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SPIN^h STRUCTURES IN LOW DIMENSION

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Abstract. All compact orientable manifolds of dimension ≤ 7 admit a spin^h structure. We discuss conditions under which the compactness assumption can be removed.

A natural quaternionic analog to the spin group is the group $Spin^h(n) = Spin(n) \times Sp(1)/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ denotes the two-element group) defined in analogy with the spin^c group $Spin^c(n) = (Spin(n) \times U(1))/\mathbb{Z}_2$. Writing the spin and spin^c groups as

$$Spin(n) = (Spin(n) \times Spin(1))/\mathbb{Z}_2, \qquad Spin^c(n) = (Spin(n) \times Spin(2))/\mathbb{Z}_2,$$

one notes that

$$Spin^h(n) = (Spin(n) \times Spin(3))/\mathbb{Z}_2$$

naturally fits in this sequence by way of low-dimensional accidental isomorphisms of Lie groups. The present authors studied this group and manifolds whose tangent bundle admits a spin^h structure, along with further analogues, in [AM21].

Therein, we made the following statement ([AM21, Theorem 1.3], [AM21, Corollary 3.10]): *Every orientable manifold of dimension ≤ 7 is spin^h.* The argument for manifolds of dimension 6 and 7 relied on invoking Cohen's immersion theorem [Coh85], and as such we should have qualified the statement with a compactness assumption. The result of [Coh85] was used in order to obtain a codimension 4 immersion in Euclidean space, followed by an application of [AM21, Proposition 3.9] that such an immersion guarantees the existence of a spin^h structure.

In the present note, we prove the following theorem, using only results chronologically preceding [Coh85], due to Atiyah, Dupont, and Hirsch:

Theorem. *The following hold:*

(1) *Every (not necessarily compact) orientable manifold of dimension ≤ 5 is spin^h.*

(2) *Compact orientable manifolds of dimension 6 and 7 are spin^h.*

(3) *A non-compact orientable manifold M of dimension 6 or 7 is spin^h if and only if $W_5(TM) = 0$.*

(4) *A non-compact orientable manifold M of dimension 6 or 7 with no elements of order exactly four in $H^5(M;\mathbb{Z})$ is spin^h.*

By "no elements of order exactly four" we mean that any $x \in H^5(M;\mathbb{Z})$ satisfying $4x = 0$ also satisfies $2x = 0$.

Proof. Part (1) is already proved in [AM21, p.5] without appealing to [Coh85]. It is a corollary of [AM21, Corollary 2.6], which states that the primary obstruction to the existence of a spin^h structure on an oriented manifold is the fifth integral Stiefel–Whitney class W_5. This is an integral class of order two and hence vanishes on all orientable manifolds of dimension ≤ 5. Here and throughout, we use the fact that the top cohomology (with any coefficients) of a non-compact manifold vanishes, e.g. see [Wh61, Lemma 2.1]; see also [Br62, Theorem 2] that every (not necessarily compact) manifold with boundary admits a collar neighborhood of its boundary, and hence it deformation retracts onto its interior.

Compact orientable six–manifolds immerse in $\mathbb{R}^{10}$, as proved in [Hir61, Corollary 9], and hence they admit spin^h structures by [AM21, Proposition 3.9]. Note that the statement of [Hir61, Corollary 9] does not explicitly assume compactness, though it is clear from the proof that the manifold is assumed to be closed; the general compact case then follows by taking the double if the boundary is non-empty. This establishes part (2) for six–manifolds.

For compact orientable seven–manifolds M, we use the result listed in the second table of [AtDu72, p.25] (see also the footnote (1) in loc. cit.), that the only obstruction to a compact seven–manifold admitting three linearly independent vector fields is the integral Bockstein of w_4, i.e. W_5. Again, the result is stated for closed manifolds, and the compact-with-boundary case follows by considering the double. (Note, for orientable seven–manifolds, W_5 is a priori the single obstruction to finding three linearly independent sections over the five–skeleton.) This class vanishes by [Mas62, Theorem 3]. Hence the tangent bundle of M splits off a trivial rank three bundle, giving an orientable rank four bundle with the same w_2 as TM, and we again apply [AM21, Proposition 3.9]. This proves part (2) for seven–manifolds.

Now let M be a non-compact orientable six–manifold. Since $H^6(M;\mathbb{Z}) = 0$, there are no secondary or higher obstructions to admitting a spin^h structure beyond W_5; this establishes part (3) for six–manifolds. As for part (4), choose an increasing exhaustion $\{M_i\}$ by compact manifolds with boundary. For an abelian group A and integer $k > 1$, we have the short exact Milnor sequence [Sw17, Proposition 7.66]

$$0 \to \varprojlim{}^1 H^{k-1}(M_i; A) \to H^k(M; A) \to \varprojlim H^k(M_i; A) \to 0.$$

From the long exact sequence in cohomology associated to the short exact coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\bmod 2} \mathbb{Z}_2 \to 0$$

we have the following commutative diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow \\
\varprojlim H^5(M_i;\mathbb{Z}) & \longrightarrow & \varprojlim H^5(M_i;\mathbb{Z}) & \longrightarrow & \varprojlim H^5(M_i;\mathbb{Z}_2) \\
\uparrow & & \uparrow & & \uparrow \\
H^5(M;\mathbb{Z}) & \xrightarrow{\cdot 2} & H^5(M;\mathbb{Z}) & \xrightarrow{\bmod 2} & H^5(M;\mathbb{Z}_2) \\
\uparrow & & \uparrow & & \uparrow \\
\varprojlim^1 H^4(M;\mathbb{Z}) & \longrightarrow & \varprojlim^1 H^4(M;\mathbb{Z}) & \longrightarrow & \varprojlim^1 H^4(M;\mathbb{Z}_2) \\
\uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}$$

For each M_i, we have $W_5(M_i) = 0$ by taking the double and applying [Mas62, Theorem 2] (or crossing the double with a circle and applying [Mas62, Theorem 3] again). For an orientable manifold, the mod 2 reduction of W_5 is w_5. From here and by naturality, $w_5(M) \in H^5(M;\mathbb{Z}_2)$ maps to the zero element in $\varprojlim H^5(M_i;\mathbb{Z}_2)$. By [MiSt74, Lemma 10.3], the term $\varprojlim^1 H^4(M;\mathbb{Z}_2)$ vanishes. Therefore $w_5(M)$ must be zero as well. Now, $W_5(M) \in H^5(M;\mathbb{Z})$ is an element of order two which maps to $w_5(M) = 0$ by mod 2 reduction. Therefore it is in the image of the map $H^5(M;\mathbb{Z}) \xrightarrow{\cdot 2} H^5(M;\mathbb{Z})$. Since by assumption there are no elements of order exactly four in $H^5(M;\mathbb{Z})$, it follows that $W_5(M)$ must be the zero class. This establishes part (4) for six–manifolds.

Now let M be an orientable non-compact seven–manifold. We will show that the secondary obstruction to the existence of a spin^h structure vanishes, establishing parts (3) and (4). Take an exhaustion $\{M_i\}$ by compact seven–manifolds with boundary. If $W_5(M) = 0$, which will for instance be true given the torsion condition on $H^5(M;\mathbb{Z})$ by the argument above, we can choose a lift of the classifying map of the tangent bundle $M \to BSO(7)$ to E_1, the second stage of the relative Postnikov tower of $BSO(4) \to BSO(7)$,

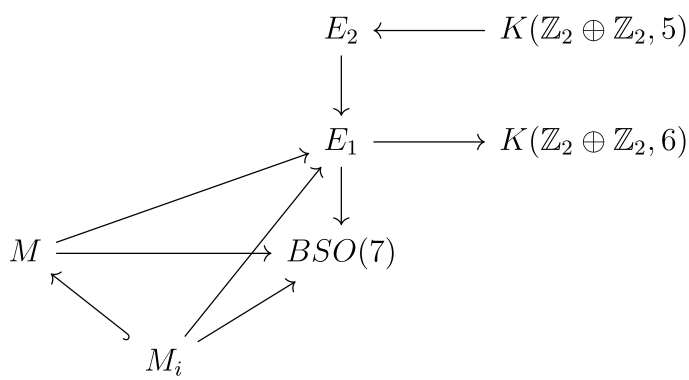
$$\begin{array}{ccccc}
 & & BSO(4) & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 & & E_1 & \longleftarrow & K(\mathbb{Z},4) \\
 & \nearrow & \downarrow & & \\
M & \longrightarrow & BSO(7) & \xrightarrow{W_5} & K(\mathbb{Z},5)
\end{array}$$

Restricting to the M_i gives a compatible system of lifts to E_1. We consider now the secondary obstruction $\mathfrak{o}(M)$ to admitting three linearly independent vector fields. This is a class in $H^6(M;\pi_5(V(3,7)))$, where $V(3,7)$ is the Stiefel manifold of 3–frames in $\mathbb{R}^7$. We have the following exact sequence of homotopy groups:

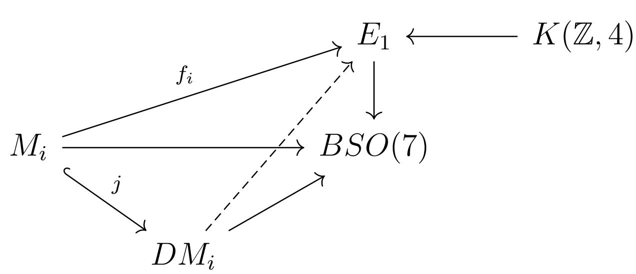
$$\pi_6(BSO(7)) \to \pi_5(V(3,7)) \to \pi_5(BSO(4)) \to \pi_5(BSO(7)).$$

The natural map $BSO(7) \to BSO$ is an isomorphism on $\pi_{\leq 6}$, and hence we have $\pi_5(BSO(7)) = \pi_6(BSO(7)) = 0$. Furthermore, $\pi_5(BSO(4)) \cong \pi_4(SO(4)) \cong \pi_4(S^3 \times S^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Since we are fixing the lifts $M \to E_1$ and $M_i \to E_1$, and they are compatible, the secondary obstruction to lifting further to E_2 is natural, i.e. $\mathfrak{o}(M_i)$ is the restriction of $\mathfrak{o}(M)$.



Let us now argue that $\mathfrak{o}(M_i) = 0$. We will use [Du74, Theorem 1.1], which gives us that for *any* choice of lift to E_1 on a *closed* orientable seven–manifold, the secondary obstruction vanishes. In order to apply this to M_i, we consider the double DM_i. We will argue that the lift $M_i \xrightarrow{f_i} E_1$ (obtained by restricting $M \xrightarrow{f} E_1$) extends to a lift $DM_i \to E_1$. Then, by applying loc. cit., we will have $\mathfrak{o}(DM_i) = 0$ and hence $\mathfrak{o}(M_i) = 0$.



First, choose any lift $DM_i \xrightarrow{G} E_1$ of $DM_i \to BSO(7)$; this exists since W_5 vanishes on any closed orientable seven–manifold. Now, f_i and the restriction of G to M_i differ by the action of an element x in $[M_i, K(\mathbb{Z},4)] = H^4(M_i;\mathbb{Z})$ (this group acts simply transitively on the homotopy classes of lifts to E_1). Let us denote this by $[f_i] = x \cdot [G|_{M_i}]$.

Now observe that x is the restriction of a class $X \in H^4(DM_i;\mathbb{Z})$. Namely, consider the Mayer–Vietoris sequence for the double:

$$\cdots \to H^4(DM_i;\mathbb{Z}) \to H^4(M_i;\mathbb{Z}) \oplus H^4(M_i;\mathbb{Z}) \to H^4(\partial M_i;\mathbb{Z}) \to \cdots$$

The element (x,x) maps to zero, and hence $x = j^*X$ for some $X \in H^4(DM_i;\mathbb{Z})$.

Therefore, if we consider the (class of the) lift $X \cdot [G]$ on DM_i instead of $[G]$, by naturality we have that its restriction to M_i is $x \cdot [G|_{M_i}]$, i.e. $[f_i]$.

Now we have that $\mathfrak{o}(M_i) = 0$ for all i. Consider the short exact sequence

$$0 \to \varprojlim{}^1 H^3(M_i;\mathbb{Z}_2 \oplus \mathbb{Z}_2) \to H^4(M;\mathbb{Z}_2 \oplus \mathbb{Z}_2) \to \varprojlim H^4(M;\mathbb{Z}_2 \oplus \mathbb{Z}_2) \to 0.$$

Since $H^*(-;\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is naturally isomorphic to $H^*(-;\mathbb{Z}_2) \oplus H^*(-;\mathbb{Z}_2)$, the $\varprojlim^1$ term vanishes. Further, since $\mathfrak{o}(M)$ maps to $(\mathfrak{o}(M_i))_i$, which is the zero element, by injectivity we have that $\mathfrak{o}(M) = 0$. Therefore M admits three linearly independent vector fields, and we conclude that M admits a spin^h structure. □

Remark. The primary obstruction to a spin^c structure on an orientable manifold is W_3, and compact orientable four–manifolds are spin^c. An analogous argument to the above then shows that non-compact orientable four–manifolds with no elements of order exactly four in $H^3(-;\mathbb{Z})$ are spin^c. The four–torsion assumption can be removed in this case [TV], and it is not clear whether one should expect this in the theorem above.

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