ON K_0 OF A SURFACE WITH CYCLIC QUOTIENT SINGULARITIES

MARC LEVINE

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

MPI/SFB 83-28

INTRODUCTION

Let X be a normal, quasi-projective surface over an algebraically closed field k, and let $f:\tilde{X}\to X$ be a resolution of singularities of X. Let $F_0K_0(X)$ denote the subgroup of $K_0(X)$ generated by the residue classes of the smooth points of X, and similarly define $F_0K_0(\tilde{X})$. From the works [L2], [L3] [L-W] and [P-W], we know that $F_0K_0(X)$ is isomorphic to $H^2(X, \mathcal{K}_2)$, where $\mathcal{K}_2$ is the Zariski K-sheaf $\mathcal{K}_{2,x} = K_2(\mathcal{O}_{X,x})$. The corresponding result in the smooth case was proved by Bloch in [B]. The map f induces a surjective homomorphism $f^*:F_0K_0(X)\to F_0K_0(\tilde{X})$. Bloch and Srinivas have conjectured that the kernel of f^* can be computed as

$$\ker f^* = \operatorname{coker}(H^1(\tilde{X}, \mathcal{K}_2)\to \varprojlim_n H^1(E_n, \mathcal{K}_2)) ,$$

where E_n is the n-fold thickening of the exceptional divisor of the map f. Loosely speaking, this says that the kernel of f comes from analytic invariants of the singular local rings of X, together with a global invariant from the resolution $\tilde{X}$. In particular, if X has only rational singularities, the analytic invariant $\varprojlim_n H^1(E_n, \mathcal{K}_2)$ vanishes, hence f^* should be an isomorphism.

M.P. Murthy and N. Mohan Kumar([M] and [M-M]) have studied the algebraic local rings on rational surfaces with a given analytic type. From their work it follows that, if X is rational, with a single singularity of type A_n ($n\neq 7,8$) or D_n ($n\neq 8$) then $F_0K_0(X) = 0$, in agreement with the conjecture above. Together with V. Srinivas ([L-S]), we have analyzed a special type of rational singularity, whose analvtic type has for minimal resoluition a single rational curve with self-intersection -4, and we have verified the above conjecture for a surface possessing singularities only of this special type.

One very nice class of rational singularities are the quotient singularities, i.e. those singularities with the analytic type of $(\mathbb{C}^2/G)_0$, where G is a finite subgroup of $GL(2,\mathbb{C})$, such that each element $g \neq id$ of G has only 0 as a fixed point. In many ways, these singularities are "essentially non-singular", and therefore should be a good place to start in attacking the conjecture of Bloch and Srinivas. In this work, we verify the conjecture for singularities which have the analytic type of a quotient singularity as above, with G cyclic. This includes for example, all the singularities with analytic type A_n, and all singularities with analytic type $\mathbb{C}[[x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n]]$. The rational double points of analytic type D_n, E_6, E_7, and E_8 are not of this form.

The proof is very simple, and consists of two main steps. The first is to notice that it is essentially enough to dominate the singularity by a regular ring via a finite map, and the second is to construct such a map for singularities of the type described above. We would like to thank A. Collino for suggesting the use of the first step, and K. Behnke, H. Esnault, and E. Viehweg for kindly and patiently teaching me some of the rudiments of quotient singularities.

We assume throughout that $\mathrm{char}(k)=0$.

Lemma 1. Let R be a normal, two-dimensional local ring, $R \supseteq k$, let $X = \mathrm{Spec}(R)$, and let $p:\tilde{X} \to X$ be a resolution of singularieties of X. Suppose there is a smooth scheme Y, and a finite surjective map $f:Y \to X$ of degree n. Let $E_1, \ldots, E_s$ be the irreducible exceptional curves of p. Then the cokernel of the map $\coprod_{E_i} k^* \to H^1(\tilde{X}, \mathcal{K}_2)$ is n-torsion.

Proof. Let Z be a resolution of singularities of the fiber product $Y \times_X \tilde{X}$. Let $F_1, \ldots, F_m$ be the irreducible exceptional curves of the map $q:Z \to Y$. As Y is smooth and semi-local, we have $H^1(Y, \mathcal{K}_2) = 0$, hence $\coprod_{F_i} k^*$ generates $H^1(Z, \mathcal{K}_2)$. We have the following diagram:

$$\begin{array}{ccc} Z & \xrightarrow{g} & \tilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X \end{array} .$$

Since Y is finite over X, Z is proper over $\tilde{X}$. The map g induces maps

$$g^*: H^1(\tilde{X}, \mathcal{K}_2) \to H^1(Z, \mathcal{K}_2)$$

$$g_*: H^1(Z, \mathcal{K}_2) \to H^1(\tilde{X}, \mathcal{K}_2) .$$

The map g_* is induced by the map which sends a curve C on Z, and a function h on C to the pair $(g(C), Nm_{C/g(C)}(h))$, if C is finite over g(C), and to the identity if g(C) is a point. For z in $H^1(\tilde{X}, \mathcal{K}_2)$, we can represent z as a collection $\{(C_i, h_i)\}$,where

C_i is a curve of $\tilde{X}$, h_i is in $k(C_i)^*$, and $\sum_i (h_i) = 0$. Altering the collection $\{(C_i,h_i)\}$ by a tame symbol if necessary, we may assume that each C_i avoids the finite set of points of $\tilde{X}$ over which Z is not finite. Then $g^*(z)$ is represented by the collection $\{(g^*(C_i),g^*(h_i))\}$. From this description, it follows that $g_* \circ g^* = n \times id$, and $g_*(F_i,a)$, a in k^*, is of the form (E_j,b), for suitable E_j, and suitable b in k^*. This proves the lemma.

q.e.d.

Let R be a normal, two-dimensional local ring, containing k. We call R a <u>cyclic quotient singularity</u> if the completion $\hat{R}$ of R is isomorphic to the completion of the local ring at 0 of a variety of the form $\mathbb{A}^2/G$, where G is a finite, cyclic subgroup of GL(2,k), such that 0 is the only fixed point of each element $g \neq id$ of G. Fix an embedding of k into $\mathbb{C}$, let X = Spec(R), $X_{\mathbb{C}} = Spec(R \otimes_k \mathbb{C})$. We may suppose that R is the local ring of a point p on an affine variety $X^* = Spec(A)$, and we let $X^*_{\mathbb{C}} = Spec(A \otimes_k \mathbb{C})$. Let X_p be a small analytic neighborhood of p in $X^*_{\mathbb{C}}$. Then R (or by abuse of notation, X) is a cyclic quotient singularity if and only if $\pi_1(X_p - p)$ is a finite cyclic group. Given a cyclic quotient singularity X, we construct a smooth scheme Y, and a finite surjective map $f: Y \to X$ as in the lemma. We proceed in five steps:

<u>Step 1</u>: Let ω be the dualizing sheaf on X, and let $j: X-p \to X$ be the inclusion. Since p is a quotient singularity, there is a minimal $r \geq 1$ such that $\omega^{[r]} \underset{def}{=} j_*(j^*(\omega)^{\otimes r})$ is free. Choosing an isomorphism $\omega^{[r]} \cong \mathcal{O}_X$ makes $B = \bigoplus_{i=0}^{r-1} \omega^{[i]}$ into a finite $\mathcal{O}_X$ algebra. Letting $X' = \mathrm{Spec}_{\mathcal{O}_X}(B)$, X' is local, Gorenstein, and X' is etale over X-p. Letting p' be the closed point of X',

and letting $X'_{p'}$ be a small analytic neighborhood of p' in a suitable affine model of X', it follows that $\pi_1(X'_{p'}-p')$ is again cyclic, and hence p' is a rational double point of type A_n. Thus we may replace X with X', and assume that p is a rational double point of type A_n.

<u>Step 2</u>: Let $\bullet\!-\!\bullet \cdots \bullet\!-\!\bullet$ be the dual graph of the exceptional curves in a minimal resolution $\tilde{X}$ of X, and let E_i be the irreducible exceptional curve corresponding to the i^{th} vertex from the left. Let H be a general hyperplane section of X* passing through p, and let $\bar{H}$ be the proper transform of H up to $\tilde{X}$. Then

$$\begin{aligned} &\bar{H}\cdot E_i = 0 \quad \text{for } i=2,\dots,n-1 \\ &\bar{H}\cdot E_1 = \bar{H}\cdot E_n = 1 \quad \text{if } n \geq 2 \\ &\bar{H}\cdot E_1 = 2 \quad \text{if } n=1 \quad , \end{aligned}$$

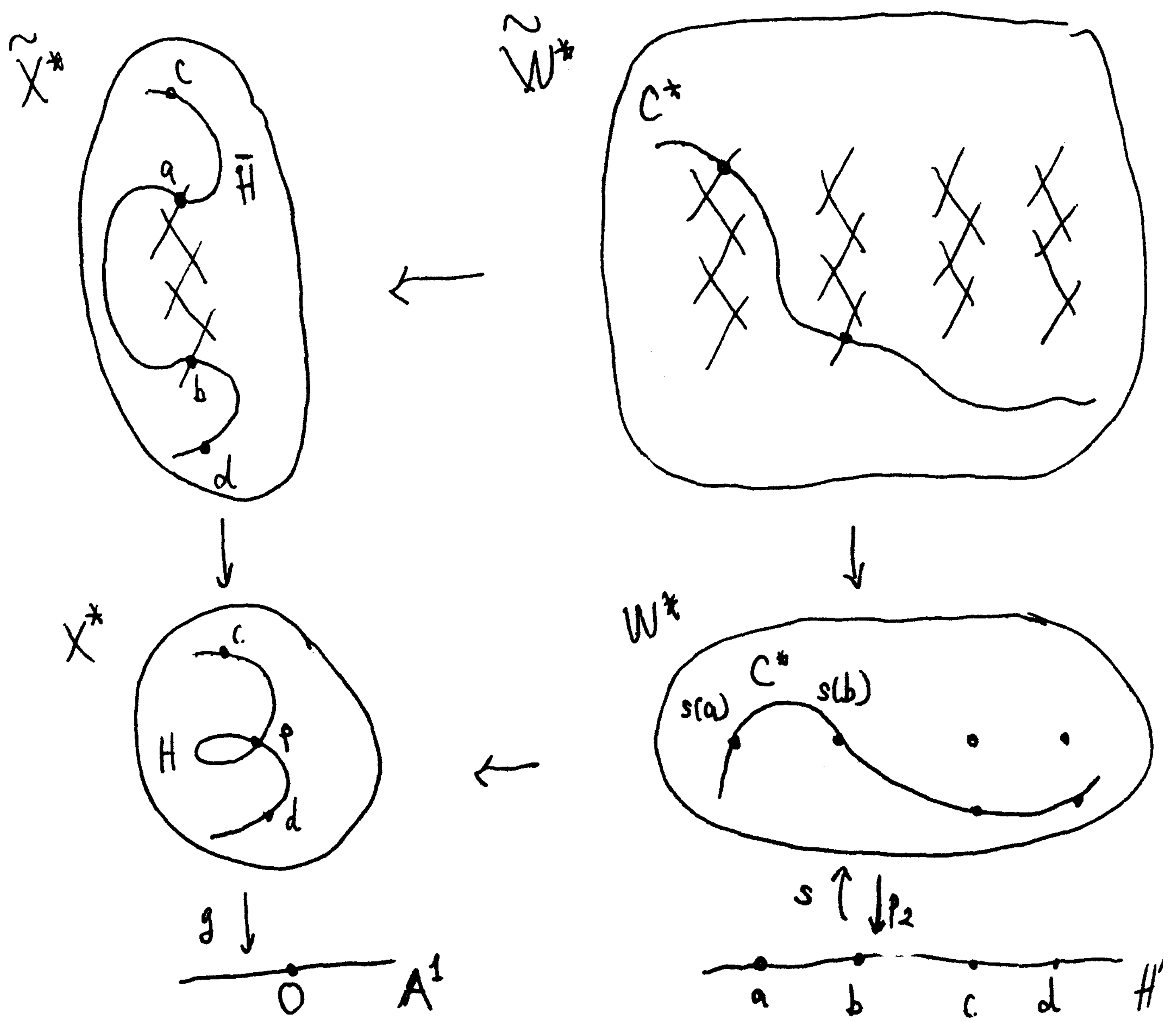
and $\bar{H}$ is smooth. In case n=1, we may choose H so that $\bar{H}$ intersects E_1 at two points transversely. Let $\tilde{X}_p$ be the inverse image of X_p under $h:\tilde{X} \to X$. Then $\bar{H} \cap \tilde{X}_p = \bar{H}_p$ breaks up into two analytically irreducible branches, L and L' with

$$\begin{aligned} &L\cdot E_1 = 1 = L'\cdot E_n \\ \text{and} \qquad &L\cdot E_n = 0 = L'\cdot E_1 \quad \text{if } n \geq 2 \quad . \end{aligned}$$

From an easy computation, it follows that both L and L' are generators of $\mathrm{Pic}(X_p-p) \cong \mathrm{Cl}(X_p) \cong \pi_1(X_p-p) \cong \mathbb{Z}/(n+1)\mathbb{Z}$.

Step 3: Take a general linear projection $g: X^* \to \mathbb{A}^1$, so that H is finite over $\mathbb{A}^1$, and the normalization H^N of H is etale over $0 = g(p)$. This can be done, as we may choose H so that H is smooth except for an ordinary double point at p. We may then choose g so that each branch of H at p is etale over 0.

Let W* be the fiber product $X^* \times_{\mathbb{A}^1} H^N$, and let a and b be the two points of H lying over p. Then W* is finite over X*, and etale over a neighborhood of p. Let $s: H^N \to W^*$ be the section to p_2 induced by the inclusion of H in X*, and let $C^* = s(H^N)$. Then C*generates $Cl(\hat{\mathcal{O}}_{W^*,x})$ for $x = s(a), s(b)$, and does not pass through the other points of W* lying over p. The following picture may be helpful: ($\tilde{X}^*$ and $\tilde{W}^*$ are minimal resolut of X* and W*, respectively)



Step 4: Let $H^{N'}$ be the Spec of the semi-local ring of the points of H^N lying over $\mathcal{O}$, and let Z be the Galois closure of $H^{N'}$ over $\mathbb{A}^1$. Then Z is finite, etale, and Galois over $\mathrm{Spec}(\mathcal{O}_{\mathbb{A}^1,0})$. Let U be the fiber product $X \times_{\mathbb{A}^1} Z$ and let C be the inverse image of C* in $\mathcal{U}$. By conjugating C by elements of $\mathrm{Gal}(Z/\mathbb{A}^1)$, we construct curves $C = C_1,\ldots,C_t$ on U such that

1) $\mathrm{Sing}(U) \subseteq \bigcup_i C_i$

2) if q is in $\mathrm{Sing}(U) \cap C_i$ then C_i is a generator of $\mathrm{Cl}(\hat{\mathcal{O}}_{U,q}) = \mathbb{Z}/(n+1)\mathbb{Z}$.

In addition, U is finite and etale over X.

Step 5: Since U is semi-local, $(n+1)C_1$ is a principal divisor. Choosing an isomorphism $\mathcal{O}_U((n+1)C_1) \cong \mathcal{O}_U$ gives an algebra structure to the finite $\mathcal{O}_U$ module $\bigoplus_{i=0}^{n} \mathcal{O}_U(iC_1) = B$. Let $f_1:U_1 \to U$ be the normalization of $\mathrm{Spec}_{\mathcal{O}_U}(B)$. Let $p_1,\ldots,p_m$ be the points of Sing(U) lying on C_1. Then U_1 is finite over U, etale over $U - \bigcup_i p_i$, and the points of U_1 lying over $p_1,\ldots,p_m$ are smooth on U_1. We now repeat this procedure with the curve $f_1^{-1}(C_2)$, and so on, constructing a regular, semi-local scheme $Y = U_t$, finite over U. Since U is finite over X, we have constructed a morphism $f:Y \to X$ as required by lemma 1.

We can now prove our main result.

<u>Theorem 2</u>. Let X be a normal, quasi-projective surface over k, $f:\tilde{X}\to X$ a resolution of singularities. Suppose that each point of Sing(X) is a cyclic quotient singularity. Then

$$H^2(X,\mathcal{K}_2)\xrightarrow{f^*}H^2(\tilde{X},\mathcal{K}_2)$$

is an isomorphism.

<u>Proof</u>. Let X' be a projective closure of X, with Sing(X')=Sing(X). Let $\tilde{X}'$ be the corresponding projective closure of $\tilde{X}$, with morphism $f':\tilde{X}'\to X'$. We have the five term exact sequences coming from the Leray spectral sequences $H^p(X,R^qf_*(\mathcal{K}_2))\Rightarrow H^{p+q}(\tilde{X},\mathcal{K}_2)$ and $H^p(X',R^qf'_*(\mathcal{K}_2))\Rightarrow H^{p+q}(\tilde{X}',\mathcal{K}_2)$:

$$\begin{array}{ccccccccccc}
0 & \to & H^1(X,f_*(\mathcal{K}_2)) & \to & H^1(\tilde{X},\mathcal{K}_2) & \xrightarrow{a} & H^0(X,R^1f_*(\mathcal{K}_2)) & \to & H^2(X,f_*(\mathcal{K}_2)) & \xrightarrow{f^*} & H^2(\tilde{X},\mathcal{K}_2) \\
 & & \uparrow i^* & & \uparrow i^* & & \uparrow i^* \wr & & \uparrow i^* & & \uparrow i^* \\
0 & \to & H^1(X',f'_*(\mathcal{K}_2)) & \to & H^1(\tilde{X}',\mathcal{K}_2) & \xrightarrow{a'} & H^0(X',R^1f'_*(\mathcal{K}_2)) & \to & H^2(X',f'_*(\mathcal{K}_2)) & \xrightarrow{f'^*} & H^2(\tilde{X}',\mathcal{K}_2)
\end{array}$$

Since $\mathcal{K}_{2,X}$ and $f_*(\mathcal{K}_{2,\tilde{X}})$ are isomorphic off a codimension two subset of X, we have

$$H^2(X,f_*\mathcal{K}_2) = H^2(X,\mathcal{K}_2)$$

and similarly

$$H^2(X',f'_*\mathcal{K}_2) = H^2(X',\mathcal{K}_2) .$$

By [L3] , or [P-W] , we have

$$H^2(X, \mathcal{K}_2) = CH^2(X, Sing(X))$$

and $$H^2(X', \mathcal{K}_2) = CH^2(X', Sing(X')) \quad ,$$

where $CH^2(X,Sing(X))$ is the free abelian group on the smooth points of X, modulo divisors of functions on curves C, with $C \cap Sing(X) = \emptyset$, and $CH^2(X',Sing(X'))$ is defined similarly. By [B] we have a similar isomorphism for the smooth varieties $\tilde{X}$ and $\tilde{X}'$. Thus f* and f'* are surjective.

By lemma 1, the cokernels of a and a' are torsion groups. On the other hand, by [R] for the smooth case, and [L1] in the singular case, we know that

$$CH^2(X',Sing(X'))_{tor} \cong Alb(\tilde{X}')_{tor} \cong CH^2(\tilde{X}')_{tor} .$$

Thus a' is surjective. Therefore a is also surjective, and $f^*: H^2(X, \mathcal{K}_2) \to H^2(\tilde{X}, \mathcal{K}_2)$ is an isomorphism.

q.e.d.

Corollary 3. Let $f:\tilde{X} \to X$ be as in Theorem 2. Let $F_0K_0(X)$ be the subgroup of $K_0(X)$ generated by the classes of the residue fields k(x) for smooth points x of X, and define $F_0K_0(\tilde{X})$ similarly. Then

$$f^*: K_0(X) \to K_0(\tilde{X}) \qquad \text{is injective}$$

and

$$f^*: F_0K_0(X) \to F_0K_0(\tilde{X})$$
$$f^*: CH^2(X,Sing(X)) \to CH^2(\tilde{X})$$

are isomorphisms.

Proof. As $CH^2(X,\mathrm{Sing}(X)) = H^2(X,\mathcal{K}_2)$, and $CH^2(\tilde{X}) = H^2(\tilde{X},\mathcal{K}_2)$, the last statement follows from theorem 2. From [L2] in the singular case, and [B-S] in the smooth case, we have

$$F_0K_0(\tilde{X}) = CH^2(\tilde{X})$$
$$F_0K_0(X) = CH^2(X,\mathrm{Sing}(X)) \quad ,$$

whence the second statement. Let $F_1K_0(X)$ be the subgroup of $K_0(X)$ generated by $F_0K_0(X)$ and the classes of Cartier divisors on X, and define $F_1K_0(\tilde{X})$ similarly. Then $K_0(X) = F_1K_0(X) \oplus \mathbb{Z}$, $K_0(\tilde{X}) = F_1K_0(\tilde{X}) \oplus \mathbb{Z}$ with the $\mathbb{Z}$ generated by the class of the trivial module. We also have the short exact sequences (by [B-S] and [L2])

$$\begin{array}{ccccccccc} 0 & \to & F_0K_0(\tilde{X}) & \to & F_1K_0(\tilde{X}) & \to & \mathrm{Pic}(\tilde{X}) & \to & 0 \\ & & f^*\uparrow & & f^*\uparrow & & f^*\uparrow & & \\ 0 & \to & F_0K_0(X) & \to & F_1K_0(X) & \to & \mathrm{Pic}(X) & \to & 0 \end{array}$$

The map $f^*:\mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ is injective since X is normal, hence $f^*:K_0(X) \to K_0(\tilde{X})$ is injective, as desired.

q.e.d.

As an application, we have the following corollary.

Corollary 4. Let X be a rational, affine surface over k, X = Spec(R). Suppose that X is normal, and has only cyclic quotient singularities. Let p be a smooth point of X, then the maximal ideal m_p of p is a complete intersection, i.e., there exist f,g, in R with $m_p = (f,g)$.

Proof. Following Serre [S], there is a rank two projective P and a short exact sequence $0 \to R \to P \to m_p \to 0$. In $K_0(X)$, we have

$$[P] = -[m_p] = -[R/m_p] \quad .$$

On the other hand, letting $\tilde{X}$ be a resolution of singularities of

X, it is easy to show that $CH^2(\tilde{X}) = 0$. By corollary 3, $CH^2(X, Sing(X))$ is also 0, hence the class of $[R/m_p]$ in $K_0(X)$ is zero as well. By the cancellation theorem of Murthy-Swan [M-S], P is a free R module, hence m_p is two-generated, as desired.

q.e.d.

The same argument as above also shows that any height two ideal I of R, locally a complete intersection, is globally a complete intersection.

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