{3n})^2 < \infty \).

So, it remains to consider only the three following cases, when \( \Sigma^{12} = \Sigma^{13} = \infty \):

1. (1.1.0)
2. (1.2.0)
3. (1.3.0)

**4.4.5. Approximation of \( D_{2n} \) and \( D_{3n} \), 2.**

Recall that by Lemma 5.13 we have \( D_{3n} \eta \mathfrak{A}^3 \Leftrightarrow \Sigma_3(\mu) = \infty \) where \( \Sigma_3(\mu) \) is defined by (5.21)

\[ \Sigma_3(\mu) := \sum_{k \in \mathbb{Z}} \frac{1}{2b_{3k}^2} \left( C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2 \right), \quad C_k = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}. \]

Similarly, by analogue of Lemma 5.13 we can prove that \( D_{2n} \eta \mathfrak{A}^3 \Leftrightarrow \Sigma_2(\mu) = \infty \), where \( \Sigma_2(\mu) \) is defined as follows

\[ \Sigma_2(\mu) := \sum_{k \in \mathbb{Z}} \frac{1}{2b_{2k}^2} \left( C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2 \right). \]

If one of \( \Sigma_2(\mu) \) or \( \Sigma_3(\mu) \) is infinite, we can approximate \( D_{2n} \) or \( D_{3n} \) and the proof is finished. If \( \Sigma_2(\mu) + \Sigma_3(\mu) < \infty \), we conclude that

\[ \Sigma_{23}(\mu) := \sum_{k \in \mathbb{Z}} \frac{1}{2b_{2k}^2} + \frac{1}{2b_{3k}^2} < \infty. \quad (4.61) \]

46
4.4.6. Approximation of $D_{2n}$ and $D_{3n}$, 3

By Lemmas 5.11–5.12 we have

\[
D_{2l}1 \in \langle x_{2k}A_{kl}1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_{22}, Y_{23}, Y_{21}) = \infty, \\
D_{3l}1 \in \langle x_{3k}A_{kl}1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_{33}, Y_{31}, Y_{32}) = \infty,
\]

where vectors $Y_{rs}$ for $2 \leq r \leq 3$, $1 \leq s \leq 3$ are defined by (5.17)–(5.20). We can not prove that $\Delta(Y_{22}, Y_{23}, Y_{21}) = \infty$ or $\Delta(Y_{33}, Y_{31}, Y_{32}) = \infty$. We can try to approximate $D_{3n}$ using Lemma 5.14 or to approximate $D_{2n}$ using an analogue of Lemma 5.14, but it does not work. Therefore, to approximate $D_{3n}$ we are forced to prove Lemma 5.15 the refinement of Lemma 5.14 and its analogue for $D_{2n}$, see Remark 4.6 below.

4.4.7. Two technical lemmas

**Lemma 4.10.** Let $a_1, a_2 \notin l_2(\mathbb{Z})$ and $C_1a_1 + C_2a_2 \in l_2$ for some $(C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}$, $C_2 \neq 0$, where $a_r = (a_{rk})_{k \in \mathbb{Z}}$, $1 \leq r \leq 2$. Then we have

\[
\sum_{k \in \mathbb{Z}} \frac{a_{1k}^2}{1 + a_{2k}^2} = \infty. \tag{4.62}
\]

**Proof.** We set $Y_r = a_r$, in the case (1.1.0) when $C_1Y_1 + C_2Y_2 = h \in l_2(\mathbb{Z})$ with $C_1C_2 > 0$ (we have $C_1C_2 \neq 0$) we should take $a_2 = -a_1 + h$, in the case when $C_1C_2 < 0$ we take $a_2 = a_1 + h$. The series $\sum_{k \in \mathbb{Z}} \frac{a_{1k}^2}{1 + a_{2k}^2}$ will remain equivalent with the initial one, if we replace $(C_1, C_2)$ with $(\pm1, 1)$ in the expression for $h$. Fix a small $\varepsilon > 0$ and a large $N \in \mathbb{N}$. Since $|\pm a + b| \leq |a| + |b|$, we get

\[
\sum_{k \in \mathbb{Z}} \frac{a_{1k}^2}{1 + a_{2k}^2} = \sum_{k \in \mathbb{Z}} \frac{a_{1k}^2}{1 + (\pm a_{1k} + h_k)^2} \geq \sum_{k \in \mathbb{Z}} \frac{a_{1k}^2}{1 + a_{1k}^2 + 2|a_{1k}|h_k + h_k^2} \overset{(2.22)}{=} \sum_{k \in \mathbb{Z}} \frac{a_{1k}^2}{1 + 2|a_{1k}|h_k + h_k^2} \overset{(*)}{=} \sum_{k \in \mathbb{Z}_N} \frac{a_{1k}^2}{1 + 2|a_{1k}|\varepsilon + \varepsilon^2} \approx \sum_{k \in \mathbb{Z}_N} a_{1k}^2 \overset{2.20}{=} \infty,
\]

where $\mathbb{Z}_N := \{n \in \mathbb{Z} \mid |n| > N\}$. The inequality (*) holds, since $h \in l_2(\mathbb{Z})$ and we have $\sum_{k \in \mathbb{Z}_N} h_k^2 < \varepsilon^2$ for sufficiently large $N \in \mathbb{N}$. \hfill \Box

**Lemma 4.11.** Let $a_1, a_2, a_3 \notin l_2(\mathbb{Z})$ and $C_1a_1 + C_2a_2 + C_3a_3 \in l_2(\mathbb{Z})$ for some $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$, $C_3 \neq 0$, where $a_r = (a_{rk})_{k \in \mathbb{Z}}$, for $1 \leq r \leq 3$. Then we have

\[
\sum_{k \in \mathbb{Z}} \frac{a_{1k}^2 + a_{2k}^2}{1 + a_{3k}^2} = \infty. \tag{4.63}
\]
Proof. We set $Y_{r} = a_{r}$, in the case (1.3.0), we have $C_{1}a_{1} + C_{2}a_{2} + C_{3}a_{3} = h \in l_{2}(\mathbb{Z})$ for some $(C_{1}, C_{2}, C_{3}) \in \mathbb{R}^{3}$, see Remark 4.4. We can take $C_{3} = 1$, then $a_{3} = -C_{1}a_{1} - C_{2}a_{2} + h$. When $C_{1} = 0$ or $C_{2} = 0$ lemma is reduced to Lemma 4.10. Suppose $C_{1}C_{2} \neq 0$. The series $\sum_{k \in \mathbb{Z}} \frac{a_{1k} + a_{2k}}{1 + a_{3k}}$ will remain equivalent with the initial one, if we replace $(C_{1}, C_{2}, C_{3})$ with $(\pm 1, \pm 1, 1)$ in the expression for $h$. Fix a small $\varepsilon > 0$ and a large $N \in \mathbb{N}$. Suppose the opposite, i.e.,

$$\infty > \sum_{k \in \mathbb{Z}} \frac{a_{1k}^2 + a_{2k}^2}{1 + (\pm a_{1k} \pm a_{2k} + h_{k})^2},$$

then $\infty > \sum_{k \in \mathbb{Z}} \frac{|a_{1k}| + |a_{2k}|^2}{1 + (\pm a_{1k} \pm a_{2k} + h_{k})^2}$

$$\geq \sum_{k \in \mathbb{Z}} \frac{(|a_{1k}| + |a_{2k}|^2)}{1 + 2|a_{1k}||a_{2k}| + 2|a_{1k}||h_{k}| + 2|a_{2k}||h_{k}| + h_{k}^2} \sim (2.22)$$

$$\sum_{k \in \mathbb{Z}} \frac{(|a_{1k}| + |a_{2k}|^2)}{1 + 2|a_{1k}||h_{k}| + 2|a_{2k}||h_{k}| + h_{k}^2} \geq \sum_{k \in \mathbb{Z}, N} \frac{(|a_{1k}| + |a_{2k}|^2)}{1 + 2(|a_{1k}| + |a_{2k}|) \varepsilon + \varepsilon^2} \sim (2.20)$$

$$\sum_{k \in \mathbb{Z}_{N}} (|a_{1k}| + |a_{2k}|^2) = \infty,$$

where $\mathbb{Z}_{N} := \{n \in \mathbb{Z} \mid |n| > N\}$, contradiction. The inequality (*) holds, since $h \in l_{2}(\mathbb{Z})$ and we have $\sum_{k \in \mathbb{Z}_{N}} h_{k}^2 < \varepsilon^2$ for sufficiently large $N \in \mathbb{N}$. □

Remark 4.6. It is possible to prove an analogue of Lemma 5.15 to approximate $D_{2n}$ with corresponding expressions $\Sigma_{2}(D, s), \Sigma_{2}^{\nu}(D, s)$ and $\Sigma_{3}(D), \Sigma_{3}^{\nu}(D)$. If one of the expressions $\Sigma_{2}(D, s), \Sigma_{2}^{\nu}(D, s), \Sigma_{3}(D, s)$ or $\Sigma_{3}^{\nu}(D, s)$ is divergent for some sequence $s = (s_{k})_{k \in \mathbb{Z}}$, we can approximate $D_{2k}$ or $D_{3k}$ and the proof is finished when $S = (0.1.1)$ in the cases (a) and (b). Suppose that for all sequence $s = (s_{k})_{k \in \mathbb{Z}}$ we have

$$\Sigma_{2}(D, s) + \Sigma_{2}^{\nu}(D, s) + \Sigma_{3}(D, s) + \Sigma_{3}^{\nu}(D, s) < \infty.$$ 

Then, in particular, we have for $s^{(r)} = (s_{rk})_{k \in \mathbb{Z}}, 2 \leq r \leq 3$ with $\frac{s_{rk}^2}{h_{rk}} \equiv 1$

$$\infty > \Sigma_{2}(D, s^{(2)}) + \Sigma_{2}^{\nu}(D, s^{(2)}) + \Sigma_{3}(D, s^{(3)}) + \Sigma_{3}^{\nu}(D, s^{(3)}) \sim$$

$$\Sigma_{2}(D) + \Sigma_{2}^{\nu}(D) + \Sigma_{3}(D) + \Sigma_{3}^{\nu}(D) = (4.64)$$

$$\sum_{k} \frac{1}{C_k} + \frac{a_{2k}^2 + a_{3k}^2}{a_{1k}^2} \sim \sum_{k} \frac{1}{2b_{2k}^2} + \frac{a_{2k}^2 + a_{3k}^2}{a_{1k}^2} \equiv: \Sigma_{23}^{\nu}(D)$$

$$\sim \sum_{k} \frac{b_{1k} + 2b_{1k}a_{2k}^2 + b_{1k} + 2b_{1k}a_{3k}^2}{1 + 2b_{1k}a_{1k}^2} \equiv: \Sigma_{23}(D).$$

48
Remark 4.7. In the case (1.1.0) (resp. the case (1.2.0)) we have $\|Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} a_{3n}^2 < \infty$ (resp. $\|Y_2\|^2 \sim \sum_{n \in \mathbb{Z}} a_{2n}^2 < \infty$) and therefore,\[
\Sigma_{23}^a(D) \sim \sum_k \frac{a_{2k}^2}{1 + a_{1k}^2} = \infty, \quad \text{resp.} \quad \Sigma_{23}^a(D) \sim \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2} = \infty,\]
by Lemma 4.10, contradicting (4.64). In the case (1.3.0) we have four cases:

(0) when $C_1 C_2 C_3 \neq 0$, $C_1 a_1 + C_2 a_2 + C_3 a_3 = h \in l_2(\mathbb{Z})$,

(1) when $C_1 = 0$ hence, $C_2 C_3 \neq 0$, $C_2 a_2 + C_3 a_3 = h \in l_2(\mathbb{Z})$,

(2) when $C_2 = 0$ hence, $C_1 C_3 \neq 0$, $C_1 a_1 + C_3 a_3 = h \in l_2(\mathbb{Z})$,

(3) when $C_3 = 0$ hence, $C_1 C_2 \neq 0$ $C_1 a_1 + C_2 a_2 = h \in l_2(\mathbb{Z})$.

In the case (0) we have $\Sigma_{23}^a(D) = \infty$ by Lemma 4.11, contradicting (4.64). In the cases (2) and (3) we get $\Sigma_{23}^a(D) = \infty$ by Lemma 4.10, contradicting (4.64). Therefore, one of the expressions $\Sigma_2(D, s)$, $\Sigma_3^\prime(D, s)$, $\Sigma_3(D, s)$ or $\Sigma_4^\prime(D, s)$ is convergent hence, we can approximate $D_{2n}$ or $D_{3n}$ and the proof is finished. To study the case (1) we need the following statement.

Lemma 4.12. Let $C_2 Y_2 + C_3 Y_3 = h_{23} \in l_2$ for some $(C_2, C_3) \in (\mathbb{R} \setminus \{0\})^2$ and $C_1 Y_1 + C_2 Y_2 \notin l_2$ or $C_1 Y_1 + C_3 Y_3 \notin l_2$ for all $(C_1, C_r) \in (\mathbb{R} \setminus \{0\})^2$, then
\[
\Delta(Y_1, Y_2, Y_3) = \infty. \tag{4.65}
\]

Proof. To prove (4.65) we have by (8.15)
\[
\begin{align*}
\Delta(Y_1, Y_2, Y_3) &= \frac{\Gamma(Y_1) + \Gamma(Y_1, Y_2) + \Gamma(Y_1, Y_3) + \Gamma(Y_1, Y_2, Y_3)}{1 + \Gamma(Y_2) + \Gamma(Y_3) + \Gamma(Y_2, Y_3)} \quad (>) \\
&\geq \frac{\Gamma(Y_1) + \Gamma(Y_1, Y_2) + \Gamma(Y_1, Y_3)}{1 + (1 + c_2)\Gamma(Y_2) + \Gamma(Y_3)} \sim \frac{\Gamma(Y_1, Y_2) + \Gamma(Y_1, Y_3)}{\Gamma(Y_2) + \Gamma(Y_3)} \quad (4.67) \\
&\sim \frac{\Gamma(Y_1, Y_2) + \Gamma(Y_1, Y_3)}{2\Gamma(Y_2)} \quad (4.67) \\
&\geq \frac{\Gamma(Y_1, Y_3)}{\Gamma(Y_3)} \sim \Gamma(Y_3), \quad \text{since} \quad C_2 Y_2 + C_3 Y_3 = h \in l_2. \quad (4.66) \\
&\geq \frac{\Gamma(Y_3)}{\Gamma(Y_2)} \quad (4.67) \\
&\geq \frac{\|Y_3\|^2}{\|Y_2\|^2} = \frac{\|Y_2 + h\|^2}{\|Y_2\|^2} \leq \left( \frac{\|Y_2\| + \|h\|}{\|Y_2\|} \right)^2 = 1,
\end{align*}
\]

The relation (*) holds by the inequality $\Gamma(Y_2, Y_3) \leq c_2 \Gamma(Y_2)$, since $C_2 Y_2 + C_3 Y_3 \in l_2$ for some $(C_1, C_3) \in (\mathbb{R} \setminus \{0\})^2$, the relation (4.66) holds by Lemma 8.11. To prove (4.67) we get since $Y_2 \notin l_2$ and $h \in l_2$,
If $C_1Y_1 + C_2Y_2 \not\in l_2$ for all $(C_1, C_2) \in (\mathbb{R} \setminus \{0\})^2$, or $C_1Y_1 + C_3Y_3 \not\in l_2$ for all $(C_1, C_3) \in (\mathbb{R} \setminus \{0\})^2$, by Lemma 4.12 we get $\Delta(Y_1, Y_2, Y_3) = \infty$ hence, we can approximate $D_{1n}$ using Lemma 5.10 and the proof is finished. If $C_1Y_1 + C_2Y_2 = h_{12} \notin l_2$ for some for $(C_1, C_2) \in (\mathbb{R} \setminus \{0\})^2$ or $C_1Y_1 + C_3Y_3 = h_{13} \notin l_2$ for some $(C_1, C_3) \in (\mathbb{R} \setminus \{0\})^2$, then we have $h_{12} + \alpha h_{23} = C_1Y_1 + C_2Y_2 + C_3Y_3 \in l_2$ or $h_{12} + \beta h_{13} = C_1Y_1 + C_2Y_2 + C_3Y_3 \in l_2$ with $C_1C_2C_3 \neq 0$ for an appropriate $\alpha\beta \neq 0$, and we are in the case (0).

4.4.8. Case (c)

In this case both $\rho_2(C_2, -C_3)$ and $\rho_3(C_2, -C_3)$ are finite, i.e., we are in the case (2) therefore, we can not approximate $x_{2n}x_{2r}, x_{3n}x_{3r}$ by Lemmas 5.2–5.3. By Lemma 4.8 $\Sigma_{23}(s) < \infty$ and hence, $\Sigma_{23}(C_2, C_3) = \infty$. Indeed, reasoning as in Remark 2.5 we see that

$$\mu^{L_{23}(\phi, s)} \perp \mu, \; \phi \in [0, 2\pi), \; s > 0 \iff \Sigma_{23}(s) + \Sigma_{23}(C_2, C_3) = \infty, \; s > 0,$$ (4.68)

for $(C_2, C_3) \in \mathbb{R}^2 \setminus \{0\}$, where $\tau_{23}(\phi, s)$, $\Sigma_{23}(s)$ and $\Sigma_{23}(C_2, C_3)$ are defined as follows:

$$\tau_{23}(\phi, s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & s^2 \sin \phi \\ 0 & s^{-2} \sin \phi & -\cos \phi \end{pmatrix},$$ (4.69)

$$\Sigma_{ij}(s) = \sum_{n \in \mathbb{Z}} \left( s^2 \sqrt{\frac{b_{in}}{b_{jn}}} - s^{-2} \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2, \; s \in \mathbb{R} \setminus \{0\},$$ (4.70)

$$\Sigma_{ij}(C_i, C_j) = \sum_{n \in \mathbb{Z}} (C_i^2b_{in} + C_j^2b_{jn})(C_i a_{in} + C_j a_{jn})^2.$$. (4.71)

In this case there are four possibilities for the pair $(\Sigma^{12}, \Sigma^{13})$:

(2.1) $(\Sigma^{12}, \Sigma^{13}) = (0, 0)$, i.e., $\Sigma^{12} < \infty$ and $\Sigma^{13} < \infty$,

(2.2) $(\Sigma^{12}, \Sigma^{13}) = (0, 1)$, i.e., $\Sigma^{12} < \infty$, but $\Sigma^{13} = \infty$,

(2.3) $(\Sigma^{12}, \Sigma^{13}) = (1, 0)$, i.e., $\Sigma^{12} = \infty$, but $\Sigma^{13} < \infty$,

(2.4) $(\Sigma^{12}, \Sigma^{13}) = (1, 1)$, i.e., $\Sigma^{12} = \infty$ and $\Sigma^{13} = \infty$.

**Lemma 4.13.** In the case (2.1), i.e., when $(\Sigma^{12}, \Sigma^{13}) = (0, 0)$, we can approximate $D_{rn}$ for $1 \leq r \leq 3$, hence the representation is irreducible.
Proof. Let $\Sigma_{12} < \infty$ and $\Sigma_{13} < \infty$ we have by (4.41)

$$\nu(C_1, C_2, C_3) \sim \sum_{k \in \mathbb{Z}} b_{1k}(C_1a_{1k} + C_2a_{2k} + C_3a_{3k})^2$$

$$\sim \sum_{k \in \mathbb{Z}} \left[ \frac{t^2 b_{1k}}{4 b_{2k}} + \frac{s^2 b_{1k}}{4 b_{3k}} + \frac{b_{1k}}{2} \left(-2a_{1k} + ta_{2k} + sa_{3k}\right)^2 \right] \sim \Sigma_{12} \Sigma_{13} = \infty.$$ (2.44)

Hence, $D_{1n}, D_{2n}, D_{3n} \eta \mathfrak{A}^3$ and the proof is finished. \[
\square
\]

Remark 4.8. The cases (2.2) and (2.3) do not occur.

Indeed, by Lemma 4.8 the three series $\Sigma_{23}(s)$ (defined by (4.70)), $\sum_{n \in \mathbb{Z}} c_n^2$ and $\sum_{n \in \mathbb{Z}} c_n^2$ are equivalent where $\frac{s_{4n} b_{3n}}{b_{3n}} = (1 + c_n)$, see Lemma 4.9. In the case (c) we have $\sum_{n \in \mathbb{Z}} c_n^2 < \infty$, therefore, $\lim_n c_n = 0$ and hence, $\lim_n d_{1n}^{-1} = \lim_n \frac{b_{2n}}{b_{3n}} = s^{-4} > 0$. Recall that $d_n = \frac{d_{3n}}{d_{2n}} = \frac{b_{2n}}{b_{3n}}$. But this contradicts $\left(\Sigma_{12}, \Sigma_{13}\right) = (0, 1)$, or $\left(\Sigma_{12}, \Sigma_{13}\right) = (1, 0)$, since the two series

$$\Sigma_{12} = \sum_n d_{2n}^{-1} \text{ and } \Sigma_{13} = \sum_n d_{3n}^{-1}$$

should be equivalent by $\lim_n \frac{d_{2n}}{d_{3n}} = s^{-4} > 0$.

In the case (2.4) we have

$$\Sigma_{23}(s) < \infty, \quad \Sigma_{23}(C_2, C_3) = \infty, \quad \Sigma_{12} = \Sigma_{13} = \infty. \quad (4.72)$$

To approximate $D_{rn}$ we need to estimate $\nu(C_1, C_2, C_3)$ defined by (4.39). By (4.41) we have

$$\nu(C_1, C_2, C_3) \sim \sum_{n \in \mathbb{Z}} b_{1n} \left(\sum_{r=1}^{3} C_r a_{rn}\right)^2.$$ Since $\|Y_1\|^2 = \sum_{n \in \mathbb{Z}} b_{1n} a_{1n}^2 \sim S_{11}^L(\mu) = \infty$, in the case (2.4) we have four possibilities for $y_{23} := (y_2, y_3) \in \{0, 1\}^2$:

<table>
<thead>
<tr>
<th></th>
<th>(2.4.1)</th>
<th>(2.4.2)</th>
<th>(2.4.3)</th>
<th>(2.4.4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$y_3$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Remark 4.9. The cases (2.4.1)–(2.4.3) are not compatible with the condition \( \Sigma_23(C_2, C_3) = \infty \) for all \((C_2, C_3) \in \mathbb{R}^2 \setminus \{0\}\).

So it suffices to consider only the case **(2.4.4)** when \( y_{123} = (1, 1, 1) \).

The case (2.4.4) splits into two subcases:
1. (2.4.4.1) when \( \Sigma_{12}(s_{12}) < \infty \) (resp. \( \Sigma_{13}(s_{13}) < \infty \)) for some \( s_{12}, s_{13} > 0 \),
2. (2.4.4.2) when both \( \Sigma_{12}(s_{12}) = \Sigma_{13}(s_{13}) = \infty \) for all \( s_{12}, s_{13} > 0 \).

The case (2.4.4.1) does not occur. Indeed, we have in this case \( \Sigma_{13}(s_{12}s_{23}) < \infty \) (resp. \( \Sigma_{12}(s_{13}s_{23}^{-1}) < \infty \)) since

\[
\Sigma_{12}(s_{12}) < \infty \Leftrightarrow \mu(s_{12}b_{1,0}) \sim \mu(b_{1,0}), \quad \Sigma_{23}(s_{23}) < \infty \Leftrightarrow \mu(s_{23}b_{2,0}) \sim \mu(b_{2,0})
\]

where \( \mu(b_{r,0}) = \otimes_{n \in \mathbb{Z}} \mu(b_{r,n,0}) \) for \( 1 \leq r \leq 3 \). Therefore,

\[
\mu((s_{12}s_{23})t_{b_{1,0}}) \sim \mu(b_{1,0}) \Leftrightarrow \Sigma_{13}(s_{12}s_{23}) < \infty.
\]

Similarly, if \( \Sigma_{13}(s_{13}) < \infty \) and \( \Sigma_{23}(s_{23}) < \infty \) we have

\[
\mu(s_{13}b_{1,0}) \sim \mu(b_{1,0}), \quad \mu(s_{23}b_{2,0}) \sim \mu(b_{2,0}) \Rightarrow \mu((s_{13}s_{23}^{-1})t_{b_{1,0}}) \sim \mu(b_{2,0})
\]

hence, \( \Sigma_{12}(s_{13}s_{23}^{-1}) < \infty \). But condition \( \Sigma_{13}(s_{12}s_{23}) + \Sigma_{12}(s_{13}s_{23}^{-1}) < \infty \) contradicts the first condition of (4.33). Indeed, we have by Lemma 4.9

\[
\Sigma_{12}(s) = \sum_{n \in \mathbb{Z}} \left( s^2 \sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2} \sqrt{\frac{b_{2n}}{b_{1n}}} \right)^2 \sim \sum_{n \in \mathbb{Z}} c_n^2 < \infty, \quad s^2 \sqrt{\frac{b_{1n}}{b_{2n}}} = 1 + c_n,
\]

\[
\Sigma_{13}(s) = \sum_{n \in \mathbb{Z}} \left( s^2 \sqrt{\frac{b_{1n}}{b_{3n}}} - s^{-2} \sqrt{\frac{b_{3n}}{b_{1n}}} \right)^2 \sim \sum_{n \in \mathbb{Z}} f_n^2 < \infty, \quad s^2 \sqrt{\frac{b_{1n}}{b_{3n}}} = 1 + f_n,
\]

and \( \lim_n c_n = \lim_n f_n = 0 \). This contradicts \( S_1(3) \sim \sum_n \frac{b_{1n}^2}{b_{2n}b_{3n}} < \infty \). Indeed,

\[
\lim_{n \to \infty} \frac{b_{1n}^2}{b_{2n}b_{3n}} = s^{-4} \lim_{n \to \infty} (1 + c_n)^2 (1 + f_n)^2 = s^{-4} > 0.
\]

Finally, to finish the case \( S = (0, 1, 1) \), we need to consider only the case (2.4.4.2) when \( \Sigma_{12}(s_{12}) = \Sigma_{13}(s_{13}) = \infty \) for all \( s_{12}, s_{13} > 0 \).
By (4.33) and all the previous considerations we have the conditions:

\[ S_1(3) \sim \sum_n \frac{b_{1n}^2}{b_{2n} b_{3n}} < \infty, \quad S_2(3) \sim \sum_n \frac{b_{2n}}{b_{3n}} = \infty, \quad S_3(3) \sim \sum_n \frac{b_{3n}}{b_{2n}} = \infty, \]

\[ \Sigma_{23}(C_2, C_3) = \infty, \quad \Sigma_{12} = \sum_n \frac{b_{1n}}{b_{2n}} = \infty, \quad \Sigma_{13} = \sum_n \frac{b_{1n}}{b_{3n}} = \infty, \quad (4.73) \]

\[ \Sigma_{12}(s_{12}) = \Sigma_{13}(s_{13}) = \infty \text{ for all } s_{12}, s_{13} > 0, \quad \Sigma_{23}(s_{23}) < \infty \text{ for some } s_{23} > 0. \]

Remark 4.10. By (4.20) without loss of generality we can suppose that \((b_{1n}, b_{2n}, b_{3n})\) is replaced with \((1, d_{2n}, d_{3n})\). Since \(\Sigma_{23}(s) < \infty\), using notations (4.47) and (4.48) of Lemma 4.8

\[ \Sigma_{23}(s) = \sum_{n \in \mathbb{Z}} \left( \frac{s^2}{d_n} - \frac{s \sqrt{d_n}}{s^2} \right)^2 = \sum_{n \in \mathbb{Z}} \left( s^2 \sqrt{\frac{d_{2n}}{d_{3n}}} - s^2 \sqrt{\frac{d_{3n}}{d_{2n}}} \right)^2, \]

and taking into consideration (4.73), we can chose \(d_{2n}\) and \(d_{3n}\) as follows:

\[ d_n = \frac{d_{3n}}{d_{2n}} = s^4(1 + c_n), \quad \sum_n c_n^2 < \infty, \quad \sum_n \frac{1}{d_{2n}^2} < \infty, \quad \sum_n \frac{1}{d_n} = \sum_n d_n = \infty. \quad (4.74) \]

Since \(\sum_n c_n^2 < \infty\) we have \(\sum_n \frac{b_{1n}^2}{b_{2n} b_{3n}} \sim \sum_n \frac{1}{d_{2n}}\) and the measures \(\mu(d_{2n}^s, 0)\) and \(\mu(d_{3n}, 0)\) are equivalent, where

\[ \mu(d_{2n}^s, 0) = \bigotimes_n \mu(s^4 d_{2n}(1+c_n), 0), \quad \mu(d_{3n}, 0) = \bigotimes_n \mu(s^4 d_{2n}, 0), \]

hence, we can choose \(c_n \equiv 0\) and \(s = 1\). So, to finish the case \(S = (0, 1, 1)\) we should prove the irreducibility for \(b = (1, d_{2n}, d_{2n})_{n \in \mathbb{Z}}\) with the only condition:

\[ \sum_n d_{2n}^{-2} < \infty. \quad (4.75) \]

As usual, \(a = (a_{1n}, a_{2n}, a_{3n})_{n \in \mathbb{Z}}\) should satisfy the orthogonality condition:

\[ (\mu_{(b, a)}^3)^L \perp \mu_{(b, a)}^3 \text{ for all } t \in \text{GL}(3, \mathbb{R}) \setminus \{e\}. \]

Example 4.2. The pairwise conditions

\[ \|C_r Y_r + C_s Y_s\|^2 = \infty \text{ for } 1 \leq r < s \leq 3 \text{ do not imply } \|\sum_{r=1}^3 C_r Y_r\|^2 = \infty. \]
Let \( a_{r,n} = a_{r,-n} \) for \( n \in \mathbb{N} \) and \( a_{1,0} = 1, \ a_{2,0} = 2, \ a_{3,0} = 3 \). We define \( a_{r,n} \) for \( n \in \mathbb{N} \) as follows

\[
 a_{1n} = \begin{cases} 
 2 & n = 2k + 1 \\
 1 & n = 2k
\end{cases}, \quad a_{2n} = \begin{cases} 
 1 & n = 2k + 1 \\
 2 & n = 2k
\end{cases}, \quad a_{3n} \equiv 3. \quad (4.76)
\]

Then we have clearly for arbitrary \( (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\} \)

\[
 ||C_1a_1 + C_2a_2||^2 = \infty, \quad ||C_1a_1 + C_3a_3||^2 = \infty, \quad ||C_2a_2 + C_3a_3||^2 = \infty, \quad (4.77)
\]

but \( a_1 + a_2 - a_3 = 0 \) hence, \( ||a_1 + a_2 - a_3||^2 = 0 \). \quad (4.78)

**Example 4.3.** Let us consider the measure \( \mu^3_{(b,a)} \) with \( a = (a_{r,n})_{r,n} \) from Example 4.2 and \( b = (b_{1n}, b_{2n}, b_{3n}) \) defined as follows:

\[
b_{1n} \equiv 1, \quad d_{2n} = d_{3n} = |n| \text{ for } n \in \mathbb{Z} \setminus \{0\}, \quad d_{20} = d_{30} = 1. \quad (4.79)
\]

**Lemma 4.14.** *In Example 4.2 we have (we consider only \( n \in \mathbb{N} \))

\[
 \Delta(a_1, a_2, a_3) = 2, \quad \Delta(a_2, a_3, a_1) = 2, \quad \Delta(a_3, a_1, a_2) = 2, \quad (4.80)
\]

where \( a_r = (a_{r,n})_{n \in \mathbb{N}}, \ 1 \leq r \leq 3 \).

**Proof.** Set \( a_r(n) = (a_{r,n})_{r=1}^n \) for \( 1 \leq r \leq 3 \) and \( n \in \mathbb{N} \), then for \( 1 \leq k < r \leq 3 \)

\[
 \Gamma(a_k(n)) \sim \Gamma(a_1(n) + a_2(n)) \sim n, \quad \Gamma(a_k(n), a_r(n)) \sim \frac{n(n-1)}{2}, \quad \Gamma(a_1, a_2, a_3) = 0.
\]

We observe that \( \Gamma(a_k, a_k + a_r) = \Gamma(a_k, a_r) \) for \( k \neq r \). Since \( a_3 = a_1 + a_2 \) we get

\[
 \Delta(a_1, a_2, a_3) = \frac{\Gamma(a_1) + \Gamma(a_1, a_2) + \Gamma(a_1, a_3) + \Gamma(a_1, a_2, a_3)}{1 + \Gamma(a_2) + \Gamma(a_3) + \Gamma(a_2, a_3)} = \frac{\Gamma(a_1) + \Gamma(a_1, a_2) + \Gamma(a_1, a_3) + \Gamma(a_1, a_2, a_3)}{1 + \Gamma(a_2) + \Gamma(a_1) + \Gamma(a_1, a_2)} = 2,
\]

\[
 \Delta(a_2, a_3, a_1) = \frac{\Gamma(a_2) + \Gamma(a_2, a_3) + \Gamma(a_2, a_1) + \Gamma(a_2, a_3, a_1)}{1 + \Gamma(a_3) + \Gamma(a_1) + \Gamma(a_2, a_3)} = \frac{\Gamma(a_2) + \Gamma(a_2, a_3) + \Gamma(a_2, a_1) + \Gamma(a_2, a_1, a_3)}{1 + \Gamma(a_1) + \Gamma(a_1, a_2)} = 2,
\]

54
\[ \Delta(a_3, a_1, a_2) = \frac{\Gamma(a_3) + \Gamma(a_3, a_1) + \Gamma(a_3, a_2) + \Gamma(a_3, a_1, a_2)}{1 + \Gamma(a_1) + \Gamma(a_2) + \Gamma(a_1, a_2)} = \frac{\Gamma(a_1 + a_2) + \Gamma(a_1 + a_2, a_1) + \Gamma(a_1 + a_2, a_2) + \Gamma(a_1 + a_2, a_1, a_2)}{1 + \Gamma(a_1) + \Gamma(a_2) + \Gamma(a_1, a_2)} = \frac{\Gamma(a_1 + a_2) + 2\Gamma(a_1, a_2)}{1 + \Gamma(a_1) + \Gamma(a_2) + \Gamma(a_1, a_2)} = 2. \]

We use two facts for \( 1 \leq r \leq 2): \[ \Gamma(a_1, a_2) \leq \Gamma(a_1) + \Gamma(a_2) \]

The first relation follows from Lemma 8.11 since \( \|C_1a_1 + C_2a_2\|^2 = \infty \). We get \[ \Gamma(a_1, a_2) = \lim_{n \to \infty} \Gamma(a_1(n), a_2(n)) = \infty. \]

Recall that \( \Gamma(a) = \|a\|^2 \). The inequality follows from \( \|a_1 + a_2\| \leq \|a_1\| + \|a_2\| \), i.e., \( \sqrt{\Gamma(a_1 + a_2)} \leq \sqrt{\Gamma(a_1)} + \sqrt{\Gamma(a_2)} \). \( \square \)

By Lemma 2.7 we have \[(\mu^2_{(b,a)})^L \perp \mu^2_{(b,a)} \iff \Sigma^\pm(t) := \Sigma^+_1(t) + \Sigma_2(t) = \infty, \]

where \[ \Sigma^+_1(t) = \sum_{n \in \mathbb{Z}} \sum_{1 \leq i < j \leq 3} \left( t_i \sqrt{\frac{b_{in}}{b_{jn}}} - A^i_j(t) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2, \]

\[ \Sigma^-_1(t) = \sum_{n \in \mathbb{Z}} \sum_{1 \leq i < j \leq 3} \left( t_i \sqrt{\frac{b_{in}}{b_{jn}}} + A^i_j(t) \sqrt{\frac{b_{jn}}{b_{in}}} \right)^2, \]

\[ \Sigma_2(t^{-1}) = \sum_{n \in \mathbb{Z}} \left[ b_{1n}((t_{11} - 1)a_{1n} + t_{12}a_{2n} + t_{13}a_{3n})^2 + b_{2n}(t_{21}a_{1n} + (t_{22} - 1)a_{2n} + t_{23}a_{3n})^2 + b_{3n}(t_{31}a_{1n} + t_{32}a_{2n} + (t_{33} - 1)a_{3n})^2 \right. \]

In Example 4.3 we can not approximate \( x_{2n}, x_{3n} \) since in this case we have \[ \Delta(Y_2^{(2)}, Y_3^{(2)}) = 1, \quad \Delta(Y_3^{(3)}, Y_2^{(3)}) = 1. \] (4.81)
Indeed, by (4.37) we have
\[
\Delta(Y_2^{(2)}, Y_3^{(2)}) = \frac{\Gamma(Y_2^{(2)}) + \Gamma(Y_2^{(3)}, Y_3^{(3)})}{1 + \Gamma(Y_2^{(3)})}, \quad \Delta(Y_3^{(3)}, Y_2^{(3)}) = \frac{\Gamma(Y_3^{(3)}) + \Gamma(Y_3^{(3)}, Y_2^{(3)})}{1 + \Gamma(Y_2^{(3)})}.
\]

In Example 4.3 we have \( d_n = \frac{d_m}{d_{2n}} \equiv 1 \) and hence, by (4.43) we have
\[
\|Y_2^{(2)}\|^2 \sim \sum_{n \in \mathbb{Z}} \frac{1}{1 + 2d_n} = \sum_{n \in \mathbb{Z}} \frac{1}{3}, \quad \|Y_3^{(3)}\|^2 \sim \sum_{n \in \mathbb{Z}} \frac{d_n^2}{1 + 2d_n} = \sum_{n \in \mathbb{Z}} \frac{1}{3},
\]
\[
\|Y_2^{(3)}\|^2 \sim \sum_{n \in \mathbb{Z}} \frac{1}{d_n^2 + 2d_n} = \sum_{n \in \mathbb{Z}} \frac{1}{3}, \quad \|Y_3^{(3)}\|^2 \sim \sum_{n \in \mathbb{Z}} \frac{d_n^2}{d_n^2 + 2d_n} = \sum_{n \in \mathbb{Z}} \frac{1}{3}.
\]
Therefore, \( \Gamma(Y_2^{(2)}, Y_3^{(2)}) = \Gamma(Y_3^{(3)}, Y_2^{(3)}) = 0 \), and
\[
\Delta(Y_2^{(2)}, Y_3^{(2)}) = \frac{\Gamma(Y_2^{(2)})}{1 + \Gamma(Y_2^{(3)})} = 1, \quad \Delta(Y_3^{(3)}, Y_2^{(3)}) = \frac{\Gamma(Y_3^{(3)})}{1 + \Gamma(Y_2^{(3)})} = 1.
\]

Since \( b_{1n} \equiv 1 \), by (4.44) we get
\[
\|Y_1\|^2 \sim \sum_{n \in \mathbb{Z}} a_{1n}^2, \quad \|Y_2\|^2 \sim \sum_{n \in \mathbb{Z}} a_{2n}^2, \quad \|Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} a_{3n}^2,
\]
so by (4.41) we have
\[
\nu(C_1, C_2, C_3) \sim \sum_{n \in \mathbb{Z}} b_{1n} \left( \sum_{r=1}^{3} C_r a_{rn} \right)^2 = \sum_{n \in \mathbb{Z}} \left( \sum_{r=1}^{3} C_r a_{rn} \right)^2.
\]
But in Example 4.2 there does not exist \( t \in \pm \text{SL}(3, \mathbb{R}) \setminus \{e\} \) such that \( \nu(C_1, C_2, C_3) = \infty \) for all \( (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\} \) to approximate some \( D_{rn} \).

4.4.9. Approximations of \( x_{2k}x_{2r} + x_{3k}x_{3r} \) in the case (c)
Since we can not approximate \( x_{2n}x_{2r}, x_{3n}x_{3r} \) using Lemmas 5.2–5.3 in the case (c), we shall try to approximate \( x_{2k}x_{2r} + s^4x_{3k}x_{3r} \) by an appropriate combinations of \( A_{kn}A_{rn} \) for \( n \in \mathbb{Z} \). Let \( s = 1 \), the general case is similar.

Lemma 4.15. For any \( k, r \in \mathbb{Z} \) one has
\[
(x_{2k}x_{2r} + x_{3k}x_{3r})1 \in \langle A_{kn}A_{rn} \mid n \in \mathbb{Z} \rangle \iff \Delta(Y^{(2)}, Y^{(1)}) = \infty, \quad (4.83)
\]
where \( Y^{(r)} = \left( \frac{b_{rn}}{\sqrt{\lambda_n}} \right)_{n \in \mathbb{Z}} \), \( 1 \leq r \leq 2 \), \( \lambda_n = (b_{1n} + b_{2n} + b_{3n})^2 - b_{1n}^2 \).
Proof. The proof of Lemma 4.15 is based on Lemma 7.2. We study when 
\((x_{2k}x_{2r} + x_{3k}x_{3r}) \mathbf{1} \in \langle A_{kn}A_{rn} \mathbf{1} \mid n \in \mathbb{Z} \rangle\). Since

\[
A_{kn}A_{rn} = (x_{1k}D_{1n} + x_{2k}D_{2n} + x_{3k}D_{3n})(x_{1r}D_{1n} + x_{2r}D_{2n} + x_{3r}D_{3n})
= x_{1k}x_{1r}D_{1n}^2 + x_{2k}x_{2r}D_{2n}^2 + x_{3k}x_{3r}D_{3n}^2 + (x_{1k}x_{2r} + x_{2k}x_{1r})D_{1n}D_{2n}
+ (x_{1k}x_{3r} + x_{3k}x_{1r})D_{1n}D_{3n} + (x_{2k}x_{3r} + x_{3k}x_{2r})D_{2n}D_{3n},
\]

and \(MD_{rn}^2 \mathbf{1} = -\frac{b_{rn}}{2}\), for \(2 \leq r \leq 3\) we take \(t = (t_n)_{n=-m}^m\) as follows:

\((t, b_2) = (t, b_3) = 1\), where \(t = (t_n)_{k=-m}^m\), \(b_2 = -(b_{2n}/2)_{n=-m}^m\), \(b_3 = -(b_{3n}/2)_{n=-m}^m\).

We have

\[
\left\| \left[ \sum_{n=-m}^m t_n A_{kn} A_{rn} - (x_{2k}x_{2r} + x_{3k}x_{3r}) \right] \mathbf{1} \right\|^2 =
\left\| \sum_{n=-m}^m t_n \left[ x_{1k}x_{1r}D_{1n}^2 + x_{2k}x_{2r}D_{2n}^2 + x_{3k}x_{3r}D_{3n}^2 + (x_{1k}x_{2r} + x_{2k}x_{1r})D_{1n}D_{2n}
+ (x_{1k}x_{3r} + x_{3k}x_{1r})D_{1n}D_{3n} + (x_{2k}x_{3r} + x_{3k}x_{2r})D_{2n}D_{3n} \right] \mathbf{1} \right\|^2
= \sum_{-m \leq n, l \leq m} (f_n, f_l) t_n t_l =: (A_{2m+1} t, t),
\]

where \(A_{2m+1} = (f_n, f_l)_{n, l = -m}^m\) and

\[
f_n = \sum_{i=1}^3 f_n^i + \sum_{1 \leq i < j \leq 3} f_n^{ij}, \quad \text{with} \quad (4.84)
\]

\[
f_n^i = x_{ik}x_{ir} \left( D_{in}^2 + \frac{b_{in}}{2} (1 - \delta_{i1}) \right) \mathbf{1}, \quad f_n^{ij} = (x_{ik}x_{jr} + x_{jk}x_{ir}) D_{in} D_{jn} \mathbf{1}
\]

for \(1 \leq i \leq 3\), \(1 \leq i < j \leq 3\). Since \(f_n^{i'} \perp f_n^{ij}\), \(f_n^{ij} \perp f_n^{i'j'}\) for different \((ij), (i'j')\), writing \(c_{kn} = \|x_{kn}\|^2 = \frac{1}{2 b_{kn}} + a_{kn}^2\), we get

\[
(f_n, f_n) = \sum_{i=1}^3 \|f_n^i\|^2 + \sum_{1 \leq i < j \leq 3} \|f_n^{ij}\|^2 =
\]

\[
c_{1k} c_{1r} 3 \left( \frac{b_{1n}}{2} \right)^2 + c_{2k} c_{2r} 2 \left( \frac{b_{2n}}{2} \right)^2 + c_{3k} c_{3r} 2 \left( \frac{b_{3n}}{2} \right)^2 +
\]

57
\[
(c_{1k}c_{2r} + c_{2k}c_{1r} + 2a_{1k}a_{2r}a_{2k}a_{1r}) \frac{b_{1n} b_{2n}}{2} + (c_{1k}c_{3r} + c_{3k}c_{1r} + 2a_{1k}a_{3r}a_{3k}a_{1r}) \\
\times \frac{b_{1n} b_{3n}}{2} + (c_{2k}c_{3r} + c_{3k}c_{2r} + 2a_{2k}a_{3r}a_{3k}a_{2r}) \frac{b_{2n} b_{3n}}{2} \sim (b_{1n} + b_{2n} + b_{3n})^2;
\]
\[
(f_n, f_l) = (f_n^1, f_l^1) = c_{1k}c_{1r} \frac{b_{1n} b_{1l}}{2} \sim b_{1n}b_{1l}.
\]

Finally, we get
\[
(f_n, f_n) \sim (b_{1n} + b_{2n} + b_{3n})^2, \quad (f_n, f_l) \sim b_{1n}b_{1l}, \quad n \neq l. \quad (4.85)
\]

Set
\[
\lambda_n = (b_{1n} + b_{2n} + b_{3n})^2 - b_{1n}^2, \quad g_n = (b_{1n}),
\]
then
\[
(f_n, f_n) \sim \lambda_n + (g_n, g_n), \quad (f_n, f_l) \sim (g_n, g_l). \quad (4.87)
\]

For \(A_{2m+1} = ((f_n, f_l))_{n,l=-m}^m\) and \(b_2 = b_3 = -(b_{2n}/2)_{n=-m}^m \in \mathbb{R}^{2m+1}\) we have
\[
A_{2m+1} = \sum_{n=-m}^m \lambda_n E_{nn} + \gamma(g_{-m}, \ldots, g_0, \ldots, g_m).
\]

To finish the proof, it suffices to use Lemma 7.2. \qed

**Remark 4.11.** In the case (c) we can approximate \(x_{2k}x_{2r} + x_{3k}x_{3r}\) since \(\Delta(Y^{(2)}, Y^{(1)}) = \infty\).

Indeed, by (4.83) we have
\[
\Delta(Y^{(2)}, Y^{(1)}) = \frac{\Gamma(Y^{(2)}) + \Gamma(Y^{(2)}, Y^{(1)})}{1 + \Gamma(Y^{(1)})} > \frac{\Gamma(Y^{(2)})}{1 + \Gamma(Y^{(1)})} =
\]
\[
\sum_{n \in \mathbb{Z}} \frac{b_{2n}^2}{\lambda_n} = \sum_{n \in \mathbb{Z}} \frac{d_{2n}^2}{1 + \sum_{n \in \mathbb{Z}} \frac{1}{(1+2d_{2n})^2-1}} \sim \sum_{n \in \mathbb{Z}} d_{n}^2 + d_{2n}^2 = \infty
\]
since by (4.74) we have \(\sum_n \frac{1}{d_{2n}^2} < \infty\). Therefore,
\[
\Gamma(Y^{(1)}) = \sum_{n \in \mathbb{Z}} \frac{1}{(1+2d_{2n})^2-1} \sim \sum_{n \in \mathbb{Z}} \frac{1}{d_{n}^2 + d_{2n}^2} < \infty.
\]
Lemma 4.16. We have for all \( k \in \mathbb{Z} \)
\[
x_{2k} 1 \in \langle (x_{2k} x_{2n} + x_{3k} x_{3n}) 1 \mid n \in \mathbb{Z} \rangle \iff \sigma_2(\mu) = \sum_{n \in \mathbb{Z}} \frac{a_{2n}^2}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{3n}^2 = \infty.
\] (4.88)

Proof. Recall the notation \( c_{rn} = \frac{1}{2b_{rn}} + a_{rn}^2 \). Since \( M_{x_{2n}} 1 = a_{2n} \) we take \( t = (t_n)_{n=-m}^m \) as follows: \( (t, a_2) = 1 \), where \( a_2 = (a_{2n})_{n=-m}^m \). We have
\[
\| \left[ \sum_{n=-m}^m t_n (x_{2k} x_{2n} + x_{3k} x_{3n}) - x_{2k} x_{2n} \right] 1 \|^2 = \\
\| \left[ \sum_{n=-m}^m t_n (x_{2k} (x_{2n} - a_{2n}) + x_{3k} x_{3n}) \right] 1 \|^2 = \|x_{2k} 1\|^2 \| \sum_{n=-m}^m t_n (x_{2n} - a_{2n}) 1 \|^2 \\
+ \|x_{3k} 1\|^2 \sum_{n=-m}^m t_n x_{3n} 1 \|^2 = c_{2k} \sum_{n=-m}^m t_n^2 \left( \frac{1}{2b_{2n}} + a_{2n}^2 \right) + c_{3k} \sum_{n=-m}^m t_n^2 \left( \frac{1}{2b_{3n}} + a_{3n}^2 \right) \\
\sim \sum_{n=-m}^m t_n^2 \left( \frac{1}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{3n}^2 \right).
\]
By (6.3) we get (4.88). \( \square \)

Similarly, we prove the following lemma.

Lemma 4.17. We have for all \( k \in \mathbb{Z} \)
\[
x_{3k} 1 \in \langle (x_{2k} x_{2n} + x_{3k} x_{3n}) 1 \mid n \in \mathbb{Z} \rangle \iff \sigma_3(\mu) = \sum_{n \in \mathbb{Z}} \frac{a_{3n}^2}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{2n}^2 = \infty.
\] (4.89)

Remark 4.12. Suppose that \( \sigma_2(\mu) + \sigma_3(\mu) < \infty \), this contradicts \( \Sigma_{23}(C_2, C_3) = \infty \) for \((C_2, C_3) \in \mathbb{R}^2 \setminus \{0\}\), where \( \Sigma_{23}(C_2, C_3) \) is defined by (4.71):
\[
\Sigma_{23}(C_2, C_3) = \sum_{n \in \mathbb{Z}} (C_{2}^2 b_{2n} + C_{3}^2 b_{3n})(C_{2} a_{2n} + C_{3} a_{3n})^2.
\]

Proof. Indeed, we have
\[
\infty > \sigma_2(\mu) + \sigma_3(\mu) \sim \sum_{n \in \mathbb{Z}} \frac{a_{2n}^2}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{3n}^2 + \sum_{n \in \mathbb{Z}} \frac{a_{3n}^2}{2b_{2n}} + \frac{1}{2b_{3n}} + a_{2n}^2 \\
\sim \sum_{n \in \mathbb{Z}} 2b_{2n} a_{2n}^2 + a_{3n}^2 + a_{2n}^2 + \frac{2}{1 + s^{-4}} \sum_{n \in \mathbb{Z}} b_{2n} (a_{2n}^2 + a_{3n}^2).
\]
This contradicts \( \Sigma_{23}(C_2, C_3) = \infty \). Indeed, by \( b_{3n} = s^4 b_{2n} \) (see (4.74)) we have
\[
\Sigma_{23}(C_2, C_3) = \sum_{n \in \mathbb{Z}} \left( C_2^2 + C_3^2 s^4 \right) b_{2n} \left( C_2 a_{2n} + C_3 a_{3n} \right)^2 < \infty. \tag*{\square}
\]

Finally, we have \( \sigma_2(\mu) + \sigma_3(\mu) = \infty \), and therefore we have \( x_{rn} \eta \mathfrak{A}^3 \) for some \( 2 \leq r \leq 3 \). Let \( x_{3n} \eta \mathfrak{A}^3 \), then we can approximate \( x_2n \) by combinations of \( x_{2n}x_{2k}, k \in \mathbb{Z} \) using an analogue of Lemma 3.3. To approximate \( D_{rn} \), \( 1 \leq r \leq 3 \) we again follows Section 4.4. As in (4.53) we get
\[
\|Y_1\|^2 \sim \sum_{n \in \mathbb{Z}} a_{1n}^2, \quad \|Y_2\|^2 \sim \sum_{n \in \mathbb{Z}} a_{2n}^2, \quad \|Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} a_{3n}^2.
\]

Indeed, for example, by (4.6) we get
\[
\|Y_1\|^2 = \sum_{n \in \mathbb{Z}} \frac{a_{1n}^2}{\frac{1}{2} b_{1n}} = \sum_{n \in \mathbb{Z}} \frac{a_{1n}^2}{1 + \frac{1}{2} \frac{1}{2} \frac{1}{2}} \sim \sum_{n \in \mathbb{Z}} a_{1n}^2.
\]

Again, as in (4.54) we have four possibilities: (1.0), (1.1), (1.2) and (1.3). The corresponding expressions in (4.58), (4.59), (4.60) becomes as follows:
\[
\nu_{12}(C_1, C_2) := \|C_1 Y_1 + C_2 Y_2\|^2 \sim \sum_{n \in \mathbb{Z}} \left( C_1 a_{1n} + C_2 a_{2n} \right)^2,
\]
\[
\nu_{13}(C_1, C_3) := \|C_1 Y_1 + C_3 Y_3\|^2 \sim \sum_{n \in \mathbb{Z}} \left( C_1 a_{1n} + C_3 a_{3n} \right)^2,
\]
\[
\nu(C_1, C_2, C_3) = \sum_{n \in \mathbb{Z}} \left( C_1 a_{1n} + C_2 a_{2n} + C_3 a_{3n} \right)^2.
\]

To study the cases (1.1.1)–(1.3.1) we should use Remark 4.4. We can approximate in these cases respectively \( D_{1n} \) and \( D_{2n} \) in (4.55), \( D_{1n} \) and \( D_{3n} \) in (4.56) all \( D_{1n}, D_{2n}, D_{3n} \) in (4.57). The proof of irreducibility is finished in these cases because we have \( D_{rn}, x_{2n}, x_{3n} \eta \mathfrak{A}^3 \) for some \( 1 \leq r \leq 3 \). Following Remark 4.6 we can use Lemma 5.15 and its analogue to approximate \( D_{2n} \) and \( D_{3n} \) with corresponding expressions \( \Sigma_2(D, s), \Sigma_3^\omega(D, s) \) and \( \Sigma_3(D), \Sigma_3^\nu(D) \). If one of the expressions \( \Sigma_2(D, s), \Sigma_3^\omega(D, s), \Sigma_3(D, s) \) or \( \Sigma_3^\nu(D, s) \) is divergent for some sequence \( s = (s_k)_{k \in \mathbb{Z}} \), we can approximate \( D_{2k} \) or \( D_{3k} \) and the proof is finished. Suppose that for all sequence \( s = (s_k)_{k \in \mathbb{Z}} \) we have
\[
\Sigma_2(D, s) + \Sigma_3^\omega(D, s) + \Sigma_3(D, s) + \Sigma_3^\nu(D, s) < \infty.
\]
Then, by (4.64) we have
\[\infty > \Sigma_{23}^\vee(D) = \sum_k \frac{1}{2b_{2k}} + a_{2k}^2 + \frac{1}{2b_{3k}} + a_{3k}^2 = \sum_k \frac{1}{2b_{1k}} + a_{2k}^2 + a_{3k}^2 \quad (4.75)\]
\[\sum_k \frac{a_{2k}^2 + a_{3k}^2}{1 + a_{1k}^2} =: \Sigma_{23}^a(D).
\]

To study the cases (1.1.0)–(1.3.0) we should follow Remark 4.7.

4.5. Case \(S = (1,1,1)\)

Denote by
\[\Sigma_{123}(s) = (\Sigma_{12}(s_1), \Sigma_{23}(s_2), \Sigma_{13}(s_3)), \quad (4.90)\]
where \(s = (s_1, s_2, s_3)\) and \(\Sigma_{ij}(s)\) are defined by (4.70) for \(1 \leq i < j \leq 3\). In terms of Remark 3.2, we have \(2^3\) possibilities for \(\Sigma_{123}(s) \in \{0, 1\}^3\):

\[
\begin{array}{cccccccc}
(0) & (1) & (2) & (3) & (4) & (5) & (6) & (7) \\
\Sigma_{12}(s_1) & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\Sigma_{23}(s_2) & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\Sigma_{13}(s_3) & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

The cases (1), (2) and (4) and respectively the cases (3), (5) and (6) result from cyclic permutations of three measures \(\mu^{(1)}, \mu^{(2)}, \mu^{(3)}\) defined as follows:

\[\mu^{(r)} = \otimes_{n \in \mathbb{Z}} \mu_{(br_n, ar_n)}, \quad 1 \leq r \leq 3, \quad \mu_0^{(r)} = \otimes_{n \in \mathbb{Z}} \mu_{(br_n, 0)}, \quad 1 \leq r \leq 3. \quad (4.91)\]

The case (1), (2) and (4) can not be realized. We prove this only in the case (1). By Lemma 8.1 we have \(\Sigma_{12}(s_1) < \infty \iff \mu_0^{(1)} \sim \mu_0^{(2)}\) and \(\Sigma_{23}(s_2) < \infty \iff \mu_0^{(2)} \sim \mu_0^{(3)}\) hence, \(\mu_0^{(1)} \sim \mu_0^{(3)}\), that contradicts \(\Sigma_{13}(s_2) = \infty \iff \mu_0^{(1)} \perp \mu_0^{(3)}\). Finally, we are left with the three cases (0), (3) and (7):

the case (0), i.e., \(\Sigma_{123}(s) = (0,0,0)\),
the case (3), i.e., \(\Sigma_{123}(s) = (0,1,1)\),
the case (7), i.e., \(\Sigma_{123}(s) = (1,1,1)\).

4.5.1. Case \(\Sigma_{123}(s) = (0,0,0)\)

In the case (0), we have for some \(s = (s_1, s_2, s_3) \in (\mathbb{R}_+)^3\)
\[\Sigma_{12}(s_1) < \infty, \quad \Sigma_{23}(s_2) < \infty, \quad \Sigma_{13}(s_3) < \infty.
\]
In this case we get \( \mu_0^{(1)} \sim \mu_0^{(2)} \sim \mu_0^{(3)} \). By (4.20) we can make the following change of the variables:

\[
\left( \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3} \right) \rightarrow \left( \frac{b'_1}{a'_1}, \frac{b'_2}{a'_2}, \frac{b'_3}{a'_3} \right) = \left( \frac{b_1}{a_1 \sqrt{b_1}}, \frac{b_2}{a_2 \sqrt{b_2}}, \frac{b_3}{a_3 \sqrt{b_3}} \right),
\]

**Remark 4.13.** By Lemma 8.2, we can suppose that

\[
b = (b_1, b_2, b_3)_{n \in \mathbb{Z}} = (1, 1 + c_n, 1 + e_n)_{n \in \mathbb{Z}}, \quad \sum_n c_n^2 < \infty, \quad \sum_n e_n^2 < \infty. \tag{4.92}
\]

But the two measures \( \mu_{(b,a)} \) and \( \mu_{(I,a)} \) are equivalent, where \( b \) is defined by (4.92) and

\[
I := (1, 1)_{n \in \mathbb{Z}}. \tag{4.93}
\]

Finally, it is sufficient to consider the measure \( \mu_{(1,a)} \).

**Example 4.4.** Let \( b_1 = b_2 = b_3 \equiv 1, \ n \in \mathbb{Z} \).

(a) Take \( a_n = (a_1, a_2, a_3), \ n \in \mathbb{Z} \) as it was defined in Example 4.2:

\[
a_1 = \begin{cases} 2n = 2k + 1 & \text{if } n = 2k + 1 \\ 1 \end{cases}, \quad a_2 = \begin{cases} 2n = 2k + 1 & \text{if } n = 2k \\ 1 \end{cases}, \quad a_3 \equiv 3.
\]

Then \( a_1 + a_2 - a_3 = 0 \), where \( a_r = (a_{rn})_{n \in \mathbb{Z}} \).

(b) Take any \( a_r = (a_{rn})_{n \in \mathbb{Z}} \) such that \( a_1, a_2, a_3 \not\in \mathbb{l}_2 \), but \( C_1a_1 + C_2a_2 + C_3a_3 \in \mathbb{l}_2(\mathbb{Z}) \) for some \( (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\} \).

**Example 4.5.** Let \( b_1 = b_2 = b_3 \equiv 1, \ n \in \mathbb{Z} \) and \( a = (a_1, a_2, a_3)_{n \in \mathbb{Z}} \) such that \( a_1, a_2, a_3 \not\in \mathbb{l}_2 \), but the measure \( \mu_{(3)}^{(b,a)} \) satisfies the orthogonality conditions. The case \( \Sigma_{123}(s) = (0, 0, 0) \) is reduced to this example.

**Remark 4.14.** Since the measure \( \mu_{(3)}^{(b,0)} \) is standard in Example 4.4 and 4.5, i.e., it is invariant under rotations \( \pm \mathbb{O}(3) \), we have

\[
(\mu_{(b,0)}^3)^{L_t} = \mu_{(b,0)}^3 \quad \text{for all} \quad t \in \pm \mathbb{O}(3). \tag{4.94}
\]

By Lemma 2.8, the orthogonality condition \( (\mu_{(b,a)}^3)^{L_t} \perp \mu_{(b,a)}^3 \) for \( t \in \pm \mathbb{O}(3) \setminus \{e\} \), is equivalent to

\[
\Sigma_1^+(t) + \Sigma_2(t) = \infty,
\]
where $\Sigma_1^+(t)$, $\Sigma_1^-(t)$ are defined by (2.31), (2.32) and $\Sigma_2(t)$ is defined by (2.28). By (4.94) we get $\Sigma_1^+(t) < \infty$ in Example 4.4 and 4.5 hence the orthogonality condition $(\mu_{(b,a)}^3)^{Lt} \perp \mu_{(b,a)}^3$ for $t \in \pm O(3) \setminus \{e\}$ is equivalent to $\Sigma_2(t) = \infty$. Further, to prove the irreducibility in Example 4.4 and 4.5 we should show that $\Sigma_2(t) = \infty$ for all $t \in \pm O(3) \setminus \{e\}$ implies

$$\|C_1Y_1 + C_2Y_2 + C_3Y_3\|^2 = \infty \quad \text{for all} \quad (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}.$$ 

**Lemma 4.18.** (1) The representations corresponding to the measures in Example 4.4 (a) and (b) are reducible.

(2) The representations corresponding to the measures in Example 4.5 are irreducible.

**Proof.** To prove the part (1) of theorem, by Remark 4.14 and (4.94), we should find for the measure in Example 4.4 an element $t \in \pm O(3) \setminus \{e\}$ such that $\Sigma_2(t) < \infty$. This will imply $(\mu_{(b,a)}^3)^{Lt} \sim \mu_{(b,a)}^3$ hence, the reducibility.

Finally, it is sufficient to find $t \in \pm O(3) \setminus \{e\}$ such that

$$t - 1 = \left( \frac{\lambda_1 C_1 \lambda_2 C_2 \lambda_3 C_3}{\lambda_1 C_1 \lambda_2 C_2 \lambda_3 C_3} \right),$$

(4.95)

where $(C_1, C_2, C_3) = (1, 1, -1)$, in part (a), or for an arbitrary $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$ in the part (b). Such an element exists by Lemma 4.19 below. For such an element $t$ we get respectively in the cases (a), (b) and Example 4.5 (see (2.28)):

$$\Sigma_2(t^{-1}) = \sum_{n \in \mathbb{Z}} (b_{1n} \lambda_1^2 + b_{2n} \lambda_2^2 + b_{3n} \lambda_3^2)(a_{1n} + a_{2n} - a_{3n})^2 = 0,$$

$$\Sigma_2(t^{-1}) = \sum_{n \in \mathbb{Z}} (b_{1n} \lambda_1^2 + b_{2n} \lambda_2^2 + b_{3n} \lambda_3^2)(C_1 a_{1n} + C_2 a_{2n} + C_3 a_{3n})^2 < \infty,$$

$$\Sigma_2(t^{-1}) = \sum_{n \in \mathbb{Z}} (b_{1n} \lambda_1^2 + b_{2n} \lambda_2^2 + b_{3n} \lambda_3^2)(C_1 a_{1n} + C_2 a_{2n} + C_3 a_{3n})^2 = \infty.$$ (4.96)

Note that the measure in Example 4.4 does not satisfy the orthogonality conditions.

(2) *Irreducibility.* In Example 4.5 we can not approximate $x_{rn}$ by Lemmas 5.1–5.3, since all the expressions

$$\Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}), \quad \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}), \quad \Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)})$$
are bounded. To approximate $D_{rn}$ using Lemmas 5.4–5.6, we should estimate the following expressions:

$$\Delta(Y_1, Y_2, Y_3), \quad \Delta(Y_2, Y_3, Y_1), \quad \Delta(Y_3, Y_1, Y_2).$$

By Lemma 8.15, all these expressions are infinite, if for all $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$ holds

$$\nu(C_1, C_2, C_3) := \|C_1 Y_1 + C_2 Y_2 + C_3 Y_3\|^2 = \sum_{n \in \mathbb{Z}} \left( \frac{C_1 a_{1n} + C_2 a_{2n} + C_3 a_{3n}}{1/2b_{1n} + 1/2b_{2n} + 1/2b_{3n}} \right)^2 = \infty.$$

In Examples 4.5 we have

$$\nu(C_1, C_2, C_3) = \|C_1 Y_1 + C_2 Y_2 + C_3 Y_3\|^2 \sim \sum_{k \in \mathbb{Z}} b_{1n}(C_1 a_{1k} + C_2 a_{2k} + C_3 a_{3k})^2$$

$$\sim \sum_{n \in \mathbb{Z}} (b_{1n} \lambda_1^2 + b_{2n} \lambda_2^2 + b_{3n} \lambda_3^2)(C_1 a_{1n} + C_2 a_{2n} + C_3 a_{3n})^2 = \Sigma_2(t^{-1}) = \infty. \quad \square$$

**Lemma 4.19.** For an arbitrary $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$, and an arbitrary $D_3(s) = \text{diag}(s_1, s_2, s_3)$ with $(s_1, s_2, s_3) \in (\mathbb{R}_+)^3$, there exists a unique element $t \in \pm O(3) \setminus \{e\}$ and $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \setminus \{0\}$ such that

$$D_3(s) t D_3^{-1}(s) - I = \begin{pmatrix} \lambda_1 & \lambda_1 C_1 & \lambda_1 C_2 \\ \lambda_2 & \lambda_2 C_1 & \lambda_2 C_2 \\ \lambda_3 & \lambda_3 C_1 & \lambda_3 C_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{pmatrix}. \quad (4.97)$$

**Proof.** By (4.97) we get

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} := t = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} = \begin{pmatrix} C_1 \lambda_1 + \frac{2\pi}{3} C_2 \lambda_2 & \frac{2\pi}{3} C_3 \lambda_1 \\ \frac{2\pi}{3} C_1 \lambda_2 & C_2 \lambda_2 + \frac{2\pi}{3} C_3 \lambda_2 \\ \frac{2\pi}{3} C_1 \lambda_3 & \frac{2\pi}{3} C_2 \lambda_3 & C_3 \lambda_3 + 1 \end{pmatrix}, \quad (4.98)$$

where \(\|e_k\|^2 = 1\) and \(e_k \perp e_r, \quad 1 \leq k < r \leq 3. \quad (4.99)\)

By (4.98) and the first relations in (4.99) we get

$$\lambda_k = -\frac{2s_k^2 C_k}{s_1^2 C_1^2 + s_2^2 C_2^2 + s_3^2 C_3^2}, \quad 1 \leq k \leq 3. \quad (4.100)$$
Then the matrix elements \( t = (t_{kr})_{k,r=1}^3 \) are defined by (4.98). To verify \( e_k \perp e_r \) we need to show that

\[
(e_1, e_2) = \frac{(s_1^2 C_1^2 + s_2^2 C_2^2 + s_3^2 C_3^2) \lambda_1 \lambda_2}{s_1 s_2} + \frac{s_1^2 C_1 \lambda_2 + s_2^2 C_2 \lambda_1}{s_1 s_2} = 0,
\]

\[
(e_1, e_3) = \frac{(s_1^2 C_1^2 + s_2^2 C_2^2 + s_3^2 C_3^2) \lambda_3}{s_1 s_3} + \frac{s_1^2 C_1 \lambda_3 + s_3^2 C_3 \lambda_1}{s_1 s_3} = 0,
\]

\[
(e_2, e_3) = \frac{(s_2^2 C_2^2 + s_3^2 C_3^2) \lambda_2 \lambda_3}{s_2 s_3} + \frac{s_2^2 C_2 \lambda_3 + s_3^2 C_3 \lambda_2}{s_2 s_3} = 0.
\]

Indeed, for example, for \((e_1, e_2)\) we have

\[
(e_1, e_2) = \frac{(s_1^2 C_1^2 + s_2^2 C_2^2 + s_3^2 C_3^2) \lambda_1 \lambda_2}{s_1 s_2} + \frac{s_1^2 C_1 \lambda_2 + s_2^2 C_2 \lambda_1}{s_1 s_2} = \frac{1}{s_1 s_2(s_1^2 C_1^2 + s_2^2 C_2^2 + s_3^2 C_3^2)} \left( 4s_1^2 s_2^2 - (2s_1^2 s_2^2 + 2s_1^2 s_2^2) \right) C_1 C_2 = 0.
\]

The proofs of \( e_1 \perp e_3 \) and \( e_2 \perp e_3 \) are similar. \( \square \)

Similarly, for any \( m \geq 2 \) we can prove the following lemma:

**Lemma 4.20.** For an arbitrary \((C_k)_{k=1}^m \in \mathbb{R}^m \setminus \{0\}\), and \(D_m(s) = \text{diag}(s_k)_{k=1}^m\) with \(s_k \in \mathbb{R}_+\), \(1 \leq k \leq m\) there exists a unique element \( t \in \pm \mathbb{O}(m) \setminus \{e\}\) and \((\lambda_k)_{k=1}^m \in \mathbb{R}^m \setminus \{0\}\) such that

\[
D_m(s) t D_m^{-1}(s) - I = \begin{pmatrix}
\lambda_1 C_1 & \lambda_1 C_2 & \ldots & \lambda_1 C_m \\
\lambda_2 C_1 & \lambda_2 C_2 & \ldots & \lambda_2 C_m \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_m C_1 & \lambda_m C_2 & \ldots & \lambda_m C_m
\end{pmatrix}.
\]

(4.101)

The formulas for the corresponding \( \lambda_k \) are as follows:

\[
\lambda_k = -\frac{2s_k^2 C_k}{\sum_{r=1}^m s_r^2 C_r}, \quad 1 \leq k \leq m.
\]

(4.102)

**Lemma 4.21.** For an arbitrary \((C_1, C_2) \in \mathbb{R}^2 \setminus \{0\}\) and arbitrary \(D_2(s) = \text{diag}(s_1, s_2)\) with \(s_1, s_2 \neq 0\) there exists a unique element \( t \in \pm \mathbb{O}(2) \setminus \{e\}\) and \((\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{0\}\) such that

\[
D_2(s) t D_2^{-1}(s) - I = \begin{pmatrix}
\lambda_1 C_1 & \lambda_2 C_2 \\
\lambda_2 C_1 & \lambda_1 C_2
\end{pmatrix}.
\]

(4.103)

The formulas for the corresponding \( \lambda_k \) are as follows:

\[
\lambda_1 = -\frac{2s_1^2 C_1}{s_1^2 C_1^2 + s_2^2 C_2^2}, \quad \lambda_2 = -\frac{2s_2^2 C_2}{s_1^2 C_1^2 + s_2^2 C_2^2}.
\]

(4.104)
In particular, we have
\[
\begin{pmatrix}
  \frac{t_1}{t_2} & \frac{s_2 t_1}{t_2} \\
  \frac{s_1 t_1}{t_2} & t_2
\end{pmatrix}
= \begin{pmatrix}
  C_1 \lambda_1 + 1 & C_2 \lambda_1 \\
  C_1 \lambda_2 & C_2 \lambda_2 + 1
\end{pmatrix}
= \begin{pmatrix}
  \frac{s_2^2 C_2^2 - s^2 C_1^2}{s_1 C_1^2 + s_2 C_2^2} & \frac{-2 s^2 C_1 C_2}{s_1 C_1^2 + s_2 C_2^2} \\
  \frac{s_1 C_1^2 - s_2 C_2^2}{s_1 C_1^2 + s_2 C_2^2} & \frac{-2 s_2^2 C_2}{s_1 C_1^2 + s_2 C_2^2}
\end{pmatrix}
= \tau_{12}(\phi, s_1, s_2).
\]

We can verify that
\[
\tau_{12}(\phi, s, s^{-1}) = \tau_-(\phi, s)
\]
where \( \tau_-(\phi, s) \) is defined by (2.4). We just set
\[
\cos \phi = \frac{s^{-2} C_2^2 - s^2 C_1^2}{s^2 C_1^2 + s^{-2} C_2^2} \quad \text{and} \quad \sin \phi = \frac{-2 C_1 C_2}{s^2 C_1^2 + s^{-2} C_2^2}.
\]

In addition \( \det t = -1 \).

4.5.2. Case \( \Sigma_{123}(s) = (0, 1, 1) \)

We have for some \( s_1 \in \mathbb{R}_+ \) and all \( (s_2, s_3) \in (\mathbb{R}_+)^2 \)
\[
\Sigma_{23}(s_2) = \infty, \quad \Sigma_{13}(s_3) = \infty.
\]

Remark 4.15. Since \( \Sigma_{12}(s_1) < \infty \), by (4.20) and Lemma 8.2, we can suppose that
\[
b = (b_{1n}, b_{2n}, b_{3n})_{n \in \mathbb{Z}} = (1, s_1^2(1 + c_n), b_{3n})_{n \in \mathbb{Z}}, \quad \sum_n c_n^2 < \infty,
\]
therefore, we can take \( b = (1, 1, b_{3n})_{n \in \mathbb{Z}}, \quad s = 1, \quad c_n \equiv 0. \)

Since \( \Sigma_{13}(s) = \sum_{n \in \mathbb{Z}} \left( \frac{s^2}{\sqrt{b_{3n}}^2} - \frac{\sqrt{b_{3m}}^2}{s^2} \right)^2 = \infty \), we have as in (4.50) three cases:
\[
\lim_n b_{3n} = \begin{cases}
(a) \infty \\
(b) b > 0 \quad \text{with} \quad \sum_n b_n^2 = \infty, \\
(c) 0
\end{cases}
\]
where \( b_{3n} = b(1 + b_n) \) with \( \lim_n b_n = 0 \) in the case (b). Note that condition \( S_3(3) = \infty \), implies \( \sum_n b_{3n}^2 = \infty \). Indeed, by (4.8) we have for \( 1 \leq r \leq 3 \)
\[
S_r(3) = \sum_{n \in \mathbb{Z}} \frac{b_{rn}^2}{b_{1n} b_{2n} + b_{1n} b_{3n} + b_{2n} b_{3n}}, \quad S_1(3) = \sum_{n \in \mathbb{Z}} \frac{1}{1 + 2 b_{3n}} = \infty,
\]
\[
S_2(3) = \sum_{n \in \mathbb{Z}} \frac{1}{1 + 2 b_{3n}} = \infty, \quad \infty = S_3(3) = \sum_{n \in \mathbb{Z}} \frac{b_{3n}^2}{1 + 2 b_{3n}} \stackrel{(2.20)}{\sim} \sum_{n \in \mathbb{Z}} b_{3n}^2. \quad (4.106)
\]
By (4.5) we have
\[ \|Y_r^{(r)}\|^2 = \sum_{k \in \mathbb{Z}} b_{rk}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}), \]
\[ \|Y_r^{(s)}\|^2 = \sum_{k \in \mathbb{Z}} b_{sk}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}), \quad s \neq r. \]

Let us denote
\[
\begin{pmatrix}
Y_{1n}^{(1)} & Y_{2n}^{(1)} & Y_{3n}^{(1)} \\
Y_{1n}^{(2)} & Y_{2n}^{(2)} & Y_{3n}^{(2)} \\
Y_{1n}^{(3)} & Y_{2n}^{(3)} & Y_{3n}^{(3)}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{3+4b_{3n}}} & \frac{1}{\sqrt{3+4b_{3n}}} & \frac{b_{1n}}{\sqrt{3+4b_{3n}}} \\
\frac{1}{\sqrt{3+4b_{3n}}} & \frac{1}{\sqrt{3+4b_{3n}}} & \frac{b_{1n}}{\sqrt{3+4b_{3n}}} \\
\frac{b_{2n}}{\sqrt{3+4b_{3n}+2}} & \frac{b_{2n}}{\sqrt{3+4b_{3n}+2}} & \frac{b_{2n}}{\sqrt{3+4b_{3n}+2}}
\end{pmatrix}
\]

(4.107)

We have \( \Delta(Y_{1}^{(1)}, Y_{2}^{(1)}, Y_{3}^{(1)}) = \Delta(Y_{2}^{(2)}, Y_{3}^{(2)}, Y_{1}^{(2)}) < \infty \), indeed, since \( Y_{1}^{(2)} = Y_{2}^{(2)} \) we get for example
\[
\Delta(Y_{2}^{(2)}, Y_{3}^{(2)}, Y_{1}^{(2)}) = \frac{\Gamma(Y_{2}^{(2)}) + \Gamma(Y_{3}^{(2)}, Y_{1}^{(2)})}{1 + \Gamma(Y_{3}^{(2)}) + \Gamma(Y_{1}^{(2)}, Y_{2}^{(2)})} < 1 < \infty.
\]

**Lemma 4.22.** In the cases (a), (b) and (c) given by (4.105) we have
\[ \Delta(Y_{1}^{(3)}, Y_{2}^{(3)}, Y_{2}^{(3)}) = \infty. \] (4.108)

**Proof.** In all these cases we have \( Y_{1}^{(3)} = Y_{2}^{(3)} \) hence, \( \Gamma(Y_{3}^{(3)}, Y_{1}^{(3)}, Y_{2}^{(3)}) = 0 \) and \( \Gamma(Y_{1}^{(3)}, Y_{2}^{(3)}) = 0 \). Therefore, by (8.15)
\[
\Delta(Y_{3}^{(3)}, Y_{1}^{(3)}, Y_{2}^{(3)}) = \frac{\Gamma(Y_{3}^{(3)}) + \Gamma(Y_{3}^{(3)}, Y_{1}^{(3)}) + \Gamma(Y_{3}^{(3)}, Y_{2}^{(3)})}{1 + \Gamma(Y_{3}^{(3)}) + \Gamma(Y_{1}^{(3)}, Y_{2}^{(3)})}
\]
\[ = \frac{\Gamma(Y_{3}^{(3)}) + 2\Gamma(Y_{3}^{(3)}, Y_{1}^{(3)})}{1 + 2\Gamma(Y_{3}^{(3)})} \sim \Delta(Y_{3}^{(3)}, Y_{1}^{(3)}), \] (4.109)

We have two cases:
(a.1) when \( \|Y_{1}^{(3)}\| < \infty \), and (a.2) when \( \|Y_{1}^{(3)}\| = \infty \).

In the case (a.1) we have \( \Delta(Y_{3}^{(3)}, Y_{1}^{(3)}) \sim \Gamma(Y_{3}^{(3)}) = \infty \). Therefore, (4.108) holds. In the case (a.2) we should verify that
\[ \|C_{1}Y_{1}^{(3)} + C_{3}Y_{3}^{(3)}\|^2 = \infty \quad \text{for all} \quad (C_{1}, C_{3}) \in \mathbb{R}^2 \setminus \{0\}. \] (4.110)
Then this will imply \((4.108)\). We have
\[
\|C_1 Y_1^{(3)} + C_3 Y_3^{(3)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{(C_1 + C_3 b_{3n})^2}{b_{3n}^2 + 4b_{3n} + 2} =: \sum_{n \in \mathbb{Z}} g_n.
\]

If \(C_1 = 0\) or \(C_3 = 0\) the later expression is divergent since \(Y_1^{(3)} = Y_3^{(3)} = \infty\). Let \(C_1 C_3 \neq 0\). In this case \(\lim_n g_n = C_3^2 > 0\) since \(\lim_n b_{3n} = \infty\), case (a). Therefore, \(\sum_{n \in \mathbb{Z}} g_n = \infty\). By Lemma 8.11 this implies \(\Delta(Y_3^{(3)}, Y_1^{(3)}) = \infty\) therefore, \((4.108)\). In the case (b) we have by \((4.109)\)
\[
\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = \Delta(Y_3^{(3)}, Y_1^{(3)}).
\]

To prove that \(\Delta(Y_3^{(3)}, Y_1^{(3)}) = \infty\) using Lemma 8.11 we should verify \((4.110)\). We have \(\|Y_3^{(3)}\|^2 = \infty\) since \(S = (0, 1, 1)\). By \((4.107)\)
\[
\|Y_1^{(3)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{1}{b_{3n}^2 + 4b_{3n} + 2} \sim \sum_{n \in \mathbb{Z}} \frac{1}{b^2 + 4b + 2} = \infty.
\]

The expression \(\|C_1 Y_1^{(3)} + C_3 Y_3^{(3)}\|^2\) can be finite only for \((C_1, C_3) = \lambda(b, -1)\). Take \(\lambda = 1\), we get in the case (b)
\[
\|C_1 Y_1^{(3)} + C_3 Y_3^{(3)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{(b - b_{3n})^2}{b_{3n}^2 + 4b_{3n} + 2} = \sum_{n \in \mathbb{Z}} \frac{b^2 b_{3n}^2}{b_{3n}^2(1 + b_{3n})^2 + 4b(1 + b_{3n}) + 2}
\]
\[
\sim \sum_{n \in \mathbb{Z}} \frac{b_{3n}^2}{(4b + 2b^2)b_{3n} + b^2 + 4b + 2} \sim \sum_{n \in \mathbb{Z}} \frac{b_{3n}^2}{b_{3n}^2} = \infty.
\]

In the case (c), we have by \((4.109)\)
\[
\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) \sim \Delta(Y_3^{(3)}, Y_1^{(3)}).
\]

To prove that \(\Delta(Y_3^{(3)}, Y_1^{(3)}) = \infty\) using Lemma 8.11 we should verify \((4.110)\). Again, we have \(\|Y_3^{(3)}\|^2 = \infty\) since \(S = (0, 1, 1)\). Because of \(\lim_n b_{3n} = 0\), we have by \((4.107)\)
\[
\|Y_1^{(3)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{1}{b_{3n}^2 + 4b_{3n} + 2} \sim \sum_{n \in \mathbb{Z}} \frac{1}{2} = \infty.
\]

Let \(C_1 C_3 \neq 0\), then since \(\lim_n b_{3n} = 0\) we get
\[
\|C_1 Y_1^{(3)} + C_3 Y_3^{(3)}\|^2 = \sum_{n \in \mathbb{Z}} \frac{(C_1 + C_3 b_{3n})^2}{b_{3n}^2 + 4b_{3n} + 2} = \sum_{n \in \mathbb{Z}} \frac{C_1^2 (b_{3n} + C_1 C_3^{-1})^2}{b_{3n}^2 + 4b_{3n} + 2} = \infty. \quad \square
\]
By Lemma 4.22 we can approximate $x_{3n}$. By (4.6) we have

$$\|Y_1\|^2 = \sum_{n \in \mathbb{Z}} \frac{a_{1n}^2}{2b_{3n}} + \frac{1}{2b_{2n}} + \frac{1}{2b_{1n}} = \sum_{k \in \mathbb{Z}} \frac{a_{1n}^2}{1 + \frac{1}{2b_{2n}}} = \sum_{k \in \mathbb{Z}} \frac{2b_{3n}a_{1n}^2}{1 + 2b_{3n}},$$

$$\|Y_2\|^2 = \sum_{n \in \mathbb{Z}} \frac{a_{2n}^2}{1 + \frac{1}{2b_{3n}}} = \sum_{k \in \mathbb{Z}} \frac{2b_{3n}a_{2n}^2}{1 + 2b_{3n}}, \quad \|Y_3\|^2 = \sum_{n \in \mathbb{Z}} \frac{a_{3n}^2}{1 + \frac{1}{2b_{3n}}} = \sum_{k \in \mathbb{Z}} \frac{2b_{3n}a_{3n}^2}{1 + 2b_{3n}}.$$
Further, in the case (a), (b) we have four possibilities: (1.1.1), (1.3.1) and (1.1.0), (1.3.0), see Remark 4.4. In the case (1.1.1) we can approximate $D_{1n}$, $D_{2n}$, in the case (1.3.1) we can approximate all $D_{rn}$, $1 \leq r \leq 3$. In these cases the proof is finished, since we get respectively $D_{1n}, D_{2n}, x_{3n} \eta \mathfrak{A}^3$. The cases (a), (b) subcases (1.1.0), (1.3.0) we consider below.

In the case (c) subcase (1.0) we can approximate $D_{3n}$ using Lemma 5.6, since $\Delta(Y_3, Y_2, Y_1) \sim ||Y_3||^2 = \infty$, so we have $D_{3n}, x_{3n} \eta \mathfrak{A}^3$, and the proof is finished.

Further, in the case (c) we have six cases (1.1.1), (1.2.1), (1.3.1), (1.1.0), (1.2.0), (1.3.0), according to whether corresponding expressions are divergent (see analogue in Remark 4.4). We can approximate in the three first cases by respectively $D_{1n}$ and $D_{3n}$ in the case (1.1.1), $D_{2n}$ and $D_{3n}$ in the case (1.1.2) and all $D_{1n}, D_{2n}, D_{3n}$ in (1.1.3). The proof of irreducibility is finished in these cases because we have respectively $D_{1n}, D_{3n}, x_{3n} \eta \mathfrak{A}^3$, $D_{2n}, D_{3n}, x_{3n} \eta \mathfrak{A}^3$, or $D_{1n}, D_{2n}, D_{3n}, x_{3n} \eta \mathfrak{A}^3$.

If the opposite holds, in the cases (a), (b) or (c), i.e., we are in the cases (1.1.0), (1.2.0) and (1.3.0) respectively, we try to approximate $D_{3n}$ using Lemma 5.15. If one of the expressions $\Sigma_3(D, s)$ or $\Sigma_3'(D, s)$ is divergent, we can approximate $D_{3n}$ and the proof is finished, since we have $x_{3n}, D_{3n} \eta \mathfrak{A}^3$. Let us suppose, as in Remark 4.2, that for every sequence $s = (s_k)_{k \in \mathbb{Z}}$ holds

$$\Sigma_3(D, s) + \Sigma_3'(D, s) < \infty.$$ 

Then, in particular, we have for $s^{(3)} = (s_k)_{k \in \mathbb{Z}}$ with $s_{3k}^2 b_{3k}^{-1} \equiv 1$

$$\infty > \Sigma_3(D, s^{(3)}) + \Sigma_3'(D, s^{(3)}) \sim \Sigma_3(D) + \Sigma_3'(D) = \sum_k \frac{1}{a_{1k}} + a_{3k}^2 + a_{2k}^2 + a_{3k}^2 \notag$$

$$\sim (2.22) \sum_k \frac{1}{2b_{3k}} + a_{1k}^2 + \frac{1}{2b_{2k}} + a_{2k}^2 = \sum_k \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2 \equiv \Sigma_3^{3,+}(D). \quad (4.113)$$

In the case (a), (b) and (c) we have respectively

$$\Sigma_3^{3,+}(D) \sim \Sigma_3^{3,+}(D) = \sum_k \frac{2a_{3k}^2}{1 + 2a_{1k}^2 + 2a_{2k}^2}, \quad \Sigma_3^{3,+}(D) = \sum_k \frac{1}{1 + a_{1k}^2 + a_{2k}^2}.$$ 

In particular, in the case (c) we have by (4.113)

$$\infty > \sum_k \frac{1}{2b_{3k}} + a_{3k}^2 \sim \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} \sim \Sigma_3^{3,+}(D). \quad (4.114)$$

70
The cases (a), subcase (1.1.0), where ∥Y₃∥² < ∞ can not occur, because conditions Σ₁₂(s₁) < ∞ and ν₁₂(C₁, C₂) < ∞ defined by (4.58), contradict the orthogonality condition for the matrix τ₁₂(φ, s):

\[
\tau₁₂(φ, s) = \begin{pmatrix}
\cos φ & s² \sin φ & 0 \\
-\sin φ & -\cos φ & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

Indeed, recall Remark 2.5 (instead of \(\mu₂(b,a)\) we can write \(\mu₃(b,a)\))

\[
(\mu₃(b,a))^{L₁₂(φ,s)} \perp \mu₃(b,a) \Leftrightarrow Σ₁₂(s) + Σ₁₂(C₁, C₂) = ∞,
\]

where Σ₁₂(s) = Σ₁(s) is defined by (2.17) and Σ₁₂(C₁, C₂) = Σ₂(C₁, C₂) is defined by (2.19)

\[
Σ₁₂(C₁, C₂) := \sum_{n∈Z} (C₁ b₁n + C₂ b₂n)(C₁ a₁n + C₂ a₂n)² ∼ ν₁₂(C₁, C₂).
\]

We get contradiction:

\[
∞ > Σ₁₂(s) + ν₁₂(C₁, C₂) ∼ Σ₁₂(s) + Σ₁₂(C₁, C₂) = ∞.
\]

In the case (a), (b), subcase (1.3.0) we get \(\Sigma₃⁺(D) = ∞\) by Lemma 4.11, contradiction with (4.113) hence, \(D₃n, 3\mathfrak{A}\)

In the case (c), subcases (1.1.0) and (1.2.0) we have respectively \(∥Y₂∥² < ∞\) and \(∥Y₁∥² < ∞\) hence,

\[
Σ₃⁺(D) ∼ \sum_k \frac{a_{3k}²}{1 + a_{1k}²} = ∞, \quad Σ₃⁺(D) ∼ \sum_k \frac{a_{3k}²}{1 + a_{2k}²} = ∞.
\]

by Lemma 4.10, contradiction with (4.113) hence, \(D₃n, 3\mathfrak{A}\). In the case (c), subcase (1.3.0) we get

\[
Σ₃⁺(D) = \sum_k \frac{a_{3k}²}{1 + a_{1k}² + a_{2k}²} = ∞
\]

by Lemma 4.11, contradiction with (4.113) hence, \(D₃n, 3\mathfrak{A}\).
4.5.3. Case $\Sigma_{123}(s) = (1, 1, 1)$

We have for all $s = (s_{12}, s_{23}, s_{13}) \in \mathbb{R}^3 \setminus \{0\}$

$$\Sigma_{12}(s_{12}) = \infty, \quad \Sigma_{23}(s_{23}) = \infty, \quad \Sigma_{13}(s_{13}) = \infty, \quad (4.116)$$

$b = (b_{1n}, b_{2n}, b_{3n})_{n \in \mathbb{Z}} \overset{(4.20)}{=} (1, d_{2n}, d_{3n})_{n \in \mathbb{Z}}$.

Recall (4.34), that we denote $D_n := d_{2n}^{-1} + d_{3n}^{-1} + 1$ and $d_n = \frac{d_{4n}}{d_{3n}}$. Set

$$
\begin{pmatrix}
Y^{(1)}_{1n} & Y^{(1)}_{2n} & Y^{(1)}_{3n} \\
Y^{(2)}_{1n} & Y^{(2)}_{2n} & Y^{(2)}_{3n} \\
Y^{(3)}_{1n} & Y^{(3)}_{2n} & Y^{(3)}_{3n}
\end{pmatrix} = 
\begin{pmatrix}
\frac{1}{\sqrt{1 + 2D_n}d_{2n}d_{3n}} & \frac{d_{2n}}{\sqrt{1 + 2D_n}d_{2n}d_{3n}} & \frac{d_{4n}}{\sqrt{1 + 2D_n}d_{2n}d_{3n}} \\
\frac{1}{\sqrt{d_{2n}^2 + 2D_n}d_{2n}d_{3n}} & \frac{d_{2n}}{\sqrt{d_{2n}^2 + 2D_n}d_{2n}d_{3n}} & \frac{d_{4n}}{\sqrt{d_{2n}^2 + 2D_n}d_{2n}d_{3n}} \\
\frac{1}{\sqrt{d_{3n}^2 + 2D_n}d_{2n}d_{3n}} & \frac{d_{2n}}{\sqrt{d_{3n}^2 + 2D_n}d_{2n}d_{3n}} & \frac{d_{4n}}{\sqrt{d_{3n}^2 + 2D_n}d_{2n}d_{3n}}
\end{pmatrix} \tag{4.117}
$$

**Remark 4.16.** For $(r, s)$ such that $1 \leq r < s \leq 3$ the following equivalence hold

$$
\Sigma_{rs}(s_{rs}) < \infty \iff \sum_{n \in \mathbb{Z}} c_{rs,n}^2 < \infty \iff \sum_{n \in \mathbb{Z}} c_{sr,n}^2 < \infty, \quad \text{where} \quad (4.118)
$$

$$
b_{rn} := s_{rs}^{-4}(1 + c_{rs,n}), \quad b_{sn} := s_{rs}^{-4}(1 + c_{sr,n}), \quad \lim_{n} \frac{b_{rn}}{b_{sn}} \in (0, \infty). \quad (4.119)
$$

**Proof.** By Lemma 8.2 we have

$$
\Sigma_{rs}(s_{rs}) = \sum_{n \in \mathbb{Z}} \left( s_{rs}^2 \sqrt{b_{rn}^2/b_{sn}^2 - s_{rs}^2} \right)^2 = \sum_{n \in \mathbb{Z}} c_{rs,n}^2 + \sum_{n \in \mathbb{Z}} c_{rs,n}^2,
$$

$$
\Sigma_{sr}(s_{rs}^{-1}) = \sum_{n \in \mathbb{Z}} \left( s_{rs}^{-2} \sqrt{b_{rn}^2/b_{sn}^2 - s_{rs}^{-2}} \right)^2 = \sum_{n \in \mathbb{Z}} c_{sr,n}^2 + \sum_{n \in \mathbb{Z}} c_{sr,n}^2,
$$

note also that $1 = \frac{b_{rn}}{b_{sn}}\frac{b_{sn}}{b_{rn}} = (1 + c_{rs,n})(1 + c_{sr,n}). \quad (4.120)$

By Remark 4.16, the condition $\Sigma_{rs}(s_{rs}) = \infty$ means the following:

$$
l_{sr} := \lim_{n} \frac{b_{sn}}{b_{rn}} = \begin{cases} 
(a) & \infty \\
(b) & s_{rs}^4 > 0 \quad \text{with} \quad \sum_{n \in \mathbb{Z}} c_{sr,n}^2 = \infty \\
(c) & 0 \\
(d) & \lim \text{does not exist}
\end{cases} \tag{4.121}
$$

72
Remark 4.17. In the case (d) we can use the fact that some subsequence of $(\frac{b_{sn}}{b_{rn}})_{n \in \mathbb{Z}}$ has property (a), (b) or (c). We can avoid the case (c). Namely, if $l_{sr} = 0$ for some pair $(r, s)$ with $1 \leq r < s \leq 3$, we can exchange the two lines $(b_{sn}, a_{sn})$ and $(b_{rn}, a_{rn})$ to obtain $l_{sr} = \infty$.

Formally, we have $3^3 = \#(A)^2(B)$ possibilities where $A = \{(21), (32), (31)\}$ and $B = \{(a), (b), (d)\}$. Since $l_{32}l_{21} = l_{31}$ we get only the following cases:

$$\begin{array}{cccc}
\text{e \setminus (rs)} & (21) & (32) & (31) \\
(1) & b & b & b \\
(2) & a & a & a \\
(3) & a & b & a \\
(4) & b & a & b
\end{array}$$

To be able to approximate $x_{rn}$ for $1 \leq r \leq 3$ we should study when the following expressions are infinite:

$$\rho_r(C_1, C_2, C_3) = \|C_1Y_1^{(r)} + C_2Y_2^{(r)} + C_3Y_3^{(r)}\|^2. \quad (4.122)$$

By (4.117) we have

$$\rho_r(C_1, C_2, C_3) =: \sum_n \frac{|C_1 + C_2d_{2n} + C_3d_{3n}|^2}{C_{rn}}, \quad (4.123)$$

where $C_{1n} = 1 + 2D_n d_{2n} d_{3n}$, $C_{2n} = d_{2n}^2 + 2D_n d_{2n} d_{3n}$, $C_{3n} = d_{3n}^2 + 2D_n d_{2n} d_{3n}$.

Consider the case $(1)=(bbb)$. We prove the analogue of Lemma 3.13 for the case $m = 3$.

Lemma 4.23. Let for all $s = (s_{12}, s_{23}, s_{13}) \in (\mathbb{R}_+)^3$ holds (4.116). Then

$$\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}) = \Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}) = \infty. \quad (4.124)$$

Proof. For $1 \leq r < s \leq 3$ set

\[ \frac{b_{sn}}{b_{rn}} = s_{rs}^4 (1 + c_{sr,n}) \quad \text{with} \quad \sum_{n \in \mathbb{Z}} c_{sr,n}^2 = \infty, \quad \lim_{n \to \infty} c_{sr,n} = 0. \]

For $b_{1n} \equiv 1$ we have

\[ b_{2n} = s_{12}^4 (1 + c_{21,n}), \quad b_{3n} = s_{13}^4 (1 + c_{31,n}), \]

\[ b_{2n} = \frac{s_{13}^4}{s_{12}^4} \frac{1 + c_{31,n}}{1 + c_{21,n}} = s_{23}^4 (1 + c_{32,n}), \quad c_{32,n} = \frac{1 + c_{31,n}}{1 + c_{21,n}} - 1, \quad s_{23} = \frac{s_{13}}{s_{12}}, \]

\[ \sum_n c_{32,n}^2 = \sum_n \left( \frac{1 + c_{31,n}}{1 + c_{21,n}} - 1 \right)^2 = \sum_n \left( \frac{c_{31,n} - c_{31,n}}{1 + c_{21,n}} \right)^2 \sim \sum_n (c_{21,n} - c_{31,n})^2 = \infty. \]

73
Finally, we get
\[ \sum_{n} c_{21,n}^2 = \infty, \quad \sum_{n} c_{31,n}^2 = \infty, \quad \sum_{n} (c_{21,n} - c_{31,n})^2 = \infty. \]  
(4.125)

By (4.122) and (4.123) we get
\[ \rho_r(C_1, C_2, C_3) = \|C_1 Y_1^{(r)} + C_2 Y_2^{(r)} + C_3 Y_3^{(r)}\|^2 = \sum_{n} \frac{|C_1 + C_2 d_{2n} + C_3 d_{3n}|^2}{C_{rn}} \]
\[ = \sum_{n} \frac{|C_1 + C_2 s_{12}(1 + c_{21,n}) + C_3 s_{13}(1 + c_{31,n})|^2}{C_{rn}}. \]

The latter expression is divergent if \( C_1 + C_2 s_{12}^4 + C_3 s_{13}^4 \neq 0 \) since \( \lim_{n \to \infty} c_{21,n} = \lim_{n \to \infty} c_{31,n} = 0 \) and \( A_1 \leq C_{rn} \leq A_2 \).

In the case when \( C_1 + C_2 s_{12}^4 + C_3 s_{13}^4 = 0 \) we get
\[ \rho_r(C_1, C_2, C_3) = \sum_{n \in \mathbb{Z}} \frac{|C_2 s_{12}^4 c_{21,n} + C_3 s_{13}^4 c_{31,n}|^2}{C_{rn}} =: \rho_r(C_2, C_3). \]  
(4.126)

The latter expression is divergent by the first two relations in (4.125) when
1) \( C_2 C_3 > 0 \), 2) \( C_2 = 0 \) and \( C_3 \neq 0 \), 3) \( C_2 \neq 0 \) and \( C_3 = 0 \).

If \( C_2 C_3 < 0 \) we have by the last relation in (4.125)
\[ \sum_{n \in \mathbb{Z}} \frac{|C_2 s_{12}^4 c_{21,n} - C_3 s_{13}^4 c_{31,n}|^2}{C_{rn}} \sim \sum_{n \in \mathbb{Z}} \frac{|C_2 s_{12}^4 c_{21,n} - C_3 s_{13}^4 c_{31,n}|^2}{C_{rn}} = \infty, \]
since \( (s_{12}, s_{13}) = \frac{1}{s_1} (s_2, s_3) \in (\mathbb{R}^*)^2 \) are arbitrary.

Consider the case (2)=(aaa). Now, see (4.121), we have
\[ l_{21} = \lim_{n} \frac{b_{2n}}{b_{1n}} = \infty, \quad l_{32} = \lim_{n} \frac{b_{3n}}{b_{2n}} = \infty, \quad \text{therefore,} \quad l_{31} = \lim_{n} \frac{b_{3n}}{b_{1n}} = \infty. \]  
(4.127)

Since \( b_{1n} \equiv 1 \) we conclude that
\[ l_{21} = \lim_{n} d_{2n} = \infty \quad \text{and} \quad l_{31} = \lim_{n} d_{3n} = \infty. \]  
(4.128)

Therefore, we get for some \( C > 0 \) and all \( n \in \mathbb{Z} \)
\[ 1 \leq D_n = 1 + \frac{1}{d_{2n}} + \frac{1}{d_{3n}} \leq C. \]  
(4.129)
By (4.122) and (4.123) we obtain

\[\rho_r(C_1, C_2, C_3) = \|C_1 Y_1^{(r)} + C_2 Y_2^{(r)} + C_3 Y_3^{(r)}\|^2 = \sum_n \frac{|C_1 + C_2 d_{2n} + C_3 d_{3n}|^2}{C_{rn}'}\]

\[\sim \sum_n \frac{|C_1 + C_2 d_{2n} + C_3 d_{3n}|^2}{C_{rn}'} =: \rho'_r(C_1, C_2, C_3),\]

where \(C_{rn}' = 1 + 2d_{2n}d_{3n}, \quad C_{rn}' = d_{2n}' + 2d_{2n}d_{3n}, \quad \rho_r = d_{3n}' + 2d_{2n}d_{3n} \).

We should study when \(\rho'_r(C_1, C_2, C_3) = \infty\) for some \(1 \leq r \leq 3\):

\[\sum_n \frac{|C_1 + C_2 d_{2n}' + C_3 d_{3n}'|^2}{1 + 2d_{2n}'d_{3n}'}, \quad \sum_n \frac{|C_1 + C_2 d_{2n}' + C_3 d_{3n}'|^2}{d_{2n}'^2 + 2d_{2n}'d_{3n}'}, \quad \sum_n \frac{|C_1 + C_2 d_{2n}' + C_3 d_{3n}'|^2}{d_{3n}'^2 + 2d_{2n}'d_{3n}'}.
\]

Denoting as before \(d_n = \frac{d_{2n}}{d_{2n}'} =: l_n^{-1}\), we get

\[\rho'_1(C_1, C_2, C_3) = \sum_n \frac{|C_1 + C_2 d_{2n} + C_3 d_{3n}|^2}{1 + 2d_{2n}d_{3n}} = \sum_n \frac{|C_1 + C_2 + C_3 d_n|^2}{1 + 2d_n}
\]

\[= \sum_n \frac{|C_2 + C_3 d_n|^2}{2d_n}, \quad \rho'_2(C_1, C_2, C_3) = \sum_n \frac{|C_1 + C_2 d_{2n} + C_3 d_{3n}|^2}{d_{2n}^2 + 2d_{2n}d_{3n}} = \sum_n \frac{|C_1 + C_2 + C_3 d_n|^2}{d_{2n}^2 + 2d_{2n}},
\]

\[\rho'_3(C_1, C_2, C_3) = \sum_n \frac{|C_1 + C_2 d_{2n} + C_3 d_{3n}|^2}{d_{3n}^2 + 2d_{3n}} = \sum_n \frac{|C_1 + C_2 + C_3 d_n|^2}{d_{3n}^2 + 2d_{3n}} = \sum_n \frac{|C_1 + C_2 + C_3 d_n|^2}{1 + 2l_n}.
\]

By Lemma 4.8 we get when \(C_C^{-1} > 0\)

\[\rho'_1(C_1, C_2, -C_3) \sim \sum_n \frac{|C_2 - C_3 d_n|^2}{2d_n} > \sum_n \frac{|C_2 - C_3 d_n|^2}{1 + 2d_n} \sim \sum_n c_n^2 \sim \Sigma_{23}(s),
\]

\[\rho'_2(C_1, C_2, -C_3) \sim \sum_n \frac{|C_2 - C_3 d_n|^2}{1 + 2d_n} \sim \sum_n c_n^2 \sim \Sigma_{23}(s), \quad (4.130)
\]

\[\rho'_3(C_1, C_2, -C_3) \sim \sum_n \frac{|C_2 l_n - C_3|^2}{1 + 2l_n} \sim \sum_n c_n^2 \sim \Sigma_{23}(s),
\]

where \(d_n = C_2 C_3^{-1}(1 + c_n), \quad l_n = C_3 C_2^{-1}(1 + e_n), \quad s^4 = C_2 C_3^{-1} > 0\).

75
But $\Sigma_{23}(s) = \infty$ for all $s > 0$ therefore, for $C_2C_3^{-1} > 0$ we have
\[ \rho_r(C_1, C_2, -C_3) \sim \Sigma_{23}(s) = \infty. \] (4.131)

If $C_2C_3^{-1} > 0$, by (4.130) we get
\[
\sum_n \frac{|C_2 + C_3d_n|^2}{1 + 2d_n} > \sum_n \frac{|C_2 - C_3d_n|^2}{1 + 2d_n} \sim \sum_n c_n^2 \sim \Sigma_{23}(s) = \infty, \\
\sum_n \frac{|C_2 + C_3d_n|^2}{1 + 2d_n} \sum_n \frac{|C_2 - C_3d_n|^2}{1 + 2d_n} \sim \sum_n c_n^2 \sim \Sigma_{23}(s) = \infty, \\
\sum_n \frac{|C_2l_n + C_3|^2}{1 + 2l_n} > \sum_n \frac{|C_2l_n - C_3|^2}{1 + 2l_n} \sim \sum_n e_n^2 \sim \Sigma_{23}(s) = \infty.
\]

Therefore, $\rho_r(C_1, C_2, C_3) = \infty$ for every $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$.

Consider the case (3) = (aba). In this case, see (4.121), we have
\[ l_{21} = \lim_n \frac{b_{2n}}{b_{1n}} = \infty, \quad l_{32} = \lim_n \frac{b_{3n}}{b_{2n}} < \infty, \quad \text{therefore,} \quad l_{31} = \lim_n \frac{b_{3n}}{b_{1n}} = \infty. \]

So, we have again, see (4.128)
\[ l_{21} = \lim_n d_{2n} = \infty, \quad \text{and} \quad l_{31} = \lim_n d_{3n} = \infty. \]

We are reduced to the case (2).

Consider the case (4) = (baa). Now, see (4.121), we have
\[ l_{21} = \lim_n d_{2n} < \infty, \quad \text{and} \quad l_{31} = \lim_n d_{3n} = \infty. \]

Hence (4.129) holds too and we can use all estimations of the case (1).

**Remark 4.18.** In the cases (1)–(4) by Lemma 4.23 and Lemmas 5.1–5.3 we can approximate $x_{rn}$ for all $1 \leq r \leq 3$ and $n \in \mathbb{Z}$ and the irreducibility is proved.

5. Approximation of $D_{kn}$ and $x_{kn}$

5.1. Approximation of $x_{kn}$ by $A_{nk}A_{tk}$

For $m = 3$, consider three rows as follows
\[
\begin{pmatrix}
\vdots & b_{11} & b_{12} & \cdots & b_{1n} & \cdots \\
\vdots & b_{21} & b_{22} & \cdots & b_{2n} & \cdots \\
\vdots & b_{31} & b_{32} & \cdots & b_{3n} & \cdots \\
\end{pmatrix}.
\]
Set
\[
\lambda_k^{(r)} = (b_{1k} + b_{2k} + b_{3k})^2 - (b_{1k}^2 + b_{2k}^2 + b_{3k}^2), \quad r = 1, 2, 3, \quad k \in \mathbb{Z}. \tag{5.1}
\]

Denote by \( Y_r^{(s)} \) the following vectors:
\[
x_r^{(s)} = b_{rk}/\sqrt{\lambda_k^{(s)}}, \quad k \in \mathbb{Z}, \quad Y_r^{(s)} = (x_r^{(s)})_{k \in \mathbb{Z}}. \tag{5.2}
\]

**Lemma 5.1.** For any \( n, t \in \mathbb{Z} \) one has
\[
x_{1n}x_{1t}1 \in \langle A_{nk}A_{tk}1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}) = \infty.
\]

Similarly, using the cyclic permutation of vectors, and changing \( \lambda_k^{(r)} \) we arrive at the following lemma.

**Lemma 5.2.** For any \( n, t \in \mathbb{Z} \) one has
\[
x_{2n}x_{2t}1 \in \langle A_{nk}A_{tk}1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}) = \infty.
\]

**Lemma 5.3.** For any \( n, t \in \mathbb{Z} \) one has
\[
x_{3n}x_{3t}1 \in \langle A_{nk}A_{tk}1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = \infty.
\]

**Proof.** The proof of Lemma 5.1 is also based on Lemma 7.3. We study when \( x_{1n}x_{1t}1 \in \langle A_{nk}A_{tk}1 \mid k \in \mathbb{Z} \rangle. \) Since
\[
A_{nk}A_{tk} = (x_{1n}D_{1k} + x_{2n}D_{2k} + x_{3n}D_{3k})(x_{1t}D_{1k} + x_{2t}D_{2k} + x_{3t}D_{3k})
\]
\[
= x_{1n}x_{1t}D_{1k}^2 + x_{2n}x_{2t}D_{2k}^2 + x_{3n}x_{3t}D_{3k}^2 + (x_{1n}x_{2t} + x_{2n}x_{1t})D_{1k}D_{2k} +
\]
\[
(x_{1n}x_{3t} + x_{3n}x_{1t})D_{1k}D_{3k} + (x_{2n}x_{3t} + x_{3n}x_{2t})D_{2k}D_{3k}
\]
and \( MD_{1k}^2 1 = -b_{1k}^2 \), we take \( t = (t_k) \) as follows: \(-\sum_{k=-m}^{m} t_k b_{k+1}^{1k} = (t, b') = 1, \) where \( t = (t_k)_{k=-m}^{m} \) and \( b' = -(b_{1k})_{k=-m}^{m} \sim b = (b_{1k})_{k=-m}^{m} \). We have
\[
\|\left[ \sum_{k=-m}^{m} t_k A_{nk}A_{tk} - x_{1n}x_{1t} \right] 1 \|^2 =
\]
\[
\| \sum_{k=-m}^{m} t_k \left[ x_{1n}x_{1t} 1 \left( D_{1k}^2 + \frac{b_{1k}^2}{2} \right) + x_{2n}x_{2t} D_{2k}^2 + x_{3n}x_{3t} D_{3k}^2 + (x_{1n}x_{2t} + x_{2n}x_{1t}) \times
\right.
\]
\[
D_{1k}D_{2k} + (x_{1n}x_{3t} + x_{3n}x_{1t})D_{1k}D_{3k} + (x_{2n}x_{3t} + x_{3n}x_{2t})D_{2k}D_{3k} \left] 1 \right\|^2
\]
\[
= \sum_{-m \leq k, r \leq m} (f_k, f_r)t_k t_r =: (A_{2m+1}t, t),
\]

77
where \( A_{2m+1} = ((f_k, f_r))_{k,r=-m}^m \) and \( f_k = \sum_{r=1}^3 f_k^r + \sum_{1 \leq i < j \leq 3} f_k^{ij} \).

\[
\begin{align*}
  f_k^r &= x_{rn}x_{rt}(D_{rk}^2 + \frac{v_{rk}}{2} \delta_{1r})1, \\
  f_k^{ij} &= (x_{in}x_{jt} + x_{jn}x_{it})D_{ik}D_{jk}1 
\end{align*}
\] (5.3)

for \( 1 \leq r \leq 3, \ 1 \leq i < j \leq 3 \). Since

\[
\begin{align*}
  f_k^r &\perp f_k^{ij}, \\
  f_k^r &\perp f_k^{i'j'}
\end{align*}
\]

for different \((ij), (i'j')\), writing \( c_{kn} = \|x_{kn}\|^2 = \frac{1}{2b_{kn}} + a_{kn}^2 \), we get

\[
(f_k, f_k) = \sum_{r=1}^3 \|f_k^r\|^2 + \sum_{1 \leq i < j \leq 3} \|f_k^{ij}\|^2 = \]

\[
\begin{align*}
  c_{1n}c_{1t}2\left(\frac{b_{1k}}{2}\right)^2 + c_{2n}c_{2t}3\left(\frac{b_{2k}}{2}\right)^2 + c_{3n}c_{3t}3\left(\frac{b_{3k}}{2}\right)^2 \]

\[
\begin{align*}
  &\left( c_{1n}c_{2t} + c_{1t}c_{2n} + 2a_{1n}a_{2t}a_{1t}a_{2n} \right) \frac{b_{1k}b_{2k}}{2} + \left( c_{1n}c_{3t} + c_{3t}c_{1n} + 2a_{1n}a_{3t}a_{3t}a_{1n} \right) \frac{b_{1k}b_{3k}}{2} \]

\[
\begin{align*}
  &\left( c_{2n}c_{3t} + c_{3t}c_{2n} + 2a_{2n}a_{3t}a_{3t}a_{2n} \right) \frac{b_{2k}b_{3k}}{2} \sim (b_{1k} + b_{2k} + b_{3k})^2,
\end{align*}
\]

\[
(f_k, f_r) = (f_k^2, f_r^2) + (f_k^3, f_r^3) = c_{2n}c_{2t} \frac{b_{2k}b_{2r}}{2} + c_{3n}c_{3t} \frac{b_{3k}b_{3r}}{2} \sim b_{2k}b_{2r} + b_{3k}b_{3r}.
\]

Finally, we have

\[
(f_k, f_k) \sim (b_{1k} + b_{2k} + b_{3k})^2, \ (f_k, f_r) \sim b_{2k}b_{2r} + b_{3k}b_{3r}, \ k \neq r. \quad (5.4)
\]

Set

\[
\lambda_k = (b_{1k} + b_{2k} + b_{3k})^2 - (b_{2k}^2 + b_{3k}^2), \quad g_k = (b_{2k}, b_{3k}), \quad (5.5)
\]

then

\[
(f_k, f_k) \sim \lambda_k + (g_k, g_k), \quad (f_k, f_r) \sim (g_k, r_r). \quad (5.6)
\]

For \( A_{2m+1} = ((f_k, f_r))_{k,r=-m}^m \) and \( b = -(b_{1k}/2)_{k=-m}^m \in \mathbb{R}^{2m+1} \) we have

\[
A_{2m+1} = \sum_{k=-m}^m \lambda_k E_{kk} + \gamma(g_{-m}, \ldots, g_0, \ldots, g_m).
\]

To finish the proof, it suffices to invoke Lemma 7.3. \(\square\)
5.2. Approximation of \( D_{rn} \) by \( A_{kn} \)

We will formulate several lemmas, which will be useful for approximation of the independent variables \( x_{kn} \) and operators \( D_{kn} \) by combinations of the generators \( A_{kn} \). The generators \( A_{kn} \) have the following form:

\[
A_{kn} = x_{1k} D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}, \quad k, n \in \mathbb{Z}.
\]

For \( m = 3 \), consider three rows as follows

\[
\begin{pmatrix}
... & a_{11} & a_{12} & \ldots & a_{1n} & \ldots \\
... & a_{21} & a_{22} & \ldots & a_{2n} & \ldots \\
... & a_{31} & a_{32} & \ldots & a_{2n} & \ldots \\
\end{pmatrix}
\]

Set

\[
\lambda_k = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}. \quad (5.7)
\]

Denote by \( Y_1, Y_2 \) and \( Y_3 \) the three following vectors:

\[
x_{rk} = \frac{a_{rk}}{\sqrt{\lambda_k}}, \quad k \in \mathbb{Z}, \quad Y_r = (x_{rk})_{k \in \mathbb{Z}}. \quad (5.8)
\]

The proofs of Lemmas 5.4–5.6 and 5.1–5.3 are based on Lemma 7.3.

**Lemma 5.4.** For any \( l \in \mathbb{Z} \) we have

\[
D_{1l} 1 \in \langle A_{kl} 1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_1, Y_2, Y_3) = \infty.
\]

Similarly, by cyclic permutation of the vectors, we obtain the following two lemmas.

**Lemma 5.5.** For any \( l \in \mathbb{Z} \) we have

\[
D_{2l} 1 \in \langle A_{kl} 1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_2, Y_3, Y_1) = \infty.
\]

**Lemma 5.6.** For any \( l \in \mathbb{Z} \) we have

\[
D_{3l} 1 \in \langle A_{kl} 1 \mid k \in \mathbb{Z} \rangle \iff \Delta(Y_3, Y_1, Y_2) = \infty.
\]

**Proof.** We determine when the inclusion

\[
D_{1n} 1 \in \langle A_{kn} 1 = (x_{1k} D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}) 1 \mid k \in \mathbb{Z} \rangle
\]

79
holds. Fix \( m \in \mathbb{N} \), since \( Mx_1 = a_1 \), we put \( \sum_{k=-m}^{m} t_k a_{1k} = (t, b) = 1 \), where \( t = (t_k)_{k=-m}^{m} \) and \( b = (a_{1k})_{k=-m}^{m} \). We have

\[
\| \left[ \sum_{k=-m}^{m} t_k (x_{1k} D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}) - D_{1n} \right] 1 \|^{2} = \| \sum_{k=-m}^{m} t_k [(x_{1k} - a_{1k}) D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}] 1 \|^{2}
\]

\[
= \sum_{-m \leq k, r \leq m} (f_k, f_r) t_k t_r =: (A_{2m+1}, t), \text{ where } A_{2m+1} = ((f_k, f_r))_{k, r=-m}^{m}, \text{ and } f_k = [(x_{1k} - a_{1k}) D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}] 1.
\]

We get

\[
(f_k, f_k) = \| [(x_{1k} - a_{1k}) D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}] 1 \|^2 = \frac{1}{2b_{1k}} \| b_{1n} \| + \left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) \| b_{2n} \| + \left( \frac{1}{2b_{3k}} + a_{3k}^2 \right) \| b_{3n} \|
\]

\[
\sim \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2.
\]

\[
(f_k, f_r) = \left( [(x_{1k} - a_{1k}) D_{1n} + x_{2k} D_{2n} + x_{3k} D_{3n}] 1, [(x_{1r} - a_{1r}) D_{1n} + x_{2r} D_{2n} + x_{3r} D_{3n}] 1 \right)
\]

\[
= (x_{2k} 1, x_{2r} 1) (D_{2n} 1, D_{2n} 1) + (x_{3k} 1, x_{3r} 1) (D_{3n} 1, D_{3n} 1)
\]

\[
= a_{2k} a_{2r} b_{2n} \frac{b_{2n}}{2} + a_{3k} a_{3r} b_{3n} \frac{b_{3n}}{2} \sim a_{2k} a_{2r} + a_{3k} a_{3r}.
\]

Finally, we have

\[
(f_k, f_k) \sim \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2, \quad (f_k, f_r) \sim a_{2k} a_{2r} + a_{3k} a_{3r}, \quad k \neq r.
\]

(5.9)

If we denote

\[
\lambda_k = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}, \quad g_k = (a_{2k}, a_{3k}),
\]

(5.10)

then we have

\[
(f_k, f_k) \sim \lambda_k + (g_k, g_k), \quad (f_k, f_r) \sim (g_k, g_r).
\]

(5.11)
For $A_{2m+1} = ((f_k, f_r))_{k, r = -m}^m$, and $b = (a_{1k})_{k = -m}^m \in \mathbb{R}^{2m+1}$ we have

$$A_{2m+1} = \sum_{k = -m}^m \lambda_k E_{kk} + \gamma (g_{-m}, \ldots, g_0, \ldots, g_m).$$

To finish the proof, it suffices to apply Lemma 7.3.

The proofs of Lemmas 5.5 and 5.6 are exactly the same.

5.3. Approximation of $x_{rk}$ by $D_{rn}A_{kn}$

**Lemma 5.7.** For any $k \in \mathbb{Z}$ we get

$$x_{1k}1 \in \langle D_{1n}A_{kn}1 \mid n \in \mathbb{Z} \rangle \iff \sum_{n \in \mathbb{Z}} \frac{b_{1n}}{b_{1n} + b_{2n} + b_{3n}} = \infty.$$

**Proof.** Since

$$D_{1n}A_{kn} = x_{1k}D_{1n}^2 + x_{2k}D_{1n}D_{2n} + x_{3k}D_{1n}D_{3n}$$

and $MD_{1k}^21 = \frac{b_{1k}}{2}$, we take $t = (t_k)_{k = -m}^m$ as follows: $(t, b') = 1$, where $t = (t_k)_{k = -m}^m$ and $b' = -(b_{1k})_{k = -m}^m \sim b = -(b_{1k})_{k = -m}^m$. We have

$$\left\| \left[ \sum_{n = -m}^m t_n D_{1n}A_{kn} - x_{1k} \right] 1 \right\|^2$$

$$= \left\| \sum_{n = -m}^m t_n \left[ x_{1k} \left( D_{1n}^2 + \frac{b_{1n}}{2} \right) + x_{2k}D_{1n}D_{2n} + x_{3k}D_{1n}D_{3n} \right] 1 \right\|^2$$

$$= \sum_{-m \leq n, r \leq m} (f_n, f_r)t_k t_r =: (A_{2m+1}t, t),$$

where $A_{2m+1} = ((f_n, f_r))_{n, r = -m}^m$ and

$$f_n = \left[ x_{1k} \left( D_{1n}^2 + \frac{b_{1n}}{2} \right) + x_{2k}D_{1n}D_{2n} + x_{3k}D_{1n}D_{3n} \right] 1.$$

We have

$$(f_n, f_n) \sim b_{1k}(b_{1n} + b_{2n} + b_{3n}) \quad \text{and} \quad (f_n, f_k) = 0 \quad \text{for} \quad n \neq k.$$

Therefore, by (6.3)

$$\min_{t \in \mathbb{R}^{2m+1}} \left\| \sum_{n = -m}^m t_n D_{1n}A_{kn} - x_{1k} \right\|^2 = \left( \sum_{n = -m}^m \frac{b_{1k}}{b_{1n} + b_{2n} + b_{3n}} \right)^{-1}. \quad \Box$$
Lemma 5.8. For any $k \in \mathbb{Z}$ we have
\[ x_{2k}1 \in \langle D_{2n}A_{kn}1 \mid n \in \mathbb{Z} \rangle \Leftrightarrow \sum_{n \in \mathbb{Z}} \frac{b_{2n}}{b_{1n} + b_{2n} + b_{3n}} = \infty. \]

Lemma 5.9. For any $k \in \mathbb{Z}$ we have
\[ x_{3k}1 \in \langle D_{3n}A_{kn}1 \mid n \in \mathbb{Z} \rangle \Leftrightarrow \sum_{n \in \mathbb{Z}} \frac{b_{3n}}{b_{1n} + b_{2n} + b_{3n}} = \infty. \]

5.4. Approximation of $D_{kn}$ by $x_{rk}A_{kn}$

Set
\[ \lambda_k^{(1)} = \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2 \right) - a_{1k}^2(a_{2k}^2 + a_{3k}^2)(5.12) \]
\[ Y_{11} = \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) \sqrt{\lambda_k^{(1)}} \] \( k \in \mathbb{Z} \) \( Y_{12} = \left( \frac{a_{1k}a_{2k}}{\sqrt{\lambda_k^{(1)}}} \right) \) \( k \in \mathbb{Z} \) \( Y_{13} = \left( \frac{a_{1k}a_{3k}}{\sqrt{\lambda_k^{(1)}}} \right) \) \( k \in \mathbb{Z} \) \( (5.13) \)

Lemma 5.10. For any $n \in \mathbb{Z}$ we have
\[ D_{1n}1 \in \langle x_{1k}A_{kn}1 \mid k \in \mathbb{Z} \rangle \Leftrightarrow \Delta(Y_{11}, Y_{12}, Y_{13}) = \infty. \]

PROOF. We determine when the inclusion
\[ D_{1n}1 \in \langle x_{1k}A_{kn}1 = (x_{1k}D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n})1 \mid k \in \mathbb{Z} \rangle \]
holds. Fix $m \in \mathbb{N}$, since $Mx_{1k}^2 = \frac{1}{2b_{1k}} + a_{1k}^2$, we put
\[ \sum_{k=-m}^{m} t_k \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) = (t, b) = 1, \]
where $t = (t_k)_{k=-m}^{m}$ and $b = (\frac{1}{2b_{1k}} + a_{1k}^2)_{k=-m}^{m}$. We have
\[ \| \left[ \sum_{k=-m}^{m} t_k \left( x_{1k}^2 D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n} \right) - D_{1n} \right] 1 \|^2 \]
\[ = \| \sum_{k=-m}^{m} t_k \left[ (x_{1k}^2 - \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) ) D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n} \right] 1 \|^2 \]
\[ = \sum_{-m \leq k, r \leq m} (f_k, f_r) t_k t_r =: (A_{2m+1}, t), \text{ where } A_{2m+1} = ((f_k, f_r)_{k,r=-m}^{m}, \]
and \[ f_k = \left[ (x_{1k}^2 - \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) ) D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n} \right] 1. \]
Since $M|\psi - M|\psi|^2 = M\psi^2 - |M\psi|^2$ we have
\[
M \left| x_{1k}^2 - \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) \right|^2 = M x_{1k}^4 - \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right)^2
= \frac{3}{(2b_{1k})^2} + 6 \frac{1}{2b_{1k}} a_{1k}^2 + a_{1k}^4 - \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right)^2 = \frac{1}{2b_{1k}} \left( \frac{2}{2b_{1k}} + 4a_{1k}^2 \right),
\]
we get
\[
(f_k, f_k) = \frac{1}{2b_{1k}} \left( \frac{2}{2b_{1k}} + 4a_{1k}^2 \right) b_{ln} + \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) \left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) b_{2n} + \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) b_{3n}^2
\times \left( \frac{1}{2b_{3k}} + a_{3k}^2 \right)
\times \frac{1}{2b_{1r}} b_{2n} + \left( \frac{1}{2b_{1r}} + a_{1r}^2 \right) \left( \frac{1}{2b_{2r}} + a_{2r}^2 \right) b_{2n}^2 + \left( \frac{1}{2b_{3k}} + a_{3k}^2 \right) b_{3n}^2
\times \left( \frac{1}{2b_{1r}} + a_{1r}^2 \right) \left( \frac{1}{2b_{2r}} + a_{2r}^2 \right) b_{2n}^2 + \left( \frac{1}{2b_{3k}} + a_{3k}^2 \right) b_{3n}^2
= (x_{1k} \mathbf{1}, x_{1r} \mathbf{1})(x_{2k} \mathbf{1}, x_{2r} \mathbf{1})(D_{2n} \mathbf{1}, D_{2n} \mathbf{1}) + (x_{1k} \mathbf{1}, x_{1r} \mathbf{1})(x_{3k} \mathbf{1}, x_{3r} \mathbf{1})(D_{3n} \mathbf{1}, D_{3n} \mathbf{1})
= a_{1k} a_{1r} a_{2k} a_{2r} \frac{b_{2n}}{2} + a_{1k} a_{3k} a_{3r} \frac{b_{3n}}{2} \simeq a_{1k} a_{2r} (a_{2k} + a_{3k} a_{3r}).
\]
Finally, we have
\[
(f_k, f_k) \sim \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + a_{2k} + a_{3k}^2 \right), \quad (f_k, f_r) \sim a_{1k} a_{1r} (a_{2k} a_{2r} + a_{3k} a_{3r}), \quad k \neq r.
\]
Set
\[
\lambda_k^{(1)} = \left( \frac{1}{2b_{1k}} + a_{2k}^2 \right) \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + a_{2k} + a_{3k}^2 \right) - a_{2k} (a_{2k} + a_{3k}^2),
\]
\[
g_k = a_{1k} (a_{2k}, a_{3k}),
\]
then
\[
(f_k, f_k) = \lambda_k^{(1)} + (g_k, g_k) \quad (f_k, f_r) \sim (g_k, g_r), \quad k \neq r.
\]
For $A_{2m+1} = ((f_k, f_r))_{k,r=-m}^m$ and $b = (a_{1k})_{k=-m}^m \in \mathbb{R}^{2m+1}$ we have
\[
A_{2m+1} = \sum_{k=-m}^m \lambda_k E_{kk} + \gamma (g_{-m}, \ldots, g_0, \ldots, g_m).
\]
To finish the proof, it suffices to apply Lemma 7.3. □
Lemma 5.13. For any $l \in \mathbb{Z}$ we have
\[ D_{2l} \mathbf{1} \in \langle x_{2k} A_{kl} \mathbf{1} | k \in \mathbb{Z} \rangle \quad \Leftrightarrow \quad \Delta(Y_{22}, Y_{23}, Y_{21}) = \infty. \]

Lemma 5.12. For any $l \in \mathbb{Z}$ we have
\[ D_{3l} \mathbf{1} \in \langle x_{3k} A_{kl} \mathbf{1} | k \in \mathbb{Z} \rangle \quad \Leftrightarrow \quad \Delta(Y_{33}, Y_{31}, Y_{32}) = \infty. \]

Here we set
\[ \lambda_k^{(2)} = \left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{3k}^2 \right) - a_{2k}^2(a_{1k}^2 + a_{3k}^2), \] \[ Y_{21} = \left( \frac{a_{1k} a_{2k}}{\sqrt{\lambda_k^{(2)}}} \right)_{k \in \mathbb{Z}}, \quad \lambda_k^{(3)} = \left( \frac{1}{2b_{3k}} + a_{3k}^2 \right) \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2 \right) - a_{3k}^2(a_{1k}^2 + a_{2k}^2), \]
\[ Y_{32} = \left( \frac{a_{1k} a_{3k}}{\sqrt{\lambda_k^{(3)}}} \right)_{k \in \mathbb{Z}}, \quad Y_{33} = \left( \frac{1}{2b_{1k}} + a_{3k}^2 \right)_{k \in \mathbb{Z}}. \]

5.5. Approximation of $D_{rn}$ by $(x_{3k} - a_{3k})A_{kn}$ and $\exp(is_k x_{rk})A_{kn}$

Lemma 5.13. For any $n \in \mathbb{Z}$ we have
\[ D_{3n} \mathbf{1} \in \langle (x_{3k} - a_{3k})A_{kn} \mathbf{1} | k \in \mathbb{Z} \rangle \quad \Leftrightarrow \quad \Sigma_3(\mu) := \sum_{k \in \mathbb{Z}} \frac{1}{2b_{3k}} \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2 + a_{3k}^2 \right)^{-1} = \infty. \]

PROOF. We determine when the following inclusion holds
\[ D_{3n} \mathbf{1} \in \langle (x_{3k} - a_{3k})A_{kn} \mathbf{1} \rangle = (x_{1k}(x_{3k} - a_{3k})D_{1n} + x_{2k}(x_{3k} - a_{3k})D_{2n} + x_{3k}(x_{3k} - a_{3k})D_{3n}) \mathbf{1} | k \in \mathbb{Z}. \]

Set $\xi_{3k} = x_{3k}(x_{3k} - a_{3k})$. Fix $m \in \mathbb{N}$. We have
\[ M\xi_{3k} \mathbf{1} = M(x_{3k} - a_{3k})^2 \mathbf{1} = \frac{1}{2b_{3k}}, \quad \text{chose } (t_k) \text{ as } \sum_{k = -m}^{m} t_k \frac{1}{2b_{3k}} = (t, b) = 1, \]

84
we denote \( t = (t_k)_{k=-m}^m \) and \( b = (\frac{a}{2b_{3k}})_{k=-m}^m \). We have

\[
\| \sum_{k=-m}^m t_k (x_{1k}(x_{3k} - a_{3k})D_{1n} + x_{2k}(x_{3k} - a_{3k})D_{2n} + \xi_{3k}D_{3n}) - D_{3n} \|_1^2
\]

\[
= \sum_{k=-m}^m (f_k, f_r) t_k t_r = \sum_{-m \leq k, r \leq m} (f_k, f_k)^2, \ 	ext{since} \ f_k \perp f_r, \ \text{where} \ (5.22)
\]

\[
f_k = x_{1k}(x_{3k} - a_{3k})D_{1n} + x_{2k}(x_{3k} - a_{3k})D_{2n} + (\xi_{3k} - M\xi_{3k})D_{3n} \ 1.
\]

To calculate \( M|\xi_{3k} - M\xi_{3k}|^2 \) set

\[
d\mu_{(b,a)}(x) = \sqrt{\frac{b}{\pi}} \exp(-b(x-a)^2)dx \ \text{and} \ d\mu_{(b,0)}(x) = \sqrt{\frac{b}{\pi}} \exp(-bx^2)dx,
\]

then we get \( M^2x^2 = (\frac{3}{(2b_{3k})^2} + a_{3k}^2/2b_{3k}). \) Indeed,

\[
Mx^2(x-a)^2 = \int x^2(x-a)^2d\mu_{(b,a)}(x) = \int x^2(x+a)^2d\mu_{(b,0)}(x)
\]

\[
= \int (x^4 + 2ax^3 + a^2x^2)d\mu_{(b,0)}(x) = \frac{3}{(2b)^2} + \frac{a^2}{2b}
\]

Since \( M|\xi - M|\xi||^2 = M\xi^2 - |M\xi|^2 \) we get

\[
M_3^2 = \frac{3}{(2b_{3k})^2} + \frac{a_{3k}^2}{2b_{3k}} - \frac{1}{(2b_{3k})^2} = \frac{1}{2b_{3k}} \left( \frac{2}{2b_{3k}} + a_{3k}^2 \right).
\]

Set \( f_{sk} = x_{sk}(x_{3k} - a_{3k})D_{1n} 1, \ 1 \leq s \leq 2 \) and \( f_{3k} = (\xi_{3k} - M\xi_{3k})D_{3n} 1, \) then

\[
(f_k, f_k) = \| x_{1k}(x_{3k} - a_{3k})D_{1n} + x_{2k}(x_{3k} - a_{3k})D_{2n} + (\xi_{3k} - M\xi_{3k})D_{3n} 1 \|^2
\]

\[
= \left( \frac{1}{2b_{1k}} + \frac{a_{1k}^2}{2b_{3k}} \right) \frac{1}{2b_{1k}} b_{1n} + \left( \frac{1}{2b_{2k}} + \frac{a_{2k}^2}{2b_{3k}} \right) \frac{1}{2b_{2k}} b_{2n} + \left( \frac{2}{2b_{3k}} + \frac{a_{3k}^2}{2b_{3k}} \right) \frac{1}{2b_{3k}} b_{3n}
\]

\[
\sim \left( \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2 + a_{3k}^2 \right) \frac{1}{2b_{3k}}, \ \text{since} \ f_{lk} \perp f_{sk}, \ l \neq s,
\]

\[
(f_k, f_r) = \left[ x_{1k}(x_{3k} - a_{3k})D_{1n} + x_{2k}(x_{3k} - a_{3k})D_{2n} + (\xi_{3k} - M\xi_{3k})D_{3n} 1, \right.
\]

\[
\left[ x_{1r}(x_{3r} - a_{3r})D_{1n} + x_{2r}(x_{3r} - a_{3r})D_{2n} + (\xi_{3r} - M\xi_{3r})D_{3n} 1 \right) = 0.
\]

85
The previous equality holds since $f_{lr} \perp f_{sk}$ for $1 \leq l, s \leq 3$ and $r \neq k$. For $l \neq s$ this follows from $(D_{ln}1, D_{sn}1) = 0$. For $l = s$ it follows from the equalities:
\[(x_{sk}1, x_{sr}1) = a_{sk}a_{sr}, \ 1 \leq s \leq 2, \ (\xi_{3k} - M\xi_{3k})1, (\xi_{3r} - M\xi_{3r})1 = 0.\]
and $((x_{3k} - a_{3k})1, (x_{3r} - a_{3r})1) = 0$. Finally, we have
\[(f_k, f_k) \sim \left(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a^2_{1k} + a^2_{2k} + a^2_{3k}\right) \frac{1}{2b_{3k}}, \ (f_k, f_r) = 0, \ k \neq r.

Set $a_k = (f_k, f_k)$ and $b_k = M\xi_k$, then by Lemma 6.1, (6.3) and (5.22) the proof is completed since
\[
\sum_k b_k^2\frac{1}{a_k} = \sum_{k \in \mathbb{Z}} \left(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a^2_{1k} + a^2_{2k} + a^2_{3k}\right)^{-1}. \quad \square
\]

Now we would like to approximate $D_{3n}$ by combinations of exp ($is_k(x_{3k} - a_{3k})iA_{kn}$).
Set $s = (s_k)_{k \in \mathbb{Z}}$
\[
\lambda_k^{(3)}(s_k) = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a^2_{3k} - \left(\frac{s^2}{4b_{3k}^2} + a^2_{3k}\right) \exp \left(-\frac{s^2}{2b_{3k}}\right), \quad (5.24)
\]
\[
Y_{31}(s) = \left(\frac{a_{1k}}{\sqrt{\lambda_k^{(3)}(s_k)}}\right)_{k \in \mathbb{Z}}, \quad Y_{32}(s) = \left(\frac{a_{2k}}{\sqrt{\lambda_k^{(3)}(s_k)}}\right)_{k \in \mathbb{Z}}, \quad (5.25)
\]
\[
Y_{32}(s) = \left(\frac{-\frac{s_k}{2b_{3k}} + ia_{3k}}{\sqrt{\lambda_k^{(3)}(s_k)}}\right)_{k \in \mathbb{Z}}.
\]

In particular for $s_k = \sqrt{2b_{3k}}$ we get
\[
\lambda_k^{(3)}(s_k) = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \left(\frac{1}{2b_{3k}} + a^2_{3k}\right) (1 - \frac{1}{e}). \quad (5.26)
\]

**Lemma 5.14.** For any $n \in \mathbb{Z}$ we have
$D_{3n}1 \in \langle \exp (is_k(x_{3k} - a_{3k})iA_{kn}1 \mid k \in \mathbb{Z} \rangle \Leftrightarrow \Delta(Y_{33}(s), Y_{31}(s), Y_{32}(s)) = \infty.$

**Proof.** We determine when the inclusion
\[
D_{3n}1 \in \langle \exp (is_k(x_{3k} - a_{3k})iA_{kn}1 = \left(ix_{1k}\exp (is_k(x_{3k} - a_{3k})) D_{1n} + ix_{2k}\exp (is_k(x_{3k} - a_{3k})) D_{2n} + ix_{3k}\exp (is_k(x_{3k} - a_{3k})) D_{3n}1 \mid k \in \mathbb{Z}\rangle
\]

86
holds. Set \( \xi_{rk}(s_k) = ix_{rk} \exp \left( is_k(x_{3k} - a_{3k}) \right) \) for \( 1 \leq r \leq 3 \) and
\[
 f_k(s_k) = \left( \xi_{1k}(s_k)D_{1n} + \xi_{2k}(s_k)D_{2n} + [\xi_{3k}(s_k) - M\xi_{3k}(s_k)]D_{3n} \right) \mathbf{1} 
\]
\( = \left( ix_{1k} \exp \left( is_k(x_{3k} - a_{3k}) \right) D_{1n} + ix_{2k} \exp \left( is_k(x_{3k} - a_{3k}) \right) D_{2n} + [ix_{3k} \exp \left( is_k(x_{3k} - a_{3k}) - M\xi_{3k}(s_k) \right) D_{3n} \right) \mathbf{1}.
\]

We show that
\[
M\xi_{3k}(s) = \left( -\frac{s}{2b_{3k}} + ia_{3k} \right) \exp \left( -\frac{s^2}{4b_{3k}} \right),
\]
\( f_k, f_r \sim \lambda_k^{(3)}(s_k) + (g_k, g_k), \)
\( f_k, f_r \sim a_{1k}a_{1r} + a_{2k}a_{2r} = (g_k, g_r), \)
where \( g_k = (a_{1k}, a_{2k}) \in \mathbb{R}^2 \). Indeed, set \( F_b(s) = \int_\mathbb{R} \exp \left( is(x - a) \right) d\mu_{(b,a)}(x), \) then
\[
 F_b(s) = \int_\mathbb{R} \exp \left( isx \right) d\mu_{(b,0)}(x) = \exp \left( -\frac{s^2}{4b} \right),
\]
where \( d\mu_{(b,a)}(x) \) and \( d\mu_{(b,0)}(x) \) are defined by (5.23). Therefore,
\[
 H_{a,b}(s) = \int_\mathbb{R} i x \exp \left( is(x - a) \right) d\mu_{(b,a)}(x) = \int_\mathbb{R} (x + a) \exp \left( isx \right) d\mu_{(b,0)}(x),
\]
\[
 = \frac{dF_b(s)}{ds} + iaF_b(s) = \left( -\frac{s}{2b} + ia \right) \exp \left( -\frac{s^2}{4b} \right). \tag{5.33}
\]
This implies (5.28). Further, to obtain (5.29) and (5.30) we write
\[
 (f_k, f_r) = \sum_{1 \leq t \leq 2} (x_{tk}, x_{tr})(D_{tn} \mathbf{1}, D_{tn} \mathbf{1}) = \sum_{1 \leq t \leq 2} (x_{tk}, x_{tr})(D_{tn} \mathbf{1}, D_{tn} \mathbf{1})
\]
\[
 = a_{1k}a_{1r} \frac{b_{1n}}{2} + a_{2k}a_{2r} \frac{b_{2n}}{2} \sim a_{1k}a_{1r} + a_{2k}a_{2r} = (g_k, g_r),
\]
\[
 (f_k, f_k) = \sum_{1 \leq t \leq 2} \|x_{tk}\|^2 \|D_{tn} \mathbf{1}\|^2 + \left( M|\xi_{3k}(s_k)|^2 - |M\xi_{3k}(s_k)|^2 \right) \|D_{3n} \mathbf{1}\|^2
\]
\[
 = \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) \frac{b_{1n}}{2} + \left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) \frac{b_{2n}}{2} + \left( M|\xi_{3k}(s_k)|^2 - |M\xi_{3k}(s_k)|^2 \right) \frac{b_{3n}}{2}
\]
\[
 \sim \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) + \left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) + \left( M|\xi_{3k}(s_k)|^2 - |M\xi_{3k}(s_k)|^2 \right)
\]
\[
 = \lambda_k^{(3)}(s_k) + (g_k, g_k).
\]
Set for brevity $\xi(s) = ix \exp (is(x - a))$, then

$$f(s) := M|\xi(s)|^2 - |M\xi(s)|^2 = \left( \frac{1}{2b} + a^2 \right) - \left( \frac{s^2}{4b^2} + a^2 \right) \exp \left( - \frac{s^2}{2b} \right),$$

$$\min_s f(s) = \begin{cases} 
\frac{1 - e^{-\left(1 - 2ba^2\right)}}{2b} + a^2, & \text{if } 1 - 2ba^2 \geq 0 \\
\frac{1}{2b}, & \text{if } 1 - 2ba^2 < 0.
\end{cases}$$

(5.34)

Indeed, consider the function $g_{a,b}(x) = \left( \frac{x^2}{2b} + a^2 \right) \exp (-x^2)$ and set $x_0 = \sqrt{1 - 2ba^2}$. We have

$$\max_{x \in \mathbb{R}} g_{a,b}(x) = \begin{cases} 
g_{a,b}(x_0) = \exp\left(-\left(1 - 2ba^2\right)\right), & \text{if } 1 - 2ba^2 \geq 0, \\
g_{a,b}(0) = a^2, & \text{if } 1 - 2ba^2 < 0.
\end{cases}$$

(5.35)

To calculate $\lambda_k^{(3)}(s_k)$, we get finally

$$\lambda_k^{(3)}(s_k) = \frac{1}{2b_{1k}} + a_{1k}^2 + \frac{1}{2b_{2k}} + a_{2k}^2 + \left( M|\xi_{3k}(s_k)|^2 - |M\xi_{3k}(s_k)|^2 \right) - (g_k, g_k)$$

$$= \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2 + a_{3k}^2 - \left( \frac{s_k^2}{4b_{3k}^2} + a_{3k}^2 \right) \exp \left( - \frac{s_k^2}{2b_{3k}} \right)$$

$$- (g_k, g_k) = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{3k}^2 - \left( \frac{s_k^2}{4b_{3k}^2} + a_{3k}^2 \right) \exp \left( - \frac{s_k^2}{2b_{3k}} \right) = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \left( \frac{1}{2b_{3k}} + a_{3k}^2 \right) (1 - e^{-1}), \quad \text{for } \frac{s_k^2}{2b_{3k}} = 1.$$

Therefore, we get (5.24) and (5.26)

$$\lambda_k^{(3)}(s_k) = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \left( \frac{1}{2b_{3k}} + a_{3k}^2 \right) (1 - \frac{1}{e}).$$

For $A_{2m+1} = ((f_k(s_k), f_r(s_r)))_{k,r=-m}^m$, and $b = (M\xi_{3k}(s_k))_{k=-m}^m \in \mathbb{R}^{2m+1}$ we have

$$A_{2m+1} = \sum_{k=-m}^m \lambda_k E_{kk} + \gamma(g_{-m}, \ldots, g_0, \ldots, g_m).$$

The proof is now finished on invoking Lemma 7.3. \qed

88
Lemma 5.15. We have

\[ D_{3k} \mathbf{1} \in \langle \sin(s_k(x_{3k} - a_{3k})) A_{kn} \mathbf{1} \mid k \in \mathbb{Z} \rangle \iff \Sigma_3(D, s) = \infty, \quad (5.36) \]

\[ D_{3k} \mathbf{1} \in \langle \cos(s_k(x_{3k} - a_{3k})) A_{kn} \mathbf{1} \mid k \in \mathbb{Z} \rangle \iff \Sigma_3^\ast(D, s) = \infty, \quad (5.37) \]

where \( \Sigma_3(D, s) = \sum_{k \in \mathbb{Z}} \frac{|M\eta_{3k}(s_k)|^2}{\|g_k(s_k)\|^2}, \quad \Sigma_3^\ast(D, s) = \sum_{k \in \mathbb{Z}} \frac{|M\eta_{3k}^\ast(s_k)|^2}{\|g_k^\ast(s_k)\|^2}, \quad (5.38) \]

moreover, \( \Sigma_3(D, s^{(3)}) \sim \Sigma_3(D) := \sum_k \frac{1}{2b_{ak}} C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2, \quad (5.39) \]

and \( \Sigma_3^\ast(D, s^{(3)}) \sim \Sigma_3^\ast(D) := \sum_k \frac{a_{1k}^2}{C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2}, \quad (5.40) \]

where \( s^{(3)} = (s_{3k})_k \) with \( \frac{s_{3k}^2}{\eta_{3k}} \equiv 1, \ k \in \mathbb{Z}. \)

Proof. We shall try to obtain separately the real part and imaginary part of \( M\xi_{3k}(s), \) where \( \xi_{3k}(s_k) = ix_{3k} \exp(is_k(x_{3k} - a_{3k})). \) Using Lemma 5.14 formulas (5.28) and (5.33) we get

\[ H_{a,b}(s) = \int_{\mathbb{R}} ix \exp(is(x-a))d\mu_{(b,a)}(x) = \int_{\mathbb{R}} i(x+a) \exp(isx)d\mu_{(b,0)}(x) \]

\[ = \frac{dF_b(s)}{ds} + iaF_b(s) = (-\frac{s}{2b} + ia) \exp(-\frac{s^2}{4b}) = M\xi_{3k}(s), \]

Recall the Euler formulas

\[ e^{it} = \cos t + i \sin t, \quad e^{-it} = \cos t - i \sin t, \quad (5.41) \]

\[ \cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}. \]

More precisely, we denote for \( 1 \leq r \leq 3 \)

\[ \eta_{rk}(s) = x_{rk} \cos(s_k(x_{3k} - a_{3k})), \quad \eta_{rk}^\ast(s) = x_{rk} \cos(s_k(x_{3k} - a_{3k})). \quad (5.42) \]

We determine when the inclusion holds:

\[ D_{3k} \mathbf{1} \in \langle \sin(s_k(x_{3k} - a_{3k})) A_{kn} \mathbf{1} \mid k \in \mathbb{Z} \rangle \iff \Sigma_3(D, s) = \infty, \]

\[ +x_{2k} \sin(s_k(x_{3k} - a_{3k})) D_{2n} + x_{3k} \sin(s_k(x_{3k} - a_{3k})) D_{3n} \mathbf{1} \mid k \in \mathbb{Z}, \]

\[ D_{3k} \mathbf{1} \in \langle \cos(s_k(x_{3k} - a_{3k})) A_{kn} \mathbf{1} \mid k \in \mathbb{Z} \rangle \iff \Sigma_3^\ast(D, s) = \infty, \]

\[ +x_{2k} \cos(s_k(x_{3k} - a_{3k})) D_{2n} + x_{3k} \cos(s_k(x_{3k} - a_{3k})) D_{3n} \mathbf{1} \mid k \in \mathbb{Z}. \]
Set
\[ g_k(s_k) = \left( \eta_{1k}(s_k)D_{1n} + \eta_{2k}(s_k)D_{2n} + \left[ \eta_{3k}(s_k) - M\eta_{3k}(s_k) \right] D_{3n} \right) 1, \quad (5.43) \]
\[ g_k^\vee(s_k) = \left( \eta_{1k}^\vee(s_k)D_{1n} + \eta_{2k}^\vee(s_k)D_{2n} + \left[ \eta_{3k}^\vee(s_k) - M\eta_{3k}^\vee(s_k) \right] D_{3n} \right) 1, \quad (5.44) \]

We show that (compare with (5.28))
\[ M\eta_{3k}(s) = -\frac{1}{2} \left( H_{a,b}(s) + \overline{H_{a,b}(s)} \right) = \frac{s}{2b_{3k}} \exp \left( -\frac{s^2}{4b_{3k}} \right), \quad (5.45) \]
\[ M\eta_{3k}^\vee(s) = \frac{1}{2i} \left( H_{a,b}(s) - \overline{H_{a,b}(s)} \right) = a_{3k} \exp \left( -\frac{s^2}{4b_{3k}} \right). \quad (5.46) \]

Recall the definition of the function \( F_b(s) \) defined by (5.31):
\[ F_b(s) = \int_{\mathbb{R}} \exp \left( is(x-a) \right) d\mu_{(b,a)}(x) = \int_{\mathbb{R}} \exp \left( isx \right) d\mu_{(b,0)}(x) = \exp \left( -\frac{s^2}{4b} \right). \quad (5.47) \]

We have
\[ M\eta(s) = \int_{\mathbb{R}} x \sin \left( s(x-a) \right) d\mu_{(b,a)}(x) = \int_{\mathbb{R}} (x+a) \sin(sx) d\mu_{(b,0)}(x) =
\[ \int_{\mathbb{R}} (x+a) \frac{e^{isx} - e^{-isx}}{2i} d\mu_{(b,0)}(x) = -\frac{1}{2} \int_{\mathbb{R}} i(x+a) \left( e^{isx} - e^{-isx} \right) d\mu_{(b,0)}(x) =
\[ -\frac{1}{2} \left( H_{a,b}(s) + \overline{H_{a,b}(s)} \right) = \frac{s}{2b} \exp \left( -\frac{s^2}{4b} \right), \]
this implies (5.45). Similarly we get
\[ M\eta^\vee(s) = \int_{\mathbb{R}} x \cos \left( s(x-a) \right) d\mu_{(b,a)}(x) = \int_{\mathbb{R}} (x+a) \cos(sx) d\mu_{(b,0)}(x) =
\[ \frac{1}{2i} \int_{\mathbb{R}} i(x+a) \left( e^{isx} + e^{-isx} \right) d\mu_{(b,0)}(x) = \frac{1}{2i} \left( H_{a,b}(s) - \overline{H_{a,b}(s)} \right) \]
\[ = a \exp \left( -\frac{s^2}{4b} \right) \]
this implies (5.45). Fix \( m \in \mathbb{N} \), we put \( \sum_{k=-m}^{m} t_k M\eta_{3k}(s_k) = (t, b) = 1, \)
where \( t = (t_k)_{k=-m}^m \) and \( b = (M\eta_3k(s_k))_{k=-m}^m \). We have

\[
\begin{align*}
\| &\sum_{k=-m}^m t_k \sin \left( s_k (x_{3k} - a_{3k}) \right) A_{kn} - D_{3n} \|_1^2 \\
= &\left\| \sum_{k=-m}^m t_k \left( \eta_{1k}(s_k) D_{1n} + \eta_{2k}(s_k) D_{2n} + \left[ \eta_{3k}(s_k) - M\eta_{3k}(s_k) \right] D_{3n} \right) \right\|_1^2 \\
= &\sum_{k=-m}^m t_k^2 \| g_k(s_k) \|^2, \quad \text{since} \quad (D_{rn} \mathbf{1}, D_{ln} \mathbf{1}) = 0, \quad 1 \leq r < l \leq 3, \quad (5.48)
\end{align*}
\]

where the \( g_k(s_k) \) are defined by (5.43). To calculate \( \| g_k(s_k) \|^2 \) we have

\[
\begin{align*}
\| g_k(s_k) \|^2 &= (g_k(s_k), g_k(s_k)) = \\
&= \left( \left( \eta_{1k}(s_k) D_{1n} + \eta_{2k}(s_k) D_{2n} + \left[ \eta_{3k}(s_k) - M\eta_{3k}(s_k) \right] D_{3n} \right) \mathbf{1} , \\
\right. \\
&\left. \left( \eta_{1k}(s_k) D_{1n} + \eta_{2k}(s_k) D_{2n} + \left[ \eta_{3k}(s_k) - M\eta_{3k}(s_k) \right] D_{3n} \right) \mathbf{1} \right) = \\
&\| x_{1k} \mathbf{1} \|^2 \| \sin \left( s_k (x_{3k} - a_{3k}) \right) \mathbf{1} \|_1^2 \| D_{1n} \mathbf{1} \|^2 + \\
&\| x_{2k} \mathbf{1} \|^2 \| \sin \left( s_k (x_{3k} - a_{3k}) \right) \mathbf{1} \|_1^2 \| D_{2n} \mathbf{1} \|^2 + \\
&\left( M |\eta_{kn}(s_k)|^2 - M |\eta_{kn}(s_k)|^2 \right) \| D_{3n} \mathbf{1} \|^2 = \frac{1}{2b_{1k}} + a_{1k}^2 I_3 b_{1n} \frac{1}{2} + \\
&\left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) I_3 b_{2n} \frac{1}{2} + \left( M |\eta_{kn}(s_k)|^2 - M |\eta_{kn}(s_k)|^2 \right) b_{3n} \frac{1}{2}. \quad (5.49)
\end{align*}
\]

We need to calculate \( I_3 = \| \sin \left( s_k (x_{3k} - a_{3k}) \right) \mathbf{1} \|^2, \quad M |\eta_{kn}(s_k)|^2 \) and \( M |\eta_{kn}(s_k)|^2 \).

If we set \( a := a_{3k}, \quad b := b_{3k} \), we get

\[
I_3 = \| \sin \left( s_k (x_{3k} - a_{3k}) \right) \mathbf{1} \|^2 = \int_R \frac{e^{ix} - e^{-ix} e^{ix} - e^{ix}}{2i} d\mu_{(b,0)}(x) = \\
\frac{1}{2} \int_R \left( 1 - \frac{e^{2ix} + e^{-2ix}}{2} \right) d\mu_{(b,0)}(x) = 1 - \frac{e^{2ix}}{2i}, \quad (5.48) \\
\]

\[
M |\eta_{kn}(s_k)|^2 = \frac{s_{2k}^2}{4b_{3k}^2} \exp \left( - \frac{s_{2k}^2}{2b_{3k}} \right), \quad (5.51)
\]

\[
M |\eta_{kn}(s_k)|^2 = \int_R (x^2 + 2xa + a^2) \frac{e^{ix} - e^{-ix} e^{-ix} - e^{ix}}{2i} d\mu_{(b,0)}(x) = \\
\frac{1}{2} \int_R (x^2 + 2xa + a^2) \left( 1 - \frac{e^{2ix} + e^{-2ix}}{2} \right) d\mu_{(b,0)}(x) = \\
91
\]
Finally, we get

\[
\frac{1}{2} \left[ \int_{\mathbb{R}} (x^2 + a^2) d\mu_{(b,0)}(x) - \int_{\mathbb{R}} (x^2 + a^2) e^{2i\pi x} + e^{-2i\pi x} d\mu_{(b,0)}(x) \right] = \\
\frac{1}{2} \left[ \frac{1}{2b} + a^2 - \frac{d^2 F_b(2s)}{ds^2} - a^2 F_b(2s) \right] \quad (5.47)
\]

\[
= \frac{1}{2} \left[ \frac{1}{2b} + a^2 - \frac{1}{(2b)^2} \left( \frac{2s}{b} \right)^2 - \frac{2}{b} \right] \times \\
(5.52)
\]

\[
e^{-\frac{s^2}{b}} - a^2 e^{-\frac{s^2}{b^2}} = \frac{1}{2} \left[ \left( \frac{1}{2b} + a^2 \right) (1 - e^{-\frac{s^2}{b^2}}) + \frac{s^2}{b^2} e^{-\frac{s^2}{b^2}} \right].
\]

By (5.58), (5.49), (5.50), (5.53) and (6.3) we prove (5.36), where

\[
\Sigma_3(D, s) = \sum_{k \in \mathbb{Z}} \frac{\left| M \eta_{kn}(s_k) \right|^2}{\|g_k(s_k)\|^2} = \\
\sum_{k \in \mathbb{Z}} \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) I_3 \frac{b_{3k}}{2} + \left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) I_3 \frac{b_{3k}}{2} + \left( M \eta_{kn}(s_k) \right|^2 - |M \eta_{kn}(s_k)|^2) \frac{b_{3k}}{2} = \\
\sum_{k \in \mathbb{Z}} \left( 1 - e^{-\frac{s^2}{b_{3k}}} \right) (c_{1k} + c_{2k}) + \frac{1}{2} \left[ c_{3k} \left( 1 - e^{-\frac{s^2}{b_{3k}}} \right) + \frac{s^2}{b_{3k}} e^{-\frac{s^2}{b_{3k}}} \right] - \frac{s^2}{4b_{3k}} e^{-\frac{s^2}{b_{3k}}} = \\
\Sigma_3(D, x),
\]

where \( x_k = \frac{s^2}{b_{3k}} \) and \( c_{rk} = \frac{1}{2b_{rk}} + a_{rk}^2 \). For \( x^{(3)} = (x_k)_k \) with \( x_k \equiv 1 \) we get

\[
\Sigma_3(D, x^{(3)}) = \sum_{k \in \mathbb{Z}} \frac{1}{2} e^{-1} \left( c_{1k} + c_{2k} \right) + \frac{1}{2} \left[ c_{3k} \left( 1 - e^{-1} \right) + \frac{1}{b_{3k}} e^{-1} \right] - \frac{1}{4b_{3k}} e^{-\frac{1}{2}} = \\
\sum_{k \in \mathbb{Z}} \frac{1}{2b_{3k}} e^{-\frac{1}{2}} \left( c_{1k} + c_{2k} + c_{3k} \right) + \frac{1}{b_{3k}} \left( e^{-1} - e^{-\frac{1}{2}} \right) \quad (2.6)
\]

\[
\sum_{k \in \mathbb{Z}} \frac{1}{2b_{3k}} e^{-\frac{1}{2}} c_{1k} + c_{2k} + c_{3k} = \sum_{k \in \mathbb{Z}} \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{1k}^2 + a_{2k}^2 + a_{3k}^2 = \Sigma_3(D). (5.55)
\]
So, we have proved (5.39) for $x = (x_k)_k$ with $x_k \equiv 1$. To approximate $D_{3n}$ in terms of functions involving the cosine, fix $m \in \mathbb{N}$, and put $\sum_{k=-m}^{m} t_k M \eta_{3k}^\vee(s_k) = (t, b) = 1$, where $t = (t_k)_{k=-m}^{m}$ and $b = (M \eta_{3k}^\vee(s_k))_{k=-m}^{m}$. We have

$$
\| \left[ \sum_{k=-m}^{m} t_k \cos \left( s_k(x_{3k} - a_{3k}) \right) A_{kn} - D_{3n} \right] \| \leq \sum_{k=-m}^{m} t_k^2 \| g_k^\vee(s_k) \|^2,
$$

since $D_{rn}1, D_{ln}1 = 0$,  $1 \leq r < l \leq 3$, (5.56)

where the $g_k^\vee(s_k)$ are defined by (5.43). To calculate $\| g_k^\vee(s_k) \|^2$ we have

$$
\| g_k^\vee(s_k) \|^2 = (g_k^\vee(s_k), g_k^\vee(s_k)) =
$$

$$
\left( \left( \eta_{1k}^\vee(s_k)D_{1n} + \eta_{2k}^\vee(s_k)D_{2n} + [\eta_{3k}^\vee(s_k) - M \eta_{3k}^\vee(s_k)]D_{3n} \right)1, \right.
$$

$$
\left. \left( \eta_{1k}^\vee(s_k)D_{1n} + \eta_{2k}^\vee(s_k)D_{2n} + [\eta_{3k}^\vee(s_k) - M \eta_{3k}^\vee(s_k)]D_{3n} \right)1 \right)
$$

$$
= \| x_{1k}1 \|^2 \| \cos \left( s_k(x_{3k} - a_{3k}) \right) 1 \|^2 \| D_{1k}1 \|^2 + \| x_{2k}1 \|^2 \| \cos \left( s_k(x_{3k} - a_{3k}) \right) 1 \|^2 \| D_{2k}1 \|^2 +
$$

$$
\left( M|\eta_{kn}^\vee(s_k)|^2 - |M\eta_{kn}^\vee(s_k)|^2 \right) \| D_{3k}1 \|^2 = \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) I_3 \frac{b_{1n}}{2} + \left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) I_3 \frac{b_{2n}}{2} + \left( M|\eta_{kn}^\vee(s_k)|^2 - |M\eta_{kn}^\vee(s_k)|^2 \right) \frac{b_{3n}}{2}.
$$

(5.57)

We need to calculate $I_3^\vee = \| \cos \left( i s_k(x_{3k} - a_{3k}) \right) \|^2$, $M|\eta_{kn}^\vee(s_k)|^2$ and $|M\eta_{kn}^\vee(s_k)|^2$. Finally, we get (we set $b := b_{3k}$) To approximate $D_{3n}$ in terms of functions involving cos, fix $m \in \mathbb{N}$, and put $\sum_{k=-m}^{m} t_k M \eta_{3k}^\vee(s_k) = (t, b) = 1$, where $t = (t_k)_{k=-m}^{m}$ and $b = (M \eta_{3k}^\vee(s_k))_{k=-m}^{m}$. We have

$$
\| \left[ \sum_{k=-m}^{m} t_k \cos \left( s_k(x_{3k} - a_{3k}) \right) A_{kn} - D_{3n} \right] \| \leq \sum_{k=-m}^{m} t_k^2 \| g_k^\vee(s_k) \|^2,
$$

since $D_{rn}1, D_{ln}1 = 0$,  $1 \leq r < l \leq 3$, (5.58)
where the \( g_k^\nu(s_k) \) are defined by (5.43). To calculate \( \|g_k^\nu(s_k)\|^2 \) we have

\[
\|g_k^\nu(s_k)\|^2 = (g_k^\nu(s_k),g_k^\nu(s_k)) = \\
\left( \eta_k^\nu(s_k)D_{1n} + \eta_k^\nu(s_k)D_{2n} + [\eta_k^\nu(s_k) - M\eta_k^\nu(s_k)]D_{3n} \right) \mathbf{1},
\]

\[
\left( \eta_k^\nu(s_k)D_{1n} + \eta_k^\nu(s_k)D_{2n} + [\eta_k^\nu(s_k) - M\eta_k^\nu(s_k)]D_{3n} \right) = \\
\|x_1k\|^2 \| \cos (s_k(x_3k - a_3k)) \|^2 + \|x_2k\|^2 \| \cos (s_k(x_3k - a_3k)) \|^2 + \\
\left( M|\eta_k^\nu(s_k)|^2 - |\eta_k^\nu(s_k)|^2 \right) \|D_{3k}\|^2 = \left( \frac{1}{2b_{1k}} + a_{1k}^2 \right) I_3 b_{1n} + \\
\left( \frac{1}{2b_{2k}} + a_{2k}^2 \right) I_3 b_{2n} + \left( M|\eta_k^\nu(s_k)|^2 - |\eta_k^\nu(s_k)|^2 \right) b_{3n}. \quad (5.59)
\]

We need to calculate \( I_3^\nu = \| \cos (s_k(x_3k - a_3k)) \|^2, \ M|\eta_k^\nu(s_k)|^2 \) and \( |\eta_k^\nu(s_k)|^2 \).
If we set \( b = b_{3k} \), we get

\[
I_3^\nu = \| \cos (s_k(x_3k - a_3k)) \|^2 = \int_{\mathbb{R}} \frac{e^{ixx} + e^{-ixx}}{2} \frac{e^{-ixx} + e^{ixx}}{2} d\mu(b_{0,0})(x) = \\
\frac{1}{2} \int_{\mathbb{R}} \left( 1 + \frac{e^{2ixx} + e^{-2ixx}}{2} \right) d\mu(b_{0,0})(x) \quad (5.47) \quad 1 + e^{-\frac{s_k^2}{b}}, \quad (5.60)
\]

\[
|\eta_k^\nu(s_k)|^2 = a_{3k}^2 \exp \left( -\frac{s_k^2}{2b_{3k}} \right), \quad (5.61)
\]

\[
M|\eta_k^\nu(s_k)|^2 = \int_{\mathbb{R}} (x^2 + 2xa + a^2) \frac{e^{ixx} + e^{-ixx}}{2} \frac{e^{-ixx} + e^{ixx}}{2} d\mu(b_{0,0})(x) = \\
\frac{1}{2} \int_{\mathbb{R}} (x^2 + 2xa + a^2) \left( 1 + \frac{e^{2ixx} + e^{-2ixx}}{2} \right) d\mu(b_{0,0})(x) = \\
\frac{1}{2} \int_{\mathbb{R}} (x^2 + 2xa + a^2) d\mu(b_{0,0})(x) + \int_{\mathbb{R}} (x^2 + 2xa + a^2) \frac{e^{2ixx} + e^{-2ixx}}{2} d\mu(b_{0,0})(x) = \\
\frac{1}{2} \left[ \frac{1}{2b} + a^2 + \frac{d^2 F_b(2s)}{ds^2} + a^2 F_b(2s) \right] \quad (5.47) \quad \frac{1}{2} \left[ \frac{1}{2b} + a^2 + \frac{1}{(2i)^2} \left( \frac{2s}{b} \right)^2 \right] \times e^{-\frac{s^2}{b}} + a^2 e^{-\frac{s^2}{b}} = \frac{1}{2} \left[ \left( \frac{1}{2b} + a^2 \right) (1 + e^{-\frac{s^2}{b}}) - \frac{s^2}{b^2} e^{-\frac{s^2}{b}} \right]. \quad (5.62)
\]

Finally, we get

\[
M|\eta_k^\nu(s_k)|^2 - |\eta_k^\nu(s_k)|^2 = \frac{1}{2} \left[ \left( \frac{1}{2b} + a^2 \right) (1 + e^{-\frac{s^2}{b}}) - \frac{s^2}{b^2} e^{-\frac{s^2}{b}} \right] - a^2 e^{-\frac{s^2}{b^2}}. \quad (5.63)
\]
By (5.56), (5.57), (5.60), (5.63) and (6.3) we prove (5.37), where
\[
\Sigma^y_3(D, s) = \sum_{k \in \mathbb{Z}} \frac{|M\eta_{kn}^y(s_k)|^2}{\|g_k^y(s_k)\|^2} = \\
\sum_{k \in \mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{s_k^2}{2\sigma_{3k}^2}}}{(\frac{1}{2b_{1k}} + a_{1k}^2)I_3^y b_{1n}^y + (\frac{1}{2b_{2k}} + a_{2k}^2)I_3^y b_{2n}^y + (M|\eta_{kn}^y|^2 - |M\eta_{kn}^y|^2)^{\frac{b_{1n}^y}{2}}} \\
\sim \sum_{k \in \mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{s_k^2}{2\sigma_{3k}^2}}}{\frac{1+e^{-\frac{s_k^2}{2}}}{2}(c_{1k} + c_{2k}) + \frac{1}{2}[c_{3k}(1 + e^{-\frac{s_k^2}{2}}) - \frac{s_k^2}{2b_{1k}} e^{-\frac{s_k^2}{2}}] - a_{3k}^2 e^{-\frac{s_k^2}{2}}} = \Sigma_3(D, x),
\]
where \(x_k^2 = \frac{s_k^2}{b_{3k}}\) and \(c_{rk} = \frac{1}{2b_{rk}} + a_{rk}^2\). For \(x^{(3)} = (x_k)_k\) with \(x_k \equiv 1\) we get
\[
\Sigma^y_3(D, x^{(3)}) = \sum_{k \in \mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{1}{2}}}{\frac{1+e^{-1}}{2}(c_{1k} + c_{2k}) + \frac{1}{2}[c_{3k}(1 + e^{-1}) - \frac{1}{b_{3k}} e^{-1}] - a_{3k}^2 e^{-\frac{1}{2}}} = \\
\sum_{k \in \mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{1}{2}}}{\frac{1+e^{-1}}{2}(c_{1k} + c_{2k} + c_{3k})} - \left(\frac{1}{2b_{3k}} e^{-1} + a_{3k}^2 e^{-\frac{1}{2}}\right)^{(2.6)} \\
\sum_{k \in \mathbb{Z}} \frac{a_{3k}^2}{c_{1k} + c_{2k} + c_{3k}} = \sum_{k \in \mathbb{Z}} c_k^* + a_{1k}^2 + a_{2k}^2 + a_{3k}^2 = \Sigma^y_3(D).
\] (5.65)
So, we have proved (5.40) for \(x = (x_k)_k\) with \(x_k \equiv 1\).

6. How far is a vector from a hyperplane?

6.1. Some estimates

We recall some material from [27], Section 1.4.1, pp.24–25.

Lemma 6.1 [24]. For a strictly positive operator \(A\) (i.e., \((Af, f) > 0\) for \(f \neq 0\) acting in \(\mathbb{R}^n\) and a vector \(b \in \mathbb{R}^n \setminus \{0\}\), we have
\[
\min_{x \in \mathbb{R}^n} \left(\langle Ax, x \rangle \mid (x, b) = 1\right) = \frac{1}{(A^{-1}b, b)}. \tag{6.1}
\]
The minimum is assumed for $x = \frac{A^{-1}b}{(A^{-1}b,b)}$.

Lemma 6.1 is a direct generalization of the well known result (see, for example, [4], Chap. I, §52), stating that for $a_k > 0$, $1 \leq k \leq n$ we have

$$\min_{x \in \mathbb{R}^n} \left( \sum_{k=1}^{n} a_k x_k^2 \mid \sum_{k=1}^{n} x_k = 1 \right) = \left( \sum_{k=1}^{n} \frac{1}{a_k} \right)^{-1}.$$  \hfill (6.2)

We will also use the same result in a slightly different form:

$$\min_{x \in \mathbb{R}^n} \left( \sum_{k=1}^{n} a_k x_k^2 \mid \sum_{k=1}^{n} x_k b_k = 1 \right) = \left( \sum_{k=1}^{n} b_k^2 \left( \sum_{k=1}^{n} \frac{1}{a_k} \right) \right)^{-1},$$  \hfill (6.3)

with the minimum being assumed for $x_k = b_k \left( \sum_{k=1}^{n} \frac{b_k}{a_k} \right)^{-1}$.

6.2. The distance of a vector from a hyperplane

We follow closely the exposition [30]. We start with a classical result, see, e.g. [9]. Consider the hyperplane $V_n$ generated by $n$ arbitrary vectors $f_1, \ldots, f_n$ in some Hilbert space $H$.

**Lemma 6.2.** The square of the distance $d(f_0, V_n)$ of a vector $f_0$ from the hyperplane $V_n$ is given by the ratio of two Gram determinants:

$$d^2(f_0, V_n) = \frac{\Gamma(f_0, f_1, f_2, \ldots, f_n)}{\Gamma(f_1, f_2, \ldots, f_n)}.$$  \hfill (6.4)

6.3. Gram determinants and Gram matrices

**Definition 6.1.** Let us recall the definition of the Gram determinant and the Gram matrix (see [9], Chap IX, §5). Given the vectors $x_1, x_2, \ldots, x_m$ in some Hilbert space $H$ the Gram matrix $\gamma(x_1, x_2, ..., x_m)$ is defined by the formula

$$\gamma(x_1, x_2, ..., x_m) = \left( \langle x_k, x_n \rangle \right)_{k,n=1}^{m}.$$  

The determinant of this matrix is called the **Gram determinant** for the vectors $x_1, x_2, ..., x_m$ and is denoted by $\Gamma(x_1, x_2, ..., x_m)$. Thus,

$$\Gamma(x_1, x_2, \ldots, x_m) := \det \gamma(x_1, x_2, \ldots, x_m).$$  \hfill (6.5)
6.4. The generalized characteristic polynomial and its properties

**Notations.** For a matrix $C \in \text{Mat}(n, \mathbb{R})$ and $1 \leq i_1 < i_2 < \ldots < i_r \leq n$, $1 \leq j_1 < j_2 < \ldots < j_r \leq n$, $r \leq n$ denote by

$$M_{i_1i_2\ldots i_r}^j(C) \quad \text{and} \quad A_{i_1i_2\ldots i_r}^j(C)$$

the corresponding minors and cofactors of the matrix $C$.

**Definition 6.2.** ([27, Ch.1.4.3]) For the matrix $C \in \text{Mat}(m, \mathbb{C})$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$ define the generalization of the characteristic polynomial, $p_C(t) = \det(tI - C)$, $t \in \mathbb{C}$ as follows:

$$P_C(\lambda) = \det(C(\lambda)), \quad \text{where} \quad C(\lambda) = \text{diag}(\lambda_1, \ldots, \lambda_m) + C. \quad (6.6)$$

**Lemma 6.3.** ([27, Ch.1.4.3]) For the generalized characteristic polynomial $P_C(\lambda)$ of $C \in \text{Mat}(m, \mathbb{C})$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^m$ we have

$$P_C(\lambda) = \det(C) + \sum_{r=1}^{m} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq m} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_r} A_{i_1i_2\ldots i_r}^j(C). \quad (6.7)$$

**Remark 6.1.** If we set $\lambda_\alpha = \lambda_1 \lambda_2 \ldots \lambda_i$, where $\alpha = (i_1, i_2, \ldots, i_r)$ and $A_\alpha^j(C) = A_{i_1i_2\ldots i_r}^j(C)$, $M_\alpha^j(C) = M_{i_1i_2\ldots i_r}^j(C)$, $\lambda_\emptyset = 1$, $A_\emptyset^j(C) = \det(C)$ we may write (6.7) as follows:

$$P_C(\lambda) = \det(C(\lambda)) = \sum_{\emptyset \subseteq \alpha \subseteq \{ 1, 2, \ldots, m \}} \lambda_\alpha A_\alpha^j(C), \quad (6.8)$$

$$P_C(\lambda) = \det(C) = \left( \prod_{k=1}^{n} \lambda_k \right) \sum_{\emptyset \subseteq \alpha \subseteq \{ 1, 2, \ldots, m \}} \frac{M_\alpha^j(C)}{\lambda_\alpha}, \quad (6.9)$$

Let

$$X = X_{mn} = \begin{pmatrix}
    x_{11} & x_{12} & \ldots & x_{1n} \\
    x_{21} & x_{22} & \ldots & x_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{m1} & x_{m2} & \ldots & x_{mn}
\end{pmatrix}, \quad (6.10)$$

Setting

$$x_k = (x_{1k}, x_{2k}, \ldots, x_{mk}) \in \mathbb{R}^m, \quad y_r = (x_{r1}, x_{r2}, \ldots, x_{rn}) \in \mathbb{R}^n, \quad (6.11)$$

97
we get
\[ X^*X = \begin{pmatrix} (x_1, x_1) & (x_1, x_2) & \cdots & (x_1, x_n) \\ (x_2, x_1) & (x_2, x_2) & \cdots & (x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ (x_n, x_1) & (x_n, x_2) & \cdots & (x_n, x_n) \end{pmatrix} = \gamma(x_1, x_2, \ldots, x_n), \quad (6.12) \]
\[ XX^* = \begin{pmatrix} (y_1, y_1) & (y_1, y_2) & \cdots & (y_1, y_m) \\ (y_2, y_1) & (y_2, y_2) & \cdots & (y_2, y_m) \\ \vdots & \vdots & \ddots & \vdots \\ (y_m, y_1) & (y_m, y_2) & \cdots & (y_m, y_m) \end{pmatrix} = \gamma(y_1, y_2, \ldots, y_m), \quad (6.13) \]
therefore, we obtain
\[ \Gamma(x_1, x_2, \ldots, x_n) = \det(X^*X) = \det(XX^*) = \Gamma(y_1, y_2, \ldots, y_m). \quad (6.14) \]

7. Explicit expressions for \( C^{-1}(\lambda) \) and \( (C^{-1}(\lambda)a, a) \)

In this section we follow [30]. Fix \( C \in \text{Mat}(n, \mathbb{R}), a \in \mathbb{R}^n \) and \( \lambda \in \mathbb{C}^n \). Our aim is to find the explicit formulas for \( C^{-1}(\lambda) \) and \( (C^{-1}(\lambda)a, a) \), where \( C(\lambda) \) is defined by (6.6). Set \( M(i_1i_2\ldots i_r)(C) = M_{i_1i_2\ldots i_r}^{i_1i_2\ldots i_r}(C) \) and \( a_{i_1i_2\ldots i_r} = (a_{i_1}, a_{i_2}, \ldots, a_{i_r}) \). Let also \( C_{i_1i_2\ldots i_r} \) be the corresponding submatrix of the matrix \( C \). The elements of this matrix are on the intersection of \( i_1, i_2, \ldots, i_r \) rows and column of the matrix \( C \). Denote by \( A(C_{i_1i_2\ldots i_r}) \) the matrix of the cofactors of the first order of the matrix \( C_{i_1i_2\ldots i_r} \), i.e.
\[ A(C_{i_1i_2\ldots i_r}) = (A_j(C_{i_1i_2\ldots i_r}))_{1 \leq i, j \leq r} \quad (7.1) \]
Let \( n = 3 \), then \( A(C_{123}) = A(C) \) is the following matrix:
\[ A(C) = A(C_{123}) = \begin{pmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{pmatrix} = \begin{pmatrix} M_{23}^{23} & -M_{13}^{23} & M_{12}^{23} \\ -M_{13}^{13} & M_{13}^{13} & -M_{13}^{13} \\ M_{13}^{23} & -M_{13}^{12} & M_{12}^{12} \end{pmatrix}, \quad (7.2) \]
where we write \( M_{i_1j_1}^{i_2j_2} \) instead of \( M_{i_1j_1}^{i_2j_2}(C) \) and \( A_j^i \) instead of \( A_j(C) \).

**Remark 7.1.** Let \( A^T \) be the transposed matrix of \( A \). Then
\[ A^T(C_{i_1i_2\ldots i_r}) = \det C_{i_1i_2\ldots i_r}^{-1}(C_{i_1i_2\ldots i_r}^{-1}), \quad (7.3) \]
In what follows we will consider the submatrix $C_{i_1i_2...i_r}$ of the matrix $C \in \text{Mat}(n, \mathbb{R})$ as an appropriate element of $\text{Mat}(n, \mathbb{R})$.

**Theorem 7.1.** For the matrix $C(\lambda)$ defined by (6.6) $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}^n$ we have

$$P_C(\lambda) = \left( \prod_{k=1}^{n} \lambda_k \right) \sum_{r=1}^{n} \sum_{1 \leq i_1 < i_2 < ... < i_r \leq n} \frac{M(i_1i_2...i_r)}{\lambda_{i_1}\lambda_{i_2}...\lambda_{i_r}},$$  \hspace{1cm} (7.4)

$$C^{-1}(\lambda) = \frac{1}{P_C(\lambda)} \left( \prod_{k=1}^{n} \lambda_k \right) \sum_{r=1}^{n} \sum_{1 \leq i_1 < i_2 < ... < i_r \leq n} \frac{A(C_{i_1i_2...i_r})}{\lambda_{i_1}\lambda_{i_2}...\lambda_{i_r}},$$  \hspace{1cm} (7.5)

$$(C^{-1}(\lambda)a, a) = \frac{1}{P_C(\lambda)} \left( \prod_{k=1}^{n} \lambda_k \right) \sum_{r=1}^{n} \sum_{1 \leq i_1 < i_2 < ... < i_r \leq n} \frac{A(C_{i_1i_2...i_r})a_{i_1i_2...i_r}}{\lambda_{i_1}\lambda_{i_2}...\lambda_{i_r}}.$$  \hspace{1cm} (7.6)

### 7.1. The case where $C$ is the Gram matrix

Fix the matrix $X_{mn}$ defined by (6.10). Denote by $C$ the Gram matrix $\gamma(x_1, x_2, ..., x_n)$, i.e.,

$$C = \gamma(x_1, x_2, ..., x_n),$$  \hspace{1cm} (7.7)

where $(x_1, x_2, ..., x_n)$ are defined by (6.11) and $\gamma(x_1, x_2, ..., x_n)$ by (6.12). In what follows we consider the operator $C(\lambda)$ defined by (6.6).

**Remark 7.2.** In this case we have

$$P_C(\lambda) = \det \left( \sum_{k=1}^{n} \lambda_k E_{kk} + \gamma(x_1, x_2, ..., x_n) \right)$$  \hspace{1cm} (7.8)

$$= \prod_{k=1}^{n} \lambda_k \left( 1 + \sum_{r=1}^{n} \sum_{1 \leq i_1 < i_2 < ... < i_r \leq m} \left( \lambda_{i_1}\lambda_{i_2}...\lambda_{i_r} \right)^{-1} \Gamma(x_{i_1}, x_{i_2}, ..., x_{i_r}) \right)$$

$$= \prod_{k=1}^{n} \lambda_k \left( 1 + \sum_{r=1}^{n} \sum_{1 \leq i_1 < i_2 < ... < i_r \leq m; \ 1 \leq j_1 < j_2 < ... < j_r \leq n} \left( \lambda_{i_1}\lambda_{i_2}...\lambda_{i_r} \right)^{-1} \left( M_{i_1j_1i_2j_2...i_rj_r}(X) \right)^2 \right),$$

where we have used the following formula (see [9], Chap IX, §5 formula (25)):

$$\Gamma(x_{i_1}, x_{i_2}, ..., x_{i_r}) = \sum_{1 \leq j_1 < j_2 < ... < j_r \leq m} \left( M_{i_1j_1i_2j_2...i_rj_r}(X) \right)^2.$$  \hspace{1cm} (7.9)
7.2. Case $m = 2$

Fix two natural numbers $n, m \in \mathbb{N}$ with $m \leq n$, two matrices $A_{mn}$ and $X_{mn}$, vectors $g_k \in \mathbb{R}^{m-1}$, $1 \leq k \leq n$ and $a \in \mathbb{R}^n$ as follows

$$A_{mn} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad g_k = \begin{pmatrix} a_{2k} \\ a_{3k} \\ \vdots \\ a_{mk} \end{pmatrix} \in \mathbb{R}^{m-1}, \quad a = (a_{1k})_{k=1}^n \in \mathbb{R}^n.$$

(7.10)

Set $C = \gamma(g_1, g_2, \ldots, g_n)$. We calculate $\Delta_n(\lambda, C)$ and $(C^{-1}(\lambda)a, a)$ for an arbitrary $n$. Consider the matrix (6.10)

$$X_{mn} = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{pmatrix}, \quad \text{where } x_{kr} = \frac{a_{1k}}{\sqrt[2]{\lambda_k}}, \quad y_r = (x_{rk})_{k=1}^n \in \mathbb{R}^n.$$

(7.11)

Lemma 7.2 ([30], Lemma 2.2). For $m = 2$ we have

$$(C^{-1}(\lambda)a, a) = \Delta(y_1, y_2) = \frac{\Gamma(y_1) + \Gamma(y_1, y_2)}{1 + \Gamma(y_2)},$$

(7.12)

where $y_1$ and $y_2$ are defined as follows

$$y_1 = \left(\frac{a_{1k}}{\sqrt[2]{\lambda_k}}\right)_{k=1}^n, \quad y_2 = \left(\frac{a_{2k}}{\sqrt[2]{\lambda_k}}\right)_{k=1}^n.$$

(7.13)

7.3. Case $m = 3$

By (7.8) we get

$$\Delta(y_1, y_2, y_3) = \frac{\Gamma(y_1) + \Gamma(y_1, y_2) + \Gamma(y_1, y_3) + \Gamma(y_1, y_2, y_3)}{1 + \Gamma(y_2) + \Gamma(y_3) + \Gamma(y_2, y_3)}.$$

Lemma 7.3. For $m = 3$ we have

$$(C^{-1}(\lambda)a, a) = \Delta(y_1, y_2, y_3),$$

(7.14)

where the $y_r$ are defined as follows:

$$y_r = \left(\frac{a_{rk}}{\sqrt[3]{\lambda_k}}\right)_{k=1}^n \in \mathbb{R}^n, \quad 1 \leq r \leq 3.$$

(7.15)
8. Appendix

8.1. Comparison of two Gaussian measures

For two centered Gaussian measures $\mu_{(b,0)}$ and $\mu_{(b',0)}$ on the real line $\mathbb{R}$ defined by (1.5) it is well known that

$$H(\mu_{(b,0)}, \mu_{(b',0)}) = \left( \frac{4bb'}{(b+b')^2} \right)^{1/4}. \quad (8.1)$$

By Kakutani’s criterion for product measures on $\mathbb{R}^N \ [13]$, and (8.1) we see that the following lemma holds true.

Lemma 8.1. Two Gaussian measures $\mu_{(b,0)} = \otimes_{n \in \mathbb{Z}} \mu_{(b,0)}$ and $\mu_{(b',0)} = \otimes_{n \in \mathbb{Z}} \mu_{(b',0)}$ are equivalent if and only if the product

$$\prod_{n \in \mathbb{Z}} \frac{4b_n b'_n}{(b_n + b'_n)^2} \quad (8.2)$$

does not converge to 0. The equivalent condition is

$$\sum_{n \in \mathbb{Z}} \left( \sqrt{\frac{b_n}{b'_n}} - \sqrt{\frac{b'_n}{b_n}} \right)^2 < \infty. \quad (8.3)$$

Consider two measures: $\mu_{(1,0)} = \otimes_{n \in \mathbb{Z}} \mu_{(1,0)}$ and $\mu_{(1+c,0)} = \otimes_{n \in \mathbb{Z}} \mu_{(1+c_n,0)}$ on the space $X_1$, where the measure $\mu_{(b,a)}$ on the real line $\mathbb{R}$ is defined by (1.5).

Lemma 8.2. Two measures $\mu_{(1,0)}$ and $\mu_{(1+c,0)}$ are equivalent if and only if

$$\sum_{n \in \mathbb{Z}} c_n^2 < \infty \quad (8.4)$$

Proof. By Lemma 8.1 and (8.3), the measures $\mu_{(1,0)}$ and $\mu_{(1+c,0)}$ are equivalent if and only if

$$\sum_{n \in \mathbb{Z}} \left( \frac{1}{\sqrt{1+c_n}} - \sqrt{1+c_n} \right)^2 = \sum_{n \in \mathbb{Z}} \frac{c_n^2}{1+c_n} < \infty.$$

By Lemma 2.5, two series $\sum_{n \in \mathbb{Z}} \frac{c_n^2}{1+c_n}$ and $\sum_{n \in \mathbb{Z}} c_n^2$ are equivalent. \qed

101
The next lemma is also a consequence of Kakutani’s criterion [13].

Lemma 8.3. Two Gaussian measures \( \mu_{m, (b,0)} \) and \( \mu_{m, (b',0)} \) are equivalent if and only if the product

\[
\prod_{r=1}^{m} \prod_{n \in \mathbb{Z}} \frac{4 b_{r n} b'_{r n}}{(b_{r n} + b'_{r n})^2}
\]

does not converge to 0. The equivalent condition is

\[
\sum_{r=1}^{m} \sum_{n \in \mathbb{Z}} \left( \frac{b_{r n}}{b'_{r n}} - \sqrt{\frac{b_{r n} b'_{r n}}{b_{r n}^2}} \right)^2 < \infty.
\]

Lemma 1.2 follows from Lemmas 8.4 – 8.7.

Lemma 8.4. For \( t \in \text{GL}(m, \mathbb{R})\backslash \{e\} \) we have \( (\mu_{m, (b,0)})^L_t \perp \mu_{m, (b,0)} \) if and only if

\[
(\mu_{m, (b,0)})^L_t \perp \mu_{m, (b,0)} \quad \text{or} \quad \mu_{m, (b,t a)} \perp \mu_{m, (b,a)}.
\]

Let us define the following measures on the spaces \( \mathbb{R}^m \) and \( X_m \):

\[
\mu_m^{(B_n,0)} = \otimes_{k=1}^{m} \mu_m^{(b_{kn},0)}, \quad \mu_m^{(B_n,a_n)} = \otimes_{k=1}^{m} \mu_m^{(b_{kn},a_{kn})},
\]

where \( a_n = (a_{1n}, ..., a_{mn}) \in \mathbb{R}^m \) and \( B_n = \text{diag}(b_{1n}, ..., b_{mn}) \in \text{Mat}(m, \mathbb{R}) \). Since

\[
\mu_m^{(b,a)} = \otimes_{n \in \mathbb{Z}} \mu_m^{(B_n,a_n)}, \quad \mu_m^{(b,0)} = \otimes_{n \in \mathbb{Z}} \mu_m^{(B_n,0)},
\]

\[
(\mu_m^{(b,a)})^L_t = \otimes_{n \in \mathbb{Z}} (\mu_m^{(B_n,a_n)})^L_t, \quad (\mu_m^{(b,0)})^L_t = \otimes_{n \in \mathbb{Z}} (\mu_m^{(B_n,0)})^L_t,
\]

and

\[
\mu_m^{(b,t a)} = \otimes_{n \in \mathbb{Z}} \mu_m^{(B_n,L_t a_n)},
\]

by the Kakutani criterion [13], we derive the following two lemmas:

Lemma 8.5. For the measures \( \mu_m^{(b,0)}, m \in \mathbb{N} \) and \( t \in \text{GL}(m, \mathbb{R})\backslash \{e\} \), we obtain

\[
(\mu_m^{(b,0)})^L_t \perp \mu_m^{(b,0)} \iff \prod_{n \in \mathbb{Z}} H\left( (\mu_m^{(B_n,0)})^L_t, \mu_m^{(B_n,0)} \right) = 0.
\]

Lemma 8.6. For the measures \( \mu_m^{(b,0)}, m \in \mathbb{N} \) and \( t \in \text{GL}(m, \mathbb{R})\backslash \{e\} \), we get

\[
\mu_m^{(b,t a)} \perp \mu_m^{(b,a)} \iff \prod_{n \in \mathbb{Z}} H\left( \mu_m^{(B_n,L_t a_n)}, \mu_m^{(B_n,a_n)} \right) = 0.
\]
To prove Lemma 1.2 it is sufficient to show, by Lemma 8.4, that
\[ H_{m,n}(t) = H \left( \left( \mu_m(B_n,0) \right)^{L_t}, \mu_m(B_0,0) \right) = \left( \frac{1}{2^m |\det t|} \det \left( I + X_n^*(t)X_n(t) \right) \right)^{-1/2}, \]
(8.8)
to prove the equivalence
\[ \prod_{n \in \mathbb{Z}} H \left( \mu_m(B_n, L_{ta_n}), \mu_m(B_n, a_n) \right) = 0 \iff \sum_{n \in \mathbb{Z}} \sum_{r=1}^{m} b_{rn} \left( \sum_{s=1}^{m} (t_{rs} - \delta_{rs}) a_{sn} \right)^2 = \infty, \]
(8.9)
and to apply the following lemma.

**Lemma 8.7.** For \( X \in \text{Mat}(m, \mathbb{R}) \) we have
\[ \det (I + X^*X) = 1 + \sum_{r=1}^{m} \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq m; 1 \leq j_1 < j_2 < \cdots < j_r \leq m} \left( M_{i_1j_1i_2j_2}^{i_3j_3} \cdots M_{i_rj_r}^{i_{r+1}j_{r+1}}(X) \right)^2. \]
(8.10)

The proof of the equivalence (8.9) is based on the following theorem that one can find, e.g., in [35, Ch. III, §16, Theorem 2].

**Theorem 8.8.** Two Gaussian measures \( \mu_{B,a} \) and \( \mu_{B,b} \) in a Hilbert space \( H \) are equivalent if and only of \( B^{-1/2}(a - b) \in H \).

Indeed, we have
\[ \|C^{-1/2}(ta - a)\|_H^2 = \sum_{n \in \mathbb{Z}} \|C_n^{-1/2}(t - I)a_n\|_{H_n}^2 = 2 \sum_{n \in \mathbb{Z}} \sum_{r=1}^{m} \frac{b_{kn}}{d_{kn}} \left( \sum_{s=1}^{m} (t_{rs} - \delta_{rs}) a_{sn} \right)^2 d_{kn}. \]

To explain the latter equality let us describe \( H \) and \( C \). To find an operator \( C \) we present the measure \( \mu_{m(B,a)} \) in the canonical form \( \mu_{C,a} \) defined by its Fourier transform:
\[ \int_H \exp i(y,x) d\mu_{C,a}(x) = \exp \left( i(a, y) - \frac{1}{2} (Cy, y) \right), \quad y \in H, \]
(8.11)
where \( C \) is a positive nuclear operator (called the covariance operator) on the Hilbert space \( H \), and \( a \in H \) is the mathematical expectation or mean.

Recall the Kolmogorov zero-one law. Let us consider in the space \( \mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots \) the infinite tensor product \( \mu_b = \otimes_{n \in \mathbb{N}} \mu_{b_n} \) of one-dimensional Gaussian measures \( \mu_{b_n} \) on \( \mathbb{R} \) defined as follows:
\[ d\mu_{b_n}(x) = \sqrt{\frac{b}{\pi}} \exp(-bx^2)dx. \]
(8.12)
Consider a Hilbert space \( l_2(a) \) defined by

\[
l_2(a) = \{ x \in \mathbb{R}^\infty : \| x \|_{l_2(a)}^2 = \sum_{k \in \mathbb{N}} a_k x_k < \infty \},
\]

where \( a = (a_k)_{k \in \mathbb{N}} \) is an infinite sequence of positive numbers.

**Theorem 8.9** (Kolmogorov’s zero-one law, [34]). We have

\[
\mu_b(l_2(a)) = \begin{cases} 0, & \text{if } \sum_{k \in \mathbb{N}} \frac{a_k}{b_k} = \infty; \\ 1, & \text{otherwise}. \end{cases}
\]

### 8.2. Properties of two vectors \( f, g \notin l_2 \)

In what follows we will use systematically the following notation. For \( k \) vectors \( f_1, f_2, \ldots, f_k \in \mathbb{R}^n \) with \( k \leq n \) we set

\[
\Delta(f_1, f_2, \ldots, f_k) = \frac{\det(I + \gamma(f_1, f_2, \ldots, f_k))}{\det(I + \gamma(f_2, \ldots, f_k))} - 1.
\]

For \( k = 2 \) and \( k = 3 \) we get respectively:

\[
\Delta(f_1, f_2) = \frac{\det(I + \gamma(f_1, f_2))}{\det(I + \gamma(f_2))} - 1 = \frac{I + \Gamma(f_1) + \Gamma(f_2) + \Gamma(f_1, f_2)}{I + \Gamma(f_2)}, \quad (8.13)
\]

\[
\Delta(f_1, f_2, f_3) = \frac{\det(I + \gamma(f_1, f_2, f_3))}{\det(I + \gamma(f_2, f_3))} - 1 = \frac{\Gamma(f_1) + \Gamma(f_1, f_2) + \Gamma(f_1, f_3) + \Gamma(f_1, f_2, f_3)}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)}. \quad (8.14)
\]

**Lemma 8.10** ([28], and [27], Ch.10). Let \( f = (f_k)_{k \in \mathbb{N}} \) and \( g = (g_k)_{k \in \mathbb{N}} \) be two real vectors such that \( \| f \|^2 = \infty \), where \( \| f \|^2 = \sum_k f_k^2 \). Denote by \( f(n) \), \( g(n) \in \mathbb{R}^n \) their projections to the subspace \( \mathbb{R}^n \), i.e., \( f(n) = (f_k)_{k=1}^n, \quad g(n) = (g_k)_{k=1}^n \). Then

\[
\Delta(f, g) := \lim_{n \to \infty} \Delta(f(n), g(n)) = \infty, \quad (8.16)
\]

where \( \Delta(f(n), g(n)) \) is defined by (8.14), in the following cases:

\[
\begin{align*}
(a) & \quad \| g \|^2 < \infty, \\
(b) & \quad \| g \|^2 = \infty, \quad \text{and} \quad \lim_{n \to \infty} \frac{\| f(n) \|}{\| g(n) \|} = \infty, \\
(c) & \quad \| f \|^2 = \| g \|^2 = \| f + sg \|^2 = \infty, \quad \text{for all} \quad s \in \mathbb{R} \setminus \{0\}.
\end{align*}
\]
Lemma 8.11. Let \( f = (f_k)_{k \in \mathbb{N}} \) and \( g = (g_k)_{k \in \mathbb{N}} \) be two real vectors such that \( \| f \|^2 = \| g \|^2 = \| C_1 f + C_2 g \|^2 = \infty \) for all \( (C_1, C_2) \in \mathbb{R}^2 \setminus \{0\} \), (8.17)

then \( \lim_{n \to \infty} \frac{\Gamma(f(n), g(n))}{\Gamma(g(n))} = \infty \) and \( \lim_{n \to \infty} \frac{\Gamma(f(n), g(n))}{\Gamma(f(n))} = \infty \). (8.18)

Lemma 8.12. Let \( f_1, f_2 \notin l_2 \) and \( \Delta(f_1, f_2) < \infty \), then for some \( (C_1, C_2) \in \mathbb{R}^2 \setminus \{0\} \) we have \( C_1 f_1 + C_2 f_2 \notin l_2 \).

Proof. Let assume the opposite, i.e., \( C_1 f_1 + C_2 f_2 \notin l_2 \) for all \( (C_1, C_2) \in \mathbb{R}^2 \setminus \{0\} \). Then by Lemma 8.11

\[ \Delta(f_1, f_2) = \frac{\Gamma(f_1) + \Gamma(f_2)}{1 + \Gamma(f_2)} > \frac{\Gamma(f_1) + \Gamma(f_2)}{1 + \Gamma(f_2)} \sim \frac{\Gamma(f_1) + \Gamma(f_2)}{\Gamma(f_2)} = \infty. \]

Lemma 8.13. Let \( f_1, f_2, f_3 \notin l_2 \) and \( \Delta(f_1, f_2, f_3) < \infty \), then for some \( (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\} \) we have \( C_1 f_1 + C_2 f_2 + C_3 f_3 \notin l_2 \).

Proof. Let assume the opposite, i.e., \( C_1 f_1 + C_2 f_2 + C_3 f_3 \notin l_2 \) for all \( (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\} \). Then by Lemmas 8.15–8.11 and (8.15) we get

\[ \Delta(f_1, f_2, f_3) = \frac{\Gamma(f_1) + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_1, f_2, f_3)}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)} > \frac{\Gamma(f_1) + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_1, f_2, f_3)}{1 + \Gamma(f_2) + \Gamma(f_3) + \Gamma(f_2, f_3)} \sim \frac{\Gamma(f_1) + \Gamma(f_2)}{\Gamma(f_2)} = \infty. \]

8.3. Properties of three vectors \( f, g, h \notin l_2 \)

Lemma 8.14. Let \( f = (f_k)_{k \in \mathbb{N}} \), \( g = (g_k)_{k \in \mathbb{N}} \) and \( h = (h_k)_{k \in \mathbb{N}} \) be three real vectors such that \( \| f \|^2 = \infty \) where \( \| f \|^2 = \sum_k f_k^2 \). Denote by \( f(n), g(n), h(n) \in \mathbb{R}^n \) their projections to the subspace \( \mathbb{R}^n \), i.e., \( f(n) = (f_k)_{k=1}^n, g(n) = (g_k)_{k=1}^n, h(n) = (h_k)_{k=1}^n \). Then

\[ \Delta(f, g, h) := \lim_{n \to \infty} \Delta(f(n), g(n), h(n)) = \infty, \] (8.19)

where \( \Delta(f, g, h) \) is defined by (8.15), in the following cases:

(a) \( \| g \|^2 < \infty \) and \( \| h \|^2 < \infty \),
(b) \( \| g \|^2 < \infty \) and \( \| h \|^2 = \infty \), or \( \| g \|^2 = \infty \) and \( \| h \|^2 < \infty \),
(c) \( \| C_1 f + C_2 g + C_3 h \|^2 = \infty \), for all \( (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\} \).
Proof. (a) In this case \( \|g(n)\|_2 \leq C \), \( \|h(n)\|_2 \leq C \) and \( \Gamma(g(n), h(n)) \leq C \) and therefore,
\[
\lim_{n \to \infty} \Delta(f(n), g(n), h(n)) \geq \lim_{n \to \infty} \frac{\|f(n)\|_2^2}{1 + 3C} = \infty.
\]
(b) Let \( \|g\|_2 < \infty \), \( \|h\|_2 = \infty \). In this case we have
\[
\Gamma(g(n), h(n)) \leq \|g(n)\|_2 \|h(n)\|_2 \sin^2(\alpha_n) \leq C_1 \Gamma(h(n)),
\]
where \( \alpha_n \) is the angle between two vectors \( g(n) \) and \( h(n) \). Therefore,
\[
1 + \Gamma(g(n)) + \Gamma(h(n)) + \Gamma(g(n), h(n)) \leq (1 + C)(1 + \Gamma(h(n))),
\]
\[
\Delta(f(n), g(n), h(n)) \geq \frac{\Gamma(f(n)) + \Gamma(f(n), h(n))}{(1 + C)(1 + \Gamma(h(n)))} \sim \Delta(f(n), h(n)).
\]
So, this case is reduced to the case \( m = 2 \), see Lemma 8.11.

The implication \((c) \Rightarrow (8.19)\) is based on the following lemma (the proof of which will appear in [29]. Compare with Lemma 8.10. \( \square \)

Lemma 8.15 (see [29]). Let \( f_0, f_1, f_2 \) be three infinite real vectors \( f_r = (f_{rk})_{k \in \mathbb{N}}, \ 0 \leq r \leq 2 \) such that for all \( (C_0, C_1, C_2) \in \mathbb{R}^3 \ \setminus \ \{0\} \) holds
\[
\sum_{r=0}^{2} C_r f_r \notin l_2, \ \text{i.e.,} \ \sum_{k \in \mathbb{N}} |C_0 f_{0k} + C_1 f_{1k} + C_2 f_{2k}|^2 = \infty. \tag{8.20}
\]

Denote by \( f_r(n) = (f_{rk})_{k=1}^{n} \in \mathbb{R}^n \) the projections of the vectors \( f_r \) on the subspace \( \mathbb{R}^n \). Then
\[
\frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)} := \lim_{n \to \infty} \frac{\Gamma(f_0(n), f_1(n), f_2(n))}{\Gamma(f_1(n), f_2(n))} = \infty. \tag{8.21}
\]

The idea of the proof (for details see [29]). We assume there exists a real number \( C \) such that
\[
\frac{\Gamma(f_0(n), f_1(n), f_2(n))}{\Gamma(f_1(n), f_2(n))} \leq C, \tag{8.22}
\]
for every integer \( n \) and will show that this leads to a contradiction.
For \( t \in \mathbb{R}^2 \) and \( f_0, f_1, f_2 \in H \) we define the function

\[
F_{20}(t) = \| \sum_{k=1}^{2} t_k f_k - f_0 \|^2 = \sum_{k,r=1}^{2} t_k t_r (f_k, f_r) - 2 \sum_{k=1}^{2} t_k (f_k, f_0) + (f_0, f_0)
\]

\[
= (At, t) - 2(t, b) + (f_0, f_0),
\]

where \( b = (f_k, f_0)^2_{k=1} \in \mathbb{R}^2 \) and \( A \) is the Gram matrix

\[
A = \gamma(f_1, f_2) = ((f_k, f_r))_{k,r=1}^2.
\]

Suppose that \( At_0 = b \), then we have

\[
(A, t) - 2(t, b) + (f_0, f_0) = (At - t_0, (t - t_0)) + \frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)},
\]

and therefore

\[
F_{20}(t) = (At - t_0, (t - t_0)) + \frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)},
\]

\[
F_{20}(t_0) = \min_{(t_1, t_2) \in \mathbb{R}^2} \| \sum_{k=1}^{2} t_k f_k - f_0 \|^2 = \frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)}, \tag{8.23}
\]

Consider the matrix

\[
X_{3n} = \begin{pmatrix}
  f_{11} & f_{12} & \ldots & f_{1n} \\
  f_{21} & f_{22} & \ldots & f_{2n} \\
  f_{01} & f_{02} & \ldots & f_{0n}
\end{pmatrix} \tag{8.24}
\]

and its minors

\[
M_{krs}^{123} = \begin{vmatrix}
  f_{1k} & f_{1r} & f_{1s} \\
  f_{2k} & f_{2r} & f_{2s} \\
  f_{0k} & f_{0r} & f_{0s}
\end{vmatrix}, \quad M_{k}^{12} = \begin{vmatrix}
  f_{1k} & f_{1r} \\
  f_{2k} & f_{2r}
\end{vmatrix}.
\]

Then by [9] we have

\[
\Gamma(f_1(n), f_2(n)) = \sum_{1 \leq r < s \leq n} |M_{rs}^{12}|^2,
\]

\[
\Gamma(f_0(n), f_1(n), f_2(n)) = \sum_{1 \leq k < r < s \leq n} |M_{krs}^{123}|^2.
\]
Therefore, the inequality (8.22) will have the following form
\[
\frac{\Gamma(f_0(n), f_1(n), f_2(n))}{\Gamma(f_1(n), f_2(n))} = \frac{\sum_{1 \leq k < r < s \leq n} |M_{krs}|^2}{\sum_{1 \leq r < s \leq n} |M_{rss}|^2} \leq C
\]  
(8.25)
for all \( n \in \mathbb{N} \). Set now
\[
a_{2n} = \gamma(f_1(n), f_2(n), f_0(n)), \quad A_{2n} = \gamma(f_1(n), f_2(n)), \quad A_{2n} t_0^{(n)} = b_{2n},
\]  
(8.26)
\[
b_{2n} = (f_k(n), f_0(n))_{k=1}^2 \in \mathbb{R}^2, \quad t_0^{(n)} = (t_{0r}^{(n)})_{r=1}^2.
\]  
(8.27)
More explicitly
\[
a_{2n} = \begin{pmatrix} (f_1(n), f_1(n)) & (f_1(n), f_2(n)) & (f_1(n), f_0(n)) \\ (f_2(n), f_1(n)) & (f_2(n), f_2(n)) & (f_2(n), f_0(n)) \\ (f_0(n), f_1(n)) & (f_0(n), f_2(n)) & (f_0(n), f_0(n)) \end{pmatrix},
\]  
(8.28)
\[
A_{2n} = \begin{pmatrix} (f_1(n), f_1(n)) & (f_1(n), f_2(n)) \\ (f_2(n), f_1(n)) & (f_2(n), f_2(n)) \end{pmatrix}.
\]  
(8.29)
If we replace the vectors \( f_0, f_1, f_2 \) with \( f_0(n), f_1(n), f_2(n) \), the equality (8.23) then becomes
\[
F_{20}^{(n)}(t_0^{(n)}) = \min_{(t_1, t_2) \in \mathbb{R}^2} \| \sum_{k=1}^2 t_k f_k(n) - f_0(n) \|^2 = \frac{\Gamma(f_0(n), f_1(n), f_2(n))}{\Gamma(f_1(n), f_2(n))}.
\]  
(8.30)
For \( t \in \mathbb{R}^2 \) and \( f_0(n), f_1(n), f_2(n) \in \mathbb{R}^n \) and \( 0 \leq s \leq 2 \), define the functions
\[
F_{2s}^{(n)}(t) = \| \sum_{0 \leq r \leq 2, r \neq s} t_r f_r(n) - f_s(n) \|^2.
\]  
(8.31)
The minimum of the corresponding expressions \( F_{2s}^{(n)}(t) \) for \( 0 \leq s \leq 2 \) is attained respectively at \( t_0^{(n)}, t_1^{(n)}, t_2^{(n)} \). The proof of the fact that one of the sequences \( t_0^{(n)}, t_1^{(n)}, t_2^{(n)} \) is bounded, is based on the positive definiteness of the matrices \( \gamma(f_1(n), f_2(n), f_0(n)) \), for the details see [29]. Let for example, the sequence \( t_0^{(n)} \in \mathbb{R}^2 \) is bounded. Therefore, there exists a subsequence \( (t_{0k})_{k \in \mathbb{N}} \) that converges to some \( t \in \mathbb{R}^2 \). This contradics (8.22). Indeed
\[
\lim_{n \to \infty} F_{20}^{(n)}(t) = \infty, \quad F_{20}^{(n)}(t_{0k}^{(n)}) \leq C, \quad \lim_{k \to \infty} t_{0k}^{(n)} = t.
\]  
(8.32)
Acknowledgement.

A. Kosyak is very grateful to Prof. K.-H. Neeb, Prof. M. Smirnov and Dr Moree for their personal efforts to make academic stays possible at their respective institutes. A. Kosyak visited: MPIM from March to April 2022 and from January to April 2023, University of Augsburg from June to July 2022, and University of Erlangen-Nuremberg from August to December 2022, all during the Russian invasion in Ukraine. Also, Prof. R. Kashaev kindly invited A. Kosyak to Geneva.

Further, both authors would like to pay their respect to Prof. P. Teichner at MPIM, for his immediate efforts started to help mathematicians in Ukraine after the Russian invasion.

Since the spring of 2023 A. Kosyak is an Arnold Fellow at the London Institute for Mathematical Sciences, and he would like to express his gratitude to Mrs Myers Cornaby and to Miss Ker Mercer and especially to the Director of LIMS Dr T. Fink and Prof. Y.-H. He.

P. Moree, not an expert in this area, thanks A. Kosyak for his patient clarifications of the big picture, thus making him see the forest for the trees.

References


[29] A.V. Kosyak, The height of an infinite parallelootope is infinite, 25 p. (in progress)


