The Last Lecture: Computability questions in the sphere packing problem

talk by Matilde Marcolli

final closing lecture of
Manin's Seminar
"Algebra, Geometry, and Physics"
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Yuri's last mathematical work (and our last joint work)

- Yuri I. Manin and Matilde Marcolli, *Computability questions in the sphere packing problem*, arXiv:2212.05119 (December 9, 2022)
- Yuri I. Manin and Matilde Marcolli, *Cohn-Elkies functions from Gabor frames*, arXiv:2212.06778 (December 13, 2022)

These were supposed to be parts of a longer project.
I will focus only on the first part, that Yuri liked most

The sphere packing problem:

- Dimension n: identify packings $\mathcal{P} \subset \mathbb{R}^n$ of spheres S^{n-1} that achieve maximal possible density $\Delta_{\mathcal{P}}$

Special types of packings:

- *lattice packings* $\mathcal{P}_L$: centers of spheres at points of a lattice $L \subset \mathbb{R}^n$
- *periodic packings* $\mathcal{P}_\Sigma$: spheres centered at the points of a *periodic set* (finite collection of translates of a lattice)

$$\Sigma = \cup_{i=1}^N v_i + L$$

size $N =: \sigma(\Sigma)$ min number N of translations describing Σ

densities $\delta_{\mathcal{P}} := \Delta_{\mathcal{P}}/\mathrm{Vol}(B_1^n(0))$

$$\delta_L = \left(\frac{\ell_L}{2}\right)^n \frac{1}{|L|}$$

ℓ_L shortest length of L; assume covolume $|L|$ fixed

$$\delta_\Sigma = \frac{N\,\ell_\Sigma^n}{2^n|L|} \quad \text{with} \quad \ell_\Sigma = \min_{\lambda\in L, i,j=1,\ldots,N} \|\lambda + v_i - v_j\|\,.$$

Examples of Explicit Solutions (very rare)

- dimensions 1, 2, 3, 8, and 24: optimal lattice packing is also optimal packing
- $n = 3$ Kepler conjecture (Hales); $n = 8$ E_8-lattice (Vlazovska); $n = 24$ Leech lattice (Vlazovska et al.)
- dimension $n = 10$ known that the max density realized by periodic packing with $N = 40$ translations

The question

Sets of dimensions OPT_A:

- $\mathrm{OPT}_{\mathrm{Latt}}$ set of dimensions $n \in \mathbb{N}$ where max sphere packing density realized by a lattice $L \subset \mathbb{R}^n$
- $\mathrm{OPT}_{\mathrm{Per}}$ set of dimensions $n \in \mathbb{N}$ where max sphere packing density realized by a periodic set $\Sigma \subset \mathbb{R}^n$
- $\mathrm{OPT}_{\mathrm{Per,N}}$ set of dimensions $n \in \mathbb{N}$ where max sphere packing density realized by a periodic set of uniformly bounded size $\sigma(\Sigma) \leq N$
- $\mathrm{OPT}_{\mathrm{Per,F}}$ set of dimensions $n \in \mathbb{N}$ where max sphere packing density realized by a periodic set $\Sigma \subset \mathbb{R}^n$ with $\sigma(\Sigma) \leq F(n)$ for a total recursive function $F : \mathbb{N} \to \mathbb{N}$

Computability properties of these sets?

Expected behavior (Conjectural)

- set $\mathrm{OPT}_{\mathrm{Latt}}$ is expected to be finite
- (Zassenhaus conjecture) $\mathrm{OPT}_{\mathrm{Per}}$ is expected to be all $\mathbb{N}$

if these hold then these two sets are computable, and question only for intermediate sets $\mathrm{OPT}_{\mathrm{Per}\leq N}$

however not much is known about this expected behavior

Hales' observation: when sphere packing solved, it is through sufficient but very non-necessary condition... does not provide a priori proof of decidability of sphere packing problem in a given dimension

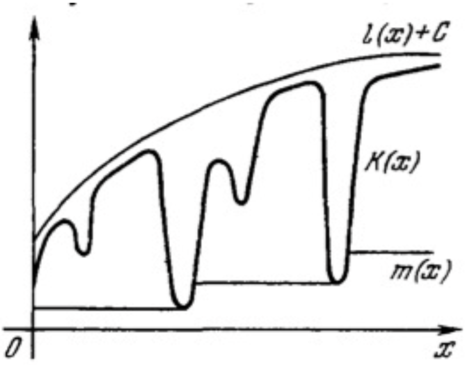
Undecidable problems related to tilings (and to formal languages: e.g. comparing context free grammars) can affect comparing and optimizing (classes of) sphere packings

how to investigate the computability properties of these sets?

Kolmogorov complexity

- Kolmogorov complexity $K(x)$ is minimal length of a program generating x in a Turing machine
- any program that produces a description of x is an upper bound on Kolmogorov complexity $K(x)$: shortest description of x is most compressed form
- good computable upper bounds for Kolmogorov complexity: using any data compression algorithms
- but not lower bounds: non-computability, halting problem
- list programs P_k (increasing lengths) and run through Turing machine: if machine halts on P_k with output x then $\ell(P_k)$ is an upper bound; but there can be earlier P_j on which the machine hasn't halted yet: $K(x)$ computable if can tell exactly on which programs P_k the Turing machine halts (undecidable halting problem)
- mild form of non-computability: (class Σ^0_1 in the arithmetical hierarchy, same non-computability as the halting problem)

the problem of lower bounds for Kolmogorov complexity



with $m(x) = \min_{y \geq x} \mathcal{K}(y)$

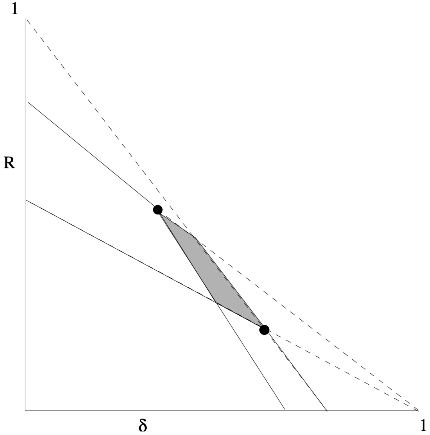
Computability question for error correcting codes

- Yu.I. Manin, *What is the maximum number of points on a curve over $\mathbb{F}_2$?* J. Fac. Sci. Tokyo, IA, Vol. 28 (1981), 715–720.
- Yu.I. Manin, M. Marcolli, *Error-correcting codes and phase transitions*, Mathematics in Computer Science (2011) 5:133–170.
- Yu.I. Manin, *A computability challenge: asymptotic bounds and isolated error-correcting codes*, in "Computation, physics and beyond," pp.174–182, Lecture Notes in Comput. Sci., 7160, Springer, 2012.
- Yu.I. Manin, M. Marcolli, *Kolmogorov complexity and the asymptotic bound for error-correcting codes*, Journal of Differential Geometry, Vol.97 (2014) 91–108

- (Manin 1981): existence of asymptotic bound in the geography of error correcting codes (code parameters (R, δ): good encoding/decoding)
- (Manin-M. 2011): characterization of asymptotic bound as a phase transition (dense code points with infinite multiplicity below, isolated code points with finite multiplicity above)
- (Manin 2011): computability question for asymptotic bound
- (Manin-M. 2012): non-computability of asymptotic bound can only be as bad as Kolmogorov complexity (bound computable given an oracle that orders codes by their Kolmogorov complexity)

Key idea: use the same method to show that any non-computability that may occur in the sphere packing problem is of the same nature (oracle-computable given Kolmogorov complexity) key idea: asymptotic bound existence

spoiling operation on codes and controlling quadrangles



$R = \alpha_q(\delta)$ continuous decreasing function with $\alpha_q(0) = 1$ and $\alpha_q(\delta) = 0$ for $\delta \in [\frac{q-1}{q}, 1]$

key argument in the case of codes

- given computable sets X, Y with computable enumerations ν_X, ν_Y, and a computable (total recursive) function $f : X \to Y$
- sets Y_{fin} and Y_∞ with finite/infinite preimage are oracle-computable given an oracle that orders the points of X by increasing Kolmorogov complexity
- can algorithmically construct these sets by replacing at each step an infinite search for a next preimage with a finite search among points of X with Kolmogorov complexity bounded by a function of $\nu_Y(f(x))$: for

$$n(x) := \#\{x' \in X \,|\, \nu_X^{-1}(x') \leq \nu_X^{-1}(x) \text{ and } f(x') = f(x)\}$$

there is a unique $x_m \in X$ with $y = f(x_m)$ and $n(x_m) = m$, and a constant $c > 0$, such that

$$K(x_m) \leq c\, \nu_Y^{-1}(y)\, m \log(\nu_Y^{-1}(y)\, m)$$

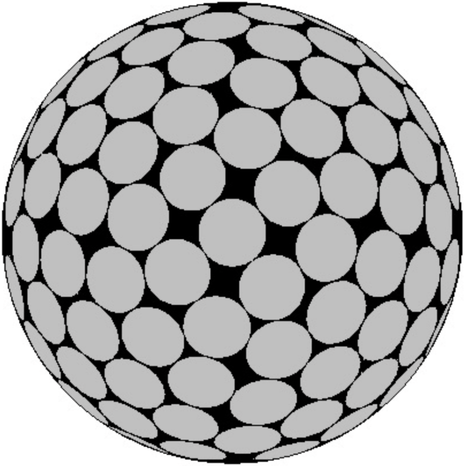
Spherical Codes: intermediate step between codes and sphere packings

- spherical code: finite set X of points on unit sphere $S^{n-1} \subset \mathbb{R}^n$
- spherical code X has minimal angle ϕ if $\forall x \neq y \in X$
$$\langle x, y \rangle \leq \cos \phi$$
- $A(n, \phi) =$ max number of points on S^{n-1} minimal angle ϕ

Space of code parameters

- usual error correcting codes $(R, \delta) \in [0,1]^2 \cap \mathbb{Q}$
- spherical codes:
  - code rate $R = n^{-1} \log_2 \#X$
  - minimum angle $\phi = \phi_X$ (or $\cos \phi$)
  - space $\mathbb{R}_+ \times [0, \pi]$
- unbounded: ϕ smaller maximal number of points $A(n, \phi)$ grows, so R unbounded near $\phi \to 0$

spherical codes



asymptotic behavior for large sphere dimension n

Asymptotic bound

- Yuri I. Manin, Matilde Marcolli, *Asymptotic bounds for spherical codes*, Izv. Math. 83 (2019), no. 3, 540–564.

There is an asymptotic bound for spherical codes

$$\Gamma = \{(R = \alpha(\phi), \phi) \,|\, \alpha(\phi) = \sup\{R \in \mathbb{R}_+ \,:\, (R, \phi) \in \mathcal{U}\}\,\}$$

with $\alpha(\phi) = 0$ if $\{R \in \mathbb{R}_+ \,:\, (R, \phi) \in \mathcal{U}\} = \emptyset$ boundary of region of points surrounded by a 2-ball densely filled by code parameters

$$\mathcal{U} = \{P = (R, \phi) \,|\, \exists \epsilon > 0 \,:\, B(P, \epsilon) \subset \mathcal{A}\}$$

similar argument (spoiling operations on codes)

characterization of region $\mathcal{U}$

- code point $P = (R, \phi) \notin \Gamma$ is in region $\mathcal{U}$ if and only if it there exists a sequence X_k of spherical codes $X_k \subset S^{n_k - 1}$ with $n_k \to \infty$ and $(R_{X_k}, \phi_{X_k}) = (R, \phi)$
- equivalent description: $\mathcal{SC}$ set of spherical codes, $P\mathcal{SC} := \mathbb{R}_+ \times [0, \pi]$ set of code parameters, $\mathbb{P} : \mathcal{SC} \to P\mathcal{SC}$ maps spherical codes to code parameters $\mathbb{P}(X) = (R_X, \phi_X)$, dimension function $D(X \subset S^n) = n$

$$\mathcal{U} = P\mathcal{SC}_{\infty,\mathbb{N}} = \{(R, \phi) \in P\mathcal{SC} \,|\, \#D(\mathbb{P}^{-1}(R, \phi)) = \infty\}$$

Main difference: spherical codes have continuous parameters unlike ordinary q-ary error correcting codes

- Γ not boundary of full region of accumulation points
- asymptotic bound only nontrivial in the "small angle region"
    - small angles region: $0 \leq \phi \leq \pi/2$
    - large angle region: $\pi/2 < \phi \leq \pi$
- in large angle region Rankin bound just gives for $n \to \infty$

$$R = \frac{\log_2 \# X}{n} \leq \frac{\log_2 A(n,\phi)}{n} \to 0, \qquad \pi/2 \leq \phi \leq \pi$$

- in small angle region *Kabatiansky–Levenshtein bound*

$$R \leq \frac{\log_2 A(n,\phi)}{n} \leq H(\phi)$$

$$H(\phi) = \frac{1+\sin\phi}{2\sin\phi}\log_2\left(\frac{1+\sin\phi}{2\sin\phi}\right) - \frac{1-\sin\phi}{2\sin\phi}\log_2\left(\frac{1-\sin\phi}{2\sin\phi}\right)$$

so asymptotic bound in this undergraph (unbounded region for $\phi \to 0$)

Spherical codes and sphere packings

- code density Δ_X of spherical code $X \subset S^{n-1}$ fraction of area of S^{n-1} covered by $\#X$ spherical caps of angular radius $\phi_X/2$

$$\Delta_X = \frac{\#X \cdot S(n, \phi_X)}{S_n}$$

with $S_n = n\pi^{n/2}/\Gamma(1+n/2) = \mathrm{Area}(S^{n-1})$ and spherical cap area

$$S(n,\phi) = S_{n-1} \int_0^{\phi/2} \sin^{n-2}(x)dx$$

- Δ_X depends on X through code parameters (R_X, ϕ_X)

- max density given ϕ is

$$\Delta(n,\phi) = A(n,\phi)\frac{S(n,\phi)}{S_n}$$

- limit density

$$\Delta_n^{codes} := \lim_{\phi\to 0} \Delta(n,\phi)$$

- related to maximal sphere packing density

$$\Delta_n^{codes} = \Delta_{n-1}^{\max}$$

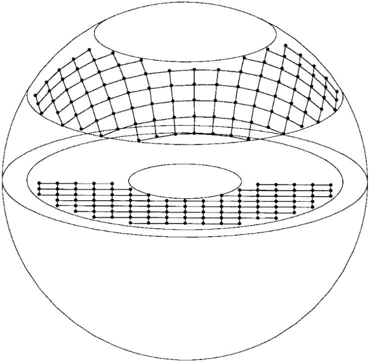
- family X_k of spherical codes $X_k \subset S^{n-1}$ with $\phi_{X_k} \to 0$ as $k \to \infty$ is an asymptotically optimal family if

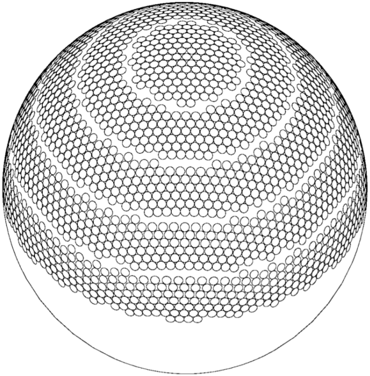
$$\lim_{k\to\infty} \frac{\# X_k}{A(n,\phi_k)} = 1 \quad \text{or equivalently} \quad \lim_{k\to\infty} \frac{\Delta_{X_k}}{\Delta(n,\phi_k)} = 1 \,.$$

constructions of optimal families: wrapped spherical codes

- given a sphere paking $\mathcal{P} \subset \mathbb{R}^{n-1}$ can map annular regions of $\mathbb{R}^{n-1}$ to annular regions on S^{n-1} with low distortion (depending on a choice of angles ϑ)
- when applied to a rescaled family $\mathcal{P}_d$ with $d \to 0$, with constant density $\Delta_{\mathcal{P}}$ gives a family of spherical codes $X_{\mathcal{P}_d,\vartheta_d}$ with
$$\lim_{d\to 0} \Delta_{X_{\mathcal{P}_d,\vartheta_d}} = \Delta_{\mathcal{P}}$$
- if $\mathcal{P}$ is a sphere packing that realizes the maximal density then $X_{\mathcal{P}_d,\vartheta_d}$ is an asymptotically optimal family
$$\lim_{d\to 0} \Delta_{X_{\mathcal{P}_d,\vartheta_d}} = \Delta_{n-1}^{\max}$$

J. Hamkins, K. Zeger, *Asymptotically dense spherical codes. I. Wrapped spherical codes*, IEEE Trans. Inform. Theory, 43:6 (1997), 1774–1785.





Computability in metric spaces

Can still ask computability question for the asymptotic bound for spherical codes: but *continuous parameters*

- metric space (M, d) open set $U \subseteq M$ recursively enumerable if $\exists$ computable sequences $\{x_k\}_{k\in\mathbb{N}} \subset M$ and $\{r_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^*_+$

$$U = \bigcup_{k\in\mathbb{N}} B_d(x_k, r_k)$$

- closed subset $S \subseteq M$ recursively enumerable if $\exists$ computable sequence $\{x_k\}_{k\in\mathbb{N}} \subset M$ dense in S
- open or closed A computable if A and $M \smallsetminus A$ recursively enumerable

- computable metric space (M, d, μ) with metric space (M, d) and sequence $\mu : \mathbb{N} \to M$ dense in M with computable

$$d_\mu : \mathbb{N}^2 \to \mathbb{R} \quad \text{with} \quad d_\mu(i,j) = d(\mu_i, \mu_j)$$

- in (M, d, μ) Cauchy name for a point $x \in M$: function $p : \mathbb{N} \to \mathbb{N}$ such that $d(x, \mu_{p(k)}) < 2^{-k}$, for all $k \in \mathbb{N}$
- Cauchy names determine partially defined function $\delta_X : \mathrm{Dom}(\delta_X) \subset \mathbb{N}^{\mathbb{N}} \to X$ with $\delta_X(p) = x$ iff p is a Cauchy name for x
- function $f : (M, d, \mu) \to (M', d', \mu')$ is computable if there is a computable function $\Phi_f : \mathrm{Dom}(\Phi_f) \subset \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ that maps a Cauchy name for x to a Cauchy name for $f(x)$

useful facts of computable analysis:

Note: computable function $f : \mathbb{N} \to M$, with (M, d, μ) a computable metric space, if open or closed computable subset $A \subset M$ then $f^{-1}(A) \subset \mathbb{N}$ computable set in the usual sense

- effectively locally connected computable metric space (M, d, μ): any $x \in M$ any open ball $B_d(x, r)$, effectively (through algorithm/computable function) find connected open set U with $x \in U$ and $U \subseteq B(x, r)$
- graph theorem $f : (M, d, \mu) \to (M', d', \mu')$ with (M, d, μ) effectively locally connected then f computable iff $\Gamma(f) \subset M \times M'$ computable

oracle-computability with Kolmogorov complexity

- (X, d_X, μ_X), (Y, d_Y, μ_Y) and (Z, d_Z, μ_Z) computable metric spaces
- computable functions $f : X \to Y$ and $g : X \to Z$ (metric sense) with $f(\mu_X) \subset \mu_Y$ and $g(\mu_X) \subset \mu_Z$
- sets
$$Y_{fin,Z} = \{y \in f(X) \,|\, \#g(f^{-1}(y)) < \infty\}$$
$$Y_{\infty,Z} = \{y \in f(X) \,|\, \#g(f^{-1}(y)) = \infty\}$$
- closed sets $\bar{\mu}_{Y,fin,Z} = \overline{\mu_Y \cap Y_{fin,Z}}$ and $\bar{\mu}_{Y,\infty,Z} := \overline{\mu_Y \cap Y_{\infty,Z}}$ are computable, given the existence of an oracle that orders the points of μ_X by increasing Kolmogorov complexity.

computable metric space of spherical codes

- space of all spherical codes

$$\mathcal{SC} = \bigsqcup_n \mathcal{SC}_n = \bigsqcup_n \mathrm{Config}(S^{n-1}) = \bigsqcup_{n,N} \mathrm{Config}_N(S^{n-1})$$

$$\mathrm{Config}(S^{n-1}) = \bigsqcup_{N\geq 1} \mathrm{Config}_N(S^{n-1})$$

$$\mathrm{Config}_N(S^{n-1}) = ((S^{n-1})^N \smallsetminus \Delta_N)/S_n$$

- round metric $d_{S^{n-1}}(x_i, y_i)$ on S^{n-1} normalized to diameter 1

$$d_{n,N}(X,Y) = \frac{1}{N}\sum_{i=1}^N d_{S^{n-1}}(x_i, y_i)$$

then $d(X,Y) = 1$ if X, Y not same N or not same n and $d(X,Y) = d_{n,N}(X,Y)$

- $\mu_{\mathcal{SC}}$ codes with rational angular coordinates

computable spherical code parameters

- space $\mathcal{PSC} = \mathbb{R}_+ \times [0, \pi]$ of code parameters of spherical codes, Euclidean metric and $\mu_{\mathcal{PSC}}$ computable subset of points $P = (R, \phi)$ with $R \in \mathbb{Q} \log_2 \mathbb{N}$ and $\phi \in \mathbb{Q}$
- function $\mathbb{P} : \mathcal{SC} \to \mathcal{PSC}$ mapping a code X to its code parameters $\mathbb{P}(X) = (R_X, \phi_X)$ is a computable (in metric sense)
- dimension function $D : \mathcal{SC} \to \mathbb{N}$ code dimension $D(X \subset S^n) = n$ also computable

oracle computability of the asymptotic bound for spherical codes

- use characterization of $\mathcal{U}$ in terms of computable functions $\mathbb{P}$ and D

$$\mathcal{U} = \{(R,\phi) \in P\mathcal{SC} \,|\, \#D(\mathbb{P}^{-1}(R,\phi)) = \infty\}$$

- oracle that orders the codes in $\mu_{\mathcal{SC}}$ by Kolmogorov complexity
- get oracle computability of $\mathcal{U}$ and of its boundary, the asymptotic bound

$$\Gamma = \{(R,\phi) \,|\, R = \alpha(\phi)\}$$

- also oracle computability of sublevel sets $\mathcal{U} \cap \Gamma_\epsilon$

$$\Gamma_\epsilon = \{(R,\phi) \,|\, R \geq \alpha(\phi) - \epsilon\}$$

Sphere packings and the asymptotic bound of spherical codes

- $\mathcal{P} \subset \mathbb{R}^{n-1}$ non-optimal sphere packing: *discrepancy* $1-\gamma$

$$\gamma := \frac{\Delta_{\mathcal{P}}}{\Delta_{n-1}^{\max}}$$

- for small d and large n code points of wrapped codes $\mathbb{P}(X_{\mathcal{P}_d,\vartheta_d})$ are in $\Gamma_{\frac{-\log_2\gamma}{n}+\epsilon}$

$$R_{X_{\mathcal{P}_d,\vartheta_d}} \sim \frac{\log_2 A(n,\phi_{X_{\mathcal{P}_d,\vartheta_d}})}{n} + \frac{\log_2\gamma}{n} \sim \alpha(\phi_{X_{\mathcal{P}_d,\vartheta_d}}) + \frac{\log_2\gamma}{n}$$

Sphere packings and metric computability

- space $\mathcal{SP}$ of sphere packings is a computable metric space, with choice of metric and $\mu_{\mathcal{SP}}$ compatible with wrapped codes (using a computable dense set of ϑ)

$$d_{\mathcal{SP}}(\mathcal{P}, \mathcal{P}') = \sup_{\vartheta} d_{\mathcal{SC}}(X_{\mathcal{P},\vartheta}, X_{\mathcal{P}',\vartheta})$$

- collection of maps $\mathbb{P}_{n,\vartheta_d} : \mathcal{SP}_{n-1} \to \mathbb{R}_+ \times [0, \pi]$ (sphere packings to code point of wrapped code)

$$\mathbb{P}_{n,\vartheta_d}(\mathcal{P}) = \mathbb{P}(X_{\mathcal{P}_d,\vartheta_d}) = (R_{X_{\mathcal{P}_d,\vartheta_d}}, \phi_{X_{\mathcal{P}_d,\vartheta_d}})$$

- these maps a computable (in the metric sense)

- computable sequence $\vartheta_{d_k} \to 0$
- oracle-computable closed set $\Gamma_\epsilon = \{R \geq \alpha(\phi) - \epsilon\}$
- also an oracle-computable set

$$\mathfrak{P}_n := \bigcap_{k \geq k_0} \mathbb{P}^{-1}_{n,\vartheta_{d_k}}(\Gamma_\epsilon) \subset \mathcal{SP}_{n-1}$$

for computable maps $\mathbb{P}_{n,\vartheta_{d_k}} : \mathcal{SP}_{n-1} \to \mathcal{PSC}$

- $A \in \{\mathrm{Latt}, \mathrm{Per} \leq N, \mathrm{Per}_F, \mathrm{Per}\}$ and $\mathcal{SP}^A_{n-1}$ be the space of sphere packings of type A
- $\mathcal{SP}^{A,\max}_{n-1} \subset \mathcal{SP}^A_{n-1}$ subset of packings that maximize density among those of type A, with $\mathcal{SP}^{A,\max} = \cup_n \mathcal{SP}^{A,\max}_{n-1}$
- $\mathcal{SP}^{\mathrm{Latt},\max}_{n-1}$ identified with algorithmically computable (Voronoi algorithm) set of vertices of Ryshkov polytope $\mathcal{R}_n$
- for N-periodic sets generalized Ryshkov polytopes $\mathcal{R}_{N,n}$
- conjecture (Andreanov-Kallus): generalized Ryshkov polytope $\mathcal{R}_{N,n}$ has finitely many vertices, that can be algorithmically determined

oracle computability of the OPT_A sets

- for $A \in \{\mathrm{Latt}, \mathrm{Per} \leq N, \mathrm{Per}_F, \mathrm{Per}\}$

$$\mathrm{OPT}_A = \{n \in \mathbb{N} \,|\, \mathcal{SP}^{A,\max} \cap \mathfrak{P}_n \neq \emptyset\}$$

- computability of $\mathcal{SP}^{A,\max}$ (conditional to Ryshkov polytope computability) and oracle computabiity of $\mathfrak{P}_n$ give oracle computability of image under computable dimension function $D : \mathcal{SP} \to \mathbb{N}$ hence of OPT_A

Computability or non-computability? need to distinguish different A, case of $\mathrm{Per} \leq N$ seems most interesting to look at

SUNT LACRIMAE RERUM