Anderson finiteness for RCD spaces

by

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1 Introduction

For $K \in \mathbb{R}$, $N \in [1, \infty)$, the class of $RCD^*(K, N)$ spaces consists of proper metric measure spaces that satisfy a synthetic condition of having Ricci curvature bounded below by $K$ and dimension bounded above by $N$. This class is closed under measured Gromov–Hausdorff convergence and contains the class of complete Riemannian manifolds of Ricci curvature $\geq K$ and dimension $\leq N$.

$RCD^*(K, N)$ spaces have a well defined notion of dimension called \textit{rectifiable dimension} (see Theorem 16), which is always an integer between 0 and $N$, and is lower semi-continuous with respect to pointed Gromov–Hausdorff convergence (see Theorem 20). This motivates the following definition.

\textbf{Definition 1.} Let $K \in \mathbb{R}$, $N \in [1, \infty)$, and $(X, p)$ a pointed $RCD^*(K, N)$ space. We define the \textit{collapsing volume} of $(X, p)$ as

$$\text{vol}^*_K N(X, p) := \inf d_{GH}((X, p), (Y, q)),$$

where the infimum is taken among all pointed $RCD^*(K, N)$ spaces $(Y, q)$ whose rectifiable dimension is strictly less than the one of $X$.

\textbf{Proposition 2.} Let $(X_i, p_i)$ be a sequence of pointed $RCD^*(K, N)$ spaces that converges in the Gromov–Hausdorff sense to a pointed $RCD^*(K, N)$ space $(X, p)$. If the rectifiable dimension of $X_i$ is $n$ for each $i$, then the following are equivalent:

1. The rectifiable dimension of $X$ is strictly less than $n$.
2. $\text{vol}^*_K N(X_i, p_i) \to 0$.
3. $\text{vol}^*_K N(X_{i_k}, p_{i_k}) \to 0$ for a subsequence.

\textit{Proof.} (3 $\Rightarrow$ 1) By hypothesis, there is a sequence of $RCD^*(K, N)$ spaces $(Y_{i_k}, q_{i_k})$ of rectifiable dimension strictly less than $n$ and converging to $(X, p)$ as $k \to \infty$. From the fact that rectifiable dimension is lower semi-continuous, we deduce 1. The implications 1 $\Rightarrow$ 2 $\Rightarrow$ 3 are tautological. \hfill \Box

\textbf{Corollary 3.} Let $X_i$ be a sequence of $RCD^*(K, N)$ spaces, and $p_i, p'_i \in X_i$ pairs of points with $\limsup_{i \to \infty} d(p_i, p'_i) < \infty$. Then $\text{vol}^*_K N(X_i, p_i) \to 0$ if and only if $\text{vol}^*_K N(X_i, p'_i) \to 0$.

The main result of this note is a generalization to $RCD^*(K, N)$ spaces of a classical finiteness result of Anderson [1].
Theorem 4. For each $K \in \mathbb{R}$, $N \in [1, \infty)$, $D > 0$, $\nu > 0$, the class of pointed $RCD^*(K, N)$ spaces of diameter $\leq D$ and $\text{vol}_{K,N}^* \geq \nu$ contains finitely many fundamental group isomorphism types.

Theorem 4 will be obtained from the following result which states that a lower bound on the collapsing volume of the quotient of an $RCD^*(K, N)$ space by a discrete group yields a uniform discreteness gap on the corresponding group (see Equation 1 below for the definition of $d_p$).

Theorem 5. For each $K \in \mathbb{R}$, $N \in [1, \infty)$, $\nu > 0$, there is $\varepsilon > 0$ such that the following holds. If $(X, p)$ is a pointed $RCD^*(K, N)$ space and $\Gamma \leq Iso(X)$ is a discrete group of measure preserving isometries with $\text{vol}_{K,N}^*(X/\Gamma, [p]) \geq \nu$, then

$$\{ g \in \Gamma | d_p(g, Id_X) \leq \varepsilon \} = \{ Id_X \}.$$

The remainder of this note contains the proof of theorems 4 and 5. In Section 2 we cover the required material and in Section 3 we present the proofs.

2 Preliminaries

2.1 Notation

For a metric space $X$, $p \in X$, $r > 0$, the closed ball of radius $r$ centered at $p$ will be denoted as $B(p, r, X)$. We denote by $X \cup \{ * \}$ the metric space obtained by adjoining to $X$ a point $*$ with $d(x, *) = \infty$ for all $x \in X$.

2.2 RCD spaces and isometries

A $CD^*(K, N)$ space is a proper metric space $(X, d)$ equipped with a fully supported Radon measure $m$ for which an appropriate entropy in its space of probability measures is in a suitable sense concave with respect to the $L^2$-Wasserstein distance. For a $CD^*(K, N)$ space $(X, d, m)$, if its Sobolev space $W^{1,2}$ is a Hilbert space, we say that it is an $RCD^*(K, N)$ space. See [7] for a precise definition and different reformulations.

Remark 6. If $(X, d, m)$ is an $RCD^*(K, N)$ space, then for any $c > 0$, $(X, d, cm)$ is also an $RCD^*(K, N)$ space, and for any $\lambda > 0$, $(X, \lambda d, m)$ is an $RCD^*(\lambda^{-2}K, N)$ space. Also, if a metric measure space $(X, d, m)$ is an $RCD^*(K - \varepsilon, N)$ space for all $\varepsilon > 0$, then it is also an $RCD^*(K, N)$ space.

Along this note we are interested only in topological properties of $RCD^*(K, N)$ spaces, so by an abuse of notation, we say that a proper metric space $(X, d)$ is an $RCD^*(K, N)$ space if it admits a Radon measure that makes it an $RCD^*(K, N)$ space. Any two such
measures are equivalent, so we can still talk about full (or zero) measure sets even after this abuse.

Even though we don’t know much about the global topology of $RCD^*(K,N)$ spaces, we know they are semi-locally-simply-connected [16], and their universal cover is still an $RCD^*(K,N)$ space [14].

**Theorem 7.** (Wang) Let $X$ be an $RCD^*(K,N)$ space. Then $X$ is semi-locally-simply-connected, so for any $p \in X$ we can identify the fundamental group $\pi_1(X,p)$ with the group of deck transformations of the universal cover $\tilde{X}$.

**Theorem 8.** (Mondino–Wei) If $X$ is an $RCD^*(K,N)$ space, then its universal cover $\tilde{X}$ admits a $\pi_1(X)$-invariant measure that makes it an $RCD^*(K,N)$ space.

Recall that for Riemannian manifolds, if an isometry coincides with the identity up to first order at a point, then it is necessarily the identity. An analogue of this statement for $RCD^*(K,N)$ spaces is the following [11].

**Lemma 9.** Let $X$ be an $RCD^*(K,N)$ space and $f : X \to X$ a non-trivial isometry. Then the set of fixed points of $f$ has zero measure.

For a pointed proper metric space $(X,p)$, we define the compact-open distance between two functions $h_1, h_2 : X \to X$ as

$$d_p(h_1, h_2) := \inf_{r>0} \left\{ \frac{1}{r} + \sup_{x \in B(p,r,X)} d(h_1x, h_2x) \right\}.$$  \hspace{1cm} (1)

When we restrict this metric to the group of isometries $Iso(X)$, we get a left invariant (not necessarily geodesic) metric that induces the compact open topology (independent of $p$) and makes $Iso(X)$ a proper metric group. However, this distance is defined on the full class of functions $X \to X$, where it is not left invariant nor proper anymore.

Recall that if $X$ is a proper geodesic space and $\Gamma \leq Iso(X)$ is a closed group of isometries, the metric $d'$ on $X/\Gamma$ defined as $d'([x], [y]) := \inf_{g \in \Gamma} d(gx, y)$ makes it a proper geodesic space. By the work of Galaz–Kell–Mondino–Sosa, the class of $RCD^*(K,N)$ spaces is closed under quotients by discrete groups [9].

**Theorem 10.** (Galaz–Kell–Mondino–Sosa) Let $(X,d,m)$ be an $RCD^*(K,N)$ space and $\Gamma \leq Iso(X)$ a discrete group of measure preserving isometries. Then the metric space $(X/\Gamma, d')$ admits a measure $m'$ that makes it an $RCD^*(K,N)$ space. Moreover, if $\rho : X \to X/\Gamma$ denotes the projection, $m'$ can be taken so that $m(A) = m'(\rho(A))$ for all Borel subsets $A$ of $X$ sent isometrically to $X/\Gamma$ by $\rho$. 

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2.3 Gromov–Hausdorff topology

The Gromov–Hausdorff topology in the class of pointed proper metric spaces quantifies how much two spaces are from being isometric.

**Definition 11.** Let \((X,p),(Y,q)\) be pointed proper metric spaces and \(\varepsilon > 0\). We say that a function \(f : X \to Y \cup \{\ast\}\) is an \(\varepsilon\)-approximation if \(d(f(p),q) \leq \varepsilon\) and for \(R = 1/\varepsilon\) one has

\[
\begin{align*}
& f^{-1}(B(q,R,Y)) \subset B(p,2R,X), \\
& \sup_{x_1,x_2 \in B(p,2R,X)} |d(f(x_1),f(x_2)) - d(x_1,x_2)| \leq \varepsilon, \\
& \sup_{y \in B(q,R,Y)} \inf_{x \in B(p,2R,X)} d(f(x),y) \leq \varepsilon.
\end{align*}
\]

The **pointed Gromov–Hausdorff distance** between \((X,p),(Y,q)\) is defined as

\[d_{GH}((X,p),(Y,q)) := \inf\{\varepsilon > 0 | \text{there is an } \varepsilon\text{-approximation } f : X \to Y \cup \{\ast\}\}.\]

Strictly speaking, \(d_{GH}\) as defined above is not a distance as it is not symmetric. However, it still generates a first countable Hausdorff topology in the class of pointed proper metric spaces (see [4], Chapter 8). This is called the **Gromov–Hausdorff topology**.

**Proposition 12.** (Gromov) For pointed proper metric spaces \((X,p),(X_i,p_i)\), we have \(d_{GH}((X_i,p_i),(X,p)) \to 0\) as \(i \to \infty\) if and only if \(d_{GH}((X,p),(X_i,p_i)) \to 0\) as \(i \to \infty\).

Moreover, in either case there are sequences \(\phi_i : X_i \to X \cup \{\ast\}\) and \(\psi_i : X \to X_i \cup \{\ast\}\) of \(\varepsilon_i\)-approximations with \(\varepsilon_i \to 0\) and such that

\[\lim_{i \to \infty} d_p(\phi_i \circ \psi_i, Id_X) = 0.\]

**Corollary 13.** Let \((X_i,p_i)\) be a sequence of pointed proper metric spaces that converges in the Gromov–Hausdorff sense to a pointed proper metric space \((X,p)\). Then for each \(R > 0, \varepsilon > 0\), there is \(M \in \mathbb{N}\) such that any set \(S \subset B(p_i,R,X_i)\) with \(d(s_1,s_2) \geq \varepsilon\) for each \(s_1,s_2 \in S\), one has \(|S| \leq M\).

One of the main features of the class of \(RCD^*(K,N)\) spaces is the Gromov–Hausdorff compactness property [2].

**Theorem 14.** (Bacher–Sturm) If \((X_i,p_i)\) is a sequence of pointed \(RCD^*(K,N)\) spaces, then one can find a subsequence that converges in the Gromov–Hausdorff sense to a pointed \(RCD^*(K,N)\) space \((X,p)\).

**Definition 15.** Let \(X\) be an \(RCD^*(K,N)\) space and \(n \in \mathbb{N}\). We say that \(p \in X\) is an \(n\)-**regular** point if for each \(\lambda_i \to \infty\), the sequence \((\lambda_i X, p)\) converges in the Gromov–Hausdorff sense to \((\mathbb{R}^n,0)\).
Mondino–Naber showed that the set of regular points in an $RCD^*(K, N)$ space has full measure \cite{13}. This result was refined by Brué–Semola who showed that almost all points have the same local dimension \cite{3}.

**Theorem 16.** (Brué–Semola) Let $X$ be an $RCD^*(K, N)$ space. Then there is a unique $n \in \mathbb{N} \cap [0, N]$ such that the set of $n$-regular points in $X$ has full measure. This number $n$ is called the *rectifiable dimension* of $X$.

**Definition 17.** Let $X_i$ be a sequence of $RCD^*(K, N)$ spaces of rectifiable dimension $n$. A choice of points $x_i \in X_i$ is said to be a *Reifenberg sequence* if for any $\lambda_i \to \infty$, the sequence $(\lambda_i X_i, x_i)$ converges in the Gromov–Hausdorff sense to $(\mathbb{R}^n, 0)$.

**Theorem 18.** (Mondino–Naber) For each $i \in \mathbb{N}$, let $(X_i, d_i, m_i)$ be an $RCD^*(-\varepsilon_i, N)$ space with $\varepsilon_i \to 0$ of rectifiable dimension. Assume that for some choice $p_i \in X_i$, the sequence $(X_i, p_i)$ converges in the Gromov–Hausdorff sense to $(\mathbb{R}^n, 0)$. Then there is a sequence of subsets $U_i \subset B(p_i, 1, X_i)$ with $m_i(U_i)/m_i(B(p_i, 1, X_i)) \to 1$ such that any sequence $x_i \in U_i$ is a Reifenberg sequence.

From the work of Mondino–Naber and Brué–Semola, we can conclude that the rectifiable dimension is a topological invariant.

**Corollary 19.** Let $X$ be an $RCD^*(K, N)$ space of rectifiable dimension $n$. There is an open dense subset $U \subset X$ locally bi-Lipschitz homeomorphic to $\mathbb{R}^n$.

Using the results above, Kitabeppu showed that the rectifiable dimension is lower semi-continuous with respect to Gromov–Hausdorff convergence \cite{12}.

**Theorem 20.** (Kitabeppu) Let $(X_i, p_i)$ be a sequence of pointed $RCD^*(K, N)$ spaces of rectifiable dimension $n$ converging in the Gromov–Hausdorff sense to the space $(X, p)$. Then the rectifiable dimension of $X$ is at most $n$.

The well known Cheeger–Gromoll splitting theorem \cite{6} was extended by Cheeger–Colding for limits of Riemannian manifolds with lower Ricci curvature bounds \cite{5}, and later by Gigli to this setting \cite{10}.

**Theorem 21.** (Gigli) Let $(X, d, m)$ be an $RCD^*(0, N)$ space of rectifiable dimension $n$. If $(X, d)$ contains an isometric copy of $\mathbb{R}^m$, then there is $c > 0$ and a metric measure space $(Y, d_Y, n)$ such that $(X, d, c m)$ is isomorphic to the product $(\mathbb{R}^m \times Y, d_{\mathbb{R}^m} \times d_Y, H^n \otimes n)$. Moreover, $(Y, d_Y, n)$ is an $RCD^*(0, N - m)$ space of rectifiable dimension $n - m$.

### 2.4 Equivariant Gromov–Hausdorff convergence

Recall from Proposition 12 that if a sequence of pointed proper metric spaces $(X_i, p_i)$ converges in the Gromov–Hausdorff sense to the pointed proper metric space $(X, p)$, one has $\varepsilon_i$-approximations $\phi_i : X_i \to X \cup \{\ast\}$ and $\psi_i : X \to X_i \cup \{\ast\}$ with $\varepsilon_i \to 0$ and satisfying Equation 5.
Definition 22. Consider a sequence of pointed proper metric spaces \((X_i, p_i)\) that converges in the Gromov–Hausdorff sense to a pointed proper metric space \((X, p)\) and a sequence of closed groups of isometries \(\Gamma_i \leq \text{Iso}(X_i)\). We say that the sequence \(\Gamma_i\) converges \textit{equivariantly} to a closed group \(\Gamma \leq \text{Iso}(X)\) if:

- For each \(g \in \Gamma\), there is a sequence \(g_i \in \Gamma_i\) with \(d_p(\psi_i \circ g_i \circ \phi_i, g) \to 0\) as \(i \to \infty\).
- For a sequence \(g_i \in \Gamma_i\) and \(g \in \text{Iso}(X)\), if there is a subsequence \(g_{i_k}\) with \(d_p(\psi_{i_k} \circ g_{i_k} \circ \phi_{i_k}, g) \to 0\) as \(k \to \infty\), then \(g \in \Gamma\).

We say that a sequence of isometries \(g_i \in \text{Iso}(X_i)\) converges to an isometry \(g \in \text{Iso}(X)\) if

\[d_p(\psi_i \circ g_i \circ \phi_i, g) \to 0\] as \(i \to \infty\).

Equivariant convergence allows one to take limits before or after taking quotients [8].

Lemma 23. Let \((Y_i, q_i)\) be a sequence of proper metric spaces that converges in the Gromov–Hausdorff sense to a proper space \((Y, q)\), and \(\Gamma_i \leq \text{Iso}(Y_i)\) a sequence of closed groups of isometries that converges equivariantly to a closed group \(\Gamma \leq \text{Iso}(Y)\). Then the sequence \((Y_i/\Gamma_i, [q_i])\) converges in the Gromov–Hausdorff sense to \((Y/\Gamma, [q])\).

Fukaya–Yamaguchi obtained an Arzelá–Ascoli type result for equivariant convergence ([8], Proposition 3.6).

Theorem 24. (Fukaya–Yamaguchi) Let \((Y_i, q_i)\) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space \((Y, q)\), and take a sequence \(\Gamma_i \leq \text{Iso}(Y_i)\) of closed groups of isometries. Then there is a subsequence \((Y_{i_k}, q_{i_k}, \Gamma_{i_k})_{k \in \mathbb{N}}\) such that \(\Gamma_{i_k}\) converges equivariantly to a closed group \(\Gamma \leq \text{Iso}(Y)\).

Proposition 25. Let \((X_i, p_i)\) be a sequence of pointed proper metric spaces that converges in the Gromov–Hausdorff sense to a pointed proper metric space \((X, p)\). Assume a sequence of closed groups \(\Gamma_i \leq \text{Iso}(X_i)\) converges equivariantly to a closed group \(\Gamma \leq \text{Iso}(X)\). Then the sequence of pointed metric spaces \((\Gamma_i, d_{p_i}, \text{Id}_{X_i})\) converges in the Gromov–Hausdorff sense to \((\Gamma, d_p, \text{Id}_X)\).

Proof. Recall that one has \(\varepsilon_i\)-approximations \(\phi_i : X_i \to X \cup \{\ast\}\) and \(\psi_i : X \to X_i \cup \{\ast\}\) with \(\varepsilon_i \to 0\) and satisfying Equation 5. With these functions, one could define \(f_i : \Gamma_i \to \Gamma \cup \{\ast\}\) in the following way: for each \(g \in \Gamma_i\), if there is an element \(\gamma \in \Gamma\) with \(d_p(\phi_i \circ g \circ \psi_i, \gamma) \leq 1\), choose \(f_i(g)\) to be an element of \(\Gamma\) that minimizes \(d_p(\phi_i \circ g \circ \psi_i, f_i(g))\). Otherwise, set \(f_i(g) = \ast\). Now we verify that \(f_i\) are \(\delta_i\)-approximations for some \(\delta_i \to 0\). The fact that \(f_i(\text{Id}_{X_i}) \to \text{Id}_X\) follows from Equation 5.

The fourth condition, corresponding to Equation 4, follows directly from the first condition in the definition of equivariant convergence and our construction.

To check that \(f_i\) satisfy the second condition, corresponding to Equation 2, assume by contradiction that after taking a subsequence we can find \(g_i \in \Gamma_i\) such that \(d_{p_i}(g_i, \text{Id}_{X_i}) \to \ast\).
\( \infty \) but \( d_p(f_i(g_i), Id_X) \) is bounded. This implies that \( d(g_i(p_i), p_i) \to \infty \) and since \( g_i \) is an isometry, \( d(g_i \circ \psi_i(p), p_i) \to \infty \). As \( \phi_i \) are \( \varepsilon_i \)-approximations for \( \varepsilon_i \to 0 \), \( d(\phi_i \circ g_i \circ \psi_i(p), p) \to \infty \) as well. On the other hand, as \( d_p(f_i(g_i), Id_X) \) is bounded, \( f_i(g_i) \neq * \) and \( d_p(\phi_i \circ g_i \circ \psi_i, f_i(g_i)) \leq 1 \) for all \( i \), meaning that \( d_p(\phi_i \circ g_i \circ \psi_i, Id_X) \) is bounded, which is a contradiction.

To verify the third condition, corresponding to Equation 3, assume by contradiction that after taking a subsequence we can find two sequences \( g_i, h_i \in \Gamma_i \) such that the sequences \( d_p(g_i, Id_{X_i}), d_p(h_i, Id_{X_i}) \) are bounded but \( |d_p(g_i, h_i) - d_p(f_i(g_i), f_i(h_i))| \geq \eta \) for some \( \eta > 0 \). After again taking a subsequence we can assume \( g_i \) converges to \( g \in \Gamma \) and \( h_i \) converges to \( h \in \Gamma \). This means that \( d_p(f_i(g_i), \phi_i \circ g_i \circ \psi_i) \to 0 \), \( d_p(f_i(h_i), \phi_i \circ h_i \circ \psi_i) \to 0 \). Hence for \( i \) large enough one has \( |d_p(g_i, h_i) - d_p(\phi_i \circ g_i \circ \psi_i, \phi_i \circ h_i \circ \psi_i)| \geq \eta/2 \).

We first deal with the case when

\[
d_p(\phi_i \circ g_i \circ \psi_i, \phi_i \circ h_i \circ \psi_i) \geq d_p(g_i, h_i) + \eta/2
\] (6)

for infinitely many \( i \). By definition of \( d_{p_i} \), there is a sequence \( \rho_i \) with

\[
d_{p_i}(g_i, h_i) + \eta \geq 1 + \sup_{x \in B(p, \rho_i, X_i)} d(g_i, h_i, x).
\]

Setting \( r_i := \min\{\rho_i, 12/\eta\} \), we obtain a bounded sequence such that

\[
d_{p_i}(g_i, h_i) + \eta/3 \geq 1 + \sup_{x \in B(p, r_i, X_i)} d(g_i, h_i, x).
\]

For \( i \) large enough and \( x \in B(p, r_i - 2 \varepsilon_i, X) \),

\[
d(\phi_i \circ g_i \circ \psi_i(x), \phi_i \circ h_i \circ \psi_i(x)) \leq \varepsilon_i + d(g_i(\psi_i(x)), h_i(\psi_i(x))) \leq \varepsilon_i - \frac{1}{r_i} + d_p(g_i, h_i) + \frac{\eta}{3}.
\]

Again from the definition of \( d_p \) we deduce

\[
d_p(\phi_i \circ g_i \circ \psi_i, \phi_i \circ h_i \circ \psi_i) \leq \frac{1}{r_i - 2 \varepsilon_i} + \sup_{x \in B(p, r_i - 2 \varepsilon_i, X)} d(\phi_i \circ g_i \circ \psi_i(x), \phi_i \circ h_i \circ \psi_i(x)) \leq \frac{1}{r_i - 2 \varepsilon_i} + \frac{1}{r_i} + d_p(g_i, h_i) + \frac{\eta}{3} + \varepsilon_i.
\]

From the fact that \( d_p(g_i, Id_{X_i}), d_p(h_i, Id_{X_i}) \) are bounded, we get that \( r_i \) is bounded away from 0. Then the right hand side is less than \( d_p(g_i, h_i) + \eta/2 \) for \( i \) large enough, contradicting Equation 6. The other case when

\[
d_p(g_i, h_i) \geq d_p(\phi_i \circ g_i \circ \psi_i, \phi_i \circ h_i \circ \psi_i) + \eta/2
\]

for infinitely many \( i \) is analogous. \( \Box \)
As a consequence of Theorem 21, it is easy to understand the situation when the quotients of a sequence converge to \( \mathbb{R}^n \).

**Lemma 26.** For each \( i \in \mathbb{N} \), let \((X_i, p_i)\) be a pointed \( RCD^*(\epsilon_i, N) \) space of rectifiable dimension \( n \) with \( \epsilon_i \to 0 \). Assume \((X_i, p_i)\) converges in the Gromov–Hausdorff sense to a pointed \( RCD^*(0, N) \) space \((X, p)\), there is a sequence of closed groups of isometries \( \Gamma_i \leq Iso(X_i) \) that converges equivariantly to \( \Gamma \leq Iso(X) \), and the sequence of pointed proper metric spaces \((X_i/\Gamma_i, [p_i])\) converges in the Gromov–Hausdorff sense to \((\mathbb{R}^n, 0)\). Then \( \Gamma \) is trivial.

**Proof.** One can use the submetry \( X \to X/\Gamma = \mathbb{R}^n \) to lift the lines of \( \mathbb{R}^n \) to lines in \( X \) passing through \( p \). By iterated applications of Theorem 21, we get that \( X = \mathbb{R}^n \times Y \) for some \( RCD^*(0, N - n) \) space \( Y \). But from Theorem 20, the rectifiable dimension of \( X \) is at most \( n \), so \( Y \) is a point. Since \( \Gamma \leq Iso(\mathbb{R}^n) \) satisfies \( \mathbb{R}^n/\Gamma = \mathbb{R}^n \), it must be trivial. \( \square \)

### 2.5 Group norms

Let \((X, p)\) be a pointed proper geodesic space and \( \Gamma \leq Iso(X) \) a group of isometries. The norm \( \| \cdot \|_p : \Gamma \to \mathbb{R} \) associated to \( p \) is defined as \( \|g\|_p := d(gp, p) \). The spectrum \( \sigma(\Gamma, X, p) \) is defined as the set of \( r \geq 0 \) such that

\[
\{g \in \Gamma \|g\|_p \leq r\} \neq \{g \in \Gamma \|g\|_p \leq r - \epsilon\} \quad \text{for all } \epsilon > 0.
\]

This spectrum is closely related to the covering spectrum introduced by Sormani–Wei in \([15]\), and it also satisfies a continuity property.

**Proposition 27.** Let \((X, p_i)\) be a sequence of pointed proper metric spaces that converges in the Gromov–Hausdorff sense to \((X, p)\) and consider a sequence of closed isometry groups \( \Gamma_i \leq Iso(X_i) \) that converges equivariantly to a closed group \( \Gamma \leq Iso(X) \). Then for any convergent sequence \( r_i \to \tau \), we have \( \lim_{i \to \infty} r_i \in \sigma(\Gamma, X, p) \).

**Proof.** Let \( r = \lim_{i \to \infty} r_i \). By definition, there is a sequence \( g_i \in \Gamma_i \) with \( \|g_i\|_p = r_i \), and \( g_i \notin \{\gamma \in \Gamma_i \|\gamma\|_p \leq r_i - \epsilon\} \) for all \( \epsilon > 0 \). Up to subsequence, we can assume that \( g_i \) converges to some \( g \in Iso(X) \) with \( \|g\|_p = r \).

If \( r \notin \sigma(\Gamma, X, p) \), it would mean that there are \( h_1, \ldots, h_k \in \Gamma \) with \( \|h_j\|_p < r \) for each \( j \in \{1, \ldots, k\} \), and \( h_1 \cdots h_k = g \). For each \( j \), choose sequences \( h_j^i \in \Gamma_i \) that converge to \( h_j \). As the norm is continuous with respect to convergence of isometries, for \( i \) large enough one has \( \|h_j^i\|_p < r_i \) for each \( j \).

The sequence \( g_i(h_1^i \cdots h_k^i)^{-1} \in \Gamma_i \) converges to \( g(h_1 \cdots h_k)^{-1} = e \in \Gamma \), so its norm is less than \( r_i \) for \( i \) large enough, allowing us to write \( g_i \) as a product of \( k + 1 \) elements with norm \( < r_i \), thus a contradiction. \( \square \)

**Definition 28.** Let \( G \) be a group and \( S \subset G \) a generating subset containing the identity. We say that \( S \) is a determining set if \( G \) has a presentation \( G = \langle S | R \rangle \) with \( R \) consisting of words of length 3 using as letters the elements of \( S \cup S^{-1} \).
Proposition 29. For each \( M \in \mathbb{N} \), there are only finitely many isomorphism types of groups admitting a determining set of size \( \leq M \).

The following lemma can be found in ([17], Section 2.12).

Lemma 30. Let \( D > 0 \), \((Y, q)\) a pointed proper geodesic space and \( G \leq \text{Iso}(Y) \) a closed group of isometries with \( \text{diam}(Y/G) \leq D \). If \( \{g \in G | \|g\|_q \leq 10D\} \) is not a determining set, then there is a non-trivial covering map \( \tilde{Y} \to Y \).

3 Proof of main theorems

Proof of Theorem 5. By contradiction, assume there is a sequence \((X_i, p_i)\) of pointed \( RCD^*(K, N) \) spaces, discrete groups \( \Gamma_i \leq \text{Iso}(X_i) \) of measure preserving isometries such that \( \text{vol}_{K,N}^\ast (X_i/\Gamma_i, [p_i]) \geq \nu \), and elements \( g_i \in \Gamma_i \setminus \{Id_{X_i}\} \) with \( d_{p_i}(g, Id_{X_i}) \to 0 \). After taking a subsequence, we can assume the spaces \( X_i \) have dimension \( n \) for some \( n \in \mathbb{N} \cap [0, N] \), the sequence \((X_i/\Gamma_i, [p_i])\) converges to a pointed \( RCD^*(K, N) \) space \((Y, q)\) of rectifiable dimension \( n \). By Corollary 3, we can also assume that \( q \) is \( n \)-regular.

Choose \( \eta_i \to \infty \) diverging so slowly that \((\eta_i X_i/\Gamma_i, [p_i])\) converges to \((\mathbb{R}^n, 0)\) and \( \eta_i d_{p_i}(g, Id_{X_i}) \to 0 \), and set \( Y_i := \eta_i X_i \). By Theorem 18, we can find a Reifenberg sequence \( y_i \in Y_i/\Gamma_i \) and \( \tilde{y}_i \in B(p_i, 1, Y_i) \) with \([\tilde{y}_i] = y_i\), which by Lemma 9 can be taken so that \( \|g\|_{\tilde{y}_i} \neq 0 \).

Notice that by Corollary 3, we still have \( d_{p_i}(g_i, Id_{Y_i}) \to 0 \). Hence we can find \( 1/\lambda_i \in \sigma(\Gamma_i, Y_i, \tilde{y}_i) \) with \( \lambda_i \to \infty \). As \( y_i \) is a Reifenberg sequence, \((\lambda_i Y_i/\Gamma_i, y_i)\) converges to \((\mathbb{R}^n, 0)\), so by Lemma 26, the actions of \( \Gamma_i \) on \( \lambda_i Y_i \) converge equivariantly to the trivial group. This contradicts Proposition 27, as by construction we have \( 1 \in \sigma(\Gamma_i, \lambda_i Y_i, \tilde{y}_i) \) for all \( i \).

Proof of Theorem 4: Assuming the contrary, we could find a sequence \((X_i, p_i)\) of pointed \( RCD^*(K, N) \) spaces of diameter \( \leq D \) and collapsing volume \( \geq \nu \) whose fundamental groups are pairwise non-isomorphic. After taking a subsequence, we may assume their universal covers \((\tilde{X}_i, \tilde{p}_i)\) converge to an \( RCD^*(K, N) \) space \((\tilde{X}, \tilde{p})\), and the actions of \( \pi_1(X_i) \) converge to the action of a group \( \Gamma \) in \( \tilde{X} \).

By Theorem 5, there is \( \varepsilon > 0 \) such that the elements of \( \Gamma_i \) are at pairwise \( d_{\tilde{p}_i} \)-distance \( \geq \varepsilon \). By Lemma 30, for each \( i \) the set \( S_i := \{g \in \Gamma_i | \|g\|_{\tilde{p}_i} \leq 10D\} \) is determining in \( \Gamma_i \), and by plugging \( r = 1 \) in Equation 1, we get \( S_i \subset \{g \in \Gamma_i | d_{\tilde{p}_i}(g, Id_{\tilde{X}_i}) \leq 10D + 3\} \). As \((\Gamma_i, d_{\tilde{p}_i}, Id_{\tilde{X}_i})\) converges in the Gromov–Hausdorff sense to \((\Gamma, d_{\tilde{p}}, Id_{\tilde{X}})\), Corollary 13 implies that \( |S_i| \leq M \) for some \( M \in \mathbb{N} \), so by Proposition 29 there are only finitely many isomorphism types in the sequence \( \{\pi_1(X_i)\}_{i \in \mathbb{N}} \), contradicting our initial assumption.
References


