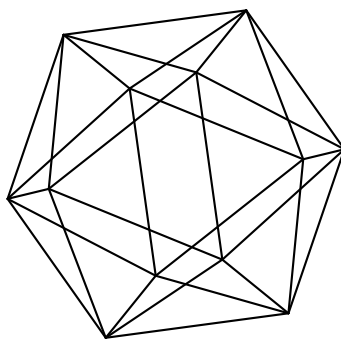


# Max-Planck-Institut für Mathematik Bonn

Automorphisms of Veronese subalgebras of polynomial  
algebras and free Poisson algebras

by

Bakhyt Aitzhanova  
Leonid Makar-Limanov  
Ualbai Umirbaev



Max-Planck-Institut für Mathematik  
Preprint Series 2023 (14)

Date of submission: July 15, 2023

# Automorphisms of Veronese subalgebras of polynomial algebras and free Poisson algebras

by

Bakhyt Aitzhanova  
Leonid Makar-Limanov  
Ualbai Umirbaev

Max-Planck-Institut für Mathematik  
Vivatsgasse 7  
53111 Bonn  
Germany

Department of Mathematics  
Wayne State University  
Detroit, MI 48202  
USA

Faculty of Mathematics and Computer Science  
Weizmann Institute of Science  
Rehovot 7610001  
Israel

Institute of Mathematics and  
Mathematical Modeling  
Almaty 050010  
Kazakhstan

# AUTOMORPHISMS OF VERONESE SUBALGEBRAS OF POLYNOMIAL ALGEBRAS AND FREE POISSON ALGEBRAS

Bakhyt Aitzhanova<sup>1</sup>, Leonid Makar-Limanov<sup>2</sup>, and Ualbai Umirbaev<sup>3</sup>

ABSTRACT. The Veronese subalgebra  $A_0$  of degree  $d \geq 2$  of the polynomial algebra  $A = K[x_1, x_2, \dots, x_n]$  over a field  $K$  in the variables  $x_1, x_2, \dots, x_n$  is the subalgebra of  $A$  generated by all monomials of degree  $d$  and the Veronese subalgebra  $P_0$  of degree  $d \geq 2$  of the free Poisson algebra  $P = P\langle x_1, x_2, \dots, x_n \rangle$  is the subalgebra spanned by all homogeneous elements of degree  $kd$ , where  $k \geq 0$ .

If  $n \geq 2$  then every derivation and every locally nilpotent derivation of  $A_0$  and  $P_0$  over a field  $K$  of characteristic zero is induced by a derivation and a locally nilpotent derivation of  $A$  and  $P$ , respectively. Moreover, we prove that every automorphism of  $A_0$  and  $P_0$  over a field  $K$  closed with respect to taking all  $d$ -roots of elements is induced by an automorphism of  $A$  and  $P$ , respectively.

**Mathematics Subject Classification (2020):** 14R10, 14J50, 13F20.

**Key words:** Automorphism, derivation, free Poisson algebra, polynomial algebra.

## 1. INTRODUCTION

Let  $K$  be an arbitrary field and let  $\mathbb{A}^n$  and  $\mathbb{P}^n$  be the affine and the projective  $n$ -spaces over  $K$ , respectively. The *Veronese map* of degree  $d$  is the map

$$\nu_{n,d} : \mathbb{P}^n \rightarrow \mathbb{P}^m$$

that sends  $[x_0 : \dots : x_n]$  to all  $m + 1$  possible monomials of total degree  $d$ , where

$$m = \binom{n+d}{d} - 1.$$

It is well known that the image  $V_{n,d}$  of the Veronese map  $\nu_{n,d}$  is a projective variety and is called the *Veronese variety* [7].

If  $Y_{i_0 \dots i_n} = x_0^{i_0} \dots x_n^{i_n}$ ,  $i_0 + \dots + i_n = d$ , then the Veronese variety is determined by the set of quadratic relations

$$(1) \quad Y_{i_0 \dots i_n} Y_{j_0 \dots j_n} = Y_{k_0 \dots k_n} Y_{r_0 \dots r_n},$$

where  $i_0 + j_0 = k_0 + r_0, \dots, i_n + j_n = k_n + r_n$  [19].

The *affine cone* of  $V_{n,d}$  or the *affine Veronese variety* is the affine subvariety  $\mathbb{A}^{m+1}$  determined by the set of relations (1). The algebra of polynomial functions on the affine

---

<sup>1</sup>Department of Mathematics, Wayne State University, Detroit, MI 48202, USA, e-mail: [aitzhanova.bakhyt01@gmail.com](mailto:aitzhanova.bakhyt01@gmail.com)

<sup>2</sup>Department of Mathematics, Wayne State University, Detroit, MI 48202, USA and Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot, 7610001, Israel, e-mail: [lml@wayne.edu](mailto:lml@wayne.edu)

<sup>3</sup>Department of Mathematics, Wayne State University, Detroit, MI 48202, USA and Institute of Mathematics and Mathematical Modeling, Almaty, 050010, Kazakhstan, e-mail: [umirbaev@wayne.edu](mailto:umirbaev@wayne.edu)

Veronese cone  $V_{n,d}$  is isomorphic to the subalgebra of  $K[x_0, \dots, x_n]$  generated by all monomials of degree  $d$  [21].

Veronese surfaces play an important role in the description of quasihomogeneous affine surfaces given by M.H. Gizatullin [4] and V.L. Popov [18]. They form one of the main examples of the so-called *Gizatullin surfaces* [10]. The structure of the automorphism groups of Veronese surfaces are studied in [1, 2, 5, 6, 11, 12]. The derivations and locally nilpotent derivations of affine Veronese surfaces are described in [1].

Recently J. Kollar devoted two interesting papers [8, 9] to the study of automorphism groups of some more general affine varieties. In particular, he described the group of automorphisms of all affine Veronese varieties.

Let  $A = K[x_1, x_2, \dots, x_n]$  be the polynomial algebra over a field  $K$  in the variables  $x_1, x_2, \dots, x_n$ . Consider the grading

$$A = A_0 \oplus A_1 \oplus \dots \oplus A_{d-1},$$

where  $d \geq 2$  and  $A_i$  is the subspace of  $A$  generated by all monomials of degree  $kd + i$  for all  $k \geq 0$ . This is a  $\mathbb{Z}_d$ -grading of  $A$ , i.e.,  $A_i A_j \subseteq A_{i+j}$  for all  $i, j \in \mathbb{Z}_d$ . The subalgebra  $A_0$  is called the *Veronese subalgebra of  $A$  of degree  $d$* .

Similarly, let  $P = P\langle x_1, x_2, \dots, x_n \rangle$  be the free Poisson algebra over  $K$  in the variables  $x_1, x_2, \dots, x_n$ . The grading

$$P = P_0 \oplus P_1 \oplus \dots \oplus P_{d-1},$$

where  $P_i$  is the subspace of  $P$  generated by all homogeneous elements of degree  $kd + i$  is a  $\mathbb{Z}_d$ -grading of  $P$ , i.e.,  $P_i P_j \subseteq P_{i+j}$  and  $\{P_i, P_j\} \subseteq P_{i+j}$  for all  $i, j \in \mathbb{Z}_d$ . The subalgebra  $P_0$  is called the *Veronese subalgebra of  $P$  of degree  $d$* .

We prove that every derivation and every locally nilpotent derivation of  $P_0$  (of  $A_0$ ) over a field  $K$  of characteristic zero is induced by a derivation and a locally nilpotent derivation of  $P$  (of  $A$ ), respectively. Moreover, we prove that every automorphism of  $P_0$  (of  $A_0$ ) over a field  $K$ , closed with respect to taking all  $d$ -roots of elements, is induced by an automorphism of  $P$  (of  $A$ ).

This paper is organized as follows. In Section 2 we describe a basis of free Poisson algebras and give some elementary definitions that are necessary in the future. Derivations of the Veronese subalgebras are studied in Section 3 and automorphisms are studied in Section 4. All results are proven for Poisson algebras. Analogues of these results for polynomial algebras are formulated in Section 5. In the same section we give some counter-examples that show the analogous results do not hold for polynomial algebras in one variable and for free associative algebras.

## 2. FREE POISSON ALGEBRA $P\langle x_1, \dots, x_n \rangle$

A vector space  $P$  over a field  $K$  endowed with two bilinear operations  $x \cdot y$  (a multiplication) and  $\{x, y\}$  (a Poisson bracket) is called a *Poisson algebra* if  $P$  is a commutative associative algebra under  $x \cdot y$ ,  $P$  is a Lie algebra under  $\{x, y\}$ , and  $P$  satisfies the following identity (the Leibniz identity):

$$\{x, y \cdot z\} = y \cdot \{x, z\} + \{x, y\} \cdot z.$$

Symplectic Poisson algebras  $P_n$  appear in many areas of algebra. For each natural  $n \geq 1$  the symplectic Poisson algebra  $P_n$  of index  $n$  is a polynomial algebra  $K[x_1, y_1, \dots, x_n, y_n]$  endowed with the Poisson bracket defined by

$$\{x_i, y_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0,$$

where  $\delta_{ij}$  is the Kronecker symbol and  $1 \leq i, j \leq n$ .

Let  $P$  be a Poisson algebra. A linear map  $D : P \rightarrow P$  is called a *derivation* of the Poisson algebra  $P$  if

$$(2) \quad D(xy) = D(x)y + xD(y)$$

and

$$(3) \quad D(\{x, y\}) = \{D(x), y\} + \{x, D(y)\}$$

for all  $x, y \in P$ .

We refer to a linear map  $D : P \rightarrow P$  satisfying (2) as an *associative derivation* of  $P$  and to a linear map  $D : P \rightarrow P$  satisfying (3) as a *Lie derivation* of  $P$ . Thus, a derivation of a Poisson algebra is simultaneously an associative and a Lie derivation. We often refer to them as Poisson derivations.

The Leibniz identity implies that for any  $x \in P$  the map

$$\text{ad}_x : P \rightarrow P \quad (y \mapsto \{x, y\})$$

is an associative derivation of  $P$ . The Jacobi identity implies that the map  $\text{ad}_x$  is also a Lie derivation of  $P$ . So for any  $x \in P$  the map  $\text{ad}_x$  is a Poisson derivation of  $P$ .

Let  $L$  be a Lie algebra with Lie bracket  $[\cdot, \cdot]$  over a field  $K$  and let  $e_1, e_2, \dots, e_k, \dots$  be a linear basis of  $L$ . Then there exists a unique bracket  $\{\cdot, \cdot\}$  on the polynomial algebra  $K[e_1, e_2, \dots, e_k, \dots]$  defined by

$$\{e_i, e_j\} = [e_i, e_j]$$

for all  $i, j$  and satisfying the Leibniz identity. With this bracket

$$P(L) = \langle K[e_1, e_2, \dots], \cdot, \{, \} \rangle$$

becomes a Poisson algebra. This Poisson algebra  $P(L)$  is called the *Poisson enveloping algebra* [23] (or *Poisson symmetric algebra* [15]) of  $L$ . Note that the bracket  $\{\cdot, \cdot\}$  of the algebra  $P(L)$  depends on the structure of  $L$  but does not depend on a chosen basis.

Let  $L = \text{Lie}\langle x_1, \dots, x_n \rangle$  be the free Lie algebra with free generators  $x_1, \dots, x_n$ . It is well-known (see, for example [20]) that  $P(L)$  is the free Poisson algebra over  $K$  in the variables  $x_1, \dots, x_n$ . We denote it by  $P\langle x_1, \dots, x_n \rangle$ .

Let us choose a multihomogeneous linear basis

$$x_1, \dots, x_n, [x_1, x_2], \dots, [x_1, x_n], \dots, [x_{n-1}, x_n], [[x_1, x_2], x_3], \dots$$

of a free Lie algebra  $L$  and denote the elements of this basis by

$$(4) \quad e_1, e_2, \dots, e_s, \dots$$

The algebra  $P = P\langle x_1, \dots, x_n \rangle$  coincides with the polynomial algebra on the elements (4). Consequently, the monomials

$$(5) \quad u = e_{i_1} e_{i_2} \dots e_{i_s},$$

where  $i_1 \leq i_2 \leq \dots \leq i_s$  form a linear basis of  $P$ .

Denote by  $\deg$  the standard degree function on  $P$ , i.e.,  $\deg(x_i) = 1$  for all  $1 \leq i \leq n$ . If  $u$  is an element of the form (5) then

$$\deg u = \deg e_{i_1} + \deg e_{i_2} + \dots + \deg e_{i_s}.$$

Set also  $d(u) = s$  and call it the *polynomial length* of  $u$ . Note that

$$\deg\{f, g\} = \deg f + \deg g$$

if  $f$  and  $g$  are homogeneous and  $\{f, g\} \neq 0$ .

Denote by  $Q(P) = P(x_1, \dots, x_n)$  the field of fractions of the polynomial algebra  $K[e_1, e_2, \dots]$  in the variables (4). The Poisson bracket  $\{\cdot, \cdot\}$  on  $K[e_1, e_2, \dots] = P$  can be uniquely extended to a Poisson bracket on the field of its fractions  $Q(P)$  and

$$(6) \quad \left\{ \frac{a}{b}, \frac{c}{d} \right\} = \frac{\{a, c\}bd - \{a, d\}bc - \{b, c\}ad + \{b, d\}ac}{b^2d^2}$$

for all  $a, b, c, d \in P$  with  $bd \neq 0$ .

The field  $Q(P) = P(x_1, x_2, \dots, x_n)$  with this Poisson bracket is called the *free Poisson field* over  $K$  in variables  $x_1, \dots, x_n$  [16].

Several combinatorial results on the structure of free Poisson algebras and free Poisson fields are proven in [13, 14, 15, 16, 17, 22].

We fix a grading

$$(7) \quad P = P_0 \oplus P_1 \oplus \dots \oplus P_{d-1}$$

of the free Poisson algebra  $P = P\langle x_1, \dots, x_n \rangle$ , where  $P_i$  is the linear span of all elements of degree  $i + ds$ ,  $i = 0, 1, \dots, d-1$ , and  $s$  is an arbitrary nonnegative integer. This is a  $\mathbb{Z}_d$ -grading of  $P$ , i.e.,

$$P_i P_j \subseteq P_{i+j}, \quad \{P_i, P_j\} \subseteq P_{i+j},$$

where  $i, j \in \mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$ . For shortness we will refer to this grading as the  $d$ -grading.

An automorphism  $\phi \in \text{Aut } P$  is called a *graded automorphism* with respect to grading (7) if  $\phi(x_1), \phi(x_2), \dots, \phi(x_n) \in P_1$ . A graded automorphism is called *graded tame* if it is a product of graded elementary automorphisms.

We will call a graded automorphism of  $P$  with respect to grading (7) a  *$d$ -graded automorphism* for shortness. Obviously, every  $d$ -graded automorphism induces an automorphism of the algebra  $P_0$ . A derivation  $D$  of  $P$  will be called a  *$d$ -graded derivation* if  $D(x_1), D(x_2), \dots, D(x_n) \in P_1$ .

### 3. DERIVATIONS OF $P_0$

In this section we assume that  $K$  is an arbitrary field of characteristic zero.

**Lemma 1.** *Every derivation of the Poisson algebra  $P = P\langle x_1, x_2, \dots, x_n \rangle$  over  $K$  can be uniquely extended to a derivation of the Poisson field  $Q(P) = P(x_1, x_2, \dots, x_n)$ .*

*Proof.* Let  $D$  be an arbitrary derivation of the free Poisson algebra  $P$ . In particular,  $D$  is a derivation of the polynomial algebra  $P = K[e_1, \dots, e_s, \dots]$ . It is well known [25, p. 120] that  $D$  can be uniquely extended to an associative derivation  $S$  of the quotient field

$Q(P) = K(e_1, \dots, e_s, \dots)$ . We will show that  $S$  is a Lie derivation of  $Q(P)$ , i.e., that  $S$  satisfies (3). We have to check that

$$(8) \quad S\left(\left\{\frac{a}{b}, \frac{c}{d}\right\}\right) = \left\{S\left(\frac{a}{b}\right), \frac{c}{d}\right\} + \left\{\frac{a}{b}, S\left(\frac{c}{d}\right)\right\}$$

for all  $a, b, c, d \in P\langle x_1, \dots, x_n \rangle$  with  $bd \neq 0$ . Using (6) we get

$$\begin{aligned} S\left(\left\{\frac{a}{b}, \frac{c}{d}\right\}\right) &= S\left(\frac{\{a, c\}bd - \{a, d\}bc - \{b, c\}ad + \{b, d\}ac}{b^2d^2}\right) \\ &= S\left(\frac{\{a, c\}}{bd}\right) - S\left(\frac{\{a, d\}c}{bd^2}\right) - S\left(\frac{\{b, c\}a}{b^2d}\right) + S\left(\frac{\{b, d\}ac}{b^2d^2}\right) \\ &= \frac{S(\{a, c\})bd - \{a, c\}S(bd)}{b^2d^2} - \frac{S(\{a, d\}c)bd^2 - \{a, d\}cS(bd^2)}{b^2d^4} \\ &\quad - \frac{S(\{b, c\}a)b^2d - \{b, c\}aS(b^2d)}{b^4d^2} + \frac{S(\{b, d\}ac)b^2d^2 - \{b, d\}acS(b^2d^2)}{b^4d^4} \\ &= \frac{S(\{a, c\})}{bd} - \frac{\{a, c\}S(b)}{b^2d} - \frac{\{a, c\}S(d)}{bd^2} - \frac{S(\{a, d\}c)}{bd^2} - \frac{\{a, d\}S(c)}{bd^2} + \frac{\{a, d\}cS(b)}{b^2d^2} \\ &\quad + \frac{2\{a, d\}cS(d)}{bd^3} - \frac{S(\{b, c\}a)}{b^2d} - \frac{\{b, c\}S(a)}{b^2d} + \frac{2\{b, c\}aS(b)}{b^3d} + \frac{\{b, c\}aS(d)}{b^2d^2} \\ &\quad + \frac{S(\{b, d\}ac)}{b^2d^2} + \frac{\{b, d\}S(a)c}{b^2d^2} + \frac{\{b, d\}aS(c)}{b^2d^2} - \frac{2\{b, d\}acS(b)}{b^3d^2} - \frac{2\{b, d\}acS(d)}{b^2d^3} \\ &= \frac{\{S(a), c\}}{bd} + \frac{\{a, S(c)\}}{bd} - \frac{\{a, c\}S(b)}{b^2d} - \frac{\{a, c\}S(d)}{bd^2} - \frac{\{S(a), d\}c}{bd^2} \\ &\quad - \frac{\{a, S(d)\}c}{bd^2} - \frac{\{a, d\}S(c)}{bd^2} + \frac{\{a, d\}cS(b)}{b^2d^2} + \frac{2\{a, d\}cS(d)}{bd^3} - \frac{\{S(b), c\}a}{b^2d} \\ &\quad - \frac{\{b, S(c)\}a}{b^2d} - \frac{\{b, c\}S(a)}{b^2d} + \frac{2\{b, c\}aS(b)}{b^3d} + \frac{\{b, c\}aS(d)}{b^2d^2} + \frac{\{S(b), d\}ac}{b^2d^2} \\ &\quad + \frac{\{b, S(d)\}ac}{b^2d^2} + \frac{\{b, d\}S(a)c}{b^2d^2} + \frac{\{b, d\}aS(c)}{b^2d^2} - \frac{2\{b, d\}acS(b)}{b^3d^2} - \frac{2\{b, d\}acS(d)}{b^2d^3}. \end{aligned}$$

Direct calculations give that

$$\begin{aligned} \left\{S\left(\frac{a}{b}\right), \frac{c}{d}\right\} + \left\{\frac{a}{b}, S\left(\frac{c}{d}\right)\right\} &= \left\{\frac{S(a)b - aS(b)}{b^2}, \frac{c}{d}\right\} + \left\{\frac{a}{b}, \frac{S(c)d - cS(d)}{d^2}\right\} \\ &= \left\{\frac{S(a)}{b}, \frac{c}{d}\right\} - \left\{\frac{aS(b)}{b^2}, \frac{c}{d}\right\} + \left\{\frac{a}{b}, \frac{S(c)}{d}\right\} - \left\{\frac{a}{b}, \frac{cS(d)}{d^2}\right\} \\ &= \frac{\{S(a), c\}}{bd} - \frac{\{S(a), d\}c}{bd^2} - \frac{\{b, c\}S(a)}{b^2d} + \frac{\{b, d\}S(a)c}{b^2d^2} - \frac{\{a, c\}S(b)}{b^2d} \\ &\quad - \frac{\{S(b), c\}a}{b^2d} + \frac{\{a, d\}S(b)c}{b^2d^2} + \frac{\{S(b), d\}ac}{b^2d^2} + \frac{2\{b, c\}aS(b)}{b^3d} - \frac{2\{b, d\}aS(b)c}{b^3d^2} \\ &\quad + \frac{\{a, S(c)\}}{bd} - \frac{\{a, d\}S(c)}{bd^2} - \frac{\{b, S(c)\}a}{b^2d} + \frac{\{b, d\}aS(c)}{b^2d^2} - \frac{\{a, c\}S(d)}{bd^2} \\ &\quad - \frac{\{a, S(d)\}c}{bd^2} + \frac{2\{a, d\}cS(d)}{bd^3} + \frac{\{b, c\}S(d)a}{b^2d^2} + \frac{\{b, S(d)\}ac}{b^2d^2} - \frac{2\{b, d\}acS(d)}{b^2d^3}. \end{aligned}$$

These two equalities imply (8).  $\square$

Consider the grading (7) of  $P$ . A Poisson derivation  $D$  of  $P$  will be called a  $d$ -graded Poisson derivation if  $D(x_i) \in P_1$  for all  $i = 1, \dots, n$ . Obviously, every  $d$ -graded Poisson derivation of  $P$  induces a Poisson derivation of  $P_0$ . The reverse is also true.

**Lemma 2.** *Every Poisson derivation of  $P_0$  can be uniquely extended to a  $d$ -graded Poisson derivation of  $P = P\langle x_1, x_2, \dots, x_n \rangle$ .*

*Proof.* Let  $D$  be a Poisson derivation of  $P_0$ . In particular,  $D$  is a derivation of the associative and commutative algebra  $P_0$ . Since  $P_0$  is a domain,  $D$  can be uniquely extended [25, p. 120] to a derivation  $T$  of the field of fractions  $Q(P_0)$  of  $P_0$ . The field extension

$$Q(P_0) \subseteq Q(P)$$

is algebraic since every  $e_i$  is a root of the polynomial  $p(t) = t^d - e_i^d \in Q(P_0)[t]$  for all  $i$ . This extension is separable since  $K$  is a field of characteristic zero. By Corollaries 2 and 2' in [25, pages 124–125], the associative derivation  $T$  of the field  $Q(P_0)$  can be uniquely extended to an associative derivation  $S$  of the field  $Q(P)$ .

Suppose that

$$S(e_j) = \frac{f_j}{g_j},$$

where  $f_j \in P$ ,  $0 \neq g_j \in P$ , and the pairs  $f_j, g_j$  are relatively prime for all  $j$ . Notice that  $e_j^d \in P_0$  and

$$D(e_j^d) = S(e_j^d) = de_j^{d-1} \frac{f_j}{g_j} \in P_0.$$

Consequently,

$$(9) \quad g_j | e_j^{d-1}$$

since the pair  $f_j, g_j$  is relatively prime, that is,  $g_j$  is a power of  $e_j$ .

If  $d | \deg e_j$  then  $e_j \in P_0$  and  $S(e_j) = D(e_j) \in P_0$ , i.e., we may assume that  $g_j = 1$ . If  $d \nmid \deg e_j$  then there exist  $1 \leq i \leq n$  and  $1 \leq k < d$  such that  $e_i = x_i \neq e_j$  and  $x_i^k e_j \in P_0$ . Then

$$D(x_i^k e_j) = S(x_i^k e_j) = kx_i^{k-1} \frac{f_i}{g_i} e_j + x_i^k \frac{f_j}{g_j} \in P_0.$$

Consequently,

$$g_i g_j | kx_i^{k-1} f_i g_j e_j + x_i^k f_j g_i.$$

This implies that  $g_j | x_i^k f_j g_i$ . Since  $f_j, g_j$  are relatively prime, we get  $g_j | x_i^k g_i$ . By (9)  $g_i$  is a power of  $e_i = x_i$  and  $g_j$  is a power of  $e_j$ . Since  $i \neq j$ , this implies that  $g_j \in K$  for all  $j$ . Consequently,  $S(e_j) \in P$  and  $S(P) \subseteq P$ .

Let us now show that the restriction of  $S$  to  $P$  is a Lie derivation, i.e.,

$$(10) \quad S(\{u, v\}) = \{S(u), v\} + \{u, S(v)\}$$



for all  $u, v$  of the form (5). We prove (10) by induction on the polynomial length  $d(u)+d(v)$ . Suppose that  $u = e_i$  and  $v = e_j$ . Since  $e_i^d, e_j^d \in P_0$ , we get

$$\begin{aligned} S(\{e_i^d, e_j^d\}) &= D(\{e_i^d, e_j^d\}) = \{D(e_i^d), e_j^d\} + \{e_i^d, D(e_j^d)\} \\ &= \{S(e_i^d), e_j^d\} + \{e_i^d, S(e_j^d)\} = \{de_i^{d-1}S(e_i), e_j\}de_j^{d-1} + de_i^{d-1}\{e_i, de_j^{d-1}S(e_j)\} \\ &= d^2(d-1)e_i^{d-2}e_j^{d-1}S(e_i)\{e_i, e_j\} + d^2e_i^{d-1}e_j^{d-1}\{S(e_i), e_j\} \\ &\quad + d^2(d-1)e_i^{d-1}e_j^{d-2}S(e_j)\{e_i, e_j\} + d^2e_i^{d-1}e_j^{d-1}\{e_i, S(e_j)\}. \end{aligned}$$

On the other hand,

$$\{e_i^d, e_j^d\} = d^2e_i^{d-1}e_j^{d-1}\{e_i, e_j\}$$

and

$$\begin{aligned} S(\{e_i^d, e_j^d\}) &= S(d^2e_i^{d-1}e_j^{d-1}\{e_i, e_j\}) \\ &= d^2S(e_i^{d-1})e_j^{d-1}\{e_i, e_j\} + d^2e_i^{d-1}S(e_j^{d-1})\{e_i, e_j\} + d^2e_i^{d-1}e_j^{d-1}S(\{e_i, e_j\}) \\ &= d^2(d-1)e_i^{d-2}e_j^{d-1}S(e_i)\{e_i, e_j\} + d^2(d-1)e_i^{d-1}e_j^{d-2}S(e_j)\{e_i, e_j\} + d^2e_i^{d-1}e_j^{d-1}S(\{e_i, e_j\}). \end{aligned}$$

Comparing two values of  $S(\{e_i^d, e_j^d\})$ , we get

$$S(\{e_i, e_j\}) = \{S(e_i), e_j\} + \{e_i, S(e_j)\}.$$

Suppose that  $d(v) \geq 2$  and  $v = v_1v_2$ . Then

$$\begin{aligned} S(\{u, v\}) &= S(\{u, v_1v_2\}) = S(v_1\{u, v_2\} + \{u, v_1\}v_2) \\ &= S(v_1)\{u, v_2\} + v_1S(\{u, v_2\}) + S(\{u, v_1\})v_2 + \{u, v_1\}S(v_2). \end{aligned}$$

By the induction proposition, (10) is true for pairs  $u, v_1$  and  $u, v_2$ , i.e.,

$$S(\{u, v_1\}) = \{S(u), v_1\} + \{u, S(v_1)\}, \quad S(\{u, v_2\}) = \{S(u), v_2\} + \{u, S(v_2)\}.$$

Then

$$\begin{aligned} S(\{u, v\}) &= S(v_1)\{u, v_2\} + v_1\{S(u), v_2\} + v_1\{u, S(v_2)\} \\ &\quad + \{S(u), v_1\}v_2 + \{u, S(v_1)\}v_2 + \{u, v_1\}S(v_2) \\ &= \{S(u), v_1v_2\} + \{u, S(v_1)v_2 + v_1S(v_2)\} = \{S(u), v\} + \{u, S(v)\}. \end{aligned}$$

Consequently,  $S$  is a derivation of a Poisson algebra  $P$  and induces  $D$  on  $P_0$ .  $\square$

**Lemma 3.** *Every locally nilpotent derivation of the Poisson algebra  $P_0$  is induced by a locally nilpotent  $d$ -derivation of the Poisson algebra  $P = P\langle x_1, x_2, \dots, x_n \rangle$ .*

*Proof.* Let  $D$  be a locally nilpotent derivation of  $P_0$  and let  $S$  be a unique extension of  $D$  to  $P$ . We have to show that  $S$  is a locally nilpotent derivation of  $P$ . Notice that

$$P_0 \subset P$$

is an integral extension of domains since  $e_i^d \in P_0$  for all  $i \geq 1$ . According to a result of W.V. Vasconcelos [24] (see also Proposition 1.3.37 from [3, p. 41]),  $S$  is locally nilpotent.  $\square$

#### 4. AUTOMORPHISMS OF $P_0$

As we noticed above, every  $d$ -graded automorphism of  $P\langle x_1, x_2, \dots, x_n \rangle$  induces an automorphism of  $P_0$ . In this section we prove the reverse of this statement for  $n > 1$ .

**Theorem 1.** *Let  $K$  be a field closed with respect to taking all  $d$ -roots of elements. Then every automorphism of  $P_0$  over  $K$  is induced by a  $d$ -graded automorphism of  $P = P\langle x_1, x_2, \dots, x_n \rangle$  if  $n > 1$ .*

*Proof.* Let  $\alpha$  be an automorphism of  $P_0$ . Denote the extension of  $\alpha$  to the quotient field  $Q(P_0)$  by the same symbol. We have  $\frac{x_2}{x_1} \in Q(P_0)$ . Suppose that

$$(11) \quad \alpha \left( \frac{x_2}{x_1} \right) = \frac{f_2}{f_1},$$

where  $f_1, f_2$  are relatively prime. Then

$$\alpha \left( \frac{x_2^d}{x_1^d} \right) = \alpha \left( \frac{x_2}{x_1} \right)^d = \frac{f_2^d}{f_1^d}.$$

Since  $f_1, f_2$  are relatively prime it follows that  $\alpha(x_1^d) = v f_1^d$  and  $\alpha(x_2^d) = v f_2^d$  for some  $v \in P$ . Moreover,  $\alpha(x_1^i x_2^{d-i}) = v f_1^i f_2^{d-i}$  for all  $0 \leq i < d$ .

We have  $v f_1^d, v f_2^d \in P_0$ . If  $K$  is a field of characteristic  $p > 0$  and  $p$  divides  $d$ , then  $f_1^d, f_2^d \in P_0$ . Consequently,  $v \in P_0$ . Assume that  $K$  is a field of characteristic 0 or of characteristic  $p > 0$  and  $p$  does not divide  $d$ . Let  $\epsilon$  be a primitive  $d$ -root of unity. Consider the automorphism  $\epsilon$  of  $Q(P)$  such that  $\epsilon(x_i) = \epsilon x_i$  for all  $i$ . Notice that for any  $f \in Q(P)$  we have  $f \in Q(P_0)$  if and only if  $\epsilon(f) = f$ . Then

$$\frac{f_2}{f_1} = \epsilon \left( \frac{f_2}{f_1} \right) = \frac{\epsilon(f_2)}{\epsilon(f_1)}$$

and  $\epsilon(f_1), \epsilon(f_2)$  are relatively prime. Hence  $f_1 \epsilon(f_2) = \epsilon(f_1) f_2$  and, say,  $f_1$  divides  $\epsilon(f_1)$  and  $\epsilon(f_1)$  divides  $f_1$ . Consequently  $f_1$  and  $\epsilon(f_1)$  are proportional which is possible only if  $f_1$  is a  $d$ -homogeneous element. Similarly,  $f_2$  is a  $d$ -homogeneous element. Then  $f_1^d, f_2^d \in P_0$  and, consequently,  $v \in P_0$ .

This implies that

$$x_1^d = \alpha^{-1}(v) \alpha^{-1}(f_1^d), \quad x_2^d = \alpha^{-1}(v) \alpha^{-1}(f_2^d).$$

Since  $x_1^d$  is irreducible in  $P_0$ , this is possible only if  $v \in K$ . Let  $\mu \in K$  be a  $d$ -root of  $v$ , i.e.,  $\mu^d = v$ . Replacing  $f_1$  and  $f_2$  by  $\mu f_1$  and  $\mu f_2$ , we may assume that

$$(12) \quad \alpha(x_1^d) = f_1^d, \quad \alpha(x_2^d) = f_2^d.$$

By (11) and (12), we get

$$\alpha(x_1^{i_1} x_2^{i_2}) = f_1^{i_1} f_2^{i_2}, \text{ if } x_1^{i_1} x_2^{i_2} \in P_0.$$

Consider an arbitrary  $e_i$  with  $i \geq 3$ . Suppose that  $\deg e_i = s$ . Then  $y_i = \frac{e_i}{x_1^s} \in Q(P_0)$ . Suppose that

$$\alpha(y_i) = \alpha \left( \frac{e_i}{x_1^s} \right) = \frac{f_i}{g_i},$$

where  $f_i, g_i$  are relatively prime. Then

$$\alpha\left(\frac{e_i^d}{x_1^{sd}}\right) = \alpha\left(\frac{e_i}{x_1^s}\right)^d = \frac{f_i^d}{g_i^d}.$$

Again  $\alpha(e_i^d) = v f_i^d$  and  $\alpha(x_1^{sd}) = v g_i^d$  for some  $v \in P$ . As above, we get that  $f_i^d, g_i^d \in P_0, v \in K$ , and we can assume that

$$\alpha(e_i^d) = f_i^d, \quad \alpha(x_1^{sd}) = g_i^d.$$

Then  $f_1^{sd} = g_1^d$  and  $g_i = \lambda f_1^s$ , where  $\lambda$  is a  $d$ -root of unity. After rescaling, we can assume that  $g_i = f_1^s$  and

$$(13) \quad \alpha(e_i^d) = f_i^d, \quad \alpha(y_i) = \alpha\left(\frac{e_i}{x_1^s}\right) = \frac{f_i}{f_1^s},$$

where  $s = \deg e_i$  and  $i \geq 3$ . This is true for  $i = 2$  by (11) and (12).

Let  $u = e_{i_1} \dots e_{i_k}$  be an arbitrary element of  $P$  of the form (5). We have

$$(14) \quad u = x_1^s \frac{e_{i_1}}{x_1^{s_{i_1}}} \dots \frac{e_{i_k}}{x_1^{s_{i_k}}} = x_1^s y_{i_1} \dots y_{i_k},$$

where  $s = s_{i_1} + \dots + s_{i_k}$ . We have  $d|s$  since  $u \in P_0$ . Then

$$\alpha(u) = f_1^s \frac{f_{i_1}}{f_1^{s_{i_1}}} \dots \frac{f_{i_k}}{f_1^{s_{i_k}}} = f_{i_1} \dots f_{i_k}$$

by (11), (12), and (13).

Consequently, the polynomial endomorphism  $\beta$  of  $P$ , determined by  $\beta(e_i) = f_i$  for all  $i \geq 1$ , induces  $\alpha$  on  $P_0$ . First we show that  $\beta$  is a polynomial automorphism of  $P$ . The elements (4) are algebraically independent and, consequently, the elements  $e_1^d, \dots, e_s^d, \dots$  are algebraically independent. Since  $\alpha$  is an automorphism and  $\alpha(e_i^d) = f_i^d$  for all  $i$  by (12) and (13), the elements  $f_1^d, \dots, f_s^d, \dots$  are algebraically independent. Therefore the elements  $f_1, \dots, f_s, \dots$  are algebraically independent and  $\beta$  is an injective endomorphism. Then  $\beta$  can be uniquely extended to an endomorphism of the quotient field  $P(x_1, x_2, \dots, x_n)$  and we denote this extension also by  $\beta$ .

The restriction of  $\beta$  on  $Q(P_0)$  is an automorphism since it coincides with the  $\alpha$ . Consider the space

$$V = Q(P_0)P\langle x_1, x_2, \dots, x_n \rangle.$$

By (14) every element  $f \in P$  can be written as

$$f = f_0 + f_1 x_1 + \dots + f_{d-1} x_1^{d-1},$$

where  $f_0, f_1, \dots, f_{d-1} \in K[t, y_2, \dots, y_s, \dots]$  and  $t = x_1^d$ . Hence  $V$  is the  $Q(P_0)$ -span of the elements  $1, x_1, x_1^2, \dots, x_1^{d-1}$ . If

$$V = b_1 Q(P_0) \oplus \dots \oplus b_k Q(P_0),$$

then

$$\beta(V) = \beta(b_1)Q(P_0) + \dots + \beta(b_k)Q(P_0)$$

since  $\beta(Q(P_0)) = Q(P_0)$ . Notice that  $\beta(V) \subseteq V$ . If  $\beta(V) \neq V$  then  $\dim_{Q(P_0)} V < k$  and  $\text{Ker } \beta \neq 0$ . It is impossible for nonzero field endomorphisms. Consequently,  $\beta(V) = V$

and  $e_i \in \beta(V)$  for all  $i$ . Therefore  $\beta$  is an automorphism of the field  $P(x_1, x_2, \dots, x_n)$  and of the polynomial algebra  $P\langle x_1, x_2, \dots, x_n \rangle$ .

It remains to show that  $\beta$  is a Lie automorphism of  $P$ , i.e.,

$$(15) \quad \beta(\{u, v\}) = \{\beta(u), \beta(v)\}$$

for all  $u, v$  of the form (5). We prove (15) by induction on the polynomial length  $d(u)+d(v)$ . Suppose that  $u = e_i$  and  $v = e_j$ . Since  $e_i^d, e_j^d \in P_0$ , we get

$$\begin{aligned} \beta(\{e_i^d, e_j^d\}) &= \alpha(\{e_i^d, e_j^d\}) = \{\alpha(e_i^d), \alpha(e_j^d)\} = \{\beta(e_i^d), \beta(e_j^d)\} \\ &= \{\beta(e_i)^d, \beta(e_j)^d\} = d^2 \beta(e_i)^{d-1} \beta(e_j)^{d-1} \{\beta(e_i), \beta(e_j)\}. \end{aligned}$$

On the other hand,

$$\beta(\{e_i^d, e_j^d\}) = \beta(d^2 e_i^{d-1} e_j^{d-1} \{e_i, e_j\}) = d^2 \beta(e_i)^{d-1} \beta(e_j)^{d-1} \beta(\{e_i, e_j\}).$$

Comparing two values of  $\beta(\{e_i^d, e_j^d\})$ , we get that (15) holds for  $u = e_i$  and  $v = e_j$ .

Suppose that  $d(v) \geq 2$  and  $v = v_1 v_2$ . Then

$$\begin{aligned} \beta(\{u, v\}) &= \beta(\{u, v_1 v_2\}) = \beta(v_1 \{u, v_2\} + \{u, v_1\} v_2) \\ &= \beta(v_1) \beta(\{u, v_2\}) + \beta(\{u, v_1\}) \beta(v_2). \end{aligned}$$

By the induction proposition, we may assume that (15) is true for pairs  $u, v_1$  and  $u, v_2$ . Then

$$\begin{aligned} \beta(\{u, v\}) &= \beta(v_1) \{\beta(u), \beta(v_2)\} + \{\beta(u), \beta(v_1)\} \beta(v_2) \\ &= \{\beta(u), \beta(v_1) \beta(v_2)\} = \{\beta(u), \beta(v)\}. \end{aligned}$$

Consequently,  $\beta$  is an automorphism of  $P$  and induces  $\alpha$  on  $P_0$ .  $\square$

Let  $\text{Aut}_d P$  be the group of all  $d$ -graded automorphisms of the free Poisson algebra  $P$ .

**Corollary 1.** *Let  $K$  be a field closed with respect to taking all  $d$ -roots of elements and let  $E = \{\lambda \text{id} \mid \lambda^d = 1, \lambda \in K\}$ , where  $\text{id}$  is the identity automorphism of  $P$ . Then*

$$\text{Aut } P_0 \cong \text{Aut}_d P / E.$$

*Proof.* Consider the homomorphism

$$(16) \quad \psi : \text{Aut}_d P \rightarrow \text{Aut } P_0$$

defined by  $\psi(\alpha) = \bar{\alpha}$ , where  $\bar{\alpha}$  is the automorphism of  $P_0$  induced by the  $d$ -graded automorphism  $\alpha$  of  $P$ .

By Theorem 1,  $\psi$  is an epimorphism. Let  $\alpha \in \text{Ker } \psi$ . Then  $\alpha(x_1)^d = x_1^d$ . Consequently,  $\alpha(x_1) = \lambda x_1$  for some  $d$ th root of unity  $\lambda \in K$ . Extending  $\alpha$  to  $Q(P_0)$ , we get  $\alpha(x_i/x_1) = x_i/x_1$ . Consequently,  $\alpha(x_i) = \lambda x_i$  for all  $i$  and  $\alpha = \lambda \text{id}$ , i.e.,  $\alpha \in E$ . Obviously,  $E \subseteq \text{Ker } \psi$ .  $\square$

## 5. VERONESE SUBALGEBRAS OF POLYNOMIAL ALGEBRAS

Let  $A = K[x_1, x_2, \dots, x_n]$  be the polynomial algebra over a field  $K$  in the variables  $x_1, x_2, \dots, x_n$ . Consider the grading

$$A = A_0 \oplus A_1 \oplus \dots \oplus A_{d-1},$$

where  $d \geq 2$  and  $A_i$  is the subspace of  $A$  generated by all monomials of degree  $kd + i$  for all  $k \geq 0$ . This is a  $\mathbb{Z}_d$ -grading of  $A$ , i.e.,  $A_i A_j \subseteq A_{i+j}$  for all  $i, j \in \mathbb{Z}_d$ . The subalgebra  $A_0$  is called the *Veronese subalgebra of  $A$  of degree  $d$* .

**Corollary 2.** *Let  $A = K[x_1, x_2, \dots, x_n]$  be the polynomial algebra over a field  $K$  of characteristic zero in  $n \geq 2$  variables  $x_1, x_2, \dots, x_n$ . Then every derivation of the Veronese subalgebra  $A_0$  can be uniquely extended to a  $d$ -graded derivation of  $K[x_1, x_2, \dots, x_n]$ .*

**Corollary 3.** *Let  $A = K[x_1, x_2, \dots, x_n]$  be the polynomial algebra over a field  $K$  of characteristic zero in  $n \geq 2$  variables  $x_1, x_2, \dots, x_n$ . Then every locally nilpotent derivation of the Veronese subalgebra  $A_0$  is induced by a locally nilpotent  $d$ -derivation of the polynomial algebra  $K[x_1, x_2, \dots, x_n]$ .*

**Corollary 4.** *Let  $A = K[x_1, x_2, \dots, x_n]$  be the polynomial algebra in  $n \geq 2$  variables  $x_1, x_2, \dots, x_n$  over a field  $K$  closed with respect to taking all  $d$ -roots of elements. Then every automorphism of the Veronese subalgebra  $A_0$  of degree  $d$  is induced by a  $d$ -graded automorphism of  $K[x_1, x_2, \dots, x_n]$ .*

This result is also proven in [8].

Let  $\text{Aut}_d A$  be the group of all  $d$ -graded automorphisms of the polynomial algebra  $A$ .

**Corollary 5.** *Let  $K$  be a field closed with respect to taking all  $d$ -roots of elements and let  $E = \{\lambda \text{id} \mid \lambda^d = 1, \lambda \in K\}$ , where  $\text{id}$  is the identity automorphism of  $A$ . Then*

$$\text{Aut } A_0 \cong \text{Aut}_d A / E.$$

The proofs of Corollary 2, Corollary 3, Corollary 4, and Corollary 5 repeat the polynomial parts of the proofs of Lemma 2, Lemma 3, Theorem 1, and Corollary 1, respectively.

Notice that these statements are not true for the polynomial algebra  $A = K[x]$  in one variable  $x$ . In this case, the Veronese subalgebra  $A_0$  of degree  $d$  is the polynomial algebra in one variable  $x^d$ . Then the locally nilpotent derivation of  $A_0$  determined by

$$x^d \mapsto 1$$

cannot be induced by any derivation of  $A$  and the automorphism of  $A_0$  determined by

$$x^d \mapsto x^d + 1$$

cannot be induced by any automorphism of  $A$ .

In addition, analogues of these results are not true for free associative algebras. In fact, if  $B = K\langle x, y \rangle$  is the free associative algebra in the variables  $x, y$  and  $d = 2$  then the Veronese subalgebra  $B_0$  of degree  $d$  is the free associative algebra in the variables  $x^2, xy, yx, y^2$ . It is easy to check that the locally nilpotent derivation of  $B_0$  determined by

$$x^2 \mapsto 1, xy \mapsto 0, yx \mapsto 0, y^2 \mapsto 0$$

cannot be induced by any derivation of  $B$  and the automorphism of  $B_0$  determined by

$$x^2 \mapsto x^2 + 1, xy \mapsto xy, yx \mapsto yx, y^2 \mapsto y^2$$

cannot be induced by any automorphism of  $B$ .

## Acknowledgments

The second and third authors are grateful to Max-Planck Institute für Mathematik for hospitality and excellent working conditions, where part of this work has been done.

The third author is supported by the grant of the Ministry of Education and Science of the Republic of Kazakhstan (project AP14872073).

## REFERENCES

- [1] B. Aitzhanova, U. Umirbaev, Automorphisms of affine Veronese surfaces. *Internat. J. Algebra Comput.* 33 (2023), no. 2, 351–367.
- [2] I. Arzhantsev, M. Zaidenberg, *Acyclic curves and group actions on affine toric surfaces. Affine algebraic geometry*, 1–41, World Sci. Publ., Hackensack, NJ, 2013.
- [3] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture. Progress in Mathematics*, 190. Birkhäuser Verlag, Basel, 2000.
- [4] M.H. Gizatullin, Quasihomogeneous affine surfaces. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 35 (1971), 1047–1071.
- [5] M.H. Gizatullin, V.I. Danilov, Automorphisms of affine surfaces. I. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 39 (1975), no. 3, 523–565.
- [6] M.H. Gizatullin, V.I. Danilov, Automorphisms of affine surfaces. II. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 41 (1977), no. 1, 54–103.
- [7] J. Harris, *Algebraic geometry. A first course. Graduate Texts in Mathematics*, 133. Springer-Verlag, New York, 1995.
- [8] J. Kollár, Automorphisms and twisted forms of quotient singularities; notes on a paper of Bresciani–Vistoli. 2022, <https://doi.org/10.48550/arXiv.2210.16265>
- [9] J. Kollár, Automorphisms of rings of invariants. 2022, <https://doi.org/10.48550/arXiv.2212.03772>
- [10] S. Kovalenko, A. Perepechko, M. Zaidenberg, On automorphism groups of affine surfaces. (English summary) *Algebraic varieties and automorphism groups*, 207–286, *Adv. Stud. Pure Math.*, 75, Math. Soc. Japan, Tokyo, 2017.
- [11] L. Makar-Limanov, On groups of automorphisms of a class of surfaces. *Israel J. of Math.* 69 (1990), 250–256.
- [12] L. Makar-Limanov, On the group of automorphisms of a surface  $x^n y = P(z)$ . *Israel J. Math.* 121 (2001), 113–123.
- [13] L. Makar-Limanov, U. Umirbaev, Centralizers in free Poisson algebras. *Proc. Amer. Math. Soc.* 135 (2007), no. 7, 1969–1975.
- [14] L. Makar-Limanov, U. Turusbekova, U. Umirbaev, Automorphisms and derivations of free Poisson algebras in two variables. *J. Algebra* 322 (2009), no. 9, 3318–3330.
- [15] L. Makar-Limanov, U. Umirbaev, The Freiheitssatz for Poisson algebras. *J. Algebra* 328 (2011), 495–503.
- [16] L. Makar-Limanov, I. Shestakov, Polynomial and Poisson dependence in free Poisson algebras and free Poisson fields, *J. Algebra* 349 (2012), 372–379.
- [17] L. Makar-Limanov, U. Umirbaev, Free Poisson fields and their automorphisms. *J. Algebra Appl.* 15 (2016), no. 10, 1650196, 13 pp.
- [18] V.L. Popov, Quasihomogeneous affine algebraic varieties of the group  $SL(2)$ . (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 37 (1973), 792–832.
- [19] I.R. Shafarevich, *Basic algebraic geometry*. (Translated from Russian by K.A. Hirsch). Springer-Verlag Berlin Heidelberg. 1977.
- [20] I.P. Shestakov, Quantization of Poisson superalgebras and speciality of Jordan Poisson of superalgebras, *Algebra Logic* 32(5) (1993) 309–317.
- [21] T. Gateva-Ivanova, Veronese subalgebras and Veronese morphisms for a class of Yang-Baxter algebras, 2022.

- [22] U. Umirbaev, Universal enveloping algebras and universal derivations of Poisson algebras. *J. Algebra* 354 (2012), pp. 77–94.
- [23] U. Umirbaev, V. Zhelyabin, A Dixmier theorem for Poisson enveloping algebras. *J. Algebra* 568 (2021), 576–600.
- [24] W.V. Vasconcelos, Derivations of commutative noetherian rings. *Math. Z.* 112 (1969), 229–233.
- [25] O. Zariski, P. Samuel, *Commutative algebra. Volume I.* D.Van Nostrand Company. Princeton. New Jersey. 1958.