Eisenstein Cohomology & Special Values of L-functions.

Notes of a mini-course of three lectures in Günter Harder's 85^{th} b'day Conference.

Lecture - 1:

Eisenstein Cohomology

Lecture - 2:

Automorphic L-functions.

Lecture - 3:

Motivic L-functions

§1. The General Context.

- F . number field $\begin{cases} \text{Totally real} \\ \text{Totally imaginary} \begin{cases} \text{"CM"} \\ \text{"TR"} \end{cases} \end{cases}$

("CM" : F contains a CM subfield
"TR" : F does not contain CM-subfield.)

- G_0 : (connected) reductive quasi-split group $/F$

G : $\mathrm{Res}_{F/\mathbb{Q}}(G_0)$

$$\begin{array}{ccccccc} G_0 & \supseteq & B_0 & \supseteq & T_0 & \supseteq & Z_0 \\ | & & & & & & \\ G & \supseteq & B & \supseteq & T & \supseteq & Z \end{array}$$

Borel Subgp $\supset$ Torus $\supset$ Center

<u>Examples</u>: $G_0 = GL_n$, (Harder-R, R)
$O(2n)$ (Bhagwat-R)
$GU(n,n)$ (Krishnamurthy-R)
G_2 (Hosseinijafari)

If $G_0 = GL_n$, $Z = \mathrm{Res}_{F/\mathbb{Q}}(GL_1/F) \supset GL_1/\mathbb{Q} = S$

S = maximal $\mathbb{Q}$-split central torus of G.

- $G(\mathbb{R})$ = group of $\mathbb{R}$-points.

  (e.g. F - tot. real, $G_0 = GL_n$, $G(\mathbb{R}) = \prod_{\eta: F \to \mathbb{C}} GL_n(\mathbb{R})$.)

  C_∞ = maximal compact subgroup of $G(\mathbb{R})$.

  (e.g: $C_\infty = \prod_{\eta: F \to \mathbb{C}} O(n)$.)

  $K_\infty = C_\infty \cdot S(\mathbb{R})$

  K_∞° = connected component of the identity elt.

  $K_f \subseteq G(\mathbb{A}_f)$ open-compact subgroup

- $S^G_{K_f} = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty^\circ K_f$

  "locally symmetric space".

- $$\begin{tikzcd} & E \\ F \arrow[ur, hook] \arrow[d, dash] & \\ \mathbb{Q} \arrow[uur, dash, bend right] & \end{tikzcd}$$

  finite Galois extn. E = field of coefficients.

- Weights:

  $\lambda \in X^*(T \times E) = \mathrm{Hom}(\mathrm{Res}_{F/\mathbb{Q}}(T_0/F), \mathbb{G}_m)$

  $$X^*(T \times E) = \bigoplus_{\tau: F \to E} X^*(T_0 \times_{F,\tau} E)$$

  $$\lambda = (\lambda^\tau)_{\tau: F \to E}$$

  $X^+(T \times E)$ = dominant-integral weights.

  (e.g., $G_0 = GL_n$, $\lambda = (\lambda^\tau)$, $\lambda^\tau = (\lambda^\tau_1, \dots, \lambda^\tau_n)$

  $\lambda^\tau_j \in \mathbb{Z}$, $\lambda^\tau_1 \geq \lambda^\tau_2 \geq \dots \geq \lambda^\tau_n$.

• $(\rho_\lambda, M_{\lambda,E})$ = finite-dimensional, absolutely-irreducible representation of $G \times E$ with h.w. λ.

(e.g.: $G_0 = GL_2$, $\lambda = (\lambda^z)$, $\lambda^z = (\lambda^z_1, \lambda^z_2)$, $\lambda^z_1 \geq \lambda^z_2$

$$M_{\lambda,E} = \bigotimes_{z : F \to E} \mathrm{Sym}^{\lambda^z_1 - \lambda^z_2}(E^2) \otimes (\det)^{\lambda^z_2}.$$

(e.g: Hilbert modular forms of weight $(k_1, \dots, k_d) = (k^z)$

Assume k^z - even;

$$\lambda^z = (k^z - 2) \cdot \rho_{GL_2} = \left(\frac{k^z-2}{2}, -\frac{(k^z-2)}{2} \right). \quad)$$

• $\widetilde{M}_{\lambda,E}$ = sheaf of E-vector spaces on $S^G_{K_f}$.

$$\begin{tikzcd}
G(\mathbb{A})/K^\circ_\infty K_f \arrow[d, "\pi"] & \pi^{-1}(U) \arrow[d, dashed] \\
G(\mathbb{Q})\backslash G(\mathbb{A})/K^\circ_\infty K_f & U \arrow[l, phantom, "\underset{\text{open}}{\supseteq}"]
\end{tikzcd}$$

$$\widetilde{M}_{\lambda,E}(U) = \Big\{ s : \pi^{-1}(U) \longrightarrow M_{\lambda,E} \;\Big|\; s\text{-locally const.}, \; s(\gamma \cdot \underline{x}) = \rho_\lambda(\gamma) \cdot s(\underline{x}) \;\; \forall \gamma \in G(\mathbb{Q}), \; \forall \underline{x} \in G(\mathbb{A})/K^0_\infty K_f \Big\}$$

- Even if $\mathcal{M}_{\lambda,E} \neq 0$, the sheaf $\widetilde{\mathcal{M}}_{\lambda,E} = 0$.

e.g. $G_0 = GL_n/_F$, F-tot. real. for $\widetilde{\mathcal{M}}_{\lambda,E} \neq 0$ we need

$$\rho_\lambda\big|_{S(\mathbb{Q}) \cap K_\infty^\circ K_f} = \mathbb{1}$$

$\Rightarrow$ ω_{ρ_λ} is the infinity-type of an algebraic Hecke character.

$\Rightarrow$ $\lambda_1^2 + \dots + \lambda_n^2$ is independent of τ.

Assume henceforth that $\widetilde{\mathcal{M}}_{\lambda,E} \neq 0$.

- **Basic object of interest:**

$$H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})$$

  - Sheaf cohomology.
  - It is an E-vector space.

- **Hecke action:**

$$\mathcal{H}^G_{K_f} = \{ f : G(\mathbb{A}_f)//K_f \longrightarrow \mathbb{Q} \}$$

it is an algebra under convolution for a $\mathbb{Q}$-valued measure

$$\pi_0(G(\mathbb{R})) \times \mathcal{H}^G_{K_f} \circlearrowright H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})$$

§2 A long exact sequence:

- Borel-Serre compactification:

$$\overline{S}^G_{K_f} = S^G_{K_f} \cup \partial S^G_{K_f}$$

The Boundary $\partial S^G_{K_f} = \cup \, \partial_P S^G_{K_f}$ as P runs over $G(\mathbb{Q})$-conjugacy classes of parabolic subgroups$/\mathbb{Q}$ of G.

The sheaf construction extends to ∂_P, ∂ & $\overline{S}^G_{K_f}$.

- The topological pair $(\overline{S}^G_{K_f}, \partial S^G_{K_f})$ induces a long exact sequence:

$$\cdots \to H^\bullet_c(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) \xrightarrow{i^\bullet} H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) \xrightarrow{r^\bullet} H^\bullet(\partial S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) \to \cdots$$

- Inner / Interior Cohomology:

$$H^\bullet_!(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) = \mathrm{Image}\,(i^\bullet)$$

- Eisenstein Cohomology:

$$H^\bullet_{\mathrm{Eis}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) = \mathrm{Image}\,(r^\bullet)$$

§3 Boundary Cohomology

- Spectral sequence $\Longrightarrow$ Bdry. Cohomology

$$E_1^{p,q} = \bigoplus_{d(P)=p} H^q(\partial_P S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})$$

$d(P) =$ parabolic rank $\quad (d(\text{max. parabolic}) = 1)$

- Cohomology of a single boundary stratum $\partial_P S^G_{K_f}$.

(i) $H^\bullet(\partial_P S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) = H^\bullet(P(\mathbb{Q})\backslash G(\mathbb{A})/K^\circ_\infty K_f, \widetilde{\mathcal{M}}_{\lambda,E})$

(ii)

$$\begin{array}{c} U_P(\mathbb{Q})\backslash U_P(\mathbb{A})/K_f^{U_P} \\ \big\downarrow \\ P(\mathbb{Q})\backslash P(\mathbb{A})/K^P_\infty K^P_f \\ \big\downarrow \\ M_P(\mathbb{Q})\backslash M_P(\mathbb{A})/K^{M_P}_\infty \cdot K^{M_P}_f \end{array}$$

$\longrightarrow$ builds up $P(\mathbb{Q})\backslash G(\mathbb{A})/K^\circ_\infty K_f$

(iii) van-Est theorem

$$H^\bullet(U_P(\mathbb{Q})\backslash U_P(\mathbb{A})/K^{U_P}, \widetilde{\mathcal{M}}_{\lambda,E}) = H^\bullet(\mathfrak{u}_P, \mathcal{M}_{\lambda,E})$$

(iv) Kostant's theorem:

$$H^\bullet(\mathfrak{u}_P, \mathcal{M}_{\lambda,E}) = \bigoplus_{\substack{w \in W^P \\ l(w) = \bullet}} \mathcal{M}_{w\cdot\lambda,E}$$

as M_P-modules

(It is also convenient to pass to the limit over all open-cpct K_f

$$G(\mathbb{A}_f) \circlearrowright H^\bullet(S^G, \widetilde{\mathcal{M}}_{\lambda,E}) := \varinjlim_{K_f} H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})$$

then $H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) = H^\bullet(S^G, \widetilde{\mathcal{M}}_{\lambda,E})^{K_f}$.)

$$\boxed{\begin{aligned} & H^{q}(\partial_P S^G, \widetilde{\mathcal{M}}_{\lambda,E}) \\ &= \bigoplus_{w \in W^P} {}^a\mathrm{Ind}_{\pi_0(P(\mathbb{R}))\times P(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)} \left(H^{q-\ell(w)}(S^{M_P}, \widetilde{\mathcal{M}}_{w\cdot\lambda,E})^{\pi_0(K_\infty^{M_P})} \right) \end{aligned}}$$

- $W^P = \{ w \in W_G \mid w^{-1}\alpha > 0 \ \forall \text{ simple roots } \alpha \text{ for } M_P \}$

$$0 \to \pi_0(K_\infty^{M_P}) \to \pi_0(P(\mathbb{R})) \to \pi_0(G(\mathbb{R})) \to 0$$

In summary: the cohomology of $\partial_P S^G_{K_f}$ is parabolically induced from the cohomology of $S^{M_P}_{K_f^{M_P}}$.

§4 Inner Cohomology$/\mathbb{C}$ $\supseteq$ Cuspidal Cohomology$/\mathbb{C}$ ——— .

- $$\begin{tikzcd} & & \mathbb{C} \\ & E \arrow[ur, hook, "\iota"] & \\ F \arrow[ur, hook, "\tau"] \arrow[d, dash] & & \\ \mathbb{Q} \arrow[uur, dash, bend right] \arrow[uuurr, dash, bend right] & & \end{tikzcd}$$

$$\iota_*: \ \mathrm{Hom}(F, E) \xrightarrow{\sim} \mathrm{Hom}(F, \mathbb{C}). \qquad \iota_*(\tau) = \iota\circ\tau = \eta.$$

$$\iota_*: \ X^*(T \times E) \longrightarrow X^*(T\times\mathbb{C}) \qquad \tau = \iota^{-1}\circ\eta$$

$$\lambda \longmapsto {}^{\iota}\lambda$$

$$\lambda = (\lambda^{\tau})_{\tau: F\to E} \qquad {}^{\iota}\lambda = ({}^{\iota}\lambda^{\eta})_{\eta: F\to\mathbb{C}}$$

$${}^{\iota}\lambda^{\eta} = \lambda^{\iota^{-1}\circ\eta}$$

- $H^\bullet(S^G_{K_f}, \mathcal{M}_{{}^\iota\lambda,\mathbb{C}}) \cong H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ;\ \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \otimes \mathcal{M}_{{}^\iota\lambda,\mathbb{C}})$

Sheaf cohomology $\qquad\qquad$ Relative Lie algebra cohomology

Use. de Rham resolution

- Inner cohomology $H^\bullet_!(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})$ is semisimple as a Hecke module because after base-change to $\mathbb{C}$, it lands inside the $(\mathfrak{g}_\infty, K_\infty^\circ)$-cohomology of the discrete spectrum of G.

- Cuspidal cohomology is defined by:

$$H^\bullet(S^G_{K_f}, \mathcal{M}_{{}^\iota\lambda,\mathbb{C}}) \cong H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \otimes \mathcal{M}_{{}^\iota\lambda,\mathbb{C}})$$

$$\uparrow \qquad\qquad \uparrow \text{ Borel.}$$

$$H^\bullet_{cusp}(S^G_{K_f}, \mathcal{M}_{{}^\iota\lambda,\mathbb{C}}) :\cong H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; \mathcal{C}^\infty_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \otimes \mathcal{M}_{{}^\iota\lambda,\mathbb{C}})$$

- $H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; \mathcal{C}^\infty_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \otimes \mathcal{M}_{{}^\iota\lambda,\mathbb{C}})$

$$= \bigoplus_\sigma m(\sigma) \cdot H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; \sigma_\infty \otimes \mathcal{M}_{{}^\iota\lambda,\mathbb{C}}) \otimes \sigma_f^{K_f}.$$

- For $G = GL_n$, $m(\sigma)=1$, σ_∞ - is "unique".

  Use (i) Wigner's Lemma

  (ii) Harish-Chandra

  (iii) Unique tempered/generic subquotient.

  Shape of $\sigma_\infty := \bigotimes_{v\in S_\infty} \sigma_v$, and σ_v looks like:

- n-even: $\mathrm{Ind}_{P_{(2,\dots,2)}}^{GL_n(\mathbb{R})}(D_{\ell_1} \otimes |\cdot|^{-d} \times \dots \times D_{\ell_{n/2}} \otimes |\cdot|^{-d})$

- n-odd: $\mathrm{Ind}_{P_{(2,\dots,2,1)}}^{GL_n(\mathbb{R})}(D_{\ell_1} \otimes |\cdot|^{-d} \times \dots \times D_{\ell_{n/2}} \otimes |\cdot|^{-d} \times (\mathrm{sgn})^\varepsilon |\cdot|^{-d})$

- $\mathrm{Ind}_{B_n(\mathbb{C})}^{GL_n(\mathbb{C})}(z^{\alpha_1}\bar{z}^{\beta_1} \times \dots \times z^{\alpha_n}\bar{z}^{\beta_n})$

${}^\iota\lambda$ explicitly determines all the "cuspidal parameters"

• **Purity & Strong-Purity:** $G_0 = GL_n$

Fact ① (Clozel) $H^\bullet_{cusp}(S^G_{K_f}, \widetilde{M}_{\iota\lambda,\mathbb{C}}) \neq 0 \implies {}^\iota\lambda$ satisfies a purity condition.

${}^\iota\lambda \in X^*(\mathrm{Res}_{F/\mathbb{Q}}(T_n) \times \mathbb{C})$, ${}^\iota\lambda = ({}^\iota\lambda^\eta)_{\eta : F \to \mathbb{C}}$, is pure if

F - totally real:

$$\exists\, w \in \mathbb{Z} \text{ s.t. } \lambda_j^{\iota^{-1}\circ\eta} + \lambda_{n-j+1}^{\iota^{-1}\circ\eta} = w$$

F - totally imaginary

$$\exists\, w \in \mathbb{Z} \text{ s.t. } \lambda_j^{\iota^{-1}\circ\eta} + \lambda_{n-j+1}^{\iota^{-1}\circ\bar\eta} = w \quad \forall\, 1 \le j \le n,\ \forall\, \eta : F \to \mathbb{C}$$

Fact ② cuspidal cohomology admits a rational structure.
(Clozel)

$$\begin{tikzcd} & E \arrow[r, "\iota"] & \overline{\mathbb{Q}} \\ F \arrow[ur, "\iota"] \arrow[d, dash] & & \\ \mathbb{Q} \arrow[uur, dash, bend right] \arrow[uurr, dash, bend right, "\gamma"'] & & \end{tikzcd}$$

$\gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

①+② $\implies$

$$H^\bullet_{cusp}(S^G_{K_f}, \widetilde{M}_{\iota\lambda,\mathbb{C}}) \neq 0 \implies \lambda \text{ is strongly-pure}$$

i.e., ${}^{\gamma\circ\iota}\lambda$ is pure $\forall\, \iota : E \to \overline{\mathbb{Q}}$, $\forall\, \gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

If F is totally imaginary:

$$\exists\, w \in \mathbb{Z} \text{ s.t. } \lambda_j^{\gamma^{-1}\circ\iota^{-1}\circ\eta} + \lambda_{n-j+1}^{\gamma^{-1}\circ\iota^{-1}\circ\eta} = w$$

$\forall\, 1 \le j \le n$, $\forall\, \eta : F \to \mathbb{C}$, $\forall\, \iota : E \to \overline{\mathbb{Q}} \subset \mathbb{C}$, $\forall\, \gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

§§ Inner-structure of F & strong-purity:

F - tot. imaginary field, $G_0 = GL_n/F$.

F_0 = maximal totally real subfield of F.

∃ at most one totally imaginary quadratic extⁿ of F_0 contained inside F.

$$\begin{array}{c} F \\ | \\ F_1 \\ |\ \le 2 \\ F_0 \\ | \\ \mathbb{Q} \end{array}$$

CM-Case: F_1 = tot. imag. quad. extⁿ. of F_0 inside F.
= maximal CM-subfield of F.

TR-case: $F_1 := F_0$ = maximal tot. real subfield of F.

Lemma:

$$\boxed{\begin{array}{l} \lambda \in X^*(\mathrm{Res}_{F/\mathbb{Q}}(T_n/F)\times E) \text{ is strongly-pure, then} \\ \exists\ \lambda_1 \in X^*(\mathrm{Res}_{F_1/\mathbb{Q}}(T_n/F_1)\times E) \text{ strongly-pure s.t} \\ \lambda = BC_{F/F_1}(\lambda_1), \text{ i.e., } \lambda^{\tau} = \lambda_1^{\tau|_{F_1}} \end{array}}$$

$X^*(\mathrm{Res}_{F/\mathbb{Q}}(T_n/F)\times E)$ - weights

$\cup$

$X^*_+(\mathrm{Res}_{F/\mathbb{Q}}(T_n/F)\times E)$ - dominant-integral

$\cup$

$X^*_0(\mathrm{Res}_{F/\mathbb{Q}}(T_n/F)\times E)$ - pure weights

$\cup$

$X^*_{00}(\mathrm{Res}_{F/\mathbb{Q}}(T_n/F)\times E)$ - strongly-pure.

(Ex: F = tot. real or CM   Pure = Strongly-pure.)

§ Strongly-inner cohomology:

Theorem

F - totally real, $G_0 = GL_n$

$\lambda \in X_0^+(T \times E)$.

$\exists$ Hecke-stable subspace

$$H^\bullet_{!!}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) \subset H^\bullet_!(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})$$

such that $\forall\ \iota : E \to \mathbb{C}$,

$$H^\bullet_{!!}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) \otimes_{E,\iota} \mathbb{C} = H^\bullet_{cusp}(S^G_{K_f}, \widetilde{\mathcal{M}}_{{}^\iota\lambda,\mathbb{C}})$$

By semisimplicity:

$$H^\bullet_!(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) = \bigoplus_{\pi_f \in \mathrm{Coh}_!(G, K_f, \lambda)} H^\bullet_!(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})(\pi_f)$$

Define:

$$\mathrm{Coh}_{!!}(G, K_f, \lambda) = \{ \pi_f \in \mathrm{Coh}_! \mid H^\bullet_!(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})(\pi_f) = H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})(\pi_f) \}$$

Define:

$$H^\bullet_{!!}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) = \bigoplus_{\pi_f \in \mathrm{Coh}_{!!}(G, K_f, \lambda)} H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})(\pi_f)$$

Lecture #2

Special Values of Automorphic L-functions.

§ 2.1 Motivating examples:

- **Theorem** (Manin, Shimura - 1977)

$\varphi \in S_k(\Gamma_0(N), \omega)_{Prim.}$ Primitive — Eigenform, Newform, $a_1 = 1$.

$\mathbb{Q}(\varphi) = \mathbb{Q}(\{a_n(\varphi)\})$ - # field.

Then $\exists\ u^{\pm}(\varphi) \in \mathbb{C}^*$ such that

$\forall\ 1 \leq m \leq k-1$, $\forall$ Dirichlet characters χ

$$L_f(m, \varphi, \chi) \approx_{\mathbb{Q}(f,\chi)} (2\pi i)^m u^{\pm}(\varphi)\, g(\chi)$$

$\chi(-1) = \pm(-1)^m$

$$\varphi(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}, \qquad L_f(s, \varphi, \chi) = \sum_{n=1}^{\infty} \frac{a_n(\varphi)\chi(n)}{n^s}.$$

Suppose $1 \leq m < m+1 \leq k-1$ $(k \geq 3)$

$$\frac{L_f(m, \varphi, \chi)}{L_f(m+1, \varphi, \chi)} \approx \frac{1}{2\pi i} \cdot \frac{u^{\pm}(\varphi)}{u^{\mp}(\varphi)}.$$

$L(s, \varphi, \chi)$ = completed L-function. $(2\pi)^{-s}\Gamma(s) \cdot L_f(s, \varphi, \chi)$

$$\frac{1}{i}\frac{u^+(\varphi)}{u^-(\varphi)} = \omega(\varphi) \quad \text{— "relative period"}$$

Then

$$\boxed{\frac{L(m, \varphi, \chi)}{L(m+1, \varphi, \chi)} \approx \omega(\varphi)^{\varepsilon_m \varepsilon_\chi}}$$

Theorem (Shimura)

Let $f \in S_k(\Gamma_0(N), \chi)$, $g \in S_\ell(\Gamma_0(N), \psi)$

$$D(s, f, g) = \sum \frac{a_n(f)\, a_n(g)}{n^s}$$

$$L_f(s, f \times g) = L_N(2s+2-k-\ell, \chi\psi) \cdot D(s, f, g)$$

(finite part of the degree 4 Rankin-Selberg L-function.)

Assume $\ell < k$. Suppose $\ell \leq m \leq k-1$. Then

- $L_f(m, f \times g) \underset{\mathbb{Q}(f,g)}{\approx} (2\pi i)^{2m+1-\ell} \cdot \mathfrak{g}(\psi) \cdot u_f^+ \cdot u_f^-$
- $u_f^+ u_f^- \approx i^{1-k} \cdot \pi \cdot \mathfrak{g}(\chi) \cdot \langle f, f \rangle$
- $L_f(m, f \times g) \approx (2\pi)^{2m+1-\ell} \cdot i^{k+\ell} \cdot \mathfrak{g}(\chi)\, \mathfrak{g}(\psi) \cdot \langle f, f \rangle$

Now suppose $\ell \leq m < m+1 \leq k-1$

Then:

$$\frac{L_f(m, f \times g)}{L_f(m+1, f \times g)} \approx \frac{1}{(2\pi)^2}$$

Suppose $L(s, f \times g)$ = completed L-function then

$$\boxed{L(\ell, f \times g) \approx L(\ell+1, f \times g) \approx \cdots \approx L(k-1, f \times g).}$$

Principle:

Eisenstein Cohomology allows one to prove rationality results for ratios of critical values of automorphic L-functions.

"Cohomology of arithmetic groups and L-values have a symbiotic relationship.".

§2.2 Automorphic L-functions.

G connected reductive group/$\mathbb{Q}$.

P maximal parabolic subgroup.

$P = M_P \cdot U_P$.

π - cuspidal automorphic repn. of $M_P(\mathbb{A})$.

${}^LG^\circ$ = complex reductive group - 'Langlands dual'.

${}^LP$ = parabolic subgroup of ${}^LG^\circ$ corresponding P.

${}^LP^\circ = {}^LM_P^\circ \cdot {}^LN_P$

${}^L\mathfrak{n}_P = \mathrm{Lie}({}^LN_P)$. $\circlearrowleft$ Adjoint repn of ${}^LM_P^\circ$.

${}^L\mathfrak{n}_P = r_1 \oplus \cdots \oplus r_m$ multiplicity free direct sum of repns

$L^S(s, \pi, r_i)$ = Langlands L-function attached to σ & r_i $1 \leq i \leq m$.

$$L^S(s,\pi,r_i) = \prod_{v \notin S} L(s, \pi_v, r_i)$$

v - unramified $\leftrightsquigarrow$ ϑ_v - Satake parameters.

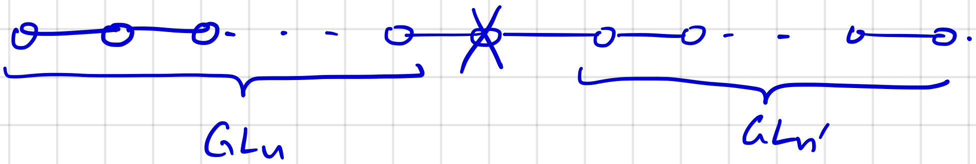
$\vartheta_v \in {}^LT^\circ \subset {}^LM_P^\circ \subset {}^LG^\circ$

$$L_v(s,\pi_v, r_i) = \det(1 - r_i(\vartheta_v)\cdot q_v^{-s})^{-1}.$$

(Note: $L^S(s,\pi,r)$ is attached to a
- a cuspidal repn of a reductive grp M
- an algebraic f.d. repn of ${}^LM^\circ$.)

Examples of automorphic L-functions

(i) $G = GL_N$



$G = GL_N/\mathbb{Q}$

$P = P_{(n,n')}/\mathbb{Q}.$ $\qquad M_P = GL_n \times GL_{n'}.$

$\pi = \sigma \otimes \sigma'$ $\qquad \sigma$ - cusp. rep^n of GL_n

σ' - " " " $GL_{n'}$.

v - unramified

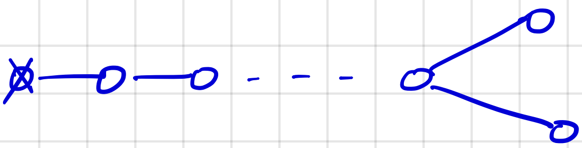
$$\mathcal{V}_v = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{bmatrix} \qquad \mathcal{V}_v' = \begin{bmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_{n'} \end{bmatrix}$$

$\alpha_i, \beta_j \in \mathbb{C}.$

$$L(s, \sigma_v \otimes \sigma_v', r) = \prod_{i,j} (1 - \alpha_i \beta_j q_v^{-s})^{-1}$$

$L^S(s, \sigma \times \sigma', r)$ = Rankin-Selberg L-fn.

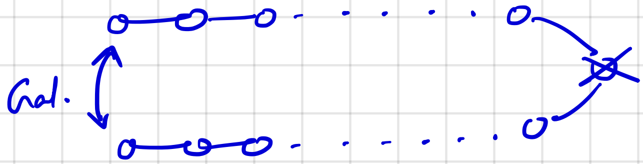
(ii) G = even orthogonal group.



$M_P = O(2n) \times GL(1) \subset G = O(2n+2)$

degree-$2n$ standard L-function of $O(2n)$ twisted by characters (of $GL(1)$).

(iii) $G = GU(n,n)$., E/F - quadratic extension.
n- even



$$M_P = \mathrm{Res}_{E/F}(GL_n/E) \times GL_1/F.$$

$\pi = \sigma \otimes \chi$ $\quad$ σ - cuspidal aut. rep^n of $GL_n(\mathbb{A}_E)$

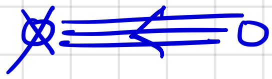
χ - Hecke character of $GL_1(\mathbb{A}_F)$.

$$L(s,\pi,r) = L(s, \sigma\otimes\chi, As\otimes P_1)$$

$$= L(s, As(\sigma)\otimes\chi)$$

Twisted Asai L-function (degree n^2 L-fn. over F.)

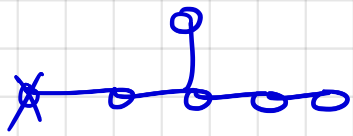
(iv) $G = G_2$



$M_P = GL_2$, $\quad L(s,\sigma,r_1) = L(s,\sigma,\mathrm{Sym}^3)$

$L(s,\sigma,r_2) = L(s,\omega_\sigma)$.

(v) $G = E_6$


,

$M_P \approx \mathrm{Spin}(10)$

$L(s,\sigma,r)$ = degree-16 attached to the

½- spin irreducible rep^n of $PSO(10,\mathbb{C})$

§2.3 Rankin-Selberg L-functions

$$\begin{tikzcd}
 & & \mathbb{C} \\
 & E \arrow[hook, ur, "\iota"] & \\
F \arrow[hook, ur] \arrow[d, dash] & & \\
\mathbb{Q} \arrow[uur, dash] \arrow[uuurr, dash] & &
\end{tikzcd}$$

F - totally imaginary number field.

$G_n = \mathrm{Res}_{F/\mathbb{Q}}(GL_n/F)$.

$\mu \in X^*_{00}(\mathrm{Res}_{F/\mathbb{Q}}(T_n/F) \times E)$

$\sigma_f \in \mathrm{Coh}_{!!}(GL_n/F, \mu/E)$ - strongly inner spectrum.

$\iota : E \longrightarrow \mathbb{C}$

${}^\iota\sigma_f = \sigma_f \otimes_{E,\iota} \mathbb{C}$ is the finite part of a cuspidal automorphic representation ${}^\iota\sigma$ of $G_n(\mathbb{A}_\mathbb{Q}) = GL_n(\mathbb{A}_F)$.

$G_{n'} = \mathrm{Res}_{F/\mathbb{Q}}(GL_{n'}/F)$

$\mu' \in X^*_{00}(\mathrm{Res}_{F/\mathbb{Q}}(T_{n'}/F) \times E)$

$\sigma_f' \in \mathrm{Coh}_{!!}(GL_{n'}/F, \mu'/E)$.

${}^\iota\sigma'$ - cusp. aut. repn. of $G_{n'}(\mathbb{A}_\mathbb{Q}) = GL_{n'}(\mathbb{A}_F)$.

$L(s, {}^\iota\sigma \times {}^\iota\sigma') =$ Rankin-Selberg L-function.

Goal: Critical Values of $L(s, {}^\iota\sigma \times {}^\iota\sigma')$.

Defn:

A half-integer $m \in \frac{N}{2} + \mathbb{Z}$ is critical for $L(s, {}^{?}\sigma \times {}^{?}\sigma')$ if $L_\infty(s, {}^{?}\sigma \times {}^{?}\sigma')$ and $L_\infty(1-s, {}^{?}\sigma^\vee \times {}^{?}\sigma')$ are finite at $s = m$.

Defn:

(i) abelian width b/w μ & μ' : $a(\mu, \mu') = \frac{w - w'}{2}$

(ii) cuspidal parameters at $v \in S_\infty$

$$\alpha^v = -w_0 {}^{?}\mu^{\eta_v} + \rho_n \quad , \quad \beta^v = -{}^{?}\mu^{\bar{\eta}_v} - \rho_n$$

$${}^{?}\sigma_v = \mathrm{Ind}_{B_n(\mathbb{C})}^{GL_n(\mathbb{C})} \left(z^{\alpha_1^v} \bar{z}^{\beta_1^v} \otimes \cdots \otimes z^{\alpha_n^v} \cdot \bar{z}^{\beta_n^v} \right)$$

(iii) cuspidal width b/w μ & μ'

$$\ell(\mu, \mu') = \min \left\{ \left| \alpha_i^v - \alpha_j'^v - \beta_i^v - \beta_j'^v \right| : \begin{array}{l} v \in S_\infty \\ 1 \le i \le n \\ 1 \le j \le n' \end{array} \right\}$$

Proposition:

$$\boxed{\mathrm{Critical}\left(L(s, {}^{?}\sigma \times {}^{?}\sigma'^\vee)\right) = \left\{ m \in \frac{N}{2} + \mathbb{Z} \;\middle|\; 1 - \frac{\ell(\mu, \mu')}{2} + a(\mu, \mu') \le m \le \frac{\ell(\mu, \mu')}{2} + a(\mu, \mu') \right\}}$$

- # Critical set $= \ell(\mu, \mu')$.
- Critical set is centered at $\frac{1}{2} + a(\mu, \mu')$.

THEOREM: (R)

$\sigma_f \in \mathrm{Coh}_{!!}(GL_n/F, \mu/E)$, $\sigma_f' \in \mathrm{Coh}_{!!}(GL_{n'}/F, \mu'/E)$

Assume: $\ell(\mu,\mu') \geq 2$. Suppose $m, m+1 \in \mathrm{Crit}(L(s, {}^\iota\sigma \times {}^\iota\sigma'^\vee))$

(i) If $L(m+1, {}^\iota\sigma \times {}^\iota\sigma'^\vee) = 0$ for some ι then it vanishes $\forall \iota$

(ii) CM-case: F contains a CM-subfield ; $\delta_{F/\mathbb{Q}}$ = abs. discriminant of F.

$$\boxed{|\delta_{F/\mathbb{Q}}|^{nn'/2} \cdot \frac{L(m, {}^\iota\sigma \times {}^\iota\sigma'^\vee)}{L(m+1, {}^\iota\sigma \times {}^\iota\sigma'^\vee)} \in \iota(E) \subset \bar{\mathbb{Q}}}$$

and $\forall \gamma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$$\gamma\left(|\delta_{F/\mathbb{Q}}|^{nn'/2} \cdot \frac{L(m, {}^\iota\sigma \times {}^\iota\sigma'^\vee)}{L(m+1, {}^\iota\sigma \times {}^\iota\sigma'^\vee)}\right) =$$

$$\varepsilon(\mu,\mu',\iota,\gamma) \cdot \tilde{\varepsilon}(\mu,\mu',\iota,\gamma) \cdot |\delta_{F/\mathbb{Q}}|^{\frac{nn'}{2}} \cdot \frac{L(m, {}^{\gamma\circ\iota}\sigma \times {}^{\gamma\circ\iota}\sigma'^\vee)}{L(m+1, {}^{\gamma\circ\iota}\sigma \times {}^{\gamma\circ\iota}\sigma'^\vee)} ;$$

$\varepsilon(\ldots), \tilde{\varepsilon}(\ldots) \in \{\pm 1\}$.

(iii) If F does not contain a CM subfield, then nn' is even and

$$\boxed{\frac{L(m, {}^\iota\sigma \times {}^\iota\sigma'^\vee)}{L(m+1, {}^\iota\sigma \times {}^\iota\sigma'^\vee)} \in \iota(E) \subset \bar{\mathbb{Q}}}$$

and $\forall \gamma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$$\gamma\left(\frac{L(m, {}^\iota\sigma \times {}^\iota\sigma'^\vee)}{L(m+1, {}^\iota\sigma \times {}^\iota\sigma'^\vee)}\right) = \frac{L(m, {}^{\gamma\circ\iota}\sigma \times {}^{\gamma\circ\iota}\sigma'^\vee)}{L(m+1, {}^{\gamma\circ\iota}\sigma \times {}^{\gamma\circ\iota}\sigma'^\vee)} .$$

- The signatures $\varepsilon(\mu,\mu',\iota,\gamma)$ and $\tilde{\varepsilon}(\mu,\mu',\iota,\gamma)$ are complicated.

  $\varepsilon(\ldots) \cdot \tilde{\varepsilon}(\ldots) = 1$ if F itself is a CM-field.

§§ "Some" literature related to this theorem:

- Moeglin
- $n = n' = 1$, Blasius, Harder
- $n = 2$, $n' \leq 2$, Hida
- Unitary groups: Michael Harris.
- $n' = n-1$: $GL(n) \times GL(n-1)$;

  Kazhdan-Mazur-Schmidt, Mahnkopf

  R, Grobner-Harris, Januszewski
- $n' = n$: Grenie
- Grobner-Harris-Lin-Sachdeva
- $GL_n \times GL_{n'} / F$-totally real: Harder-R

Langlands' Constant Term Theorem:

$G = GL_N$

$P = P_{(n,n')}$

$M_P = GL_n \times GL_{n'}$

$Q = P_{(n',n)}$

$$\begin{tikzcd}
 & \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \arrow[dr, "\mathcal{F}_Q"] & \\
\mathcal{C}^\infty(P(\mathbb{Q})\backslash G(\mathbb{A})) \arrow[ur, "\mathrm{Eis}_P"] & & \mathcal{C}^\infty(Q(\mathbb{Q})\backslash G(\mathbb{A})) \\
\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(s, \sigma\otimes\sigma') \arrow[u, hook] \arrow[rr, "T_{st}(s)"] & & \mathrm{Ind}_{Q(\mathbb{A})}^{G(\mathbb{A})}(-s, \sigma'\otimes\sigma) \arrow[u, hook]
\end{tikzcd}$$

$$\mathrm{Eis}_P(f)(\underline{x}) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma \underline{x})$$

$$\mathcal{F}_Q(f)(\underline{x}) = \int_{U_Q(\mathbb{Q})\backslash U_Q(\mathbb{A})} f(\underline{u}\,\underline{x})\, d\underline{u}$$

$$T_{st}(s)(f)(\underline{x}) = \int_{U_Q(\mathbb{A})} f(w_0^{-1}\,\underline{u}\,\underline{x})\, d\underline{u}, \quad T_{st}(s) = \bigotimes_v T_{st,v}(s)$$

$$\forall\, v \notin S, \quad T_{st,v}(s)(f_v^\circ) = \frac{L(s, {}^{?}\sigma_v \times {}^{?}\sigma_v'^{\vee})}{L(1+s, {}^{?}\sigma_v \times {}^{?}\sigma_v'^{\vee})}\, \tilde{f}_v^\circ$$

Point of evaluation: $s = -N/2$

$$a\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma\times\sigma') = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(s, \sigma\times\sigma')\Big|_{s=-\frac{N}{2}}$$

$$
\begin{tikzcd}
H^{b_N}(S^{GL_N}, \widetilde{\mathcal{M}}_{\lambda,E}) \arrow[d] \arrow[dd, bend right=60, "4"'] \\
H^{b_N}(\partial S^{GL_N}, \widetilde{\mathcal{M}}_{\lambda,E}) \arrow[d, "2"] \\
I_P \oplus I_Q
\end{tikzcd}
$$

$$
\begin{tikzcd}
H^{b_N}(\partial_P S^{GL_N}, \widetilde{\mathcal{M}}_{\lambda,E}) \quad 1 & & H^{b_N}(\partial_Q S^{GL_N}, \widetilde{\mathcal{M}}_{\lambda,E}) \\
I_P = {}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\sigma_f \times \sigma_f') \arrow[u, hook] \arrow[rr, "T_{st} \; 3"] & & {}^a\mathrm{Ind}_{Q(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\sigma_f'(n) \times \sigma_f(-n')) \approx I_Q \arrow[u, hook]
\end{tikzcd}
$$

1 **Combinatorial lemma:** $\exists\, w \in W^P$ s.t. $\lambda = w^{-1} \cdot (\mu \times \mu')$ is dominant & $\ell(w) = \frac{1}{2}\dim(U_P)$.

- $b_N = b_n + b_{n'} + \frac{1}{2}\dim(U_P)$.

2 **Manin-Drinfeld principle:** $I_P \oplus I_Q$ splits off as an isotypical summand from $H^\bullet(\partial S^{GL_N}, \widetilde{\mathcal{M}}_{\lambda,E})$.

3 **Standard intertwining operator** -

- Langlands: $T_{st,v}(f_v^\circ) = \dfrac{L_v(-N/2, {}^\vee\sigma \times {}^\vee\sigma'^\vee)}{L_v(1-N/2, {}^\vee\sigma \times {}^\vee\sigma'^\vee)} \cdot \tilde{f}_v^\circ$
- Local subproblems for $v \in S_\infty$ and $v \in S_{\mathrm{ram}}$

4 **Main Technical Theorem:**

$$\mathrm{Image}\left(H^\bullet(S^{GL_N}, \mathcal{M}_{\lambda,E}) \longrightarrow I_P \oplus I_Q\right)$$
$$= \{(\xi, T_{Eis}(\xi)) : \xi \in I_P\}.$$

• ① Combinatorial Lemma:

The following three statements are equivalent:

① $s = -N/2$ and $1 - \frac{N}{2}$ are critical for $L(s, {}^{\gamma}\sigma \times {}^{\gamma}\sigma'^{\vee})$

② $-\frac{N}{2} + 1 - \frac{\ell(\mu,\mu')}{2} \le a(\mu,\mu') \le -\frac{N}{2} - 1 + \frac{\ell(\mu,\mu')}{2}$

(abelian width is bounded by the cuspidal width.)

③ $\exists\, w \in W^P$ s.t. $w^{-1} \cdot (\mu \times \mu')$ is dominant &

$\ell(w) = \frac{1}{2} \dim(U_P)$.

• ④ Main Technical Theorem of Eisenstein Cohomology.

The image of Eisenstein cohomology in

$$I_P \oplus I_Q = {}^a\mathrm{Ind}_P^G (\sigma_f \times \sigma_f')^{K_f} \oplus {}^a\mathrm{Ind}_Q^G (\sigma'(n) \times \sigma(-n'))^{K_f}$$

is "like a line in a two dimensional space."

The slope of this line is a ratio of L-values.

- Image has dimension ≤ 1

  Poincaré duality for boundary cohomology

- Image has dimension ≥ 1

  Base-change to $\mathbb{C}$ and produce enough cohomology classes.

  by using Langlands' theorem.

• ⑤ Signatures arise from Galois action on $H^\bullet(\partial_P S^G_{K_f}, \mathcal{M}_{\lambda,\bar{\mathbb{Q}}})$

Lecture-3

Special Values of motivic L-functions.

§3.1 Langlands Program

Automorphic forms $\leftrightsquigarrow$ Motives

To make this correspondence precise we need a working definition of motives.

What is a motive?

$$\begin{tikzcd} & M_B \arrow[d, leftrightsquigarrow, "\text{comparison}"] \\ M \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & M_{dR} \\ & M_\ell \arrow[loop right, "\text{Galois}"] \end{tikzcd}$$

comparison $\mapsto$ Periods c^+, c^-

Galois $\mapsto$ $L(s, M)$

Deligne's Conjecture: $L(0, M) \approx c^+(M)$.

§3.2

References:
(Deligne - Corvallis ; Blasius, Inventiones '87)

A pure motive M over $\mathbb{Q}$ of rank n with coefficients in a number field E consists of : $(M_B, M_{dR}, M_\ell, I, I_\ell)$

Betti

- M_B is an E-vector space of dim n.
  - * F_∞ involution on M_B (real Frobenius)

    $M_B = M_B^+ \oplus M_B^-$, ± 1 eigenspaces for F_∞.
  - * Hodge decomposition:

$$M_B \otimes_{E,\iota} \mathbb{C} = \bigoplus_{p+q=w} M_{B,\iota}^{p,q} \quad \text{(purity)} \qquad w = \text{weight}(M).$$

de Rham

- M_{dR} is an E-vector space of dim n,
  - * Hodge filtration $\mathcal{F}^\bullet$

ℓ-adic

- M_λ is an E_λ-vector space of dim n
  - * Action of $\mathrm{Galois}(\overline{\mathbb{Q}}/\mathbb{Q})$.

$$\begin{tikzcd}
\lambda \arrow[d, dash] & E \arrow[d, dash] \arrow[r, dash] & E_\lambda \arrow[d, dash] \\
\ell & \mathbb{Q} \arrow[r, dash] & \mathbb{Q}_\ell
\end{tikzcd}$$

Comparison

- $I : M_B \otimes_{E,\iota} \mathbb{C} \xrightarrow{\sim} M_{dR} \otimes_{E,\iota} \mathbb{C}$

$$F_\infty \otimes c_B \longleftrightarrow 1 \otimes c_{dR}$$

$$\bigoplus_{p \geq a} M_{B,\iota}^{p,q} \longleftrightarrow \mathcal{F}^a \otimes_{E,\iota} \mathbb{C}$$

$$M^{p,q} \longleftrightarrow \mathcal{F}^p \cap \overline{\mathcal{F}}^q \qquad (p+q=w)$$

- $I_\lambda : M_B \otimes E_\lambda \xrightarrow{\sim} M_\lambda$.

§§ Motivic L-functions.

M is a pure motive of rank n over $\mathbb{Q}$ with coeffs. in E.

- Take $\iota : E \longrightarrow \mathbb{C}$

$$L_f(s, M, \iota) = \prod_p L_p(s, M, \iota)$$

$\lambda \nmid p$

$$L_p(s, M, \iota) = \iota\left(\det\left(1 - \mathrm{Frob}_p \cdot t \; M_\lambda^{I_p}\right)^{-1}\right)\Big|_{t = p^{-s}}$$

Artin L-function attached to Galois rep?

modulo conjectural ℓ (or λ) - independence.

- $L_f(s, M) = \{L_f(s, M, \iota)\}_{\iota : E \to \mathbb{C}}$

Motivic L-fn. is an $E \otimes \mathbb{C} = \prod_{\iota : E \to \mathbb{C}} \mathbb{C}$ - valued fn.

- Γ- factors at infinity (Serre)

$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$ $\qquad$ $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$

$h_\iota^{p,q} = \dim(M_{B,\iota}^{p,q})$ - Hodge numbers

$h_\iota^{p,p}(\varepsilon) = \dim(M_{B,\iota}^{p,p}(\varepsilon \cdot (-1)^p))$ - eigenspace for $F_\infty \circlearrowright M_\iota^{p,p}$

$$L_\infty(s, M, \iota) =$$

$$\prod_{p < q} \Gamma_{\mathbb{C}}(s - p)^{h_\iota^{p,q}} \cdot \Gamma_{\mathbb{R}}(s - p)^{h_\iota^{p,p}(+1)} \cdot \Gamma_{\mathbb{R}}(s - p + 1)^{h_\iota^{p,p}(-1)}$$

We can talk about the completed L-fn.

and the functional equation.

$$L(s, M) \approx L(1 - s, M^\vee)$$

§§ Langlands Correspondence:

Automorphic forms $\rightsquigarrow$ Motives

$\mu \in X^{*}_{00}(\mathrm{Res}_{F/\mathbb{Q}}(T_n) \times E)$

$\sigma_f \in \mathrm{Coh}_{!!}(GL_n/_F, \mu/E) \quad \longleftrightarrow \quad M = M(\sigma_f).$

$\iota : E \to \mathbb{C}$

Pure rank n motive over F with coefficients in E.

- Langlands parameters of the rep^n ${}^{\iota}\sigma_\infty$ $\longleftrightarrow$ Hodge types of $M_{B,\iota}$

- Local Langlands parameter at a finite place ${}^{\iota}\sigma_v$ $\longleftrightarrow$ Local Galois rep^n. deduced from M_λ.

- $$L\left(s + \frac{1-n}{2}, {}^{\iota}\sigma\right) = L(s, M, \iota).$$

(References: Clozel - Ann Arbor ; [HR]. Chapter 7.)

§§ **Example**:

X = smooth projective variety $/\mathbb{Q}$.

$M = H^i(X, E)$

$M_B = H^i_{Betti}(X(\mathbb{C}), E) \circlearrowleft F_\infty$

$M_{dR} = H^i_{deRham}(X/\mathbb{Q}, E)$

$M_\lambda = H^i_{et}(X/\overline{\mathbb{Q}}, E_\lambda) \circlearrowleft Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

e.g. $H^2(\mathbb{P}^1)$ is a pure motive of rank 1 over $\mathbb{Q}$ and coefficient in $\mathbb{Q}$.

$L(s, H^2(\mathbb{P}^1)) = \zeta(s-1)$.

$H^2(\mathbb{P}^1) = \mathbb{Q}(-1)$. – "Dual of the Tate motive"

$\mathbb{Q}(m) = \overset{m}{\otimes}\, \mathbb{Q}(1)$

$\mathbb{Q}(-m) = \mathbb{Q}(m)^\vee$.

Tate - twist: $M(m) = M \otimes \mathbb{Q}(m)$.

§§ **Defn:** We say M is critical if comparison induces the isomorphism in the bottom arrow.

$$\begin{tikzcd} M_B \otimes_E \mathbb{C} \arrow[r, "I"] & M_{dR} \otimes_E \mathbb{C} \arrow[d] \\ M_B^+ \otimes_E \mathbb{C} \arrow[u, hook] \arrow[r, "I^+", "\cong"'] & M_{dR}/F^0 \otimes_E \mathbb{C} \end{tikzcd}$$

Note: M is critical $\iff$ $L_\infty(0, M)$ & $L_\infty(1, M^\vee)$ are finite

Defn: $m \in \mathbb{Z}$, we say m is critical for $L(s, M)$ if $M \otimes \mathbb{Q}(m)$ is critical

$L(s, M(m)) = L(s+m, M)$

Defn: Suppose M is critical, its period is defined by:

$$\boxed{c^+(M) = \det(I^+)}$$

computed wrt E-bases for M_B^+ & M_{dR}/F^0.

$c^+(M) \in (E \otimes \mathbb{C})^* / E^*$.

or $c^+(M) = \{c^+(M_\eta)\}_{\eta : E \to \mathbb{C}}$

Conjecture (Deligne Conwallis)

$$\boxed{\begin{array}{l} \text{If } L(M,0) \neq 0 \text{ then} \\ \qquad L(0, M) = c^+(M) \quad \text{in } (E \otimes \mathbb{C})^*/E^* \end{array}}$$

§§ The periods $c^{\pm}(M)$ & critical values:

$$\begin{tikzcd} M_B \otimes \mathbb{C} \arrow[r, "I", "\approx"'] & M_{dR} \otimes \mathbb{C} \arrow[d] \\ M_B^{\pm} \otimes \mathbb{C} \arrow[u, hook] \arrow[r, "I^{\pm}", "\approx"'] & M_{dR}/_{\mathcal{F}^{\mp}} \otimes \mathbb{C} \end{tikzcd}$$

for suitable $\mathcal{F}^{\pm}(M)$.

$$c^{\pm}(M) = \det(I^{\pm}) \; .$$

Deligne's Conjecture:

$$\boxed{L_f(m, M) = (1 \otimes 2\pi i)^{m \dim(M_B^{\pm})} \cdot c^{\pm}(M) \qquad (-1)^m = \pm 1 \quad \text{in } (E \otimes \mathbb{C})/E^*}$$

Question:

Are theorems on ratios of critical values obtained from Eisenstein cohomology compatible with Deligne's conjecture (admitting the conjectural correspondence in the Langlands program)?

Note:

$$\frac{L(m, M)}{L(m+1, M)} = (1 \otimes i)^{\dim(M_B^{\pm})} \cdot \left(\frac{c^{+}(M)}{c^{-}(M)}\right)^{\varepsilon_m}$$

for the completed L-function.

§ 3.3 When is $c^+(M) = c^-(M)$? (Jt. work with Deligne)

• **Motivic analogue of Langlands transfer**

Suppose $\{M_\alpha\}$ is a finite family of motives, M_α of wt w_α.

$$T_{r,s} = \otimes M_\alpha^{\otimes r_\alpha} \otimes M_\alpha^{\vee \otimes s_\alpha}$$

a multilinear algebra structure S on (M_α) is a collection of morphisms $\mathbb{Q}(0) \longrightarrow T_{r_i, s_i}((M_\alpha))$

$$G(S) = \{ g \in \prod_\alpha GL(M_\alpha) \mid g \cdot S = S \}$$

V = f.d. repn of $G(S)$

Then we can construct a motive M^V

$$\big((M_\alpha),\ S,\ V\text{-rep}^n \text{ of } G(S) \big) \longmapsto M^V.$$

Think of this as a motivic analogue of Langlands transfer & for L-functions:

$$L(s, M^V) = L(s, \pi, r)$$

Example:

(i) (M', M''), ϕ, $G(S) = GL(M') \times GL(M'')$, $V = \otimes$-product)

Rankin-Selberg L-function.

(ii) M pure motive of wt. 0 & rk $2n$

$S = \{ B : M \otimes M \longrightarrow \mathbb{Q}$ symmetric nondegenerate,

$\delta : \wedge^{top} M \xrightarrow{\sim} \mathbb{Q} \}$

$G[S] = SO(M)$ L-functions for even orthogonal groups.

(iii) "Asai motives" or "Twisted tensor motives"

K | F — M is a rank n motive $/K$ with coeffs. in E.

Then $\bigotimes_{K|F} M$ — twisted tensor motive of rank $n^{[K:F]}$ with coeffs. in E.

- e.g. $M = H^q(X)$ $\quad X/K$

Then $\bigotimes_{K|F} M$ is a piece in $H^{q[K:F]}(\mathrm{Res}_{K/F}(X))$

Let $\sigma : F \longrightarrow \mathbb{C}$ & $\{\eta_i : K \to \mathbb{C}\}$ be its extensions to K.

$$\begin{tikzcd} K \arrow[r, "\eta_i"] & \mathbb{C} \\ F \arrow[u, dash] \arrow[r, hook, "\sigma"] & \mathbb{C} \arrow[u, equal] \end{tikzcd}$$

$$H^p_B(\mathrm{Res}_{K/F}(X)_\sigma(\mathbb{C})) = H^p_B\left(\prod_i X_{\eta_i}(\mathbb{C})\right)$$
$$= \bigoplus_{\sum p_i = p} \bigotimes_i H^{p_i}_B(X_{\eta_i}(\mathbb{C}))$$

If $p = [K:F]q$ then we take the summand for $p_i = q \ \forall i$

$$\left(\bigotimes_{K|F} M\right)_{B,\sigma} = \bigotimes_i M_{B,\eta_i}$$

- If $M \longleftrightarrow \pi$ - cusp. aut. rep^n of GL_n/K.

Then $L(s, \bigotimes_{K|F} M) = L(s, \pi, As)$ $\quad$ Asai L-function.

Theorem

Let M be a pure rank n motive with coefficients in E

Suppose, M is of the form M^{ν}, for some $((M_\alpha), S, \nu)$.

Assume, there is no middle Hodge type ($M^{p,p} = 0$.)

$Z(F_\infty) =$ centralizer of F_∞. ; $Z(F_\infty) \subset G(S)$

$V = V^+ \oplus V^-$ for the action of F_∞

$$\begin{tikzcd} Z(F_\infty) \arrow[r] \arrow[dr, "\chi_E^{\pm}"'] & GL(V^{\pm}) \arrow[d, "\det"] \\ & E^* \end{tikzcd}$$

Suppose $G(S)$ is connected.

If $\chi_E^+ = \chi_E^-$ then $C^+(M)/C^-(M) \in E^*$

- The essence of the proof is to look at how large is
$$P_{dR}^{\pm} \backslash \mathrm{Isom}(M_B, M_{dR}) / Z(F_\infty).$$

$\mathrm{Isom}(M_B, M_{dR})$ = scheme of isomorphisms$/E$

$P_{dR}^{\pm}$ = parabolic subgroup that stabilizes $F^{\pm} \subset M_{dR}$.

- The ratio of χ_E^+/χ_E^- governs $C^+(M)/C^-(M)$

Example: F - totally imaginary base field.

M = pure motive of rank n over F with coeffs. in E

Suppose M has no middle Hodge type.

F_0 = maximal totally real subfield.

$$\begin{array}{c} F \\ | \\ F_1 \\ |\,{\scriptstyle \leq 2} \\ F_0 \\ | \\ \mathbb{Q} \end{array}$$

"CM-case": F_1 - totally imaginary quadratic $/F_0$

Suppose $F_1 = F_0(\sqrt{D})$ $D \ll 0$.

Define $\Delta_F = \sqrt{N_{F_0/\mathbb{Q}}(D)}^{\,[F:F_1]} \in \mathbb{C}^*$

"TR-case": $F_1 = F_0$. (F does not contain a CM subfield)

Define: $\Delta_F = 1$

Then:

$$\boxed{c^+(M)/c^-(M) = (1 \otimes \Delta_F)^n \quad \text{in } (E \otimes \mathbb{C})^*/E^*.}$$

Corollary (To this period rel^n + Deligne's conjecture)

Suppose m and $m+1$ are critical then:

$$\frac{L(m, M, \imath)}{L(m+1, M, \imath)} \in \imath(E) \subset \bar{\mathbb{Q}}$$

$\forall\, \gamma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$

$$\gamma\left(\frac{L(m, M, \imath)}{L(m+1, M, \imath)}\right) = \left(\frac{\gamma(i^{[F:\mathbb{Q}]/2} \cdot \Delta_F)}{i^{[F:\mathbb{Q}]/2} \cdot \Delta_F}\right)^n \frac{L(m, M, \imath)}{L(m+1, M, \imath)}$$

Applying this to the case $M = M(\sigma_f) \otimes M(\sigma_{f'})$

(cuspidal width $> 0 \Rightarrow$ no middle Hodge type.)

$\left\{\begin{array}{l}\text{Ratio of L-values from} \\ \text{Eisenstein cohomology}\end{array}\right\} \xleftrightarrow{\text{compatible}} \left\{\text{Deligne's conjecture}\right\}$

Lemma:

$$\left(\frac{\gamma(i^{[F:\mathbb{Q}]/2} \cdot \Delta_F)}{i^{[F:\mathbb{Q}]/2} \cdot \Delta_F}\right)^{nn'} \left(\frac{\gamma(\delta_{F/\mathbb{Q}}^{1/2})}{\delta_{F/\mathbb{Q}}^{1/2}}\right)^{nn'} = \varepsilon(\mu, \mu'; \gamma) \cdot \tilde{\varepsilon}(\mu, \mu'; \gamma)$$

LHS = depends only the base field

RHS = Galois action on the coefficients in the unipotent cohomology which appeared in boundary cohomology.

Goal: Understand Galois action on Eisenstein Cohomology

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