Max-Planck-Institut für Mathematik Bonn

On the behavior of Massey products under field extension

by

Aleksandar Milivojević

Max-Planck-Institut für Mathematik
Preprint Series 2023 (8)

Date of submission: April 15, 2023

On the behavior of Massey products under field extension

by

Aleksandar Milivojević

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

MPIM 23-8

ON THE BEHAVIOR OF MASSEY PRODUCTS UNDER FIELD EXTENSION

ALEKSANDAR MILIVOJEVIĆ

Abstract. We show that global vanishing of Massey products on a commutative differential graded algebra is not invariant under field extension. The non-vanishing of any triple Massey product is invariant under field extension, while higher Massey products can generally vanish upon field extension. If the field being extended is algebraically closed, all non-vanishing Massey products remain non-vanishing on a finite type commutative differential graded algebra.

1. Introduction

Massey products are higher-order multi-valued operations on the cohomology of a differential graded algebra which provide obstructions to formality, i.e. the existence of a chain of quasi-isomorphisms between the differential graded algebra and its cohomology equipped with trivial differential. We will be working with $\Bbbk$-cdga's, i.e. commutative (though this property will largely be inessential) differential graded algebras over fields $\Bbbk$. For purposes of discussion we will assume the fields are of characteristic zero and the cdga's connected; however, our general results (Theorem 1.1 (2) and (4)) do not require these assumptions.

Following the convention of [4], [1], for homogeneous cohomology classes $z_1, \ldots, z_n$, we first define a *defining system* for the n-fold Massey product $\langle z_1, \ldots, z_n \rangle$ to be a choice of representatives $a_{i,i}$ for z_i, and for each pair $0 \leq i < j \leq n$ other than $(i,j) = (1,n)$, a choice of element $a_{i,j}$ (if it exists) such that

$$d(a_{i,j}) = \sum_{k=i}^{j-1} (-1)^{|a_{i,k}|+1} a_{i,k} a_{k+1,j}.$$

Then the n-fold Massey product is the set of cohomology classes

$$\Big\{ \Big[\sum_{k=1}^{n-1} (-1)^{|a_{1,k}|+1} a_{1,k} a_{k+1,n} \Big] \Big\}$$

obtained by running over all defining systems[1]. The Massey product is well-defined if the above set is non-empty, i.e. if there exists at least one defining system. We say the Massey product is *trivial*, or *vanishes*, if $0 \in \langle z_1, \ldots, z_n \rangle$. The above set is a quasi-isomorphism invariant of differential graded algebras.

Though formality of a cdga implies that all Massey products vanish, the converse does not hold (see for example [3, 1.5] for a systematic study of this phenomenon on sufficiently highly connected rational Poincaré duality algebras). Here we illustrate one more shortcoming of the property "all Massey products vanish": unlike what is clearly true for formality, the validity of this property is not preserved under field extension.

We then investigate how individual Massey products behave under field extension. Concretely, non-trivial triple Massey products, like cup products, remain non-trivial upon field extension. For higher Massey products this need not be the case, unless the starting field is algebraically closed:

Theorem 1.1. Let $\Bbbk$ be a field. We have the following:

[1] The triple Massey product $\langle z_1, z_2, z_3 \rangle$ enjoys the special property that it can equivalently be described, when defined, as a single element in the quotient of the cohomology modulo the ideal generated by z_1, z_3.

(1) There are examples of $\Bbbk$-cdga's on which all Massey products vanish, but such that upon field extension not all Massey products vanish (Section 2).
(2) Non-trivial triple Massey products remain non-trivial upon field extension (Section 3.1).
(3) A quadruple (or higher) Massey product can in general become trivial upon field extension (Section 3.2).
(4) If $\Bbbk$ is algebraically closed, non-trivial Massey products of any order remain non-trivial upon extension of the field $\Bbbk$ on a degree-wise finite-dimensional cdga. (Proposition 3.4).

In order to detect the non-formality of a cdga via Massey products, one may thus benefit from looking at both larger and smaller ground fields. The example used in (1), and its minimal models, provide examples of non-formal real cdga's with "uniformly" (i.e. simultaneously, consistently, in a precise sense) vanishing real Massey products (Remark 2.6).

In the category of A_∞- (or C_∞-) algebras, there are quasi-isomorphisms $H(A) \to A$, where the cohomology $H(A)$ is an A_∞-algebra with trivial differential, the multiplication it inherits from A, and n-to-1 operations $\{m_n\}_{n\geq 3}$. The higher operations on $H(A)$ are also sometimes referred to as Massey products, and enjoy the property of being genuine n-to-1 operations. However, the A_∞-algebra structure on $H(A)$ is not unique; automorphisms of this structure will in general change the operations m_n, and after collecting the outputs of all possible m_n on a given input, one again ends up with a multi-valued operation as in the "ad hoc" definition given before. Note also that the ad hoc Massey products are not even well-defined on all inputs, unlike the operations m_n. We refer the reader to [1] for an investigation of the relation between these two notions of Massey products. In what follows we will be considering only the ad hoc notion, defined directly on the cdga level.

Acknowledgements. This work was inspired by a conversation with Scott Wilson about [13, Section 6]; I thank him and Jonas Stelzig for numerous helpful discussions and comments, together with the Max Planck Institute for Mathematics in Bonn for its generous hospitality.

2. Global vanishing of Massey products is not preserved under field extension

In this section we show by example that having all Massey products vanish over a given field does not imply all Massey products vanish over a larger field. Upon field extension, elements in an algebra generally may become decomposable, allowing for substantially new Massey products to be considered; cf. [12, p. 203f.] on how the *tensor* Massey products, arising from the Eilenberg–Moore spectral sequence for an augmented dga, generally have a larger domain of definition than the ad hoc (*tuple*, in the terminology of loc. cit.) Massey products. Our example will have all Massey products vanishing over the real numbers, in fact *uniformly* vanishing in a sense to be explained below, while its non-formality will be detected by a non-trivial triple Massey product on its complexification.

The example is based on one often considered at the interface of rational homotopy theory and complex geometry. Take the complex Lie group G consisting of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{C}$. Quotienting by the subgroup of matrices with entries in the Gaussian integers $\mathbb{Z}[i]$ yields a compact complex threefold known as the Iwasawa manifold. The holomorphic one-forms $\phi_1 = dx, \phi_2 = dy, \phi_3 = xdy - dz$ on $\mathbb{C}^3$ are left G-invariant and hence descend to the Iwasawa manifold. The induced map from the exterior algebra generated by these forms and their conjugates $\Lambda(\phi_1, \overline{\phi_1}, \phi_2, \overline{\phi_2}, \phi_3, \overline{\phi_3})$, equipped with the de Rham differential determined by $d\phi_3 = \partial\phi_3 = \phi_1\phi_2$, into the de Rham algebra of smooth complex-valued forms on the Iwasawa manifold, is a quasi-isomorphism. From this finite-dimensional model of the Iwasawa manifold one easily sees that it carries a non-trivial triple Massey product over the complex numbers. Namely consider $\langle[\phi_1],[\phi_1],[\phi_2]\rangle$. For any choice of primitive of $\phi_1\phi_1=0$ and of $\phi_1\phi_2$, the resulting class in the Massey product[2] is $[\phi_1\phi_3]\neq 0$.

Now let us consider the above over the real numbers. As a real Lie group, the group G consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & x_1 & -y_1 & x_3 & -y_3 \\ 0 & 1 & y_1 & x_1 & y_3 & x_3 \\ 0 & 0 & 1 & 0 & x_2 & -y_2 \\ 0 & 0 & 0 & 1 & y_2 & x_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_i, y_i \in \mathbb{R}$.

By looking at the entries of $A^{-1}dA$ for a generic matrix A of this form, we compute a real basis of left-invariant one-forms to be given by

$$\eta_1 = dx_1, \eta_2 = dy_1, \eta_3 = dx_2, x_4 = dy_2, \eta_5 = dx_3 + y_1dy_2 - x_1dx_2, \eta_6 = x_1dy_2 + y_1dx_2 - dy_3,$$

and so by Nomizu's theorem the natural inclusion of the $\mathbb{R}$-cdga

$$A = (\Lambda(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6), d\eta_1 = d\eta_2 = d\eta_3 = d\eta_4 = 0, d\eta_5 = \eta_1\eta_3 - \eta_2\eta_4, d\eta_6 = \eta_2\eta_3 + \eta_1\eta_4))$$

into the smooth real-valued forms on the Iwasawa manifold is a quasi-isomorphism, cf. [13, Example 6.24] and [2, Section 6]. The complexification of this cdga is identified with the complex model given above via $\phi_j = \eta_{2j-1} + i\eta_{2j}, \overline{\phi_j} = \eta_{2j-1} - i\eta_{2j}$ for $j = 1, 2, 3$.

Lemma 2.1. Let A be the real model of the Iwasawa manifold given above. If $z_1z_2 = 0$ for non-zero classes $z_1, z_2 \in H^1(A)$, then $z_2 = cz_1$ for some real number c.

Proof. Note that $H^1(A)$ is spanned by $[\eta_1],[\eta_2],[\eta_3],[\eta_4]$. Choosing representatives

$$\alpha_1\eta_1 + \alpha_2\eta_2 + \alpha_3\eta_3 + \alpha_4\eta_4, \quad \beta_1\eta_1 + \beta_2\eta_2 + \beta_3\eta_3 + \beta_4\eta_4$$

of z_1, z_2, we have that z_1z_2 is represented by

$$(\alpha_1\beta_2 - \alpha_2\beta_1)\eta_1\eta_2 + (\alpha_1\beta_3 - \alpha_3\beta_1)\eta_1\eta_3 + (\alpha_1\beta_4 - \alpha_4\beta_1)\eta_1\eta_4 + (\alpha_2\beta_3 - \alpha_3\beta_2)\eta_2\eta_3 \\ + (\alpha_2\beta_4 - \alpha_4\beta_2)\eta_2\eta_4 + (\alpha_3\beta_4 - \alpha_4\beta_3)\eta_3\eta_4.$$

The image of d in degree two is spanned by $\eta_1\eta_3 - \eta_2\eta_4$ and $\eta_2\eta_3 + \eta_1\eta_4$, so $z_1z_2 = 0$ is equivalent to

$$\alpha_1\beta_3 - \alpha_3\beta_1 + \alpha_2\beta_4 - \alpha_4\beta_2 = 0, \tag{1}$$

$$\alpha_1\beta_4 - \alpha_4\beta_1 - \alpha_2\beta_3 + \alpha_3\beta_2 = 0, \tag{2}$$

$$\alpha_1\beta_2 - \alpha_2\beta_1 = 0, \tag{3}$$

$$\alpha_3\beta_4 - \alpha_4\beta_3 = 0. \tag{4}$$

Case 1: $\alpha_1 = 0$. Then (3) gives $\alpha_2\beta_1 = 0$.

Case 1.1: If $\alpha_2 = 0$, then (1) and (2) become

$$\alpha_3\beta_1 + \alpha_4\beta_2 = 0, \\ -\alpha_4\beta_1 + \alpha_3\beta_2 = 0,$$

i.e. the scalar product of (β_1, β_2) with both (α_3, α_4) and $(-\alpha_4, \alpha_3)$ is zero. Since the latter two are orthogonal, we conclude $(\alpha_3, \alpha_4) = 0$ or $(\beta_1, \beta_2) = 0$. In the first case we would have $z_1 = 0$ so we are done. In the second case, we have that both z_1 and z_2 are represented by elements in the span of η_3, η_4, and the claim clearly holds.

[2]The Massey product $\langle z_1, \ldots, z_n\rangle$ does not depend on the choice of representatives of the classes z_i [7, Theorem 3].

Case 1.2: If $\beta_1 = 0$, then z_1 and z_2 are represented by elements in the span of η_2, η_3, η_4, and the claim again clearly holds (since there is no η_1 involved, which necessarily shows up in any non-trivial differential).

Case 2: $\alpha_1 \neq 0$. We can assume $\alpha_1 = 1$. So, by (3), $\beta_2 = \alpha_2\beta_1$. Now (1) gives us

$$\beta_3 = (\alpha_3 + \alpha_2\alpha_4)\beta_1 - \alpha_2\beta_4.$$

Plugging this into (2), we get $(1 + \alpha_2^2)\beta_4 = \alpha_4(1 + \alpha_2^2)\beta_1$, hence $\beta_4 = \alpha_4\beta_1$. Lastly, (4) gives us $\alpha_4\beta_3 = \alpha_3\alpha_4\beta_1$. If $\alpha_4 \neq 0$, we conclude $\beta_3 = \alpha_3\beta_1$; if $\alpha_4 = 0$, then from (1) and (2) we see $\beta_3 = \alpha_3\beta_1$. Hence $z_2 = \beta_1 z_1$. □

Corollary 2.2. Every real n-fold Massey product $\langle z_1, \dots, z_n\rangle$ on the Iwasawa manifold M, for $n \geq 3$ with $z_i \in H^1(M;\mathbb{R}) \cong H^1(A)$, is trivial.

Proof. Since $z_i z_{i+1} = 0$ by assumption, by the above lemma z_{i+1} is a scalar multiple of z_i (we may assume all the z_i to be non-zero, since otherwise the Massey product automatically vanishes). Note that the differential is trivial on degree zero, so each class in $H^1(A)$ in fact has a unique representative. Therefore the representatives of z_i are scalar multiples of each other. In particular, the pairwise products of representatives of z_i and z_{i+1} are zero, and we can choose the zero element as primitive. Inductively choosing zero for all primitive elements, we are done. □

Remark 2.3. There are non-trivial real Massey products on A landing in $H^{\geq 3}(A)$, cf. [2, Section 6], for example $\langle [\eta_1], [\eta_3\eta_4], [\eta_2]\rangle$.

We can truncate the Iwasawa manifold's real minimal model A in order to obtain an $\mathbb{R}$-cdga with cohomology concentrated in degrees up to two, and with vanishing Massey products. Namely, consider the differential ideal $A_{\geq 3}$ of elements of degrees ≥ 3, and consider the quotient

$$B = A/A_{\geq 3}.$$

It is immediate that the quotient map $A \xrightarrow{f} B$ is a 1-quasi-isomorphism, i.e. an isomorphism on H^1 and an injection on H^2.

Corollary 2.4. All real Massey products on the real cdga B vanish.

Proof. Since the quotient map $A \xrightarrow{f} B$ is a 1-quasi-isomorphism, all real Massey products involving only degree 1 classes can be computed on A [14, Section 3.6], where they vanish by Corollary 2.2. For degree reasons, all other Massey products (involving at least one element of degree at least two) trivially vanish, since $H^{\geq 3}(B) = 0$. □

Proposition 2.5. B is not formal.

Before giving the argument, let us recall some concepts and results. Let $\mathbb{k}$ be a field of characteristic zero. A 1-minimal model (over $\mathbb{k}$) [10, Definition 5.3] of a $\mathbb{k}$-cdga A is a minimal cdga generated in degree 1, with a 1-quasi-isomorphism to A. The 1-minimal model is unique up to isomorphism [10, Theorem 5.6]. We say a cdga is 1-formal if there is a 1-quasi-isomorphism from its 1-minimal model to its $\mathbb{k}$-cohomology algebra [8, Lemma 2.2]; we refer the reader to [8], where this notion is also referred to as *1-stage formality*, for further discussion (including a comparison with the different notion of i-formality considered in [6]).

Formality famously satisfies a descent property: a degree-wise cohomologically finite-dimensional connected $\mathbb{k}$-cdga A is formal in the category of $\mathbb{k}$-cdga's if and only if $A \otimes_{\mathbb{k}} \mathbb{K}$ is formal in the category of $\mathbb{K}$-cdga's, where $\mathbb{k}$ is any field extension of $\mathbb{k}$. The analogous statement holds for 1-formality, and more generally, i-formality (or *i-stage formality*, in the terminology of [8]). We refer the reader to [16, Theorem 12.1] and [15, Theorem 4.19], and to [14, Section 3] for a nice overview.

Back to the truncated model of the Iwasawa manifold, there is clearly a $\mathbb{Q}$-cdga B' such that $B' \otimes_{\mathbb{Q}} \mathbb{R} \cong B$, as the same holds for the model A of the Iwasawa manifold. So, B is formal as an $\mathbb{R}$-cdga if and only if B' is formal as a $\mathbb{Q}$-cdga. We remark that since B is obtained by extending a rational cdga, by taking nilpotent models we can realize the phenomena in this section by topological spaces.

Proof of Proposition 2.5. If B were formal, then it would be 1-formal. Note that A is a real 1-minimal model of B via the quotient map; the 1-minimal model of a cdga is unique up to isomorphism. Therefore we would have a 1-quasi-isomorphism $A \xrightarrow{f} H(A)$ [8, Lemma 2.2]. Tensoring with $\mathbb{C}$, we would have a 1-quasi-isomorphism from the complex minimal model of the Iwasawa manifold to its $\mathbb{C}$-valued cohomology. Hence all Massey products landing in $H^2(A \otimes \mathbb{C})$ would be trivial [14, Proposition 3.15], a contradiction. □

For the reader's convenience, we spell the last sentence out for triple Massey products. Consider a triple Massey product $\langle [a], [b], [c] \rangle$ where $[a], [b], [c] \in H^1(A \otimes \mathbb{C})$. Choose primitives x and y of ab and bc respectively. Then there are closed elements $x', y' \in A$ such that $[x'] = f(x)$ and $[y'] = f(y)$. Since f is a 1-quasi-isomorphism, there are *closed* elements $\tilde{x}, \tilde{y} \in A$ such that $f(\tilde{x}) = x', f(\tilde{y}) = y'$. Then $d(x - \tilde{x}) = ab, d(y - \tilde{x}) = bc$ and $f(x - \tilde{x}) = 0, f(y - \tilde{y}) = 0$. With this new choice of primitives, namely $x - \tilde{x}$ and $y - \tilde{y}$ instead of x and y, the Massey product $\langle [a], [b], [c] \rangle$ is represented by $[(x - \tilde{x})c + a(y - \tilde{y})]$. Now,

$$f^*[(x - \tilde{x})c + a(y - \tilde{y})] = [f(x - \tilde{x})f(c) + f(a)f(y - \tilde{y})] = 0.$$

Since f is injective on H^2, we conclude that the Massey product is trivial.

Remark 2.6. This real cdga B satisfies more than just vanishing of all real Massey products: There exists a choice of representing forms for all classes such that in any well-defined (real) Massey product, one can uniformly choose the zero element for all primitives. This illustrates that the explanation of the criterion for formality given in [5, Theorem 4.1][3] as "a way of saying that *one may make uniform choices so that the forms representing all Massey products and higher order Massey products are exact*" is not meant to go both ways.

Choosing any section for the projection $\ker(d) \to H(B)$ that sends $[\eta_i]$ to η_i, we can take the zero element whenever a choice of primitive must be made when constructing Massey products, and all Massey products will be exact with this "uniform" (or, "simultaneous") choice. Of course, B is not minimal, so the criterion in [5] does not directly apply. However, choosing any minimal model $M(B) \xrightarrow{\sim} B$, we can still make uniform choices making all Massey product representatives exact, in the following sense: There is a section of $\ker(d) \to H(M(B))$ and a section $d^{-1} : \operatorname{im} d \to A$ of the differential, such that $d \circ d^{-1} = \mathrm{id}$ and for a Massey product $\langle [a_{0,1}], ..., [a_{r-1,r}] \rangle$ one can inductively build a defining system yielding the zero class by setting $a_{i,j} := d^{-1} \sum_{i<l<j} (-1)^{|a_{i,l}|+1} a_{i,l} a_{l,j}$, where $a_{i,i+1}$ are the representatives of their cohomology classes given by the splitting. Namely, first note that $\ker(d)$ in degree one of any $M(B)$ maps isomorphically to $\ker(d)$ in degree one of B. We choose the section of $\ker(d) \to H(M(B))$ so that $[\eta_i]$ maps to the element corresponding to η_i under the above isomorphism, and for degree reasons we can choose d^{-1} to be any section of the differential.

Example 2.7. The Iwasawa manifold and the product $H \times H$ of the Heisenberg manifold with itself have the same complex homotopy type, but distinct real homotopy types. Said differently, their de Rham algebras of real-valued forms are not connected by a chain of $\mathbb{R}$-cdga quasi-isomorphisms, while their de Rham algebras of complex-valued forms are connected by a chain of $\mathbb{C}$-cdga quasi-isomorphisms.

[3]One should assume in this criterion that the minimal cdga is furthermore in *normal form* as in [11].

Indeed, a complex minimal model of $H \times H$ is given by complexifying the real minimal model given by

$$(\Lambda(x_1, x_2, x_3, y_1, y_2, y_3), dx_3 = x_1x_2, dy_3 = y_1y_2)\,,$$

and so relabelling x_i to ϕ_i and y_i to $\overline{\phi_i}$ identifies this with the complex minimal model of the Iwasawa manifold given earlier. The real homotopy types are distinct, since the real minimal model of H has a non-trivial Massey product in H^2, e.g. $\langle [x_1], [x_1], [x_2]\rangle$.

3. Behavior of individual Massey products under field extension

3.1. **Triple Massey products persist under field extension.** For simplicity of notation let us consider the field extension $\mathbb{R} \subset \mathbb{C}$. Suppose a triple Massey product $\langle [x], [y], [z]\rangle$ in a real cdga A becomes trivial in $A \otimes \mathbb{C}$. That is, there are $\alpha, \beta, \Psi \in A \otimes \mathbb{C}$ such that

$$d\alpha = xy,\ d\beta = yz,\ d\Psi = \alpha z - (-1)^{|x|} x\beta.$$

Choose a homogeneous real vector space basis $\{u_j\}$ for A; then a real basis for $A \otimes \mathbb{C}$ is given by $\{u_j, iu_j\}$. Write

$$\alpha = \sum_j c_j u_j + i \sum_j c'_j u_j, \quad \beta = \sum_j \tilde{c}_j u_j + i \sum_j \tilde{c}'_j u_j,$$

where $c_j, c'_j, \tilde{c}_j, \tilde{c}'_j \in \mathbb{R}$. Since d is real, we conclude $d(\sum_j c_j u_j) = xy$ and $d(\sum_j \tilde{c}_j u_j) = yz$. Since $\alpha z - (-1)^{|x|} x\beta = d\Psi$, we similarly conclude that $(\sum_j c_j u_j)z - (-1)^{|x|} x(\sum_j \tilde{c}_j u_j)$ is exact by some real element. That is, the real Massey product $\langle [x], [y], [z]\rangle$ is trivial.

The above argument works just as well for any field extension $\Bbbk \subset \mathbb{K}$, by choosing a $\Bbbk$-basis $\{1, c_1, c_2, \ldots\}$ of $\mathbb{K}$ and taking the $\Bbbk$-basis $\{u_i, c_j u_i\}_{i,j}$ for $A \otimes_{\Bbbk} \mathbb{K}$. We give an alternative argument in Proposition 3.5. However, the result does not generalize to quadruple (or higher) Massey products.

3.2. **Non-trivial quadruple Massey products can become trivial upon field extension.** Consider the real cdga

$$\left(\Lambda(x, y, a, b, u, v, w), dx = dy = db = 0, da = xy, du = ay, dv = by, dw = 2xu - a^2 - b^2\right),$$

where $\deg(x) = 2, \deg(y) = 3, \deg(a) = \deg(b) = 4, \deg(u) = \deg(v) = 6, \deg(w) = 7$. Consider the Massey product $\langle [x], [y], [y], [x]\rangle$. For the unique representatives x and y, a generic choice of primitives is given by $xy = d(a + k_1 b), y^2 = d(k_2 xy), yx = d(a + k_3 b)$ for some scalars k_i. Then the triple Massey product representatives are made exact via

$$(a + k_1 b)y - k_2 x^2 y = d(u + k_1 v - k_2 xa + k_4 x^3 + k_5 xb),$$
$$k_2 x^2 y + ya + k_3 by = d(k_2 xa + u + k_3 v + k_6 x^3 + k_7 xb).$$

The resulting element in the quadruple Massey product is then represented by

$$\begin{aligned}&(u + k_1 v - k_2 xa + k_4 x^3 + k_5 xb)x - (a + k_1 b)(a + k_3 b) + x(k_2 xa + u + k_3 v + k_6 x^3 + k_7 xb)\\ &= (2xu - a^2 - k_1 k_3 b^2) + (k_1 + k_3)(xv - ab) + (k_4 + k_6)x^4 + (k_5 + k_7)x^2 b.\end{aligned}$$

We compute that H^8 is spanned by $\{[x^4], [2xu - a^2], [x^2 b], [xv - ab]\}$. For the Massey product to be trivial, we need to choose k_i so that $k_4 + k_6 = 0, k_5 + k_7 = 0, k_1 + k_3 = 0, k_1 k_3 = 1$. This can be solved over $\mathbb{C}$ by choosing $k_1 = i, k_3 = -i$, but cannot be solved over $\mathbb{R}$.

Completely analogously we have the following: let $\Bbbk \subset \Bbbk(\sqrt{\theta})$ be a proper extension of fields of characteristic zero. Then the $\Bbbk$-cdga

$$\left(\Lambda(x_2, y_3, a_4, b_4, u_6, v_6, w_7), dx = dy = db = 0, da = xy, du = ay, dv = by, dw = 2xu - a^2 + \theta b^2\right)$$

has a non-trivial quadruple Massey product, namely $\langle [x], [y], [y], [x]\rangle$, which vanishes upon field extension to $\Bbbk(\sqrt{\theta})$.

Note that there is a non-trivial triple Massey product in the above examples, on the unextended cdga, given by $\langle [x],[y],[b]\rangle$.

3.3. **Extending from an algebraically closed field.** The above example indicates that triviality of a Massey product comes down to solvability of a system of polynomial equations in the coefficients along some vector space basis of the considered cdga.

Proposition 3.4. Let $\mathbb{K}$ be an algebraically closed field, and $\mathbb{K}\subset\mathbb{L}$ any extension. Let A be a $\mathbb{K}$-cdga which is degree-wise finite-dimensional. If a Massey product $\langle [x_1],\dots,[x_n]\rangle$ is non-trivial on A, then it remains non-trivial on $A\otimes_{\mathbb{K}}\mathbb{L}$.

Proof. In each graded piece A_i of A choose a complement C_i to $\mathrm{image}(d)$ inside $\ker(d)$, and choose a complement I_i to $\ker(d)$ in A_i. This gives us a splitting $A_i=\mathrm{image}(d)\oplus C\oplus I$ for each i. Choose graded $\mathbb{K}$-vector space bases $\{u_i\},\{v_i\},\{w_i\}$ of $\mathrm{image}(d), C=\oplus_i C_i, I=\oplus_i I$. For convenience let us denote the union of these bases by $\{b_i\}$. Here and throughout, indices on lower-case letters are for enumerative purposes and do not correspond to the degree. There are scalars $\beta^k_{i,j}\in\mathbb{K}$ such that $b_ib_j=\sum_k\beta^k_{i,j}b_k$.

Now, d is an isomorphism $I\to\mathrm{image}(d)$; we denote its inverse by δ. Extending the field to $\mathbb{L}$ respects the above splitting; $\{u_i\},\{v_i\},\{w_i\}$ still form bases and (the extended) d is an isomorphism $I\otimes_{\mathbb{K}}\mathbb{L}\to\mathrm{image}(d)\otimes_{\mathbb{K}}\mathbb{L}$ with inverse the extension of δ.

We go through the procedure of building a generic representative of $\langle [x_1],\dots,[x_n]\rangle$. First of all, a generic primitive for x_ix_{i+1} is given by

$$\delta(x_ix_{i+1})+\sum_j\alpha^{u_j}_{i,i+1}u_j+\sum_j\alpha^{v_j}_{i,i+1}v_j.$$

Then a generic representative of $\langle [x_i],[x_{i+1}],[x_{i+2}]\rangle$ is given by

$$\begin{aligned}&\Big(\delta(x_ix_{i+1})+\sum_j\alpha^{u_j}_{i,i+1}u_j+\sum_j\alpha^{v_j}_{i,i+1}v_j\Big)x_{i+2}\\&\qquad-(-1)^{|x_i|}x_i\Big(\delta(x_{i+1}x_{i+2})+\sum_{j'}\alpha^{u_{j'}}_{i+1,i+2}u_{j'}+\sum_{j'}\alpha^{v_{j'}}_{i+1,i+2}v_{j'}\Big).\end{aligned}$$

Expanding x_i,x_{i+1},x_{i+2} in terms of the basis, we see that exactness of this expression is equivalent to the vanishing of the coefficients along $\{v_i\}$ and $\{w_i\}$, which are $\mathbb{K}$-linear expressions in the α's. Given that this element is exact, a generic primitive is given by the following: we apply δ (which writes each u_i in terms of w_j) and add an element in $\ker(d)$, i.e. a linear combination of $\{u_i,v_j\}$, whose coefficients we also label with α and treat as variables. When considering the generic representative of the fourfold product $\langle [x_i],[x_{i+1}],[x_{i+2}],[x_{i+3}]\rangle$, exactness will be equivalent to the existence of a zero of a system of polynomial equations, in the variables α with coefficients in $\mathbb{K}$, and of degree ≤ 2 (α terms will be multiplied with other α terms when multiplying the primitive of x_ix_{i+1} with that of $x_{i+2}x_{i+3}$).

Repeating the above, we see that triviality of the Massey product over $\mathbb{L}$ is equivalent to the existence of a zero $\{a_j\}$ of a system of $\mathbb{K}$-polynomial equations $\{P_i(\{\alpha_j\})\}$ over $\mathbb{L}$ (recall, the coefficients $\beta^k_{i,j}$ and the coefficients in the expansion of each z_i are in $\mathbb{K}$). By the degree-wise finite-dimensionality assumption, the set of variables $\{\alpha_j\}$ and the set of polynomials $\{P_i\}$ under consideration are finite. If there were a zero over $\mathbb{L}$, then there would be one over its algebraic closure $\overline{\mathbb{L}}$. Now by the weak Nullstellensatz, this is equivalent to the ideal generated by the P_i in $\mathbb{K}[\{\alpha_j\}]$ being proper. Since $\mathbb{K}$ is itself algebraically closed, this is in turn equivalent to the existence of a zero over $\mathbb{K}$. □

In fact, the above is showing a bit more: if a Massey product of $\mathbb{K}$-classes is trivial over $\mathbb{L}$, then it is first of all *well-defined*, and furthermore trivial over $\mathbb{K}$.

For Massey products of length $2n$ or $2n+1$, the degrees of the polynomials that appear above are bounded from above by n. In this regard the persistence of non-triviality of triple Massey products upon field extension has the same explanation as the persistence of the non-triviality of a cup product upon field extension; namely, a $\mathbb{k}$-linear system has a zero over an extension $\mathbb{K}$ only if it has a zero over $\mathbb{k}$.

Due to quasi-isomorphism invariance of Massey products, the degree-wise finite-dimensionality assumption above can be relaxed to cohomological degree-wise finite-dimensionality in common situations, e.g. if we are in characteristic zero and the cdga is modelling a simply connected space with degree-wise finite-dimensional cohomology.

Let us now turn back to the persistence of triple Massey products under field extension. One can extend any cohomologically finite-dimensional and connected $\mathbb{k}$–cdga A to one satisfying n-dimensional Poincaré duality, which we call its Poincaré dualization $P_n(A)$. Furthermore, a degree one map of n–dimensional Poincaré duality cdga's preserves triple Massey products [17]. Using this we give a topologically inspired argument for Section 3.1. We can take A to be degree-wise finite-dimensional, and truncate it above a sufficiently high degree, not altering the triviality of a given Massey product.

Proposition 3.5. Non-trivial triple Massey products remain non-trivial under field extension $\mathbb{k} \subset \mathbb{K}$.

Proof. Let $\langle z_1, z_2, z_3 \rangle$ be a non-trivial triple Massey product on the $\mathbb{k}$-cdga A. We will use the notation of [9, Section 4]. Consider the $\mathbb{k}$-linear map $A \xrightarrow{i} A \otimes_{\mathbb{k}} \mathbb{K}$ sending $a \mapsto a \otimes 1$. For any n we obtain a $\mathbb{k}$-linear map Φ from the shifted dual complex $D_n(A)$ to the shifted dual complex $D_n(A \otimes_{\mathbb{k}} \mathbb{K})$, where the latter consists of the $\mathbb{K}$-linear functionals, thought of as a $\mathbb{k}$-complex. Namely, we send φ to the $\mathbb{K}$-linear functional $\Phi(\varphi)$ determined by $\Phi(\varphi)(a \otimes u) = \varphi(a)u$. Now the $\mathbb{k}$-linear map $A \oplus D_n(A) \xrightarrow{i \oplus \Phi} (A \otimes_{\mathbb{k}} \mathbb{K}) \oplus (D_n(A \otimes \mathbb{K}))$ is a map of $\mathbb{k}$-cdga's. The verification is similar to [9, Lemma 4.6]; we carry it out here. For clarity we denote the cup product on the Poincaré dualization by $\wedge$. The only non-trivial check of multiplicativity is for elements of the form $\varphi \wedge a$, where φ is in the dual complex and $a \in A$. On the one hand, for $b \in A$ and $u \in \mathbb{K}$, we have

$$\begin{aligned} \left((i \oplus \Phi)(\varphi \wedge a)\right)(b \otimes u) &= \Phi(\varphi \wedge a)(b \otimes u) \\ &= \left((\varphi \wedge a)(b)\right) u = \left(\varphi(ab)\right) u \\ &= \Phi(\varphi)(ab \otimes u) = \Phi(\varphi)(i(a)(b \otimes u)). \end{aligned}$$

On the other hand we have

$$\left((i \oplus \Phi)(\varphi) \wedge (i \oplus \Phi)(a)\right)(b \otimes u) = \left(\Phi(\varphi) \wedge i(a)\right)(b \otimes u) = \Phi(\varphi)(i(a)(b \otimes u)).$$

Therefore, for large enough n we have a map of $\mathbb{k}$-cdga's which satisfy cohomological Poincaré duality $P_n^{\mathbb{k}}(A) \to P_n^{\mathbb{K}}(A \otimes_{\mathbb{k}} \mathbb{K})$, where the superscript indicates the category in which the Poincaré dualization is performed. Note also that since i maps 1 to 1, the volume class of $P_n^{\mathbb{k}}(A)$ is mapped to the volume class of $P_n^{\mathbb{K}}(A \otimes_{\mathbb{k}} \mathbb{K})$.

Now we argue that non-trivial triple Massey products remain non-trivial under such a map, following an argument due to Taylor [17]. A triple product involving cohomology classes on a cdga algebra is non-trivial if and only if it is non-trivial on the Poincaré dualization [9, Proposition 4.9]. For simplicity let us denote by f the map induced by $i \oplus \Phi$ on $\mathbb{k}$-cohomology followed by the inclusion of $\mathbb{k}$-cohomology into $\mathbb{K}$-cohomology.

Consider the ideal $\mathcal{J}_{z_1,z_3}$ generated by z_1, z_3 in $H(P_n^{\mathbb{k}}(A))$, and the vector space $\mathcal{A}_{z_1,z_3}$ of classes z_0 such that $z_1 z_0 = z_0 z_3 = 0$, [17, Notation 1.2]. We then have a map

$$\langle z_1, -, z_3 \rangle : \mathcal{A}_{z_1,z_3} \to H(P_n^{\mathbb{k}}(A))/\mathcal{J}_{z_1,z_3}.$$

For $z_0 \in \mathcal{A}_{z_1,z_3}$, the product $z_0\langle z_1, z_2, z_3\rangle$ is a single class [17, Theorem 2.1], since z_0 kills the indeterminacy. As a consequence, $z_0\langle z_1, z_2, z_3\rangle$ being a non-zero class implies that $\langle z_1, z_2, z_3\rangle$ is non-trivial. Now, the Poincaré duality pairing on cohomology induces a non-degenerate pairing [17, Proposition 5.1]

$$(H(P_n^{\mathbb{k}}(A))/\mathcal{J}_{z_1,z_3})^r \otimes_{\mathbb{k}} (\mathcal{A}_{z_1,z_3})^{n-r} \to \mathbb{k}.$$

Therefore, given the non-trivial Massey product $\langle z_1, z_2, z_3\rangle$, there is a $t \in \mathcal{A}_{z_1,z_3}$ such that

$$\int t\langle z_1, z_2, z_3\rangle = 1$$

under the Poincaré duality pairing [17, Theorem 5.2]. Now consider the triple Massey product $\langle f(z_1), f(z_2), f(z_3)\rangle$ in $A \otimes_{\mathbb{k}} \mathbb{K}$. We have that $f(t)$ is in the $\mathbb{K}$-vector space $\mathcal{A}_{f(z_1),f(z_3)}$, and so $f(t)\langle f(z_1), f(z_2), f(z_3)\rangle$ is a single class which is non-zero, since f maps the volume class of $P_n^{\mathbb{k}}(A)$ to that of $P_n^{\mathbb{K}}(A \otimes_{\mathbb{k}} \mathbb{K})$. Therefore the Massey product $\langle f(z_1), f(z_2), f(z_3)\rangle$ is non-trivial on $A \otimes_{\mathbb{k}} \mathbb{K}$. □

One can compare the general non-persistence of quadruple and higher products under field extension with the non-preservation of such products under non-zero degree maps of rational Poincaré duality algebras as investigated in [9].

References

[1] Buijs, U., Moreno-Fernández, J.M. and Murillo, A., 2020. *A_∞ structures and Massey products.* Mediterranean Journal of Mathematics, 17(1), p.31.

[2] Cordero, L.A., Fernández, M. and Gray, A., 1986. *Symplectic manifolds with no Kähler structure.* Topology, 25(3), pp.375-380.

[3] Crowley, D. and Nordström, J., 2020. *The rational homotopy type of $(n-1)$-connected manifolds of dimension up to $5n-3$.* Journal of Topology, 13(2), pp.539-575.

[4] Cavalcanti, G.R., Fernández, M. and Muñoz, V., 2008, June. *On Non-Formality of a Simply-Connected Symplectic 8-Manifold.* In AIP Conference Proceedings (Vol. 1023, No. 1, pp. 82-91). American Institute of Physics.

[5] Deligne, P., Griffiths, P., Morgan, J. and Sullivan, D., 1975. *Real homotopy theory of Kähler manifolds.* Inventiones mathematicae, 29, pp.245-274.

[6] Fernández, M. and Muñoz, V., 2005. *Formality of Donaldson submanifolds.* Mathematische Zeitschrift, 250, pp.149-175.

[7] Kraines, D., 1966. *Massey higher products.* Transactions of the American Mathematical Society, 124(3), pp.431-449.

[8] Mačinic, A.D., 2010. *Cohomology rings and formality properties of nilpotent groups.* Journal of Pure and Applied Algebra, 214(10), pp.1818-1826.

[9] Milivojević, A., Stelzig, J. and Zoller, L., 2022. *Poincaré dualization and formal domination.* arXiv preprint arXiv:2203.15098v2.

[10] Morgan, J.W., 1978. *The algebraic topology of smooth algebraic varieties.* Publications Mathématiques de l'IHÉS, 48, pp.137-204.

[11] Papadima, S., 1985. *The cellular structure of formal homotopy types.* Journal of Pure and Applied Algebra, 35, pp.171-184.

[12] Positselski, L., 2017. *Koszulity of cohomology $= K(\pi,1)$-ness + quasi-formality.* Journal of Algebra, 483, pp.188-229.

[13] Stelzig, J. and Wilson, S.O., 2022. *A dd^c-type condition beyond the Kähler realm.* arXiv preprint arXiv:2208.01074.

[14] Suciu, A.I., 2022. *Formality and finiteness in rational homotopy theory.* arXiv preprint arXiv:2210.08310.

[15] Suciu, A.I. and Wang, H., 2019. *Formality properties of finitely generated groups and Lie algebras.* In Forum Mathematicum (Vol. 31, No. 4, pp. 867-905). De Gruyter.

[16] Sullivan, D., 1977. *Infinitesimal computations in topology.* Publications Mathématiques de l'IHÉS, 47, pp.269-331.

[17] Taylor, L.R., 2010. *Controlling indeterminacy in Massey triple products.* Geometriae Dedicata, 148, pp.371-389.

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn
Email address: milivojevic@mpim-bonn.mpg.de