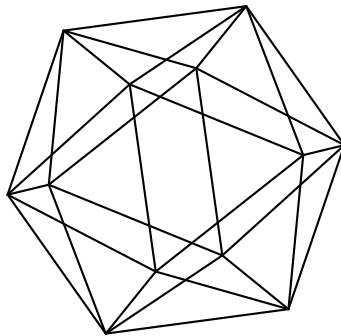


# Max-Planck-Institut für Mathematik Bonn

## Resurgence of Habiro elements

by

Samuel Crew  
Ankush Goswami  
Robert Osburn



Max-Planck-Institut für Mathematik  
Preprint Series 2023 (10)

Date of submission: May 31, 2023

# Resurgence of Habiro elements

by

Samuel Crew  
Ankush Goswami  
Robert Osburn

Max-Planck-Institut für Mathematik  
Vivatsgasse 7  
53111 Bonn  
Germany

Faculty of Computer Science  
Ruhr University  
Universitätsstr. 150  
44801 Bochum  
Germany

Max Planck Institute for Security and Privacy  
Universitätsstr. 140  
44799 Bochum  
Germany

School of Mathematical and  
Statistical Sciences  
The University of Texas Rio Grande Valley  
1201 W. University Dr.  
Edinburg, TX 78539  
USA

# RESURGENCE OF HABIRO ELEMENTS

SAMUEL CREW, ANKUSH GOSWAMI, AND ROBERT OSBURN

ABSTRACT. We prove resurgence properties for the Borel transform of elements in the Habiro ring which satisfy a general type of strange identity. As an application, we provide evidence for (and against) conjectures in quantum topology due to Costin and Garoufalidis.

## 1. INTRODUCTION

The modern theory of resurgence began with the voluminous works of Écalle in 1981 [14, 15] and 1985 [16]. This theory now plays a vital rôle in a remarkable number of diverse areas, to name just a few examples: nonlinear systems of ODEs and difference equations [9, 32], algebraic combinatorics [7, 8], enumerative combinatorics and quantum field theory [6, 36], period integrals and string theory scattering amplitudes [13], wall-crossing phenomena [33] and matrix models [38]. In this paper, we are interested in a conjecture due to Costin and Garoufalidis [11] which connects resurgence and quantum topology. For other such interactions, see [2] and the references therein.

Let us recall the general setup from [11]. For further details, see [10, 12, 37, 39] or [42, Section 2] for an excellent concise review of the basic notions of resurgence and alien calculus. A formal power series

$$\sum_{n=0}^{\infty} a_n x^{-n} \in \mathbb{C}[[1/x]] \quad (1.1)$$

is called Gevrey-1 if there exists positive real numbers  $A, B$  such that

$$|a_n| \leq AB^n n!$$

for all  $n \geq 0$ . Consider a Gevrey-1 formal power series given by (1.1) and its Borel transform  $\mathcal{B}: \mathbb{C}[[1/x]] \rightarrow \mathbb{C}[[p]]$  defined by

$$\mathcal{B}\left(\sum_{n=0}^{\infty} a_n x^{-n}\right) = \sum_{n=0}^{\infty} a_{n+1} \frac{p^n}{n!} =: G(p). \quad (1.2)$$

Here,  $G(p)$  has a positive radius of convergence as it arose from a Gevrey-1 power series. In a standard abuse of notation,  $G(p)$  denotes the formal power series in (1.2) and the analytic continuation of the associated germ at the origin. In addition,  $G(p)$  is called a *resurgent function* if it is endlessly analytically continuable. This property means  $G(p)$  extends to a (possibly multi-valued) holomorphic function along unbounded paths that need only circumvent a discrete set

---

*Date:* May 31, 2023.

*2020 Mathematics Subject Classification.* 57K16, 11F37, 30B40, 40A05.

*Key words and phrases.* Resurgence, Habiro ring, quantum topology, strange identities.

of singularities. In the present work, we consider the case where  $G(p)$  defines a multi-valued function on  $\mathbb{C} \setminus \mathcal{N}$ , with  $\mathcal{N}$  a discrete set of singularities lying on a ray.

For  $x \in \mathbb{C}$ , we now define the left and right Borel resummations as

$$S^{\mathbf{L}}(x) = \int_{\gamma_{\mathbf{L}}} e^{-px} G(p) dp$$

and

$$S^{\mathbf{R}}(x) = \int_{\gamma_{\mathbf{R}}} e^{-px} G(p) dp$$

where  $\gamma_{\mathbf{L}}$  and  $\gamma_{\mathbf{R}}$  are contours in  $\mathbb{C} \setminus \mathcal{N}$  from 0 to  $\infty$  that turn left (respectively, right) at each singularity in  $\mathcal{N}$  (see Figure 1) and  $G(p)$  satisfies suitable growth conditions to ensure convergence of the integrals.

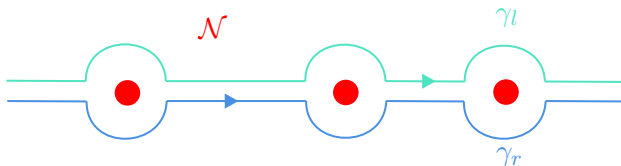


FIGURE 1. A singularity set  $\mathcal{N} \subset \mathbb{C}$  and the contours  $\gamma_{\mathbf{L}}$  and  $\gamma_{\mathbf{R}}$ .

Following [11, Sections 4.2–4.3], one may define the median resummation as

$$S^{\mathbf{med}}(x) = \frac{1}{2} \left( S^{\mathbf{L}}(x) + S^{\mathbf{R}}(x) \right). \quad (1.3)$$

In general, care is required when defining median summations, see [25, Section 2]. Under favourable circumstances, for example when  $G(p)$  is the Borel transform of a formal (divergent) solution in powers of  $1/x$  to a differential equation, the median resummation  $S^{\mathbf{med}}(x)$  is a well-defined holomorphic solution in  $x$ . This is the origin of the terminology “resummation”. In this paper, we will study the median resummation of the Borel transform for certain formal power series associated to knots.

Let  $K$  be a knot and  $J_N(K; q)$  be the usual colored Jones polynomial, normalized to be 1 for the unknot. If  $N = 2$ , then we recover the Jones polynomial [30]. As a knot invariant, the colored Jones polynomial is of fundamental importance in several open problems in quantum topology (see, e.g., [17, 19, 31, 41, 44]). We now consider the Habiro ring [24]

$$\mathcal{H} := \varprojlim_n \mathbb{Z}[q] / \langle (q)_n \rangle$$

where

$$(a)_n = (a; q)_n := \prod_{k=1}^n (1 - aq^{k-1})$$

is the standard  $q$ -hypergeometric notation. Given  $K$ , there exists an element  $\Phi_K(q) \in \mathcal{H}$  which matches  $J_N(K; q)$  when  $q = \zeta_N := e^{\frac{2\pi i}{N}}$  [29, Theorem 2]. Finally, let

$$F_K(x) := \Phi_K(e^{-\frac{1}{x}}).$$

To emphasize the dependence on  $K$ , we write  $G_K(p)$  for  $G(p)$  and  $S_K^{\text{med}}(x)$  for  $S^{\text{med}}(x)$ . We now state the main conjecture from [11] which is part of a much larger program to understand the analytic continuation of invariants of “knotted objects” arising in Chern-Simons theory [18, 20].

**Conjecture 1.1.** *For every knot  $K$ ,*

- (1)  $F_K(x)$  has a resurgent Borel transform  $G_K(p)$ .
- (2)  $S_K^{\text{med}}(x)$  is an analytic function defined on  $\Re(x) > 0$  with radial limits at the points  $\frac{1}{2\pi i}\mathbb{Q}$  of its natural boundary.
- (3) For  $0 \neq \alpha \in \mathbb{Q}$ , we have  $S_K^{\text{med}}(-\frac{1}{2\pi i\alpha}) = \Phi_K(e^{2\pi i\alpha})$ .

In [11, Theorem 2.5, Theorem 3.1], Conjecture 1.1 (1) and (2) were proven for  $K = 3_1$ , the trefoil knot. In this case, we have [23, 34]

$$J_N(3_1; q) = q^{1-N} \sum_{n=0}^{\infty} q^{-nN} (q^{1-N})_n \quad (1.4)$$

and (after a slight normalization)

$$F_{3_1}(x) = e^{-\frac{1}{24x}} \sum_{n=0}^{\infty} (e^{-\frac{1}{x}})_n$$

where

$$\Phi_{3_1}(q) := \sum_{n=0}^{\infty} (q)_n \quad (1.5)$$

is the Kontsevich-Zagier “strange” series [44]. From (1.4) and (1.5), we observe

$$\Phi_{3_1}(\zeta_N)\zeta_N = J_N(3_1; \zeta_N).$$

The Borel transform  $G_{3_1}(p)$  of  $F_{3_1}(x)$  is explicitly given by [11, Theorem 3.1]

$$G_{3_1}(p) = \frac{3\pi}{2\sqrt{2}} \sum_{n=1}^{\infty} \frac{n \left(\frac{12}{n}\right)}{\left(-p + \frac{n^2\pi^2}{6}\right)^{\frac{5}{2}}} \quad (1.6)$$

where  $\left(\frac{12}{*}\right)$  is the quadratic character of conductor 12. The key to establishing Conjecture 1.1 (1) and (2) for the trefoil knot is the “strange identity”

$$\Phi_{3_1}(q) \text{ “ = ” } - \frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}} \quad (1.7)$$

where “ = ” means that the two sides agree to all orders at every root of unity (for further details, see Sections 2 and 5 in [43]). Moreover, a close inspection of the proofs of Theorems 2.5 and 3.1 in [11] reveals that it is actually the periodic function  $\left(\frac{12}{*}\right)$  which determines the analytic nature of  $G_{3_1}(p)$  and  $S_{3_1}^{\text{med}}(x)$ . Our main goal in this paper is to prove resurgence properties for elements in  $\mathcal{H}$  which satisfy a general type of strange identity. Here, we emphasize the importance of periodic functions. We will also observe that Conjecture 1.1 (3) is false for the trefoil knot (contrary to Theorem 5.2 in [11]) and provide a correct reformulation (see Theorem 1.2 (3), Theorem 3.1 and (4.1)). Before stating our main result, we introduce some notation.

Let  $f : \mathbb{Z} \rightarrow \mathbb{C}$  be a function of period  $M \geq 2$  with mean value zero. For integers  $a \geq 0$  and  $b > 0$ , consider the partial theta series

$$\theta_{a,b,f}^{(\nu)}(q) := \sum_{n=0}^{\infty} n^{\nu} f(n) q^{\frac{n^2-a}{b}} \quad (1.8)$$

where  $q = e^{2\pi iz}$ ,  $z \in \mathbb{H}$ , and  $\nu \in \{0, 1\}$ . Suppose there exists

$$\Phi_f(q) := \sum_{n=0}^{\infty} A_{n,f}(q) (q)_n \in \mathcal{H}$$

where  $A_{n,f}(q) \in \mathbb{Z}[q]$  such that

$$\Phi_f(q) = \theta_{a,b,f}^{(\nu)}(q). \quad (1.9)$$

We now make a choice of periodic function. Let  $k_1, k_2 \in \mathbb{Z}$  with  $k_1 < k_2$ ,  $0 \neq c \in \mathbb{R}$  and  $g$  be the function

$$g(n) := \begin{cases} c & \text{if } n \equiv k_1, M - k_1 \pmod{M}, \\ -c & \text{if } n \equiv k_2, M - k_2 \pmod{M}, \\ 0 & \text{otherwise.} \end{cases} \quad (1.10)$$

Note that  $g$  is an even function of period  $M$ . For  $\ell \in \mathbb{N}$ , set

$$\begin{aligned} \mathcal{S}(k_1, k_2, \ell, M) &:= \sin\left(\frac{(k_2 - k_1)\ell\pi}{M}\right) \sin\left(\frac{(M - k_1 - k_2)\ell\pi}{M}\right), \\ \tilde{g}(\ell) &:= \begin{cases} (-1)^\ell \mathcal{S}(k_1, k_2, \ell, M) & \text{if } \frac{M}{\gcd(M, k_2 - k_1)} \nmid \ell \text{ or } \frac{M}{\gcd(M, M - k_2 - k_1)} \nmid \ell, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (1.11)$$

and

$$C_M := \frac{2Mc}{\pi^2} \sum'_{\ell \in \mathbb{N}} \frac{(-1)^\ell}{\ell^2} \mathcal{S}(k_1, k_2, \ell, M) \quad (1.12)$$

where, here and throughout,  $\sum'$  denotes the condition  $\frac{M}{\gcd(M, k_2 - k_1)} \nmid \ell$  or  $\frac{M}{\gcd(M, M - k_2 - k_1)} \nmid \ell$ . Finally, define

$$\mathcal{F}_g(x) := e^{-\frac{a}{bx}} \Phi_g(e^{-\frac{1}{x}})$$

and let  $G_g(p)$  denote the Borel transform of  $\mathcal{F}_g(x)$ . Here, we write  $S_g^{\text{med}}(x)$  for  $S^{\text{med}}(x)$  and  $\theta_{a,b,g}^{(\nu)}(z)$  for  $\theta_{a,b,g}^{(\nu)}(q)$ . Our main result is now the following.

**Theorem 1.2.** *Assume (1.9) is true. Then*

- (1)  $\mathcal{F}_g(x)$  has a resurgent Borel transform  $G_g(p)$ .
- (2)  $S_g^{\text{med}}(x)$  is an analytic function defined on  $\Re(x) > 0$  with radial limits at the points  $\frac{1}{2\pi i} \mathbb{Q}$  of its natural boundary.

(3) For  $0 \neq \alpha \in \mathbb{Q}$ , we have

$$S_g^{\text{med}}\left(-\frac{1}{2\pi i\alpha}\right) = -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}}\int_0^{i\infty}\frac{\theta_{0,4M^2,\tilde{g}}^{(0)}(bp)}{\left(p+\frac{1}{\alpha}\right)^{\frac{3}{2}}}dp - C_M + \tilde{\delta}_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right)$$

where

$$\tilde{\delta}_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right) := \left(\frac{b}{\alpha}\right)^{\frac{3}{2}}\frac{\sqrt{2}ce^{\frac{\pi i}{4}}}{M^2}\sum'_{\ell \in \mathbb{N}}(-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i\frac{b\ell^2\pi}{2M^2\alpha}}. \quad (1.13)$$

As an application of Theorems 1.2 and 3.1, we provide evidence in support of Conjecture 1.1 (1) and (2) for two families of knots. For coprime integers  $r$  and  $s$ , let  $T(r, s)$  denote the family of torus knots. As the trefoil knot  $3_1 = T(2, 3) = T(3, 2)$ , the following result generalizes Theorems 2.5 and 3.1 in [11].

**Corollary 1.3.** *Let  $m, k \geq 1$  be integers. Conjecture 1.1 (1) and (2) are true for the families of torus knots  $T(2, 2m+1)$  and  $T(3, 2^k)$ .*

The paper is organized as follows. In Section 2, we prove Theorem 1.2 by first providing an explicit evaluation of  $G_g(p)$  which generalizes (1.6). Next, we compute  $S_g^{\text{med}}(x)$  directly and then  $S_g^{\text{L}}(x)$  and  $S_g^{\text{R}}(x)$  via a careful contour deformation argument. In Section 3, we prove Theorem 3.1 which is an important (and technical) special case of Theorem 1.2 (3). In Section 4, we utilize Theorem 3.1 to prove a correct reformulation of Conjecture 1.1 (3) for the trefoil knot and Corollary 1.3. In Section 5, we make some concluding remarks.

## 2. PROOF OF THEOREM 1.2

*Proof of Theorem 1.2.* We begin by considering the expansion

$$e^{-\frac{ta}{b}}\Phi_f(e^{-t}) = \sum_{n=0}^{\infty}\frac{C_{n,f}}{n!}\left(\frac{t}{b}\right)^n. \quad (2.1)$$

The strange identity (1.9) and a standard calculation using the Mellin transform and inversion of

$$\mathcal{P}_{a,b,f}^{(\nu)}(q) := q^{\frac{a}{b}}\theta_{a,b,f}^{(\nu)}(q)$$

combined with a complex analytic computation shows that (see, e.g., [1, 21, 44])

$$C_{n,f} = (-1)^n L(-2n - \nu, f) \quad (2.2)$$

where

$$L(s, f) := \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

is the  $L$ -series associated to  $f$ . Using (2.2) and [1, Lemma 3.2], we obtain

$$C_{n,f} = (-1)^{n+1}\frac{M^{2n+\nu}}{2n+\nu+1}\sum_{m=1}^M f(m)B_{2n+\nu+1}\left(\frac{m}{M}\right) \quad (2.3)$$

where for  $k \geq 0$ ,  $B_k(x)$  denotes the  $k$ th Bernoulli polynomial. Equation (2.3) now yields

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{C_{n,f}}{(2n+\nu)!} t^{2n+\nu} &= (-1)^{\nu+1} i^{\nu} \sum_{m=1}^M f(m) \sum_{n=0}^{\infty} \frac{B_{2n+\nu+1}\left(\frac{m}{M}\right)}{(2n+\nu+1)!} (iMt)^{2n+\nu} \\ &= \frac{(-1)^{\nu+1} i^{\nu-1}}{Mt} \sum_{m=1}^M f(m) \sum_{n=0}^{\infty} \frac{B_{2n+\nu+1}\left(\frac{m}{M}\right)}{(2n+\nu+1)!} (iMt)^{2n+\nu+1} \end{aligned}$$

and so

$$\sum_{n=0}^{\infty} \frac{C_{n,f}}{(2n+\nu)!} t^{2n+\nu} = \begin{cases} \frac{i}{Mt} \sum_{m=1}^M f(m) \sum_{n=0}^{\infty} \frac{B_{2n+1}\left(\frac{m}{M}\right)}{(2n+1)!} (iMt)^{2n+1} & \text{if } \nu = 0, \\ \frac{1}{Mt} \sum_{m=1}^M f(m) \sum_{n=0}^{\infty} \frac{B_{2n+2}\left(\frac{m}{M}\right)}{(2n+2)!} (iMt)^{2n+2} & \text{if } \nu = 1. \end{cases} \quad (2.4)$$

From the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n,$$

where  $|z| < 2\pi$ , we deduce

$$\sum_{n=0}^{\infty} \frac{B_{2n+1}(x)}{(2n+1)!} z^{2n+1} = \frac{z}{2} \left( \frac{e^{xz}}{e^z - 1} + \frac{e^{-xz}}{e^{-z} - 1} \right) = \frac{z(e^{xz} - e^{-(x-1)z})}{2(e^z - 1)} \quad (2.5)$$

and

$$\sum_{n=0}^{\infty} \frac{B_{2n}(x)}{(2n)!} z^{2n} = \frac{z}{2} \left( \frac{e^{xz}}{e^z - 1} - \frac{e^{-xz}}{e^{-z} - 1} \right) = \frac{z(e^{xz} + e^{-(x-1)z})}{2(e^z - 1)}. \quad (2.6)$$

Assuming  $|t| < \frac{2\pi}{M}$ ,  $z = iMt$ , and taking  $x = \frac{m}{M}$ , where  $1 \leq m \leq M$ , we use (2.4)–(2.6) to obtain

$$\sum_{n=0}^{\infty} \frac{C_{n,f}}{(2n+\nu)!} t^{2n+\nu} = \begin{cases} -\frac{1}{2(e^{iMt} - 1)} \sum_{m=1}^M f(m) (e^{imt} - e^{-i(m-M)t}) & \text{if } \nu = 0, \\ \frac{i}{2(e^{iMt} - 1)} \sum_{m=1}^M f(m) (e^{imt} + e^{-i(m-M)t}) & \text{if } \nu = 1. \end{cases} \quad (2.7)$$

Now, let  $f = g$  where  $g$  is given by (1.10). One can then check that (2.7) becomes

$$\sum_{n=0}^{\infty} \frac{C_{n,g}}{(2n+\nu)!} t^{2n+\nu} = \begin{cases} 0 & \text{if } \nu = 0, \\ -2c \cdot \frac{\sin\left(\frac{(k_2 - k_1)t}{2}\right) \sin\left(\frac{(M - k_1 - k_2)t}{2}\right)}{\sin\left(\frac{Mt}{2}\right)} & \text{if } \nu = 1. \end{cases} \quad (2.8)$$



Define the sequence  $(\tilde{C}_{n,g})_{n \geq 0}$  by

$$\mathcal{F}_g(x) := e^{-\frac{a}{bx}} \Phi_g(e^{-\frac{1}{x}}) = \sum_{n=0}^{\infty} \frac{\tilde{C}_{n,g}}{b^n} \frac{1}{x^n}. \quad (2.9)$$

From (2.1) and (2.9), we have  $\tilde{C}_{n,g} = \frac{C_{n,g}}{n!}$ . Let  $G_g(p)$  and  $H_g(p)$  denote the Borel transforms of  $\mathcal{F}_g(x)$  and  $\mathcal{F}_g\left(\frac{x}{b}\right)$ , respectively. Then we have

$$G_g(p) = \sum_{n=0}^{\infty} \frac{\tilde{C}_{n+1,g}}{b^{n+1}n!} p^n$$

and

$$H_g(p) = \mathcal{B} \left( 1 + \sum_{n=0}^{\infty} \tilde{C}_{n+1,g} \frac{1}{x^{n+1}} \right) = \sum_{n=0}^{\infty} \frac{\tilde{C}_{n+1,g}}{n!} p^n \quad (2.10)$$

which imply

$$G_g(p) = \frac{1}{b} H_g\left(\frac{p}{b}\right). \quad (2.11)$$

From (2.8), we have

$$C_{0,g}p + \sum_{n=0}^{\infty} \frac{\tilde{C}_{n+1,g}(n+1)!}{(2n+3)!} p^{2n+3} = -2c \cdot \frac{\sin\left(\frac{(k_2-k_1)p}{2}\right) \sin\left(\frac{(M-k_1-k_2)p}{2}\right)}{\sin\left(\frac{Mp}{2}\right)}$$

and so

$$\sum_{n=0}^{\infty} \frac{\tilde{C}_{n+1,g}}{n!} \frac{n!(n+1)!}{(2n+3)!} p^{2n} = -\frac{1}{p^3} \left( 2c \cdot \frac{\sin\left(\frac{(k_2-k_1)p}{2}\right) \sin\left(\frac{(M-k_1-k_2)p}{2}\right)}{\sin\left(\frac{Mp}{2}\right)} + C_{0,g}p \right). \quad (2.12)$$

As

$$\sum_{n=0}^{\infty} \frac{(2n+3)!}{n!(n+1)!} p^n = \frac{6}{(1-4p)^{\frac{5}{2}}}, \quad (2.13)$$

we conclude from (2.10), (2.12) and (2.13)

$$H_g(p) = (f_{1,g} \otimes f_{2,g})(p)$$

where

$$f_{1,g}(p) = -\frac{1}{p^{3/2}} \left( 2c \cdot \frac{\sin\left(\frac{(k_2-k_1)\sqrt{p}}{2}\right) \sin\left(\frac{(M-k_1-k_2)\sqrt{p}}{2}\right)}{\sin\left(\frac{M\sqrt{p}}{2}\right)} + C_{0,g}\sqrt{p} \right), \quad (2.14)$$

$$f_{2,g}(p) = \frac{6}{(1-4p)^{\frac{5}{2}}}$$

and  $\otimes$  denotes the Hadamard product of two formal power series

$$\left( \sum_{n=0}^{\infty} a_n p^n \right) \otimes \left( \sum_{n=0}^{\infty} b_n p^n \right) := \sum_{n=0}^{\infty} a_n b_n p^n.$$

It now follows that

$$H_g(p) = \frac{1}{2\pi i} \int_{\gamma} f_{1,g}(s) f_{2,g}\left(\frac{p}{s}\right) \frac{ds}{s} \quad (2.15)$$

where  $\gamma$  is a circle with center at the origin and a small radius. Now, we consider the poles of  $f_{1,g}(s)$ . Note that there is no pole at  $s = 0$ . By (2.14), they are supported where  $p = \frac{4\ell^2\pi^2}{M^2}$ ,  $\ell \in \mathbb{Z}$ . However,

$$\sin\left(\frac{(k_2 - k_1)\sqrt{p}}{2}\right) \sin\left(\frac{(M - k_1 - k_2)\sqrt{p}}{2}\right) = 0$$

when  $p = \frac{4k^2\pi^2}{(k_2 - k_1)^2}$  or  $p = \frac{4k^2\pi^2}{(M - k_2 - k_1)^2}$  for  $k \in \mathbb{Z}$ . Thus, if for some  $k, \ell \in \mathbb{Z}$

$$\ell = \frac{Mk}{k_2 - k_1} \quad \text{or} \quad \ell = \frac{Mk}{M - k_2 - k_1}$$

which is true if and only if

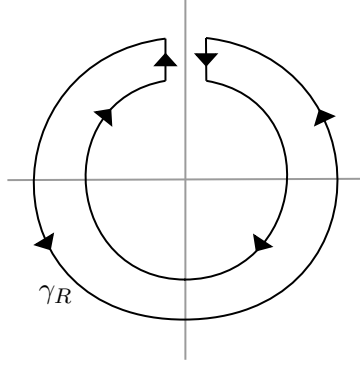
$$\ell = \frac{\frac{M}{\gcd(M, k_2 - k_1)} k}{\gcd(M, k_2 - k_1)} \quad \text{or} \quad \ell = \frac{\frac{M}{\gcd(M, M - k_2 - k_1)} k}{\gcd(M, M - k_2 - k_1)},$$

then  $f_{1,g}(s)$  does not have poles at  $p = \frac{4\ell^2\pi^2}{M^2}$ . Thus, the set of poles of  $f_{1,g}(s)$  is given by

$$\mathcal{N}_{1,g} = \left\{ \frac{4\ell^2\pi^2}{M^2} : \ell \in \mathbb{N}, \frac{M}{\gcd(M, k_2 - k_1)} \nmid \ell \text{ or } \frac{M}{\gcd(M, M - k_2 - k_1)} \nmid \ell \right\}. \quad (2.16)$$

We next carefully calculate the residues of the poles of the integrand (2.15) which are given by (2.16). Take a circle  $\gamma_R$  with center at the origin and of large radius  $R$  which encloses a finite number of points  $N$  in  $\mathcal{N}_{1,g}$ . Thus,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma_R} f_{1,g}(s) f_{2,g}\left(\frac{p}{s}\right) \frac{ds}{s} &= \sum_{\substack{p_i \in \mathcal{N}_{1,g} \\ 1 \leq i \leq N}} \text{Res}_{s \rightarrow p_i} \left( \frac{f_{1,g}(s) f_{2,g}\left(\frac{p}{s}\right)}{s} \right) \\ &= \sum_{\ell \in \mathbb{N}} \iota_N \frac{f_{2,g}\left(\frac{pM^2}{4\ell^2\pi^2}\right)}{\left(\frac{4\ell^2\pi^2}{M^2}\right)} \\ &\quad \times \lim_{s \rightarrow \frac{4\ell^2\pi^2}{M^2}} \left( s - \frac{4\ell^2\pi^2}{M^2} \right) \left[ -\frac{1}{s^{\frac{3}{2}}} \left( 2c \cdot \frac{\sin\left(\frac{(k_2 - k_1)\sqrt{s}}{2}\right) \sin\left(\frac{(M - k_1 - k_2)\sqrt{s}}{2}\right)}{\sin\left(\frac{M\sqrt{s}}{2}\right)} + C_{0,g}\sqrt{s} \right) \right] \end{aligned}$$

FIGURE 2. Contour deformation of  $\gamma_R$ 

$$\begin{aligned}
&= -\frac{96\pi c}{M^2} \sum'_{\ell \in \mathbb{N}} \frac{(-1)^\ell \ell}{\left(\frac{M^2}{4\ell^2\pi^2}\right)^{\frac{3}{2}} \left(\frac{4\ell^2\pi^2}{M^2} - 4p\right)^{\frac{5}{2}}} \left(\frac{M^2}{4\ell^2\pi^2}\right)^{\frac{3}{2}} \mathcal{S}(k_1, k_2, \ell, M) \\
&= -\frac{3\pi c}{M^2} \sum'_{\ell \in \mathbb{N}} \frac{(-1)^\ell \ell}{\left(\frac{\ell^2\pi^2}{M^2} - p\right)^{\frac{5}{2}}} \mathcal{S}(k_1, k_2, \ell, M)
\end{aligned} \tag{2.17}$$

where  $\sum'^N$  means that  $\ell$  runs over the first  $N$  elements of  $\mathcal{N}_{1,g}$ . It follows that

$$\left| \int_{\gamma_R} f_{1,g}(s) f_{2,g}\left(\frac{p}{s}\right) \frac{ds}{s} \right| \rightarrow 0 \tag{2.18}$$

as  $R \rightarrow \infty$ . Thus, (2.18) together with (2.15), (2.17) and the contour deformation given in Figure 2 imply that

$$H_g(p) = \frac{3\pi c}{M^2} \sum'_{\ell \in \mathbb{N}} \frac{(-1)^\ell \ell}{\left(\frac{\ell^2\pi^2}{M^2} - p\right)^{\frac{5}{2}}} \mathcal{S}(k_1, k_2, \ell, M). \tag{2.19}$$

Thus, (2.11) and (2.19) yield

$$G_g(p) = \frac{3\pi c}{M^2 b} \sum'_{\ell \in \mathbb{N}} \frac{(-1)^\ell \ell}{\left(\frac{\ell^2\pi^2}{M^2} - \frac{p}{b}\right)^{\frac{5}{2}}} \mathcal{S}(k_1, k_2, \ell, M). \tag{2.20}$$

The set of poles of  $G_g(p)$  is then

$$\mathcal{N}_g := \left\{ \frac{b\ell^2\pi^2}{M^2} : \ell \in \mathbb{N}, \frac{M}{\gcd(M, k_2 - k_1)} \nmid \ell \text{ or } \frac{M}{\gcd(M, M - k_2 - k_1)} \nmid \ell \right\}.$$

Since  $G_g(p)$  has two branches, by considering the cut complex plane  $\mathbb{C} \setminus \mathcal{N}_g$ , it can be made analytic in this region. This implies (1).

To prove the first part of (2), we first note the following formula, valid for  $\Re(x) > 0$ :

$$\int_{\gamma_m} \frac{e^{-px}}{(1-p)^{\frac{5}{2}}} dp = -\frac{2}{3} + \frac{4}{3}\sqrt{\pi} \cdot \mathcal{E}(\sqrt{x}) \tag{2.21}$$

where

$$\mathcal{E}(y) := \frac{2y^3 e^{-y^2}}{\sqrt{\pi}} \int_0^y e^{t^2} dt - \frac{y^2}{\sqrt{\pi}}$$

and  $\gamma_m = \frac{1}{2}(\gamma_l + \gamma_r)$ . From (2.20), we have

$$\begin{aligned} S_g^{\text{med}}(x) &= \int_{\gamma_m} e^{-px} G(p) dp = \frac{3\pi c}{M^2 b} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \int_{\gamma_m} \frac{e^{-px}}{\left(\frac{\ell^2 \pi^2}{M^2} - \frac{p}{b}\right)^{\frac{5}{2}}} dp \\ &= b \left(\frac{M}{\pi}\right)^3 \frac{3\pi c}{M^2 b} \sum'_{\ell \in \mathbb{N}} \frac{(-1)^\ell}{\ell^2} \mathcal{S}(k_1, k_2, \ell, M) \int_{\gamma_m} \frac{e^{-\frac{bp\ell^2\pi^2 x}{M^2}}}{(1-p)^{\frac{5}{2}}} dp \end{aligned} \quad (2.22)$$

where we have made the change of variable  $p \rightarrow \frac{bp\ell^2\pi^2}{M^2}$ . Thus, (2.21) and (2.22) imply

$$\begin{aligned} S_g^{\text{med}}(x) &= \frac{3Mc}{\pi^2} \sum'_{\ell \in \mathbb{N}} \frac{(-1)^\ell}{\ell^2} \mathcal{S}(k_1, k_2, \ell, M) \left\{ -\frac{2}{3} + \frac{4}{3} \sqrt{\pi} \cdot \mathcal{E}\left(\frac{\ell\pi}{M} \sqrt{bx}\right) \right\} \\ &= \frac{4Mc}{\pi^{\frac{3}{2}}} \sum'_{\ell \in \mathbb{N}} \frac{(-1)^\ell}{\ell^2} \mathcal{S}(k_1, k_2, \ell, M) \mathcal{E}\left(\frac{\ell\pi}{M} \sqrt{bx}\right) - 2 \cdot C_M, \end{aligned}$$

where  $C_M$  is given by (1.12). This implies that  $S_g^{\text{med}}(x)$  is analytic on  $\Re(x) > 0$ .

To prove the second part of (2) and (3), we explicitly evaluate  $S_g^{\mathbf{L}}(x)$  and  $S_g^{\mathbf{R}}(x)$  at  $x = -\frac{1}{2\pi i\alpha}$  with  $0 \neq \alpha \in \mathbb{Q}$ . First, we consider the case  $\alpha < 0$ . Noting that  $G_g(p)$  does not have poles in  $\mathbb{H}$ , and has branch point singularities at every point in  $\mathcal{N}_g$ , we move the contour  $\gamma_l$  to the positive imaginary axis, and  $\gamma_r$  to loop around every point in  $\mathcal{N}_g$  as shown in Figure 3.

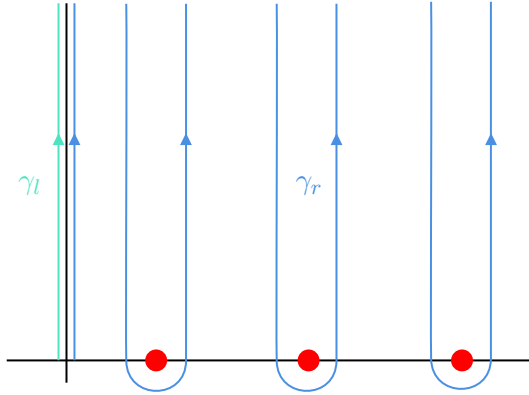


FIGURE 3. Contour deformation of  $\gamma_l$  and  $\gamma_r$

Thus (2.20) and integration by parts imply that

$$S_g^{\mathbf{L}}\left(-\frac{1}{2\pi i\alpha}\right) = \int_{\gamma_l} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp = \frac{3\pi c}{M^2 b} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \int_0^{i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2 \pi^2}{M^2} - \frac{p}{b}\right)^{\frac{5}{2}}} dp$$

$$\begin{aligned}
&= \frac{3\pi c}{M^2 b} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \left\{ \frac{bi}{3\pi\alpha} \int_0^{i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2 \pi^2}{M^2} - \frac{p}{b}\right)^{\frac{3}{2}}} dp - \frac{2bM^3}{3\pi^3 \ell^3} \right\} \\
&= \frac{ic}{M^2 \alpha} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \int_0^{i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2 \pi^2}{M^2} - \frac{p}{b}\right)^{\frac{3}{2}}} dp - C_M \\
&= -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \mathcal{S}(k_1, k_2, \ell, M) \int_0^{i\infty} \frac{e^{\frac{i\ell^2 \pi b p}{2M^2}}}{\left(p + \frac{1}{\alpha}\right)^{\frac{3}{2}}} dp - C_M \\
&= -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \int_0^{i\infty} \frac{\theta_{0,4M^2, \tilde{g}}^{(0)}(bp)}{\left(p + \frac{1}{\alpha}\right)^{\frac{3}{2}}} dp - C_M \tag{2.23}
\end{aligned}$$

where  $\tilde{g}$  is given by (1.11). Next, we consider the integral  $S_g^{\mathbf{R}}(x)$ . This equals the integral over the deformed contour  $\gamma_r$  as shown as in Figure 3. Let  $\mathcal{H}_\ell$  denote the Hankel contour around the point  $p_\ell \in \mathcal{N}_g$ . Then, we have

$$S_g^{\mathbf{R}}\left(-\frac{1}{2\pi i\alpha}\right) = \int_{\gamma_r} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp = \int_0^{i\infty} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp + \int_{\cup_\ell \mathcal{H}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp. \tag{2.24}$$

We compute the contribution of the Hankel contour in (2.24). Using (2.20), we get

$$\int_{\cup_\ell \mathcal{H}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp = \frac{3\pi c}{M^2 b} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \int_{\mathcal{H}_\ell} \frac{e^{\frac{p_\ell}{2\pi i\alpha}}}{\left(\frac{\ell^2 \pi^2}{M^2} - \frac{p_\ell}{b}\right)^{\frac{5}{2}}} dp_\ell. \tag{2.25}$$

Making the change of variables  $p_\ell \rightarrow bt + \frac{b\ell^2 \pi^2}{M^2}$  followed by  $t \rightarrow -\frac{2\pi i\alpha s}{b}$  to the integral in (2.25), we obtain

$$\begin{aligned}
\int_{\cup_\ell \mathcal{H}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp &= \frac{3\pi c}{M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \int_{\mathcal{H}^*} e^{\frac{bt + \frac{b\ell^2 \pi^2}{M^2}}{2\pi i\alpha}} (-t)^{-\frac{5}{2}} dt \\
&= \frac{3\pi c}{M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i\frac{b\ell^2 \pi}{2M^2 \alpha}} \int_{\mathcal{H}^*} e^{\frac{bt}{2\pi i\alpha}} (-t)^{-\frac{5}{2}} dt \\
&= \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{3c \cdot e^{\frac{3\pi i}{4}}}{2\sqrt{2\pi} M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i\frac{b\ell^2 \pi}{2M^2 \alpha}} \int_{\mathcal{H}} e^{-s} (-s)^{-\frac{5}{2}} ds \\
&= \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{2\sqrt{2}c \cdot e^{\frac{\pi i}{4}}}{M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i\frac{b\ell^2 \pi}{2M^2 \alpha}} \tag{2.26}
\end{aligned}$$

where  $\mathcal{H}^*$  (respectively,  $\mathcal{H}$ ) denotes the Hankel contour which encircles the origin and goes up and around the positive imaginary axis (respectively, going around the positive real axis). Note that in the last step of (2.26), we used Hankel's formula

$$\int_{\mathcal{H}} e^{-t} (-t)^{-z} dt = -\frac{2\pi i}{\Gamma(z)} \tag{2.27}$$

where  $\Gamma(z)$  is the usual Gamma function. Finally, for  $\alpha > 0$ , we move the contour  $\gamma_r$  to the negative imaginary axis, and  $\gamma_l$  to loop around every point in  $\mathcal{N}_g$  (this is simply the reflection of Figure 3 about the positive real axis with the labels  $\gamma_l$  and  $\gamma_r$  exchanged). Thus (2.20) and integration by parts imply that

$$\begin{aligned}
S_g^{\mathbf{R}}\left(-\frac{1}{2\pi i\alpha}\right) &= \int_{\gamma_r} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp \\
&= \frac{3\pi c}{M^2 b} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \int_0^{-i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2 \pi^2}{M^2} - \frac{p}{b}\right)^{\frac{5}{2}}} dp \\
&= \frac{3\pi c}{M^2 b} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \left\{ \frac{bi}{3\pi\alpha} \int_0^{-i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2 \pi^2}{M^2} - \frac{p}{b}\right)^{\frac{3}{2}}} dp - \frac{2bM^3}{3\pi^3 \ell^3} \right\} \\
&= \frac{ic}{M^2 \alpha} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \int_0^{-i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2 \pi^2}{M^2} - \frac{p}{b}\right)^{\frac{3}{2}}} dp - C_M \\
&= \frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \mathcal{S}(k_1, k_2, \ell, M) \int_0^{-i\infty} \frac{e^{-\frac{i\ell^2 \pi b p}{2M^2}}}{\left(-p + \frac{1}{\alpha}\right)^{\frac{3}{2}}} dp - C_M \\
&= -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \int_0^{i\infty} \frac{\theta_{0,4M^2, \tilde{g}}^{(0)}(bp)}{\left(p + \frac{1}{\alpha}\right)^{\frac{3}{2}}} dp - C_M. \tag{2.28}
\end{aligned}$$

Next, we consider the integral  $S_g^{\mathbf{L}}(x)$ . This equals the integral over the deformed contour as described above. Let  $\tilde{\mathcal{H}}_\ell$  denote the Hankel contour around the point  $p_\ell \in \mathcal{N}_g$ . Then we have

$$S_g^{\mathbf{L}}\left(-\frac{1}{2\pi i\alpha}\right) = \int_{\gamma_l} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp = \int_0^{-i\infty} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp + \int_{\cup_\ell \tilde{\mathcal{H}}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp. \tag{2.29}$$

We compute the contribution of the Hankel contour in (2.29). Using (2.20), we get

$$\int_{\cup_\ell \tilde{\mathcal{H}}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp = \frac{3\pi c}{M^2 b} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \int_{\tilde{\mathcal{H}}_\ell} \frac{e^{\frac{p_\ell}{2\pi i\alpha}}}{\left(\frac{\ell^2 \pi^2}{M^2} - \frac{p_\ell}{b}\right)^{\frac{5}{2}}} dp_\ell. \tag{2.30}$$

Making the change of variables  $p_\ell \rightarrow bt + \frac{b\ell^2 \pi^2}{M^2}$  followed by  $t \rightarrow -\frac{2\pi i\alpha s}{b}$  to the integral (2.30), we obtain

$$\begin{aligned}
\int_{\cup_\ell \tilde{\mathcal{H}}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p) dp &= \frac{3\pi c}{M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) \int_{\tilde{\mathcal{H}}^*} e^{\frac{bt + \frac{b\ell^2 \pi^2}{M^2}}{2\pi i\alpha}} (-t)^{-\frac{5}{2}} dt \\
&= \frac{3\pi c}{M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i\frac{b\ell^2 \pi}{2M^2 \alpha}} \int_{\tilde{\mathcal{H}}^*} e^{\frac{bt}{2\pi i\alpha}} (-t)^{-\frac{5}{2}} dt \\
&= \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{3c \cdot e^{\frac{3\pi i}{4}}}{2\sqrt{2\pi} M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i\frac{b\ell^2 \pi}{2M^2 \alpha}} \int_{\mathcal{H}} e^{-s} (-s)^{-\frac{5}{2}} ds
\end{aligned}$$

$$= \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{2\sqrt{2}c \cdot e^{\frac{\pi i}{4}}}{M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i \frac{b\ell^2 \pi}{2M^2 \alpha}} \quad (2.31)$$

where  $\tilde{\mathcal{H}}^*$  is the Hankel contour which encircles the origin and goes down and around the negative imaginary axis and (2.27) has been employed in the last line of (2.31). From (1.3), we have for  $0 \neq \alpha \in \mathbb{Q}$

$$S_g^{\text{med}} \left( -\frac{1}{2\pi i \alpha} \right) = \int_{\gamma_m} e^{\frac{p}{2\pi i \alpha}} G_g(p) dp = \frac{1}{2} \left( S_g^{\mathbf{L}} \left( -\frac{1}{2\pi i \alpha} \right) + S_g^{\mathbf{R}} \left( -\frac{1}{2\pi i \alpha} \right) \right). \quad (2.32)$$

By (2.24), (2.26), (2.29) and (2.31), we now define

$$\begin{aligned} \delta_{b,c,g} \left( -\frac{1}{2\pi i \alpha} \right) &= \frac{1}{2} \left( S_g^{\mathbf{L}} \left( -\frac{1}{2\pi i \alpha} \right) - S_g^{\mathbf{R}} \left( -\frac{1}{2\pi i \alpha} \right) \right) \\ &= \begin{cases} -\left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{\sqrt{2}c \cdot e^{\frac{\pi i}{4}}}{M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i \frac{b\ell^2 \pi}{2M^2 \alpha}} & \text{if } \alpha < 0, \\ \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{\sqrt{2}c \cdot e^{\frac{\pi i}{4}}}{M^2} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i \frac{b\ell^2 \pi}{2M^2 \alpha}} & \text{if } \alpha > 0. \end{cases} \end{aligned} \quad (2.33)$$

Thus, (2.23), (2.28), (2.32) and (2.33) yield

$$S_g^{\text{med}} \left( -\frac{1}{2\pi i \alpha} \right) = \begin{cases} -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \int_0^{i\infty} \frac{\theta_{0,4M^2, \tilde{g}}^{(0)}(bp)}{\left(p + \frac{1}{\alpha}\right)^{\frac{3}{2}}} dp - C_M - \delta_{b,c,g} \left( -\frac{1}{2\pi i \alpha} \right) & \text{if } \alpha < 0, \\ -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \int_0^{i\infty} \frac{\theta_{0,4M^2, \tilde{g}}^{(0)}(bp)}{\left(p + \frac{1}{\alpha}\right)^{\frac{3}{2}}} dp - C_M + \delta_{b,c,g} \left( -\frac{1}{2\pi i \alpha} \right) & \text{if } \alpha > 0. \end{cases}$$

This proves the second part of (2) and (3) upon letting  $\tilde{\delta}_{b,c,g} \left( -\frac{1}{2\pi i \alpha} \right) = \text{sgn}(\alpha) \delta_{b,c,g} \left( -\frac{1}{2\pi i \alpha} \right)$  where  $\text{sgn}$  is the usual sign function.  $\square$

### 3. A SPECIAL CASE OF THEOREM 1.2

In this section, we state an important special case of Theorem 1.2 (3). First, we introduce the following notations and conventions. Let  $h : \mathbb{Z} \rightarrow \mathbb{C}$  be a function of period  $\mathcal{M} = M$  or  $2M$  where  $M \geq 2$ . For any fixed  $1 \leq k_0 < 2\mathcal{M}$ , consider the set

$$\mathcal{S}(k_0) := \left\{ 1 \leq k \leq \frac{\mathcal{M}}{2} : k^2 \equiv k_0 \pmod{2\mathcal{M}} \right\}.$$

Let  $\mathcal{M}(k_0) \subseteq \mathcal{S}(k_0)$  be non-empty and such that  $h(j) = 0$  whenever

$$j \notin \mathcal{M}(k_0) \cup \{\mathcal{M} - k : k \in \mathcal{M}(k_0)\}.$$

Clearly,  $\mathcal{S}_h(k_0) := \mathcal{M}(k_0) \cup \{\mathcal{M} - k : k \in \mathcal{M}(k_0)\}$  is the support of  $h$ . For  $N \in \mathbb{N}$ , let

$$\Gamma_1(N) := \left\{ \begin{pmatrix} o & r \\ s & t \end{pmatrix} \in SL_2(\mathbb{Z}) : s \equiv 0 \pmod{N}, o \equiv t \equiv 1 \pmod{N} \right\}$$

and let  $\Gamma_{\mathcal{M}}$  be defined as  $\Gamma_1(2\mathcal{M})$  if  $\mathcal{M}$  is even and

$$\left\{ \begin{pmatrix} o & r \\ s & t \end{pmatrix} \in \Gamma_1(2\mathcal{M}) : r \equiv 0 \pmod{2} \right\}$$

if  $\mathcal{M}$  is odd. For  $\alpha_1, \alpha_2 \in \mathbb{Q}$ , we say that  $\alpha_1$  is  $\Gamma_{\mathcal{M}}$ -equivalent to  $\alpha_2$  if there exists a  $\gamma \in \Gamma_{\mathcal{M}}$  such that  $\gamma\alpha_1 = \alpha_2$  and write  $\alpha_1 \sim_{\mathcal{M}} \alpha_2$ . By adapting the proof of [22, Lemma 2.1], it follows that

$$\theta_{0,2M,h}^{(0)}(u\gamma z) = \chi(\gamma)(sz + t)^{\frac{1}{2}} \theta_{0,2M,h}^{(0)}(uz) \quad (3.1)$$

for all  $\gamma = \begin{pmatrix} o & r \\ s & t \end{pmatrix} \in \Gamma_{\mathcal{M}}$  and  $z \in \mathbb{H}$  where  $u = 1$  or  $2$  according as  $\mathcal{M} = M$  or  $2M$ ,  $\varepsilon_t = 1$  or  $i$  according as  $t \equiv 1$  or  $3 \pmod{4}$  and  $\chi(\gamma)$  is the multiplier defined by  $\chi(\gamma) = e^{\frac{\pi i \text{ork}_0}{\mathcal{M}}} \left(\frac{2s\mathcal{M}}{t}\right) \varepsilon_t^{-1}$  where  $(\cdot)$  is the extended Jacobi symbol. Next, we note that  $m := [\Gamma_{\mathcal{M}} : SL_2(\mathbb{Z})] < \infty$  as  $\Gamma_{\mathcal{M}} \subseteq \Gamma_1(2\mathcal{M})$ . Let  $K_m := \{\alpha_j \in \mathbb{Q} : 1 \leq j \leq m\}$  be a set of inequivalent cusps of  $\Gamma_{\mathcal{M}}$  and denote by

$$K'_m := \{\alpha_\ell \in K_m : \theta_{0,2M,\tilde{g}}^{(0)}(u\tau) \text{ decays exponentially as } \tau \rightarrow \alpha_\ell\}.$$

Finally, consider the set

$$A_{m,u} := \left\{ 0 \neq \alpha \in \mathbb{Q} : \exists (\alpha_\ell, \alpha_j) \in K'_m \times K'_m, \alpha \sim_{\mathcal{M}} \alpha_\ell \text{ and } \alpha_j \sim'_{\mathcal{M}} -\frac{1}{u\alpha} \right\}$$

where  $\sim'_{\mathcal{M}}$  means that if  $-\frac{1}{u\alpha} \sim_{\mathcal{M}} i\infty$ , then we choose a representative  $\alpha_j$  in the  $i\infty$  class with  $\alpha_j \neq i\infty$ .

**Theorem 3.1.** *Assume (1.9) is true and  $b = 2M$ . Let  $\alpha \in A_{m,u}$  so that there exists  $\alpha_j \in K'_m$  and  $\gamma_\alpha \in \Gamma_{\mathcal{M}}$  such that  $\gamma_\alpha \alpha_j = -\frac{1}{u\alpha}$ . Then we have*

$$S_g^{\text{med}} \left( -\frac{1}{2\pi i \alpha} \right) = \frac{2c\chi(\gamma_\alpha)(s\alpha_j + t)^{\frac{3}{2}}}{\alpha^{\frac{3}{2}}} \left( \frac{2e^{\frac{\pi i}{4}}}{\sqrt{M}} \cdot \theta_{0,2M,\tilde{g}}^{(1)}(u\alpha) - \tilde{R}_{\gamma_\alpha}(\alpha, \alpha_j) \right) - C_M$$

where

$$\tilde{R}_{\gamma_\alpha}(\alpha, \alpha_j) := \frac{2e^{\frac{\pi i}{4}}}{\sqrt{M}} R_{\gamma_\alpha}(\alpha, \alpha_j) + \frac{i}{\pi\sqrt{u}} \int_{C_\alpha} \frac{\theta_{0,2M,\tilde{g}}^{(0)}(u\tau)}{(\tau - \alpha_j)^{\frac{3}{2}}} d\tau$$

and

$$R_{\gamma_\alpha}(\alpha, \alpha_j) = -\frac{e^{\frac{\pi i}{4}}}{2\pi} \sqrt{\frac{M}{u}} \left( \int_{\alpha}^{i\infty} \frac{\theta_{0,2M,\tilde{g}}^{(0)}(u\tau)}{(\tau - \alpha)^{\frac{3}{2}}} d\tau - \int_{\alpha_j}^{\gamma^{-1}(i\infty)} \frac{\theta_{0,2M,\tilde{g}}^{(0)}(u\tau)}{(\tau - \alpha_j)^{\frac{3}{2}}} d\tau \right).$$

Here  $C_\alpha$  is any path from  $\gamma_\alpha^{-1}(0)$  to  $\gamma_\alpha^{-1}(i\infty)$  for which the integrand does not have any poles in the closed contour formed by  $C_\alpha$  and the positive imaginary axis.



*Proof of Theorem 3.1.* We choose  $b = 2M$  in Theorem 1.2 (3). This yields for any  $\alpha \in A_{m,u}$  that<sup>1</sup>

$$\mathcal{S}_g^{\text{med}} \left( -\frac{1}{2\pi i \alpha} \right) = -\frac{2ic}{\pi \alpha^{\frac{3}{2}}} \int_0^{i\infty} \frac{\theta_{0,2M,\tilde{g}}^{(0)}(p)}{\left(p + \frac{1}{\alpha}\right)^{\frac{3}{2}}} dp - C_M + \tilde{\delta}_{2M,c,g} \left( -\frac{1}{2\pi i \alpha} \right). \quad (3.2)$$

Note that  $\tilde{g}$  is function of period  $\leq 2M$  and so we take  $h = \tilde{g}$  (where we are assuming  $\mathcal{M} = M$  or  $2M$ ). Next, there exists  $\gamma_\alpha = \begin{pmatrix} o & r \\ s & t \end{pmatrix} \in \Gamma_{\mathcal{M}}$  and  $\alpha_j \in K'_m$  such that  $\gamma_\alpha \alpha_j = -\frac{1}{u\alpha}$ . Thus, by changing variables, using

$$d(\gamma_\alpha p) = \frac{dp}{(sp+t)^2}, \quad \gamma_\alpha p - \gamma_\alpha z = \frac{p-z}{(sp+t)(sz+t)} \quad (3.3)$$

for  $z \in \mathbb{C}$  and (3.1), we obtain

$$\begin{aligned} \int_0^{i\infty} \frac{\theta_{0,2M,\tilde{g}}^{(0)}(p)}{\left(p + \frac{1}{\alpha}\right)^{\frac{3}{2}}} dp &= \frac{1}{\sqrt{u}} \int_0^{i\infty} \frac{\theta_{0,2M,\tilde{g}}^{(0)}(up)}{\left(p + \frac{1}{u\alpha}\right)^{\frac{3}{2}}} dp = \frac{1}{\sqrt{u}} \int_{C_\alpha} \frac{\theta_{0,2M,\tilde{g}}^{(0)}(u\gamma_\alpha p)}{(\gamma_\alpha p - \gamma_\alpha \alpha_j)^{\frac{3}{2}}} d(\gamma_\alpha p) \\ &= \frac{\chi(\gamma_\alpha)(s\alpha_j + t)^{\frac{3}{2}}}{\sqrt{u}} \int_{C_\alpha} \frac{\theta_{0,2M,\tilde{g}}^{(0)}(up)}{(p - \alpha_j)^{\frac{3}{2}}} dp. \end{aligned} \quad (3.4)$$

Next, by (1.13) we have

$$\begin{aligned} \tilde{\delta}_{2M,c,g} \left( -\frac{1}{2\pi i \alpha} \right) &= \frac{4ce^{\frac{\pi i}{4}}}{\sqrt{M\alpha^{\frac{3}{2}}}} \sum'_{\ell \in \mathbb{N}} (-1)^\ell \ell \mathcal{S}(k_1, k_2, \ell, M) e^{-i\frac{\ell^2 \pi}{M\alpha}} \\ &= \frac{4ce^{\frac{\pi i}{4}}}{\sqrt{M\alpha^{\frac{3}{2}}}} \cdot \theta_{0,2M,\tilde{g}}^{(1)} \left( -\frac{1}{\alpha} \right) = \frac{4ce^{\frac{\pi i}{4}}}{\sqrt{M\alpha^{\frac{3}{2}}}} \cdot \theta_{0,2M,\tilde{g}}^{(1)}(u\gamma_\alpha \alpha_j). \end{aligned} \quad (3.5)$$

Consider the following holomorphic Eichler integral:

$$\tilde{\theta}_{2M,\tilde{g}}(z) := \int_z^{i\infty} \frac{\theta_{0,2M,\tilde{g}}^{(0)}(u\tau)}{(\tau - \bar{z})^{\frac{3}{2}}} d\tau. \quad (3.6)$$

Clearly,  $\tilde{\theta}_{2M,\tilde{g}}(z)$  is well-defined on  $\mathbb{H} \cup A_{m,u}$ . Thus, for  $z = x + iy \in \mathbb{H} \cup A_{m,u}$ , it follows using contour integration that

$$\tilde{\theta}_{2M,\tilde{g}}(z) = \sqrt{\frac{\pi u}{M}} e^{-\frac{i\pi}{4}} \sum_{\ell=0}^{\infty} \ell \tilde{g}(\ell) \Gamma \left( -\frac{1}{2}, \frac{2\pi u \ell^2 y}{M} \right) e^{\frac{\pi i u \ell^2}{M} \bar{z}} \quad (3.7)$$

where  $\Gamma(a, x)$  is the upper incomplete Gamma function given by

$$\Gamma(a, x) := \int_x^{\infty} w^{a-1} e^{-w} dw.$$

For  $\alpha \in A_{m,u}$ , we see from (3.7) and  $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$  that

$$\tilde{\theta}_{2M,\tilde{g}}(\alpha) = -2\pi \sqrt{\frac{u}{M}} e^{-\frac{i\pi}{4}} \theta_{0,2M,\tilde{g}}^{(1)}(u\alpha). \quad (3.8)$$

<sup>1</sup>Here, we can actually take  $0 \neq \alpha \in \mathbb{Q}$ .

Thus, for  $\alpha \in A_{m,u}$ , one can check that first using (3.8), then (3.6) followed by (3.1) and (3.3) yields

$$\theta_{0,2M,\tilde{g}}^{(1)}(u\alpha) - \overline{\chi(\gamma_\alpha)}(s\alpha_j + t)^{-\frac{3}{2}}\theta_{0,2M,\tilde{g}}^{(1)}(u\gamma_\alpha\alpha_j) = R_{\gamma_\alpha}(\alpha, \alpha_j). \quad (3.9)$$

Finally, (3.2), (3.4), (3.5) and (3.9) imply the result.  $\square$

#### 4. REFORMULATION AND PROOF OF COROLLARY 1.3

We first apply Theorems 1.2 and 3.1 to the case of the trefoil knot. If we take  $c = -\frac{1}{2}$ ,  $M = 12$ ,  $k_1 = 1$  and  $k_2 = 5$  in (1.10), then  $\Phi_g(q) = \Phi_{3_1}(q)$ , the Kontsevich-Zagier series given by (1.5). In this case, (2.16) becomes

$$\left\{ \frac{\ell^2 \pi^2}{6^2} : \ell \in \mathbb{N}, 2 \nmid \ell \text{ or } 3 \nmid \ell \right\}$$

which is (3.5) in [11]. In addition, as the conditions  $2 \nmid \ell$  or  $3 \nmid \ell$  are equivalent to  $\ell \equiv 1, 5, 7, 11 \pmod{12}$ , then after taking  $g = \left(\frac{12}{*}\right)$  and  $b = 24$  in (2.20) we have

$$G(p) = G_g(p) = -\frac{\sqrt{6}\pi}{2} \sum_{r \in \{1,5,7,11\}} \sum_{k=0}^{\infty} \frac{(-1)^{12k+r}(12k+r)}{\left(-p + \frac{(12k+r)^2 \pi^2}{6}\right)^{\frac{5}{2}}} \mathcal{S}(1, 5, 12k+r, 12).$$

As

$$(-1)^{12k+r} \mathcal{S}(1, 5, 12k+r, 12) = \begin{cases} -\frac{\sqrt{3}}{2} & \text{if } r \in \{1, 11\}, \\ \frac{\sqrt{3}}{2} & \text{if } r \in \{5, 7\}, \end{cases}$$

then (1.6) follows. By Theorem 1.2 (1) and (2),  $F_{3_1}(x) = \mathcal{F}_g(x)$  has a resurgent Borel transform  $G(p)$  and  $S^{\text{med}}(x)$  is an analytic function for  $\Re(x) > 0$  with radial limits at points  $\frac{1}{2\pi i}\mathbb{Q}$ . Next, we compute the radial limits. Notice that  $b = 2M$  and a short computation using (1.11) shows that  $\tilde{g}(n) = -\frac{\sqrt{3}}{2} \left(\frac{12}{n}\right)$  which has period  $\mathcal{M} = M = 12$ . Thus,  $\theta_{0,24,\tilde{g}}^{(0)}(z) = -\frac{\sqrt{3}}{2}\eta(z)$  where  $\eta(z)$  is the Dedekind eta-function. For  $0 \neq \alpha \in \mathbb{Q}$ , we choose  $K_1 = K'_1 = \{\alpha\}$ . Noting that  $\gamma\alpha = -\frac{1}{\alpha}$  for  $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $u = 1$ , we have  $C_{12} = 1$ ,  $\chi(\gamma) = e^{-\frac{i\pi}{4}}$  and

$$R_\gamma(\alpha, \alpha) = \frac{3e^{\frac{\pi i}{4}}}{2\pi} \int_0^{i\infty} \frac{\eta(\tau)}{(\tau - \alpha)^{\frac{3}{2}}} d\tau, \quad \tilde{R}_\gamma(\alpha, \alpha) = \frac{i\sqrt{3}}{\pi} \int_0^{i\infty} \frac{\eta(\tau)}{(\tau - \alpha)^{\frac{3}{2}}} d\tau.$$

Thus, the strange identity (1.7) and Theorem 3.1 yield

$$\begin{aligned} S^{\text{med}}\left(-\frac{1}{2\pi i\alpha}\right) &= \frac{\theta_{0,24,\tilde{g}}^{(1)}(\alpha)}{2} + \frac{e^{\frac{\pi i}{4}}\sqrt{3}}{\pi} \int_0^{i\infty} \frac{\eta(\tau)}{(\tau - \alpha)^{\frac{3}{2}}} d\tau - 1 \\ &= -e^{\frac{\pi i\alpha}{12}} \Phi_{3_1}(e^{2\pi i\alpha}) + \frac{e^{\frac{\pi i}{4}}\sqrt{3}}{\pi} \int_0^{i\infty} \frac{\eta(\tau)}{(\tau - \alpha)^{\frac{3}{2}}} d\tau - 1. \end{aligned} \quad (4.1)$$

Equation (4.1) corrects the statement of Theorem 5.2 in [11]. We now prove Corollary 1.3.

*Proof of Corollary 1.3.* For  $k \in \mathbb{N}$ , consider

$$\begin{aligned} \mathfrak{F}_k(q) &= (-1)^{h''(k)} q^{-h'(k)} \sum_{n=0}^{\infty} (q)_n \\ &\quad \times \sum_{3 \sum_{\ell=1}^{m(k)-1} j_\ell \equiv 1 \pmod{m(k)}} (-1)^{\sum_{\ell=1}^{m(k)-1} j_\ell} q^{\frac{-a(k) + \sum_{\ell=1}^{m(k)-1} j_\ell \ell}{m(k)} + \sum_{\ell=1}^{m(k)-1} \binom{j_\ell}{2}} \\ &\quad \times \sum_{i=0}^{m(k)-1} \prod_{\ell=1}^{m(k)-1} \begin{bmatrix} n + I(\ell \leq i) \\ j_\ell \end{bmatrix} \end{aligned} \quad (4.2)$$

where

$$h''(k) = \begin{cases} \frac{2^k-1}{3}, & \text{if } k \text{ is even,} \\ \frac{2^k-2}{3}, & \text{if } k \text{ is odd,} \end{cases} \quad h'(k) = \begin{cases} \frac{2^k-4}{3}, & \text{if } k \text{ is even,} \\ \frac{2^k-5}{3}, & \text{if } k \text{ is odd,} \end{cases}$$

$$a(k) = \begin{cases} \frac{2^{k-1}+1}{3}, & \text{if } k \text{ is even,} \\ \frac{2^k+1}{3}, & \text{if } k \text{ is odd,} \end{cases}$$

$m(k) = 2^{k-1}$ ,  $I(*)$  is the characteristic function and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_{n-k}(q)_k}$$

is the  $q$ -binomial coefficient. The expression  $\mathfrak{F}_k(q)$  matches the  $N$ th colored Jones polynomial for  $T(3, 2^k)$  at a root of unity  $q = e^{\frac{2\pi i}{N}}$  and is an element of  $\mathcal{H}$  (see [4] for further details). Choose  $c = -\frac{1}{2}$ ,  $M = 3 \cdot 2^{k+1}$ ,  $k_1 = 2^{k+1} - 3$  and  $k_2 = 2^{k+1} + 3$  in (1.10) and  $a = k_1^2$ ,  $b = 2M$  and  $\nu = 1$  in (1.8). In this case, we have the strange identity [4, Proposition 2.4]<sup>2</sup>

$$\mathfrak{F}_k(q) \text{ “} = \text{” } \theta_{(2^{k+1}-3)^2, 3 \cdot 2^{k+2}, g}^{(1)}(q). \quad (4.3)$$

Using (1.11), we obtain

$$\tilde{g}(\ell) = \begin{cases} (-1)^\ell \mathcal{S}(2^{k+1} - 3, 2^{k+1} + 3, \ell, 3 \cdot 2^{k+1}) & \text{if } 2^k \nmid \ell \text{ or } 3 \nmid \ell, \\ 0 & \text{otherwise} \end{cases}$$

which has period  $\mathcal{M} = M = 3 \cdot 2^k$ . By Theorem 1.2 (1) and (2),  $F_{T(3, 2^k)}(x) = \mathcal{F}_g(x)$  has a resurgent Borel transform  $G_g(p)$  and  $S_g^{\text{med}}(x)$  is an analytic function for  $\Re(x) > 0$  with radial limits at points  $\frac{1}{2\pi i} \mathbb{Q}$ . To compute the radial limits, first observe that  $b = 2M$ . From [22, Corollary 2.5], we choose  $K'_m = \{0, i\infty\}$  and

$$A_{m,1} = \left\{ 0 \neq \alpha \in \mathbb{Q} : \alpha \sim_M 0 \text{ or } i\infty, -\frac{1}{\alpha} \sim_M i\infty \text{ or } 0 \right\}.$$

<sup>2</sup>For  $k = 1$ , one may define the sum over the  $j_\ell$  to be 1 in (4.2) to recover (1.5). Thus, taking  $k = 1$  in (4.3) yields (1.7).

Clearly,  $A_{m,1}$  is non-empty. Indeed, if  $\alpha = 2Mn$  for  $n \in \mathbb{N}$ , we have  $\begin{pmatrix} 1 & -2Mn \\ 0 & 1 \end{pmatrix} (2Mn) = 0$  and  $\begin{pmatrix} 1 & 0 \\ 2Mn & 1 \end{pmatrix} (-\frac{1}{2Mn}) = i\infty$ . Set  $c_t := -\frac{2^{\frac{t-1}{2}} \sqrt{3} e^{\frac{\pi i}{4}}}{\pi}$  and  $d_t := \frac{e^{\frac{\pi i}{4}}}{\sqrt{3} \cdot 2^{\frac{t-1}{2}}}$ . For simplicity, put  $\theta_{k,\tilde{g}}(\tau) := \theta_{0,3 \cdot 2^{k+2},\tilde{g}}^{(0)}(\tau)$ . Then for  $\alpha \in A_{m,1}$ , we have

$$R_{\gamma_\alpha}(\alpha, \alpha_j) = c_t \cdot \begin{cases} \left( \int_\alpha^{i\infty} \frac{\theta_{k,\tilde{g}}(\tau)}{(\tau - \alpha)^{\frac{3}{2}}} d\tau - \int_0^{\gamma^{-1}(i\infty)} \frac{\theta_{k,\tilde{g}}(\tau)}{\tau^{\frac{3}{2}}} d\tau \right) & \text{if } \alpha \sim_M i\infty, \alpha_j = 0, \\ \left( \int_\alpha^{i\infty} \frac{\theta_{k,\tilde{g}}(\tau)}{(\tau - \alpha)^{\frac{3}{2}}} d\tau - \int_{\alpha_j}^{\gamma^{-1}(i\infty)} \frac{\theta_{k,\tilde{g}}(\tau)}{(\tau - \alpha_j)^{\frac{3}{2}}} d\tau \right) & \text{if } \alpha \sim_M 0, \alpha_j \sim_M i\infty \end{cases} \quad (4.4)$$

where in the first (respectively, second) case of (4.4)  $\gamma_\alpha \in \Gamma_M$  is such that  $\gamma_\alpha(0) = -\frac{1}{\alpha}$  (respectively,  $\gamma_\alpha(\alpha_j) = -\frac{1}{\alpha}$  and  $\alpha_j \neq i\infty$ ). Thus,

$$\tilde{R}_{\gamma_\alpha}(\alpha, \alpha_j) = d_t \cdot R_{\gamma_\alpha}(\alpha, \alpha_j) + \frac{i}{\pi} \begin{cases} \int_{C_\alpha} \frac{\theta_{k,\tilde{g}}(\tau)}{\tau^{\frac{3}{2}}} d\tau & \text{if } \alpha \sim_M i\infty, \alpha_j = 0, \\ \int_{C_\alpha} \frac{\theta_{k,\tilde{g}}(\tau)}{(\tau - \alpha_j)^{\frac{3}{2}}} d\tau & \text{if } \alpha \sim_M 0, \alpha_j \sim_M i\infty \end{cases}$$

and so by Theorem 3.1, we have an expression for  $S_g^{\text{med}}(-\frac{1}{2\pi i\alpha})$  where  $\alpha \in A_{m,1}$ . This proves the result for  $T(3, 2^k)$ .

Let  $m \in \mathbb{N}$ . For  $0 \leq \ell \leq m-1$ , define

$$X_m^{(\ell)}(q) := \sum_{k_1, k_2, \dots, k_m=0}^{\infty} (q)_{k_m} q^{k_1^2 + \dots + k_{m-1}^2 + k_{\ell+1} + \dots + k_{m-1}} \prod_{i=1}^{m-1} \begin{bmatrix} k_{i+1} + \delta_{i,\ell} \\ k_i \end{bmatrix} \quad (4.5)$$

where  $\delta_{i,\ell}$  is the characteristic function. The expression  $X_m^{(0)}(q)$  matches the  $N$ th colored Jones polynomial for  $T(2, 2m+1)$  when  $q = e^{\frac{2\pi i}{N}}$  and is an element of  $\mathcal{H}$ . Choose  $c = -\frac{1}{2}$ ,  $M = 8m+4$ ,  $k_1 = 2m - 2\ell - 1$  and  $k_2 = 2m + 2\ell + 3$  in (1.10) and  $a = k_1^2$ ,  $b = 2M$  and  $\nu = 1$  in (1.8). In this case, we have Hikami's strange identity [26, Eqn. (15)]<sup>3</sup>

$$X_m^{(\ell)}(q) = \theta_{(2m-2\ell-1)^2, 8(2m+1), g}^{(1)}(q). \quad (4.6)$$

Thus, by Theorem 1.2 (1) and (2),  $F_{T(2,2m+1)}(x) = \mathcal{F}_g(x)$  has a resurgent Borel transform  $G_g(p)$  and  $S_g^{\text{med}}(x)$  is an analytic function for  $\Re(x) > 0$  with radial limits at points  $\frac{1}{2\pi i}\mathbb{Q}$ . To compute the radial limits, first observe that  $b = 2M$ . From [22, Corollary 2.5], we choose  $K'_m = \{0, i\infty\}$  and

$$A_{m,1} = \left\{ 0 \neq \alpha \in \mathbb{Q} : \alpha \sim_M 0 \text{ or } i\infty, -\frac{1}{\alpha} \sim_M i\infty \text{ or } 0 \right\}.$$

<sup>3</sup>Taking  $m = 1$  and  $\ell = 0$  in (4.5) and (4.6) recovers (1.7).

Clearly,  $A_{m,1}$  is non-empty. Indeed, if  $\alpha = 2Mn$  for  $n \in \mathbb{N}$ , we have  $\begin{pmatrix} 1 & -2Mn \\ 0 & 1 \end{pmatrix} (2Mn) = 0$  and  $\begin{pmatrix} 1 & 0 \\ 2Mn & 1 \end{pmatrix} (-\frac{1}{2Mn}) = i\infty$ . Set  $c_m := -\frac{e^{\frac{\pi i}{4}} \sqrt{2m+1}}{\pi}$  and  $d_m := \frac{e^{\frac{\pi i}{4}}}{\sqrt{2m+1}}$ . For simplicity, put  $\theta_{m,\ell,\tilde{g}}(\tau) := \theta_{(2m-2\ell-1)^2, 8(2m+1), \tilde{g}}^{(0)}(\tau)$ . Then for  $\alpha \in A_{m,1}$ , we have

$$R_{\gamma_\alpha}(\alpha, \alpha_j) = c_m \cdot \begin{cases} \left( \int_{\alpha}^{i\infty} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{(\tau - \alpha)^{\frac{3}{2}}} d\tau - \int_0^{\gamma^{-1}(i\infty)} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{\tau^{\frac{3}{2}}} d\tau \right) & \text{if } \alpha \sim_M i\infty, \alpha_j = 0, \\ \left( \int_{\alpha}^{i\infty} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{(\tau - \alpha)^{\frac{3}{2}}} d\tau - \int_{\alpha_j}^{\gamma^{-1}(i\infty)} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{(\tau - \alpha_j)^{\frac{3}{2}}} d\tau \right) & \text{if } \alpha \sim_M 0, \alpha_j \sim_M i\infty \end{cases} \quad (4.7)$$

where in the first (respectively, second) case of (4.7)  $\gamma_\alpha \in \Gamma_M$  is such that  $\gamma_\alpha(0) = -\frac{1}{\alpha}$  (respectively,  $\gamma_\alpha(\alpha_j) = -\frac{1}{\alpha}$  and  $\alpha_j \neq i\infty$ ). Thus,

$$\tilde{R}_{\gamma_\alpha}(\alpha, \alpha_j) = d_m \cdot R_{\gamma_\alpha}(\alpha, \alpha_j) + \frac{i}{\pi} \begin{cases} \int_{C_\alpha} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{\tau^{\frac{3}{2}}} d\tau & \text{if } \alpha \sim_M i\infty, \alpha_j = 0, \\ \int_{C_\alpha} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{(\tau - \alpha_j)^{\frac{3}{2}}} d\tau & \text{if } \alpha \sim_M 0, \alpha_j \sim_M i\infty \end{cases}$$

and so by Theorem 3.1, we have an expression for  $S_g^{\text{med}}(-\frac{1}{2\pi i\alpha})$  where  $\alpha \in A_{m,1}$ . This proves the result for  $T(2, 2m+1)$ .  $\square$

## 5. CONCLUDING REMARKS

In order to apply Theorem 1.2 (and thus verify Conjecture 1.1 (1) and (2)) and Theorem 3.1 to all torus knots  $T(r, s)$ , one needs to prove the relevant strange identity (1.9). The first obstruction in this task is finding an explicit “non-cyclotomic” expansion for  $J_N(T(r, s); q)$  from which an element in  $\mathcal{H}$  such as (4.2) or (4.5) can be extracted. The Rosso-Jones formula for  $J_N(T(r, s); q)$  [40, page 132] does not appear to be sufficient. Instead, one should consider the walks along braids method in [3, 5]. To obtain the right-hand side of (1.9) in this case, one takes  $\nu = 1$ ,  $a = k_1^2$ ,  $b = 2M$  and  $f = g$  where  $k_1 = rs - r - s$ ,  $k_2 = rs + r - s$ ,  $c = -\frac{1}{2}$  and  $M = 2rs$  [27, 28]. The second obstruction is in determining the underlying  $q$ -series identity which implies the strange identity. Recently, Lovejoy [35] used the theory of Bailey pairs to not only give a streamlined proof of such a  $q$ -series identity for  $T(2, 2m+1)$  [26], but provide a wealth of new examples. Thus, it would be of substantial interest to further develop his techniques. Finally, what can one say about Conjecture 1.1 for satellite or hyperbolic knots?

## ACKNOWLEDGEMENTS

The first author thanks Dan Zhang, Nick Dorey and Phil Trinh for many interesting discussions. He was supported by EPSRC grant EP/V012479/1 as a postdoc at the University of Bath while part of this work was completed. The third author was partially supported by Enterprise Ireland CS20212030. He would like to thank the Okinawa Institute of Science and Technology for their hospitality and support during his visit from June 9 to September 1, 2022 as part of

their Theoretical Sciences Visiting Program. In particular, the first and third author are very grateful to Professor Reiko Toriumi (OIST) for her firm instance for us to converse after the first author's talk at OIST on August 23, 2022. He would also like to thank the Max-Planck-Institut für Mathematik for their hospitality and support during the completion of this paper. Finally, the authors thank David Sauzin for his helpful comments on median resummation.

## REFERENCES

- [1] S. Ahlgren, B. Kim and J. Lovejoy, *Dissections of strange  $q$ -series*, Ann. Comb. **23** (2019), no. 3-4, 427–442.
- [2] J. Andersen, W. Mistegård, *Resurgence analysis of quantum invariants of Seifert fibered homology spheres*, J. Lond. Math. Soc. (2) **105** (2022), no. 2, 709–764.
- [3] C. Armond, *Walks along braids and the colored Jones polynomial*, J. Knot Theory Ramifications **23** (2014), no. 2, 1450007, 15pp.
- [4] C. Bijaoui, H.U. Boden, B. Myers, R. Osburn, W. Rushworth, A. Tronsgard and S. Zhou, *Generalized Fishburn numbers and torus knots*, J. Combin. Theory Ser. A **178** (2021), 105355.
- [5] H.U. Boden, M. Shimoda, *Braid representatives minimizing the number of simple walks*, Ars Math. Contemp. **23** (2023), no. 1, Paper No. 10, 27 pp.
- [6] M. Borinsky, G. V. Dunne, *Non-perturbative completion of Hopf-algebraic Dyson-Schwinger equations*, Nuclear Phys. B **957** (2020), 115096, 17 pp.
- [7] F. Chapoton, F. Fauvet, C. Malvenuto and J-Y, Thibon, *Algebraic combinatorics, resurgence, moulds and applications (CARMA). Volume 1*, IRMA Lectures in Mathematics and Theoretical Physics, **31**. EMS Publishing House, Berlin 2020.
- [8] F. Chapoton, F. Fauvet, C. Malvenuto and J-Y, Thibon, *Algebraic combinatorics, resurgence, moulds and applications (CARMA). Volume 2*, IRMA Lectures in Mathematics and Theoretical Physics, **32**. EMS Publishing House, Berlin 2020.
- [9] O. Costin, *On Borel summation and Stokes phenomena for rank-1 nonlinear systems of ordinary differential equations*, Duke Math. J. **93** (1998), no. 2, 289–344.
- [10] O. Costin, *Asymptotics and Borel summability*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, **141** CRC Press, Boca Raton, FL, 2009.
- [11] O. Costin, S. Garoufalidis, *Resurgence of the Kontsevich-Zagier series*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 3, 1225–1258.
- [12] E. Delabaere, F. Pham, *Resurgent methods in semi-classical asymptotics*, Ann. Inst. H. Poincaré Phys. Théor. **71** (1999), no. 1, 1–94.
- [13] D. Dorigoni, A. Kleinschmidt, *Resurgent expansion of Lambert series and iterated Eisenstein integrals*, Commun. Number Theory Phys. **15** (2021), no. 1, 1–57.
- [14] J. Écalle, *Les fonctions résurgentes. Tome I. Les algèbres de fonctions résurgentes*, Publications Mathématiques d'Orsay, vol. 81, p. 5, Université de Paris-Sud, Département de Mathématique, Orsay, 1981, 247 pp.
- [15] J. Écalle, *Les fonctions résurgentes. Tome II. Les fonctions résurgentes appliquées à l'itération*, Publications Mathématiques d'Orsay, vol. 81, p. 6, Université de Paris-Sud, Département de Mathématique, Orsay, 1981, pp. 248–531.
- [16] J. Écalle, *Les fonctions résurgentes. Tome III. L'équation du pont et la classification analytique des objets locaux*, Publications Mathématiques d'Orsay, vol. 85, p. 5, Université de Paris-Sud, Département de Mathématique, Orsay, 1985, 587 pp.
- [17] S. Garoufalidis, *On the characteristic and deformation varieties of a knot*, Proceedings of the Casson Fest, 291–309, Geom. Topol. Monogr., **7**, Geom. Topol. Publ., Coventry, 2004.
- [18] S. Garoufalidis, *Chern-Simons theory, analytic continuation and arithmetic*, Acta Math. Vietnam. **33** (2008), no. 3, 335–362.
- [19] S. Garoufalidis, *The Jones slopes of a knot*, Quantum Topol. **2** (2011), no. 1, 43–69.
- [20] S. Garoufalidis, T. T. Q. Lê, *Gevey series in quantum topology*, J. Reine Angew. Math. **618** (2008), 169–195.
- [21] A. Goswami, A. Jha, B. Kim and R. Osburn, *Asymptotics and sign patterns for coefficients in expansions of Habiro elements*, Math. Z., to appear.

- [22] A. Goswami, R. Osburn, *Quantum modularity of partial theta series with periodic coefficients*, Forum Math. **33** (2021), no. 2, 451–463.
- [23] K. Habiro, *On the colored Jones polynomial of some simple links*, in: Recent progress toward the volume conjecture (Kyoto, 2000), Sūrikaiseikikenkyūsho Kōkyūroku **1172** (2000), 34–43.
- [24] K. Habiro, *Cyclotomic completions of polynomials rings*, Publ. Res. Inst. Math. Sci. **40** (2004), no. 4, 1127–1146.
- [25] L. Han, Y. Li, D. Sauzin and S. Sun, *Resurgence and partial theta series*, Funct. Anal. Appl., to appear.
- [26] K. Hikami, *q-series and L-functions related to half-derivates of the Andrews-Gordon identity*, Ramanujan J. **11** (2006), no. 2, 175–197.
- [27] K. Hikami, A.N. Kirillov, *Torus knot and minimal model*, Phys. Lett. B **575** (2003), no. 3-4, 343–348.
- [28] K. Hikami, A.N. Kirillov, *Hypergeometric generating function of L-function, Slater’s identities, and quantum invariant*, St. Petersburg Math. J. **17** (2006), no. 1, 143–156.
- [29] V. Huyuh, T. T. Q. Lê, *On the colored Jones polynomial and the Kashaev invariant*, J. Math. Sci. (N.Y.) **146** (2007), no. 1, 5490–5504.
- [30] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111.
- [31] E. Kalfagianni, A. T. Tran, *Knot cabling and the degree of the colored Jones polynomial*, New York J. Math. **21** (2015), 905–941.
- [32] S. Kamimoto, *Resurgent functions and nonlinear systems of differential and difference equations*, Adv. Math. **406** (2022), Paper No. 108533, 28pp.
- [33] M. Kontsevich, Y. Soibelman, *Analyticity and resurgence in wall-crossing formulas*, Lett. Math. Phys. **112** (2022), no. 2, Paper No. 32, 56 pp.
- [34] T. T. Q. Lê, *Quantum invariants of 3-manifolds: Integrality, splitting, and perturbative expansion*, Topology Appl. **127** (2003), no. 1-2, 125–152.
- [35] J. Lovejoy, *Bailey pairs and strange identities*, J. Korean Math. Soc. **59** (2022), no. 5, 1015–1045.
- [36] A. A. Mahmoud, K. Yeats, *Connected chord diagrams and the combinatorics of asymptotic expansions*, J. Integer Seq. **22** (2022), no. 7, Art. 22.7.5, 22 pp.
- [37] B. Malgrange, *Introduction aux travaux de J. Écalle*, Enseign. Math. (2) **31** (1985), no. 3-4, 261–282.
- [38] M. Mariño, *Lectures on non-perturbative effects in large N gauge theories, matrix models and strings*, Fortschr. Phys. **62** (2014), no. 5-6, 455–540.
- [39] C. Mitschi, D. Sauzin, *Divergent series, summability and resurgence. I. Monodromy and resurgence*, Lecture Notes in Mathematics, **2153** Springer, 2016.
- [40] H.R. Morton, *The coloured Jones function and Alexander polynomial for torus knots*, Math. Proc. Cambridge Philos. Soc. **117** (1995), no. 1, 129–135.
- [41] H. Murakami, Y. Yokota, *Volume conjecture for knots*, SpringerBriefs in Mathematical Physics, 30. Springer, Singapore, 2018.
- [42] D. Sauzin, *Variations on the resurgence of the Gamma function*, preprint available at <https://arxiv.org/abs/2112.15226>
- [43] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Topology **40** (2001), no. 5, 945–960.
- [44] D. Zagier, *Quantum modular forms*, Quanta of maths, 659–675, Clay Math. Proc., **11**, Amer. Math. Soc., Providence, RI, 2010.

FACULTY OF COMPUTER SCIENCE, RUHR UNIVERSITY, BOCHUM, GERMANY  
 Email address: samuel.crew@rub.de

MAX PLANCK INSTITUTE FOR SECURITY AND PRIVACY, BOCHUM, GERMANY

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, THE UNIVERSITY OF TEXAS RIO GRANDE VALLEY,  
 1201 W. UNIVERSITY DR., EDINBURG, TX, 78539  
 Email address: ankush.goswami@utrgv.edu  
 Email address: ankushgoswami3@gmail.com

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY COLLEGE DUBLIN, BELFIELD, DUBLIN 4, IRELAND  
*Email address:* `robert.osburn@ucd.ie`

THEORETICAL SCIENCES VISITING PROGRAM, OKINAWA INSTITUTE OF SCIENCE AND TECHNOLOGY GRADUATE  
UNIVERSITY, ONNA, 904-0495, JAPAN