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ENDOMORPHISM ALGEBRAS AND AUTOMORPHISM GROUPS OF CERTAIN COMPLEX TORI

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ABSTRACT. We study the endomorphism algebra and automorphism groups of complex tori, whose second rational cohomology group enjoys a certain Hodge property introduced by F. Campana.

1. INTRODUCTION

Let X be a connected compact complex Kähler manifold of dimension ≥ 2, $\mathrm{H}^2(X,\mathbb{Q})$ its second rational cohomology group equipped with the canonical rational Hodge structure, i.e., there is the Hodge decomposition

$$\mathrm{H}^2(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{H}^2(X,\mathbb{C}) = \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^{2,0}(X)$$

where $\mathrm{H}^{2,0}(X) = \Omega^2(X)$ is the space of holomorphic 2-forms on X, $\mathrm{H}^{0,2}(X)$ is the "complex-conjugate" of $\mathrm{H}^{2,0}(X)$ and $\mathrm{H}^{1,1}(X)$ coincides with its own "complex-conjugate" (see [7, Sections 2.1–2.2], [10, Ch. VI-VII])). The following property of X was introduced and studied by F. Campana [5, Definition 3.3]. (Recently, it was used in the study of coisotropic and lagrangian submanifolds of symplectic manifolds [1].)

Definition 1.1. A manifold X is *irreducible in weight 2* (irréductible en poids 2) if it enjoys the following property.

Let H be a rational Hodge substructure of $\mathrm{H}^2(X,\mathbb{Q})$ such that

$$H_{\mathbb{C}} \cap \mathrm{H}^{2,0}(X) \neq \{0\}$$

where $H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C}$.

Then $H_{\mathbb{C}}$ contains the whole $\mathrm{H}^{2,0}(X)$.

Our aim is to study complex tori T that enjoy this property.

1.2. Let $T = V/\Lambda$ be a complex torus of positive dimension g where V is a g-dimensional complex vector space, and Λ is a discrete lattice of rank $2g$

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in V. One may naturally identify Λ with the first integral homology group $\mathrm{H}_1(T,\mathbb{Z})$ of T and

$$\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q} = \{v \in V \mid \exists n \in \mathbb{Z} \setminus \{0\} \text{ such that } nv \in \Lambda\}$$

with the first rational homology group $\mathrm{H}_1(T,\mathbb{Q})$ of T. There are also natural isomorphisms of real vector spaces

$$\Lambda \otimes \mathbb{R} = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \to V, \ \lambda \otimes r \mapsto r\lambda$$

that may be viewed as isomorphisms related to the first real cohomology group $\mathrm{H}_1(T,\mathbb{R})$ of T:

$$\mathrm{H}_1(T,\mathbb{R}) = \mathrm{H}_1(T,\mathbb{Z}) \otimes \mathbb{R} = \mathrm{H}_1(T,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \to V.$$

In particular, there is a canonical isomorphism of real vector spaces

$$\mathrm{H}_1(T,\mathbb{R}) = V, \tag{1}$$

and a canonical isomorphism of complex vector spaces

$$\mathrm{H}_1(T,\mathbb{C}) = \mathrm{H}_1(T,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{H}_1(T,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C} =: V_{\mathbb{C}} \tag{2}$$

where $\mathrm{H}_1(T,\mathbb{C})$ is the first complex homology group of T.

There are natural isomorphisms of $\mathbb{R}$-algebras

$$\mathrm{End}_{\mathbb{Z}}(\Lambda) \otimes \mathbb{R} \cong \mathrm{End}_{\mathbb{R}}(V), \ u \otimes r \mapsto ru,$$

$$\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \otimes \mathbb{R} \cong \mathrm{End}_{\mathbb{R}}(V), \ u \otimes r \mapsto ru,$$

which give rise to the natural ring embeddings

$$\mathrm{End}_{\mathbb{Z}}(\Lambda) \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \subset \mathrm{End}_{\mathbb{R}}(V) \subset \mathrm{End}_{\mathbb{R}}(V) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}}). \tag{3}$$

Here the structure of an $2g$-dimensional *complex* vector space on $V_{\mathbb{C}}$ is defined by

$$z(v \otimes s) = v \otimes zs \ \forall v \otimes s \in V \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}, \ z \in \mathbb{C}.$$

If $u \in \mathrm{End}_{\mathbb{R}}(V)$ then we write $u_{\mathbb{C}}$ for the corresponding $\mathbb{C}$-linear operator in $V_{\mathbb{C}}$, i.e.,

$$u_{\mathbb{C}}(v \otimes z) = u(v) \otimes z \ \forall u \in V, z \in \mathbb{C}, v \otimes z \in V_{\mathbb{C}}. \tag{4}$$

Remark 1.3. Sometimes, we will identify $\mathrm{End}_{\mathbb{R}}(V)$ with its image $\mathrm{End}_{\mathbb{R}}(V) \otimes 1 \subset \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ and write u instead of $u_{\mathbb{C}}$, slightly abusing notation.

As usual, one may naturally extend the complex conjugation $z \mapsto \bar{z}$ on $\mathbb{C}$ to the $\mathbb{C}$-antilinear involution

$$V_{\mathbb{C}} \to V_{\mathbb{C}}, \ w \mapsto \bar{w}, \ v \otimes z \mapsto \overline{v \otimes z} = v \otimes \bar{z},$$

which is usually called the complex conjugation on $V_{\mathbb{C}}$. Clearly,

$$u_{\mathbb{C}}(\bar{w}) = \overline{u(w)} \ \forall u \in \mathrm{End}_{\mathbb{R}}(V), w \in V_{\mathbb{C}}. \tag{5}$$

This implies easily that the set of fixed points of the involution is

$$V = V \otimes 1 \subset V_{\mathbb{C}}.$$

Let $\mathrm{End}(T)$ be the endomorphism ring of the complex commutative Lie group T and $\mathrm{End}^0(T) = \mathrm{End}(T) \otimes \mathbb{Q}$ the corresponding endomorphism algebra, which is a finite-dimensional algebra over the field $\mathbb{Q}$ of rational numbers, see [8, 4, 2]. Then it is well known that there are canonical isomorphisms

$$\mathrm{End}(T) = \mathrm{End}_{\mathbb{Z}}(\Lambda) \cap \mathrm{End}_{\mathbb{C}}(V), \ \mathrm{End}^0(T) = \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \cap \mathrm{End}_{\mathbb{C}}(V).$$

Let $g \geq 2$ and

$$\mathrm{H}^2(T, \mathbb{Q}) = \bigwedge_{\mathbb{Q}}^{2}(\Lambda_{\mathbb{Q}}, \mathbb{Q})$$

be the *second rational cohomology group* of T, which carries the natural structure of a rational Hodge structure of weight two:

$$\mathrm{H}^2(T, \mathbb{Q}) = \mathrm{H}^2(T, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{H}^{2,0}(T) \oplus \mathrm{H}^{1,1}(T) \oplus \mathrm{H}^{0,2}(T)$$

where $\mathrm{H}^{2,0}(T) = \Omega^2(T)$ is the $g(g-1)/2$-dimensional space of holomorphic 2-forms on T.

Definition 1.4. Let $g = \dim(T) \geq 2$. We say that T is 2-*simple* if it is *irreducible of weight 2*, i.e., enjoys the following property.

Let H be a rational Hodge substructure of $\mathrm{H}^2(T, \mathbb{Q})$ such that

$$H_{\mathbb{C}} \cap \mathrm{H}^{2,0}(T) \neq \{0\}$$

where $H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C}$.

Then $H_{\mathbb{C}}$ contains the whole $\mathrm{H}^{2,0}(T)$.

Remark 1.5. We call such complex tori 2-simple, because they are simple in the usual meaning of this word if $g > 2$, see Theorem 1.7(i) below.

Example 1.6. (See [5, Example 3.4(2)].) If $g = 2$ then $\dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T)) = 1$. This implies that (in the notation of Definition 1.4) if $H_{\mathbb{C}} \cap \mathrm{H}^{2,0}(T) \neq \{0\}$ then $H_{\mathbb{C}}$ contains the whole $\mathrm{H}^{2,0}(T)$. Hence, every 2-dimensional complex torus is 2-simple.

In what follows we write $\mathrm{Aut}(T) = \mathrm{End}(T)^*$ for the automorphism group of the complex Lie group T.

Our main result is the following assertion.

Theorem 1.7. *Let T be a complex torus of dimension $g \geq 3$. Suppose that T is 2-simple.*

Then T enjoys the following properties.

(i) *T is simple.*
(ii) *If E is any subfield of $\mathrm{End}^0(T)$ then it is a number field, whose degree over $\mathbb{Q}$ is either 1 or g or $2g$.*
(iii) *$\mathrm{End}^0(T)$ is a number field E such that its degree $[E : \mathbb{Q}]$ is either 1 (i.e., $\mathrm{End}^0(T) = \mathbb{Q}$, $\mathrm{End}(T) = \mathbb{Z}$) or g or $2g$.*
(iv) *If $\mathrm{End}(T) = \mathbb{Z}$ then $\mathrm{Aut}(T) = \{\pm 1\}$.*
(v) *If $[E : \mathbb{Q}] = 2g$ then E is a purely imaginary number field and $\mathrm{Aut}(T) \cong \{\pm 1\} \times \mathbb{Z}^{g-1}$*

(vi) *Suppose that* $[E:\mathbb{Q}]=g$. *Then* $\mathrm{Aut}(T)\cong\mathbb{Z}^d\times\{\pm 1\}$ *where the integer* d *satisfies* $\frac{g}{2}-1\le d\le g-1$.

In addition, if T *is a complex abelian variety then* E *is a totally real number field and* $d=g-1$.

Remark 1.8. (i) It is well known (and can be easily checked) that T is *simple* if and only if the rational Hodge structure on $\Lambda_{\mathbb{Q}}=\mathrm{H}_1(T,\mathbb{Q})$ is *irreducible*. [1]

(ii) We may view $\mathrm{H}^2(T,\mathbb{Q})$ as the $\mathbb{Q}$-vector subspace $\mathrm{H}^2(T,\mathbb{Q})\otimes 1$ of $\mathrm{H}^2(T,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}=\mathrm{H}^2(T,\mathbb{C})$. Let us consider the $\mathbb{Q}$-vector (sub)space

$$\mathrm{H}^{1,1}(T,\mathbb{Q}):=\mathrm{H}^2(T,\mathbb{Q})\cap\mathrm{H}^{1,1}(T)$$

of 2-dimensional *Hodge cycles* on T. Notice that the *irreducibility* of the rational Hodge structure on $\Lambda_{\mathbb{Q}}$ implies the *complete reducibility* of the rational Hodge structure on $\mathrm{H}^2(T,\mathbb{Q})=\mathrm{Hom}_{\mathbb{Q}}\left(\bigwedge^2_{\mathbb{Q}}\Lambda_{\mathbb{Q}},\mathbb{Q}\right)$. (It follows from the reductiveness of the Mumford-Tate group of a simple torus [6, Sect. 2.2].) In light of (i) and Theorem 1.7(i), a complex torus T of dimension >2 is 2-simple if and only if it is simple and $\mathrm{H}^2(T,\mathbb{Q})$ splits into a direct sum of $\mathrm{H}^{1,1}(T,\mathbb{Q})$ and an *irreducible* rational Hodge substructure.

We prove Theorem 1.7 in Section 3, using explicit constructions related to the Hodge structure on $\Lambda_{\mathbb{Q}}$ that will be discussed in Section 2.

This paper may be viewed as a follow up of [8] and [2].

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2. Hodge structures

2.1. It is well known that $\Lambda_{\mathbb{Q}}=\mathrm{H}_1(T,\mathbb{Q})$ carries the natural structure of a rational Hodge structure of weight -1. Let us recall the construction. Let $J:V\to V$ be the multiplication by $\mathbf{i}=\sqrt{-1}$, which is viewed as a certain element of $\mathrm{End}_{\mathbb{R}}(V)$ such that

$$J^2=-1.$$

Hence, $J_{\mathbb{C}}^2=-1$ in $\mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ and we define two mutually complex-conjugate $\mathbb{C}$-vector subspaces (of the same dimension) $\mathrm{H}_{-1,0}(T)$ and $\mathrm{H}_{0,-1}(T)$ of $V_{\mathbb{C}}$ as the eigenspaces $V_{\mathbb{C}}(\mathbf{i})$ and $V_{\mathbb{C}}(-\mathbf{i})$of $J_{\mathbb{C}}$ attached to eigenvalues $\mathbf{i}$ and $-\mathbf{i}$ respectively. Clearly,

$$V_{\mathbb{C}}=V_{\mathbb{C}}(\mathbf{i})\oplus V_{\mathbb{C}}(-\mathbf{i})=\mathrm{H}_{-1,0}(T)\oplus\mathrm{H}_{0,-1}(T),$$

which defines the rational Hodge structure on $\Lambda_{\mathbb{Q}}$, in light of $V_{\mathbb{C}}=\Lambda_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C}$. It also follows that both $\mathrm{H}_{-1,0}(T)$ and $\mathrm{H}_{0,-1}(T)$ have the same dimension $2g/2=g$.

[1] A rational Hodge structure H is called *irreducible* or *simple* if its only rational Hodge substructures are H itself and $\{0\}$ [6, Sect. 2.2].

Now it's a time to recall that V is a complex vector space. I claim that the map

$$\Psi : V \to V_{\mathbb{C}}(\mathbf{i}) = \mathrm{H}_{-1,0}(T),\ v \mapsto Jv \otimes 1 + v \otimes \mathbf{i} \tag{6}$$

is an isomorphism of complex vector spaces. Indeed, first, Ψ defines a homomorphism of real vector spaces $V \to V_{\mathbb{C}}$. Second, if $v \in V$ then

$$J_{\mathbb{C}}(Jv \otimes 1 + v \otimes \mathbf{i}) = J^2 v \otimes 1 + Jv \otimes \mathbf{i} = -v \otimes 1 + Jv \otimes \mathbf{i} = \mathbf{i}(Jv \otimes 1 + v \otimes \mathbf{i}),$$

i.e., $Jv \otimes 1 + v \otimes \mathbf{i} \in V_{\mathbb{C}}(\mathbf{i}) = \mathrm{H}_{0,-1}(T)$ and therefore the map (6) is defined correctly. Third, taking into account that J is an automorphism of V and $V_{\mathbb{C}} = V \otimes 1 \oplus V \otimes \mathbf{i}$, we conclude that Ψ is an injective homomorphism of real vector spaces and the dimension arguments imply that is is actually an isomorphism. It remains to check that Ψ is $\mathbb{C}$-linear, i.e.,

$$\Psi(Jv) = \mathbf{i}\Psi(v).$$

Let us do it. We have

$$\Psi(Jv) = J(Jv) \otimes 1 + Jv \otimes \mathbf{i} = -v \otimes 1 + Jv \otimes \mathbf{i} = \mathbf{i}(Jv \otimes 1 + v \otimes \mathbf{i}) = \mathbf{i}\Psi(v).$$

Hence, Ψ is a $\mathbb{C}$-linear isomorphism and we are done.

Now suppose that $u \in \mathrm{End}_{\mathbb{R}}(V)$ commutes with J, i.e., $u \in \mathrm{End}_{\mathbb{C}}(V)$. Then

$$\Psi \circ u = u_{\mathbb{C}} \circ \Psi. \tag{7}$$

In particular, $\mathrm{H}_{-1,0}(T)$ is $u_{\mathbb{C}}$-invariant. Indeed, if $v \in V$ then

$$\Psi \circ u(v) = Ju(v) \otimes 1 + u(v) \otimes \mathbf{i} = uJ(v) \otimes 1 + u_{\mathbb{C}}(v \otimes \mathbf{i}) = u_{\mathbb{C}}(J(v) \otimes 1) + u_{\mathbb{C}}(v \otimes \mathbf{i}) = u_{\mathbb{C}} \circ \Psi(v),$$

which proves our claim.

Similarly, there is an anti-linear isomorphism of complex vector spaces

$$V \to V_{\mathbb{C}}(-\mathbf{i}) = \mathrm{H}_{0,-1}(T),\ v \mapsto Jv \otimes 1 - v \otimes \mathbf{i}.$$

It is also well known that there is a canonical isomorphism of rational Hodge structures of weight 2

$$\mathrm{H}^2(T, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(\bigwedge_{\mathbb{Q}}^2 \mathrm{H}_1(T, \mathbb{Q}), \mathbb{Q})$$

where the Hodge components $\mathrm{H}^{p,q}(T)$ $(p, q \geq 0, p + q = 2)$ are as follows.

$$\mathrm{H}^{2,0}(T) = \mathrm{Hom}_{\mathbb{C}}(\bigwedge_{\mathbb{C}}^2 (\mathrm{H}_{-1,0}(T), \mathbb{C}), \quad \mathrm{H}^{0,2}(T) = \mathrm{Hom}_{\mathbb{C}}(\bigwedge_{\mathbb{C}}^2 (\mathrm{H}_{0,-1}(T), \mathbb{C}), \tag{8}$$

$$\mathrm{H}^{1,1}(T) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_{-1,0}(T), \mathbb{C}) \wedge \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_{0,-1}(T), \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_{-1,0}(T), \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_{0,-1}(T), \mathbb{C}).$$

Clearly,

$$\dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T)) = \frac{g(g-1)}{2}.$$

3. Endomorphism Fields and Automorphism Groups

Proof of Theorem 1.7. Let T be a 2-simple complex torus and

$$g = \dim(T) \geq 3.$$

(i) Suppose that T is *not* simple. This means that there is a proper complex subtorus $S = W/\Gamma$ where W is a complex vector subspace of V with

$$0 < d = \dim_{\mathbb{C}}(W) < \dim_{\mathbb{C}}(V) = g$$

such that

$$\Gamma = W \cap \Lambda$$

is a discrete lattice of rank $2d$ in W. Then the quotient T/S is a complex torus of positive dimension $g - d$.

Let $H \subset \mathrm{H}^2(T, \mathbb{Q})$ be the image of the canonical *injective* homomorphism of rational Hodge structures $\mathrm{H}^2(T/S, \mathbb{Q}) \hookrightarrow \mathrm{H}^2(T, \mathbb{Q})$ induced by the quotient map $T \to T/S$ of complex tori. Clearly, H is a rational Hodge substructure of $\mathrm{H}^2(T, \mathbb{Q})$ and its $(2,0)$-component

$$H^{2,0} \subset H_{\mathbb{C}}$$

has $\mathbb{C}$-dimension

$$\dim_{\mathbb{C}}(H^{2,0}) = \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T/S))) = \frac{(g-d)(g-d-1)}{2} < \frac{g(g-1)}{2} = \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T))).$$

In light of 2-simplicity of T,

$$\dim_{\mathbb{C}}(\mathrm{H}^{2,0}) = 0,$$

which implies that

$$g - d = 1.$$

On the other hand, let $\tilde{H}$ be the kernel of the canonical *surjective* homomorphism of rational Hodge structures $\mathrm{H}^2(T, \mathbb{Q}) \twoheadrightarrow \mathrm{H}^2(S, \mathbb{Q})$ induced by the inclusion map $S \subset T$ of complex tori. Clearly, $\tilde{H}$ is a rational Hodge substructure of $\mathrm{H}^2(T, \mathbb{Q})$. Notice that the induced homomorphism of $(2,0)$-components $\mathrm{H}^{2,0}(T) \to \mathrm{H}^{2,0}(S)$ is also surjective, because every holomorphic 2-form on S obviously extends to a holomorphic 2-form on T. This implies that the $(2,0)$-component

$$\tilde{H}^{2,0} \subset \tilde{H}_{\mathbb{C}}$$

of $\tilde{H}$ has $\mathbb{C}$-dimension

$$\dim_{\mathbb{C}}(\tilde{H}^{2,0}) = \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T))) - \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(S))) = \frac{g(g-1)}{2} - \frac{d(d-1)}{2} > 0.$$

In light of 2-simplicity of T,

$$\dim_{\mathbb{C}}(\tilde{H}^{2,0}) = \dim_{\mathbb{C}}(\mathrm{H}^{2,0}(T))) = \frac{g(g-1)}{2},$$

which implies that $\frac{d(d-1)}{2} = 0$, i.e., $d = 1$. Taking into account that $g - d = 1$, we get $g = 1 + 1 = 2$, which is not true. The obtained contradiction proves that T is simple and (i) is proven. In particular, $\mathrm{End}^0(T)$ is a division algebra over $\mathbb{Q}$.

In order to handle (ii), let us assume that E is a subfield of $\mathrm{End}^0(T)$. The simplicity of T implies that $1 \in E$ is the identity automorphism of T. Then $\Lambda_{\mathbb{Q}}$ becomes a faithful E-module. This implies that E is a number field and $\Lambda_{\mathbb{Q}}$ is an E-vector space of finite positive dimension

$$d_E = \frac{2g}{[E:\mathbb{Q}]}.$$

This implies that $V_{\mathbb{C}} = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ is a free $E \otimes_{\mathbb{Q}} \mathbb{C}$-module of rank d_E. Clearly, both $\mathrm{H}_{-1,0}(T)$ and $\mathrm{H}_{0,-1}(T)$ are $E \otimes_{\mathbb{Q}} \mathbb{C}$-submodules of its direct sum $V_{\mathbb{C}}$. Let

$$\mathrm{tr}_{E/\mathbb{Q}} : E \to \mathbb{Q}$$

bet the trace map attached to the field extension $E/\mathbb{Q}$ of finite degree. Let

$$\mathrm{Hom}_E(\bigwedge_E^2 \Lambda_{\mathbb{Q}}, E)$$

be the $\frac{d_E(d_E-1)}{2}$-dimensional E-vector space of alternating E-bilinear forms on $\Lambda_{\mathbb{Q}}$ that carries the natural structure of a rational Hodge structure of $\mathbb{Q}$-dimension $[E:\mathbb{Q}] \cdot \frac{d_E(d_E-1)}{2}$. There is the natural embedding of rational Hodge structures

$$\mathrm{Hom}_E\left(\bigwedge_E^2 \Lambda_{\mathbb{Q}}, E\right) \hookrightarrow \mathrm{Hom}_{\mathbb{Q}}(\bigwedge_{\mathbb{Q}}^2 \Lambda_{\mathbb{Q}}, \mathbb{Q}) = \mathrm{H}^2(T,\mathbb{Q}),\ \phi_E \mapsto \phi := \mathrm{tr}_{E/\mathbb{Q}} \circ \phi_E, \tag{9}$$

i.e.,

$$\phi(\lambda_1,\lambda_2) = \mathrm{tr}_{E/\mathbb{Q}}\big(\phi_E(\lambda_1,\lambda_2)\big)\ \forall \lambda_1,\lambda_2 \in \Lambda_{\mathbb{Q}}. \tag{10}$$

The image of $\mathrm{Hom}_E\left(\bigwedge_E^2 \Lambda_{\mathbb{Q}}, E\right)$ in $\mathrm{Hom}_{\mathbb{Q}}\left(\bigwedge_{\mathbb{Q}}^2 \Lambda_{\mathbb{Q}}, \mathbb{Q}\right) = \mathrm{H}^2(T,\mathbb{Q})$ coincides with the $\mathbb{Q}$-vector subspace

$$H_E := \{\phi \in \mathrm{Hom}_{\mathbb{Q}}\left(\bigwedge_{\mathbb{Q}}^2 \Lambda_{\mathbb{Q}}, \mathbb{Q}\right) \mid \phi(u\lambda_1,\lambda_2) = \phi(\lambda_1, u\lambda_2)\ \forall u \in E,\ \lambda_1,\lambda_2 \in \Lambda_{\mathbb{Q}}\}. \tag{11}$$

Indeed, it is obvious that the image lies in H_E. In order to check that the image coincide with the whole subspace H_E, let us construct the inverse map

$$H_E \to \mathrm{Hom}_E\left(\bigwedge_E^2 \Lambda_{\mathbb{Q}}, E\right),\ \phi \mapsto \phi_E$$

to (9) as follows. If $\lambda_1, \lambda_2 \in \Lambda_{\mathbb{Q}}$ then there is a $\mathbb{Q}$-linear map

$$\Phi : E \mapsto \mathbb{Q},\ u \mapsto \phi(u\lambda_1,\lambda_2) = \phi(\lambda_1, u\lambda_2) = -\phi(u\lambda_2,\lambda_1) = -\phi(\lambda_2, u\lambda_1). \tag{12}$$

The properties of trace map imply that there exists precisely one $\beta \in E$ such that

$$\Phi(u) = \mathrm{tr}_{E/\mathbb{Q}}(u\beta)\ \forall u \in E.$$

Let us put

$$\phi_E(\lambda_1, \lambda_2) := \beta.$$

It follows from (12) that $\phi_E \in \mathrm{Hom}_E\left(\bigwedge_E^2 \Lambda_{\mathbb{Q}}, E\right)$. In addition,

$$\mathrm{tr}_{E/\mathbb{Q}}(\phi_E(\lambda_1, \lambda_2)) = \mathrm{tr}_{E/\mathbb{Q}}(\beta) = \mathrm{tr}_{E/\mathbb{Q}}(1 \cdot \beta) = \Phi(1) = \phi(\lambda_1, \lambda_2),$$

which proves that $\phi \mapsto \phi_E$ is indeed the inverse map, in light of (10).

Clearly, H_E is a rational Hodge substructure of $\mathrm{H}^2(T, \mathbb{Q})$.

By 2-simplicity of T, the $\mathbb{C}$-dimension of the $(2,0)$-component $H_E^{(2,0)}$ of H_E is either 0 or $g(g-1)/2$. Let us express this dimension explicitly in terms of g and $[E : \mathbb{Q}]$.

In order to do that, let us consider the $[E : \mathbb{Q}]$-element set Σ_E of all field embedding $\sigma : E \hookrightarrow \mathbb{C}$. We have

$$E_{\mathbb{C}} := E \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{\sigma \in \Sigma_E} \mathbb{C}_\sigma \quad \text{where } \mathbb{C}_\sigma = E \otimes_{E,\sigma} \mathbb{C} = \mathbb{C}, \tag{13}$$

which gives us the splitting of $E_{\mathbb{C}}$-modules

$$V_{\mathbb{C}} = \oplus_{\sigma \in \Sigma_E} V_\sigma = \oplus_{\sigma \in \Sigma_E} \left(\mathrm{H}_{-1,0}(T)_\sigma \oplus \mathrm{H}_{0,-1}(T)_\sigma\right) \tag{14}$$

where for all $\sigma \in \Sigma_E$ we define

$$\mathrm{H}_{-1,0}(T)_\sigma := \mathbb{C}_\sigma \mathrm{H}_{-1,0}(T) = \{x \in \mathrm{H}_{-1,0}(T) \mid u_{\mathbb{C}} x = \sigma(u)x\ \forall u \in E\} \subset \mathrm{H}_{-1,0}(T);$$

$$n_\sigma := \dim_{\mathbb{C}}(\mathrm{H}_{-1,0}(T)_\sigma);$$

$$\mathrm{H}_{0,-1}(T)_\sigma := \mathbb{C}_\sigma \mathrm{H}_{0,-1}(T) = \{x \in \mathrm{H}_{0,-1}(T) \mid u_{\mathbb{C}} x = \sigma(u)x\ \forall u \in E\} \subset \mathrm{H}_{0,-1}(T);$$

$$m_\sigma := \dim_{\mathbb{C}}(\mathrm{H}_{0,-1}(T)_\sigma);$$

$$V_\sigma = \mathbb{C}_\sigma = \mathbb{C}_\sigma V_{\mathbb{C}} = \{x \in V_{\mathbb{C}} \mid u_{\mathbb{C}} x = \sigma(u)x\ \forall u \in E\} = \mathrm{H}_{-1,0}(T)_\sigma \oplus \mathrm{H}_{0,-1}(T)_\sigma$$

Since $\mathrm{H}_{-1,0}(T) \oplus \mathrm{H}_{0,-1}(T) = V_{\mathbb{C}}$ is a free $E_{\mathbb{C}}$-module of rank d_E, its direct summand V_σ is a $\mathbb{C}_\sigma = \mathbb{C}$-vector space of dimension d_E and therefore

$$n_\sigma + m_\sigma = d_E\ \forall \sigma \in \Sigma_E. \tag{15}$$

Since $\mathrm{H}_{-1,0}(T)$ and $\mathrm{H}_{0,-1}(T)$ are mutually complex-conjugate subspaces of $V_{\mathbb{C}}$, it follows from (5) that

$$m_\sigma = n_{\bar\sigma} \quad \text{where } \bar\sigma : E \hookrightarrow \mathbb{C},\ u \mapsto \overline{\sigma(u)}$$

is the *complex-conjugate* of σ. Therefore, in light of (15),

$$n_\sigma + n_{\bar\sigma} = d_E\ \forall \sigma. \tag{16}$$

We have

$$\sum_{\sigma \in \Sigma_E} n_\sigma = \sum_{\sigma \in \Sigma_E} \dim_{\mathbb{C}}(\mathrm{H}_{-1,0}(T)_\sigma) = \dim_{\mathbb{C}}(\mathrm{H}_{-1,0}(T)) = g. \tag{17}$$

Let us consider the complexification of H_E

$$H_{E,\mathbb{C}} := H_E \otimes_{\mathbb{Q}} \mathbb{C} \subset \mathrm{Hom}_{\mathbb{Q}}\left(\bigwedge^2 \Lambda_{\mathbb{Q}}, \mathbb{Q}\right) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge_{\mathbb{C}}^2 (\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}), \mathbb{C}\right) = \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge^2 V_{\mathbb{C}}, \mathbb{C}\right).$$

In light of (11),

$$H_{E,\mathbb{C}} = \{\phi \in \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge^2 V_{\mathbb{C}}, \mathbb{C}\right) \mid \phi(u_{\mathbb{C}} x, y) = \phi(x, u_{\mathbb{C}} y) \ \forall u \in E, ; x, y \in V_{\mathbb{C}}\} \tag{18}$$

$$= \{\phi \in \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge^2 V_{\mathbb{C}}, \mathbb{C}\right) \mid \phi(u_{\mathbb{C}} x, y) = \phi(x, u_{\mathbb{C}} y) \ \forall u \in E_{\mathbb{C}}; x, y \in V_{\mathbb{C}}\}.$$

In particular, if $\sigma, \tau \in \Sigma_E$ are *distinct* field embeddings then for all $\phi \in H_{E,\mathbb{C}}$

$$\phi(V_\sigma, V_\tau) = \phi(V_\tau, V_\sigma) = \{0\}.$$

This implies that

$$H_{E,\mathbb{C}} = \oplus_{\sigma \in \Sigma_E} \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge_{\mathbb{C}}^2 V_\sigma, \mathbb{C}\right) \tag{19}$$

$$= \oplus_{\sigma \in \Sigma_E} \mathrm{Hom}_{\mathbb{C}}\big(\bigwedge_{\mathbb{C}}^2 (\mathrm{H}_{-1,0}(T)_\sigma \oplus \mathrm{H}_{0,-1}(T)_\sigma), \mathbb{C}\big).$$

In light of (8), the $(2,0)$-Hodge component of $H_{E,\mathbb{C}}$

$$H_E^{(2,0)} = \oplus_{\sigma \in \Sigma_E} \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge_{\mathbb{C}}^2 \mathrm{H}_{-1,0}(T)_\sigma, \mathbb{C}\right) \ \text{ and } \ \dim_{\mathbb{C}}(H_E^{(2,0)}) = \sum_{\sigma \in \Sigma_E} \frac{n_\sigma(n_\sigma - 1)}{2}. \tag{20}$$

This implies that $\dim_{\mathbb{C}}(H_E^{(2,0)}) = 0$ if and only if all $n_\sigma \in \{0,1\}$. If this is the case then, in light of (16), $d_E \in \{1,2\}$, i.e., $[E:\mathbb{Q}] = 2g$ or g.

On the other hand, it follows from (17) combined with the second formula in (20) that $\dim_{\mathbb{C}}(H_E^{(2,0)}) = g(g-1)/2$ if and only if there is precisely one σ with $n_\sigma = g$ (and all the other multiplicities n_τ are 0). This implies that either $d_E = 2g$ and $E = \mathbb{Q}$ or $d_E = g$ and E an imaginary quadratic field with the pair of the field embeddings

$$\sigma, \bar{\sigma} : E \hookrightarrow: \mathbb{C}$$

such that

$$n_\sigma = g, \ n_{\bar{\sigma}} = 0.$$

Let us assume that $d_E = g$. Then E is an imaginary quadratic field; in addition,

$$u \in E \subset \mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) \subset \mathrm{End}_{\mathbb{R}}(V)$$

then $u_{\mathbb{C}}$ acts on $\mathrm{H}_{-1,0}(T)$ as multiplication by $\sigma(u) \in \mathbb{C}$. In light of (5), $u_{\mathbb{C}}$ acts on the complex-conjugate subspace $\mathrm{H}_{0,-1}(T)$ as multiplication by $\overline{\sigma(u)} = \bar{\sigma}(u) \in \mathbb{C}$. Since E is an imaginary quadratic field, there are a positive integer D and $\alpha \in E$ such that $\alpha^2 = -D$ and $E = \mathbb{Q}(\alpha)$. It follows that $\sigma(\alpha) = \pm\mathbf{i}\sqrt{D}$. Replacing if necessary α by $-\alpha$, we may and will assume that

$$\sigma(\alpha) = \mathbf{i}\sqrt{D}$$

and therefore $\alpha_{\mathbb{C}}$ acts on $\mathrm{H}_{-1,0}(T)$ as multiplication by $\mathbf{i}\sqrt{D}$. Hence, $\alpha_{\mathbb{C}}$ acts on $\mathrm{H}_{0,-1}(T)$ as multiplication by $\overline{\mathbf{i}\sqrt{D}} = -\mathbf{i}\sqrt{D}$. Since

$$V_{\mathbb{C}} = \mathrm{H}_{-1,0}(T) \oplus \mathrm{H}_{0,-1}(T),$$

we get $\alpha_{\mathbb{C}} = \sqrt{D}J_C$ and therefore

$$\alpha = \sqrt{D}J.$$

This implies that the centralizer $\mathrm{End}^0(T)$ of J in $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$ coincides with the centralizer of α in $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$, which, in turn, coincides with the centralizer $\mathrm{End}_E(\Lambda_{\mathbb{Q}})$ of E in $\mathrm{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$, i.e.,

$$\mathrm{End}^0(T) = \mathrm{End}_E(\Lambda_{\mathbb{Q}}) \cong \mathrm{Mat}_{d_E}(E).$$

This is the matrix algebra, which is not a division algebra, because $d_E = g > 1$. This contradicts to the simplicity of T. The obtained contradiction rules out the case $d_E = g$. This ends the proof of (ii).

In order to prove (iii), recall that $\mathrm{End}^0(T)$ is a division algebra of $\mathbb{Q}$, thanks to the simplicity of T [8]. Hence $\Lambda_{\mathbb{Q}}$ is a free $\mathrm{End}^0(T)$-module of finite positive rank and therefore

$$\dim_{\mathbb{Q}}(\mathrm{End}^0(T))|2g, \tag{21}$$

because $2g = \dim_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})$. We will apply several times already proven assertion (ii) to various subfields of $\mathrm{End}^0(T)$.

Suppose that $\mathrm{End}^0(T)$ is *not* a field and let $\mathcal{Z}$ be its center. Then $\mathcal{Z}$ is a number field and there is an integer $d > 1$ such that $\dim_{\mathcal{Z}}(\mathrm{End}^0(T)) = d^2$ and therefore

$$\dim_{\mathbb{Q}}(\mathrm{End}^0(T)) = d^2 \cdot [\mathcal{Z} : \mathbb{Q}]$$

divides $2g$, thanks to (21). Since $\mathcal{Z}$ is a subfield of $\mathrm{End}^0(T)$, the degree $[\mathcal{Z} : \mathbb{Q}]$ is either 1 or g or $2g$. If $[\mathcal{Z} : \mathbb{Q}] > 1$ then $2g$ is divisible by

$$d^2 \cdot [\mathcal{Z} : \mathbb{Q}] \geq 2^2 g = 4g,$$

which is nonsense. Hence, $[\mathcal{Z} : \mathbb{Q}] = 1$, i.e., $\mathcal{Z} = \mathbb{Q}$ and $\mathrm{End}^0(T)$ is a central division $\mathbb{Q}$-algebra of dimension d^2 with $d^2|2g$. Then every maximal subfield E of the division algebra $\mathrm{End}^0(T)$ has degree d over $\mathbb{Q}$. Hence $d \in \{1, g, 2g\}$. Since $d > 1$, we obtain that either $d = g$ and $g^2|2g$ or $d = 2g$ and $(2g)^2|2g$. This implies that $d = g$ and $g = 1$ or 2. Since $g \geq 3$, we get a contradiction, which implies that $\mathrm{End}^0(T)$ is a field.

It follows from already proven assertion (ii) that the degree $\dim_{\mathbb{Q}}(\mathrm{End}^0(T))$ of the number field $\mathrm{End}^0(T)$ is either 1 or g or $2g$. Assertion (iv) is obvious and was included just for the sake of completeness.

In order to handle the structure of $\mathrm{Aut}(T)$, let us check first that the only roots of unity in $\mathrm{End}^0(T)$ are 1 and -1. If this is not the case then the field $\mathrm{End}^0(T)$ contains either $\sqrt{-1}$ or a primitive pth root of unity ζ where p is a certain odd prime. In the former case $\mathrm{End}^0(T)$ contains the quadratic field $\mathbb{Q}(\sqrt{-1})$, which contradicts (ii). In the latter case $\mathrm{End}^0(T)$ contains either the quadratic field $\mathbb{Q}(\sqrt{-p})$ or the quadratic field $\mathbb{Q}(\sqrt{p})$: each of these outcomes contradicts to (ii) as well.

Now recall that $\mathrm{End}(T)$ is an order in the number field $E = \mathrm{End}^0(T)$ and $\mathrm{Aut}(T) = \mathrm{End}(T)^*$ is its group of units. By Theorem of Dirichlet about units [3, Ch. II, Sect. 4, Th. 5], the group of units

$$\mathrm{Aut}(T) \cong \mathbb{Z}^d \times \{\pm 1\} \quad \text{with} \quad d = r + s - 1 \tag{22}$$

where r is the number of real field embeddings $E \hookrightarrow \mathbb{R}$ and

$$r + 2s = [E : \mathbb{Q}], \quad \text{i.e.,} \quad s = \frac{[E : \mathbb{Q}] - r}{2}. \tag{23}$$

Let us prove (v). Assume that the number field $E := \mathrm{End}^0(T)$ has degree $2g$. The dimension arguments imply that $\Lambda_{\mathbb{Q}}$ is a 1-dimensional E-vector space and $V = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is a free $E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R}$-module of rank 1. Hence $E_{\mathbb{R}}$ coincides with its own centralizer $\mathrm{End}_{E_{\mathbb{R}}}(V)$ in $\mathrm{End}_{\mathbb{R}}(V)$. Since J commutes with $\mathrm{End}^{)}(T) = E$, it also commutes with $E_{\mathbb{R}}$ and therefore

$$J \in \mathrm{End}_{E_{\mathbb{R}}}(V) = E_{\mathbb{R}}.$$

Recall that the $\mathbb{R}$-algebra $E_{\mathbb{R}}$ is isomorphic to a product of copies of $\mathbb{R}$ and $\mathbb{C}$. Since $J^2 = -1$, the only copies of $\mathbb{C}$ appear in $E_{\mathbb{R}}$, i.e., E is purely imaginary, which means that $r = 0$ and therefore $2g = [E : \mathbb{Q}] = 2s$. This proves the first assertion of (v); the second one follows readily from (22) combined with (23).

Let us prove (vi). Assume that $[E : \mathbb{Q}] = g$. Then the first assertion follows readily from (22) combined with (23).

Assume now that T is a complex abelian variety. By Albert's classification [9], $E = \mathrm{End}^0(T)$ is either a totally real number field or a CM field. If E is a CM field then it contains a subfield E_0 of degree $[E : \mathbb{Q}]/2 = g/2$. Since E_0 is a subfield of $\mathrm{End}^0(T)$ and $1 < g/2 < g$ (recall that $g \geq 3$), the existence of E_0 contradicts to the already proven assertion (ii). This proves that E is a totally real number field, i.e., $s = 0, r = g$. Now the assertion about $\mathrm{Aut}(T)$ follows from (22). □

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