Max-Planck-Institut für Mathematik Bonn

A dichotomy between rationality and a natural boundary for Reidemeister type zeta functions

by

Wojciech Bondarewicz
Alexander Fel'shtyn
Malwina Zietek

Max-Planck-Institut für Mathematik
Preprint Series 2023 (5)

Date of submission: February 15, 2023

A dichotomy between rationality and a natural boundary for Reidemeister type zeta functions

by

Wojciech Bondarewicz
Alexander Fel'shtyn
Malwina Zietek

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Instytut Matematyki
Uniwersytet Szczecinski
ul. Wielkopolska 15
70-451 Szczecin
Poland

MPIM 23-5

A DICHOTOMY BETWEEN RATIONALITY AND A NATURAL BOUNDARY FOR REIDEMEISTER TYPE ZETA FUNCTIONS

WOJCIECH BONDAREWICZ, ALEXANDER FEL'SHTYN AND MALWINA ZIETEK

ABSTRACT. We prove a dichotomy between rationality and a natural boundary for the analytic behavior of the Reidemeister zeta function for endomorphisms of groups $\mathbb{Z}_p^d$, where $\mathbb{Z}_p$ the group of p-adic integers. We also prove the rationality of the coincidence Reidemeister zeta function for tame endomorphisms pairs of finitely generated torsion-free nilpotent groups, based on a weak commutativity condition.

0. INTRODUCTION

Let G be a group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \beta \in G$ are said to be *ϕ-conjugate* or *twisted conjugate* iff there exists $g \in G$ with $\beta = g\alpha\phi(g^{-1})$. We shall write $\{x\}_\phi$ for the *ϕ-conjugacy* or *twisted conjugacy* class of the element $x \in G$. The number of ϕ-conjugacy classes is called the *Reidemeister number* of an endomorphism ϕ and is denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G. We call the endomorphisms ϕ *tame* if the Reidemeister numbers $R(\phi^n)$ are finite for all $n \in \mathbb{N}$. Taking a dynamical point of view, we consider the iterates of a tame endomorphism ϕ, and we may define following [11] a Reidemeister zeta function of ϕ as a power series:

$$R_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n\right),$$

where z denotes a complex variable. The following problem was investigated [13]: for which groups and endomorphisms is the Reidemeister zeta function a rational function? Is this zeta function an algebraic function?

In [11, 13, 19, 14, 12], the rationality of the Reidemeister zeta function $R_\phi(z)$ was proven in the following cases: the group is finitely generated and

---

2010 *Mathematics Subject Classification.* Primary 37C25; 37C30; 22D10; Secondary 20E45; 54H20; 55M20.

Key words and phrases. Twisted conjugacy class; Reidemeister number; Reidemeister zeta function; unitary dual.

The work is funded by the Narodowe Centrum Nauki of Poland (NCN) (grant No. 2016/23/G/ST1/04280 (Beethoven 2)).

an endomorphism is eventually commutative; the group is finite; the group is a direct sum of a finite group and a finitely generated free abelian group; the group is finitely generated, nilpotent and torsion-free. In [29] the rationality of the Reidemeister zeta function was proven for endomorphisms of fundamental groups of infra-nilmanifolds under some sufficient conditions. Recently, the rationality of the Reidemeister zeta function was proven for endomorphisms of fundamental groups of infra-nilmanifolds [6]; for endomorphisms of fundamental groups of infra-solvmanifolds of type (R) [16]; for automorphisms of crystallographic groups with diagonal holonomy $\mathbb{Z}_2$ and for automorphisms of almost-crystallographic groups up to dimension 3 [7]; for the right shifts of a non-finitely generated, non-abelian torsion groups $G = \oplus_{i\in\mathbb{Z}} F_i$, $F_i \cong F$ and F is a finite non-abelian group [28].

Let G be a group and $\phi, \psi : G \to G$ two endomorphisms. Two elements $\alpha, \beta \in G$ are said to be $(\phi,\psi) - conjugate$ iff there exists $g \in G$ with

$$\beta = \psi(g)\alpha\phi(g^{-1}).$$

The number of (ϕ,ψ)-conjugacy classes is called the Reidemeister coincidence number of endomorphisms ϕ and ψ, denoted by $R(\phi,\psi)$. If ψ is the identity map then the (ϕ, id)-conjugacy classes are the ϕ - conjugacy classes in the group G and $R(\phi, id) = R(\phi)$. The Reidemeister coincidence number $R(\phi,\psi)$ has useful applications in Nielsen coincidence theory. We call the pair (ϕ,ψ) of endomorphisms *tame* if the Reidemeister numbers $R(\phi^n,\psi^n)$ are finite for all $n \in \mathbb{N}$. For such a tame pair of endomorphisms we define following [15] the *coincidence Reidemeister zeta function*

$$R_{\phi,\psi}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n,\psi^n)}{n} z^n\right).$$

If ψ is the identity map then $R_{\phi,id}(z) = R_\phi(z)$. In the theory of dynamical systems, the coincidence Reidemeister zeta function counts the synchronisation points of two maps, i.e. the points whose orbits intersect under simultaneous iteration of two endomorphisms; see [23], for instance.

In [17], in analogy to works of Bell, Miles, Ward [1] and Byszewski, Cornelissen [2, §5] about Artin–Mazur zeta function, the Pólya–Carlson dichotomy between rationality and a natural boundary for analytic behavior of the coincidence Reidemeister zeta function was proven for tame pair of commuting automorphisms of non-finitely generated torsion-free abelian groups that are subgroups of $\mathbb{Q}^d$, $d \geq 1$.

In [15] Pólya–Carlson dichotomy was proven for coincidence Reidemeister zeta function of tame pair of endomorphisms of non-finitely generated torsion-free nilpotent groups of finite Prüfer rank by means of profinite completion techniques.

In this paper we prove a dichotomy between rationality and a natural boundary for the Reidemeister zeta function of endomorphisms of the groups $\mathbb{Z}_p^d,\ d \geq 1$, where $\mathbb{Z}_p$, p-prime, is the additive group of p-adic integers.

We also prove the rationality of the coincidence Reidemeister zeta function for tame endomorphisms pairs of finitely generated torsion-free nilpotent groups, based on a weak commutativity condition .

Acknowledgments. This work was supported by the grant Beethoven 2 of the Narodowe Centrum Nauk of Poland(NCN), grant No. 2016/23/G/ST1/04280. The second author is indebted to the Max-Planck-Institute for Mathematics(Bonn) for the support and hospitality and the possibility of the present research during his visit there.

1. PÓLYA–CARLSON DICHOTOMY FOR THE REIDEMEISTER ZETA FUNCTION OF ENDOMORPHISMS OF THE GROUPS $\mathbb{Z}_p^d$

In this section we prove a Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behaviour of the Reidemeister zeta function for endomorphisms of groups $\mathbb{Z}_p^d,\ d \geq 1$, where $\mathbb{Z}_p$, p-prime, denotes the additive group of p-adic integers. The group $\mathbb{Z}_p$ is the most basic infinite pro-p group, it is totally disconnected, compact, abelian, torsion-free group.The field of p-adic numbers is denoted by $\mathbb{Q}_p$ and the p-adic absolute value (as well as its unique extension to the algebraic closure $\overline{\mathbb{Q}}_p$) by $|\cdot|_p$.

We remind the definition of a natural boundary (see [26], sec. 6.2).

Definition 1.1. Suppose that an analytic function F is defined somehow in a region D of the complex plane. If there is no point of the boundary ∂D of D over which F can be analytically continued, then ∂D is called a *natural boundary* for F .

We need the following statement

Lemma 1.2. *(cf. [1]) Let $Z(z) = \sum_{n=1}^{\infty} R(\phi^n)z^n$. If $R_\phi(z)$ is rational then $Z(z)$ is rational. If $R_\phi(z)$ has an analytic continuation beyond its circle of convergence, then so does $Z(z)$ too. In particular, the existence of a natural boundary at the circle of convergence for $Z(z)$ implies the existence of a natural boundary for $R_\phi(z)$.*

Proof. This follows from the fact that $Z(z) = z \cdot R_\phi(z)'/R_\phi(z)$. □

One of the important links between the arithmetic properties of the coefficients of a complex power series and its analytic behaviour is given by the Pólya–Carlson theorem [26].

Pólya–Carlson Theorem. *A power series with integer coefficients and radius of convergence* 1 *is either rational or has the unit circle as a natural boundary.*

Lemma 1.3. $\mathrm{End}(\mathbb{Z}_p) = \mathbb{Z}_p$ *for abelian group* $\mathbb{Z}_p$.

Proof. Let $\phi \in \mathrm{End}(\mathbb{Z}_p)$. We have $p^n\phi(x) = \phi(p^n x)$. Then $\phi(p^n\mathbb{Z}_p) \subset p^n\mathbb{Z}_p$, so ϕ is continuous. For every $x \in \mathbb{Z}_p$ there exists a sequence of integers x_n converging to x. Then

$$\phi(x) = \lim \phi(x_n) = \lim x_n\phi(1) = \phi(1)x,$$

so ϕ is a multiplication by $\phi(1)$. □

Let $\phi \in \mathrm{End}(\mathbb{Z}_p)$, then $\phi(x) = ax$, where $a \in \mathbb{Z}_p$. We have $\phi^n(x) = a^n x$. By definition,

$$y \sim_\phi x \Leftrightarrow \exists b \in \mathbb{Z}_p : y = b + x - ab = x + b(1-a) \Leftrightarrow y \equiv x(\mathrm{mod}(1-a)).$$

This implies that $R(\phi) = |\mathbb{Z}_p/(1-a)\mathbb{Z}_p|$. But

$$(1-a)\mathbb{Z}_p = p^{v_p(1-a)}\mathbb{Z}_p = |1-a|_p^{-1}\mathbb{Z}_p,$$

so we can write $R(\phi) = |1-a|_p^{-1} = |a-1|_p^{-1}$ and, more generally,

$R(\phi^n) = |1-a^n|_p^{-1} = |a^n-1|_p^{-1}$, for all $n \in \mathbb{N}$.

Now consider a group $\mathbb{Z}_p^d,\ d \geq 2$. It follows easily from Lemma 1.3, that $\mathrm{End}(\mathbb{Z}_p^d) = \mathrm{M}_d(\mathbb{Z}_p)$. For any matrix $A \in M_d(\mathbb{Z}_p)$ there exists a diagonal matrix $D \in M_d(\mathbb{Z}_p)$ and unimodular matrices $E, F \in M_d(\mathbb{Z}_p)$ such that $D = EAF$.

Lemma 1.4. *For endomorphism* $\phi_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ *we have*

$$R(\phi_p) = \#\mathrm{Coker}(1-\phi_p) = |\det(\Phi_p - \mathrm{Id})|_p^{-1},$$

where Φ_p *is a matrix of* ϕ_p.

Proof. Let matrices $D, E, F \in M_d(\mathbb{Z}_p)$ be such that $D = E(\mathrm{Id} - \Phi_p)F$, where $D = (a_i)$ is diagonal matrix, $a_i \in \mathbb{Z}_p, 1 \leq i \leq d$, and matrices E, F are unimodular. Then we have

$$R(\phi_p) = \#\mathrm{Coker}(1-\phi_p) = |\mathbb{Z}_p^d : (\mathrm{Id} - \Phi_p)\mathbb{Z}_p^d| = |\mathbb{Z}_p^d : D\mathbb{Z}_p^d| =$$
$$= |\mathbb{Z}_p : a_1\mathbb{Z}_p| \cdot |\mathbb{Z}_p : a_2\mathbb{Z}_p| \cdot \ldots \cdot |\mathbb{Z}_p : a_d\mathbb{Z}_p| = |a_1|_p^{-1} \cdot \ldots \cdot |a_d|_p^{-1} =$$
$$= |\det(D)|_p^{-1} = |\det(\mathrm{Id} - \Phi_p)|_p^{-1} = |\det(\Phi_p - \mathrm{Id})|_p^{-1}.$$

□

In order to handle the sequence $R(\phi^n) = |a^n - 1|_p^{-1},\ n \in \mathbb{N}$ more easily, we need a way to evaluate expressions of the form $|a^n - 1|_p$ when $|a|_p = 1$. The following technical lemma is useful.

Lemma 1.5. *(cf.* [21, Lemma 4.9], [1, Lemma 2]*) Let K_v be a non-archimedean local field and suppose $x \in K_v$ has $|x|_v = 1$ and infinite multiplicative order. Let $p > 0$ be the characteristic of the residue field F_v and $\gamma \in \mathbb{N}$ the multiplicative order of the image of x in F_v . Then $|x^n - 1|_v = 1$ whenever $(\gamma, n) = 1$ and $\gamma \neq 1$. Furthermore, there are constants $0 < C < 1$ and $r_0 \geq 0$ such that whenever $n = k\gamma p^r$ with $(p, k) = 1$ and $r > r_0$, then $|x^n - 1|_v = C|p|_v^r$ if $char(K_v) = 0$.*

Now we prove a Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behaviour of the Reidemeister zeta function for endomorphisms of groups $\mathbb{Z}_p^d,\ d \geq 1$.

Theorem 1.6. *Let $\phi_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ be a tame endomorphism and $\lambda_1, \lambda_2, ...\lambda_d \in \overline{\mathbb{Q}}_p$ be the eigenvalues of Φ_p, including multiplicities. Then the Reidemeister zeta function $R_{\phi_p}(z)$ is either a rational function or it has the unit circle as a natural boundary. Furthermore, the latter occurs if and only if $|\lambda_i|_p = 1$ for some $i \in \{1, \dots, d\}$.*

Proof. Firstly, we consider separately the case of the group $\mathbb{Z}_p$ as it illustrates some ideas needed for the proof of the dichotomy in general case when $d \geq 1$. Lemma 1.3 yields $\phi_p(x) = ax$, where $a \in \mathbb{Z}_p$. Hence $|a|_p \leqslant 1$. Then the Reidemeister numbers $R(\phi_p^n) = |a^n - 1|_p^{-1}$, for all $n \in \mathbb{N}$. If $|a|_p < 1$, then $R(\phi_p^n) = |a^n - 1|_p^{-1} = 1$, for all $n \in \mathbb{N}$. Hence the radius of convergence of $R_{\phi_p}(z)$ equals 1 and the Reidemeister zeta function $R_{\phi_p}(z) = \frac{1}{1-z}$ is a rational function.

From now on, we shall write $a(n) << b(n)$ if there is a constant c independent of n for which $a(n) < c \cdot b(n)$. When $|a|_p = 1$, we show that the radius of convergence of $R_{\phi_p}(z)$ equals 1 by deriving the bound

$$\frac{1}{n} << |a^n - 1|_p \leqslant 1 \tag{1}$$

Upper bound in (1) follows from the definition of the p-adic norm. We may suppose that $|a^n - 1|_p < 1$. Let F denote the smallest field which contains $\mathbb{Q}_p$ and is both algebraically closed and complete with respect to $|\cdot|_p$. The p-adic logarithm $\log_p$ is defined as

$$\log_p(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n,$$

and converges for all $z \in F$ such that $|z|_p < 1$. Setting $z = a^n - 1$ we get

$$\log_p(a^n) = (a^n - 1) - \frac{(a^n-1)^2}{2} + \frac{(a^n-1)^3}{3} - ...$$

and so $|\log_p(a^n)|_p \leqslant |a^n - 1|_p$. We always have

$$\frac{1}{n} << |n \log_p(a)|_p = |\log_p(a^n)|_p,$$

so this establishes (1).

The bound (1) implies by Cauchy - Hadamard formula that the radius of convergence of $R_{\phi_p}(z)$ equals 1. Hence it remains to show that the Reidemeister zeta function $R_{\phi_p}(z)$ is irrational if $|a|_p = 1$. Then $R_{\phi_p}(z)$ has the unit circle as a natural boundary by the Lemma 1.2 and by the Pólya–Carlson Theorem. For a contradiction, assume that Reidemeister zeta function $R_{\phi_p}(z)$ is rational. Then Lemma 1.2 implies that the function $Z_p(z) = \sum_{n=1}^{\infty} R(\phi_p^n) z^n$ is rational also. Hence the sequence $R(\phi_p^n)$ satisfies a linear recurrence relation. Define $n = \gamma p^r$, where integer constant $r \geqslant 0$. Applying Lemma 1.5, we see that

$$R(\phi_p^{kn}) = R(\phi_p^n)$$

whenever k is coprime to n. Hence the sequence $R(\phi_p^n)$ assumes infinitely many values infinitely often, and so it cannot satisfy a linear recurrence by a result of Myerson and van der Poorten [24, Prop. 2], giving a contradiction.

Now we consider the general case of a tame endomorphism $\phi_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d,\ d \geq 1$. According to the Lemma 1.4 we have

$$R(\phi_p^n) = \#\mathrm{Coker}(1 - \phi_p^n) = |\det(\Phi_p^n - \mathrm{Id})|_p^{-1} = \prod_{i=1}^{d} |\lambda_i^n - 1|_p^{-1},$$

where Φ_p is a matrix of ϕ_p and $\lambda_1, \lambda_2, ...\lambda_d \in \overline{\mathbb{Q}}_p$ are the eigenvalues of Φ_p, including multiplicities. The polynomial $\prod_{i=1}^{d}(X - \lambda_i)$ has coefficients in $\mathbb{Z}_p$; in particular, $|\lambda_i|_p \leqslant 1$ for every $i \in \{1, \dots, d\}$ (see [15]). If $|\lambda_i|_p < 1$, for every $i \in \{1, \dots, d\}$ then $R(\phi_p^n) = \prod_{i=1}^{d} |\lambda_i^n - 1|_p^{-1} = 1$, for all $n \in \mathbb{N}$. Hence the radius of convergence of $R_{\phi_p}(z)$ equals 1 and the Reidemeister zeta function $R_{\phi_p}(z) = \frac{1}{1-z}$ is a rational function. If $|\lambda_i|_p = 1$, for some $i \in \{1, \dots, d\}$ then the bound (1) implies the bound

$$\frac{1}{n^d} << R(\phi_p^n) = \prod_{i=1}^{d} |\lambda_i^n - 1|_p^{-1} \leqslant 1. \tag{2}$$

Hence the radius of convergence of $R_{\phi_p}(z)$ equals 1 by the
Cauchy– Hadamard formula and the bound (2). Now for the proof of the theorem it remains to show that the Reidemeister zeta function $R_{\phi_p}(z)$

is irrational if $|\lambda_i|_p = 1$ for some $i \in \{1, \dots, d\}$. Then $R_{\phi_p}(z)$ has the unit circle as a natural boundary by the Lemma 1.2 and by the Pólya–Carlson Theorem. For a contradiction, assume that Reidemeister zeta function $R_{\phi_p}(z)$ is rational. Then Lemma 1.2 implies that the function $Z_p(z) = \sum_{n=1}^{\infty} R(\phi_p^n) z^n$ is rational also. Hence the sequence $R(\phi_p^n)$ satifies a linear recurrence relation. Define $n = \gamma p^r$, where integer constant $r \geqslant 0$. Applying Lemma 1.5, we see that $R(\phi_p^{kn}) = R(\phi_p^n)$ whenever k is coprime to n. Hence the sequence $R(\phi_p^n)$ assumes infinitely many values infinitely often, and so it cannot satisfy a linear recurrence by a result of Myerson and van der Poorten [24, Prop. 2], giving a contradiction.

□

2. THE RATIONALITY OF THE COINCIDENCE REIDEMEISTER ZETA FUNCTION FOR ENDOMORPHISMS OF FINITELY GENERATED TORSION-FREE NILPOTENT GROUPS

Example 2.1. ([15], Example 1.3) Let $G = \mathbb{Z}$ be the infinite cyclic group, written additively, and let

$$\phi\colon \mathbb{Z} \to \mathbb{Z}, \quad x \mapsto d_\varphi x \qquad \text{and} \qquad \psi\colon \mathbb{Z} \to \mathbb{Z}, \quad x \mapsto d_\psi x$$

for $d_\phi, d_\psi \in \mathbb{Z}$. The coincidence Reidemeister number $R(\phi, \psi)$ of endomorphisms ϕ, ψ of an Abelian group G coincides with the cardinality of the quotient group $\mathrm{Coker}\,(\phi - \psi) = G/\mathrm{Im}(\phi - \psi)$ (or $\mathrm{Coker}\,(\psi - \phi) = G/\mathrm{Im}(\psi - \phi)$). Hence we have

$$R(\phi^n, \psi^n) = \begin{cases} |d_\psi^n - d_\phi^n| & \text{if } d_\phi^n \neq d_\psi^n, \\ \infty & \text{otherwise.} \end{cases}$$

Consequently, (ϕ, ψ) is tame precisely when $|d_\phi| \neq |d_\psi|$ and, in this case,

$$R_{\phi,\psi}(z) = \frac{1 - d_2 z}{1 - d_1 z} \qquad \text{where } d_1 = \max\{|d_\phi|, |d_\psi|\} \text{ and } d_2 = \frac{d_\phi d_\psi}{d_1}.$$

This simple example (or at least special cases of it) are known. The aim of the current section is to generalise this example to finitely generated torsion-free nilpotent groups. Let G be a finitely generated group and $\phi, \psi : G \to G$ two endomorphisms.

Lemma 2.2. *Let $\phi, \psi : G \to G$ are two automorphisms. Two elements x, y of G are $\psi^{-1}\phi$-conjugate if and only if elements $\psi(x)$ and $\psi(y)$ are (ψ, ϕ)-conjugate. Therefore the Reidemeister number $R(\psi^{-1}\phi)$ is equal to $R(\phi, \psi)$. For a tame pair of commuting automorphisms $\phi, \psi : G \to G$ the coicidence Reidemeister zeta function $R_{\phi,\psi}(z)$ is equal to the Reidemeister zeta function $R_{\psi^{-1}\phi}(z)$.*

Proof. If x and y are $\psi^{-1}\phi$-conjugate, then there is a $g \in G$ such that $x = gy\psi^{-1}\phi(g^{-1})$. This implies $\psi(x) = \psi(g)\psi(y)\phi(g^{-1})$. So $\psi(x)$ and $\psi(y)$ are (ϕ, ψ)-conjugate. The converse statement follows if we move in opposite direction in previous implications. □

We assume X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. The Lefschetz zeta function of a discrete dynamical system f^n is defined as $L_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n\right)$, where

$$L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \operatorname{tr}\left[f^n_{*k} : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})\right]$$

is the Lefschetz number of the iterate f^n of f. The Lefschetz zeta function is a rational function of z and is given by the formula:

$$L_f(z) = \prod_{k=0}^{\dim X} \det\left(I - f_{*k}.z\right)^{(-1)^{k+1}}.$$

In this section we consider finitely generated torsion-free nilpotent group Γ. It is well known [20] that such group Γ is a uniform discrete subgroup of a simply connected nilpotent Lie group G (uniform means that the coset space G/Γ is compact). The coset space $M = G/\Gamma$ is called a nilmanifold. Since $\Gamma = \pi_1(M)$ and M is a $K(\Gamma, 1)$, every endomorphism $\phi : \Gamma \to \Gamma$ can be realized by a selfmap $f : M \to M$ such that $f_* = \phi$ and thus $R(f) = R(\phi)$. Any endomorphism $\phi : \Gamma \to \Gamma$ can be uniquely extended to an endomorphism $F : G \to G$. Let $\tilde{F} : \tilde{G} \to \tilde{G}$ be the corresponding Lie algebra endomorphism induced from F and let $Spectr(\tilde{F})$ be the set of eigenvalues of $\tilde{F}$.

Lemma 2.3. *(cf. Theorem 23 of* [12] *and Theorem 5 of* [14]*) Let $\phi : \Gamma \to \Gamma$ be a tame endomorphism of a finitely generated torsion free nilpotent group. Then the Reidemeister zeta function $R_\phi(z) = R_f(z)$ is a rational function and is equal to*

$$R_\phi(z) = R_f(z) = L_f((-1)^p z)^{(-1)^r}, \tag{3}$$

where p the number of $\mu \in Spectr(\tilde{F})$ such that $\mu < -1$, and r the number of real eigenvalues of $\tilde{F}$ whose absolute value is > 1.

Every pair of automorphisms $\phi, \psi : \Gamma \to \Gamma$ of finitely generated torsion free nilpotent group Γ can be realized by a pair of homeomorphisms $f, g : M \to M$ such that $f_* = \phi$, $g_* = \psi$ and thus $R(g^{-1}f) = R(\psi^{-1}\phi) = R(\phi, \psi)$. An automorphism $\psi^{-1}\phi : \Gamma \to \Gamma$ can be uniquely extended to an automorphism $L : G \to G$. Let $\tilde{L} : \tilde{G} \to \tilde{G}$ be the corresponding Lie algebra automorphism induced from L.

Lemma 2.2 and Lemma 2.3 imply the following

Theorem 2.4. *Let $\phi, \psi : \Gamma \to \Gamma$ be a tame pair of commuting automorphisms of a finitely generated torsion-free nilpotent group Γ. Then the coincidence Reidemeister zeta function $R_{\phi,\psi}(z)$ is a rational function and is equal to*

$$R_{\phi,\psi}(z) = R_{\psi^{-1}\phi}(z) = R_{g^{-1}f}(z) = L_{g^{-1}f}((-1)^p z)^{(-1)^r}, \tag{4}$$

where p the number of $\mu \in Spectr(\tilde{L})$ such that $\mu < -1$, and r the number of real eigenvalues of $\tilde{L}$ whose absolute value is > 1.

For an arbitrary group G, we can define the k-fold commutator group $\gamma_k(G)$ inductively as $\gamma_1(G) := G$ and $\gamma_{k+1}(G) := [G, \gamma_k(G)]$. Let G be a group. For a subgroup $H \leqslant G$, we define the isolator $\sqrt[G]{H}$ of H in G as: $\sqrt[G]{H} = \{g \in G \mid g^n \in H \text{ for some } n \in \mathbb{N}\}$. Note that the isolator of a subgroup $H \leqslant G$ doesn't have to be a subgroup in general. For example, the isolator of the trivial group is the set of torsion elements of G.

Lemma 2.5. *(see* [5]*, Lemma 1.1.2 and Lemma 1.1.4) Let G be a group. Then*

(i) for all $k \in \mathbb{N}$, $\sqrt[G]{\gamma_k(G)}$ is a fully characteristic subgroup of G,
(ii) for all $k \in \mathbb{N}$, the factor $G/\sqrt[G]{\gamma_k(G)}$ is torsion-free,
(iii) for all $k, l \in \mathbb{N}$, the commutator $[\sqrt[G]{\gamma_k(G)}, \sqrt[G]{\gamma_l(G)}] \leq \sqrt[G]{\gamma_{k+l}(G)}$,
(iv) for all $k, l \in \mathbb{N}$ such that $k \geqslant l$ if $M := \sqrt[G]{\gamma_l(G)}$, then

$$\sqrt[G/M]{\gamma_k(G/M)} = \sqrt[G]{\gamma_k(G)}/M.$$

We define the adapted lower central series of a group G as

$$G = \sqrt[G]{\gamma_1(G)} \geqslant \sqrt[G]{\gamma_2(G)} \geqslant ... \geqslant \sqrt[G]{\gamma_k(G)} \geqslant ...,$$

where $\gamma_k(G)$ is the k-th commutator of G.

The adapted lower central series will terminate if and only if G is a torsion-free, nilpotent group. Moreover, all factors $\sqrt[G]{\gamma_k(G)}/\sqrt[G]{\gamma_{k+1}(G)}$ are torsion-free.

We are particularly interested in the case where G is a finitely generated, torsion-free, nilpotent group. In this case the factors of the adapted lower central series are finitely generated, torsion-free, abelian groups, i.e. for all $k \in \mathbb{N}$ we have that

$\sqrt[G]{\gamma_k(G)}/\sqrt[G]{\gamma_{k+1}(G)} \cong \mathbb{Z}^{d_k}$, for some $d_k \in \mathbb{N}$.

Let N be a normal subgroup of a group G and $\phi, \psi \in End(G)$ with $\phi(N) \subseteq N, \psi(N) \subseteq N$. We denote the restriction of ϕ to N by $\phi|_N$, ψ to N by $\psi|_N$ and the induced endomorphisms on the quotient G/N by ϕ', ψ' respectively. We then get the following commutative diagrams with exact rows:

(5)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi|_N,\psi|_N} & & \downarrow{\scriptstyle \phi,\psi} & & \downarrow{\scriptstyle \phi',\psi'} & & \\ 1 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \longrightarrow & 1 \end{array}$$

Note that both i and p induce functions $\hat{i}, \hat{p}$ on the set of Reidemeister classes so that the sequence

$$\mathfrak{R}[\phi|_N, \psi|_N] \xrightarrow{\hat{i}} \mathfrak{R}[\phi, \psi] \xrightarrow{\hat{p}} \mathfrak{R}[\phi', \psi'] \longrightarrow 0$$

is exact, i.e. $\hat{p}$ is surjective and $\hat{p}^{-1}[1] = im(\hat{i})$, where 1 is the identity element of G/N (see also [18]).

Lemma 2.6. *If $R(\phi|_N, \psi|_N) < \infty, R(\phi', \psi') < \infty$ and $N \subseteq Z(G)$, then $R(\phi, \psi) \leqslant R(\phi|_N, \psi|_N)R(\phi', \psi')$.*

Proof. Let $\mathfrak{R}(\phi|_\mathfrak{N}, \psi|_\mathfrak{N}) = \{[n_1]_{\phi|_N,\psi|_N}, ..., [n_{R(\phi|_N,\psi|_N)}]_{\phi|_N,\psi|_N}\}$ and $\mathfrak{R}(\phi', \psi') = \{[g_1N]_{\phi',\psi'}, ..., [g_{R(\phi',\psi')}N]_{\phi',\psi'}\}$ be the $(\phi|_N, \psi|_N)$–Reidemeister classes and (ϕ', ψ')–Reidemeister classes respectively. Let us take $g \in G$. Then $gN \in [g_iN]_{\phi',\psi'}$ for some i, so there exists $hN \in G/N$ such that

$$gN = \psi'(hN)g_iN\phi'(hN)^{-1} = \psi'(h)g_i\phi'(h)^{-1}N.$$

It follows that there exists $n \in N$ such that $g = \psi'(h)g_i\phi'(h)^{-1}n$. In turn $n \in [n_j]_{\phi|_N,\psi|_N}$ for some j, hence there exists $m \in N$ such that $n = \psi|_N(m)n_j\phi|_N(m)^{-1}$. Since $n, m, n_j, \psi|_N(m), \phi|_N(m) \in N \subset Z(G)$, it follows that

$$g = [\psi(hm)](g_in_j)[\phi(hm)^{-1}],$$

i.e $g \in [g_in_j]_{\phi,\psi}$. Since this is true for arbitrary $g \in G$ we obtain that

$$R(\phi, \psi) \leqslant R(\phi|_N, \psi|_N)R(\phi', \psi').$$

□

Theorem 2.7. *Let N be a finitely generated, torsion-free, nilpotent group and*

$$N = \sqrt[N]{\gamma_1(N)} \geqslant \sqrt[N]{\gamma_2(N)} \geqslant ... \geqslant \sqrt[N]{\gamma_c(N)} \geqslant \sqrt[N]{\gamma_{c+1}(N)} = 1$$

be an adapted lower central series of N. Suppose that $R(\phi, \psi) < \infty$ and $R(\phi_k, \psi_k) < \infty$ for a pair ϕ, ψ of endomorphisms of N and for every *pair ϕ_k, ψ_k of induced endomorphisms on the finitely generated torsion-free abelian factors*

$$\sqrt[N]{\gamma_k(N)}/\sqrt[N]{\gamma_{k+1}(N)} \cong \mathbb{Z}^{d_k},\ d_k \in \mathbb{N},\ 1 \leq k \leq c,$$

then

$$R(\phi, \psi) = \prod_{k=1}^{c} R(\phi_k, \psi_k).$$

Proof. We will prove the product formula for the coincidence Reidemeister numbers by induction on the length of an adapted lower central series. Let us denote $\sqrt[N]{\gamma_k(N)}$ as N_k. If $c = 1$, the result follows trivially. Let $c > 1$ and assume the product formula holds for a central series of length $c - 1$. Let $\phi, \psi \in End(N)$, then $\phi(N_c) \subseteq N_c, \psi(N_c) \subseteq N_c$ and hence we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_c & \xrightarrow{i} & N & \xrightarrow{p} & N/N_c & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi_c,\psi_c} & & \downarrow{\scriptstyle \phi,\psi} & & \downarrow{\scriptstyle \phi',\psi'} & & \\ 1 & \longrightarrow & N_c & \xrightarrow{i} & N & \xrightarrow{p} & N/N_c & \longrightarrow & 1 \end{array},$$

where ϕ_c, ψ_c are induced endomorphisms on the N_c. The quotient N/N_c is a finitely generated, nilpotent group with a adapted central series

$$N/N_c = N_1/N_c \geqslant N_2/N_c \geqslant ... \geqslant N_{c-1}/N_c \geqslant N_c/N_c = 1$$

of length $c - 1$.

Every factor of this series is of the form $(N_k/N_c)/(N_{k+1}/N_c) \cong N_k/N_{k+1}$ by the third isomorphism theorem, hence it is also torsion-free. Moreover, because of this natural isomorphism we know that for every induced pair of endomorphisms (ϕ'_k, ψ'_k) on $(N_k/N_c)/(N_{k+1}/N_c)$ it is true that $R(\phi'_k, \psi'_k) = R(\phi_k, \psi_k)$.

The assumptions of the theorem imply that $R(\phi', \psi') < \infty$ and $R(\phi_c, \psi_c) < \infty$.

Moreover, let $[g_1 N_c]_{\phi',\psi'}, ..., [g_n N_c]_{\phi',\psi'}$ be the (ϕ', ψ')-Reidemeister classes and $[c_1]_{\phi_c,\psi_c}, ..., [c_m]_{\phi_c,\psi_c}$ - the (ϕ'_c, ψ'_c)-Reidemeister classes. Since $N_c \subseteq Z(N)$, by Lemma 2.6 we obtain that $R(\phi, \psi) \leqslant R(\phi_c, \psi_c)R(\phi', \psi')$.

To prove the opposite inequality it suffices to prove that every class $[c_i g_j]_{\phi,\psi}$ represents a different (ϕ, ψ)-Reidemeister class. Then we obtain

$$R(\phi, \psi) = R(\phi_c, \psi_c)R(\phi', \psi')$$

and then the theorem follows from the induction hypothesis.

Suppose, that there exists some $h \in N$ such that $c_i g_j = \psi(h) c_a g_b \phi(h)^{-1}$. Then by taking the projection to N/N_c we find that

$$g_j N_c = p(c_i g_j) = p(\psi(h) c_a g_b \phi(h)^{-1}) = \psi'(hN_c)(g_b N_c)\phi'(hN_c)^{-1},$$

hence $[g_j N_c]_{\phi',\psi'} = [g_b N_c]_{\phi',\psi'}$. Now assume that $c_i g_j = \psi(h) c_a g_j \phi(h)^{-1}$. If $h \in N_c \subseteq Z(N)$, then $c_i g_j = \psi(h) c_a \phi(h)^{-1} g_j$ and consequently $[c_i]_{\phi_c,\psi_c} = [c_a]_{\phi_c,\psi_c}$, so let us assume that $h \notin N_c$ and that N_k is the smallest group in the central series which contains h. Then $c_i g_j = \psi(h) c_a g_j \phi(h)^{-1} \Leftrightarrow g_j c_i = \psi(h) c_a g_j \phi(h)^{-1} \Leftrightarrow c_i = g_j^{-1} \psi(h) c_a g_j \phi(h)^{-1}$ $\Leftrightarrow c_i = g_j^{-1} \psi(h) g_j \phi(h)^{-1} c_a \Leftrightarrow c_i c_a^{-1} = g_j^{-1} \psi(h) g_j \phi(h)^{-1}$ and therefore

$$\begin{aligned} c_i c_a^{-1} N_{k+1} &= g_j^{-1} \psi(h) g_j \phi(h)^{-1} N_{k+1} \\ &= [g_j, \psi(h)^{-1}](\psi(h)\phi(h)^{-1}) N_{k+1} \end{aligned}$$

As $c_i c_a^{-1} \in N_c \subseteq N_{k+1}$ and $[g_j, \psi(h)^{-1}] \in N_{k+1}$, we find that $(\phi_k)(hN_{k+1})$ $= (\psi_k)(hN_{k+1})$. That means that the set of coincidence points $\mathrm{Coin}(\phi'_k, \psi'_k)$ $\neq \{1\}$, which implies that $R(\phi', \psi') = \infty$ and this contradicts assumption. □

Theorem 2.8. *Let $\phi, \psi \colon N \to N$ be a tame pair of endomorphisms of a finitely generated torsion-free nilpotent group N. Let c denote the nilpotency class of N and, for $1 \leqslant k \leqslant c$, let $\phi_k, \psi_k \colon G_k \to G_k$, $1 \leq k \leq c$, denote the tame pairs of induced endomorphisms of the finitely generated torsion-free abelian factor groups $G_k = N_k/N_{k+1} =$ $= \sqrt[N]{\gamma_k(N)}/\sqrt[N]{\gamma_{k+1}(N)} \cong \mathbb{Z}^{d_k}$, for some $d_k \in \mathbb{N}$, that arise from an adapted lower central series of N. Then the following hold.*

(1) *For each $n \in \mathbb{N}$,*

$$R(\phi^n, \psi^n) = \prod_{k=1}^{c} R(\phi_k^n, \psi_k^n) \qquad \textit{for } n \in \mathbb{N}.$$

(2) *For $1 \leqslant k \leqslant c$, let*

$$\phi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}} \colon G_{k,\mathbb{Q}} \to G_{k,\mathbb{Q}}$$

denote the extensions of ϕ_k, ψ_k to the divisible hull $G_{k,\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} G_k \cong \mathbb{Q}^{d_k}$ of G_k. Suppose that each pair of endomorphisms $\phi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}}$ is simultaneously triangularisable. Let $\xi_{k,1}, \ldots, \xi_{k,d_k}$ and $\eta_{k,1}, \ldots, \eta_{k,d_k}$ be the eigenvalues of $\phi_{k,\mathbb{Q}}$ and $\psi_{k,\mathbb{Q}}$ in the field $\mathbb{C}$, including multiplicities, ordered so that, for $n \in \mathbb{N}$, the eigenvalues of $\phi_{k,\mathbb{Q}}^n - \psi_{k,\mathbb{Q}}^n$ are $\xi_{k,1}^n - \eta_{k,1}^n, \ldots, \xi_{k,d_k}^n - \eta_{k,d_k}^n$. Then for each $n \in \mathbb{N}$,

$$R(\phi_k^n, \psi_k^n) = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|; \tag{6}$$

(3) *Moreover, suppose that $|\xi_{k,i}| \neq |\eta_{k,i}|$ for $1 \leqslant k \leqslant c$, $1 \leqslant i \leqslant d_k$. If*

ϕ, ψ is a tame pair of endomorphisms of N and $\phi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}}$, $1 \leqslant k \leqslant c$, are simultaneously triangularisable pairs of endomorphisms of $G_{k,\mathbb{Q}}$, then the coincidence Reidemeister zeta function $R_{\phi,\psi}(z)$ is a rational function.

Proof. The coincidence Reidemeister number $R(\phi, \psi)$ of automorphisms ϕ, ψ of an Abelian group G coincides with the cardinality of the quotient group $\mathrm{Coker}\ (\phi - \psi) = G/\mathrm{Im}(\phi - \psi)$ (or $\mathrm{Coker}\ (\psi - \phi) = G/\mathrm{Im}(\psi - \phi)$).

For $1 \leqslant k \leqslant c$, let tame pairs $\phi_k, \psi_k \colon G_k \to G_k$ of induced endomorphisms of the finitely generated torsion-free abelian factor groups $G_k = N_k/N_{k+1} \cong \mathbb{Z}^{d_k}$, are represented by integer matrices $A_k, B_k \in M_{d_k}(\mathbb{Z})$ associated to them respectively. There is a diagonal integer matrix $C_k = \mathrm{diag}(c_1, ..., c_{d_k})$ such that $C_k = M_k(A_k - B_k)N_k$; where M_k and N_k are unimodular matrices. Now we have $\det C_k = \det(A_k - B_k)$ and the order of the cokernel of $\phi_k - \psi_k$ is the order of the group $\mathbb{Z}/c_1\mathbb{Z} \oplus ..... \oplus \mathbb{Z}/c_{d_k}\mathbb{Z}$. Thus the order of the cokernel of $\phi_k - \psi_k$ is $|\mathrm{Coker}\ (\phi_k - \psi_k)| = |c_1 \cdots c_{d_k}| = |\det C_k| = |\det(\phi_k - \psi_k)|$.

Then for each $n \in \mathbb{N}$ and $1 \leqslant k \leqslant c$, $R(\phi_k^n, \psi_k^n) = |\mathrm{Coker}\ (\phi_k - \psi_k)| = |\det(\phi_k - \psi_k)| = |\det(\phi_{k,\mathbb{Q}} - \psi_{k,\mathbb{Q}})| = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|$.

Now we will prove the rationality of $R_{\phi,\psi}(z)$. We open up the absolute values in the product $R(\phi_k^n, \psi_k^n) = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|$, $1 \leqslant k \leqslant c$. Complex eigenvalues $\xi_{k,i}$ in the spectrum of $\phi_{k,\mathbb{Q}}$, respectively $\eta_{k,i}$ in the spectrum of $\psi_{k,\mathbb{Q}}$, appear in pairs with their complex conjugate $\overline{\xi_{k,i}}$, respectively $\overline{\eta_{k,i}}$.

Moreover, such pairs can be lined up with one another in a simultaneous triangularisation as follows. Write $\phi_{k,\mathbb{C}}, \psi_{k,\mathbb{C}}$ for the induced endomorphisms of the $\mathbb{C}$-vector space $V = \mathbb{C} \otimes_{\mathbb{Q}} G \cong \mathbb{C}^{d_k}$. If $v \in V$ is, at the same time, an eigenvector of $\phi_{k,\mathbb{C}}$ with complex eigenvalue ξ_{k,d_k} and an eigenvector of $\psi_{k,\mathbb{C}}$ with eigenvalue η_{k,d_k}, then there is $w \in V$ such that w is, at the same time, an eigenvector of $\phi_{k,\mathbb{C}}$ with eigenvalue $\overline{\xi_{k,d_k}} \neq \xi_{k,d_k}$ and an eigenvector of $\psi_{k,\mathbb{C}}$ with eigenvalue $\overline{\eta_{k,d_k}}$, possibly equal to η_{k,d_k}. Thus we can start our complete flag of $\{\phi, \psi\}$-invariant subspaces of V with $\{0\} \subset \langle v \rangle \subset \langle v, w \rangle$, and proceed with $V/\langle v, w \rangle$ by induction to produce the rest of the flag in the same way, treating complex eigenvalues of $\psi_{k,\mathbb{C}}$ in the same way as they appear. If at least one of $\xi_{k,i}, \eta_{k,i}$ is complex so that these eigenvalues of $\phi_{\mathbb{Q}}$ and $\psi_{\mathbb{Q}}$ are paired with eigenvalues $\xi_{k,j} = \overline{\xi_{k,i}}, \eta_{k,j} = \overline{\eta_{k,i}}$, for suitable $j \neq i$, as discussed above, we see that

$$\left|\xi_{k,i}^n - \eta_{k,i}^n\right| \left|\xi_{k,j}^n - \eta_{k,j}^n\right| = \left|\xi_{k,i}^n - \eta_{k,i}^n\right|^2 = \left(\xi_{k,i}^n - \eta_{k,i}^n\right) \cdot \left(\overline{\xi_{k,i}}^{\,n} - \overline{\eta_{k,i}}^{\,n}\right).$$

If $\xi_{k,i}$ and $\eta_{k,i}$ are both real eigenvalues of $\phi_{\mathbb{Q}}$ and $\psi_{\mathbb{Q}}$, not paired up with another pair of eigenvalues, then exactly as in Example 2.1 above we have $|\xi_{k,i}^n - \eta_{k,i}^n| = \delta_{1,k,i}^n - \delta_{2,k,i}^n$, where $\delta_{1,k,i} = \max\{|\xi_{k,i}|, |\eta_{k,i}|\}$ and $\delta_{2,k,i} = \frac{\xi_{k,i} \cdot \eta_{k,i}}{\delta_{1,k,i}}$. Hence we can expand each product $R(\phi_k^n, \psi_k^n)$, $1 \leqslant k \leqslant c$ using an appropriate symmetric polynomial, to obtain for the Reidemeister numbers $R(\phi^n, \psi^n)$ an expression of the form

$$R(\phi^n, \psi^n) = \prod_{k=1}^{c} R(\phi_k^n, \psi_k^n) = \prod_{k=1}^{c} \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n| = \sum_{j \in J} c_j w_j^n, \tag{7}$$

where J is a finite index set, $c_j \in \{-1, 1\}$ and $\{w_j \mid j \in J\} \subseteq \mathbb{C} \setminus \{0\}$. Consequently, the coincidence Reidemeister zeta function can be written as

$$R_{\phi,\psi}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n, \psi^n)}{n} z^n\right) = \exp\left(\sum_{j \in J} c_j \sum_{n=1}^{\infty} \frac{(w_j z)^n}{n}\right).$$

and it follows immediately that $R_{\phi,\psi}(z) = \prod_{j \in J}(1 - w_j z)^{-c_j}$ is a rational function.

□

REFERENCES

[1] J. Bell, R. Miles, T. Ward, Towards a Pólya–Carlson dichotomy for algebraic dynamics, *Indag. Math.(N.S.* **25** (2014), no. 4, 652–668.

[2] J. Byszewski and G. Cornelissen, Dynamics on abelian varieties in positive characteristic, with an appendix by R. Royals and T. Ward, Algebra Number Theory **12** (2018), 2185–2235.

[3] F. Carlson, 'Über ganzwertige Funktionen', *Math. Z.* **11** (1921), no. 1-2, 1–23.

[4] V. Chothi, G. Everest, and T. Ward, S-integer dynamical systems: periodic points, *J. Reine Angew. Math.* **489** (1997), 99–132.

[5] Dekimpe K. Almost-Bieberbach groups: affine and polynomial structures. Vol. 1639. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1996, pp. x+259

[6] Dekimpe, K. and Dugardein, G.-J. Nielsen zeta functions for maps on infra-nilmanifolds are rational, J. Fixed Point Theory Appl., **17**(2)(2015), 355–370.

[7] Karel Dekimpe, Sam Tertooy, and Iris Van den Bussche, Reidemeister zeta functions of low-dimensional almost-crystallographic groups are rational, Communications in Algebra, **46** (9)(2018), 4090–4103.

[8] D. Eisenbud, Commutative algebra, volume 150 of Graduate Texts in Mathematics, Springer - Verlag, New York, 1995, With a view toward algebraic geometry.

[9] G. Everest, V. Stangoe, and T. Ward, 'Orbit counting with an isometric direction', in Algebraic and topological dynamics, in Contemp. Math. **385** (2005), pp. 293–302, Amer. Math. Soc., Providence, RI.

[10] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, Recurrence sequences, in *Mathematical Surveys and Monographs* **104**, Amer. Math. Soc., Providence, RI, 2003.

[11] A. L. Fel'shtyn, The Reidemeister zeta function and the computation of the Nielsen zeta function, Colloq. Math., **62** (1991), 153–166.

[12] A. Fel'shtyn, Dynamical zeta functions, Nielsen theory and Reidemeister torsion, Mem. Amer. Math. Soc., **699**, Amer. Math. Soc., Providence, R.I. 2000.

[13] A. L. Fel'shtyn and R. Hill, The Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion, K-theory, **8** (1994), 367–393.

[14] A. L. Fel'shtyn, R. Hill and P. Wong, Reidemeister numbers of equivariant maps, Topology Appl., **67** (1995), 119–131
[15] Alexander Fel'shtyn and Benjamin Klopsch. Pólya-Carlson dichotomy for coincidence Reidemeister zeta functions via profinite completions, Indag. Math. (N.S.), **33**(2022), No. 4, 753–767.
[16] Alexander Fel'shtyn and Jong Bum Lee, The Nielsen and Reidemeister numbers of maps on infra-solvmanifolds of type (R), Topology Appl. **181**(2015), 62–103.
[17] A. Fel'shtyn and M. Zietek, Dynamical zeta functions of Reidemeister type, Topological Methods Nonlinear Anal. **56**(2020), 433–455.
[18] Daciberg L. Gonçalves, Peter N.-S. Wong: Homogenous spaces in coincidence theory, Forum Math. **17** (2005), 297–313.
[19] L. Li, On the rationality of the Nielsen zeta function, Adv. in Math. (China), **23** (1994) no. 3, 251–256.
[20] A. Mal'cev, On a class of homogeneous spaces, Izvestiya Akademii Nauk SSSR. Seriya Matematičeskaya, **13** (1949), 9-32.
[21] R. Miles, Zeta functions for elements of entropy rank-one actions, Ergodic Theory Dynam. Systems **27** (2007), no. 2, 567–582.
[22] R. Miles, Periodic points of endomorphisms on solenoids and related groups, Bull. Lond. Math. Soc. **40** (2008), no. 4, 696–704.
[23] R. Miles, Synchronization points and associated dynamical invariants, Trans. Amer. Math. Soc. **365** (2013), 5503–5524.
[24] G. Myerson and A. J. van der Poorten, Some problems concerning recurrence sequences, Amer. Math. Monthly **102** (1995), no. 8, 698–705.
[25] G. Pólya, 'Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe', *Math. Ann.* **99** (1928), no. 1, 687–706.
[26] S. L. Segal. Nine introductions in complex analysis, volume **208** of North-Holland Mathematics Studies. Elsevier Science B.V., Amsterdam, revised edition, 2008.
[27] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc., **73** (1967), 747–817.
[28] Evgenij Troitsky, Two examples related to the twisted Burnside-Frobenius theory for infinitely generated groups, Fundam. Appl. Math. **21**(2016), No. 5, 231–239.
[29] P. Wong, Reidemeister zeta function for group extensions, J. Korean Math. Soc., **38** (2001), 1107–1116.

WOJCIECH BONDAREWICZ, INSTYTUT MATEMATYKI, UNIWERSYTET SZCZECINSKI, UL. WIELKOPOLSKA 15, 70-451 SZCZECIN, POLAND
Email address: wojciech.bondarewicz@usz.edu.pl

ALEXANDER FEL'SHTYN, INSTYTUT MATEMATYKI, UNIWERSYTET SZCZECINSKI, UL. WIELKOPOLSKA 15, 70-451 SZCZECIN, POLAND
Email address: alexander.felshtyn@usz.edu.pl

MALWINA ZIETEK, INSTYTUT MATEMATYKI, UNIWERSYTET SZCZECINSKI, UL. WIELKOPOLSKA 15, 70-451 SZCZECIN, POLAND
Email address: malwina.zietek@gmail.com