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We count the number of critical points of a modular form with real Fourier coefficients in a γ-translate of the standard fundamental domain $\mathcal{F}$ (with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$). Whereas by the valence formula the (weighted) number of zeros of this modular form in $\gamma\mathcal{F}$ is a constant only depending on its weight, we give a closed formula for this number of critical points in terms of those zeros of the modular form lying on the boundary of $\mathcal{F}$, the value of $\gamma^{-1}(\infty)$ and the weight. More generally, we indicate what can be said about the number of zeros of a quasimodular form.

1. Introduction

A valence formula for quasimodular forms For a non-zero modular form g of weight k, the (weighted) number of zeros in a fundamental domain is given by

$$\sum_{\tau\in\gamma\mathcal{F}} \frac{\nu_\tau(g)}{e_\tau} \;=\; \frac{k}{12}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where $\mathcal{F}$ denotes the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the extended complex upper half plane, $\nu_\tau(g)$ denotes the order of vanishing of g at τ (see Section 2 for the definitions) and $e_\tau = 2$ for a $\mathrm{SL}_2(\mathbb{Z})$-translate of i, $e_\tau = 3$ for a $\mathrm{SL}_2(\mathbb{Z})$-translate of $\rho = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ and $e_\tau = 1$ else, including at the cusp at infinity. Much less is known about the (weighted) number of zeros of derivatives of modular forms, or, more generally, of zeros of quasimodular forms. That is, in this paper we study the value

$$N_\lambda(f) \;:=\; \sum_{\tau\in\gamma\mathcal{F}} \frac{\nu_\tau(f)}{e_\tau} \qquad\qquad \big(\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \text{ such that } \lambda = -\tfrac{d}{c}\big)$$

for derivatives of modular forms or, more generally, for quasimodular forms f (the quantity $N_\lambda(f)$ is well-defined if f is quasimodular).

As an example, consider the critical points of the modular discriminant Δ. Note $\Delta' = \Delta E_2$ (with $f' = \frac{1}{2\pi \mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\tau}f$), where

$$E_2(\tau) \;=\; 1 - 24 \sum_{m,r\geq 1} m\, q^{mr} \qquad\qquad (\tau \in \mathfrak{h},\ \text{the complex upper half plane})$$

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is the quasimodular Eisenstein series of weight 2 transforming as

$$(E_2|\gamma)(\tau) \;=\; E_2(\tau) + \frac{12}{2\pi \mathrm{i}}\frac{c}{c\tau+d} \;=\; E_2(\tau) + \frac{12}{2\pi \mathrm{i}}\frac{1}{\tau-\lambda(\gamma)}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and with $\lambda(\gamma) = -\frac{d}{c}$. For E_2 the number of zeros in a fundamental domain depends on the choice of this domain. There are infinitely many non-equivalent zeros of E_2; in fact, two zeros are only equivalent if one is a $\mathbb{Z}$-translate of the other [EBS10]. Nevertheless, one can still count the number of zeros of E_2 in $\gamma\mathcal{F}$:

$$N_\lambda(E_2) \;=\; \begin{cases} 0 & |\lambda| \in (\frac{1}{2}, \infty] \\ 1 & |\lambda| \in [0, \frac{1}{2}), \end{cases} \tag{1}$$

as follows from [IJT14, WY14]. Recently, Gun and Oesterlé counted the number of critical points of the Eisenstein series E_k for $k > 2$ [GO22]

$$N_\lambda(E_k') \;=\; \begin{cases} \lfloor \frac{k+2}{6} \rfloor + \frac{1}{3}\delta_{k\equiv 2\,(6)} & |\lambda| \in (1, \infty] \\ \frac{1}{3}\delta_{k\equiv 2\,(6)} & |\lambda| \in [0, 1). \end{cases} \tag{2}$$

(Note E_k has a double zero at $\rho = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{i}$ for $k \equiv 2 \mod 6$. Hence, the factor $\delta_{k\equiv 2\,(6)}$ corresponds to the trivial zero of E_k' at a γ-translate of $\rho = -\frac{1}{2} + \frac{1}{2}\sqrt{3}$.)

Critical points of modular forms Modular forms admit infinitely many non-equivalent critical points [SS12]. By counting the number of critical points within a fundamental domain $\gamma\mathcal{F}$ ($\gamma \in \mathrm{SL}_2(\mathbb{Z})$), we provide a quantitative version of this statement. Notice that the only zero of Δ in the fundamental domain $\mathcal{F}$ is at the cusp, whereas the Eisenstein series have all their zeros in $\mathcal{F}$ on the unit circle [RS70]. Our main theorem expresses the number of critical points of a modular form in terms of the number of zeros of this modular form on the boundary of $\mathcal{F}$.

Write $C(g)$ for the number of distinct zeros z of g satisfying $|z| = 1$ and $-\frac{1}{2} \le \mathrm{Re}\,(z) \le 0$, where a zero z is counted with weight e_z^{-1} (i.e., a zero at $\rho = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{i}$ or at i is counted with weight $\frac{1}{3}$ or $\frac{1}{2}$ respectively). Write $L(g)$ for the number of distinct zeros z of g at the cusp or satisfying $\mathrm{Re}(z) = -\frac{1}{2}$ and $|z| > 1$.

Theorem 1.1. *Let g be a modular form of weight k with* real *Fourier coefficients. Then,*

$$N_\lambda(g') \;=\; \frac{k}{12} + \begin{cases} \;\;C(g) \,+\, \frac{1}{3}\delta_{g(\rho)=0} & |\lambda| \in (1, \infty] \\ -C(g) & |\lambda| \in (\frac{1}{2}, 1) \\ -C(g) + L(g) & |\lambda| \in [0, \frac{1}{2}). \end{cases}$$

In particular, for any sequence of cuspidal Hecke eigenforms g_k of weight k we have

$$\frac{N_\lambda(g_k')}{N_\lambda(g_k)} \to 1 \qquad \text{as } k \to \infty,$$

as by the holomorphic quantum unique ergodicity theorem the zeros of g_k are equidistributed in $\mathcal{F}$ as $k \to \infty$ [HS10]. This leads to the question whether the zeros of derivatives of Hecke eigenforms are also equidistributed.

Moreover, if g is a modular form with *all* its zeros on the interior of $\mathcal{F}$, we find

$$N_\lambda(g') \;=\; \frac{k}{12} \;=\; N_\lambda(g).$$

(In case all the zeros of g lie on the interior of $\mathcal{F}$, then $k \equiv 0 \mod 12$.) Observe that as Δ has a unique zero at the cusp, we have $C(\Delta) = 0$ and $L(\Delta) = 1$, by which we recover (1). Moreover, for modular Eisenstein series we have $C(E_k) = \frac{k}{12} - \frac{1}{3}\delta_{k\equiv 2\,(6)}$ and $L(E_k) = 0$, by which we recover (2).

Further results We aim to generalize these formulae to *all* quasimodular forms, i.e., polynomials in E_2 with modular coefficients. First of all, we show that for any quasimodular form f the number $N_\lambda(f)$ only takes finitely many different values if we vary λ.

Theorem 1.2. *Given a quasimodular form f, there exist finitely many disjoint intervals I such that $\mathbb{R} = \cup_{I\in\mathscr{I}} I$, and for each I a constant $N_I(f) \in \frac{1}{6}\mathbb{Z}$ such that*

$$N_\lambda(f) = N_I(f) \qquad \text{if } \lambda \in I.$$

For example, in Example 5.5 we will see that

$$N_\lambda(E_2') = \begin{cases} 1 & |\lambda| \in (\frac{1}{v}, \frac{1}{2}) \cup (v, \infty] \\ 0 & |\lambda| \in (0, \frac{1}{v}) \cup (\frac{1}{2}, v). \end{cases} \tag{3}$$

for some $v \in (5,6)$, which we compare to the results in [CL19].

Secondly, we study in more detail the case that $f = f_0 + f_1 E_2$ for some modular forms f_0 and f_1 of weight k and $k-2$ respectively and with real Fourier coefficients. For example, the first derivative of a modular form can be written in such a way. From now on, assume that f_0 and f_1 admit no common zeros on the extended upper half plane $\mathfrak{h}^*$. We give closed formulas for $N_\lambda(f)$ depending on the behaviour of f_1 at ρ, its zeros on the boundary of $\mathcal{F}$, and the value of f at $\mathrm{i}\infty, \rho$ and these zeros of f_1. That is, let $z_1, \ldots, z_m$ be the zeros of f_1 such that $\operatorname{Re} z_i = -\frac{1}{2}$ and $\operatorname{Im} z_i > \frac{1}{2}\sqrt{3}$, counted with multiplicity and ordered by imaginary part, and let $z_0 = \rho$. Also, let $\theta_1, \ldots, \theta_n$ be the angles of those zeros of f_1 on the unit circle satisfying $\frac{2\pi}{3} \geq \theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \geq \frac{\pi}{2}$ (counted with multiplicity). We introduce the following notation:

- $\widehat{f}(\theta) = e^{\frac{1}{2}k\mathrm{i}\theta} f(e^{\mathrm{i}\theta})$
- $r(f_1)$ denotes the sign of the first non-zero Taylor coefficient in the *natural* Taylor expansion of f_1 around ρ (see (11)); $\nu_\rho(f_1)$ denotes the order of vanishing of f_1 at ρ.
- $s(f) = \operatorname{sgn} a_0(f)$ if f does not vanish at infinity, and $s(f) = -\operatorname{sgn} a_0(f_1)$ else. Here, a_0 denotes the constant term in the Fourier expansion, i.e., $a_0(f) = \lim_{z\to -\frac{1}{2}+\mathrm{i}\infty} f(z)$.
- $w(z_0) = 2$ if z_0 equals ρ, i or $-\frac{1}{2} + \mathrm{i}\infty$, and $w(z) = 1$ for all other $z \in \mathcal{F}$.

Theorem 1.3. *Let $f = f_0 + f_1 E_2$ be a quasimodular form of weight k for which f_0 and f_1 are modular forms without common zeros on $\mathfrak{h}^*$ and with real Fourier coefficients. Then, there exist constants $N_{(1,\infty]}(f), N_{(\frac{1}{2},1)}(f), N_{[0,\frac{1}{2})}(f) \in \mathbb{Z}$ such that*

$$N_\lambda(f) = \begin{cases} N_{(1,\infty]}(f) & |\lambda| \in (1,\infty] \\ N_{(\frac{1}{2},1)}(f) & |\lambda| \in (\frac{1}{2}, 1) \\ N_{[0,\frac{1}{2})}(f) & |\lambda| \in [0, \frac{1}{2}). \end{cases}$$

Moreover, these constants are uniquely determined by

$$N_{(1,\infty]}(f) = \frac{1}{2}\left\lfloor \frac{k}{6} \right\rfloor - (-1)^{\nu_\rho(f_1)} r(f_1) \sum_{j=1}^{n} \frac{(-1)^j}{w(e^{\mathrm{i}\theta_j})} \operatorname{sgn} \widehat{f}(\theta_j) \tag{4}$$

$$N_{(\frac{1}{2},1)}(f) = \left\lfloor \frac{k}{6} \right\rfloor - N_{(1,\infty]}(f) \tag{5}$$

$$N_{[0,\frac{1}{2})}(f) = \left\lceil \frac{k}{6} \right\rceil - N_{(1,\infty]}(f) - r(f_1) \sum_{j=0}^{m} \frac{(-1)^j}{w(z_j)} \operatorname{sgn} f(z_j) - \frac{1}{2}(-1)^{m+1} r(f_1)\, s(f). \tag{6}$$

Remark 1.

(i) Observe that the conditions of the theorem guarantee that the sign functions is applied to a non-zero real number, that is, $\widehat{f}(\theta_j), f(z_j) \in \mathbb{R}^*$.

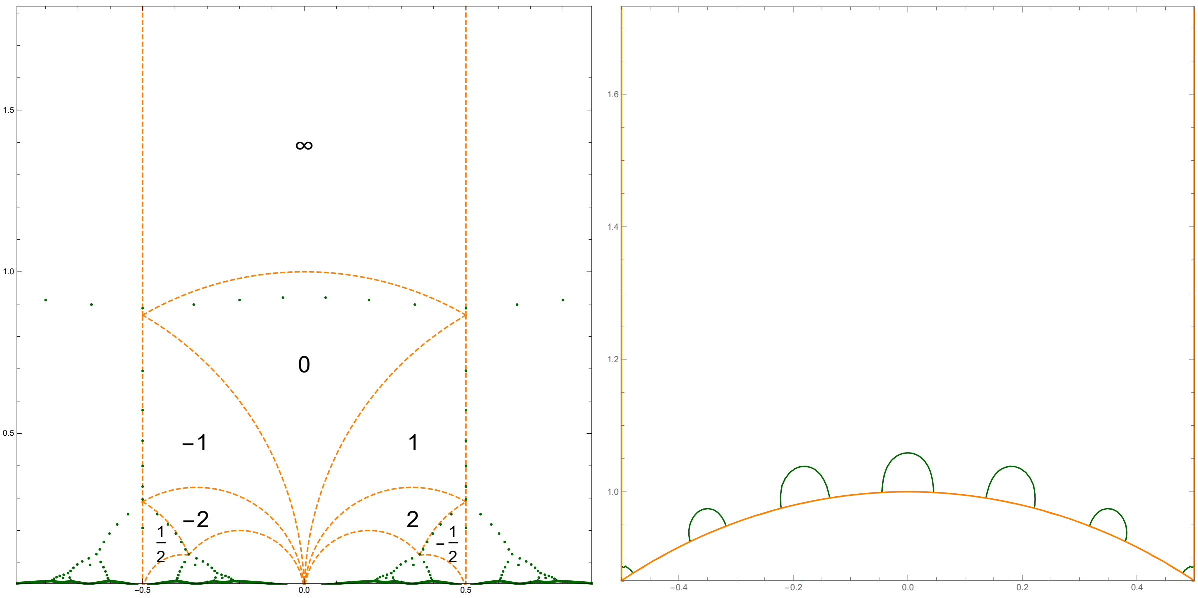
(ii) In case f_0 and f_1 do admit common zeros, there always exists a modular form g such that f_0/g and f_1/g are (holomorphic) modular forms without common zeros.

(iii) For quasimodular forms of depth > 1 (i.e., if f is a polynomial in E_2 of degree > 1 with modular coefficients), the first part of the statement is wrong. This has already been illustrated with the example in (3); for more details, see Example 5.5. △

An extreme example To illustrate some of the characteristic properties of zeros of quasimodular forms, consider the unique quasimodular form $f = f_0 + f_1 E_2$ in the 7-dimensional vector space $M_{36}^{\leq 1}$ with q-expansion $f = 1 + O(q^7)$ (as the constant coefficient is 1, the quasimodular form f cannot be the derivative of a modular form)[1]. Explicitly, f equals

$$1 + 212963830173619200q^7 + 4512225555990230800q^8 + 39202641996632255232 00q^9 + O(q^{10}).$$

Figure 1: Zeros of the unique quasimodular form of weight 36, depth 1 and of the form $1 + O(q^7)$.



(a) The approximate location of 1000 zeros of f and fundamental domains $\gamma\mathcal{F}$ for $\lambda(\gamma) = 0, \pm\frac{1}{2}, \pm 1, \infty$.

(b) The curves (7) associated to f.

The zeros of f, depicted in Fig. 1a, satisfy

$$N_{(1,\infty]} = 1, \qquad N_{(\frac{1}{2},1)} = 5, \qquad N_{[0,\frac{1}{2})}(f) = 6.$$

Moreover, in Fig. 1b, we depicted the rational curves

$$\{\gamma z \mid \gamma \in \mathrm{SL}_2(\mathbb{Z}), f(z) = 0\} \cap \mathcal{F} \;=\; \{z \in \mathcal{F} \mid h(z) \in \mathbb{Q}\}, \tag{7}$$

where the function $h : \mathfrak{h} \to \mathbb{C}$ is given by

$$h(\tau) \;=\; \tau + \frac{12}{2\pi i}\frac{f_1(\tau)}{f(\tau)}.$$

[1]Following [JP14] one could call this a *gap quasimodular form*. The theta series of an extremal lattice is a gap modular form. Do gap quasimodular forms have a similar interpretation?

In fact, h is an equivariant function, i.e.,

$$h(\tau+1)=h(\tau)+1, \qquad h\Big(-\frac{1}{\tau}\Big)=-\frac{1}{h(\tau)}, \qquad h(-\overline{\tau})=-\overline{h(\tau)}.$$

For other quasimodular forms of depth 1 the corresponding function h is also equivariant, and these transformation properties are the main ingredients for Theorem 1.3.

Extremal quasimodular forms Write $\widetilde{M}_k^{\leq p}$ is the space of holomorphic quasimodular forms of weight k and depth $\leq p$. Recall that $\widetilde{M}_4^{\leq 1}=M_4=\mathbb{C}E_4$ and E_4 has a unique zero at ρ, so that $N_\lambda(E_4)=\frac{1}{3}$. Excluding this modular form, we find the following upper bound.

Corollary 1.4. *For all $f\in\widetilde{M}_k^{\leq 1}$ such that $\frac{f}{E_4}\notin\widetilde{M}_k^{\leq 1}$, we have*

$$N_\lambda(f)\;\leq\;\dim\widetilde{M}_k^{\leq 1}\;+\;\begin{cases}-1 & \lambda\in(\frac{1}{2},\infty]\\ 0 & \lambda\in[0,\frac{1}{2}).\end{cases}$$

Observe that in any vector subspace of $\mathbb{C}[\![q]\!]$ of dimension m, there exist an element f with $v_{\mathrm{i}\infty}(f)\geq m-1$. Hence, there exist a quasimodular form f such that (i) the inequality (1.4) is sharp for $\lambda=\infty$ and (ii) f admits no zeros in $\mathcal{F}$ outside infinity.

Corollary 1.5. *There exists a quasimodular form $f=f_0+f_1E_2\in\widetilde{M}_k^{\leq 1}$ such that*

$$N_\infty(f)\;=\;v_{\mathrm{i}\infty}(f)\;=\;\dim\widetilde{M}_k^{\leq 1}-1$$

and all zeros of f_1 in $\mathcal{F}$ are located on the unit circle.

The existence of a quasimodular form for which $v_{\mathrm{i}\infty}(f)=\dim\widetilde{M}_k^{\leq 1}-1$ was proven by Kaneko and Koiko, who called such a quasimodular form *extremal* [KK06]. It is natural to generalize their question whether $v_{\mathrm{i}\infty}(f)\leq\dim\widetilde{M}_k^{\leq p}-1$ for all $f\in\widetilde{M}_k^{\leq p}$ (which has been confirmed for $p\leq 4$ in [Pel20]) to the following one.

Question 1. Let $k,p>0$. Do all $f\in\widetilde{M}_k^{\leq p}$ with $\frac{f}{E_4}\notin\widetilde{M}_k^{\leq p}$ satisfy

$$N_\lambda(f)\;\leq\;\dim\widetilde{M}_k^{\leq p}\;+\;\begin{cases}-1 & \lambda\in(\frac{1}{2},\infty]\\ 0 & \lambda\in[0,\frac{1}{2})\end{cases}\quad ?$$

Contents We start by recalling some basic properties of quasimodular forms in Section 2. In Section 4 we discuss equivariant functions h associated to quasimodular forms of depth 1 and of higher depth, which results in the proof of Theorem 1.2 in Section 5. The proof of Theorem 1.3 is obtained in Section 3 (for $\lambda=\infty$) and in Section 5 (for $\lambda<\infty$). We indicate how Theorem 1.1 and Corollary 1.4 then follow as corollaries of Theorem 1.2. Moreover, in all sections we give many additional examples.

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2. Set-up: zeros of quasimodular forms

Set-up Fix a holomorphic quasimodular form f for $\mathrm{SL}_2(\mathbb{Z})$, of weight k and depth p, and with real Fourier coefficients, i.e., let $f \in \mathbb{R}[E_2, E_4, E_6]$ of homogenous weight k and depth p. We write

$$f \;=\; \sum_{j=0}^{p} f_j\, E_2^j$$

where f_j is a modular form of weight $k-2j$ and $f_p \neq 0$.

Remark 2. For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$(f|_k\gamma)(\tau) \;:=\; (c\tau+d)^{-k} f\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; \sum_{j=0}^{p} \frac{(\mathfrak{d}^j f)(\tau)}{j!}\Big(\frac{1}{2\pi \mathrm{i}}\frac{c}{c\tau+d}\Big)^j, \tag{8}$$

where $\mathfrak{d}$ is the derivation on quasimodular forms uniquely determined by $\mathfrak{d}(E_2) = 12$ and the fact that it annihilates modular forms (see [Zag08, Section 5.3]), i.e.,

$$\frac{\mathfrak{d}^m(f)}{m!} \;=\; (12)^m \sum_{j=m}^{p} \binom{j}{m} f_j E_2^{j-m} \qquad (m \leq p). \qquad \triangle$$

In fact, one cannot understand the theory of quasimodular forms without recognizing the $\mathfrak{sl}_2$-action on quasimodular forms by the derivation $D = \frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial\tau} = q\frac{\partial}{\partial q}$, the weight derivation W, which multiplies a quasimodular form with its weight, and the derivation $\mathfrak{d}$, satisfying

$$[W, D] = 2D, \qquad [W, \mathfrak{d}] = -2\mathfrak{d}, \qquad [\mathfrak{d}, D] = W.$$

Remark 3. Restricting to quasimodular forms with real Fourier coefficients isn't that restrictive, for the following two reasons:

(i) All Hecke eigenforms for $\mathrm{SL}_2(\mathbb{Z})$ have real Fourier coefficients;

(ii) Suppose g is a quasimodular with complex, rather than real, Fourier coefficients. Then, $\tilde{g}(\tau) := \overline{g(-\overline{\tau})}$ is a quasimodular form which vanishes at τ if g vanishes at $-\overline{\tau}$. Hence, $N_\lambda(g) = N_{-\lambda}(\tilde{g})$, $N_\lambda(g\tilde{g}) = N_\lambda(g) + N_{-\lambda}(g)$ and $g\tilde{g}$ is a quasimodular form with real Fourier coefficients. $\triangle$

The fundamental domain Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the complex upper half plane, $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ be the extended upper half plane and

$$\mathcal{F} \;:=\; \{z \in \mathfrak{h} \mid |z| > 1, -\tfrac{1}{2} \leq \mathrm{Re}(z) < \tfrac{1}{2}\} \cup \{z \in \mathfrak{h} \mid |z| = 1, -\tfrac{1}{2} \leq \mathrm{Re}(z) \leq 0\} \cup \{\mathrm{i}\infty\}$$

the standard (strict) fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}^*$, where $\mathrm{i}\infty$ is the point $[1, 0] \in \mathbb{P}^1(\mathbb{Q})$ at infinity. Recall that the $\mathrm{SL}_2(\mathbb{Z})$-translates of $\rho = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{i}$ and of i have a non-trivial stabilizer, i.e., $e_\rho = 3, e_\mathrm{i} = 2$ and $e_z = 1$ if $z \in \mathfrak{h}^* \backslash \big(\mathrm{SL}_2(\mathbb{Z})\rho \cup \mathrm{SL}_2(\mathbb{Z})\mathrm{i}\big)$.

Moreover, we write $\mathcal{C}, \mathcal{L}$ and $\mathcal{R}$ for the positively oriented circular part, left vertical half-line and right vertical half-line of the boundary $\partial\mathcal{F}$ of $\mathcal{F}$, i.e., $\partial\mathcal{F} = \mathcal{L} \cup \mathcal{C} \cup \mathcal{R} \cup \{\mathrm{i}\infty\}$ with

$$\mathcal{C} \;=\; \{z \in \mathfrak{h} \mid |z| = 1, -\tfrac{1}{2} \leq \mathrm{Re}(z) \leq \tfrac{1}{2}\}, \tag{9}$$
$$\mathcal{L} \;=\; \{z \in \mathfrak{h} \mid |z| \geq 1, \mathrm{Re}(z) = -\tfrac{1}{2}\}, \tag{10}$$
$$\mathcal{R} \;=\; \{z \in \mathfrak{h} \mid |z| \geq 1, \mathrm{Re}(z) = \tfrac{1}{2}\}.$$

Order of vanishing at the cusps Note that for a quasimodular form f around $\tau_0 = -\frac{d}{c} \in \mathbb{P}^1(\mathbb{Q})$ we have

$$(c\tau + d)^k f(\tau) \;=\; \sum_{n=1}^{\infty} a_n(f, \tau, \tau_0) \exp\Big(2\pi\mathrm{i}n\, \frac{a\tau + b}{c\tau + d}\Big),$$

where $a, b \in \mathbb{Z}$ are such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and with

$$a_n(f, \tau, \tau_0) \;=\; \sum_{j=0}^{p} \frac{a_{n,j}}{j!} \Big(\frac{-c(c\tau + d)}{2\pi\mathrm{i}}\Big)^j \;\in\; \mathbb{C}[\tau],$$

where $a_{n,j}$ is the nth Fourier coefficient of $\mathfrak{d}^j f$. We define the order of vanishing as follows.

Definition 2.1. For a quasimodular form f and $\tau_0 \in \mathfrak{h}$, let $\nu_{\tau_0}(f)$ be the order of vanishing of f at τ_0. If $\tau_0 \in \mathbb{P}^1(\mathbb{Q})$, we let $\nu_{\tau_0}(f)$ be the minimal value of n for which $a_n(f, \tau, \tau_0) \in \mathbb{C}[\tau]$ is not the zero polynomial.

Remark 4. Equivalently, for a cusp τ_0 which is not the cusp at infinity we have

$$\nu_{\tau_0}(f) = \min(\nu_{\mathrm{i}\infty}(f_0), \ldots, \nu_{\mathrm{i}\infty}(f_p)). \qquad \triangle$$

The counting function

Definition 2.2. Given $\lambda \in \mathbb{P}^1(\mathbb{Q})$, denote by $N_\lambda(f)$ the weighted number of zeros of f in $\gamma\mathcal{F}$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $-\frac{d}{c} = \lambda$, i.e.,

$$N_\lambda(f) \;=\; \sum_{\tau \in \gamma\mathcal{F}} \frac{\nu_\tau(f)}{e_\tau},$$

where $\nu_\tau(f)$ is defined by Definition 2.1.

Observe that as $f(\tau+1) = f(\tau)$, the weighted number of zeros in $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\mathcal{F}$ and $\mathcal{F}$ agree. Hence, after fixing a rational number $\lambda = -\frac{d}{c}$ with c, d coprime integers, for all possible choices $a, b \in \mathbb{Z}$ such that $ad - bc = 1$ the weighted number of zeros in $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\mathcal{F}$ agree.

Without loss of generality, we often restrict to irreducible quasimodular forms:

Definition 2.3. A quasimodular form is *irreducible* if it cannot be written as the product of two quasimodular form of strictly lower weights.

Remark 5. If f is a quasimodular form $f = f_0 + f_1 E_2$ of depth 1, then f is irreducible if and only if f_0 and f_1 have no common zeros. $\triangle$

Remark 6. Suppose that f is quasimodular with *algebraic* Fourier coefficients. As noted by Gun and Oesterlé, if $a \in \mathfrak{h}$ is a zero of f, there exists an irreducible factor g of f, unique up to multiplication by a scalar, such that g has a single zero in a [GO22]. Hence, if f is irreducible, it has only single zeros. Moreover, if f has a zero at i or ρ (or one of their $\mathrm{SL}_2(\mathbb{Z})$-translates), then it has E_6 or E_4 respectively as one of its factors. In particular, if f is an irreducible quasimodular form which is not a modular form, then

$$N_\lambda(f) = \sum_{\tau \in \gamma\mathcal{F}} \nu_\tau(f) \;\in\; \mathbb{Z}_{\geq 0}\,.$$

Local behaviour of modular forms around ρ Recall $\rho=-\frac{1}{2}+\frac{1}{2}\sqrt{3}\,\mathrm{i}$. Let g be a modular form of weight k *with real Fourier coefficients*. Note that the mapping $w\mapsto\frac{\rho-\overline{\rho}w}{1-w}$ maps the unit disc to $\mathfrak{h}$. Then, the natural Taylor expansion of g on $\mathfrak{h}$ (see [Zag08, Proposition 17]) around $\tau=\rho$ is given by

$$(1-w)^{-k}\,g\Big(\frac{\rho-\overline{\rho}w}{1-w}\Big)\;=\;\sum_{n=\nu_\rho(g)}^{\infty}b_n(g)\,w^n\qquad(|w|<1),$$

for some $\nu_\rho(g)\geq 0$ and coefficients $b_n(g)\in\mathbb{C}$ with $b_{\nu_\rho(g)}\neq 0$. (This Taylor expansion is natural as the image of $w\mapsto\frac{\rho-\overline{\rho}w}{1-w}$ for $|w|<1$ equals the full domain $\mathfrak{h}$ on which g is holomorphic.) Alternatively, g admits an ordinary Taylor expansion $g(z)=\sum_{n=\nu_\rho(g)}^{\infty}c_n(g)\,(2\pi\mathrm{i})^n(z-\rho)^n$ (with $|z|$ sufficiently small, and for the same value of $\nu_\rho(g)$) and for some coefficients $c_n(g)\in\mathbb{C}$ with $c_{\nu_\rho(g)}(g)\neq 0$. Let

$$r(g)\;:=\;\operatorname{sgn}b_{\nu_\rho(g)}(g).\tag{11}$$

In the sequel we need the following relation between $r(g)$ and the limiting behaviour of g on the boundary of $\mathcal{F}$.

Lemma 2.4. *Let g be a modular form of weight k with real Fourier coefficients. Then, for all $t\in\mathbb{R}_{>0}$ and $0<\theta<\pi$ the values of $g(\rho+\mathrm{i}t)$ and $e^{k\mathrm{i}\theta/2}g(e^{\mathrm{i}\theta})$ are real. Moreover,*

$$\lim_{t\downarrow 0}\operatorname{sgn}(g(\rho+\mathrm{i}t))\;=\;r(g)\;=\;(-1)^{\nu_\rho(g)}\operatorname{sgn}(c_{\nu_\rho(g)})\;=\;(-1)^{\nu_\rho(g)}\lim_{\theta\uparrow 2\pi/3}\operatorname{sgn}(e^{k\mathrm{i}\theta/2}g(e^{\mathrm{i}\theta})),$$

where $r(g)$ is defined by (11) *and c_n are the Taylor coefficients of g around ρ as above.*

Proof. The fact that $g(\rho+\mathrm{i}t)$ is real for real t, follows directly from the assumption that the Fourier coefficients of g are real. Moreover, this assumption implies that

$$\overline{e^{k\mathrm{i}\theta/2}g(e^{\mathrm{i}\theta})}\;=\;e^{-\mathrm{i}k\theta/2}g\Big(\frac{-1}{e^{\mathrm{i}\theta}}\Big)\;=\;e^{-\mathrm{i}k\theta/2}e^{\mathrm{i}k\theta}g(e^{\mathrm{i}\theta})$$

Hence, $\operatorname{Im}e^{k\mathrm{i}\theta/2}g(e^{\mathrm{i}\theta})=0$.

Now, note $g(\rho+\mathrm{i}t)=g(\frac{\rho-\overline{\rho}w}{1-w})$ for $w=\frac{t}{\sqrt{3}+t}$. Hence,

$$\lim_{t\downarrow 0}\operatorname{sgn}(g(\rho+\mathrm{i}t))\;=\;\lim_{w\downarrow 0}\operatorname{sgn}g\Big(\frac{\rho-\overline{\rho}w}{1-w}\Big)\;=\;\operatorname{sgn}(b_{\nu_\rho(g)}).$$

Also, $g(\rho+\mathrm{i}t)=\sum_{n=\nu_\rho(g)}^{\infty}(-2\pi t)^n$, hence, $\operatorname{sgn}(b_{\nu_\rho(g)})=(-1)^{\nu_\rho(g)}\operatorname{sgn}(c_{\nu_\rho(g)})$. Finally, for the last equality, we observe that by the valence formula, we know that g has order $\nu_\rho(g)=3\ell+\delta$ at ρ for some non-negative integer ℓ. Here, $\delta\in\{0,1,2\}$ is the reduced value of $k \mod 3$. In particular, in all cases we find that

$$e^{k\mathrm{i}\theta/2}g(e^{\mathrm{i}\theta})\sim(-2\pi)^{\nu_\rho(g)}(\theta-\tfrac{2\pi}{3})^{\nu_\rho(g)}c_{\nu_\rho(g)}$$

as $\theta\uparrow 2\pi/3$. □

3. Zeros in the standard fundamental domain $(\lambda=\infty)$

Let f be a quasimodular form. In order to compute $N_\infty(f)$, we compute the contour integral of the logarithmic derivative of f over the boundary of $\mathcal{F}$ (suitably adapted with small circular arcs, if f has zeros on this boundary). For simplicity of exposition, assume f has no zeros on the circular part of the boundary $\mathcal{C}$. Then, by a standard argument

$$N_\infty(f)\;=\;\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}}\frac{f'(z)}{f(z)}\,\mathrm{d}z\;=\;-\frac{1}{2\pi\mathrm{i}}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{\mathrm{d}}{\mathrm{d}\theta}\log(f(e^{\mathrm{i}\theta}))\,\mathrm{d}\theta.\tag{12}$$

If g is quasimodular of weight k, we define $\widehat{g} : [\frac{\pi}{3}, \frac{2\pi}{3}] \to \mathbb{C}$ by

$$\widehat{g}(\theta) = e^{k\mathrm{i}\theta/2} g(e^{\mathrm{i}\theta}).$$

We express $N_\infty(f)$ in terms of $\widehat{f}$, as follows

$$N_\infty(f) \;=\; -\frac{1}{2\pi\mathrm{i}} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\mathrm{d}}{\mathrm{d}\theta} \log(e^{-k\mathrm{i}\theta/2}\widehat{f}(\theta))\, \mathrm{d}\theta \;=\; \frac{k}{12} - \frac{1}{2\pi\mathrm{i}} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\widehat{f}'(\theta)}{\widehat{f}(\theta)}\, \mathrm{d}\theta.$$

Since $N_\infty(f)$ is real-valued, we find

$$N_\infty(f) \;=\; \frac{k}{12} - \frac{1}{2\pi}\mathrm{Im}\left(\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\widehat{f}'(\theta)}{\widehat{f}(\theta)}\, \mathrm{d}\theta\right).$$

We have the following interpretation for the latter integral. Write $\widehat{f}(\theta) = r(\theta)e^{2\pi\mathrm{i}\alpha(\theta)}$, where r and α are real-valued continuous functions, i.e., r is the radius of $\widehat{f}$ and α is called the *continuous argument* of $\widehat{f}$. Recall that by assumption f has no zeros on $\mathcal{C}$, so $r(\theta) > 0$ for all θ. Then,

$$N_\infty(f) \;=\; \frac{k}{12} - \Big(\alpha\Big(\frac{2\pi}{3}\Big) - \alpha\Big(\frac{\pi}{3}\Big)\Big). \tag{13}$$

In order to compute *the variation of the argument* $\alpha(\frac{2\pi}{3}) - \alpha(\frac{\pi}{3})$, we first determine all $\theta \in [\pi/3, 2\pi/3]$ for which $\alpha(\theta) \in \frac{1}{2}\mathbb{Z}$, or equivalently, for which $\mathrm{Im}\,(\widehat{f}) = 0$. By making use of the assumption that our quasimodular form f has *real* Fourier coefficients, we obtain:

Lemma 3.1. *We have*

$$\mathrm{Im}\,(\widehat{f}) \;=\; \frac{\mathrm{i}}{2} \sum_{m\geq 1} \frac{1}{(2\pi\mathrm{i})^m m!} \widehat{\mathfrak{d}^m f}.$$

Proof. First, assume g is a modular form (rather than a *quasi*modular form) of homogeneous weight k. Then, using that g has real Fourier coefficients, have

$$\overline{\widehat{g}(\theta)} = e^{-\mathrm{i}k\theta/2} g\Big(\frac{-1}{e^{\mathrm{i}\theta}}\Big) = e^{-\mathrm{i}k\theta/2} e^{\mathrm{i}k\theta} g(e^{\mathrm{i}\theta})$$

Hence, $\mathrm{Im}\,\widehat{g} = 0$. Similarly, for $\widehat{E}_2$ one has

$$\overline{\widehat{E}_2(\theta)} = e^{-\mathrm{i}\theta} E_2\Big(\frac{-1}{e^{\mathrm{i}\theta}}\Big) = e^{-\mathrm{i}\theta}\big(e^{2\mathrm{i}\theta}E_2(e^{\mathrm{i}\theta}) + \tfrac{12}{2\pi i}e^{\mathrm{i}\theta}\big) = \widehat{E}_2(\theta) + \frac{12}{2\pi\mathrm{i}} \tag{14}$$

Hence,

$$\mathrm{Im}\,\widehat{E}_2^j = \frac{\mathrm{i}}{2}\sum_{m=1}^{j} \binom{j}{m}\Big(\frac{12}{2\pi\mathrm{i}}\Big)^m \widehat{E}_2^{j-m}$$

Applying this to the expansion $f = \sum_{j\geq 0} f_j\, E_2^j$ and using the expansion as in Remark 2, we find

$$\mathrm{Im}\,(\widehat{f}) \;=\; \frac{\mathrm{i}}{2}\sum_{m\geq 1}\sum_{j\geq m} \binom{j}{m}\Big(\frac{12}{2\pi\mathrm{i}}\Big)^m \widehat{f}_j \widehat{E}_2^{j-m} \;=\; \frac{\mathrm{i}}{2}\sum_{m=1}^{j} \frac{1}{(2\pi\mathrm{i})^m m!} \widehat{\mathfrak{d}^m f}. \qquad \square$$

Depth 1 We now restrict to irreducible quasimodular forms of depth 1, i.e., $f = f_0 + E_2 f_1$. Then, by the previous lemma we have $\mathrm{Im}\,(\widehat{f}) = \frac{3}{\pi} f_1$. We write

$$\frac{2\pi}{3} \geq \theta_1 > \ldots > \theta_n > \frac{\pi}{3}$$

for the zeros of $\theta \mapsto f_1(e^{\mathrm{i}\theta})$ on $(\frac{\pi}{3}, \frac{2\pi}{3}]$, counted with multiplicity. Recall that $\nu_\rho(f_1)$ denotes the order of vanishing of f_1 at ρ. Then, Equation (4) in Theorem 1.3 will follow from the following lemma and proposition.

Lemma 3.2. *The mapping $\iota : \theta_j \mapsto \theta_{n-j-\nu_\rho(f_1)+1}$ defines an involution on $\{\theta_j \mid \theta_j \neq \frac{2\pi}{3}\}$ such that*

$$(-1)^j \operatorname{sgn}(\widehat{f}(\theta_j)) = (-1)^{n-j-\nu_\rho(f_1)+1} \operatorname{sgn}(\widehat{f}(\theta_{n-j-\nu_\rho(f_1)+1})).$$

Proof. Note that if θ_j is the angle of an element on the unit disk for which f_1 has a zero, then $\pi - \theta_j$ also is such an angle. Leaving out the $\nu_\rho(f_1)$ angles $\theta_1 = \ldots = \theta_{\nu_\rho(f_1)} = \frac{2\pi}{3}$, we see that ι is a well-defined involution.

As $f_1(e^{\mathrm{i}\theta_j}) = 0$, we obtain

$$\widehat{f}(\theta_{n-j-\nu_\rho(f_1)+1}) \;=\; e^{\frac{1}{2}\mathrm{i}k(\pi-\theta_j)}\, f\Big(-\frac{1}{e^{\mathrm{i}\theta_j}}\Big) \;=\; e^{\frac{1}{2}\mathrm{i}k(\pi-\theta_j)}\, e^{k\mathrm{i}\theta_j}\, f(e^{\mathrm{i}\theta_j}) \;=\; (-1)^{k/2}\, \widehat{f}(\theta_j).$$

We finish the proof by showing that $\frac{k}{2} \equiv n - \nu_\rho(f_1) + 1 \mod 2$. Namely, $n - \nu_\rho(f_1)$ is odd precisely if f_1 admits a zero of odd order at i, or, equivalently, if $k - 2 \equiv 2, 6$ or $10 \mod 12$. We can exclude the case where $k \equiv 4\,(6)$. Namely, then both f_0 and f_1 are divisible by E_4, contradicting the irreducibility of f. Hence, $n - \nu_\rho(f_1)$ is odd if $k \equiv 0, 8 \mod 12$ and even if $k \equiv 2, 6 \mod 12$ as desired. □

Proposition 3.3. *For an irreducible quasimodular form f of weight k and depth 1, we have*

$$N_\infty(f) \;=\; \frac{1}{2}\left\lfloor \frac{k}{6} \right\rfloor - \frac{(-1)^{\nu_\rho(f_1)} r(f_1)}{2} \sum_j (-1)^j \operatorname{sgn}(\widehat{f}(\theta_j)).$$

Proof. The idea of the proof is to determine the value $\alpha(\frac{2\pi}{3}) - \alpha(\frac{\pi}{3})$ in (13). Denote by $A(\theta)$ the argument of $\widehat{f}$, i.e., the unique value in $(-\frac{1}{2}, \frac{1}{2}]$ such that $\alpha(\theta) \equiv A(\theta) \mod 1$.

As α is real analytic, this value can uniquely be determined by knowing $A(\frac{\pi}{3}), A(\frac{2\pi}{3})$ and all the values of θ for which $A(\theta) \in \{0, \frac{1}{2}\}$. For example, if $0 < A(\frac{2\pi}{3}) < \frac{1}{2}$, and $A(\theta_1) = \frac{1}{2}$, whereas $A(\theta_2) = 0$, then for $\frac{2\pi}{3} > \theta \geq \theta_2$, α increases by $1 - A(\frac{2\pi}{3})$. Observe that $A(\theta) \in \{0, \frac{1}{2}\}$ precisely if $\mathrm{Im}\, f(e^{\mathrm{i}\theta}) = 0$, or equivalently, $f_1(e^{\mathrm{i}\theta}) = 0$.

Now, in order to compute the value of $\alpha(\frac{2\pi}{3}) - \alpha(\frac{\pi}{3})$, first assume that all zeros of $\theta \mapsto f_1(e^{\mathrm{i}\theta})$ on $(\frac{\pi}{3}, \frac{2\pi}{3}]$ are simple and satisfy $\theta \in \{\frac{\pi}{3}, \frac{\pi}{2}\}$. Whether $f_1(e^{\mathrm{i}\theta}) = 0$ for such $\theta \in \{\frac{\pi}{3}, \frac{\pi}{2}\}$ (or, equivalently, $A(\frac{2\pi}{3}) \in \{0, \frac{1}{2}\}$) is determined by the value of k modulo 12, see below. Note that as f is *irreducible*, we have $k \not\equiv 4\,(6)$.

$k \mod 12$	$A(\frac{2\pi}{3}) \in \{0, \frac{1}{2}\}$	$A(\frac{\pi}{2}) \in \{0, \frac{1}{2}\}$
0	✓	✓
2	x	x
6	✓	x
8	x	✓

Temporarily, denote by φ_i the elements of $\{2\pi/3, \pi/2\}$ for which $\theta \mapsto f_1(e^{\mathrm{i}\theta})$ admits a zero, and such that $\varphi_1 \geq \varphi_2$. As f is irreducible, we have $\widehat{f}(\varphi_i) \neq 0$, so that $\operatorname{sgn}(\widehat{f}(\varphi_i))$ is well-defined. The sign being positive (or negative) corresponds to $\alpha(\varphi_i) \equiv 0 \mod 1$ (or $\frac{1}{2} \mod 1$

respectively). By Lemma 2.4 we have $(-1)^{\nu_\rho(f_1)} r(f_1) = \lim_{\theta\uparrow 2\pi/3} \operatorname{sgn}(e^{ki\theta/2} f_1(e^{i\theta}))$ with $\nu_\rho(f_1)$ the order of vanishing of f_1 at ρ. Hence, a case-by-case analysis using the symmetry $\operatorname{Im} \widehat{f}(\theta) = (-1)^{k/2+1} \operatorname{Im} \widehat{f}(\pi-\theta)$ shows

$$A\Big(\frac{2\pi}{3}\Big) - A\Big(\frac{\pi}{3}\Big) \;-\; \frac{(-1)^{\nu_\rho(f_1)} r(f_1)}{2} \sum_j (-1)^j \operatorname{sgn}(\widehat{f}(\varphi_j)) \;=\; \begin{cases} 0 & k \equiv 0\,(6) \\ \frac{1}{6} & k \equiv 2\,(6). \end{cases}$$

Now, in the general case, note that the contribution to the variation of the argument on each interval $[\theta_j, \theta_{j+1}]$ is

$$\frac{(-1)^{\nu_\rho(f_1)} r(f_1)}{4} \Big((-1)^j \operatorname{sgn}(\widehat{f}(\theta_j)) + (-1)^{j+1} \operatorname{sgn}(\widehat{f}(\theta_{j+1})) \Big).$$

Adding these contributions with special care at the boundary cases as above leads to the result

$$\alpha\Big(\frac{2\pi}{3}\Big) - \alpha\Big(\frac{\pi}{3}\Big) \;-\; \frac{(-1)^{\nu_\rho(f_1)} r(f_1)}{2} \sum_j (-1)^j \operatorname{sgn}(\widehat{f}(\theta_j)) \;=\; \begin{cases} 0 & k \equiv 0\,(6) \\ \frac{1}{6} & k \equiv 2\,(6). \end{cases}$$

By Equation (13) the result follows. □

Remark 7. For a mixed modular form $F = \sum_{j=0}^p f_j$ with f_j of weight $k-2j$, we analogously find

$$\operatorname{Im}(\widehat{F}) = \sum_{j\geq 1} \widehat{f}_j(\theta) \sin(j\theta).$$

From this we similarly deduce that for a mixed modular form $F = f_0 + f_j$ (with f_j of weight $k-2j$) we have

$$N_\infty(F) \;=\; \frac{1}{2}\left\lfloor \frac{k}{6} \right\rfloor - \frac{(-1)^{\nu_\rho(f_1)} r(f_j)}{2} \sum_i (-1)^i \operatorname{sgn}(\widehat{F}(\theta_i)),$$

where, accordingly, the θ_i are the zeros of $\theta \mapsto f_j(e^{i\theta})$. △

Examples in depth 1

Example 3.4. Consider $f = E_2$. In this case, $f_0 \equiv 0$ and $f_1 \equiv 1$. As f_1 has no zeros on the arc, application of Proposition 3.3 gives

$$N_\infty(E_2) = \frac{1}{2}\left\lfloor \frac{2}{6} \right\rfloor = 0.$$

Hence, E_2 has no zeros in the standard fundamental domain—a result which was discovered and proven in [EBS10, Proposition 4.2] by different means.

Example 3.5. We now return to the example in the introduction, i.e., let f be the unique quasimodular form $f = f_0 + f_1 E_2$ in the 7-dimensional vector space $M_{36}^{\leq 1}$ with q-expansion $f = 1 + O(q^7)$. In order to apply Theorem 1.3, we compute f_0 and f_1 explicitly:

$$f_0 \;=\; \frac{43976643}{108264772}\Delta^3 \left(j^3 - \frac{28903981960}{14658881} j^2 \;+\; \frac{9706007861928}{14658881} j \;+\; \frac{396402626858112}{14658881} \right)$$

and

$$f_1 \;=\; \frac{64288129}{108264772}E_4E_6\Delta^2\left(j^2 - \frac{2225338584}{1737517}j + \frac{373036607496}{1737517}\right),$$

where j is the modular j-invariant, given by $j = 1728\frac{E_4^3}{E_4^3-E_6^2}$. We find that f_1 has zeros at i and ρ, coming from the factors E_4E_6. Moreover, the roots of the degree 2 polynomial in the j-invariant are given by $j(\tau_1) \approx 198.3495\ldots$ and $j(\tau_2) \approx 1082.4083\ldots$. Recall $j(\mathcal{L}) = (-\infty, 0]$ and $j(\mathcal{C}) = [0, 1728]$, where $\mathcal{L}$ and $\mathcal{C}$ are the left vertical and circular boundary of the fundamental domain as in (9) and (10). Therefore, the zeros of f_1 in $\mathcal{F}$ are all located on $\mathcal{C}$. Similarly, the zeros of f_0 are τ_3, τ_4, τ_5, where $j(\tau_3) \approx -36.7451\ldots$, $j(\tau_4) \approx 482.1402\ldots$ and $j(\tau_5) \approx 1526.3776\ldots$, indicating that τ_4 and τ_5 lie on $\mathcal{C}$ (and τ_3 lies on $\mathcal{L}$).

As $\Delta(\rho) < 0$ and $j(\rho) = 0$, we have that

$$f(\rho) \;=\; \widehat{f}\Big(\frac{2\pi}{3}\Big) \;=\; \widehat{f_0}\Big(\frac{2\pi}{3}\Big) \;=\; f_0(\rho) \;<\; 0.$$

From the location of the zeros of f_0 on $\mathcal{C}$, we conclude (writing $\tau_j = e^{\mathrm{i}\theta_j}$ for $\frac{\pi}{2} \leq \theta_j \leq \frac{2\pi}{3}$ if τ_j is located on $\mathcal{C}$)

$$\begin{aligned}\widehat{f}(\theta_1) \;&=\; \widehat{f_0}(\theta_1) \;<\; 0,\\ \widehat{f}(\theta_2) \;&=\; \widehat{f_0}(\theta_2) \;>\; 0,\\ \widehat{f}(\tfrac{\pi}{2}) \;&=\; \widehat{f_0}(\tfrac{\pi}{2}) \;<\; 0.\end{aligned}$$

Further, $f_1(-\frac{1}{2}+\mathrm{i}\infty) > 0$ and f_1 does not change sign on $\mathcal{L}$, as it has no zeros there. Therefore, $r(f_1) = 1$ and $(-1)^{\nu_\rho(f_1)} = -1$. We now apply Proposition 3.3 (in the form of Equation (4)):

$$N_{(1,\infty]}(f) \;=\; \frac{1}{2}\left\lfloor\frac{36}{6}\right\rfloor \;+\; \Big(\frac{-1}{2}\cdot -1 + 1\cdot -1 + -1\cdot 1 + \frac{1}{2}\cdot -1\Big) \;=\; 1.$$

We come back to this example in Example 5.4.

Example 3.6. For $k = 6n$ ($n = 1, 2, 3, \ldots$), we consider the Kaneko–Zagier differential equation

$$f''(\tau) - \frac{k}{6}E_2(\tau)f'(\tau) + \frac{k(k-1)}{12}E_2'(\tau)f(\tau) = 0.$$

In [KK06, Theorem 2.1], it was shown that a solution to this equation is given by an *extremal* quasimodular form g of depth 1 and weight k, i.e.

$$g(\tau) \;=\; cq^{m-1} + \mathcal{O}(q^m),$$

where $c \neq 0$ and m is the dimension of the space of weight k forms of depth 1 (i.e., $m = n+1$). Clearly, g has a zero at $i\infty$ of order $m - 1 = \frac{k}{6}$. From Proposition 3.3 we learn that g has no other zeros in $\mathcal{F}$ (see also Corollary 1.5).

Examples in higher depth In depth 1 we have seen that the number of zeros of a quasimodular forms $f = f_0 + f_1E_2$ only depends on the sign of $\widehat{f_0}$ in the zeros of f_1 on the arc $\mathcal{C}$. The next example shows that this is not anymore the case for higher depth.

Example 3.7. Consider the following quasimodular form of weight 4 and depth 1 for a real parameter t

$$f_t = E_4 - t\,E_2^2.$$

We are interested in the value of $N_\infty(f_t)$. By Lemma 3.1 we have

$$\mathrm{Im}\,(\widehat{f_t}) \;=\; -t\frac{6}{\pi}\Big(\widehat{E}_2 + \frac{3}{\pi\mathrm{i}}\Big).$$

It can be seen that $\mathrm{Im}\,(\widehat{f_t})$ only vanishes once, at $\theta_0 = \frac{\pi}{2}$. Hence,

$$\widehat{f_t}(\theta_0) \;=\; \widehat{E}_4(\theta_0) + t\frac{9}{\pi^2}.$$

Now, first assume $t<0$. As $\widehat{E}_4 < 0$ on $(\frac{\pi}{3}, \frac{2\pi}{3})$, we have $\widehat{f_t}(\theta_0) < 0$. Also $\widehat{f_t}(\pi/3) = re^{2\pi\mathrm{i}/3}$ for some positive r and $\widehat{f_t}(2\pi/3) = re^{4\pi i/3}$. This means that $\widehat{f_t}(\theta)$ with $\theta \in (\frac{\pi}{3}, \frac{2\pi}{3})$ moves from $re^{2\pi\mathrm{i}/3}$ to $re^{4\pi\mathrm{i}/3}$, crossing the (negative) real axis exactly once. Hence, the variation of the argument $\alpha(\frac{2\pi}{3}) - \alpha(\frac{\pi}{3}) = \frac{2\pi}{3}$. Therefore,

$$N_\infty(f_t) = \frac{4}{12} - \frac{1}{2\pi}\frac{2\pi}{3} = 0,$$

for $t<0$.

For $t>0$, we have two cases. Let

$$t_1 \;:=\; -\frac{\pi^2}{9}\widehat{E}_4(\theta_0) \approx 1.596...$$

Now assume $0 < t < t_1$. Since $t>0$, we have $\widehat{f_t}(\pi/3) = se^{5\pi\mathrm{i}/3}$ for some positive s and $\widehat{f_t}(2\pi/3) = se^{\pi\mathrm{i}/3}$. Since $t<t_1$, we still have $\widehat{f_t}(\theta_0) < 0$. Hence the variation of the argument is now $-\frac{4\pi}{3}$. Therefore,

$$N_\infty(f_t) = \frac{4}{12} - \frac{1}{2\pi}\cdot\frac{-4\pi}{3} = 1.$$

For the case $t > t_1$, we have that $\widehat{f_t}(\theta_0) > 0$. So the variation of the argument equals $\frac{2\pi}{3}$ in that case, and $N_\infty(f_t) = 0$.

We conclude that

$$N_\infty(f_t) = \begin{cases} 0 & \text{if } t<0 \text{ or } t>t_1 \\ 1 & \text{if } 0<t<t_1\,. \end{cases}$$

Example 3.8. Write $f = f_0 + E_2^p$, where f_0 is a modular form of weight $2p>0$. In this case,

$$\mathrm{Im}\,(\widehat{f}) = \mathrm{Im}\,(\widehat{E}_2^p).$$

Using Proposition A.1 below, the function $\widehat{f}$ crosses the imaginary axis exactly $p-1-2\lfloor\frac{p}{3}\rfloor$ times. This means that the variation of the argument of $\widehat{f}$ along $\mathcal{C}$ is at most $\pi\big(p-2\lfloor\frac{p}{3}\rfloor\big)$. Therefore,

$$N_\infty(f) \;\leq\; \frac{1}{2}\Big\lfloor\frac{2p}{6}\Big\rfloor + \frac{1}{2}\Big(p - 2\Big\lfloor\frac{p}{3}\Big\rfloor\Big) \;\leq\; \frac{p+1}{3}.$$

4. Vector-valued equivariant forms

By the quasimodular transformation equation (8)

$$(f|_k\gamma)(\tau) \;=\; \sum_{j=0}^{p}\frac{(\mathfrak{d}^j f)(\tau)}{j!}\Big(\frac{1}{2\pi\mathrm{i}}\frac{1}{\tau-\lambda}\Big)^j,$$

where $\lambda = -\frac{d}{c}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We study the solutions $h : \mathfrak{h} \to \mathbb{C}$ of

$$0 \;=\; \sum_{j=0}^{p} \frac{(\mathfrak{d}^j f)(\tau)}{j!}\Big(\frac{1}{2\pi\mathrm{i}}\frac{1}{\tau - h(\tau)}\Big)^{j}. \tag{15}$$

The main property of the solutions h is that f has a zero at $\gamma\tau$ with $\tau \in \mathcal{F}$ if and only if there is a solution satisfying $h(\tau) = \lambda$. Another property is that if $h_1(\tau), \ldots, h_p(\tau)$ different solutions (for a fixed $\tau \in \mathfrak{h}$), we have

$$\prod_i (h_i(\tau) - \lambda) = (\tau - \lambda)^p \frac{(f|_k\gamma)(\tau)}{f(\tau)}, \tag{16}$$

where as always $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $\lambda = -\frac{d}{c}$.

We now study the invariance of the solutions h under $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 4.1. *If $h : \mathcal{F} \to \mathbb{C}$ is a solution of* (15) *for $\tau = \tau_0$, then*

(i) $\gamma h = \dfrac{ah + b}{ch + d} : \mathcal{F} \to \mathbb{C}$ *is a solution of* (15) *for $\tau = \gamma\tau_0$.*

(ii) $-\overline{h} : \mathcal{F} \to \mathbb{C}$ *is a solution of* (15) *for $\tau = -\overline{\tau_0}$.*

Proof. It suffices to show the first part for the two generators $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$. For the first, the result is almost immediate, so we only prove it for the inversion in the unit disk. This follows from the following computation:

$$\begin{aligned}
&\sum_{j=0}^{p} \frac{(\mathfrak{d}^j f)(-\tau^{-1})}{j!}\Big(\frac{1}{2\pi\mathrm{i}}\frac{1}{-\tau^{-1} + h(\tau)^{-1}}\Big)^{j} \\
&\quad= \tau^k \sum_{j=0}^{p}\sum_{m=0}^{p-j} \frac{(\mathfrak{d}^{j+m} f)(\tau)}{j!m!}\Big(\frac{1}{2\pi\mathrm{i}}\frac{1}{-\tau + \tau^2 h(\tau)^{-1}}\Big)^{j}\Big(\frac{1}{2\pi\mathrm{i}}\frac{1}{\tau}\Big)^{m} \\
&\quad= \tau^k \sum_{\ell=0}^{p} \frac{(\mathfrak{d}^\ell f)(\tau)}{\ell!}\Big(\frac{1}{2\pi\mathrm{i}}\frac{-\tau + \tau^2 h(\tau)^{-1} + \tau}{\tau(-\tau + \tau^2 h(\tau)^{-1})}\Big)^{\ell} \\
&\quad= \tau^k \sum_{\ell=0}^{p} \frac{(\mathfrak{d}^\ell f)(\tau)}{\ell!}\Big(\frac{1}{2\pi\mathrm{i}}\frac{1}{-h(\tau) + \tau}\Big)^{\ell} = 0.
\end{aligned}$$

The second statement follows from the fact that for quasimodular forms g with real Fourier coefficients one has $g(-\overline{\tau}) = g(\tau)$. □

Extend the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}$ to an action of $\mathrm{GL}_2(\mathbb{Z})$ by

$$\gamma\tau = \frac{a\overline{\tau} + b}{c\overline{\tau} + d} \qquad \text{if } \det\gamma = -1$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{Z})$ and $\tau \in \mathfrak{h}$. Then, we find that the vector $\underline{h} = (h_1, \ldots, h_p)$ is a meromorphic vector-valued equivariant form for $\mathrm{GL}_2(\mathbb{Z})$:

Corollary 4.2. *Let U be a simply connected open subset of $\mathfrak{h}$ for which the p solutions of* (15) *are distinct. Then, one can choose solutions $h_1, \ldots, h_p : U \to \mathbb{C}$ of* (15) *such that*

(i) *for all $\tau \in U$ and $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ the solutions $h_j(\gamma\tau) : \gamma U \to \mathbb{C}$ are meromorphic;*

(ii) *if* $\Gamma \leq \mathrm{GL}_2(\mathbb{Z})$ *such that* $\Gamma U \subseteq U$*, then there exists a homomorphism* $\sigma : \Gamma \to \mathfrak{S}_p$ *such that for all* $\tau \in U$ *and* $\gamma \in \Gamma$ *one has*

$$h_j(\gamma\tau) = \gamma\, h_{\sigma(\gamma)j}(\tau);$$

(iii) f *has a zero at* $\gamma\tau$ *if and only if* $h_j(\tau) = \lambda(\gamma)$ *for some* j.

Proof. By the implicit function theorem, there exist p meromorphic solutions h_j on U, which, by construction, satisfy the third property. By the previous proposition for all $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ we have $h_j(\gamma\tau) = \gamma\, h_{\sigma(\gamma)j}(\tau)$ for some $\sigma(\gamma) \in \mathfrak{S}_p$, possibly depending on τ. However, by continuity of h_j on U, we find $\sigma(\gamma)$ does not depend on τ for $\gamma \in \Gamma$. In particular, $h_j : \gamma U \to \mathbb{C}$ is meromorphic and σ is easily seen to be a homomorphism. □

We often make use of the fact that h_j is a vector-valued equivariant function in the following way. Write

$$C = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \quad T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$$

for complex *C*onjugation, *S*piegeln (reflecting) in the unit disc and *T*ranslation. Then,

$$\mathrm{PGL}_2(\mathbb{Z}) = \langle C, S, T \mid C^2 = 1, (CT)^2 = 1, (CS)^2 = 1, S^2 = 1, (ST)^3 = 1\rangle.$$

Given $\gamma \in \mathrm{PGL}_2(\mathbb{Z})$, let $\mathbb{C}^\gamma$ be the set of $\tau \in \mathbb{C}$ such that $\gamma\tau = \tau$. Then, for all $\tau \in U$ we have

$$h_j(\tau) = h_j(\gamma\tau) = \gamma h_{\sigma(\gamma)j}(\tau).$$

Hence, if $\sigma(\gamma) = e$, then $h_j(\tau) \in \mathbb{C}^\gamma$. For example, we have proven the following lemma.

Lemma 4.3. *Given the solutions* h_j *and* $\sigma : \Gamma \to \mathfrak{S}_p$ *as in Corollary 4.2, write* $\Gamma_j = \{\gamma \in \Gamma \mid \sigma(\gamma)j = j\}$*. Then,*

(i) $h_j(\frac{n}{2} + \mathbb{R}\mathrm{i}) \in \frac{n}{2} + \mathbb{R}\mathrm{i}$ *if* $n \in \mathbb{Z}$ *is such that* $CT^{2n} \in \Gamma_j$,

(ii) $|h_j(z)| = 1$ *for* $|z| = 1$ *if* $CS \in \Gamma_j$,

(iii) $h_j(\rho) \in \{\pm\rho\}$ *if* $ST \in \Gamma_j$.

Depth 1 For a quasimodular form $f = f_0 + E_2 f_1$ of depth 1, we have $h : \mathfrak{h} \to \mathbb{C}$ is a holomorphic equivariant function, i.e.,

$$h(\gamma\tau) = \gamma h(\tau)$$

for all $\tau \in \mathfrak{h}$ and $\gamma \in \mathrm{GL}_2(\mathbb{Z})$. In fact, we can write h as

$$h(\tau) \;=\; \tau + \frac{12}{2\pi\mathrm{i}}\frac{f_1(\tau)}{f(\tau)} \;=\; \tau\frac{f|S(\tau)}{f(\tau)},$$

where $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Remark 8. There are several additional interesting properties of equivariant functions h of which we do not make use in this work. Among those are:

(i) The Schwarzian derivative $\{h(\tau), \tau\}$ is a meromorphic modular form of weight 4. This follows directly from the properties of the Schwarzian derivative, see, e.g., [EBS12].

(ii) The derivative h' equals Ff^{-2} for some (holomorphic) modular form F of weight $2k$ (where k is the weight of f). Explicitly,

$$F \;=\; f_0^2 \;+\; 12\, f_0\, \vartheta(f_1) \;-\; 12\, \vartheta(f_0)\, f_1 + f_1^2\, E_4\,,$$

where ϑ denotes the Serre derivative. This was observed and proven in [GO22, Section 5.3] in the case f is the derivative of a modular form. △

Example in depth 1

Example 4.4. If $f = g' = \frac{\partial g}{\partial \tau}$ with g a modular form of weight k, then $h(\tau) = \tau + k\frac{g(\tau)}{g'(\tau)}$ (to the study of which [EBS12] is devoted). In particular, for $f = \frac{1}{2\pi \mathrm{i}}\Delta' = \Delta E_2$ we find $h(\tau) = \tau + \frac{12}{2\pi \mathrm{i}}\frac{1}{E_2}$, which is also the equivariant function associated to E_2.

Examples in higher depth

Example 4.5. Consider $f = E_4 + E_2^2$. Then the h_j for $j = 1, 2$ are solutions of the equation

$$0 = (2\pi \mathrm{i})^2(E_4(\tau) + E_2(\tau)^2)(\tau - h(\tau))^2 + 48\pi \mathrm{i} E_2(\tau)(\tau - h(\tau)) + 144.$$

From this

$$h_j(\tau) = \tau + \frac{12}{2\pi \mathrm{i}}\frac{E_2(\tau) + (-1)^j\sqrt{-E_4(\tau)}}{(E_4(\tau) + E_2(\tau)^2)},$$

which (for an appropriate choice of the square root) is meromorphic on every simply connected domain U not containing a $\mathrm{SL}_2(\mathbb{Z})$-translate of ρ. For example, for

$$U \;=\; \mathrm{int}(\mathcal{F} \cup S\mathcal{F}) \;=\; \{z \in \mathfrak{h} \mid -\tfrac{1}{2} < \mathrm{Re}\,(z) < \tfrac{1}{2}, |z-1| > 1 \text{ and } |z+1| > 1\}$$

we have that $\Gamma = \langle C, S\rangle$ satisfies $\Gamma U = U$ and $\sigma : \Gamma \to \mathfrak{S}_2$ is given by $\sigma(C) = (1\,2), \sigma(S) = (1\,2)$. In particular, $CS \in \ker \sigma$. Hence, by Lemma 4.3 we have $|h_j(z)| = 1$ if $|z| = 1$. However, it is not the case that, e.g., $h_j(-\frac{1}{2} + \mathbb{R}\mathrm{i}) \in -\frac{1}{2} + \mathbb{R}\mathrm{i}$.

5. Zeros in other fundamental domains ($\lambda < \infty$)

By definition of the functions h_i we have

$$N_\lambda(f) - N_\infty(f) \;=\; \frac{1}{2\pi \mathrm{i}}\sum_i \int_{\partial\mathcal{F}} \frac{h_i'(\tau)}{h_i(\tau) - \lambda}\,\mathrm{d}\tau \;=\; \frac{1}{2\pi}\sum_i \mathrm{Im}\int_{\partial\mathcal{F}} \frac{h_i'(\tau)}{h_i(\tau) - \lambda}\,\mathrm{d}\tau,$$

where the second equality holds as the number of zeros of a function is a real number. Hence, the value of $N_\lambda(f) - N_\infty(f)$ is determined by the variation of the argument of the $h_i - \lambda$.

Proof of Theorem 1.2. Given $\lambda \in \mathbb{R}$, we consider the variation of the argument of $h_j(\tau) - \lambda$ for all j. Note that, when moving along $\tau \in \partial\mathcal{F}$, by Corollary 4.2 the functions h_i are continuous and piecewise meromorphic. We change the contour $\partial\mathcal{F}$ to a family of contours $\mathscr{C}_\epsilon$ such that for $\epsilon > 0$ sufficiently small there are no poles on the contour of integration, and that the functions h_j only take finitely many real values. (For example, we could define $\mathscr{C}_\epsilon$ to be the shift of the contour $\partial\mathcal{F}$ by $(|\mathrm{Re}\,(z)| + \sqrt{3}\mathrm{Re}\,(z)\mathrm{i})\epsilon + \frac{1}{2}\epsilon^2$. Note that for $\epsilon \to 0$ the value of the integral over the shifted contour converges to the desired value N_λ.)

The functions $\tau \mapsto h_j(\tau)$ intersect the real axis only a finite number of times as τ goes over the (shifted) contour. Write $\lambda_1(\epsilon) < \ldots < \lambda_{n(\epsilon)}(\epsilon)$ for the intersection points for all functions $h_1, \ldots, h_p$ (here $n(\epsilon)$ may also depend on ϵ). Moreover, write $\lambda_1, \ldots, \lambda_n$ for the limiting values of $\lambda_i(\epsilon)$ as $\epsilon \to 0$. As $h_j - \lambda$ is just a horizontal shift of h_j, given $\lambda, \lambda' \in \mathbb{R}$, the functions $h_j - \lambda$ and $h_j - \lambda'$ admit the same variation of the argument if there is no ℓ such that $\lambda < \lambda_\ell < \lambda'$ or $\lambda' < \lambda_\ell < \lambda$. Hence, in that case $N_\lambda - N_\infty = N_{\lambda'} - N_\infty$. Moreover, for $\lambda > \lambda_n$ the variation of the argument is 0. Hence, $N_\lambda - N_\infty = 0$ for $\lambda > \lambda_n$. We conclude that if we define the elements of $\mathscr{I}$ to be

$$(-\infty, \lambda_1), \{\lambda_1\}, (\lambda_1, \lambda_2), \ldots, \{\lambda_n\}, (\lambda_n, \infty)$$

the statement follows. (In case $\lambda_i = \pm\infty$ simply leave out the corresponding sets.) $\square$

Depth 1 For quasimodular forms of depth 1, by Lemma 4.3 we have

- $h(\frac{1}{2} + it) \in \frac{1}{2} + i\mathbb{R}$ for $t \in \mathbb{R}$;
- $|h(z)| = 1$ if $|z| = 1$.

Hence, the only possible values of λ_i in the above proof are $\pm\frac{1}{2}, \pm 1$ and $\pm\infty$. Therefore, we obtain the following corollary of Theorem 1.2.

Corollary 5.1. *For a quasimodular form of depth* 1 *with real Fourier coefficients, there exist constants* $N_{[0,\frac{1}{2})}(f), N_{(\frac{1}{2},1)}(f), N_{(1,\infty)}(f)$ *such that*

$$N_\lambda(f) = \begin{cases} N_{[0,\frac{1}{2})}(f) & |\lambda| \in (0, \frac{1}{2}) \\ N_{(\frac{1}{2},1)}(f) & |\lambda| \in (\frac{1}{2}, 1) \\ N_{(1,\infty)}(f) & |\lambda| \in (1, \infty). \end{cases}$$

Next, by relating $N_\lambda(f)$ to $N_{-\frac{1}{\lambda}}(f)$ and by using the above properties of h, we prove the following statement, which finishes the proof of Theorem 1.3. Recall $z_1, \ldots, z_m$ are the zeros of f_1 such that $\operatorname{Re} z_i = -\frac{1}{2}$ and $\operatorname{Im} z_i > \frac{1}{2}\sqrt{3}$, counted with multiplicity and ordered by imaginary part, and $z_0 = \rho$. Moreover, recall $r(f_1)$ denotes the sign of the first non-zero Taylor coefficient of f_1 (see (11)), and $s(f) = \operatorname{sgn} a_0(f)$ if f does not vanish at infinity, and $s(f) = -\operatorname{sgn} a_0(f_1)$ else. Also, $w(z_0) = 2$ if z_0 equals ρ, i or $-\frac{1}{2} + \mathrm{i}\infty$, and $w(z) = 1$ for all other $z \in \mathcal{F}$.

Theorem 5.2. *Let* $f = f_0 + E_2 f_1$ *be an irreducible quasimodular form of depth* 1*. If* $\lambda \in \mathbb{R}$ *with* $\frac{1}{2} < |\lambda| < 2$*, then*

$$N_\lambda(f) + N_{-\frac{1}{\lambda}}(f) \;=\; \left\lfloor \frac{k}{6} \right\rfloor.$$

Moreover, if $|\lambda| < \frac{1}{2}$ *or* $|\lambda| > 2$ *we have*

$$N_\lambda(f) + N_{-\frac{1}{\lambda}}(f) \;=\; \left\lceil \frac{k}{6} \right\rceil \;-\; r(f_1) \sum_{j=0}^{m} \frac{(-1)^j}{w(z_j)} \operatorname{sgn} f(z_j) - \frac{1}{2}(-1)^{m+1} r(f_1)\, s(f).$$

Proof. As before, we have

$$N_\lambda(f) + N_{-\frac{1}{\lambda}}(f) - 2N_\infty(f) = \frac{1}{2\pi}\mathrm{Im} \int_{\partial\mathcal{F}} \frac{h'(\tau)}{h(\tau) - \lambda} + \frac{h'(\tau)}{h(\tau) + \frac{1}{\lambda}}\, \mathrm{d}\tau. \tag{17}$$

We split this integral in several pieces, and compute the contribution of each piece separately. This argument resembles the one of Proposition 3.3, as well as the proof found in [GO22, Section 5.6].

Setup.

Let $z_1, \ldots, z_m$ be the zeros of f_1 on $\mathcal{L}\backslash\{\rho\}$ and let $v_1, \ldots, v_r$ be the zeros of f on $\mathcal{L}$, all ordered by imaginary part. We assume $r \geq 1$, and at the end of the argument verify the proof goes through if $r = 0$. Moreover, without loss of generality, we assume $|\lambda| > 1$.

The finite poles of h are exactly the finite zeros of f. We fix $\epsilon > 0$ sufficiently small. Let L_ϵ be the punctured left half line

$$[\rho, v_1 - \mathrm{i}\epsilon] \cup [v_1 + \mathrm{i}\epsilon, v_2 - \mathrm{i}\epsilon] \cup \ldots \cup [v_r + \mathrm{i}\epsilon, -\tfrac{1}{2} + \mathrm{i}\infty],$$

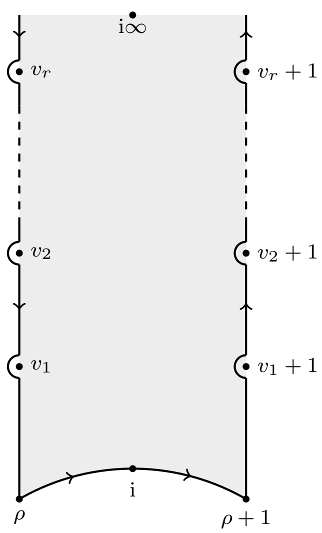

Figure 2: Contour of integration

and define the punctured right half line by $R_\epsilon = L_\epsilon + 1$. The line segment $[\rho, v_1 - \mathrm{i}\epsilon]$ (as well as its shift on R_ϵ) is referred to as the *lower vertical segment*, whereas $[v_r + \mathrm{i}\epsilon, -\frac{1}{2} + \mathrm{i}\infty]$ a called an *upper vertical segment*.

Recall that by definition of N_λ, we include all zeros/poles of $h(\tau) - \lambda$ and $h(\tau) + \frac{1}{\lambda}$ on $\mathcal{L}$ in the integral (17), but not those on $\mathcal{R}$. Hence, for each zero v_i we introduce a semicircle C_i around v_i of radius ϵ on the left of $\mathcal{L}$, as well as the semicircle $C_i + 1$. The boundary of $\mathcal{F}$ then consists of $\mathcal{C}, L_\epsilon, R_\epsilon$ and the semicircles C_i and $C_i + 1$ for all i.

Circular segment $\mathcal{C}$. Recall $|h(\tau)| = 1$ if $|\tau| = 1$. Hence, h has no poles on $\mathcal{C}$. Using the expression (16) above, we write

$$\begin{aligned}
&\frac{1}{2\pi}\mathrm{Im}\int_{\mathcal{C}} \frac{h'(\tau)}{h(\tau)-\lambda} + \frac{h'(\tau)}{h(\tau)+\frac{1}{\lambda}}\,\mathrm{d}\tau \\
&\quad= \frac{1}{2\pi}\mathrm{Im}\int_{\mathcal{C}} \frac{\partial}{\partial\tau}\log\big((h(\tau)-\lambda)(h(\tau)+\tfrac{1}{\lambda})\big)\,\mathrm{d}\tau \\
&\quad= \frac{1}{2\pi}\mathrm{Im}\int_{\mathcal{C}} \frac{\partial}{\partial\tau}\Big((\tau-\lambda)^{1-k}(\tau+\frac{1}{\lambda})^{1-k}\frac{f(\gamma\tau)f(\gamma S\tau)}{f(\tau)^2}\Big)\,\mathrm{d}\tau,
\end{aligned}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is such that $\lambda = -\frac{d}{c}$, and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$. Note that $\gamma S = \left(\begin{smallmatrix} -b & a \\ -d & c\end{smallmatrix}\right)$ and $-\frac{1}{\lambda} = -\frac{c}{-d}$.

First, we compute (recall $|\lambda| > 1$ by assumption)

$$\begin{aligned}
&\frac{1}{2\pi}\mathrm{Im}\int_{\mathcal{C}} \frac{\partial}{\partial\tau}\log\Big((\tau-\lambda)^{1-k}(\tau+\frac{1}{\lambda})^{1-k}\Big)\,\mathrm{d}\tau \\
&= \frac{k-1}{2\pi}\mathrm{Re}\int_{\pi/3}^{2\pi/3} -\frac{\frac{1}{\lambda}e^{\mathrm{i}\theta}}{1-\frac{1}{\lambda}e^{\mathrm{i}\theta}} + \frac{1}{1+\frac{1}{\lambda}e^{-\mathrm{i}\theta}}\,\mathrm{d}\theta \\
&= \frac{k-1}{2\pi}\mathrm{Re}\int_{\pi/3}^{2\pi/3} \Big(1 - 2\sum_{n\geq 1}\frac{1}{\lambda^{2n-1}}\cos((2n+1)\theta) + 2\mathrm{i}\sum_{m\geq 1}\frac{1}{\lambda^{2m}}\sin(2m\theta)\Big)\,\mathrm{d}\theta \\
&= \frac{k-1}{6}.
\end{aligned}$$

Next, we show that the following expression is actually independent of γ:

$$\begin{aligned}
&\frac{1}{2\pi}\mathrm{Im}\int_{\mathcal{C}} \frac{\partial}{\partial\tau}\log\Big(\frac{f(\gamma\tau)f(\gamma S\tau)}{f(\tau)^2}\Big)\,\mathrm{d}\tau \\
&\quad= \frac{1}{2\pi}\mathrm{Im}\int_{\mathcal{C}} \frac{1}{(c\tau+d)^2}\frac{f'(\gamma\tau)}{f(\gamma\tau)} + \frac{1}{(-d\tau+c)^2}\frac{f'(\gamma S\tau)}{f(\gamma S\tau)} - 2\frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau.
\end{aligned}$$

Applying the coordinate transformation $\tau \mapsto -\frac{1}{\tau}$ to the second term in the integrand, and using (12) for the last term, this equals

$$\frac{1}{2\pi}\mathrm{Im}\int_{\mathcal{C}} \frac{1}{(c\tau+d)^2}\frac{f'(\gamma\tau)}{f(\gamma\tau)} - \frac{1}{(c\tau+d)^2}\frac{f'(\gamma\tau)}{f(\gamma\tau)} - 2\frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau = -2N_\infty(f).$$

Hence, the contribution of $\mathcal{C}$ equals

$$\frac{k-1}{6} - 2N_\infty(f).$$

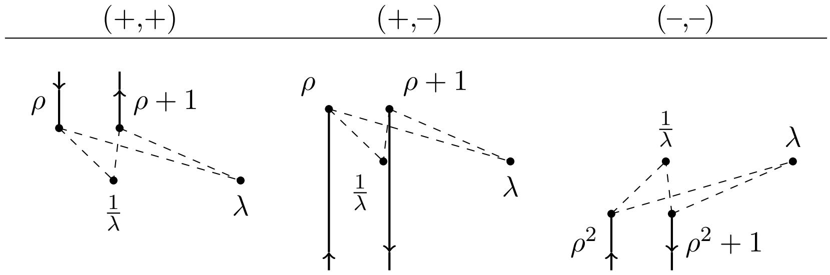

Table 1: The three possibilities for the graph of $h(\tau)$ for τ in a small neighborhood of ρ (resp. $\rho+1$) along the left (resp. right) vertical line segment of $\partial\mathcal{F}$, given $\big(\operatorname{sgn}\operatorname{Im}h(\rho), \operatorname{sgn}\lim_{\epsilon\downarrow 0}(\operatorname{Im}h(\rho+\epsilon\mathrm{i})-\rho)\big)\in\{\pm\}^2$.

	(+,+)	(+,–)	(–,–)
$\lvert\lambda\rvert > 2$	$\frac{1}{6}$	$-\frac{5}{6}$	$-\frac{1}{6}$
$1 < \lvert\lambda\rvert < 2$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$

Table 2: Variation of the argument of the lower vertical segments in several cases (see Table 1).

Lower vertical segments. Now, first assume that f_1 admits no zeros on the lower vertical segment $[\rho, v_1-\mathrm{i}\epsilon]$. By Lemma 4.3(i) we know that $h(\frac{1}{2}+\mathrm{i}t)\in\frac{1}{2}+\mathrm{i}\mathbb{R}$ for all $t\in\mathbb{R}$. Also, by Lemma 4.3(iii) we have $h(\rho)\in\{\pm\rho\}$. Hence, as τ moves on $[\rho, v_1-\mathrm{i}\epsilon]$ the function $h(\tau)$ moves from ρ or ρ^2 to $\mathrm{i}\infty$ or $-\mathrm{i}\infty$. We have three possibilities for the combined sign of $\operatorname{Im}h(\rho)$ and $\operatorname{Im}\big(h(\rho+\epsilon\mathrm{i})-\rho\big)$, depicted in Table 1. (Note that if the first is negative, the second is necessarily negative as well. As in this segment there are no zeros of f_1 or f, we know that in the case (–,–) eventually h tends to $-\mathrm{i}\infty$.).

Hence, the variation of the argument of $h-\lambda$ along $[\rho, v_1-\mathrm{i}\epsilon]$ and $[\rho, v_1-\mathrm{i}\epsilon]+1$ equals the (oriented) angle between $h(\rho+1),\lambda$ and $h(\rho)$, as shown in the same table. In particular, it is an exercise in Euclidean geometry that the sum of the oriented angles for λ and $-\frac{1}{\lambda}$ only depends on whether $|\lambda|<2$ or not (recall $|\lambda|>1$ by assumption). In Table 2 we displayed the contribution of each of the cases in Table 1. Correspondingly, the contribution to the variation of the argument equals

$$\frac{1}{6}\operatorname{sgn}\operatorname{Im}h(\rho)+\delta_{|\lambda|>2}\Big(-\frac{1}{2}\operatorname{sgn}\operatorname{Im}h(\rho)+\frac{1}{2}\operatorname{sgn}\lim\nolimits_{\epsilon\downarrow 0}(\operatorname{Im}h(\rho+\epsilon\mathrm{i})-\rho)\Big)$$

Observe that

$$\operatorname{sgn}\lim\nolimits_{\epsilon\downarrow 0}(\operatorname{Im}h(\rho+\epsilon\mathrm{i})-\rho)\;=\;-\operatorname{sgn}\lim\nolimits_{\epsilon\downarrow 0}\operatorname{sgn}(f_1(\rho+\epsilon\mathrm{i}))\operatorname{sgn}(f(\rho))$$

as $h(\tau)=\tau+\frac{12}{2\pi\mathrm{i}}\frac{f_1(\tau)}{f(\tau)}$. Hence, under the assumption f_1 has no zeros on the lower vertical segment and by Lemma 2.4, its contribution equals

$$\frac{1}{6}\operatorname{sgn}\operatorname{Im}h(\rho)\;+\;\delta_{|\lambda|>2}\Big(-\frac{1}{2}\operatorname{sgn}\operatorname{Im}h(\rho)\;-\;\frac{r(f_1)}{2}\operatorname{sgn}f(\rho)\Big).$$

Now, suppose f_1 admits p zeros on the lower vertical segment. Then, as f admits no zeros on this segment, $h(\tau)$ crosses the line τ precisely p times. If p is even, this does not alter the variation of the argument, but if p is odd, then $h(\tau)$ changes sign if τ tends to the pole v_1.

Note that this does not affect the variation of the argument of $h(\tau)-\mu$ if $|\mu|>1$. Suppose I is a line segment of $\mathcal{L}$ for which $h(\tau)-\mu$ tends to $\pm i\infty$ or to 0 on the two boundary points of I. In that case the variation of the argument on I equals v, whereas the variation of the argument on the corresponding line segment on $\mathcal{R}$ is $-v$. We conclude that if $|\mu|>1$, the only contribution for the variation of the argument is displayed in Table 2.

Hence, the contribution of the lower vertical segment equals

$$\frac{1}{6}\operatorname{sgn}\operatorname{Im} h(\rho) \;+\; \delta_{|\lambda|>2}\Big(-\frac{1}{2}\operatorname{sgn}\operatorname{Im} h(\rho) \;-\; \frac{r(f_1)}{2}\operatorname{sgn} f(\rho) \;-\; r(f_1)\sum_j (-1)^j \operatorname{sgn} f(z_j)\Big),$$

where the sum is over all j such that z_j lies in the lower vertical segment. Note that by the factor $r(f_1)(-1)^j$ we keep track of the sign of f_1 at z_j.

Vertical segments between two poles. Similar as in the previous case (now there are no boundary terms), we find

$$-\,\delta_{|\lambda|>2}\, r(f_1)\sum_j (-1)^j \operatorname{sgn} f(z_j)$$

where the sum is over all j such that z_j lies between two poles of f.

Semicircles centered at the poles. Note that for sufficiently small ϵ, we have that the value $h-\lambda$ on these semicircles is arbitrary large (say, bigger than $|\lambda|+1$ in absolute value). Moreover, the contours of $h-\lambda$ on such a semicircle C_i and C_i+1 differ only by 1. Hence, as $\epsilon\to 0$, the contributions of the corresponding semicircles C_i and C_i+1 of $\partial\mathcal{F}$ (which admit an opposite orientation) cancel in pairs.

Upper vertical segments. As before, the variation of the argument vanishes, except for $h(\tau)+\frac{1}{\lambda}$ if $|\lambda|>2$. Again, in this case, we have the contribution

$$-\,\delta_{|\lambda|>2}\, r(f_1)\sum_j (-1)^j \operatorname{sgn} f(z_j),$$

where the sum is over all j such that z_j lies in the upper vertical segment. This is also the only contribution, except in the following exceptional case. It may be that $h(\tau)$ is smaller than τ in imaginary value for $\tau=\frac{1}{2}+it$ with t tending to infinity, but still converging to $i\infty$. If this is the case, we have to add a contribution of $+1$. By considering the Fourier expansion of $h(\tau)-\tau$, we see this can only happen if f has no zero at infinity (else, h goes to $\pm i\infty$ at exponential rate). Moreover, the first non-zero Fourier coefficient of f_1/f should be positive if $h(\tau)-\tau\leq 0$ for $\tau=-\frac{1}{2}+it$. In case f does not vanish at infinity, we have

$$\lim_{t\to\infty}\operatorname{sgn}\operatorname{Im}\big(h(-\tfrac{1}{2}+it)-(-\tfrac{1}{2}+it)\big) = -\lim_{t\to i\infty}\operatorname{sgn}\big(f_1(-\tfrac{1}{2}+it)\, f(-\tfrac{1}{2}+it)\big),$$

Note that $\lim_{t\to\infty}\operatorname{sgn}(f_1(-\frac{1}{2}+it)=(-1)^m r(f_1)$. We found that this special contribution equals

$$\begin{cases}\frac{1}{2}+\frac{1}{2}(-1)^m r(f_1)\lim_{t\to\infty}\operatorname{sgn} f(-\frac{1}{2}+it) & \text{if } f\left(-\frac{1}{2}+i\infty\right)\neq 0\\ 0 & \text{if } f\left(-\frac{1}{2}+i\infty\right)=0,\end{cases}$$

As $\lim_{t\to\infty}\operatorname{sgn} f(-\frac{1}{2}+it)=\operatorname{sgn} a_0(f)$ if f does not vanish at the cusp, and $(-1)^m r(f_1)=\operatorname{sgn} a_0(f_1)$, by definition of $s(f)$, we find that the total contribution of the upper vertical segment for $|\lambda|>2$ equals

$$-\,r(f_1)\sum_{j=1}^{m}(-1)^j\operatorname{sgn} f(z_j)+\frac{1}{2}-\frac{1}{2}(-1)^{m+1}r(f_1)\,s(f)$$

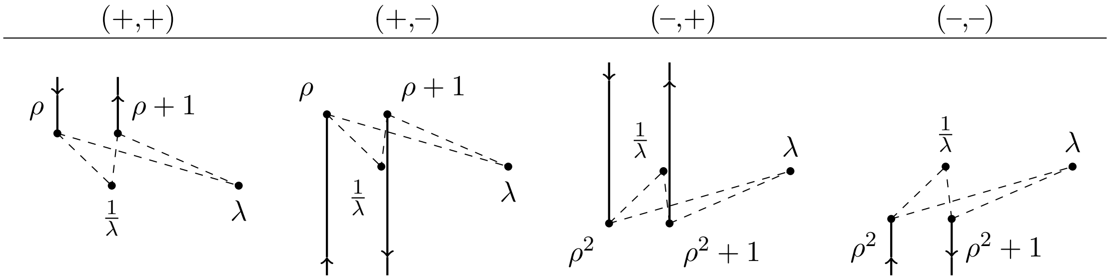

Table 3: The four possibilities for the graph of $h(\tau)$ for τ along the left (resp. right) vertical line segment of $\partial\mathcal{F}$, given $\big(\operatorname{sgn}\operatorname{Im}h(\rho), \lim_{t\to\infty}\operatorname{Im}\operatorname{sgn}h(-\frac{1}{2}+\mathrm{i}t)\big) \in \{\pm\}^2$.

	(+,+)	(+,–)	(–,+)	(–,–)
$\lvert\lambda\rvert > 2$	$\frac{1}{6}$	$-\frac{5}{6}$	$\frac{5}{6}$	$-\frac{1}{6}$
$1 < \lvert\lambda\rvert < 2$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$

Table 4: Variation of the argument along the left (resp. right) vertical line segment of $\partial\mathcal{F}$ in several cases (see Table 3).

Total contribution. Adding all contributions for $1 < |\lambda| < 2$, we obtain

$$\frac{k-1}{6} + \frac{1}{6}\operatorname{sgn}\operatorname{Im}h(\rho).$$

Note that $f_1(\rho) = 0$ if $k \equiv 0\,(6)$ and $f_0(\rho) = 0$ if $k \equiv 2\,(6)$. Hence,

$$h(\rho) = \rho + \begin{cases} 0 & k \equiv 0\,(6) \\ \frac{12}{2\pi\mathrm{i}}\frac{1}{E_2(\rho)} & k \equiv 2\,(6). \end{cases}$$

Therefore,

$$\operatorname{sgn}\operatorname{Im}h(\rho) \;=\; \begin{cases} 1 & k \equiv 0\,(6) \\ -1 & k \equiv 2\,(6), \end{cases}$$

We conclude that for $1 < |\lambda| < 2$, the variation of the argument equals $\lfloor \frac{k}{6} \rfloor$.

Adding all contributions for $|\lambda| > 2$, we obtain

$$\left\lceil \frac{k}{6} \right\rceil - \frac{r(f_1)}{2}\operatorname{sgn} f(\rho) - r(f_1)\sum_{j=1}^{m}(-1)^j \operatorname{sgn} f(z_j) - \frac{1}{2}(-1)^{m+1} r(f_1)\, s(f).$$

f has no zeros on $\mathcal{L}$. In this case h has no poles on $\mathcal{L}$ and $\mathcal{R}$. Therefore, the variation of the argument along $\mathcal{L}$ and $\mathcal{R}$ only depends on the values $h(\rho)$ and $h(-\frac{1}{2}+\mathrm{i}\infty)$. The image of h on $\mathcal{L}$ and $\mathcal{R}$ is summarized in Table 3. The contributions to the variation of the argument is given in the following Table 4. Note that the contributions in this table yield the same final formula for $N_\lambda(f) + N_{-\frac{1}{\lambda}}(f)$. □

Proof of Theorem 1.1. The result follows from Theorem 1.3, as we explain now. First, let φ be the unique modular form such that $f := g'/\varphi$ is irreducible (if g has only simple zeros, and no zeros at the cusp, then $\varphi = 1$).

Recall that the sign of the derivative of a real-valued differentiable function in two consecutive zeros of this function is opposite. Hence, for two consecutive zeros of g, the function

$$f \;=\; \frac{g'}{\varphi}$$

changes sign, i.e., $(-1)^j \operatorname{sgn} f(z_j)$ and $(-1)^j \operatorname{sgn} \widehat{f}(\theta_j)$ are independent of j. Comparing with the behaviour of f_1 around ρ one obtains

$$(-1)^j \operatorname{sgn} f(z_j) = -r(f_1) = (-1)^j \operatorname{sgn} \widehat{f}(\theta_j)$$

for all $j > 0$.

Moreover, we have

$$\operatorname{sgn} f(\rho) \;=\; \begin{cases} r(f_1) & k(f) \equiv 2 \mod 6 \\ -r(f_1) & k(f) \equiv 0 \mod 6, \end{cases}$$

where $k(f)$ denotes the weight of f. Namely, in case $k(f) \equiv 2 \mod 6$ we have $f_0(\rho) = 0$ and $f(\rho) = \operatorname{sgn} f_1(\rho)$, which is non-zero by definition of φ. Moreover, in case $k(f) \equiv 0 \mod 6$, we have

$$\operatorname{sgn} f(\rho) \;=\; \operatorname{sgn}\Big(\frac{g'}{\varphi}(\rho)\Big) \;=\; \operatorname{sgn}\Big(\frac{(f_1\varphi)'}{\varphi}\Big) \;=\; \operatorname{sgn}\Big(f_1'(\rho) + f_1(\rho)\frac{\varphi'}{\varphi}(\rho)\Big).$$

Note that $\varphi(\rho) \neq 0$ if $k(f) \equiv 0 \mod 6$, and $f_1(\rho) = 0$. Hence, we find

$$\operatorname{sgn} f(\rho) \;=\; \operatorname{sgn} f_1'(\rho) \;=\; -r(f_1).$$

Note that $a_0(f) = 0$ if and only if $a_0(g) \neq 0$. In case g has $n \geq 1$ roots of the cusps, observe that f and $n\frac{g}{\varphi}E_2$ have the same (non-zero) constant term. Also, observe that f_1 equals $\frac{g}{\varphi}$ up to a constant (being the weight of g, divided by 12). Hence, in that case we have

$$s(f) = \operatorname{sgn}(a_0(f)) = \operatorname{sgn}(a_0(g/\varphi)) = \operatorname{sgn}(a_0(f_1)) = (-1)^m r(f_1),$$

where the last equality holds as f_1 does not vanish at infinity. Hence,

$$-\frac{1}{2}(-1)^{m+1} r(f_1) s(f) \;=\; \begin{cases} -\frac{1}{2} & a_0(g) \neq 0 \\ \frac{1}{2} & a_0(g) = 0. \end{cases}$$

Finally, write δ' and ϵ' for the order of E_4 and E_6 in g'. Note that $k(f) \equiv k+2 \mod 6$ if g does not have repeated zeros at ρ and i (here k is the weight of g, or $k+2$ is the weight of k'). More generally,

$$k+2 \;\equiv\; k(f) + 4\delta' + 6\epsilon' \mod 6.$$

Observe the following identity

$$\frac{1}{2}\left\lfloor \frac{k+2}{6} - \frac{2}{3}\delta' \right\rfloor \;+\; \frac{1}{3}\delta' = \frac{k}{12} + \frac{1}{6}\delta_{g(\rho)=0}\,,$$

where $\delta_{g(\rho)=0}$ is 1 precisely if $g(\rho) = 0$ (if not, then $k \equiv 0 \mod 6$).

We conclude that

$$\begin{aligned} N_{(1,\infty]}(g') \;&=\; \frac{1}{2}\left\lfloor \frac{k+2-4\delta'-6\epsilon'}{6} \right\rfloor \;+\; \frac{2\delta'+3\epsilon'}{6} \;+\; C(g) \;+\; \frac{1}{6}\delta_{g(\rho)=0} \\ &=\; \frac{k}{12} \;+\; C(g) \;+\; \frac{1}{3}\delta_{g(\rho)=0}\,, \end{aligned}$$

$$N_{(1,\infty]}(g') + N_{(\frac{1}{2},1)}(g') \;=\; \left\lfloor \frac{k+2-4\delta'-6\epsilon'}{6} \right\rfloor \;+\; \frac{2\delta'+3\epsilon'}{3} \;=\; \frac{k}{6} + \frac{1}{3}\delta_{g(\rho)=0}$$

and $N_{(1,\infty]}(g') + N_{[0,\frac{1}{2})}(g')$ equals

$$\begin{aligned} &\left\lceil \frac{k+2-4\delta'-6\epsilon'}{6} \right\rceil \;+\; \frac{2\delta'+3\epsilon'}{3} \;+\; \frac{1}{2}\delta_{k(f)\equiv 4\,(6)} - \frac{1}{2}\delta_{k(f)\equiv 0\,(6)} + |\mathcal{L}(g)| - \frac{1}{2} \\ &= \left\lfloor \frac{k+2}{6} - \frac{2}{3}\delta' \right\rfloor \;+\; \frac{2}{3}\delta' \;+\; L(g) \\ &= \frac{k}{12} \;+\; L(g) + \delta_{g(\rho)=0}\,. \end{aligned}$$

□

Proof of Corollary 1.4. First of all, for f as in Theorem 1.3 we have

$$N_{(1,\infty]}(f) \;\leq\; \frac{1}{2}\left\lfloor \frac{k}{6} \right\rfloor + n' + \frac{1}{2}\delta_{k\equiv 0\,(6)},$$

where n' counts the weighted number of zeros of f_1 on the part of the unit circle with angle θ such that $\frac{\pi}{2} \leq \theta < \frac{2\pi}{3}$. Now $n' \leq \frac{k-2}{12} - \frac{1}{3}\delta_{k\equiv 0\,(6)}$, as f_1 has precisely $\frac{k-2}{12}$ zeros (of which at least one in ρ if $k \equiv 0 \mod 6$). Hence,

$$N_{(1,\infty]}(f) \;\leq\; \frac{1}{2}\left\lfloor \frac{k}{6} \right\rfloor + \frac{k-2}{12} + \frac{1}{6}\delta_{k\equiv 0\,(6)} \;=\; \left\lfloor \frac{k}{6} \right\rfloor \;=\; \dim \widetilde{M}_k^{\leq 1} - 1,$$

where we used that $k \equiv 0, 2 \mod 6$. Now, as f is not divisible by E_4 by assumption, this upper bound also holds true if f is reducible. Similarly, we obtain the upper bound for $N_{(\frac{1}{2},1)}(f)$.

Next, for f as in Theorem 1.3 we have

$$N_{(1,\infty]}(f) + N_{[0,\frac{1}{2})}(f) \;\leq\; \left\lceil \frac{k}{6} \right\rceil + m + \delta_{k\equiv 0\,(6)}.$$

Here, the term $\delta_{k\equiv 0\,(6)}$ comes from the fact that $-r(f_1)\frac{(-1)^0}{2}\operatorname{sgn} f(\rho)$ equals $-\frac{1}{2}$ if $k \equiv 2\,(6)$, as in that case $r(f_1) = \operatorname{sgn} f_1(\rho) = \operatorname{sgn} f(\rho)$. For $k \equiv 0\,(6)$ we have $-r(f_1)\frac{(-1)^0}{2}\operatorname{sgn} f(\rho) \leq \frac{1}{2}$. Now, similar as before, $N_{(1,\infty]}(f) \;\geq\; \frac{1}{2}\lfloor \frac{k}{6} \rfloor - n' - \frac{1}{2}\delta_{k\equiv 0\,(6)}$. Hence,

$$N_{[0,\frac{1}{2})}(f) \;\leq\; \left\lceil \frac{k}{6} \right\rceil - \frac{1}{2}\left\lfloor \frac{k}{6} \right\rfloor + n' + m + \frac{3}{2}\delta_{k\equiv 0\,(6)}$$

Now, similarly, $n' + m \leq \frac{k-2}{12} - \frac{1}{3}\delta_{k\equiv 0\,(6)}$, so that

$$N_{[0,\frac{1}{2})}(f) \;\leq\; \left\lceil \frac{k}{6} \right\rceil - \frac{1}{2}\left\lfloor \frac{k}{6} \right\rfloor + \frac{k-2}{12} + \frac{7}{6}\delta_{k\equiv 0\,(6)} \;=\; \left\lfloor \frac{k}{6} \right\rfloor + 1 \;=\; \dim \widetilde{M}_k^{\leq 1}$$

as $k \equiv 0, 2 \mod 6$. This implies the corollary. □

Examples

Example 5.3. Let f be as in Theorem 1.3 and assume f_1 has only zeros on the interior of $\mathcal{F}$ and at infinity. Write $a_0(f)$ for the constant term at infinity of f, and $a(f_1)$ for the first non-zero Fourier coefficient of f_1. Then, in case $a_0(f) = 0$ (as is the case when f is the derivative of a modular form), or if $\operatorname{sgn}(a_0(f)) = -\operatorname{sgn}(a(f_1))$, a direct evaluation of the result implies that

$$N_\lambda(f) = N_\lambda(f_1).$$

The situation alters slightly if $\operatorname{sgn}(a_0(f)) = \operatorname{sgn}(a(f_1))$ (as is the case if $f = E_2$); in that case we find

$$N_\lambda(f) \;=\; \begin{cases} N_\lambda(f_1) & |\lambda| \in (\frac{1}{2}, \infty) \\ N_\lambda(f_1) + 1 & |\lambda| \in (0, \frac{1}{2}). \end{cases}$$

Example 5.4. We return again to the example in the introduction. Applying Equations (5) and (6) in Theorem 1.3 and using the computations in Example 3.5, we find

$$N_{(\frac{1}{2},1)}(f) \;=\; -1 + \left\lfloor \frac{36}{6} \right\rfloor \;=\; 5.$$

Moreover, as $r(f_1) = 1$, $m = 0$, $f(\rho) < 0$ and $s(f) = 1$, we obtain

$$N_{[0,\frac{1}{2})}(f) \;=\; -1 + \left\lceil \frac{36}{6} \right\rceil - -\frac{1}{2} - \frac{-1}{2} \;=\; 6.$$

Example in higher depth

Example 5.5. Let $f = E_2^2 - E_4$. Its zeros are the critical points of E_2. We have

$$h_j(\tau) = \tau + \frac{12}{2\pi \mathrm{i}} \frac{E_2(\tau) + (-1)^j \sqrt{E_4(\tau)}}{(E_2(\tau)^2 - E_4(\tau))},$$

and for $U = \{z \in \mathfrak{h} \mid \operatorname{Im} z > \frac{1}{2}\sqrt{3}\}$ we find $\sigma(C) = \sigma(T) = e$ (see Corollary 4.2 for the definition of σ). In particular, $h(-\frac{1}{2} + \mathrm{i}t) \in -\frac{1}{2} + \mathrm{i}\mathbb{R}$ for all $t \in \mathbb{R}$. Using the same ideas as in the proof of Theorem 5.2, we find that the contribution of $\mathcal{C}$ equals $\frac{k-p}{6} - 2N_\infty(f)$ (with $k = 4, p = 2$). For the other contributions, we check that the proof goes through for both $E_2 \pm \sqrt{E_4}$ (which is not a *holomorphic* quasimodular form). That is, take h for $f_0 = \pm\sqrt{E_4}$ and $f_1 = 1$. Then, the function h for $E_2 + \sqrt{E_4} = 2 + O(q)$ behaves as $(-,+)$ in Table 4, whereas the function h for $E_2 - \sqrt{E_4} = -144q + O(q^2)$ behaves as $(-,-)$ in the same table. Hence,

$$N_\lambda(f) + N_{-\frac{1}{\lambda}}(f) \;=\; \begin{cases} \frac{4-2}{6} + \frac{5}{6} - \frac{1}{6} \;=\; 1 & |\lambda| < \frac{1}{2} \text{ or } |\lambda| > 2 \\ \frac{4-2}{6} - \frac{1}{6} - \frac{1}{6} \;=\; 0 & \frac{1}{2} < |\lambda| < 2. \end{cases}$$

Observe that, in contrast to the case where the depth is 1, we do not longer have that $|h_j(z)| = 1$ for $|z| = 1$. In particular, $h_1(z)$ and $h_2(z)$ intersect the real line for $z \in \mathcal{C}$ in the value v and $\frac{1}{v}$ respectively, given by

$$\frac{1}{v} = 0.180008\ldots, \qquad v = 5.555295\ldots$$

As the value of N_λ for positive λ only changes at $\lambda = \frac{1}{2}, \frac{1}{v}, v$, and $N_\infty(f) \geq 1$ (because $\mathrm{i}\infty$ is a zero of f), we conclude

$$N_\lambda(f) \;=\; \begin{cases} 1 & \frac{1}{v} < |\lambda| < \frac{1}{2} \text{ or } |\lambda| > v \\ 0 & |\lambda| < \frac{1}{v} \text{ or } \frac{1}{2} < |\lambda| < v. \end{cases}$$

Another result on the critical points of E_2 Again, let $f = E_2^2 - E_4$. Let

$$\mathcal{F}_0(2) \;:=\; \{z \in \mathfrak{h} \mid 0 \leq \operatorname{Re} z \leq 1 \text{ and } |z - \tfrac{1}{2}| \geq \tfrac{1}{2}\} \cup \{\mathrm{i}\infty\}$$

be (the closure of) a fundamental domain for $\Gamma_0(2)$. In [CL19] it is shown that

$$\sum_{\tau \in \gamma \mathcal{F}_0(2)} \nu_\tau(f) = 1 \tag{18}$$

for all $\gamma \in \Gamma_0(2)$. In particular, the number of critical points of E_2 is constant in every γ-translate of $\mathcal{F}_0(2)$, but depends on $\lambda(\gamma)$ in every γ-translate of $\mathcal{F}$ (see the previous example). Why are the zeros of a quasimodular form for $\mathrm{SL}_2(\mathbb{Z})$ 'better' distributed with respect to $\Gamma_0(2)$?

To get some more insight, we sketch how the proof of Theorem 5.2 can be adapted in order to give an alternative proof of (18). Let

$$N_\lambda^{(2)}(f) := \sum_{\tau \in \gamma\mathcal{F}_0(2)} \frac{\nu_\tau(f)}{e_{\tau,2}},$$

where $\lambda(\gamma) = \lambda$ and $e_{\tau,2} = 2$ if τ is a γ-translate of $\frac{1}{2} + \frac{1}{2}\mathrm{i}$ for $\gamma \in \Gamma_0(2)$, and $e_{\tau,2} = 1$ else. Then, we claim

$$N_\lambda^{(2)}(f) + N_{\frac{\lambda-1}{2\lambda-1}}^{(2)}(f) = 2$$

for all λ. Observe that $z \mapsto \frac{z-1}{2z-1}$ has the circle centered around $\frac{1}{2}$ with radius $\frac{1}{2}$ as its fixset. Now, the integral over this the corresponding circular segment in the upper half plane yields a contribution of

$$\frac{k-p}{2} - 2N_\infty^{(2)}(f)$$

with $k = 4, p = 2$ (namely, we integrate a function containing a $(k-p)$fold pole at λ and $\frac{\lambda-1}{2\lambda-1}$ over $\frac{1}{2}$ of the circle). Moreover, the functions h corresponding to $E_2 \pm \sqrt{E_4}$ tend to 0 and 1 as z tends to 0 and 1, and tend to $+\mathrm{i}\infty$ for $\tau \to \mathrm{i}\infty$ and $\tau \to 1 + \mathrm{i}\infty$. As exactly one of λ and $\frac{\lambda-1}{2\lambda-1}$ lies between 0 and 1, we find in both cases that the contribution to the variation of the argument is $\frac{1}{2}$, which yields the claim.

Next, let $U = \mathrm{int}\mathcal{F}_0(2)\backslash[\rho, \frac{1}{2} + i\infty)$. This is an open subset of $\mathfrak{h}$ invariant under TC and TST^2S (corresponding to $z \mapsto \frac{z-1}{2z-1}$). Moreover, $\sigma(CT) = (1\,2)$ and $\sigma(TST^2S) = (1\,2)$. As both leave the circle $\frac{1}{2} + \frac{1}{2}e^{\mathrm{i}\theta}$ invariant, we conclude

$$|h_i(\tfrac{1}{2} + \tfrac{1}{2}e^{\mathrm{i}\theta}) - \tfrac{1}{2}| = \tfrac{1}{4}.$$

Hence, find that $N_\lambda^{(2)}(f)$ as a function of λ can only change value at $\lambda = 0, 1$. Showing by other means that $N_\infty^{(2)}(f) = 1$, one could conclude that

$$N_\lambda^{(2)}(f) = 1 \text{ for all } \lambda.$$

A. The zeros of $\mathrm{Re}\,(\widehat{E}_2^n)$ and $\mathrm{Im}\,(\widehat{E}_2^n)$

We are interested in counting the number of zeros of $\mathrm{Re}\,(\widehat{E}_2^n)$ and $\mathrm{Im}\,(\widehat{E}_2^n)$ on $(\frac{\pi}{3}, \frac{2\pi}{3})$ for $n > 0$.

Proposition A.1. *The functions* $\mathrm{Re}\,(\widehat{E}_2^n)$ *and* $\mathrm{Im}\,(\widehat{E}_2^n)$ *admit*

$$n - 2\Big\lfloor \frac{n}{3} + \frac{1}{2} \Big\rfloor, \qquad \textit{resp.} \qquad n - 1 - 2\Big\lfloor \frac{n}{3} \Big\rfloor$$

zeros on $(\frac{\pi}{3}, \frac{2\pi}{3})$ *for* $n \geq 0$.

The proof almost immediately follows from the following result.

Lemma A.2. *For* $n > 0$, *write*

$$(x + \mathrm{i})^n = R_n(x) + \mathrm{i}\, Q_n(x),$$

where $R_n(x), Q_n(x) \in \mathbb{Z}[x]$. Then all roots of R_n and Q_n are real, of which

$$n - 2\Big\lfloor \frac{n}{3} + \frac{1}{2} \Big\rfloor, \qquad \textit{resp.} \qquad n - 1 - 2\Big\lfloor \frac{n}{3} \Big\rfloor$$

lie in $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

Proof. For $x > 0$, we may write

$$(x + \mathrm{i})^n = (x^2 + 1)^{n/2} e^{\mathrm{i}\, n \arctan(1/x)}.$$

Therefore, for all x, we recognize

$$\begin{aligned} R_n(x) &= (x^2 + 1)^{n/2}\, T_n(\cos(\arctan(1/x))) \\ &= (x^2 + 1)^{n/2}\, T_n\Big(\frac{x}{\sqrt{x^2 + 1}}\Big), \end{aligned}$$

where T_n is the n-th Chebyshev polynomial of the first kind, admitting the n distinct real roots $\cos\big(\frac{(k+\frac{1}{2})\pi}{n}\big)$ for $k = 0, \ldots, n - 1$ in $(-1, 1)$. Hence, R_n has n distinct real roots, of which $n - 2\lfloor \frac{n}{3} + \frac{1}{2} \rfloor$ are contained in $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

Write U_n for the n-th Chebyshev polynomial of the second kind. Then, for $n \geq 1$,

$$\begin{aligned} Q_n(x) &= (1 + x^2)^{(n-1)/2}\, U_{n-1}(\cos(\arctan(1/x))) \\ &= (1 + x^2)^{(n-1)/2}\, U_{n-1}\Big(\frac{x}{\sqrt{1 + x^2}}\Big), \end{aligned}$$

from which one deduces that all roots of Q_n are real and $n - 1 - 2\lfloor \frac{n}{3} \rfloor$ lie in $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Note that Q_n admits a zero at $\pm\frac{1}{\sqrt{3}}$ for $n \equiv 0 \mod 3$. □

Proof of Proposition A.1. We apply the previous lemma to $x = \frac{\pi}{3}\mathrm{Re}\,(\widehat{E}_2)$. Note that by (14), we have $\mathrm{Im}\,(\widehat{E}_2) = \frac{3}{\pi}$. Hence,

$$\frac{\pi^n}{3^n}\mathrm{Re}\,(\widehat{E}_2^n) = \mathrm{Re}\,\big((\frac{\pi}{3}\mathrm{Re}\,(\widehat{E}_2) + \mathrm{i})^n\big) = R_n\big(\frac{\pi}{3}\mathrm{Re}\,(\widehat{E}_2)\big)$$

and

$$\frac{\pi^n}{3^n}\mathrm{Im}\,(\widehat{E}_2^n) = Q_n\big(\frac{\pi}{3}\mathrm{Re}\,(\widehat{E}_2)\big).$$

Observe that $\mathrm{Re}\,(\widehat{E}_2)$ is a strictly decreasing function on $(\frac{\pi}{3}, \frac{2\pi}{3})$ with a unique zero at $\theta = \frac{\pi}{2}$. As on the boundary we have

$$\frac{\pi}{3}\mathrm{Re}\,(\widehat{E}_2)\Big(\frac{\pi}{3}\Big) = -\frac{\pi}{3}\mathrm{Re}\,(\widehat{E}_2)\Big(\frac{2\pi}{3}\Big) = \frac{1}{\sqrt{3}},$$

the proposition follows from the lemma above. □

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