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A NON-COMPACT CONVEX HULL IN GENERAL NON-POSITIVE CURVATURE

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Abstract. In this article, we are interested in metric spaces that satisfy a weak non-positive curvature condition in the sense that they admit a conical bicombing. Recently, these spaces have begun to be studied in more detail, and a rich theory is beginning to emerge. In this paper, we contribute to this study by constructing a complete metric space X with a conical bicombing σ such that there is a finite subset of X whose closed σ-convex hull is non-compact. In CAT(0)-geometry, the analogous statement is an open question, i.e. it is not known whether closed convex hulls of finite subsets of complete CAT(0) space are compact or not. This question goes back to Gromov. Our result shows that to obtain a positive answer to Gromov's question, more than just the convexity properties of the metric must be used. The constructed space X has the additional property that there is an integer n such that it is an initial object in the category of convex hulls of n-point sets. Thus, roughly speaking, X can be thought of as the largest possible convex hull of n-points.

1. Introduction

A family $\sigma = (\sigma_{xy})_{x,y\in X}$ of geodesics $\sigma_{xy}\colon [0,1] \to X$ of a metric space X with the property that $\sigma_{xy}(0) = x$ and $\sigma_{xy}(1) = y$ for all x, $y \in X$ is called (geodesic) bicombing. The terms combing and bicombing have been coined by Thurston [15, p. 84] and variants of it have originally been studied in the context of geometric group theory (see [1, 20, 25]). In the present article, we are mainly concerned with metric spaces that admit bicombings whose geodesics share properties with geodesics in non-positively curved spaces such as CAT(0) spaces or, more generally, Busemann spaces. Following Descombes and Lang [11], we say that a bicombing σ is *conical* if

$$d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1-t)d(x,x') + td(y,y') \tag{1.1}$$

for all x, x', y, $y' \in X$ and all $t \in [0,1]$. We remark that in CAT(0) spaces the function $t \mapsto d(\gamma(t), \eta(t))$ is convex on $[0,1]$ for all linearly reparametrized geodesics γ, $\eta\colon [0,1] \to X$. In particular, the unique geodesics of a CAT(0) space form a conical bicombing. Recently, conical bicombing have gained some interest and have begun to be studied in more detail. This is partly

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due to some applications in the context of Helly groups (see [9, 24]). Moreover, they also naturally occur as target spaces in the context of Lipschitz extension problems (see [30, 33]). Indeed, a metric space with a conical bicombing has many more properties that are usually associated to 'non-positive curvature'. See Theorem 2.2 below for a collection of some of those results. In this article, we study convex hulls in metric spaces with a conical bicombing. The starting point of our considerations is the following intriguing question regarding convex hulls in CAT(0) spaces due to Gromov (see [20, 6.B_1(f)]).

Question 1.1 (Gromov). *Let X be a complete* CAT(0) *space and $K \subset X$ a compact subset. Is it true that the closed convex hull of K is compact?*

Gromov's question has been popularized by Petrunin (see [36] and also [37, p. 77]). Since the closed convex hull of K has the same diameter as K, it is not difficult to see that Question 1.1 has a positive answer if X is proper. However, for non-proper spaces it seems to be very difficult to answer. In fact, already for three-point subsets the question is completely open. We remark that using standard techniques from CAT(0)-geometry one can show that Question 1.1 has a positive answer if and only if it has a positive answer for finite subsets; see [13, Lemma 2.18]. However, also for finite subsets the closure of the convex hull needs to be considered. Indeed, already the convex hull of three points is not closed if the points do not lie on a geodesic and are contained in a generic complete Riemannian manifold of dimension ≥ 3 (see [29, Corollary 1.2]).

Clearly, Question 1.1 can also be stated for spaces with a conical bicombing. Let σ be a conical bicombing on a complete metric space X. We say that $A \subset X$ is σ-convex if for all x, $y \in A$, the geodesic σ_{xy} is contained in A. We consider the closed σ-convex hull of A,

$$\sigma\text{–conv}(A) = \bigcap C,$$

where the intersection is taken over all closed σ-convex subsets $C \subset X$ containing A. Our main result shows that in the setting of spaces with conical bicombings the analogue of Gromov's question has a negative answer.

Theorem 1.2 (Non-compact convex hull). *There exists a complete metric space X with a conical bicombing σ such that there is a finite subset of X whose closed σ-convex hull is not compact.*

Thus, to obtain a positive answer to Gromov's question, more than just the convexity properties of the metric must be used. We remark that there is a metric space X as in Theorem 1.2 which is additionally an injective metric space, see Theorem 5.2 below. Injective metric spaces are prime examples of metric spaces with a conical bicombing (see [28, Proposition 3.8]). Descombes and Lang [11] showed that injective metric spaces of finite combinatorial dimension admit a unique bicombing which satisfies a stronger convexity property than (1.1). More precisely, such spaces admit a unique convex bicombing which is furthermore consistent. The exact definitions are recalled in Section 2.3. We do not know whether Theorem 1.2 holds also for such bicombings.

The construction of the metric space X in Theorem 1.2 is discrete in nature. Indeed, X is the metric completion of the direct limit V of a sequence of finite graphs $G_n = (V_n, E_n)$. The morphisms in question are injective maps $V_n \to V_m$, which are 1-Lipschitz with respect to an appropriate scaling of the shortest-path metric on G_n. The conical bicombing σ on X is then constructed using a midpoint map $m \colon V \times V \to V$ which satisfies a discrete version of (1.1). More details about the construction of X can be found in Section 1.1. The original idea behind this construction was to ensure the existence of the initial object X_0 in the following theorem. Indeed, the metric space X in Theorem 1.2 can be taken to be X_0 for any $n_0 \geq 2$.

Theorem 1.3 (Initial object)**.** *Let $n_0 \in \mathbb{N}$. Then there exists a complete metric space X_0 with a conical bicombing such that whenever $A \subset Y$ is an n_0-point subset of some complete* CAT(0) *space Y, then there exists a Lipschitz map $\Phi \colon X_0 \to Y$ such that $\Phi(X_0)$ is convex and contains A.*

We actually prove a stronger statement than Theorem 1.3. Instead of complete CAT(0) spaces Y, more general non-positively curved target spaces such as Busemann spaces can be considered. See Theorem 5.3 below for the exact statement. We remark that, by construction, $\operatorname{conv}(A) \subset \operatorname{closure}(\Phi(X_0))$. Therefore, if $\Phi(X_0)$ is precompact, then the closed convex hull of A is compact. Given this relation, it seems reasonable to suspect that X_0 is not compact. As it turns out, this is indeed the case for every $n_0 \geq 2$; see Theorem 5.1. In addition, it also follows immediately from the construction of X_0 that there is some finite subset $A \subset X_0$ such that $\sigma\text{–}\operatorname{conv}(A) = X_0$. Hence, Theorem 1.2 is a direct consequence of Theorem 5.1.

One may of course wonder whether there also exists such a space X_0 as above, which is in addition a complete CAT(0) space. The existence of such spaces would reduce Gromov's question to the problem of deciding whether these spaces X_0 are compact or not. If they are all compact, then Question 1.1 would have a positive answer. On the other hand, the non-compactness of X_0 for some $n_0 \in \mathbb{N}$ would give a negative answer. However, our proof does not seem to be directly amenable for generating CAT(0) spaces.

1.1. **Strategy of proof.** In the following, we give a brief overview of how the metric space X_0 in Theorem 1.3 is constructed as a direct limit of a sequence of graphs $G_n = (V_n, E_n)$. We fix $n_0 \in \mathbb{N}$ and we let G_0 denote the null graph and G_1 the complete graph on n_0 vertices. The basic idea is that we have an increasing sequence of vertex sets

$$V_0 \subset V_1 \subset \cdots$$

such that the vertex set V_n is obtained from V_{n-1} by appending all possible midpoints, i.e.,

$$V_n = V_{n-1} \cup \mathrm{midpoints}(V_{n-1}).$$

The formal definition of the midpoint construction $m(a, b)$ can be found in (3.1). Any connected graph can naturally be viewed as a metric space by equipping it with the shortest-path metric (see (2.1) for the definition). The edge set E_n is now defined such that the shortest-path metric of G_n satisfies a discrete version of (1.1) for $x = x'$. Loosely speaking, E_n is obtained by considering cones in G_{n-1}, and then the 'cone midpoints' in G_n are adjacent, and indeed every edge in G_n arises in this way. More concretely, we have $x \sim y$ in G_n if and only if there there exists a vertex $v \in V_{n-1}$ (the cone point) and an edge $u \sim w$ in G_{n-1} (the base) such that $x = \mathrm{midpoint}(v, u)$ and $y = \mathrm{midpoint}(v, w)$. Hence,

$$\mathrm{midpoint}(v, u) \sim \mathrm{midpoint}(v, w).$$

whenever $v \in V_{n-1}$ and $u \sim w$ in G_{n-1}. This is illustrated in Figure 1.1.

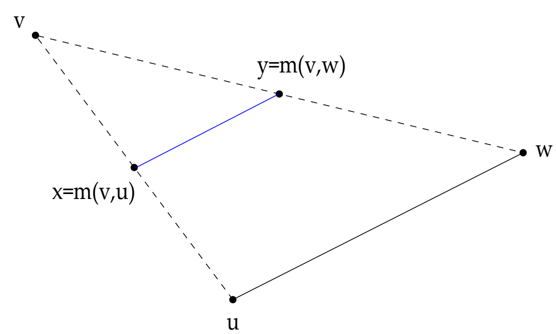

FIGURE 1.1. Cone midpoints are adjacent.

For $n_0 = 2$ and $n = 1, 2, 3, 4$, the graphs $G_n = (V_n, E_n)$ obtained by applying this rule are shown in Figure 3.1. The graph G_5 has already 68 vertices and 184 edges and quite an intricate structure.

Letting d_{G_n} denote the shortest-path metric of G_n, we find by definition of E_n that

$$d_{G_n}\big(\mathrm{midpoint}(x, y), \mathrm{midpoint}(x, z)\big) \leq d_{G_{n-1}}(y, z) \tag{1.2}$$

for all x, y, $z \in V_{n-1}$; see Lemma 3.3. We interpret this as a discrete version of (1.1) with $x = x'$. In particular, as $x = \mathrm{midpoint}(x, x)$, the inclusion $(V_{n-1}, d_{G_{n-1}}) \hookrightarrow (V_n, d_{G_n})$ is 1-Lipschitz. Hence, letting $V = \bigcup_n V_n$, it follows that $\varrho \colon V \times V \to \mathbb{R}$ given by

$$\varrho(x, y) = \lim_{n \to \infty} (\mathrm{diam}\, V_n)^{-1} \cdot d_{G_n}(x, y),$$

defines a semi-metric on V (see Section 2.1 for the definition). We remark that V is the direct limit of the sequence (V_n) with morphisms $V_n \to V_m$, for $n \leq m$, induced by the identity. By construction, V is equipped with a midpoint map $m\colon V \times V \to V$ defined by $m(x, y) = \text{midpoint}(x, y)$. Since $\operatorname{diam} V_n = 2^{n-1}$, it follows because of (1.2) that

$$\varrho(m(x,y), m(x,z)) \leq \frac{1}{2}\varrho(y,z) \tag{1.3}$$

for all x, y, $z \in V$. Now, $X_0 = (X_0, d)$ is defined as the metric completion of the metric space (X, d) associated to (V, ϱ). We prove in Lemma 3.4 that m extends to a map $m\colon X \times X \to X$ such that (1.3) still holds true. It is not difficult to show that such a map m induces a conical bicombing σ on X_0 such that $X_0 = \sigma\text{–conv}(V_1)$; see Lemmas 2.4 and 2.5.

We finish this overview with the main ideas that go into the proof of the non-compactness of X_0. As a first reduction, it is clearly sufficient to show that X is not totally bounded. Now an important observation is that to prove that X has an $(m \cdot 2^{-n})$-separated set of cardinality $r + 1$, it suffices to show that the graph $\overline{G_n^m}$ has an $(r+1)$-clique; see Lemma 4.1. Here, G_n^m denotes the m-th power of G_n and $\overline{G_n^m}$ its complement. This is standard terminology from graph theory, which is recalled in Section 2.2. Thus, by the above, the problem has been completely reduced to the existence of cliques in graph powers of G_n. This opens the field for applications of techniques from extremal graph theory. Indeed, using Turán's theorem, see Theorem 2.1, it is not difficult to show that if for a certain sequence of integers $m(n)$ one has that

$$\liminf_{n\to\infty} \frac{|E(G_n^{m(n)})|}{|V_n|^2} = 0,$$

then X is not totally bounded; see Corollary 4.2. Hence, to conclude the proof we need to show that G_n^m does not contain too many edges. This is achieved by exploiting the explicit construction of E_n. In particular, the number of edges of G_n^m is related to the edge counts of G_{n-1}^a and G_{n-1}^b with $a + b = m$; see Lemmas 4.3 and 4.4. Moreover, one has that

$$\frac{|E_n|}{|V_n|^{1+\varepsilon}} \to 0$$

as $n \to \infty$ for every $\varepsilon > 0$; see Lemma 3.1. Combining these two results, we finish the proof by a simple case distinction. This is done in Section 5. We remark that our proof does no explicitly construct ε-separated sets with arbitrarily large cardinality. We only establish their existence by an application of Turán's theorem. We believe that an explicit construction of such sets would be worthwhile but probably very difficult.

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2. Preliminaries

2.1. **Basic metric notions.** We use $\mathbb{N} = \{1, 2, \dots\}$ to denote the set of positive integers. A non-negative function $\varrho\colon X \times X \to \mathbb{R}$ is called *semi-metric* if it is symmetric, satisfies the triangle inequality and $\varrho(x, x) = 0$ for all $x \in X$. In other words, all axioms of a metric are satisfied except (possibly) the positivity axiom, that is, there might exist distinct x, $y \in X$ such that $\varrho(x, y) = 0$. In the literature, such a function is sometimes also called a pseudometric (see, for example, [8]). However, in the present article we will only use the term semi-metric. Let $X = (X, d)$ be a metric space. We use $\overline{X}$ to denote the metric completion of X. If readability demands it we will sometimes tacitly identify X with its canonical isometric copy in $\overline{X}$. A metric space is said to be *totally bounded* if for every $\varepsilon > 0$ there exists a finite subset $A \subset X$ such that for every $x \in X$ there exists $a \in A$ such that $d(x, a) < \varepsilon$. We recall that X is totally bounded if and only if $\overline{X}$ is compact.

2.2. **Graph theory.** We use standard notation from graph theory as found in [7, 12]. Let $G = (V, E)$ be a graph, that is, V is a (possibly infinite) set and $E \subset \{e \subset V : |e| = 2\}$. If $\{x, y\} \in E$ then we often write $x \sim y$. We let $\overline{G}$ denote the complement graph of G. That is, $\overline{G}$ has vertex set V and $x \sim y$ in $\overline{G}$ if and only if $x \neq y$ and x, y are not adjacent in G. We will also need to consider graph powers of G. Let $m \geq 1$ be an integer. We let G^m denote the *m-th power* of G. By definition, G^m is a graph with vertex set V and distinct vertices x, $y \in V$ are adjacent if and only if there exists a path in G of length at most m that connects x to y. We use the convention that G^0 denotes the empty graph $(V, \varnothing)$. Given an integer $r \geq 1$, we let K_{r+1} denote the complete graph on $(r + 1)$-vertices. The following theorem by Turán is a foundational result in extremal graph theory.

Theorem 2.1 (Turán's theorem)**.** *Let $G = (V, E)$ be a finite graph and $r \geq 1$ an integer. If G does not contain K_{r+1} as a subgraph, then*

$$|E| \leq \big(1 - \frac{1}{r}\big) \cdot \frac{|V|^2}{2}.$$

We will apply this theorem to graphs of the form $\overline{G^m}$ to obtain m-separated sets in G with respect to the shortest-path metric d_G; see Lemma 4.1 and Corollary 4.2. Recall that the shortest-path metric $d_G\colon V \times V \to \mathbb{R}$ is defined by

$$d_G(x, y) = \min\big\{k : (x_0, \dots, x_k) \text{ is a path in } G \text{ form } x \text{ to } y\big\} \tag{2.1}$$

for all x, $y \in V$.

2.3. **Bicombings.** In the following we introduce bicombings and the various properties one can impose on them. We decided to be a little more detailed than would be strictly necessary for the main body of this article; see in particular Theorem 2.2. All definitions appearing below are essentially due to Descombes and Lang (see [11]).

Let X be a metric space. We say that $\sigma\colon [0,1] \to X$ is a *geodesic* if $d(\sigma(s),\sigma(t)) = |s-t| \cdot d(\sigma(0),\sigma(1))$ for all s, $t \in [0,1]$. A map

$$\sigma\colon X \times X \times [0,1] \to X$$

is called *(geodesic) bicombing* if for all x, $y \in X$, the path $\sigma_{xy}(\cdot)\colon [0,1] \to X$ defined by $\sigma_{xy}(t) = \sigma(x,y,t)$ is a geodesic connecting x to y. We remark that, in contrast, a map $\sigma\colon X \times [0,1] \to X$ is called *combing* with basepoint $p \in X$ if for all $x \in X$, the path $\sigma(x,\cdot)$ is a geodesic connecting p to x. However, we will not make use of this definition. Bicombings are also called *system of good geodesics*; see [17, 19, 34]. Clearly, every geodesic metric spaces admits a bicombing. We often consider bicombings in metric spaces that have non-unique geodesics such as, for example, $\mathbb{R}^n$ equipped with the p-norm for $p = 1, \infty$. Therefore, it is useful to formalize some of the natural properties of the bicombing on a uniquely geodesic metric space. We say that σ is *reversible* if $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all x, $y \in X$ and all $t \in [0,1]$. In [5, Proposition 1.3] it is shown that any complete metric space with a conical bicombings also admits a conical reversible bicombing (see also [10] for an earlier result). Furthermore, we say that a bicombing σ is *consistent* if it is reversible and $\sigma(x,y,st) = \sigma(x,\sigma_{xy}(t),s)$ for all x, $y \in X$ and all s, $t \in [0,1]$. Consistent bicombings are used in [18, 23], and a variant of the definition that allows for a bounded error is studied in [14, Definition 2.6]. We do not know if every space with a bicombing also admits a consistent bicombing. This seemingly straightforward question does not seem to be so easy to answer on closer inspection. For proper metric spaces admitting a conical bicombing, it turns out to be true (see [3, Theorem 1.4]).

Descombes and Lang [11] introduced the following two non-positive curvature conditions for a bicombing σ:

(1) if (1.1) holds, then σ is said to be *conical*.
(2) if for all x, y, x', $y' \in X$, the map $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex on $[0,1]$, then σ is called *convex*.

There are many examples of conical bicombings that are not convex (see [11, Example 2.2] and [3, Example 3.6]). However, any consistent conical bicombings is convex. One may wonder if any convex bicombing is automatically consistent. This turns out to be not to be the case, as is demonstrated in [5, Theorem 1.1]. To the authors' knowledge, a relatively simple example of a convex non-consistent bicombings seems to be missing.

The following theorem is a 'state of the art' collection of general facts about spaces that admit a conical bicombing. All of these properties are usually associated with 'non-positive curvature'.

Theorem 2.2. *Let X be a complete metric space admitting a conical bicombing. Then the following holds true.*

(1) *X is contractible,*
(2) *X admits barycenter map in the sense of Sturm* [40]*,*
(3) *all Lipschitz homotopy groups $\pi_k^{\mathrm{Lip}}(X)$ are trivial,*

(4) *X admits an isoperimetric inequality of Euclidean type for $\mathbf{I}_k(X)$.*

Moreover, if X is proper then

(5) *X is an absolute retract,*
(6) *X admits a visual boundary which is a $\mathcal{Z}$-boundary in the sense of Bestvina* [6]*,*
(7) *any subgroup of the isometry group of X with bounded orbits has a non-empty fixed-point set.*

Proof. We prove each item separately. Fix $o \in X$. Clearly, $H\colon X \times [0,1] \to X$ defined by $H(x,t) = \sigma(x,o,t)$ is a homotopy between the identity map on X and the constant map with value o. This shows (1). A proof of (2) can be found in [3, Theorem 2.6]. We proceed by showing (3). A metric space X is called Lipschitz k-connected with constant c if for every $\ell \in \{0,\dots,k\}$, every L-Lipschitz map $f\colon S^\ell \to X$ has a cL-Lipschitz extension $\bar{f}\colon B^{\ell+1} \to X$. Here, S^ℓ, $B^{\ell+1} \subset \mathbb{R}^{\ell+1}$ denote the Euclidean unit sphere and closed Euclidean unit ball, respectively. To prove that $\pi_k^{\mathrm{Lip}}(X)$ is trivial it suffices to show that X is Lipschitz k-connected for some constant c. Therefore, the statement follows, since in [39, Proposition 6.2.2] it is proved that X is Lipschitz k-connected with constant 3. For a proof of (4) we refer to Corollary 1.4 in [41]. Next, we prove (5). Using that X admits a conical bicombing, it is not difficult to show that X is strictly equiconnected. Therefore, it follows from a result by Himmelberg [22, Theorem 4] that X is an absolute retract. The next statement, (6), follows directly from Theorem 1.5 in [3].

To finish the proof, we establish (7). Let Γ be a subgroup of the isometry group of X with bounded orbits. Fix $x_0 \in X$ and consider the orbit $A = \{f(x_0) : f \in \Gamma\}$. In the following we combine results from [3] and [4] to show that the fixed-point set of Γ is non-empty. In view of [4, Theorem 1.2] it suffices to show that X admits a Γ-equivariant conical bicombing. We now use the proof strategy of [3, Lemma 4.5] to show that such a bicombing exists. Let $\mathrm{CB}(X)$ be the set of all conical bicombings on X and for every $x \in X$ let the metric D_x on $\mathrm{CB}(X)$ be given as in [3, Section 4]. We define $\tilde{D} = \sup_{x\in A} D_x$. Clearly, $\tilde{D}$ defines a metric on $\mathrm{CB}(X)$ and by considering the proof of [3, Lemma 4.2] it is straightforward to show that $(\mathrm{CB}(X), \tilde{D})$ is a compact metric space. Let $f \in \Gamma$ and let $F\colon \mathrm{CB}(X) \to \mathrm{CB}(X)$ be defined by $F(\sigma)(x,y,t) = f^{-1}(\sigma(f(x), f(y), t))$. Since $f(A) = A$, it follows that F is distance-preserving if $\mathrm{CB}(X)$ is equipped with $\tilde{D}$. Now, one can argue exactly as in the proof of [3, Lemma 4.5] to conclude that there exists some $\sigma_* \in \mathrm{CB}(X)$ such that $F(\sigma_*) = \sigma_*$ for all $f \in \Gamma$. In other words, σ_* is a Γ-equivariant concial bicombing, as desired. We remark that additional fixed-point results for spaces with a conical bicombing can be found in [26, 27]. □

2.4. **Conical midpoint maps.** In this section we introduce conical midpoint maps and derive some of their basic properties. We are mainly interested in this notion since it can be seen as a discrete analogue of conical bicombings. Indeed, any conical midpoint map on a metric space X induces a conical bicombing on $\overline{X}$. This is discussed at the end of this section.

Definition 2.3. We say that $m\colon X \times X \to X$ is a *conical midpoint map* if for all x, y, $z \in X$, the following holds:

(1) $m(x, x) = x$,
(2) $m(x, y) = m(y, x)$,
(3) $d(m(x, y), m(x, z)) \leq \frac{1}{2} d(y, z)$.

We remark that for midpoints in Euclidean space, the inequality in (3) becomes in fact an equality. It is easy to see that if m is as in Definition 2.3, then $z = m(x_1, x_2)$ is a midpoint of x_1 and x_2. Indeed,

$$d(z, x_i) = d(z, m(x_i, x_i)) \leq \frac{1}{2} d(x_1, x_2)$$

and thus using the triangle inequality, we find that $d(z, x_i) = \frac{1}{2} d(x_1, x_2)$. Hence, a conical midpoint map is a midpoint map in the usual sense.

Furthermore, (3) can be upgraded to a more general inequality involving four points. For all x, y, x', $y' \in X$, one has

$$d(m(x, y), m(x', y')) \leq \frac{1}{2} d(x, x') + \frac{1}{2} d(y, y'). \tag{2.2}$$

This can be seen as follows. Using (2) and the triangle inequality, we get

$$d(m(x, y), m(x', y')) \leq d(m(x, y), m(x, y')) + d(m(y', x), m(y', x'))$$

and thus by virtue of (3) we obtain (2.2). Next, we show that conical midpoint maps induce conical bicombings in a natural way. The used recursive construction is well-known and goes back to Menger (see [31, Section 6]).

Let m be a concial midpoint map on X and x, $y \in X$. Further, let $\mathcal{G}_n = (2^{-n} \cdot \mathbb{Z}) \cap [0, 1]$, where $n \geq 0$, be the 2^{-n}-grid in $[0, 1]$. We define $\sigma_{xy}\colon \bigcup \mathcal{G}_n \to X$ recursively as follows. We put $\sigma_{xy}(0) = x$, $\sigma_{xy}(1) = y$ and if $t \in \mathcal{G}_n \setminus \mathcal{G}_{n-1}$, then we set

$$\sigma_{xy}(t) = m(\sigma_{xy}(r), \sigma_{xy}(s)),$$

where r, $s \in \mathcal{G}_{n-1}$ are the unique points such that $t = \frac{1}{2} r + \frac{1}{2} s$ and $|r - s| = 2^{-(n-1)}$.

Lemma 2.4. *The map σ_{xy} extends uniquely to a geodesic $\overline{\sigma}_{xy}\colon [0, 1] \to \overline{X}$. Moreover,*

$$d(\overline{\sigma}_{xy}(t), \overline{\sigma}_{x'y'}(t)) \leq (1 - t) d(x, x') + t d(y, y') \tag{2.3}$$

for all x, y, x', $y' \in X$.

Proof. To begin, we show that $\sigma_{xy}|_{\mathcal{G}_n}$ is an isometric embedding for all $n \geq 0$. We proceed by induction. Clearly, $\sigma_{xy}|_{\mathcal{G}_0}$ is an isometric embedding. Now, fix $t_i \in \mathcal{G}_n$, $i = 1, 2$ and let r_i, $s_i \in \mathcal{G}_{n-1}$ with $s_i \leq r_i$ be points such that $t_i = \frac{1}{2} s_i + \frac{1}{2} r_i$ and $\sigma_{xy}(t_i) = m(\sigma_{xy}(s_i), \sigma_{xy}(r_i))$. By construction of σ_{xy}

such points clearly exist. Without loss of generality, we may suppose that $t_1 \leq t_2$. Using the triangle inequality, we get

$$\begin{aligned} d(\sigma_{xy}(t_1), \sigma_{xy}(t_2)) \leq d(\sigma_{xy}(t_1), \sigma_{xy}(r_1)) + d(\sigma_{xy}(r_1), \sigma_{xy}(s_2)) \\ + d(\sigma_{xy}(s_2), \sigma_{xy}(t_2)), \end{aligned}$$

and so, by the induction hypothesis and because m is a midpoint map,

$$d(\sigma_{xy}(t_1), \sigma_{xy}(t_2)) \leq \Big(\frac{r_1 - s_1}{2} + |s_2 - r_1| + \frac{r_2 - s_2}{2}\Big) d(x, y).$$

But, since $t_1 \leq t_2$, it holds $r_1 \leq s_2$. Hence, by the above, $d(\sigma_{xy}(t_1), \sigma_{xy}(t_2)) \leq |t_1 - t_2| d(x, y)$. As a result,

$$\begin{aligned} d(x, y) &\leq d(x, \sigma_{xy}(t_1)) + d(\sigma_{xy}(t_1), \sigma_{xy}(t_2)) + d(\sigma_{xy}(t_2), y) \\ &\leq \big(t_1 + |t_1 - t_2| + |t_2 - 1|\big) d(x, y). \end{aligned}$$

This implies that $d(\sigma_{xy}(t_1), \sigma_{xy}(t_2)) = |t_1 - t_2| d(x, y)$, and so $\sigma_{xy}|_{\mathcal{G}_n}$ is an isometric embedding. It follows by induction that $\sigma_{xy}|_{\mathcal{G}_n}$ is an isometric embedding for every $n \geq 0$, as claimed. Now, since $\bigcup \mathcal{G}_n$ is a dense subset of $[0, 1]$, it follows that σ_{xy} can be uniquely extended to an isometric embedding $\overline{\sigma}_{xy} \colon [0, 1] \to \overline{X}$. Next, we show (2.3). Clearly,

$$d(\overline{\sigma}_{xy}(1/2), \overline{\sigma}_{x'y'}(1/2)) \leq \frac{1}{2} d(x, x') + \frac{1}{2} d(y, y'),$$

as $\overline{\sigma}_{xy}(1/2) = m(x, y)$, $\overline{\sigma}_{x'y'}(1/2) = m(x', y')$ and m is conical midpoint map and thus satisfies (2.2). We now proceed by induction and show that if (2.3) is valid for all $t \in \mathcal{G}_{n-1}$, then it is also valid for all $t \in \mathcal{G}_n$. Fix $t \in \mathcal{G}_n$ and let s, $r \in \mathcal{G}_{n-1}$ be the unique points with $s \leq r$ such that $t = \frac{1}{2}s + \frac{1}{2}t$. We compute

$$\begin{aligned} d(\overline{\sigma}_{xy}(t), \overline{\sigma}_{x'y'}(t)) &\leq \frac{1}{2} d(\overline{\sigma}_{xy}(s), \overline{\sigma}_{x'y'}(s)) + \frac{1}{2} d(\overline{\sigma}_{xy}(r), \overline{\sigma}_{x'y'}(r)) \\ &\leq \Big(\frac{1-s}{2} + \frac{1-r}{2}\Big) d(x, x') + \Big(\frac{s}{2} + \frac{r}{2}\Big) d(y, y'); \end{aligned}$$

hence, (2.3) holds for all $t \in \mathcal{G}_n$. Since $\bigcup \mathcal{G}_n$ is a dense subset of $[0, 1]$ and $\overline{\sigma}_{xy}$ and $\overline{\sigma}_{x'y'}$ are geodesics, (2.3) is valid for all $t \in [0, 1]$. $\square$

Thus, we have constructed a map $\overline{\sigma} \colon X \times X \times [0, 1] \to \overline{X}$ such that (1.1) holds for all geodesics $\overline{\sigma}_{xy}$ and $\overline{\sigma}_{x'y'}$. Now, given x, $y \in \overline{X}$, we set

$$\overline{\sigma}_{xy}(t) = \lim_{n \to \infty} \overline{\sigma}_{x_n y_n}(t)$$

where x_n, $y_n \in X$ are points such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, respectively. It follows that $\overline{\sigma}$ is a well-defined conical bicombing on $\overline{X}$. We call $\overline{\sigma}$ the conical bicombing induced by m. We point out that m is defined on an arbitrary metric space X but $\overline{\sigma}$ is always a bicombing on $\overline{X}$.

We conclude this section by giving a description of σ-convex hulls in terms of m. Indeed, as with conical bicombings, conical midpoint maps give rise to 'convex hulls'. For any $A \subset X$, we let m–conv$(A) \subset \overline{X}$ denote the closure of the set

$$\bigcup_{n \in \mathbb{N}} \mathcal{M}_n(A),$$

where $\mathcal{M}_1(A) = \big\{ m(a, a') : a, a' \in A \big\}$ and $\mathcal{M}_n(A) = \mathcal{M}_1(\mathcal{M}_{n-1}(A))$ for all $n \geq 2$.

Lemma 2.5. *Let m be a conical midpoint map on a metric space X and suppose σ denotes the conical bicombing on $\overline{X}$ induced by m. Then*

$$\sigma\text{–conv}(A) = m\text{–conv}(A)$$

for all $A \subset X$.

Proof. Clearly, m–conv$(A) \subset \sigma$–conv(A). Thus, it suffices to show that the closed set m–conv(A) is σ-convex. To this end, let $n \geq 1$ and let x, $y \in \mathcal{M}_n(A)$. By construction of σ, it follows that $\sigma_{xy}(\mathcal{G}_m) \subset \mathcal{M}_{n+m}(A)$ for all $m \in \mathbb{N}$. Hence, $\sigma_{xy}([0, 1]) \subset m$–conv$(A)$. Now, suppose that x, $y \in m$–conv(A). There exist points x_k, $y_k \in \mathcal{M}_{n_k}(A)$ such that $x_k \to x$ and $y_k \to y$ as $k \to \infty$, respectively. Moreover, $\sigma_{x_k y_k} \to \sigma_{xy}$ uniformly. This implies that $\sigma_{xy}([0, 1]) \subset m$–conv$(A)$, and so m–conv(A) is σ-convex. □

3. Appending midpoints

Throughout this section we fix $n_0 \in \mathbb{N}$. This n_0 will correspond to the parameter from Theorem 1.3. We follow the proof strategy outlined in Section 1.1 to construct the metric space X_0. To begin, we construct recursively a sequence of graphs $G_n = (V_n, E_n)$. The whole construction is quite formal. The basic idea is that V_n is obtained from V_{n-1} by appending 'midpoints' and two midpoints in V_n are adjacent if and only if they are part of a cone whose base is an edge of G_{n-1}.

We let G_0 denote the null graph and G_1 the complete graph on n_0 vertices with vertex set $V_1 = \{1, \ldots, n_0\}$. For $n \geq 2$ we set

$$V_n = V_{n-1} \cup \Big\{ \{x, y\} : x, y \in V_{n-1}, x \neq y \Big\}.$$

To formalize the notion of 'midpoint' we use the following notation

$$m(a, b) = \begin{cases} \{a, b\} & \text{if } a \neq b, \\ a & \text{otherwise.} \end{cases} \tag{3.1}$$

Notice that $V_n = m(V_{n-1} \times V_{n-1})$. Moreover, we remark that we have constructed an infinite nested sequence

$$V_0 \subset V_1 \subset V_2 \subset \cdots$$

Now, the edge set E_n is uniquely determined by $\{x, y\} \in E_n$ if and only if there exist $v \in V_{n-1}$ and $\{u, w\} \in E_{n-1}$ such that $x = m(v, u)$ and $y = m(v, w)$. Loosely speaking, x and y are adjacent in G_n if and only if x, y are the midpoints parallel to the base of a cone with vertex $v \in V_{n-1}$

and base $u \sim w$ in G_{n-1}. See Figure 1.1 for an illustration. For example, if $n_0 = 2$, one has

$$V_2 = \big\{0, 1, \{0,1\}\big\} \qquad \text{and} \qquad E_2 = \big\{\{0,\{0,1\}\}, \{\{0,1\},1\}\big\}.$$

The graphs G_n for $n_0 = 2$ and $n = 1, 2, 3, 4$ are depicted in Figure 3.1.

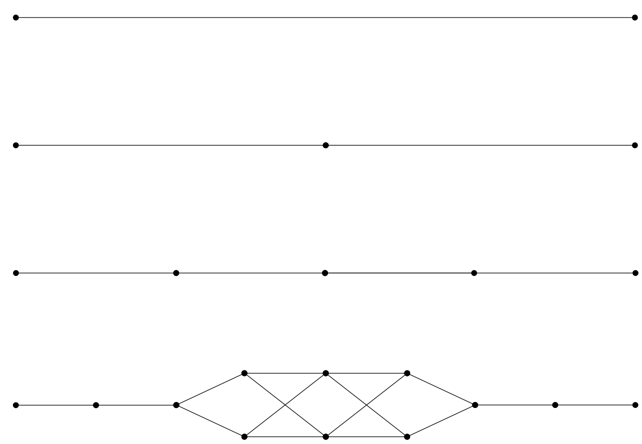

FIGURE 3.1. The graphs G_n for small n with $n_0 = 2$.

To begin, we collect some basic facts about the cardinalities of V_n and E_n that will be used later on.

Lemma 3.1. *One has* $|V_0| = 0$, $|V_1| = n_0$, *and for all* $n \geq 2$,

$$|V_n| = \frac{1}{2} \cdot \big(|V_{n-1}| + |V_{n-2}|\big) \cdot \big(|V_{n-1}| - |V_{n-2}| + 1\big) \tag{3.2}$$

Moreover, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|E_n|}{|V_n|^{1+\varepsilon}} = 0. \tag{3.3}$$

Proof. By construction, $V_{n-2} \subset V_{n-1}$. Thus, letting $W_{n-1} = V_{n-1} \setminus V_{n-2}$ and using that m is symmetric, we find

$$V_n = m(V_{n-1}\times V_{n-1}) = m(V_{n-2}\times V_{n-2}) \cup m(V_{n-2}\times W_{n-1}) \cup m(W_{n-1}\times W_{n-1}).$$

Therefore, as these sets are pairwise disjoint,

$$|V_n| = |V_{n-1}| + |V_{n-2}| \cdot |W_{n-1}| + \frac{|W_{n-1}| \cdot \big(|W_{n-1}| - 1\big)}{2}.$$

Since $|W_{n-1}| = |V_{n-1}| - |V_{n-2}|$, this yields (3.2). To finish the proof, we establish (3.3). Clearly, this is valid if $n_0 = 1$. Thus, in the following, we may suppose that $n_0 \geq 2$. Notice that $|E_0| = 0$, $|E_1| = \frac{1}{2}(n_0 - 1)n_0$ and $|E_n| \leq |V_{n-1}| \cdot |E_{n-1}|$ for all $n \geq 2$. Consequently,

$$|E_n| \leq C \cdot \prod_{i=1}^{n-1} |V_i|, \tag{3.4}$$

where $C = \frac{1}{2}(n_0 - 1)n_0$. We claim that

$$\frac{|V_{n-1}|^2}{|V_n|} \leq 3 \tag{3.5}$$

for all $n \in \mathbb{N}$. For $n = 1, 2$ this can be seen by a direct verification. Let now $n \geq 3$. Letting $\alpha_n = |V_{n-2}|/|V_{n-1}|$ and using (3.2), we find that

$$\frac{|V_{n-1}|^2}{|V_n|} \leq \frac{2}{1 - \alpha_n^2}.$$

In particular, if $\alpha_n \leq 1/\sqrt{3}$, then (3.5) follows. Now, suppose that $n_0 \geq 3$. It follows that $\alpha_3 \leq 1/\sqrt{3}$ and hence (3.5) is valid for $n = 3$. Clearly, if (3.5) holds for $n - 1$, then $\alpha_n \leq 3/|V_{n-2}|$. Thus, as $|V_2| \geq 6 \geq 3\sqrt{3}$, the desired inequality (3.5) follows by induction. This establishes (3.5) when $n_0 \geq 3$. We now treat the special case when $n_0 = 2$. We have $|V_2| = 3$, $|V_3| = 5$, $|V_4| = 12$, and $|V_5| = 68$. Hence, (3.5) holds true if $n = 3, 4, 5$. The general case now follows as before by noting that $|V_4| = 12 \geq 3\sqrt{3}$, and so (3.5) can be established by induction. This completes the proof of (3.5). By combining (3.4) with (3.5), we arrive at

$$\frac{|E_n|}{|V_n|^{1+\varepsilon}} \leq C \cdot \frac{3^n}{|V_n|^{\varepsilon}}.$$

We claim that $|V_n| \geq |V_{n-2}|^2$ for all $n \geq 6$. Letting $\beta = |V_{n-1}|/|V_{n-2}|$, we obtain

$$\frac{|V_n|}{|V_{n-2}|^2} \geq \frac{1}{2}(\beta + 1)(\beta - 1). \tag{3.6}$$

Since $|V_4| = 12$ if $n_0 = 2$, it follows that $|V_4| \geq 12$ for every $n_0 \geq 2$. Therefore, $\frac{|V_{n-2}|}{3} \geq \sqrt{3}$ for all $n \geq 6$, and thus by virtue of $|V_{n-1}| \geq \frac{1}{3}|V_{n-2}|$, we obtain $\beta^2 \geq 3$. This is equivalent to $\frac{1}{2}(\beta + 1)(\beta - 1) \geq 1$. By the use of (3.6), we can conclude that $|V_n| \geq |V_{n-2}|^2$ for all $n \geq 6$, as desired. Now, by repeated use of this inequality and using that $|V_3| \geq |V_2|$, we get

$$|V_n| \geq |V_{n-2}|^2 \geq \cdots \geq \big(|V_2|\big)^{2^{\frac{n-3}{2}}}$$

for all $n \geq 6$. Thus, letting $c = \frac{\varepsilon}{2\sqrt{2}}$ and using that $|V_2| \geq 3$, we obtain

$$\lim_{n\to\infty} \frac{|E_n|}{|V_n|^{1+\varepsilon}} \leq C \cdot \lim_{n\to\infty} \frac{3^n}{|V_n|^{\varepsilon}} \leq C \cdot \lim_{n\to\infty} |V_2|^{n - c(\sqrt{2})^n} = 0.$$

□

Let $d_n\colon V_n \times V_n \to \mathbb{R}$ denote the shortest-path metric on G_n. The definition of the shortest-path metric of a graph is recalled in (2.1). Our next result shows that any two distinct points in $V_1 \subset V_n$ realize the diameter of V_n with respect to d_n.

Lemma 3.2. *For all distinct x, $y \in V_1$,*

$$d_n(x,y) = \operatorname{diam} V_n = 2^{n-1}.$$

Proof. To begin, we show that

$$d_n(x,y) \leq 2d_{n-1}(x,y) \tag{3.7}$$

for all $n \geq 2$ and all x, $y \in V_{n-1}$. Let $(x_0, x_1, \dots, x_k)$ be a shortest-path in G_{n-1} connecting x to y. We set $x_i' = m(x_{i-1}, x_i)$ for all $i = 1, \dots, k$. Clearly, $x_{i-1} \sim x_i'$ and $x_i' \sim x_i$ in G_n for all $i = 1, \dots, k$. Hence, $(x_0, x_1', x_1, x_2', \dots, x_k', x_k)$ is a path in G_n connecting x to y, and so $d_n(x,y) \leq 2k = 2d_{n-1}(x,y)$, as desired.

By construction, $\operatorname{diam} V_1 = 1$. Hence, it follows from (3.7) that $\operatorname{diam} V_n \leq 2^{n-1}$. To finish the proof we thus need to show that $d_n(x,y) \geq 2^{n-1}$ for all distinct x, $y \in V_1$. For this we will use the following construction. We define the functions $\delta_n\colon V_n \to \Delta^{n_0-1} \cap 2^{-(n-1)} \cdot \mathbb{Z}^{n_0}$ recursively as follows. We may suppose that $V_1 = \{1, \dots, n_0\}$ and we set $\delta_1(i) = e_i$ for each $i = 1, \dots, n_0$. Here, $e_i \in \mathbb{R}^{n_0}$ is the vector with a one at the ith position and zeros everywhere else. Suppose now $n \geq 2$ and $x \in V_n$. We set

$$\delta_n(x) = \frac{1}{2}\big(\delta_{n-1}(a) + \delta_{n-1}(b)\big)$$

if $x = m(a,b)$ with $a \neq b$, and $\delta_n(x) = \delta_{n-1}(x)$ otherwise. It follows by induction that if $\{x,y\} \in E_n$, then

$$|\delta_n(x) - \delta_n(y)|_\infty = \frac{1}{2^{n-1}}, \tag{3.8}$$

where $|\cdot|_\infty$ denotes the supremum norm on $\mathbb{R}^{n_0}$. Clearly, $\delta_n(i) = e_i$ for all $n \in \mathbb{N}$ and all $i = 1, \dots, n_0$. Now, let x, $y \in V_1$ be distinct and $(x_0, x_1, \dots, x_k)$ a path in G_n connecting x to y. By the above, it follows that

$$1 = |\delta_n(x) - \delta_n(y)|_\infty \leq \sum_{i=0}^{k-1} |\delta_n(x_i) - \delta_n(x_{i+1})|_\infty = \frac{k}{2^{n-1}}.$$

Hence, $d_n(x,y) \geq 2^{n-1}$, as was to be shown. □

Our next lemma relates shortest-paths in G_n to shortest-paths in G_{n-1}. The proof follows easily from the definition of d_n and the recursive construction of E_n.

Lemma 3.3. *For all x_1, x_2, y_1, $y_2 \in V_{n-1}$,*

$$d_n(m(x_1,x_2), m(y_1,y_2)) \leq d_{n-1}(x_1,y_1) + d_{n-1}(x_2,y_2). \tag{3.9}$$

Proof. Let $(p_0, \ldots, p_k)$ and $(q_0, \ldots, q_\ell)$ be shortest-paths in G_{n-1} connecting x_1 to y_1, and x_2 to y_2, respectively. We construct a path $(r_0, \ldots, r_{k+\ell})$ in G_n as follows. We set $r_i = m(p_0, q_i)$ for all $i = 0, \ldots, \ell$ and $r_{\ell+j} = m(p_j, q_\ell)$ for all $j = 1, \ldots, k$. By construction, $r_{i-1} \sim r_i$ in G_n for all $i = 1, \ldots, k+\ell$. Hence, $(r_0, \ldots, r_{k+\ell})$ is a path in G_n connecting $r_0 = m(x_1, x_2)$ to $r_{k+\ell} = m(y_1, y_2)$, and so it follows that

$$d_n(m(x_1, x_2), m(y_1, y_2)) \leq k + \ell.$$

But $d_{n-1}(x_1, y_1) = k$ and $d_{n-1}(x_2, y_2) = \ell$. This finishes the proof of (3.9). □

We remark that (3.9) should be thought of as a discrete analogue of the conical inequality (1.1). Indeed, by considering the scaled metrics $\varrho_n = (\operatorname{diam} V_n)^{-1} \cdot d_n$ and using that $\operatorname{diam} V_n = 2^{n-1}$ by Lemma 3.2, we obtain that

$$\varrho_n(m(x_1, x_2), m(y_1, y_2)) \leq \frac{1}{2}\varrho_{n-1}(x_1, y_1) + \frac{1}{2}\varrho_{n-1}(x_2, y_2) \tag{3.10}$$

for all x_1, x_2, y_1, $y_2 \in V_{n-1}$. In particular, if x, $y \in V_{n-1}$, then $\varrho_n(x, y) \leq \varrho_{n-1}(x, y)$. In view of these inequalities, letting

$$V = \bigcup_{n \geq 1} V_n$$

we find that the map $\varrho \colon V \times V \to \mathbb{R}$ defined by

$$\varrho(x, y) = \lim_{n \to \infty} \varrho_n(x, y)$$

is a semi-metric on V (see Section 2.1 for the definition). More formally, V could also be constructed as the direct limit of the sequence of metric spaces (V_n, ϱ_n) with morphisms $V_n \to V_m$, for $n \leq m$, induced by the identity.

By the above, the semi-metric space (V, ϱ) is naturally equipped with a 'conical midpoint map'. Indeed, because of (3.10), $m \colon V \times V \to V$ defined by $(x, y) \mapsto m(x, y)$ satisfies

$$\varrho(m(x, y), m(x, z)) \leq \frac{1}{2}\varrho(y, z) \tag{3.11}$$

for all x, y, $z \in V$. It is now not difficult to upgrade m to a concial midpoint map on a metric space X. Indeed, let us denote by (X, d) the metric space induced by (V, ϱ). By definition, $X = V/\!\sim$ with $x \sim y$ if and only if $\varrho(x, y) = 0$ and d is the quotient metric on X. We recall that $d([x], [y]) = \varrho(x, y)$ for all x, $y \in V$.

Lemma 3.4. *The map $m \colon X \times X \to X$ defined by $m([x], [y]) = [m(x, y)]$ for all $[x]$, $[y] \in X$ is a concial midpoint map on X. Moreover,*

$$X = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n(A),$$

where $A = [V_1] \subset X$.

Proof. By applying (3.11), we get

$$\begin{aligned}\varrho(m(x,y),m(x',y')) &\leq \varrho(m(x,y),m(x,y'))+\varrho(m(x,y'),m(x',y'))\\ &\leq \frac{1}{2}\varrho(y,y')+\frac{1}{2}\varrho(x,x').\end{aligned}$$

Hence, if $\varrho(x,x')=0$ and $\varrho(y,y')=0$, then $\varrho(m(x,y),m(x',y'))=0$. This shows that $m\colon X\times X\to X$ defined by $m([x],[y])=[m(x,y)]$ for all $[x]$, $[y]\in X$ is well-defined. Moreover, it follows directly from the inequality above that m is a concial midpoint map on X. By construction, $m(V_{n-1}\times V_{n-1})=V_n$ for all $n\geq 2$, and so $V\subset m\text{−conv}(V_1)$. This implies the desired equality. □

In summary, we have shown that the map $m\colon V\times V\to V$ descends to conical midpoint map on X, where X denotes the metric space associated to V. For simplicity this map is also denoted by m. Due to the results in Section 2.4, m now induces a conical bicombing on $\overline{X}$. We set $X_0=\overline{X}$. In Section 5 we show that X_0 is non-compact. We achieve this by showing that X is not totally bounded. In order to work effectively with X, it seems natural to determine how much the semi-metric ϱ (and hence d) differs from the metric ϱ_n on V_n. The following lemma shows that ϱ does not collapse the distances too much.

Lemma 3.5. *For all $n\geq 2$,*

$$\varrho_n(x,y)-\frac{8}{2^n}\leq \varrho(x,y)\leq \varrho_n(x,y). \tag{3.12}$$

for all x, $y\in V_n$.

Notice that due to Lemma 3.5, if x, $y\in V_n$ satisfy $d_n(x,y)\geq 5$, then $d(x,y)>0$. In particular, ε-separated sets in (V_n,ϱ_n) induce ε'-separated sets in X. See Lemma 4.1 for the exact statement.

Proof of Lemma 3.5. The desired upper bound of $\varrho(x,y)$ follows directly from (3.7). In what follows we show the lower bound. To begin, we claim that

$$2d_{n-1}(x,y)\leq d_n(x,y)+4 \tag{3.13}$$

for all x, $y\in V_n$. Fix distinct points x, $y\in V_{n-1}$ and let $\{x,x'\}$ and $\{y,y'\}$ be edges in E_{n-1}. Since G_{n-1} is connected such edges surely exists. Because of (3.8), it follows that $p\coloneqq m(x,x')$, $q\coloneqq m(y,y')\in V_n\setminus V_{n-1}$. Moreover, since $x\sim p$ and $y\sim q$ in G_n, by the triangle inequality,

$$|d_n(x,y)-d_n(p,q)|\leq 2.$$

We claim that

$$d_n(p,q)=\min\big\{d_{n-1}(x,y)+d_{n-1}(x',y'),\, d_{n-1}(x,y')+d_{n-1}(x',y)\big\}. \tag{3.14}$$

Indeed, let $(x_0,\dots,x_\ell)$ be a shortest-path in G_n connecting p to q. For each $i=1,\dots,\ell$ there is $v_i\in V_{n-1}$ and $\{u_i,w_i\}\in E_{n-1}$ such that

$$x_{i-1}=m(v_i,u_i) \quad\text{ and }\quad x_i=m(v_i,w_i).$$

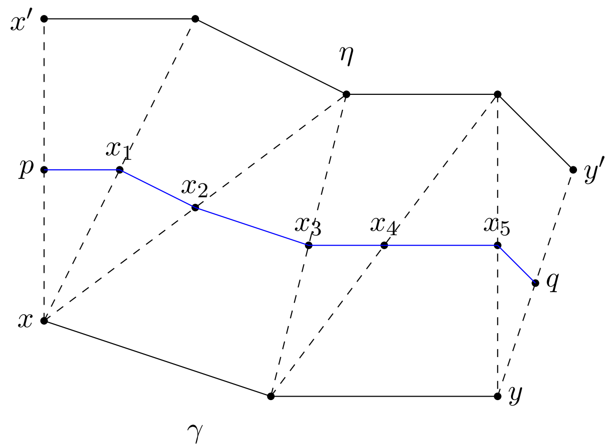

FIGURE 3.2. The construction from Lemma 3.5.

We define $a_0, \ldots, a_\ell \in V_{n-1}$ and $b_0, \ldots, b_{\ell+1} \in V_{n-1}$ by induction as follows. We put $a_0 = v_1$ and $b_0 = u_1$, $b_1 = w_1$. Now, for every $i = 1, \ldots, \ell - 1$, we set

$$\begin{cases} a_i = w_{i+1} \text{ and } b_{i+1} = b_i & \text{if } u_{i+1} = a_{i-1}, \\ a_i = a_{i-1} \text{ and } b_{i+1} = w_{i+1} & \text{if } u_{i+1} = b_i. \end{cases}$$

By construction, $m(a_0, b_0) = x_0 = p$ and $m(a_{\ell-1}, b_\ell) = x_\ell = q$. Moreover, after deleting repeated entries, $\gamma = (a_0, \ldots, a_{\ell-1})$ and $\eta = (b_0, \ldots, b_\ell)$ are (possibly degenerate) shortest-paths in G_{n-1} such that $\mathrm{length}(\gamma) + \mathrm{length}(\eta) = \ell = d_n(p, q)$. See Figure 3.2. Hence,

$$d_{n-1}(a_0, a_{\ell-1}) + d_{n-1}(b_0, b_\ell) \leq d_n(p, q).$$

Because of p, $q \notin V_{n-1}$, without loss of generality we have $a_0 = x$, $a_{\ell-1} = y$, $b_0 = x'$ and $b_\ell = y'$, and so the desired equality (3.14) now follows due to Lemma 3.3.

Having (3.14) at hand, (3.13) now follows easily. Indeed, using that $d_{n-1}(x, x') = d_{n-1}(y, y') = 1$, we have

$$d_{n-1}(x, y) - 1 \leq d_{n-1}(x, y')$$
$$d_{n-1}(x, y) - 1 \leq d_{n-1}(x', y)$$

and

$$d_{n-1}(x, y) - 2 \leq d_{n-1}(x', y'),$$

and so using (3.14), we deduce that

$$d_n(x, y) \geq d_n(p, q) - 2 \geq 2d_{n-1}(x, y) - 4.$$

This shows (3.13). Now, by dividing (3.13) by 2^{n-1}, we obtain

$$\varrho_{n-1}(x, y) \leq \varrho_n(x, y) + \frac{8}{2^n}.$$

In particular, for every $k \in \mathbb{N}$,

$$\varrho_n(x, y) \leq \varrho_{n+k}(x, y) + \frac{8}{2^n} \sum_{i=1}^{k} \frac{1}{2^i}$$

and the left inequality of (3.12) follows by taking the limit $k \to \infty$. $\square$

We remark that in (3.13) at least an additive error of 2 must occur. This is discussed further in the following example.

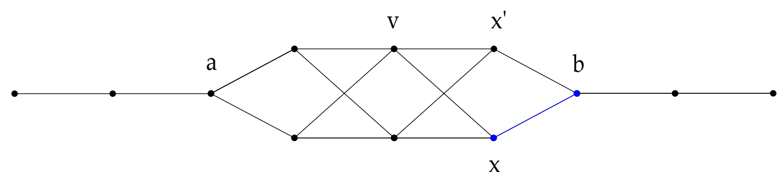

FIGURE 3.3. Illustration of the construction in Example 3.6.

Example 3.6. Let $n_0 = 2$ and consider the graph G_4 depicted in Figure 3.3. In particular, $v = m(0, 1)$, $a = m(0, v)$, $b = m(v, 1)$ and

$$x = m(a, 1) \quad \text{and} \quad x' = m(v, b).$$

Clearly, $d_4(x, x') = 2$. We claim that $d_5(x, x') = 2$ as well. Since $x \sim b$ in G_4, the points $x_0 := m(v, b)$ and $x_1 := m(v, x)$ are adjacent in G_5. Thus, as $m(x, v) \sim m(x, x)$ in G_5, it follows that (x_0, x_1, x_2) is a path in G_5 connecting x' to x. Hence, $d_5(x, x') \leq 2$. On the other hand, it is not difficult to see that $\delta_5(x) = \delta_5(x')$ and thus due to (3.8), it follows that x and x' are not adjacent in G_5. This shows that $d_5(x, x') = 2$. Hence, $2d_4(x, x') - d_5(x, x') = 2$, and so the additive error in (3.13) must be at least 2.

4. G_n^m HAS FEW EDGES

In this section, we find a sufficient condition that X is not totally bounded in terms of the number of edges of G_n^m. The basic graph theory notation that is needed in the sequel can be found in Section 2.2.

Lemma 4.1. *Let $n, r \geq 1$ and $m \geq 6$ be integers. If $\overline{G_n^m}$ has an $(r+1)$-clique, then there exist $r+1$ points $x_1, \ldots, x_{r+1} \in X$ such that*

$$d(x_i, x_j) \geq \frac{m}{2^n}$$

for all distinct $i, j = 1, \ldots, r+1$.

Proof. If $v_1, \ldots, v_{r+1} \in V_n$ are the vertices of an $(r+1)$-clique in $\overline{G_n^m}$, then by definition one has

$$d_n(v_i, v_j) \geq m + 1$$

for all distinct $i, j = 1, \dots, r+1$. Hence, by dividing by 2^{n-1} on both sides and using Lemma 3.5, we obtain

$$d(v_i, v_j) \geq \frac{2m-6}{2^n} \geq \frac{m}{2^n},$$

as desired. □

Fix an integer $k \geq 1$ sufficiently large to be determined later. We abbreviate $m(n) = 2^{n-k}$. Using Turán's theorem we obtain the following non-compactness criterion for X.

Corollary 4.2. *Let $n_0 \geq 2$ and let X be constructed as in Section 3. If*

$$\liminf_{n\to\infty} \frac{|E\big(G_n^{m(n)}\big)|}{|V_n|^2} = 0, \tag{4.1}$$

then X is not totally bounded.

Proof. We prove the contrapositive. Suppose that X is totally bounded. There exists $r \geq 1$ such that X does not contain a $\frac{1}{2^k}$-net of cardinality $r+1$. Hence, by Lemma 4.1, for $n \geq 1$ sufficiently large, the complement graph of the $m(n)$-th power G_n does not contain an $(r+1)$-clique. Thus, Turán's theorem, see Theorem 2.1, tells us that

$$|E\big(\overline{G_n^{m(n)}}\big)| \leq \big(1 - \frac{1}{r}\big) \cdot \frac{|V_n|^2}{2}$$

for all n sufficiently large. Therefore,

$$\frac{|V_n| \cdot (|V_n| - 1)}{2} - |E\big(G_n^{m(n)}\big)| \leq \big(1 - \frac{1}{r}\big) \cdot \frac{|V_n|^2}{2}$$

and it follows that

$$\liminf_{n\to\infty} \frac{|E\big(G_n^{m(n)}\big)|}{|V_n|^2} \geq \frac{1}{2r} > 0,$$

as desired. We remark that to show the lower bound on the liminf we have used that $|V_n|$ is an unbounded sequence, which is only valid if $n_0 \geq 2$. □

Thus, to prove that X not totally bounded, it suffices to establish (4.1). To this end, in the next subsection we derive some upper bounds for $|E\big(G_n^{m(n)}\big)|$.

4.1. **Upper bounds.** The following estimate is not sharp in general, but is sufficient for our purposes. It is the crucial building block for inequality (4.3), which is our key tool in the proof of Theorem 1.2.

Lemma 4.3. *Let $n, m \in \mathbb{N}$. Then there exist non-negative integers a, b such that $a + b = m$ and*

$$|E(G_n^m)| \leq 2m\,|E(G_{n-1}^a)| \cdot |E(G_{n-1}^b)|.$$

We recall that we use the convention that $|E(G^0)| = |V|$ for any finite graph $G = (V, E)$.

Proof of Lemma 4.3. Suppose that x is adjacent to y in G_n^m. By definition, there exist a shortest-path $(x_0, \ldots, x_\ell)$ in G_n of length $\leq m$ connecting x to y. For each $i = 1, \ldots, \ell$ there is $v_i \in V_{n-1}$ and $\{u_i, w_i\} \in E_{n-1}$ such that

$$x_{i-1} = m(v_i, u_i) \quad \text{and} \quad x_i = m(v_i, w_i).$$

As in the proof of Lemma 3.5, we define $a_0, \ldots, a_\ell \in V_{n-1}$ and $b_0, \ldots, b_{\ell+1} \in V_{n-1}$ by induction as follows. We put $a_0 = v_1$ and $b_0 = u_1$, $b_1 = w_1$. Now, for every $i = 1, \ldots, \ell - 1$, we set

$$\begin{cases} a_i = w_{i+1} \text{ and } b_{i+1} = b_i & \text{if } u_{i+1} = a_{i-1}, \\ a_i = a_{i-1} \text{ and } b_{i+1} = w_{i+1} & \text{if } u_{i+1} = b_i. \end{cases}$$

By construction, $m(a_0, b_0) = x_0 = x$ and $m(a_{\ell-1}, b_\ell) = x_\ell = y$. Moreover, after deleting repeated entries, $\gamma = (a_0, \ldots, a_{\ell-1})$ and $\eta = (b_0, \ldots, b_\ell)$ are (possibly degenerate) shortest-paths in G_{n-1} such that $\operatorname{length}(\gamma) + \operatorname{length}(\eta) = \ell = d_n(x, y)$. See Figure 3.2. Moreover, any two non-degenerate shortest-paths γ and η induce at most two edges in G_n^m in this way. Consequently,

$$|E(G_n^m)| \leq |V_{n-1}| \cdot |E(G_{n-1}^m)| + 2 \sum_{i=1}^{m-1} |E(G_{n-1}^i)| \cdot |E(G_{n-1}^{m-i})|.$$

We put

$$M = \max\left\{ |E(G_{n-1}^i)| \cdot |E(G_{n-1}^{m-i})| \,:\, i = 0, \ldots, m \right\}.$$

By the above, it follows that $|E(G_n^m)| \leq M + 2(m-1)C \leq 2mM$. $\square$

Recall that we have fixed an integer $k \geq 1$ which is sufficiently large to be determined later, and we use the notation

$$m(n) = 2^{n-k} \quad \text{and} \quad \bar{n} = n - k.$$

Using Lemma 4.3, it is possible to obtain an upper bound on the number of edges of $G_n^{m(n)}$ in terms of a product with factors $|E(G_k^{m_i})|$ and $|V_{n-i}|^{k_i}$.

Lemma 4.4. *Let $n \geq 1$ be sufficiently large. Then there exist an integer $K \in \{1, \ldots, m(n)\}$, positive integers $m_1, \ldots, m_K$ such that $m_1 + \cdots + m_K = m(n)$, and integers $k_i \in \{0, \ldots, 2^i - 1\}$ for $i = 1, \ldots, \bar{n}$ satisfying*

$$\sum_{i=1}^{\bar{n}} k_i \cdot 2^{\bar{n}-i} = m(n) - K, \tag{4.2}$$

such that

$$|E(G_n^{m(n)})| \leq 32^{m(n)} \Big(\prod_{i=1}^{K} |E(G_k^{m_i})| \Big) \Big(\prod_{i=1}^{\bar{n}} |V_{n-i}|^{k_i} \Big). \tag{4.3}$$

Proof. We consider the following replacement rule:

$$|E(G_n^m)| \to \begin{cases} 2m\, |E(G_{n-1}^a)| \cdot |E(G_{n-1}^b)| & \text{if } m > 0, \text{ where } a, b \text{ are as in Lemma 4.3} \\ |V_n| & \text{if } m = 0. \end{cases}$$

By using this rule and Lemma 4.3 sufficiently many times, we obtain integers $\ell_i \in \{1, \dots, 2^i\}$, for $i = 0, \dots, \bar{n} - 1$, such that

$$|E(G_n^{m(n)})| \leq \Big(\prod_{i=1}^{\bar{n}} 2^{\ell_{i-1}} \cdot A_i \cdot |V_{n-i}|^{k_i} \Big) \cdot \Big(\prod_{i=1}^{K} |E(G_k^{m_i})| \Big), \tag{4.4}$$

where

$$A_i \coloneqq \prod_{j=1}^{\ell_{i-1}} \alpha_{i,j}$$

for some positive integers $\alpha_{i,j} > 0$ satisfying $\alpha_{i,1} + \dots + \alpha_{i,\ell_i} = m(n)$. Notice that in particular $\ell_0 = 1$. Using the inequality of arithmetic and geometric means, we get

$$\prod_{j=1}^{\ell_i} \alpha_{i,j} \leq \Big(\frac{m(n)}{\ell_i} \Big)^{\ell_i} = 2^{(\bar{n} - \log_2 \ell_i) \cdot 2^{\log_2 \ell_i}}.$$

The function $f(x) \coloneqq (\bar{n} - x) \cdot 2^x$ is increasing on $[0, \bar{n} - 2]$, and

$$\max_{x \in [0, \bar{n}]} f(x) = \frac{2^{\bar{n}}}{e \log 2} \leq 2^{\bar{n}}.$$

Hence, using that $\ell_i \in \{1, \dots, 2^i\}$, we have $A_{\bar{n}} \leq 2^{\bar{n}}$ and for all $i = 1, \dots, \bar{n} - 2$,

$$A_i \leq 2^{(\bar{n} - (i-1)) \cdot 2^{i-1}}.$$

Thus, since

$$\sum_{j=0}^{\bar{n}-2} (\bar{n} - j) 2^j \leq 2^{\bar{n}-1} + \sum_{j=1}^{\bar{n}-1} 2^j \leq 2^{\bar{n}-1} + 2^{\bar{n}},$$

we find that

$$\prod_{i=1}^{\bar{n}} A_i \leq 2^{2^{\bar{n}} + 2^{\bar{n}-1} + 2^{\bar{n}}} \leq 16^{2^{\bar{n}}}.$$

Moreover,

$$\prod_{i=1}^{\bar{n}} 2^{\ell_{i-1}} \leq \prod_{i=0}^{\bar{n}-1} 2^{2^i} \leq 2^{2^{\bar{n}}},$$

and thus (4.3) follows from (4.4). □

We remark that if Lemma 4.3 were true for $a = b = \frac{m}{2}$, by exactly the same reasoning as in the proof of Lemma 4.4, we would get the following slightly more elegant upper bound in (4.3),

$$8^{m(n)} \cdot |E(G_k)|^{m(n)},$$

but we do not know how to prove this.

5. Proof of main results

In this section we prove the main results from the introduction. Theorem 1.2 is an immediate consequence of the following result.

Theorem 5.1. *Let $n_0 \in \mathbb{N}$ and let X_0 be the complete metric space constructed in Section 3. Then X_0 admits a conical bicombing σ and there is a finite subset $A \subset X_0$ such that σ–$\operatorname{conv}(A) = X_0$. Moreover, X_0 is non-compact for every $n_0 \geq 2$.*

Proof. In the following, we retain the notation of Section 3. Recall that $X_0 = \overline{X}$, where (X, d) is the metric space associated to the semi-metric space (V, ϱ). We set $A = V_1 \subset X_0$. Lemma 3.4 tells us that $m \colon X \times X \to X$ defines a conical midpoint map on X and

$$X = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n(A).$$

Let σ be the conical bicombing on X_0 induced by m. For the construction of σ we refer to Section 2.4. Because of Lemma 2.5, it follows that σ–$\operatorname{conv}(A) = X_0$.

Let now $n_0 \geq 2$. To finish the proof we show that X_0 is not compact. This is achieved by showing that X is not totally bounded, which in turn is established via Corollary 4.2. Fix $\varepsilon \in (0, 2^{-4})$ and choose $k \geq 1$ sufficiently large such that

$$\max\left\{ \frac{1}{|V_k|}, \frac{|E(G_k)|}{|V_k|^{(1+\varepsilon)}} \right\} \leq \frac{1}{(2\alpha)^{\frac{1}{\varepsilon}}}, \tag{5.1}$$

for some large constant $\alpha > 0$ to be determined later. The existence of k is guaranteed by Lemma 3.1. As in Section 4, we set $m(n) = 2^{n-k}$ and $\bar{n} = n - k$. We claim that

$$\frac{|E(G_n^{m(n)})|}{|V_n|^2} \leq \left(\frac{1}{2}\right)^{m(n)} \tag{5.2}$$

for all $n \geq 1$ sufficently large. By Lemma 4.4, there exists an integer $K \in \{1, \dots, m(n)\}$, positive integers $m_1, \dots, m_K$ such that $m_1 + \cdots + m_K = m(n)$, and $k_i \in \{0, \dots, 2^i - 1\}$ for $i = 1, \dots, \bar{n}$, such that (4.2) holds and

$$|E(G_n^{m(n)})| \leq 32^{m(n)} \Big(\prod_{i=1}^{K} |E(G_k^{m_i})| \Big) \Big(\prod_{i=1}^{\bar{n}} |V_{n-i}|^{k_i} \Big). \tag{5.3}$$

In the following, we derive an upper bound for $1/|V_n|^2$. Due to (3.5), we have

$$\frac{|V_{n-1}|^2}{|V_n|} \leq 3, \tag{5.4}$$

and so we find that

$$\frac{1}{|V_n|^2} \leq \frac{3^2}{|V_{n-1}|^4} = \frac{3^{b_0}}{|V_{n-1}|^{k_1}} \cdot \frac{1}{|V_{n-1}|^{b_1}},$$

where $b_0 = 2$ and $b_1 = 2b_0 - k_1$. We define the integers $b_0, \ldots, b_{\bar{n}}$ recursively as follows. We set $b_0 = 2$, and $b_i = 2b_{i-1} - k_i$ for all $i = 1, \ldots, \bar{n}$. Hence, by using (5.4) repeatedly, we arrive at

$$\frac{1}{|V_n|^2} \leq \Big(\prod_{i=1}^{\bar{n}} \frac{3^{b_{i-1}}}{|V_{n-i}|^{k_i}} \Big) \cdot \frac{1}{|V_k|^{b_{\bar{n}}}} \tag{5.5}$$

Via a straightforward computation, we find

$$\sum_{i=0}^{\bar{n}-1} b_i \leq \sum_{i=0}^{\bar{n}-1} 2^{i+1} \leq 2^{\bar{n}+1}, \qquad b_{\bar{n}} = 2 \cdot m(n) - \sum_{i=0}^{\bar{n}-1} k_{\bar{n}-i} \cdot 2^i.$$

Hence, because of (4.2), it follows that $b_{\bar{n}} = m(n) + K$. By combining (5.3) with (5.5), we obtain

$$\frac{|E(G_n^{m(n)})|}{|V_n|^2} \leq \alpha^{m(n)} \cdot \frac{\prod_{i=1}^{K} |E(G_k^{m_i})|}{|V_k|^{2K}} \cdot \frac{1}{|V_k|^{m(n)-K}}, \tag{5.6}$$

where $\alpha = 32 \cdot 9$. In the following, we consider the cases $K \leq (1-\varepsilon)m(n)$ and $K > (1-\varepsilon)m(n)$ separately. First, we suppose that $K \leq (1-\varepsilon)m(n)$. From (5.6), we find that

$$\frac{|E(G_n^{m(n)})|}{|V_n|^2} \leq \alpha^{m(n)} \cdot \frac{1}{|V_k|^{m(n)-K}}.$$

Since $\varepsilon \cdot m(n) \leq m(n) - K$, it follows from our assumption (5.1) on k that

$$\frac{|E(G_n^{m(n)})|}{|V_n|^2} \leq \alpha^{m(n)} \cdot \left(\frac{1}{(2\alpha)^{\frac{1}{\varepsilon}}} \right)^{\varepsilon \cdot m(n)} \leq \Big(\frac{1}{2}\Big)^{m(n)}.$$

Second, suppose that $K > (1-\varepsilon)m(n)$. Since $m_i \geq 1$ and $m_1 + \ldots + m_K = m(n)$, it follows that $m_j \geq 2$ for at most $2\varepsilon \cdot m(n)$ many indices j. To ease the notation, we may suppose $m_1 = \cdots = m_L = 1$, where $L = \lceil K - 2\varepsilon m(n) \rceil$. Hence, using (5.6) once again, we find that

$$\begin{aligned} \frac{|E(G_n^{m(n)})|}{|V_n|^2} &\leq \alpha^{m(n)} \cdot \Big(\frac{|E(G_k)|}{|V_k|^{(1+\varepsilon)}} \Big)^L \cdot \frac{1}{|V_k|^{(1-2\varepsilon)m(n)-\varepsilon L}} \\ &\leq \alpha^{m(n)} \cdot \Big(\frac{1}{2\alpha} \Big)^{(1-2\varepsilon)m(n)+(1-\varepsilon)L}, \end{aligned}$$

where in the last inequality we used (5.1), our assumption on k. By construction, $L \geq (1-3\varepsilon)m(n)$, and so we get

$$(1-2\varepsilon)m(n) + (1-\varepsilon)L \geq (1-\varepsilon)(2-5\varepsilon)m(n) \geq m(n),$$

where in the last step we used that $\varepsilon \in (0, 2^{-4})$. Therefore, it follows from the above that

$$\frac{|E(G_n^{m(n)})|}{|V_n|^2} \leq \Big(\frac{1}{2}\Big)^{m(n)}.$$

This concludes the case distinction and establishes (5.2). Finally, having (5.2) at hand we find that

$$\liminf_{n\to\infty} \frac{|E(G_n^{m(n)})|}{|V_n|^2} = 0,$$

since $m(n) \to \infty$ as $n \to \infty$. So Corollary 4.2 tells us that X is not totally bounded. Hence, X_0 is not compact. □

A metric space Y is called *injective* if whenever $A \subset B$ are metric spaces and $f\colon A \to Y$ a 1-Lipschitz map, then there exists a 1-Lipschitz extension $\bar{f}\colon B \to Y$ of f. More formally, Y is an injective object in the category of metric spaces with 1-Lipschitz maps as morphisms. Injective metric spaces have been introduced by Aronszajn and Panitchpakdi in [2] and are sometimes also called hyperconvex metric spaces by some authors. We refer to [16, 28] for an introduction to injective metric spaces. As observed by Lang in [28, Proposition 3.8], every injective metric spaces admits a conical bicombing. Indeed, given an injective metric space Y, by applying Kuratowski's embedding theorem, we may suppose that $Y \subset C_b(Y)$, and so because Y is injective, there is a 1-Lipschitz retraction $r\colon C_b(Y) \to Y$ and thus

$$\sigma(x, y, t) = r((1-t)x + ty)$$

defines a conical bicombing on Y. Using an extension result of [3], we find that Theorem 1.2 is also valid for an injective metric space.

Theorem 5.2. *There exists an injective metric space Y with a conical bicombing σ such that there is a finite subset of Y whose closed σ-convex hull is not compact.*

Proof. Let $n_0 \geq 2$ and let X_0 be constructed as in Section 3. We recall that by definition $X_0 = \overline{X}$ and X is naturally equipped with a conical midpoint map m. Let σ denote the conical bicombing on X_0 induced by m. As m is symmetric, it is not difficult to see that $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all x, $y \in X_0$. This shows that σ is a reversible conical bicombing. Hence, by virtue of [3, Theorem 1.2], there exists an injective metric space Y containing X_0, and a conical bicombing $\tilde{\sigma}$ on Y such that $\tilde{\sigma}_{xy} = \sigma_{xy}$ for all x, $y \in X_0$. As X_0 is complete, it follows that $\tilde{\sigma}$–conv$(A) = \sigma$–conv(A) for any $A \subset X_0$. Therefore, due to Theorem 5.1, Y admits a finite subset whose closed $\tilde{\sigma}$-convex hull is not compact. □

We finish this section by proving the following more general version of Theorem 1.3.

Theorem 5.3. *Let $n_0 \in \mathbb{N}$. Then there exists a complete metric space X_0 with a conical bicombing such that whenever $A \subset Y$ is an n_0-point subset of some complete metric space Y with a conical midpoint map m, then there exists a Lipschitz map $\Phi\colon X_0 \to Y$ with $A \subset \Phi(X_0)$ and furthermore $\Phi(X_0)$ is σ-convex with respect to the conical bicombing σ induced by m.*

Proof. Let $X_0 = (X_0, d)$ be the metric space constructed in Section 3. We set $A_0 = V_1 \subset X_0$. By Lemma 3.2, it follows that

$$d(x, y) = d_1(x, y) = 1 \tag{5.7}$$

for all distinct x, $y \in A_0$. In particular, $A_0 \subset X_0$ is an n_0-point subset. Now, let A be as in the statement of the theorem. Since A and A_0 are both n_0-point sets, there is a surjective map $\varphi\colon A_0 \to A$. Clearly, φ is L-Lipschitz for some $L \geq 1$. We define L-Lipschitz maps $\varphi_n\colon (V_n, \varrho_n) \to \mathcal{M}_n(A)$ recursively as follows. Because of (5.7), it follows that $\varphi_1 = \varphi$ is L-Lipschitz with respect to ϱ_1. Given $n \geq 2$ and $x \in V_n$, we set

$$\varphi_n(x) = m\big(\varphi_{n-1}(a), \varphi_{n-1}(b)\big)$$

if $x = m(a, b)$ with a, $b \in V_{n-1}$. Let x, $y \in V_n$ be such that $x \sim y$ in G_n. Hence, by definition, there is $v \in V_{n-1}$ and $u \sim w$ in G_{n-1} such that $x = m(v, u)$ and $y = m(v, w)$, and so

$$\begin{aligned} d(\varphi_n(x), \varphi_n(y)) &= d\big(m(\varphi_{n-1}(v), \varphi_{n-1}(u)), m(\varphi_{n-1}(v), \varphi_{n-1}(w))\big) \\ &\leq \frac{1}{2} d(\varphi_{n-1}(u), \varphi_{n-1}(w)) \leq L \cdot \frac{1}{2^{n-1}}, \end{aligned}$$

where in the last step we have used that φ_{n-1} is L-Lipschitz with respect to ϱ_{n-1}. Since $\varrho_n = 2^{-(n-1)} \cdot d_{G_n}$, it now follows directly from the above and the definition of the shortest-path metric d_{G_n} that φ_n is L-Lipschitz with respect to ϱ_n. By construction, $\varphi_n(x) = \varphi_m(x)$ for all $x \in V_n$ and $m \geq n$. Hence, as Y is complete these maps naturally give rise to a L-Lipschitz map $\Phi\colon X_0 \to Y$.

To finish the proof we show that $\Phi(X_0)$ is σ-convex. For simplicity, in the following we will denote the bicombings on X_0 and Y both by σ. By construction of Φ and since σ is induced by a conical midpoint map, it follows that $\Phi(\sigma(x, y, t)) = \sigma(\Phi(x), \Phi(y), t)$ for all x, $y \in \mathcal{M}_n(A_0)$ and all $t \in [0, 1]$. Let now x, $y \in X_0$ be arbitrary. Then there exists x_k, $y_k \in \mathcal{M}_{n_k}(A_0)$ such that $x_k \to x$ and $y_k \to y$ as $k \to \infty$, respectively. Moreover, $\sigma_{x_k y_k} \to \sigma_{xy}$ uniformly. Hence, as Φ is Lipschitz continuous, we have $\Phi(\sigma(x, y, t)) = \sigma(\Phi(x), \Phi(y), t)$ for all $t \in [0, 1]$. This shows that $\Phi(X_0)$ is σ-convex. □

6. Does X_0 admits a consistent conical bicombing?

In practice, it is often desirable to impose stronger properties on a bicombing than (1.1). By asserting that a conical bicombing is consistent, see Section 2.3 for the definition, one obtains an interesting class of bicombings which seem to be quite rigid. Following Haettel, we call a metric space a CUB-space if it admits a unique consistent conical bicombing (see [21]). The class of CUB-space is already quite rich and still growing. For example, in [5] it is shown that any convex body in a dual Banach space is CUB. Moreover, proper, finite-dimensional injective metric space are CUB and Deligne complexes of certain Artin groups are CUB if they are re-metrized by considering the length metric induced by the ℓ_∞-metric on each cell (see [11, 21]).

However, using a non-affine isometry first introduced by Schechtman [38], one can construct a complete metric space with two distinct consistent conical bicombings (see [5, Example 4.4]). On the other hand, up to the author's knowledge, there is no example of a metric space with a conical bicombing that does not also admit a consistent conical bicombing. In other words, the following question of Descombes and Lang [11] is still open.

Question 6.1 (Descombes–Lang)**.** *Let X be a complete metric space. Is it true that X admits a conical bicombing if and only if it admits a consistent conical bicombing.*

This question also appears in the problem list [35, p. 385]. A partial result that indicates a positive answer when X is proper has been obtained in [3, Theorem 1.4]. One difficulty in finding a negative answer to Question 6.1 lies in the fact that many know examples of metric spaces with a conical bicombing have locally a nice structure. In this situation one can then employ a generalized version of the Cartan-Hadamard theorem due to Miesch [32] to construct a consistent conical bicombing. The metric space X_0 is locally not 'nice' as it is fractal-like in nature. So we believe that it could be a potential candidate for a counterexample to Question 6.1.

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