Endomorphism algebras and automorphism groups of certain complex tori

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Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Pennsylvania State University
Department of Mathematics
University Park, PA 16802
USA
ENDOMORPHISM ALGEBRAS AND AUTOMORPHISM GROUPS OF CERTAIN COMPLEX TORI

YURI G. ZARHIN

Abstract. We study the endomorphism algebra and automorphism groups of complex tori, whose second rational cohomology group enjoys a certain Hodge property introduced by F. Campana.

1. Introduction

Let $X$ be a connected compact complex Kähler manifold of dimension $\geq 2$, $H^2(X, \mathbb{Q})$ its second rational cohomology group equipped with the canonical rational Hodge structure, i.e., there is the Hodge decomposition

$$H^2(X, \mathbb{Q}) = H^2(X, \mathbb{C}) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$$

where $H^{2,0}(X) = \Omega^2(X)$ is the space of holomorphic 2-forms on $X$, $H^{0,2}(X)$ is the “complex-conjugate” of $H^{2,0}(X)$ and $H^{1,1}(X)$ coincides with its own “complex-conjugate” (see [7, Sections 2.1–2.2], [10, Ch. VI-VII]). The following property of $X$ was introduced and studied by F. Campana [5, Definition 3.3]. (Recently, it was used in the study of coisotropic and lagrangian submanifolds of symplectic manifolds [1].)

Definition 1.1. A manifold $X$ is irreducible in weight 2 (irréductible en poids 2) if it enjoys the following property.

Let $H$ be a rational Hodge substructure of $H^2(X, \mathbb{Q})$ such that

$$H \cap H^{2,0}(X) \neq \{0\}$$

where $H_C := H \otimes_\mathbb{Q} \mathbb{C}$.

Then $H_C$ contains the whole $H^{2,0}(X)$.

Our aim is to study complex tori $T$ that enjoy this property.

1.2. Let $T = V/\Lambda$ be a complex torus of positive dimension $g$ where $V$ is a $g$-dimensional complex vector space, and $\Lambda$ is a discrete lattice of rank $2g$
in $V$. One may naturally identify $\Lambda$ with the first integral homology group $H_1(T, \mathbb{Z})$ of $T$ and

$$\Lambda_\mathbb{Q} = \Lambda \otimes \mathbb{Q} = \{ v \in V \mid \exists n \in \mathbb{Z} \setminus \{0\} \text{ such that } nv \in \Lambda \}$$

with the first rational homology group $H_1(T, \mathbb{Q})$ of $T$. There are also natural isomorphisms of real vector spaces

$$\Lambda \otimes \mathbb{R} = \Lambda_\mathbb{Q} \otimes \mathbb{Q} \otimes \mathbb{R} \rightarrow V, \lambda \otimes r \mapsto r\lambda$$

that may be viewed as isomorphisms related to the first real cohomology group $H^1(T, \mathbb{R})$ of $T$:

$$H_1(T, \mathbb{R}) = H_1(T, \mathbb{Q}) \otimes \mathbb{R} \rightarrow V.$$ 

In particular, there is a canonical isomorphism of real vector spaces

$$H_1(T, \mathbb{R}) = V, \quad (1)$$

and a canonical isomorphism of complex vector spaces

$$H_1(T, \mathbb{C}) = H_1(T, \mathbb{Q}) \otimes \mathbb{C} = H_1(T, \mathbb{R}) \otimes \mathbb{C} = V \otimes \mathbb{C} =: V_\mathbb{C}. \quad (2)$$

where $H_1(T, \mathbb{C})$ is the first complex homology group of $T$.

There are natural isomorphisms of $\mathbb{R}$-algebras

$$\text{End}_\mathbb{Z}(\Lambda) \otimes \mathbb{R} \cong \text{End}_\mathbb{R}(V), \quad u \otimes r \mapsto ru,$$

$$\text{End}_\mathbb{Q}(\Lambda_\mathbb{Q}) \otimes \mathbb{R} \cong \text{End}_\mathbb{R}(V), \quad u \otimes r \mapsto ru,$$

which give rise to the natural ring embeddings

$$\text{End}_\mathbb{Z}(\Lambda) \subset \text{End}_\mathbb{Q}(\Lambda_\mathbb{Q}) \subset \text{End}_\mathbb{R}(V) \subset \text{End}_\mathbb{R}(V) \otimes \mathbb{C} = \text{End}_\mathbb{C}(V_\mathbb{C}). \quad (3)$$

Here the structure of an $2g$-dimensional complex vector space on $V_\mathbb{C}$ is defined by

$$z(v \otimes s) = v \otimes zs \forall v \otimes s \in V \otimes \mathbb{C} = V_\mathbb{C}, \quad z \in \mathbb{C}.$$ 

If $u \in \text{End}_\mathbb{R}(V)$ then we write $u_\mathbb{C}$ for the corresponding $\mathbb{C}$-linear operator in $V_\mathbb{C}$, i.e.,

$$u_\mathbb{C}(v \otimes z) = u(v \otimes z) \forall v \in V, z \in \mathbb{C}, v \otimes z \in V_\mathbb{C}. \quad (4)$$

**Remark 1.3.** Sometimes, we will identify $\text{End}_\mathbb{R}(V)$ with its image $\text{End}_\mathbb{R}(V) \otimes 1 \subset \text{End}_\mathbb{C}(V_\mathbb{C})$ and write $u$ instead of $u_\mathbb{C}$, slightly abusing notation.

As usual, one may naturally extend the complex conjugation $z \mapsto \bar{z}$ on $\mathbb{C}$ to the $\mathbb{C}$-antilinear involution

$$V_\mathbb{C} \rightarrow V_\mathbb{C}, \quad w \mapsto \bar{w}, \quad v \otimes z \mapsto \overline{v \otimes z} = v \otimes \bar{z},$$

which is usually called the complex conjugation on $V_\mathbb{C}$. Clearly,

$$u_\mathbb{C}(\bar{w}) = \overline{u(w)} \forall u \in \text{End}_\mathbb{R}(V), w \in V_\mathbb{C}. \quad (5)$$

This implies easily that the set of fixed points of the involution is

$$V = V \otimes 1 \subset V_\mathbb{C}.$$ 

Let $\text{End}(T)$ be the endomorphism ring of the complex commutative Lie group $T$ and $\text{End}^0(T) = \text{End}(T) \otimes \mathbb{Q}$ the corresponding endomorphism
algebra, which is a finite-dimensional algebra over the field $\mathbb{Q}$ of rational numbers, see [8, 4, 2]. Then it is well known that there are canonical isomorphisms

$$\text{End}(T) = \text{End}_\mathbb{Z}(A) \cap \text{End}_\mathbb{C}(V), \quad \text{End}^0(T) = \text{End}_\mathbb{Q}(\Lambda) \cap \text{End}_\mathbb{C}(V).$$

Let $g \geq 2$ and

$$H^2(T, \mathbb{Q}) = \bigwedge^2 (\mathbb{Q}(\Lambda), \mathbb{Q})$$

be the second rational cohomology group of $T$, which carries the natural structure of a rational Hodge structure of weight two:

$$H^2(T, \mathbb{Q}) = H^2(T, \mathbb{Q}) \otimes \mathbb{C} = H^{2,0}(T) \oplus H^{1,1}(T) \oplus H^{0,2}(T)$$

where $H^{2,0}(T) = \Omega^2(T)$ is the $(g-1)/2$-dimensional space of holomorphic 2-forms on $T$.

**Definition 1.4.** Let $g = \dim(T) \geq 2$. We say that $T$ is 2-simple if it is irreducible of weight 2, i.e., enjoys the following property.

Let $H$ be a rational Hodge substructure of $H^2(T, \mathbb{Q})$ such that

$$H_C \cap H^{2,0}(T) \neq \{0\}$$

where $H_C := H \otimes \mathbb{Q} \mathbb{C}$.

Then $H_C$ contains the whole $H^{2,0}(T)$.

**Remark 1.5.** We call such complex tori 2-simple, because they are simple in the usual meaning of this word if $g > 2$, see Theorem 1.7(i) below.

**Example 1.6.** (See [5, Example 3.4(2)].) If $g = 2$ then $\dim_C(H^{2,0}(T)) = 1$. This implies that (in the notation of Definition 1.4) if $H_C \cap H^{2,0}(T) \neq \{0\}$ then $H_C$ contains the whole $H^{2,0}(T)$. Hence, every 2-dimensional complex torus is 2-simple.

In what follows we write $\text{Aut}(T) = \text{End}(T)^*$ for the automorphism group of the complex Lie group $T$.

Our main result is the following assertion.

**Theorem 1.7.** Let $T$ be a complex torus of dimension $g \geq 3$. Suppose that $T$ is 2-simple.

Then $T$ enjoys the following properties.

(i) $T$ is simple.

(ii) If $E$ is any subfield of $\text{End}^0(T)$ then it is a number field, whose degree over $\mathbb{Q}$ is either 1 or $g$ or $2g$.

(iii) $\text{End}^0(T)$ is a number field $E$ such that its degree $[E : \mathbb{Q}]$ is either 1 (i.e., $\text{End}^0(T) = \mathbb{Q}$, $\text{End}(T) = \mathbb{Z}$) or $g$ or $2g$.

(iv) If $\text{End}(T) = \mathbb{Z}$ then $\text{Aut}(T) = \{\pm 1\}$.

(v) If $[E : \mathbb{Q}] = 2g$ then $E$ is a purely imaginary number field and $\text{Aut}(T) \cong \{\pm 1\} \times \mathbb{Z}^{g-1}$.
(vi) Suppose that $[E : \mathbb{Q}] = g$. Then $\text{Aut}(T) \cong \mathbb{Z}^d \times \{\pm 1\}$ where the integer $d$ satisfies $\frac{g}{2} - 1 \leq d \leq g - 1$.

In addition, if $T$ is a complex abelian variety then $E$ is a totally real number field and $d = g - 1$.

**Remark 1.8.**

(i) It is well known (and can be easily checked) that $T$ is simple if and only if the rational Hodge structure on $\Lambda_\mathbb{Q} = H_1(T, \mathbb{Q})$ is irreducible. \(^1\)

(ii) We may view $H^2(T, \mathbb{Q})$ as the $\mathbb{Q}$-vector subspace $H^2(T, \mathbb{Q}) \otimes \mathbb{C} = H^2(T, \mathbb{C})$. Let us consider the $\mathbb{Q}$-vector (sub)space

$$H^{1,1}(T, \mathbb{Q}) := H^2(T, \mathbb{Q}) \cap H^{1,1}(T)$$

of 2-dimensional Hodge cycles on $T$. Notice that the irreducibility of the rational Hodge structure on $\Lambda_\mathbb{Q}$ implies the complete reducibility of the rational Hodge structure on $H^2(T, \mathbb{Q}) = \text{Hom}_\mathbb{Q}(\Lambda_\mathbb{Q}^2, \Lambda_\mathbb{Q})$. (It follows from the reductiveness of the Mumford-Tate group of a simple torus [6, Sect. 2.2].) In light of (i) and Theorem 1.7(i), a complex torus $T$ of dimension $> 2$ is 2-simple if and only if it is simple and $H^2(T, \mathbb{Q})$ splits into a direct sum of $H^{1,1}(T, \mathbb{Q})$ and an irreducible rational Hodge substructure.

We prove Theorem 1.7 in Section 3, using explicit constructions related to the Hodge structure on $\Lambda_\mathbb{Q}$ that will be discussed in Section 2.

This paper may be viewed as a follow up of [8] and [2].

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## 2. Hodge structures

**2.1.** It is well known that $\Lambda_\mathbb{Q} = H_1(T, \mathbb{Q})$ carries the natural structure of a rational Hodge structure of weight $-1$. Let us recall the construction. Let $J : V \to V$ be the multiplication by $i = \sqrt{-1}$, which is viewed as a certain element of $\text{End}_\mathbb{R}(V)$ such that

$$J^2 = -1.$$

Hence, $J^2_C = -1$ in $\text{End}_\mathbb{C}(V_C)$ and we define two mutually complex-conjugate $\mathbb{C}$-vector subspaces (of the same dimension) $H_{-1,0}(T)$ and $H_{0,-1}(T)$ of $V_C$ as the eigenspaces $V_C(i)$ and $V_C(-i)$ of $J_C$ attached to eigenvalues $i$ and $-i$ respectively. Clearly,

$$V_C = V_C(i) \oplus V_C(-i) = H_{-1,0}(T) \oplus H_{0,-1}(T),$$

which defines the rational Hodge structure on $\Lambda_\mathbb{Q}$, in light of $V_C = \Lambda_\mathbb{Q} \otimes \mathbb{C}$. It also follows that both $H_{-1,0}(T)$ and $H_{0,-1}(T)$ have the same dimension $2g/2 = g$.

\(^1\)A rational Hodge structure $H$ is called irreducible or simple if its only rational Hodge substructures are $H$ itself and $\{0\}$ [6, Sect. 2.2].
Now it’s a time to recall that $V$ is a complex vector space. I claim that the map
\[ \Psi : V \to V_C(i) = H_{-1,0}(T), \quad v \mapsto Jv \otimes 1 + v \otimes i \]
(6)
is an isomorphism of complex vector spaces. Indeed, first, $\Psi$ defines a homomorphism of real vector spaces $V \to V_C$. Second, if $v \in V$ then
\[ J_C(Jv \otimes 1 + v \otimes i) = J^2 v \otimes 1 + Jv \otimes i = -v \otimes 1 + Jv \otimes i = i(Jv \otimes 1 + v \otimes i), \]
i.e., $Jv \otimes 1 + v \otimes i \in V_C(i) = H_{0,1}(T)$ and therefore the map (6) is defined correctly. Third, taking into account that $J$ is an automorphism of $V$ and $V_C = V \otimes 1 + V \otimes i$, we conclude that $\Psi$ is an injective homomorphism of real vector spaces and the dimension arguments imply that it is actually an isomorphism. It remains to check that $\Psi$ is $\mathbb{C}$-linear, i.e.,
\[ \Psi(Jv) = i\Psi(v). \]
Let us do it. We have
\[ \Psi(Jv) = J(Jv) \otimes 1 + Jv \otimes i = -v \otimes 1 + Jv \otimes i = i(Jv \otimes 1 + v \otimes i) = i\Psi(v). \]
Hence, $\Psi$ is a $\mathbb{C}$-linear isomorphism and we are done.

Now suppose that $u \in \text{End}_\mathbb{R}(V)$ commutes with $J$, i.e., $u \in \text{End}_\mathbb{C}(V)$. Then
\[ \Psi \circ u = u_C \circ \Psi. \]
(7)
In particular, $H_{-1,0}(T)$ is $u_C$-invariant. Indeed, if $v \in V$ then
\[ \Psi \circ u(v) = Ju(v) \otimes 1 + u(v) \otimes i = uJ(v) \otimes 1 + u_C(v \otimes i) = u_C(J(v) \otimes 1 + u_C(v \otimes i) = u_C \circ \Psi(v), \]
which proves our claim.

Similarly, there is an anti-linear isomorphism of complex vector spaces
\[ V \to V_C(-i) = H_{0,-1}(T), \quad v \mapsto Jv \otimes 1 - v \otimes i. \]

It is also well known that there is a canonical isomorphism of rational Hodge structures of weight 2
\[ H^2(T, \mathbb{Q}) = \text{Hom}_\mathbb{Q}(\bigwedge^2 H_1(T, \mathbb{Q}), \mathbb{Q}) \]
where the Hodge components $H^{p,q}(T)$ ($p, q \geq 0, p + q = 2$) are as follows.
\[ H^{2,0}(T) = \text{Hom}_\mathbb{C}(\bigwedge^2 (H_{-1,0}(T), \mathbb{C}), \quad H^{0,2}(T) = \text{Hom}_\mathbb{C}(\bigwedge^2 (H_{0,-1}(T), \mathbb{C}), \quad H^{1,1}(T) = \text{Hom}_\mathbb{C}(H_{-1,0}(T), \mathbb{C}) \wedge \text{Hom}_\mathbb{C}(H_{0,-1}(T), \mathbb{C}) \cong \text{Hom}_\mathbb{C}(H_{-1,0}(T), \mathbb{C}) \otimes \text{Hom}_\mathbb{C}(H_{0,-1}(T), \mathbb{C}). \]

Clearly,
\[ \dim_\mathbb{C}(H^{2,0}(T)) = \frac{g(g-1)}{2}. \]
3. ENDOMORPHISM FIELDS AND AUTOMORPHISM GROUPS

Proof of Theorem 1.7. Let $T$ be a 2-simple complex torus and

$$g = \dim(T) \geq 3.$$ 

(i) Suppose that $T$ is not simple. This means that there is a proper complex subtorus $S = W/\Gamma$ where $W$ is a complex vector subspace of $V$ with

$$0 < d = \dim_{\mathbb{C}}(W) < \dim_{\mathbb{C}}(V) = g$$

such that

$$\Gamma = W \cap \Lambda$$

is a discrete lattice of rank $2d$ in $W$. Then the quotient $T/S$ is a complex torus of positive dimension $g - d$.

Let $H \subset H^2(T, \mathbb{Q})$ be the image of the canonical injective homomorphism of rational Hodge structures $H^2(T/S, \mathbb{Q}) \hookrightarrow H^2(T, \mathbb{Q})$ induced by the quotient map $T \rightarrow T/S$ of complex tori. Clearly, $H$ is a rational Hodge substructure of $H^2(T, \mathbb{Q})$ and its $(2,0)$-component

$$H^{2,0} \subset H_{\mathbb{C}}$$

has $\mathbb{C}$-dimension

$$\dim_{\mathbb{C}}(H^{2,0}) = \dim_{\mathbb{C}}(H^{2,0}(T/S))) = \frac{(g - d)(g - d - 1)}{2} < \frac{g(g - 1)}{2} = \dim_{\mathbb{C}}(H^{2,0}(T))).$$

In light of 2-simplicity of $T$,

$$\dim_{\mathbb{C}}(H^{2,0}) = 0,$$

which implies that

$$g - d = 1.$$ 

On the other hand, let $\tilde{H}$ be the kernel of the canonical surjective homomorphism of rational Hodge structures $H^2(T, \mathbb{Q}) \rightarrow H^2(S, \mathbb{Q})$ induced by the inclusion map $S \subset T$ of complex tori. Clearly, $\tilde{H}$ is a rational Hodge substructure of $H^2(T, \mathbb{Q})$. Notice that the induced homomorphism of $(2,0)$-components $H^{2,0}(T) \rightarrow H^{2,0}(S)$ is also surjective, because every holomorphic 2-form on $S$ obviously extends to a holomorphic 2-form on $T$. This implies that the $(2,0)$-component

$$\tilde{H}^{2,0} \subset \tilde{H}_{\mathbb{C}}$$

of $\tilde{H}$ has $\mathbb{C}$-dimension

$$\dim_{\mathbb{C}}(\tilde{H}^{2,0}) = \dim_{\mathbb{C}}(H^{2,0}(T))) - \dim_{\mathbb{C}}(H^{2,0}(S))) = \frac{g(g - 1)}{2} - \frac{d(d - 1)}{2} > 0.$$ 

In light of 2-simplicity of $T$,

$$\dim_{\mathbb{C}}(\tilde{H}^{2,0}) = \dim_{\mathbb{C}}(H^{2,0}(T))) = \frac{g(g - 1)}{2},$$

which implies that $\frac{d(d - 1)}{2} = 0$, i.e., $d = 1$. Taking into account that $g - d = 1$, we get $g = 1 + 1 = 2$, which is not true. The obtained contradiction proves
that $T$ is simple and (i) is proven. In particular, $\text{End}^0(T)$ is a division algebra over $\mathbb{Q}$.

In order to handle (ii), let us assume that $E$ is a subfield of $\text{End}^0(T)$. The simplicity of $T$ implies that $1 \in E$ is the identity automorphism of $T$. Then $\Lambda_Q$ becomes a faithful $E$-module. This implies that $E$ is a number field and $\Lambda_Q$ is an $E$-vector space of finite positive dimension

$$d_E = \frac{2g}{[E : \mathbb{Q}]}.$$ 

This implies that $V_C = \Lambda_Q \otimes \mathbb{C}$ is a free $E \otimes \mathbb{C}$-module of rank $d_E$. Clearly, both $H_{-1,0}(T)$ and $H_{0,-1}(T)$ are $E \otimes \mathbb{C}$-submodules of its direct sum $V_C$. Let

$$\text{tr}_{E/\mathbb{Q}} : E \to \mathbb{Q}$$

bet the trace map attached to the field extension $E/\mathbb{Q}$ of finite degree. Let

$$\text{Hom}_E \left( \bigwedge^2 \Lambda_Q, E \right)$$

be the $\frac{d_E(d_E-1)}{2}$-dimensional $E$-vector space of alternating $E$-bilinear forms on $\Lambda_Q$ that carries the natural structure of a rational Hodge structure of $\mathbb{Q}$-dimension $[E : \mathbb{Q}] \cdot \frac{d_E(d_E-1)}{2}$. There is the natural embedding of rational Hodge structures

$$\text{Hom}_E \left( \bigwedge^2 \Lambda_Q, E \right) \hookrightarrow \text{Hom}_Q \left( \bigwedge^2 \Lambda_Q, \mathbb{Q} \right) = H^2(T, \mathbb{Q}), \quad \phi_E \mapsto \phi := \text{tr}_{E/\mathbb{Q}} \circ \phi_E,$$

i.e.,

$$\phi(\lambda_1, \lambda_2) = \text{tr}_{E/\mathbb{Q}}(\phi_E(\lambda_1, \lambda_2)) \quad \forall \lambda_1, \lambda_2 \in \Lambda_Q. \quad (10)$$

The image of $\text{Hom}_E \left( \bigwedge^2 \Lambda_Q, E \right)$ in $\text{Hom}_Q \left( \bigwedge^2 \Lambda_Q, \mathbb{Q} \right) = H^2(T, \mathbb{Q})$ coincides with the $\mathbb{Q}$-vector subspace

$$H_E := \{ \phi \in \text{Hom}_Q \left( \bigwedge^2 \Lambda_Q, \mathbb{Q} \right) \mid \phi(u\lambda_1, \lambda_2) = \phi(\lambda_1, u\lambda_2) \forall u \in E, \lambda_1, \lambda_2 \in \Lambda_Q \}.$$ 

(11)

Indeed, it is obvious that the image lies in $H_E$. In order to check that the image coincide with the whole subspace $H_E$, let us construct the inverse map

$$H_E \to \text{Hom}_E \left( \bigwedge^2 \Lambda_Q, E \right), \quad \phi \mapsto \phi_E$$

to (9) as follows. If $\lambda_1, \lambda_2 \in \Lambda_Q$ then there is a $\mathbb{Q}$-linear map

$$\Phi : E \to \mathbb{Q}, \quad u \mapsto \phi(u\lambda_1, \lambda_2) = \phi(\lambda_1, u\lambda_2) = -\phi(\lambda_2, \lambda_1) = -\phi(\lambda_2, u\lambda_1). \quad (12)$$
The properties of trace map imply that there exists precisely one $\beta \in E$ such that
$$\Phi(u) = \text{tr}_{E/Q}(u\beta) \forall u \in E.$$ Let us put
$$\phi_E(\lambda_1, \lambda_2) := \beta.$$ It follows from (12) that $\phi_E \in \text{Hom}_E\left( \bigwedge^2_E \Lambda_Q, E \right)$. In addition,
$$\text{tr}_{E/Q}(\phi_E(\lambda_1, \lambda_2)) = \text{tr}_{E/Q}(\beta) = \text{tr}_{E/Q}(1 \cdot \beta) = \Phi(1) = \phi(\lambda_1, \lambda_2),$$ which proves that $\phi \mapsto \phi_E$ is indeed the inverse map, in light of (10).

Clearly, $H_E$ is a rational Hodge substructure of $H^2(T, Q)$. By 2-simplicity of $T$, the $\mathbb{C}$-dimension of the $(2, 0)$-component $H^{(2, 0)}_E$ of $H_E$ is either 0 or $g(g - 1)/2$. Let us express this dimension explicitly in terms of $g$ and $[E : \mathbb{Q}]$.

In order to do that, let us consider the $[E : \mathbb{Q}]$-element set $\Sigma_E$ of all field embedding $\sigma : E \hookrightarrow \mathbb{C}$. We have
$$E_\mathbb{C} := E \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma_E} \mathbb{C}_\sigma$$ where $\mathbb{C}_\sigma = E \otimes_{E, \sigma} \mathbb{C} = \mathbb{C},$ (13)
which gives us the splitting of $E_{\mathbb{C}}$-modules
$$V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma_E} V_\sigma = \bigoplus_{\sigma \in \Sigma_E} (H_{-1, 0}(T)_\sigma \oplus H_{0, -1}(T)_\sigma)$$ (14)
where for all $\sigma \in \Sigma_E$ we define
$$H_{-1, 0}(T)_\sigma := C_\sigma H_{-1, 0}(T) = \{ x \in H_{-1, 0}(T) \mid u_\mathbb{C} x = \sigma(u) x \forall u \in E \} \subset H_{-1, 0}(T);$$
$$n_\sigma := \dim\mathbb{C}(H_{-1, 0}(T)_\sigma);$$
$$H_{0, -1}(T)_\sigma := C_\sigma H_{0, -1}(T) = \{ x \in H_{0, -1}(T) \mid u_\mathbb{C} x = \sigma(u) x \forall u \in E \} \subset H_{0, -1}(T);$$
$$m_\sigma := \dim\mathbb{C}(H_{0, -1}(T)_\sigma);$$
$$V_\sigma = C_\sigma = C_\sigma V_\mathbb{C} = \{ x \in V_\mathbb{C} \mid u_\mathbb{C} x = \sigma(u) x \forall u \in E \} = H_{-1, 0}(T)_\sigma \oplus H_{0, -1}(T)_\sigma.$$
Since $H_{-1, 0}(T) \oplus H_{0, -1}(T) = V_\mathbb{C}$ is a free $E_{\mathbb{C}}$-module of rank $d_E$, its direct summand $V_\sigma$ is a $\mathbb{C}_\sigma = \mathbb{C}$-vector space of dimension $d_E$ and therefore
$$n_\sigma + m_\sigma = d_E \forall \sigma \in \Sigma_E.$$
(15)

Since $H_{-1, 0}(T)$ and $H_{0, -1}(T)$ are mutually complex-conjugate subspaces of $V_\mathbb{C}$, it follows from (5) that
$$m_\sigma = n_{\bar{\sigma}}$$ where $\bar{\sigma} : E \hookrightarrow \mathbb{C}, u \mapsto \overline{\sigma(u)}$
is the complex-conjugate of $\sigma$. Therefore, in light of (15),
$$n_\sigma + n_{\bar{\sigma}} = d_E \forall \sigma.$$ (16)
We have
$$\sum_{\sigma \in \Sigma_E} n_\sigma = \sum_{\sigma \in \Sigma_E} \dim\mathbb{C}(H_{-1, 0}(T)_\sigma) = \dim\mathbb{C}(H_{-1, 0}(T)) = g.$$ (17)
Let us consider the complexification of $H_E$

$$H_{E,C} := H_E \otimes \mathbb{Q} \subset \text{Hom}_\mathbb{Q} \left( \bigwedge^2 \Lambda \mathbb{Q}, \mathbb{Q} \right) \otimes \mathbb{Q} = \text{Hom}_C \left( \bigwedge^2 \Lambda \mathbb{Q} \otimes \mathbb{Q}, \mathbb{C} \right) = \text{Hom}_C \left( \bigwedge^2 V, \mathbb{C} \right).$$

In light of (11),

$$H_{E,C} = \{ \phi \in \text{Hom}_C \left( \bigwedge^2 V, \mathbb{C} \right) \mid \phi(u_{E,x,y}) = \phi(x,u_{E,y}) \forall u \in E; x,y \in V \}$$

$$= \{ \phi \in \text{Hom}_C \left( \bigwedge^2 V, \mathbb{C} \right) \mid \phi(u_{C,x,y}) = \phi(x,u_{C,y}) \forall u \in E; x,y \in V \}. \quad (18)$$

In particular, if $\sigma, \tau \in \Sigma_E$ are distinct field embeddings then for all $\phi \in H_{E,C}$

$$\phi(V_\sigma, V_\tau) = \phi(V_\tau, V_\sigma) = \{0\}.$$ 

This implies that

$$H_{E,C} = \bigoplus_{\sigma \in \Sigma_E} \text{Hom}_C \left( \bigwedge^2 V_\sigma, \mathbb{C} \right)$$

$$= \bigoplus_{\sigma \in \Sigma_E} \text{Hom}_C \left( \bigwedge^2 (H_{-1,0}(T)_\sigma \oplus H_{0,-1}(T)_\sigma), \mathbb{C} \right). \quad (19)$$

In light of (8), the (2,0)-Hodge component of $H_{E,C}$

$$H_{E}^{(2,0)} = \bigoplus_{\sigma \in \Sigma_E} \text{Hom}_C \left( \bigwedge^2 H_{-1,0}(T)_\sigma, \mathbb{C} \right) \quad \text{and} \quad \dim \mathbb{C}(H_{E}^{(2,0)}) = \sum_{\sigma \in \Sigma_E} \frac{n_\sigma(n_\sigma - 1)}{2}. \quad (20)$$

This implies that $\dim \mathbb{C}(H_{E}^{(2,0)}) = 0$ if and only if all $n_\sigma \in \{0,1\}$. If this is the case then, in light of (16), $d_E \in \{1,2\}$, i.e., $[E : \mathbb{Q}] = 2g$ or $g$.

On the other hand, it follows from (17) combined with the second formula in (20) that $\dim \mathbb{C}(H_{E}^{(2,0)}) = g(g - 1)/2$ if and only if there is precisely one $\sigma$ with $n_\sigma = g$ (and all the other multiplicities $n_\tau$ are 0). This implies that either $d_E = 2g$ and $E = \mathbb{Q}$ or $d_E = g$ and $E$ an imaginary quadratic field with the pair of the field embeddings

$$\sigma, \bar{\sigma} : E \hookrightarrow \mathbb{C}$$

such that

$$n_\sigma = g, \quad n_\bar{\sigma} = 0.$$ 

Let us assume that $d_E = g$. Then $E$ is an imaginary quadratic field; in addition,

$$u \in E \subset \text{End}_\mathbb{Q}(\Lambda \mathbb{Q}) \subset \text{End}_\mathbb{R}(V)$$

then $u_{\mathbb{C}}$ acts on $H_{-1,0}(T)$ as multiplication by $\sigma(u) \in \mathbb{C}$. In light of (5), $u_{\mathbb{C}}$ acts on the complex-conjugate subspace $H_{0,-1}(T)$ as multiplication by $\bar{\sigma(u)} = \bar{\sigma}(u) \in \mathbb{C}$. Since $E$ is an imaginary quadratic field, there are a
positive integer $D$ and $\alpha \in E$ such that $\alpha^2 = -D$ and $E = \mathbb{Q}(\alpha)$. It follows that $\sigma(\alpha) = \pm i\sqrt{D}$. Replacing if necessary $\alpha$ by $-\alpha$, we may and will assume that

$$\sigma(\alpha) = i\sqrt{D}$$

and therefore $\alpha_C$ acts on $H_{-1,0}(T)$ as multiplication by $i\sqrt{D}$. Hence, $\alpha_C$ acts on $H_{0,-1}(T)$ as multiplication by $i\sqrt{D} = -i\sqrt{D}$. Since

$$V_C = H_{-1,0}(T) \oplus H_{0,-1}(T),$$

we get $\alpha_C = \sqrt{D}J_C$ and therefore

$$\alpha = \sqrt{D}J.$$

This implies that the centralizer $\text{End}^0(T)$ of $J$ in $\text{End}_{\mathbb{Q}}(A_{\mathbb{Q}})$ coincides with the centralizer of $\alpha$ in $\text{End}_{\mathbb{Q}}(A_{\mathbb{Q}})$, which, in turn, coincides with the centralizer $\text{End}_E(A_{\mathbb{Q}})$ of $E$ in $\text{End}_{\mathbb{Q}}(A_{\mathbb{Q}})$, i.e.,

$$\text{End}^0(T) = \text{End}_E(A_{\mathbb{Q}}) \cong \text{Mat}_{d_E}(E).$$

This is the matrix algebra, which is not a division algebra, because $d_E = g > 1$. This contradicts to the simplicity of $T$. The obtained contradiction rules out the case $d_E = g$. This ends the proof of (ii).

In order to prove (iii), recall that $\text{End}^0(T)$ is a division algebra of $\mathbb{Q}$, thanks to the simplicity of $T$ [8]. Hence $A_{\mathbb{Q}}$ is a free $\text{End}^0(T)$-module of finite positive rank and therefore

$$\dim_{\mathbb{Q}}(\text{End}^0(T))|2g,$$

because $2g = \dim_{\mathbb{Q}}(A_{\mathbb{Q}})$. We will apply several times already proven assertion (ii) to various subfields of $\text{End}^0(T)$.

Suppose that $\text{End}^0(T)$ is not a field and let $\mathcal{Z}$ be its center. Then $\mathcal{Z}$ is a number field and there is an integer $d > 1$ such that $\dim_{\mathcal{Z}}(\text{End}^0(T)) = d^2$ and therefore

$$\dim_{\mathcal{Z}}(\text{End}^0(T)) = d^2 \cdot [\mathcal{Z} : \mathbb{Q}]$$

divides $2g$, thanks to (21). Since $\mathcal{Z}$ is a subfield of $\text{End}^0(T)$, the degree $[\mathcal{Z} : \mathbb{Q}]$ is either 1 or $g$ or $2g$. If $[\mathcal{Z} : \mathbb{Q}] > 1$ then $2g$ is divisible by

$$d^2 \cdot [\mathcal{Z} : \mathbb{Q}] \geq 2^2 \cdot 4g = 4g,$$

which is nonsense. Hence, $[\mathcal{Z} : \mathbb{Q}] = 1$, i.e., $\mathcal{Z} = \mathbb{Q}$ and $\text{End}^0(T)$ is a central division $\mathbb{Q}$-algebra of dimension $d^2$ with $d^2|2g$. Then every maximal subfield $E$ of the division algebra $\text{End}^0(T)$ has degree $d$ over $\mathbb{Q}$. Hence $d \in \{1, g, 2g\}$. Since $d > 1$, we obtain that either $d = g$ and $g^2|2g$ or $d = 2g$ and $(2g)^2|2g$. This implies that $d = g$ and $g = 1$ or 2. Since $g \geq 3$, we get a contradiction, which implies that $\text{End}^0(T)$ is a field.

It follows from already proven assertion (ii) that the degree $\dim_{\mathbb{Q}}(\text{End}^0(T))$ of the number field $\text{End}^0(T)$ is either 1 or $g$ or $2g$. Assertion (iv) is obvious and was included just for the sake of completeness.

In order to handle the structure of $\text{Aut}(T)$, let us check first that the only roots of unity in $\text{End}^0(T)$ are 1 and $-1$. If this is not the case then the
field $\text{End}^0(T)$ contains either $\sqrt{-1}$ or a primitive $p$th root of unity $\zeta$ where $p$ is a certain odd prime. In the former case $\text{End}^0(T)$ contains the quadratic field $\mathbb{Q}(\sqrt{-1})$, which contradicts (ii). In the latter case $\text{End}^0(T)$ contains either the quadratic field $\mathbb{Q}(\sqrt{-p})$ or the quadratic field $\mathbb{Q}(\sqrt{p})$: each of these outcomes contradicts to (ii) as well.

Now recall that $\text{End}(T)$ is an order in the number field $E = \text{End}^0(T)$ and $\text{Aut}(T) = \text{End}(T)^* \subseteq \mathbb{Z}$ is its group of units. By Theorem of Dirichlet about units [3, Ch. II, Sect. 4, Th. 5], the group of units

$$\text{Aut}(T) \cong \mathbb{Z}^d \times \{\pm 1\}$$

with $d = r + s - 1$ (22)

where $r$ is the number of real field embeddings $E \hookrightarrow \mathbb{R}$ and

$$r + 2s = [E : \mathbb{Q}], \quad \text{i.e.,} \quad s = \frac{[E : \mathbb{Q}] - r}{2}. \quad (23)$$

Let us prove (v). Assume that the number field $E := \text{End}^0(T)$ has degree $2g$. The dimension arguments imply that $\Lambda_0$ is a 1-dimensional $E$-vector space and $V = \Lambda_0 \otimes \mathbb{R}$ is a free $E_{\mathbb{R}} = E \otimes \mathbb{R}$-module of rank 1. Hence $E_{\mathbb{R}}$ coincides with its own centralizer $\text{End}_{E_{\mathbb{R}}}(V)$ in $\text{End}_{\mathbb{R}}(V)$. Since $J$ commutes with $\text{End}^0(T) = E$, it also commutes with $E_{\mathbb{R}}$ and therefore

$$J \in \text{End}_{E_{\mathbb{R}}}(V) = E_{\mathbb{R}}.$$

Recall that the $\mathbb{R}$-algebra $E_{\mathbb{R}}$ is isomorphic to a product of copies of $\mathbb{R}$ and $\mathbb{C}$. Since $J^2 = -1$, the only copies of $\mathbb{C}$ appear in $E_{\mathbb{R}}$, i.e., $E$ is purely imaginary, which means that $r = 0$ and therefore $2g = [E : \mathbb{Q}] = 2s$. This proves the first assertion of (v); the second one follows readily from (22) combined with (23).

Let us prove (vi). Assume that $[E : \mathbb{Q}] = g$. Then the first assertion follows readily from (22) combined with (23).

Assume now that $T$ is a complex abelian variety. By Albert’s classification [9], $E = \text{End}^0(T)$ is either a totally real number field or a CM field. If $E$ is a CM field then it contains a subfield $E_0$ of degree $[E : \mathbb{Q}]/2 = g/2$. Since $E_0$ is a subfield of $\text{End}^0(T)$ and $1 < g/2 < g$ (recall that $g \geq 3$), the existence of $E_0$ contradicts to the already proven assertion (ii). This proves that $E$ is a totally real number field, i.e., $s = 0, r = g$. Now the assertion about $\text{Aut}(T)$ follows from (22).

\[ \square \]

References


Pennsylvania State University, Department of Mathematics, University Park, PA 16802, USA

*Email address*: zarhin@math.psu.edu