Non-immersions of 2^k-lens spaces

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Abstract

We study the connective complex K-theoretic Euler class of a certain bundle associated to an Euclidean immersion of the lens space $L(2^k)^{2n+1}$ to show that this manifold cannot be immersed in $\mathbb{R}^{4n-2\alpha(n)}$ if $k \geq \alpha(n)$.

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1 Introduction

Given positive integers n and k, let $L(2^k)^{2n+1}$ be the $(2n+1)$-dimensional skeleton of the classifying space $L(2^k)^\infty$ for the cyclic group of order 2^k. Thus, $L(2^k)^{2n+1}$ is the orbit space associated to the usual $\mathbb{Z}/2^k$-action on the unit vectors of $\mathbb{C}^{n+1}$. We address ourselves to the problem of existence of Euclidean immersions for this manifold. We stick to the usual convention of writing $M \subseteq \mathbb{R}^N$ ($M \not\subseteq \mathbb{R}^N$) if the manifold M admits (does not admit) an immersion in the Euclidean space $\mathbb{R}^N$. Our main result is the following one, where $\alpha(n)$ is the number of ones in the dyadic expansion of n.

Theorem 1.1 *$L(2^k)^{2n+1} \not\subseteq \mathbb{R}^{4n-2\alpha(n)}$, if $k \geq \alpha(n)$.*

Remark 1.2 The immersion $L(2^k)^{2n+1} \subseteq \mathbb{R}^{4n-\alpha(n)+1}$ holds in view of Cohen's general solution to the immersion conjecture [6]. Therefore the non-immersion of Theorem 1.1 is within $\alpha(n)$ dimensions of the best possible.

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The main source of non-immersions for (high) 2-torsion lens spaces is Berrick's result in [5], which is improved by Theorem 1.1 in two ways. To begin with, our hypothesis $k \geq \alpha(n)$ replaces the much more restrictive condition in [5]: $k \geq 2n-1-\alpha(n-1)$. On the other hand, the Euclidean space in Berrick's non-immersion theorem is two dimensions smaller than the Euclidean space used in Theorem 1.1. Moreover, as indicated by Proposition 1.4 below, this second improvement is optimal in many cases, and sharper non-immersions would hold only for particular values of n (or $\alpha(n)$, in the most favorable cases).

It is profitable to compare immersions of 2^k-lens spaces for several values of k (including the case $k=1$ of real projective spaces) with immersions of complex projective spaces. This is done in Section 2, where we establish the next fact.

Lemma 1.3 **a)** *If $\mathbb{C}P^n \subseteq \mathbb{R}^M$, then $L(2^k)^{2n+1} \subseteq \mathbb{R}^{M+1}$.*

b) *If $L(2^{k+1})^{2n+1} \subseteq \mathbb{R}^M$, then $L(2^k)^{2n+1} \subseteq \mathbb{R}^M$.*

For instance, the case $\alpha(n)=1$ in Theorem 1.1 follows from Lemma 1.3 b) together with the well known (optimal) non-immersion $L(2)^{2^\ell} \not\subseteq \mathbb{R}^{2^{\ell+1}-2}$ obtained with Stiefel-Whitney classes. Likewise, Theorem 1.1 and Lemma 1.3 a) have as an immediate consequence the well known non-immersion $\mathbb{C}P^n \not\subseteq \mathbb{R}^{4n-2\alpha(n)-1}$. The idea of comparing immersions of lens spaces with those for complex projective spaces is not new (the case $k=1$ in a) above is due to Sanderson [14, Theorem 5.2]). Besides of its own interest, we believe this approach may lead to a better understanding of the immersion problem for complex projective spaces (see Remarks 2.1 and 4.6). Here we use the idea in the opposite direction; namely, [7] and [15] give conditions for the existence of immersions of the form $\mathbb{C}P^n \subseteq \mathbb{R}^{4n-2\alpha(n)}$ which, in view of Lemma 1.3, give the following result.

Proposition 1.4 *For odd n, the non-immersion in Theorem 1.1 is optimal in any of the following cases:*

a) $\alpha(n)=3$ and $n \equiv 3$ (mod 4).

b) $\alpha(n)=4$ and $n \equiv 7$ (mod 8).

c) $\alpha(n)=5$ and $n \not\equiv 3$ (mod 8).

d) $\alpha(n) = 6$ *and either* $n \equiv 11$ *(mod 16) or* $n \equiv 7$ *(mod 8).*

e) $\alpha(n) = 7$ *and* $n \equiv 7$ *(mod 8).*

The approach in this paper is based on Astey's original ideas [3] as improved by Davis in [8]. Our computations resemble those in [16] for deriving non-immersions of products of real projective spaces. Nonetheless, our results are still within the reach of K-theoretic calculations.

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2 Geometric preliminaries

A basic tool in studying immersions of a manifold M is Hirsch's theorem [12]: If τ denotes the tangent bundle of M, then M has an immersion of codimension k if, and only if, $-\tau$ is stably equivalent to a real bundle of dimension k. For $M = L(2^k)^{2n+1}$, the well known equation $\tau \oplus 1 = (n+1)\lambda^{(k)}$ helps identifying $-\tau$. Here $\lambda^{(k)}$ is the realification of $p^*(H)$, where H is the canonical complex line bundle over $\mathbb{C}P^n$ and $p : L(2^k)^{2n+1} \to \mathbb{C}P^n$ is the usual projection (note that $\lambda^{(1)}$ equals twice the canonical real bundle over the projective space $L(2)^{2n+1} = P^{2n+1}$). Thus we have

$$L(2^k)^{2n+1} \subseteq R^{2n+1+c} \quad \text{iff} \quad gd\Big((N-n-1)\lambda^{(k)}\Big) \leq c. \tag{1}$$

In this equation "*gd*" stands for (real) geometric dimension and N is any integer highly divisible by 2 [13]. In view of [10], a similar result holds for complex projective spaces, namely

$$\mathbb{C}P^n \subseteq \mathbb{R}^{2n+d} \quad \text{iff} \quad gd\Big((M-n-1)H\Big) \leq d, \tag{2}$$

where M is any multiple of the Atiyah-Todd number M_{n+1}.

Proof of Lemma 1.3. Part a) is an immediate consequence of equations (1) and (2). To show part b) some care is needed in order to compare two lens spaces. The following diagram with fiber sequences determines a map $L(2^k)^\infty \to L(2^{k+1})^\infty$.

$$\begin{tikzcd} L(2^k)^\infty \arrow[r] \arrow[d, dashed] & \mathbb{C}P^\infty \arrow[r, "2^k"] \arrow[d, equal] & \mathbb{C}P^\infty \arrow[d, "2"] \\ L(2^{k+1})^\infty \arrow[r] & \mathbb{C}P^\infty \arrow[r, "2^{k+1}"] & \mathbb{C}P^\infty. \end{tikzcd}$$

From its construction, we see that the cellular restriction $L(2^k)^{2n+1} \to L(2^{k+1})^{2n+1}$ pulls back $\lambda^{(k+1)}$ into $\lambda^{(k)}$, so that again (1) completes the proof in this case. ∎

Remark 2.1 For positive integers n and k, let $I_k(n)$ (resp. $I(n)$) denote the smallest integer m for which there is an immersion $L(2^k)^{2n+1} \subseteq \mathbb{R}^m$ (resp. $\mathbb{C}P^n \subseteq \mathbb{R}^m$). Lemma 1.3 claims that the sequence $I_1(n), I_2(n), \ldots$ is nondecreasing and bounded above by $I(n)+1$, therefore $I_k(n) = I_{k+\ell}(n)$ for any ℓ and for $k \gg 0$. In fact, Proposition 1.4 gives examples where $I_k(n) = I(n)+1$ for $k \geq \alpha(n)$. It is conceivable that the general behavior of the functions I_k stabilizes for large k, modeling the function I. Heuristically, the latter will have a simpler behavior than that for I_k with small k. For instance, I_1 — optimal immersions of odd dimensional real projective spaces — represents an extremely difficult problem; the strongest general non-immersion result known in this case is $P^{2n+2(\alpha(n)-1)} \not\subseteq \mathbb{R}^{4n-2\alpha(n)}$ [8]. However, Theorem 1.1 shows that the part "$+2(\alpha(n)-1)$" can be changed simply to "$+1$" if we are willing to consider lens spaces with large 2-torsion. (Note by the way that $L(2^k)^{2n}$ cannot be a manifold for $k \geq 2$).

From now on, $\lambda^{(k)}$ will simply be denoted by λ (k being explicit from the context) and ξ will stand for the canonical (real) bundle over the real projective space P^∞. The subindex d in η_d will be used to indicate that the bundle η has been restricted to the d-dimensional skeleton of its base space. The multiple sectioning problem indicated in (1) is better managed by means of Astey's trick of reducing it to a single sectioning problem [3]. This produces the next result. (The reader familiar with [3], [8] or [16] may skip the proof).

Proposition 2.2 *If $L(2^k)^{2n+1} \subseteq \mathbb{R}^{4n-2d}$, then $(2^L - n - 1)\lambda_{2n+1} \otimes \xi_{2m}$ has a nowhere trivial section over $L(2^k)^{2n+1} \times P^{2m}$. Here $m = 2^L - 2n + d - 1$ and $L \gg 0$.*

Proof. By (1) the hypothesis is $gd((2^L - n - 1)\lambda_{2n+1}) \leq 2(n-d) - 1$. Thus there is a bundle α over $L(2^k)^{2n+1}$ of dimension $|\alpha| = 2(n-d)-1$ such that $(2^L - n - 1)\lambda_{2n+1} \oplus \theta_M \simeq \alpha \oplus \theta_N$, where $M + 2(2^L - n - 1) = |\alpha| + N$ (θ_T is the trivial bundle of dimension T). But L and therefore N are large compared with $2n+1$, thus we may kill off M trivial bundles to get $(2^L - n - 1)\lambda_{2n+1} \simeq \alpha \oplus \theta_{2(2^L-n-1)-|\alpha|}$. This means that $(2^L - n - 1)\lambda_{2n+1}$ has $2(2^L - 2n + d - 1) + 1$ everywhere independent sections. The results follows from [3, Proposition 2.1]. ∎

In Section 4 we will discard the hypothetical immersion in Proposition 2.2 by showing that the connective complex K-theoretic (ku) Euler class of $\eta = (2^L - n - 1)\lambda_{2n+1} \otimes \xi_{2m}$ is nontrivial when $k \geq d = \alpha(n)$. The next result establishes the orientability of η and helps identifying its ku-Euler class. The proof is an easy application of the techniques in [4, Section 1.2] to extend the natural isomorphism of (real) vector spaces $V \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} W) = V \otimes_{\mathbb{R}} W$ to vector bundles, for V (W) any complex (real) vector space.

Lemma 2.3 *Let $\mu = p^*(H)$ be the pull back of the canonical complex line bundle H over $\mathbb{C}P^\infty$ under the projection map $p : L(2^k)^\infty \to \mathbb{C}P^\infty$. If $\xi^{\mathbb{C}}$ stands for the complexification of the canonical real line bundle ξ over $P^\infty = L(2)^\infty$, then $r(\mu \otimes_{\mathbb{C}} \xi^{\mathbb{C}}) = r(\mu) \otimes_{\mathbb{R}} \xi$, where r stands for realification.*

Let h^* denote a multiplicative commutative cohomology theory with a given orientation for complex bundles [1]. Let $\chi(\alpha)$ stand for the h^*-Euler class of the complex bundle α. By definition, for α and β complex line bundles, $\chi(\alpha \otimes \beta) = F_h(\chi(\alpha), \chi(\beta))$ — the h^*-formal sum of $\chi(\alpha)$ and $\chi(\beta)$. In general, this has a very complicated expression as a power series on $\chi(\alpha)$ and $\chi(\beta)$ with coefficients in $h^*(*)$. However, when $\beta = \xi^{\mathbb{C}}$ the above expression for $\chi(\alpha \otimes \xi^{\mathbb{C}})$ simplifies dramatically. The trick appeared first in [2] and has been used subsequently in [8] and [16]. The next result represents our version of Astey's trick.

Proposition 2.4 *For any complex line bundle α over a cellular complex X, there is a unit $u \in h^*(X \times P^\infty)$ (depending on α) such that*

$$\chi(\alpha \otimes \xi^{\mathbb{C}}) = u \cdot (\chi(\alpha) - \chi(\xi^{\mathbb{C}})).$$

Proof. Let $q : X \to \mathbb{C}P^\infty$ classify α and recall $\xi^{\mathbb{C}} = p^*(H)$, where $p : P^\infty \to \mathbb{C}P^\infty$ is the projection map. Then $\chi(\alpha \otimes \xi^{\mathbb{C}})$ is the h^*-pull back of $\chi(H)$ under the composite in the top row of the following commutative diagram

$$\begin{tikzcd}
X \times P^\infty \arrow[r, "q\times p"] \arrow[dr, "q\times p"'] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "mult"] & \mathbb{C}P^\infty \\
& \mathbb{C}P^\infty \times \mathbb{C}P^\infty. \arrow[u, "1\times(-1)"'] &
\end{tikzcd}$$

The assertion follows from the fact that $(mult \circ [1\times(-1)])^*(\chi(H))$ agrees, up to a unit, with $\chi(H_1) - \chi(H_2)$, where $H_i = \pi_i^*(H)$ and $\pi_i : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the i-th projection $(i = 1, 2)$. ■

3 Cohomology of a product of lens spaces.

The last ingredient in the proof of Theorem 1.1 is an explicit description of $h^*(L(2^k)^{2n+1} \times P^{2m})$. This is done in detail for $k = 1$ in Astey's thesis, although [3] manages to avoid these calculations. Here we follow [2] to analyze a slightly more general case (Proposition 3.1). The results in this section are well known to specialists, who should proceed directly to the next and final section.

Since h^* is complex oriented we have $h^*(\mathbb{C}P^\infty) = h^*[[\bar{x}]]$, where $\bar{x} = \chi(H)$ (recall H is the canonical complex line bundle over $\mathbb{C}P^\infty$). Dividing by the ideal generated by $\bar{x}^{n+1}$ yields $h^*(\mathbb{C}P^n)$. The projection $p : L(2^k)^\infty \to \mathbb{C}P^\infty$ can be embedded in the Barrat type sequence of fiber maps

$$S^1 \longrightarrow L(2^k)^\infty \longrightarrow \mathbb{C}P^\infty \xrightarrow{2^k} \mathbb{C}P^\infty.$$

Thus, p is a spherical fibration with fiber S^1 and with h^*-Euler class given by $(2^k)^*(\bar{x})$ — the 2^k-series for h^*, otherwise denoted by $[2^k](\bar{x})$. Let M stand for the Thom complex of p. In terms of the Thom isomorphism $\widetilde{h}^*(M) \simeq h^{*-2}(\mathbb{C}P^\infty)$, the cofiber sequence $L(2^k)^\infty \to \mathbb{C}P^\infty \to M$ induces the long exact sequence

$$\begin{tikzcd}
h^*(\mathbb{C}P^\infty) \arrow[rr, "\chi"] & & h^*(\mathbb{C}P^\infty) \arrow[dl] \\
& h^*(L(2^k)^\infty) \arrow[ul, "\delta"] &
\end{tikzcd}$$

where δ decreases dimension by 1. We assume from now on that 2 is not a zero divisor in $h^*(*)$. Then χ — multiplication by the 2^k-series — is an injective map so that

$$h^*\Big(L(2^k)^\infty\Big) = h^*[[x]]/([2^k](x)),$$

where $x = p^*(\bar{x})$. A similar analysis for the restricted spherical fibration $S^1 \to L(2^k)^{2n+1} \to \mathbb{C}P^n$ yields the long exact sequence

$$\begin{tikzcd} h^*(\mathbb{C}P^n) \arrow[rr, "\chi"] & & h^*(\mathbb{C}P^n) \arrow[dl] \\ & h^*(L(2^k)^{2n+1}). \arrow[ul, "\delta"] & \end{tikzcd}$$

This time χ is not injective, for $\chi(\bar{x}^n) = \bar{x}^n \cdot (2^k\bar{x} + \cdots) = 0$. In fact, $\operatorname{Ker}\chi$ is $h^*(*)$-free generated by $\bar{x}^n$. This produces a free split summand of $h^*(L(2^k)^{2n+1})$, with generator σ satisfying

$$\delta(\sigma) = \bar{x}^n, \tag{3}$$

which carries all the odd dimensional cohomology. On the other hand, the cokernel of χ gives the even dimensional part as

$$h^{ev}\Big(L(2^k)^{2n+1}\Big) = h^*[x]/(x^{n+1}, [2^k](x)). \tag{4}$$

This description is best expressed by noticing that the long exact sequence associated to the cofiber sequence $L(2^k)^{2n} \hookrightarrow L(2^k)^{2n+1} \xrightarrow{\pi} S^{2n+1}$ reduces to an isomorphism $h^*(L(2^k)^{2n+1}) = h^*(L(2^k)^{2n}) \oplus \widetilde{h}^*(S^{2n+1})$. In particular $h^*(L(2^k)^{2n})$ is given by the right hand side of (4). A parallel argument is used to prove the following result.

Proposition 3.1 *For $\delta = 0, 1$,*

$$h^{ev}\Big(L(2^k)^{2n+\delta} \times L(2^\ell)^{2m}\Big) = h^*[x, y]/(x^{n+1}, y^{m+1}, [2^k](x), [2^\ell](y)),$$

where $x = \chi(\lambda_{2n+\delta} \otimes 1)$ and $y = \chi(1 \otimes \lambda_{2m})$.

Proof. By naturality, the S^1-bundle $L(2^k)^{2n+1} \times L(2^\ell)^{2m} \xrightarrow{p\times 1} \mathbb{C}P^n \times L(2^\ell)^{2m}$ has h^*-Euler class given by $[2^k](\bar{x})$. As before, consideration of the mapping cone gives rise to the long exact sequence

$$
\begin{tikzcd}
h^*(\mathbb{C}P^n \times L(2^\ell)^{2m}) \arrow[rr, "\chi"] & & h^*(\mathbb{C}P^n \times L(2^\ell)^{2m}) \arrow[dl] \\
& h^*(L(2^k)^{2n+1} \times L(2^\ell)^{2m})). \arrow[ul, "\delta"] &
\end{tikzcd}
$$

Since $h^*(\mathbb{C}P^n)$ is h^*-free, the product $h^*(\mathbb{C}P^n) \otimes_{h^*} h^*(L(2^\ell)^{2m}) \to h^*(\mathbb{C}P^n \times L(2^\ell)^{2m})$ is an isomorphism, so that

$$
h^*\Big(\mathbb{C}P^n \times L(2^\ell)^{2m}\Big) = h^*[\bar{x}, y]/(\bar{x}^{n+1}, y^{m+1}, [2^\ell](y)).
$$

This ring is concentrated in even dimensions, so that the long exact sequence above yields

$$
h^{ev}\Big(L(2^k)^{2n+1} \times L(2^\ell)^{2m}\Big) = h^*[x, y]/(x^{n+1}, y^{m+1}, [2^k](x), [2^\ell](y)). \qquad (5)
$$

As before, let $\pi : L(2^k)^{2n+1} \to S^{2n+1}$ be the collapsing map. Note that both maps in the composition

$$
h^*\Big(S^{2n+1} \times L(2^\ell)^{2m}\Big) \stackrel{(\pi\times 1)^*}{\longrightarrow} h^*\Big(L(2^k)^{2n+1} \times L(2^\ell)^{2m}\Big) \stackrel{\delta}{\to} h^*\Big(\mathbb{C}P^n \times L(2^\ell)^{2m}\Big)
$$

are morphisms of $h^*(L(2^\ell)^{2m})$-modules. Moreover all these modules, but the middle one, are free and, by (3), the above composite sends a basis into a linearly independent set. In particular, $(\pi \times 1)^*$ as well as its reduced version $(\pi \wedge 1)^*$ are injective. Then the long exact sequence for $(L(2^k)^{2n} \stackrel{\iota}{\to} L(2^k)^{2n+1} \stackrel{\pi}{\to} S^{2n+1}) \wedge L(2^\ell)^{2m}$, reduces to a short exact sequence $0 \to \widetilde{h}^*(S^{2n+1} \wedge L(2^\ell)^{2m}) \to \widetilde{h}^*(L(2^k)^{2n+1} \wedge L(2^\ell)^{2m}) \to \widetilde{h}^*(L(2^k)^{2n} \wedge L(2^\ell)^{2m}) \to 0$. But $\widetilde{h}^{ev}(S^{2n+1} \wedge L(2^\ell)^{2m}) \simeq \widetilde{h}^{odd}(L(2^\ell)^{2m}) = 0$, so that $(\iota \wedge 1)^*$ must be an isomorphism in even dimensions. Since this is the case for ι^* too, the result follows from (5). ■

Remark 3.2 In dealing with the ring in Proposition 3.1, it is much better to consider the usual formal sum for $h = ku$, instead of the 2-typical one. Indeed, this way the 2^k-series is a well known polynomial rather than a series. It is likely that Theorem 1.1 can be pushed further for specific values of n by considering $h = BP$ or even $h = BP\langle 2\rangle$. In those cases we would be forced to use the typical formal sum and typical 2^k-series, whose 2-divisibility properties are studied in [11].

4 The ku-Euler class.

Let ku stand for the spectrum representing connective 2-local complex K-theory. Thus $\pi_*(ku) = \mathbb{Z}_{(2)}[v_1]$, where $v_1 \in \pi_2(ku)$ corresponds to Bott periodicity. The formal sum of this spectrum is the multiplicative one:

$$F(x,y) = x + y + v_1xy = \frac{(1+v_1x)(1+v_1y)-1}{v_1}.$$

In particular the corresponding 2^k-series takes the form

$$[2^k](x) = \frac{(1+v_1x)^{2^k}-1}{v_1} = \sum_{j=1}^{2^k}\binom{2^k}{j}v_1^{j-1}x^j. \tag{6}$$

The key result in the proof of Theorem 1.1 is obtained by linking Propositions 2.2 and 2.4 by means of Lemma 2.3.

Proposition 4.1 *If $L(2^k)^{2n+1} \subseteq \mathbb{R}^{4n-2d}$, then for $L \gg 0$, $(x-y)^{2^L-n-1}$ is trivial in*

$$\mathbb{Z}_{(2)}[v_1,x,y]/(x^{n+1}, y^{m+1}, [2^k](x), [2](y)), \tag{7}$$

where $m = 2^L - 2n + d - 1$.

Note that, in the ring (7),

$$(x-y)^{2^L-n-1} = \pm\sum_{j=0}^{d}(-1)^j\binom{2^L-n-1}{n-j}x^{n-j}y^{m-d+j}. \tag{8}$$

We need an analysis of the 2-divisibility properties for the binomial coefficients in (8).

Lemma 4.2 *Let $\nu(\ell)$ denote the highest power of 2 dividing the integer ℓ. Then*

a) $\nu\binom{2^L-n-1}{n} = \alpha(n)$.

b) $\nu\binom{2^L-n-1}{n-1} = \alpha(n)-1$.

c) For $2 \le j \le \alpha(n)$, $\nu\binom{2^L-n-1}{n-j} > \alpha(n)-j$.

Proof. Part a) is an easy consequence of the well know relations $\nu\binom{a}{b} = \alpha(b) + \alpha(a-b) - \alpha(a)$ and $\alpha(2^L - a - 1) = L - \alpha(a)$, for $L \gg 0$. Part b), as well as the case $j = 2$ in c), are also direct consequences of these relations together with the formula $\alpha(a-1) = \alpha(a) - 1 + \nu(a)$. For $3 \le j \le \alpha(n)$ take $i \in \mathbb{Z}$ such that $2^{i-1} \le j \le 2^i - 1$ (in particular $2 \le i < j \le \alpha(n)$), and let $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_{\alpha(n)}}$, $0 \le a_1 < a_2 < \cdots < a_{\alpha(n)}$, be the dyadic expansion of n. Since $a_s \ge s-1$, we see $2^{a_1} + \cdots + 2^{a_i} \ge 1 + \cdots + 2^{i-1} \ge j$. Therefore, the dyadic expansion of $n - j$ includes the powers $2^{a_{i+1}}, 2^{a_{i+2}}, \cdots, 2^{a_{\alpha(n)}}$. On the other hand, the dyadic expansion of $2^L - n - 1$, being "complementary" to the one for n, does not include such powers. Thus, in the dyadic subtraction of $n - j$ from $2^L - n - 1$, there are at least $\alpha(n) - i$ "borrows", so that $\nu\binom{2^L-n-1}{n-j} \ge \alpha(n) - i$, in view of Kummer's result (see for instance [17]). ■

Corollary 4.3 *Assume $d = \alpha(n)$ and $k \le \alpha(n)$, then in the ring (7), the element (8) agrees, up to a unit, with $2^{\alpha(n)-1}x^{n-1}y^{m-\alpha(n)+1}$.*

Proof. In view of (6), the 2^k-series starts as $[2^k](x) = 2^k x + \omega 2^{k-1} v_1 x^2 + \cdots$, for a unit $\omega \in \mathbb{Z}_{(2)}$. It follows that $2^k x^n = 0$. Likewise, using the fact that $[2](y) = 2y + v_1 y^2$, it is elementary to show that $2^{i+1} y^{m-i} = 0$, for $0 \le i < m$. Thus, the result follows from Lemma 4.2. ■

Proof of Theorem 1.1. We may assume $k = \alpha(n) = d$, with $\alpha(n) > 1$, in view of Lemma 1.3 and the observation after it. If $L(2^{\alpha(n)})^{2n+1} \subseteq \mathbb{R}^{4n-2\alpha(n)}$, then by Proposition 4.1 and Corollary 4.3, the monomial $2^{\alpha(n)-1}x^{n-1}y^{m-\alpha(n)+1} \in \mathbb{Z}_{(2)}[v_1, x, y]$ would lie in the ideal generated by the four elements x^{n+1}, y^{m+1}, $[2^{\alpha(n)}](x)$ and $[2](y)$, where $m = 2^L - 2n + \alpha(n) - 1$. Say

$$2^{\alpha(n)-1}x^{n-1}y^{m-\alpha(n)+1} = x^{n+1} \cdot P + y^{m+1} \cdot Q + [2^{\alpha(n)}](x) \cdot R + [2](y) \cdot S, \quad (9)$$

for polynomials $P, Q, R, S, \in \mathbb{Z}_{(2)}[v_1, x, y]$. For $i = 1, \ldots, \alpha(n)$, we look at the coefficient of the monomial $v_1^{i-1}x^{n-1}y^{m-\alpha(n)+i}$ on both sides of (9). Only R and S contribute to these monomials. Let

$$R = \sum_{r=(r_1,r_2,r_3)} a_r v_1^{r_1} x^{r_2} y^{r_3} \quad \text{and} \quad S = \sum_{s=(s_1,s_2,s_3)} b_s v_1^{s_1} x^{s_2} y^{s_3},$$

and recall $[2](y) = 2y + v_1 y^2$ and $[2^{\alpha(n)}](x) = \sum_{j=1}^{2^{\alpha(n)}} \binom{2^{\alpha(n)}}{j} v_1^{j-1} x^j$. Then, for $i = 1$ we get the equation

$$2^{\alpha(n)-1} = 2^{\alpha(n)} a_{(0,n-2,m+1-\alpha(n))} + 2b_{(0,n-1,m-\alpha(n))};$$

whereas for $2 \leq i \leq \alpha(n)$ we get the equation

$$0 = \sum_{j=1}^{i} \binom{2^{\alpha(n)}}{j} a_{(i-j,n-1-j,m-\alpha(n)+i)} + \\ +2b_{(i-1,n-1,m-\alpha(n)+i-1)} + b_{(i-2,n-1,m-\alpha(n)+i-2)}.$$

But for $1 \leq j \leq i \leq \alpha(n)$, we have $\nu\binom{2^{\alpha(n)}}{j} = \alpha(j) + \alpha(2^{\alpha(n)} - j) - 1 \geq \alpha(2^{\alpha(n)} - j) = \alpha(n) - \alpha(j-1) \geq \alpha(n) - j + 1 \geq \alpha(n) - i + 1$. Therefore, the above equations simplify to

$$\begin{array}{rcl} 2^{\alpha(n)-1}c_1 & = & 2b_0; \\ 2^{\alpha(n)-1}c_2 & = & b_0 + 2b_1; \\ 2^{\alpha(n)-2}c_3 & = & b_1 + 2b_2; \\ & \cdots & \\ 2c_{\alpha(n)} & = & b_{\alpha(n)-2} + 2b_{\alpha(n)-1}; \end{array} \tag{10}$$

for coefficients $c_1, \ldots, c_{\alpha(n)} \in \mathbb{Z}_{(2)}$ with c_1 a unit, and where b_j stands for $b_{(j,n-1,m-\alpha(n)+j)}$. Solving the last $\alpha(n)-1$ equations (from bottom to top) we see that $b_{\alpha(n)-i} = 2^{i-1}d_i$, for $2 \leq i \leq \alpha(n)$. In particular $b_0 = 2^{\alpha(n)-1}d_{\alpha(n)}$, which is incompatible with the first equation in (10) and the fact that c_1 is a unit. ∎

Remark 4.4 The condition $k \geq \alpha(n)$ on the 2-torsion of the lens spaces in Theorem 1.1 is the best we can do with the present methods. Indeed, in view of Corollary 4.3 the next result indicates that, in the case of a (possible) immersion $L(2^k)^{2n+1} \subseteq \mathbb{R}^{4n-2\alpha(n)}$, the Euler class in Proposition 4.1 does vanish if $k < \alpha(n)$.

Proposition 4.5 *Let $m = 2^L - 2n + \alpha(n) - 1$ ($L \gg 0$), then the monomial $2^{\alpha(n)-1}x^{n-1}y^{m-\alpha(n)+1}$ is trivial in the ring*

$$\mathbb{Z}_{(2)}[v_1, x, y]/(x^{n+1}, y^{m+1}, [2^{\alpha(n)-1}](x), [2](y)).$$

Proof. In view of (6), the relations $0 = x^{n+1} = [2^{\alpha(n)-1}](x)$ give

$$2^{\alpha(n)-1}x^{n-1}y^{m-\alpha(n)+1} = -u2^{\alpha(n)-2}v_1x^ny^{m-\alpha(n)+1}$$

for $u \in \mathbb{Z}_{(2)}$ a unit. Since $[2](y) = 2y + v_1y^2 = 0$, it follows that the above element agrees with $u2^{\alpha(n)-1}x^ny^{m-\alpha(n)}$; but now $2^{\alpha(n)-1}x^n = 0$. ∎

Remark 4.6 It is conjectured in [9, 10] that the optimal immersion of $\mathbb{C}P^n$ is (a bit higher, but very) close to $\mathbb{R}^{4n-2\alpha(n)}$. By Lemma 1.3 such a result would imply a corresponding one for immersions of 2^k-lens spaces (so that the non-immersion of Theorem 1.1 might be even closer to the optimal one, than what is suggested by Remark 1.2). For example, when n is odd, the conjecture is explicitly verified in [7] for $\alpha(n) \leq 7$. On the other hand, the non-immersion $\mathbb{C}P^n \not\subseteq \mathbb{R}^{4n-2\alpha(n)+1}$ is known to hold for $n \equiv 0 \pmod 2$ and $\alpha(n) \equiv 2, 3 \pmod 4$. After the success in [8], this suggests that the methods in this paper could be adapted to show non-immersions of the form $L(2^k)^{2n+1} \not\subseteq \mathbb{R}^{4n-2\alpha(n)+2}$ (or even with larger Euclidean spaces), for $k \gg 0$ and suitable values of n. We close the paper by observing (Remark 4.7 below) that this is not the case, even if we replace ku by any other oriented cohomology theory. This fact gives little extra support to the above conjecture and to the philosophy in Remark 2.1.

Remark 4.7 As in Proposition 4.1, the calculations in Section 3 show that the h^*-Euler class (8) associated to an immersion $L(2^k)^{2n+1} \subseteq \mathbb{R}^{4n-2d}$ lies in

$$h^*[x, y]/(x^{n+1}, y^{m+1}, [2^k](x), [2](y)),$$

where $m = 2^L - 2n + d - 1$, and $[2^k]$ and $[2]$ are the corresponding series for h^*. Then, the well known equation $2^{i+1}y^{m-i} = 0$ $(0 \leq i < m)$ and Lemma 4.2 imply that all the summands in the expression (8) vanish if $d < \alpha(n)$.

References

[1] J. F. Adams. *Stable Homotopy and Generalised Homology.* University of Chicago, Press, 1974.

[2] L. Astey. *Secciones de haces sobre el proyectivo real.* PhD thesis, Centro de Investigación y Estudios Avanzados del IPN, 1978.

[3] L. Astey. Geometric dimension of bundles over real projective spaces. *Quart. J. Math. Oxford Ser. (2)*, 31:139–155, 1980.

[4] M. F. Atiyah. *K-Theory.* Advanced Book Classics. Addison-Wesley, 1967.

[5] A. J. Berrick. Nonimmersions of lens spaces with 2-torsion. *Trans. Amer. Math. Soc.*, 224(2):399–405, 1976.

[6] R. L. Cohen. The immersion conjecture for differentiable manifolds. *Annals of Math.*, 122:237–328, 1985.

[7] M. C. Crabb. Immersing projective spaces in Euclidean space. *Proc. Royal Soc. Edinburgh*, 117A:155–170, 1991.

[8] D. M. Davis. A strong nonimmersion theorem for real projective spaces. *Ann. of Math. (2)*, 120(3):517–528, 1984.

[9] D. M. Davis. Immersions of projective spaces: a historical survey. *Contemp. Math.*, 146:31–37, 1993.

[10] D. M. Davis and M. E. Mahowald. Immersions of complex projective spaces and the generalized vector field problem. *Proc. London Math. Soc. (3)*, 35(2):333–344, 1977.

[11] J. González. 2-divisibility in the Brown-Peterson $[2^k]$-series. To appear in *J. Pure and Appl. Algebra*.

[12] M. W. Hirsch. Immersion of manifolds. *Trans. Amer. Math. Soc.*, 93:242–276, 1959.

[13] T. Kobayashi and M. Sugawara. Note on KO-rings of lens spaces mod 2^r. *Hiroshima Math. J.*, 8(1):85–90, 1978.

[14] B. J. Sanderson. Immersions and embeddings of projective spaces. *Proc. London Math. Soc. (3)*, 14:137–153, 1964.

[15] F. Sigrist and U. Suter. On immersions of $\mathbb{C}P^n$ in $\mathbb{R}^{4n-2\alpha(n)}$. *Lecture Notes in Math.*, 678:106–115, 1978.

[16] H. J. Song and W. S. Wilson. On the nonimmersion of products of real projective spaces. *Trans. Amer. Math. Soc.*, 318(1):327–334, 1990.

[17] W. J. Wong. Powers of a prime dividing binomial coefficients. *Amer. Math. Monthly*, 96(6):513–517, 1989.

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