Max-Planck-Institut für Mathematik
Bonn

On characteristic classes modulo torsion for spin groups

by

Nikita A. Karpenko

Max-Planck-Institut für Mathematik
Preprint Series 2022 (76)

Date of submission: December 9, 2022

On characteristic classes modulo torsion for spin groups

by

Nikita A. Karpenko

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Mathematical & Statistical Sciences
University of Alberta
Edmonton
Canada

MPIM 22-76

ON CHARACTERISTIC CLASSES MODULO TORSION FOR SPIN GROUPS

NIKITA A. KARPENKO

Abstract. We study the ring of Chow characteristic classes (also called the Chow ring of the classifying space) for the split spin group $\mathrm{Spin}(n)$ with n odd or divisible by 4. For such n up to 12, we determine this ring modulo torsion.

Contents

1. Introduction

For an affine algebraic group G over a field, the (graded) ring $\mathrm{CH}(BG)$ of its Chow characteristic classes (also called the Chow ring of the classifying space of G) has been introduced in [22].

Assume that G is reductive and has a split maximal torus T. The kernel of the ring homomorphism $\Phi\colon \mathrm{CH}(BG) \to \mathrm{CH}(BT)$, given by the inclusion $T \hookrightarrow G$, is precisely the ideal of the elements of finite order so that the ring $\mathrm{CH}(BG)$ modulo torsion is identified with the image of Φ. Every element in the image is invariant under the action of the Weyl group W of G and the quotient $\mathrm{CH}(BT)^W / \operatorname{Im}\Phi$ (as well as the kernel of Φ) is killed by the torsion index of G, [23, Theorem 1.3(1)]. Note that the ring $\mathrm{CH}(BT)$ is know to be the symmetric $\mathbb{Z}$-algebra on the character group of T. Its subring $\mathrm{CH}(BT)^W \subset \mathrm{CH}(BT)$ of the W-invariants has been computed (in the topological context) in [1, Theorem 7.1].

Let G be the split spin group $\mathrm{Spin}(n)$ over some field (on which we don't put any restriction). For arbitrary n, unlike topology, where the cohomology of the classifying

Date: 8 Dec 2022.

Key words and phrases. Affine algebraic group; spin group; classifying space; Chow ring. *Mathematical Subject Classification (2020):* 20G15; 14C25.

Author's work has been supported by a Discovery Grant from the National Science and Engineering Research Council of Canada and accomplished during his stay at the Max-Planck Institute for Mathematics in Bonn.

space of G is well-understood (see [18] and [1]), its Chow ring in algebraic geometry is *"notoriously difficult to study"* ([21, Page 43]).

For $n \leq 6$ however, the torsion index of G is 1 so that $\mathrm{CH}(BG) = \mathrm{Im}\,\Phi = \mathrm{CH}(BT)^W$. One can also mention [6, Table 16] telling that G is isomorphic to a classical group for which $\mathrm{CH}(BG)$ is computed in [22] and [16]. We do not consider these values of n any further in the paper.

For $n \geq 13$, the torsion index is divisible by 4. Note that the torsion index of $\mathrm{Spin}(n)$ has been determined for any n in [23, Theorem 0.1].

For the remaining values $7, \dots, 12$ of n, the torsion index is 2. Determination of $\mathrm{Im}\,\Phi$ in this case is equivalent to determination of the image for the modulo 2 reduction

$$\varphi\colon\ \mathrm{Ch}(BG) \to \mathrm{Ch}(BT)$$

of Φ, where $\mathrm{Ch}(-)$ is the Chow ring $\mathrm{CH}(-)/2\,\mathrm{CH}(-)$ with coefficients $\mathbb{Z}/2\mathbb{Z}$. Our main result affirms that for $n \neq 10$ the image of φ is the subring

$$(\mathrm{Ch}(BT)^W)^2 := \{a^2,\ a \in \mathrm{Ch}(BT)^W\} \subset \mathrm{Ch}(BT)^W$$

of squares in $\mathrm{Ch}(BT)^W$:

Theorem 1.1. *For the algebraic group $G = \mathrm{Spin}(n)$ with $n = 7, 8, 9, 11, 12, 13, 15, 16, 17$ and its split maximal torus T, the image of $\varphi\colon\ \mathrm{Ch}(BG) \to \mathrm{Ch}(BT)$ is the ring $(\mathrm{Ch}(BT)^W)^2$ of squares of the W-invariants.*

The consequence for the integral Chow group is captured in Corollary 3.5.

The value $n = 13$ is included into Theorem 1.1 because it does not require additional efforts. The values $n = 15, 16, 17$ were interesting to check in view of the rationality problem [15, Conjecture 4.5] for BG. For higher values of n the amount of needed computations becomes higher than the motivation available.

The values 10 and 14 of n are excluded because the statement of Theorem 1.1 fails for every $n \geq 7$ which is 2 modulo 4: the image under φ of the highest Chern class in $\mathrm{Ch}(BG)$ of the half-spin representation of G is not a square in $\mathrm{Ch}(BT)^W$ (cf. the proof of Proposition 3.4). At the same time, the odd degree Euler class (defined in Proposition 2.1), occurring for such n, creates additional complications for our approach to determination of the image of φ. For $n = 10$, the Euler class is in the image of Φ (see Lemma A.1) and, viewed modulo 2, yields another example of a non-square element in $\mathrm{Im}\,\varphi$; applying appropriate Steenrod operations to it (see §4), one can enlarge the number of such examples even further.

The ring $\mathrm{Ch}(BT)^W$ of W-invariants in the Chow ring $\mathrm{Ch}(BT)$ with coefficients $\mathbb{Z}/2\mathbb{Z}$ is easy to compute for arbitrary n – see Proposition 3.2. Note that the image of $\mathrm{CH}(BT)^W$ under the reduction modulo 2 homomorphism $\mathrm{CH}(BT) \to \mathrm{Ch}(BT)$ is in general smaller than $\mathrm{Ch}(BT)^W$ (see Remark 3.3).

Since the entire ring $\mathrm{CH}(B\,\mathrm{Spin}(n))$ (over a field of characteristic different from 2) is computed for $n = 7$ in [7] and [19] and for $n = 8$ in [19], the statement of Theorem 1.1 in these cases is not new.

Concerning the proof of Theorem 1.1, it is easy to check that $(\mathrm{Ch}(BT)^W)^2 \subset \mathrm{Im}\,\varphi$ for any n – see Proposition 3.4: the job is done by computing the images in $\mathrm{Ch}(BT)$ of the Chern classes in $\mathrm{Ch}(BG)$ for the (half-)spin and the orthogonal representations of G.

The opposite inclusion (now for the specified values of n) is obtained by combining the methods of [8] (summarized in §2) and [11] (pushed further in §4), where the second approach makes use of the Steenrod operations St^i, $i \geq 0$ on the modulo 2 Chow groups. These operations extend to the Steenrod operations Sq^{2i}, $i \geq 0$ in the motivic cohomology with coefficients $\mathbb{Z}/2\mathbb{Z}$, where, like as well in topology, the Steenrod algebra has one more generator – the Bockstein homomorphism Sq^1. We observe a formal similarity between Sq^1 and St^1. The similarity observed allows us to apply to our setting some topological techniques and computations related to the Bockstein cohomology (see the proof of Lemma 4.2). This results in a new bound on $\operatorname{Im}\varphi$ valid for arbitrary $n \geq 7$ which is odd or divisible by 4 (see Proposition 4.3). Adding on top the restrictions provided by the even Steenrod operations St^2 and St^4, we achieve the proof of Theorem 1.1.

2. Known restrictions on $\operatorname{Im}\Phi$

Here we describe a stronger than $\mathrm{CH}(BT)^W$ upper bound on $\operatorname{Im}\Phi$ obtained in [8].

Depending on parity, we write the integer $n \geq 7$ in the form $n = 2l + 1$ or in the form $n = 2l$ (with an integer l) and identify the graded ring $\mathrm{CH}(BT)$ with the polynomial ring $\mathbb{Z}[z, x_1, \dots, x_l]$ in the $l + 1$ variables modulo the homogeneous relation $2z = x_1 + \ldots + x_l$. For odd $n = 2l + 1$, the Weyl group W is a semidirect product of the symmetric group S_l and the direct product $(\mathbb{Z}/2\mathbb{Z})^{\times l}$ of l copies of $\mathbb{Z}/2\mathbb{Z}$. The action of W on $\mathrm{CH}(BT)$ is induced by its action on the polynomial ring, in which S_l acts trivially on z and permutes $x_1, \dots, x_l$, whereas the ith copy of $\mathbb{Z}/2\mathbb{Z}$ acts by $x_i \mapsto -x_i$, $z \mapsto z - x_i$, and trivially on the remaining variables. We let $\tilde{z}$ to be the product of the elements in the orbit of z:

$$\tilde{z} = \prod_{I \subset \{1,\dots,l\}} (z - \sum_{i \in I} x_i) \in \mathbb{Z}[z, x_1, \dots, x_l].$$

For even $n = 2l$, the Weyl group W is a semidirect product of S_l and the subgroup in $(\mathbb{Z}/2\mathbb{Z})^{\times l}$ of the elements with an even number of nonzero components, acting by restriction of the odd case action. We let $\check{z}$ to be the product of the elements in the orbit of z:

$$\check{z} = \prod_{\text{even } I \subset \{1,\dots,l\}} (z - \sum_{i \in I} x_i),$$

where an *even* subset is a subset with even number of elements.

It is shown in [8] that $\operatorname{Im}\Phi$ is contained in the image of $\mathbb{Z}[z, x_1, \dots, x_l]^W$ (which, in general, is strictly smaller than $\mathrm{CH}(BT)^W$). Moreover, the ring $\mathbb{Z}[z, x_1, \dots, x_l]^W$ is computed in [9, Proposition 2.4] and [13, Proposition 5.1] (see also [5]). As a part of this computation, for $n = 2l + 1$ and every $i \geq 0$, certain homogeneous W-invariant element $f_i \in \mathbb{Z}[z, x_1, \dots, x_l]^W$ of degree 2^i is constructed. The element f_0 equals $2z - (x_1 + \ldots + x_l)$ and vanishes in $\mathrm{CH}(BT)$. As a result, we get

Proposition 2.1 ([8, Theorems 2.2 and 3.2]). *Let $G = \mathrm{Spin}(n)$ with $n \geq 7$ and let $S \subset \mathbb{Z}[z, x_1, \dots, x_l]^W$ be the subring of symmetric polynomials in the squares $x_1^2, \dots, x_l^2$. For $n = 2l + 1$, the image of $\Phi\colon \mathrm{CH}(BG) \to \mathrm{CH}(BT)$ is contained in the S-subalgebra of $\mathrm{CH}(BT)$ generated by $f_1, \dots, f_{l-1}$ and the orbit product $\tilde{z}$ of z (of degree 2^l). The generator $\tilde{z}$ equals $\tilde{z}_1^2$, where $\tilde{z}_1 \in \mathrm{CH}(BT)^W$ is defined below.*

In the case of $n = 2l$, the image of Φ is contained in the S-subalgebra generated by $f_1, \dots, f_{l-2}$, the orbit product $\check{z}$ of z (now of degree 2^{l-1}), and the element $e := x_1 \ldots x_l$

(called the Euler class). If n is divisible by 4 (i.e., l is even), the generator $\check{z}$ equals $\check{z}_1^2$, where $\check{z}_1 \in \mathrm{CH}(BT)^W$ is defined below.

Remark 2.2. The subring $S \subset \mathrm{CH}(BT)$ is the image of the composition

$$\mathrm{CH}(B\,\mathrm{O}(n)) \longrightarrow \mathrm{CH}(BG) \xrightarrow{\Phi} \mathrm{CH}(BT),$$

where $\mathrm{O}(n)$ is the standard split orthogonal group. More precisely, the elementary symmetric polynomials in the squares $x_1^2, \dots, x_l^2$ are the images of the even Chern classes in $\mathrm{CH}(B\,\mathrm{O}(n))$ of the standard representation $\mathrm{O}(n) \hookrightarrow \mathrm{GL}(n)$. In particular, $S \subset \mathrm{Im}\,\Phi$. By [22] (see also [16]), the Chern classes of the standard representation generate the ring $\mathrm{CH}(B\,\mathrm{O}(n))$. The odd ones have exponent 2 and vanish in $\mathrm{CH}(BT)$.

Remark 2.3. The group $G = \mathrm{Spin}(n)$ is defined (e.g., in [14, §23A]) as a subgroup in $\mathrm{GL}_1(C_0(n))$, where $C_0(n)$ is the even Clifford algebra of the standard split n-dimensional quadratic form. For even n, the algebra $C_0(n)$ is the product of two copies of a split central simple algebra $C^+(n)$. The two representations $G \to \mathrm{GL}_1(C^+(n))$, given by the two projections $C_0(n) \to C^+(n)$, are irreducible and called the half-spin representations of G. There sum is the spin representation $G \to \mathrm{GL}_1(C_0(n))$. The image in $\mathrm{CH}(BT)$ of the highest Chern class in $\mathrm{CH}(BG)$ of one half-spin representation is equal to $\check{z}$. (The other half-spin representation yields $\prod_{\text{odd } I\subset\{1,\dots,l\}}(z - \sum_{i\in I} x_i)$.)

For odd n, $C_0(n)$ is a split central simple algebra and $G \to \mathrm{GL}_1(C_0(n))$ is the spin representation. This representation is irreducible and the image in $\mathrm{CH}(BT)$ of its highest Chern class is $\tilde{z}$.

The upper bound on $\mathrm{Im}\,\Phi$ described in Proposition 2.1 is in general smaller than the ring $\mathrm{CH}(BT)^W$, computed in [1]. For odd n, let us define

$$\tilde{z}_1 := \prod_{I\subset\{2,\dots,l\}} (z - \sum_{i\in I} x_i) \in \mathrm{CH}(BT).$$

Because of the relation $2z = x_1 + \ldots + x_l$, which holds in $\mathrm{CH}(BT)$, the element $\tilde{z}_1$ is W-invariant and $\tilde{z}_1^2 = \tilde{z}$.

Similarly, for n divisible by 4, let us define

$$\check{z}_1 := \prod_{\text{even } I\subset\{2,\dots,l\}} (z - \sum_{i\in I} x_i) \in \mathrm{CH}(BT).$$

Then $\check{z}_1$ is W-invariant and $\check{z}_1^2 = \check{z}$.

Proposition 2.4 ([1, Theorem 7.1])**.** *Assume that $n \geq 7$. For odd n, the S-algebra $\mathrm{CH}(BT)^W$ is generated by $f_1, \dots, f_{l-2}$ and $\tilde{z}_1$. For n divisible by 4, the S-algebra $\mathrm{CH}(BT)^W$ is generated by $e, f_1, \dots, f_{l-3}$ and $\check{z}_1$. For even n not divisible by 4, the S-algebra $\mathrm{CH}(BT)^W$ is generated by $e, f_1, \dots, f_{l-2}$ and $\check{z}$.*

Remark 2.5. Instead of the generators $f_1, \dots, f_{l-2}$, some different generators $q_1, \dots, q_{l-2}$ (homogeneous of degrees $2^1, \dots, 2^{l-2}$ as well) are used in [1]. However, as shown in [8, Lemma 2.3], both generate the same subring in $\mathrm{CH}(BT)$.

3. Computation of $\mathrm{Ch}(BT)^W$

To decode the statement of Theorem 1.1, we provide a description of W-invariants $\mathrm{Ch}(BT)^W$ for the modulo 2 Chow ring. First of all, the ring $\mathrm{Ch}(BT)$ itself is the polynomial ring $\mathbb{F}[z, x_1, \dots, x_l]$ in the $l+1$ variables modulo the relation $x_1 + \dots + x_l = 0$, where $\mathbb{F} := \mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. So, $\mathrm{Ch}(BT)$ is isomorphic to the polynomial ring $\mathbb{F}[z, x_2, \dots, x_l]$ in the variables other than x_1. The elementary symmetric polynomials $c_1, \dots, c_l$ in $x_1, \dots, x_l$ (where $c_1 = 0$ in $\mathrm{Ch}(BT)$) are W-invariant.

Example 3.1. The quotient ring $R := \mathbb{Z}[x_1, \dots, x_l]/(x_1 + \dots + x_l)$ can be viewed as the symmetric $\mathbb{Z}$-algebra of the character group of the standard split maximal torus in the special linear group $\mathrm{SL}(l)$. The Weyl group of $\mathrm{SL}(l)$ is the symmetric group S_l acting on R by permutations of $x_1, \dots, x_l$. By [12, Lemma 8.1] we have $R^{S_l} = \mathbb{Z}[c_2, \dots, c_n]$. It follows by [4, Théorème] that $(R \otimes \mathbb{F})^{S_l} = R^{S_l} \otimes \mathbb{F} = \mathbb{F}[c_2, \dots, c_n]$.

Proposition 3.2. *The $\mathbb{F}[c_2, \dots, c_l]$-algebra $\mathrm{Ch}(BT)^W$ is generated by the following single element: $\tilde{z}_1$ for odd n, $\check{z}_1$ for n divisible by 4, $\check{z}$ for even n not divisible by 4.*

Proof. Since the group $(\mathbb{Z}/2\mathbb{Z})^{\times l}$ acts on $\mathrm{Ch}(BT)$ trivially, the invariants under its intersection with W contain the $\mathbb{F}[x_1, \dots, x_l]$-subalgebra of $\mathrm{Ch}(BT)$ generated by the orbit product of $z \in \mathrm{Ch}(BT)$ which is equal – depending on $n \pmod 4$ – to $\tilde{z}_1$, $\check{z}_1$, or $\check{z}$. Since the linear factors of each orbit product are distinct primes of the polynomial ring $\mathbb{F}[z, x_2, \dots, x_l]$, the inclusion is actually an equality (cf. [5, Proof of Lemma 3.2]). Taking additionally into account the action of $S_l \subset W$ (trivial on the above orbit products) and Example 3.1, we come to the announced answer. ☐

Remark 3.3. Under the reduction modulo 2 homomorphism

$$\mathbb{Z}[z, x_1, \dots, x_l] \to \mathbb{F}[z, x_1, \dots, x_l]$$

of the polynomial rings, the images of the generators $f_0, f_1, \dots$ are determined as follows: the image of f_0 is $c_1 = x_1 + \dots + x_l$ and for every $i \geq 0$ the image of f_{i+1} is the sum of pairwise products of distinct monomials in the image of f_i. In particular, these images are symmetric polynomials in $x_1, \dots, x_l$ (the variable z does not intervene). The element f_0 vanishes in $\mathrm{Ch}(BT)$, whereas f_1 and f_2 map respectively to c_2 and c_4, where $c_i := 0$ for $i > l$. The formulas for f_i with $i \geq 3$ are more complicated.

We can already prove the easy inclusion of Theorem 1.1:

Proposition 3.4. *For any n, the image of φ contains $(\mathrm{Ch}(BT)^W)^2$.*

Proof. For odd n, $\tilde{z}_1^2 = \tilde{z} \in \mathrm{CH}(BT)$ is the image under Φ of the highest Chern class of the spin representation of G (see Remark 2.3). For even n, $\check{z} \in \mathrm{CH}(BT)$ is the image under Φ of highest Chern class of a half-spin representation of G and $\check{z}_1^2 = \check{z}$ for n divisible by 4 (see Remark 2.3). Finally, the squares $c_1^2, \dots, c_n^2 \in \mathrm{Ch}(BT)$ are the images under φ of the even Chern classes of the orthogonal representation of G (see Remark 2.2). ☐

Thus, Theorem 1.1 yields

Corollary 3.5. *We set $t := \tilde{z}$ for odd n and we set $t := \check{z}$ for even n. Let $S' \subset \mathrm{CH}(BT)$ be the subring generated by t, S, and $2\,\mathrm{CH}(BT)^W$. Then $\mathrm{Im}\,\Phi = S'$ for $n = 7, 8, 9, 11, 12$ and $\mathrm{Im}\,\Phi \subset S'$ for $n = 13, 15, 16, 17$.*

Proof. For any n as in Theorem 1.1, any element of $\operatorname{Im}\Phi$ is a sum of an element of S' with an element $\alpha \in 2\operatorname{CH}(BT)$. It follows that $\alpha \in 2\operatorname{CH}(BT) \cap \operatorname{CH}(BT)^W = 2\operatorname{CH}(BT)^W$. If $n \leq 12$, the the torsion index of G is 2 so that $2\operatorname{CH}(BT)^W \subset \operatorname{Im}\Phi$. □

4. Restrictions on $\operatorname{Im}\varphi$

In this section, we discuss restrictions on the image of $\varphi\colon \operatorname{Ch}(BG) \to \operatorname{Ch}(BT)$, where $G = \operatorname{Spin}(n)$ with arbitrary $n \geq 7$. First of all, an upper bound on $\operatorname{Im}\varphi$ is given by the image of the subring described in Proposition 2.1. Another restriction, already considered in [11] and pushed further below, is given by the action of the modulo 2 Steenrod algebra. Combining the two restrictions will be our ultimate strategy.

We have a commutative square

$$\begin{array}{ccc} \operatorname{Ch}(BG) & \xrightarrow{\varphi} & \operatorname{Ch}(BT) \\ \downarrow{\scriptstyle \operatorname{St}} & & \downarrow{\scriptstyle \operatorname{St}} \\ \operatorname{Ch}(BG) & \xrightarrow{\varphi} & \operatorname{Ch}(BT) \end{array} \tag{4.1}$$

where St is the total cohomological Steenrod operation, constructed for smooth algebraic varieties in characteristic $\neq 2$ in [3] and in characteristic 2 in [17]. It is also defined for classifying spaces of affine algebraic groups via their approximations by algebraic varieties introduced in [22]. The operation St is a (nonhomogeneous) ring homomorphism, determined in the case of $\operatorname{Ch}(BT) = \mathbb{F}[z, x_1, \dots, x_l]$ by the rule $\operatorname{Ch}^1(BT) \ni a \mapsto a + a^2$.

It follows from (4.1) that $\operatorname{Im}\varphi$ is stable under St. Moreover, being graded, the image of φ is stable for every $i \geq 0$ under the ith graded component St^i of St, rasing the degree by i. (The negative graded components of St are trivial.)

The image of φ is contained in the subring $\mathbb{F}[z, c_2, \dots, c_l] \subset \mathbb{F}[z, x_1, \dots, x_l] = \operatorname{Ch}(BT)$ which is also stable under St. The subring $\mathbb{F}[c_2, \dots, c_l]$ is stable under St as well. For any $i, j \geq 0$, a formula for $\operatorname{St}^i(c_j)$ (where $c_0 := 1$) is provided in [2, Théorème 7.1] and applied here below. Note that $\operatorname{St}^i(c_j)$ vanishes for $i > j$, equals c_j^2 for $i = j$, and is equal to

$$\operatorname{St}^i(c_j) = \sum_{k=0}^{i} \binom{i-j}{k} c_{i-k} c_{j+k}$$

otherwise. The binomial coefficient in this simplified formula (borrowed from [20, Proposition 3.1.12]) is taken modulo 2 and has a negative upper entry.

Remind that c_1 is trivial in our setting.

Here is the key observation of the section:

Lemma 4.2. *The kernel of* $\operatorname{St}^1\colon \mathbb{F}[c_2, \dots, c_l] \to \mathbb{F}[c_2, \dots, c_l]$ *is the* $\mathbb{F}[c_2^2, \dots, c_l^2]$*-algebra generated by* c_l *and all* c_i *with odd* i.

Proof. We have $\operatorname{St}^1(c_i) = c_{i+1}$ for even i (with the agreement $c_{l+1} := 0$) and $\operatorname{St}^1(c_i) = 0$ for odd i. Since $\operatorname{St}^1(ab) = \operatorname{St}^1(a)b + a\operatorname{St}^1(b)$, the above rules determine the additive map $\operatorname{St}^1\colon \mathbb{F}[c_2, \dots, c_l] \to \mathbb{F}[c_2, \dots, c_l]$. Note that $\operatorname{St}^1 \circ \operatorname{St}^1 = 0$ so that the kernel of St^1 contains its image. The kernel is a ring, containing the squares, and the image is an ideal in this ring, generated by all c_i with odd i. The quotient is known to be the ring generated by the squares for odd l; for even l, it is generated by c_l and the squares (see, e.g., [1, §9] dealing with the topological Sq^1 in place of St^1). □

Proposition 4.3. *Assume that $n \geq 7$ is odd or divisible by 4. If l is odd, then the image of φ is contained in the $(\mathrm{Ch}(BT)^W)^2$-subalgebra of $\mathrm{Ch}(BT)^W$ generated by all c_ic_j with odd i,j. If l is even, then the image of φ is contained in the $(\mathrm{Ch}(BT)^W)^2$-subalgebra of $\mathrm{Ch}(BT)^W$ generated by c_l and all c_ic_j with odd i,j.*

Proof. By Proposition 2.4, the assumption on n ensures that the graded ring $\operatorname{Im}\varphi$ is concentrated in even degrees. It follows that $\operatorname{Im}\varphi$ vanishes under the first Steenrod operation $\mathrm{St}^1\colon \mathrm{Ch}(BT) \to \mathrm{Ch}(BT)$.

By Proposition 2.1 and Remark 3.3, any element in $\operatorname{Im}\varphi$ is a polynomial in t^2 with coefficients in $\mathbb{F}[c_2,\dots,c_l]$, where $t := \tilde{z}_1$ for odd n and $t := \check{z}_1$ for n divisible by 4. Note that t is divisible by z in $\mathbb{F}[z, c_2,\dots,c_l]$.

Let $a \in \mathbb{F}[t^2, c_2,\dots,c_l]$ be any polynomial in t^2 with coefficients in $\mathbb{F}[c_2,\dots,c_l]$ satisfying $\mathrm{St}^1(a) = 0$. To prove Proposition 4.3 for odd l, it suffices to show that the coefficients of a are polynomials in $c_2^2,\dots,c_l^2$ and c_ic_j with odd i,j. For even l, is suffices to show that the coefficients of a are polynomials in $c_2^2,\dots,c_{l-1}^2, c_l$ and c_ic_j with odd i,j. We prove that the coefficients of a have the required form by induction on degree of a.

If a is constant (i.e., $a \in \mathbb{F}[c_2,\dots,c_l]$), the statement follows by Lemma 4.2. Otherwise, we have $a = a't^2 + b$, where a' is a polynomial in t^2 of smaller degree and b is the constant term of a. We have $0 = \mathrm{St}^1(a) = \mathrm{St}^1(a')t^2 + \mathrm{St}^1(b)$ implying that $\mathrm{St}^1(a') = 0 = \mathrm{St}^1(b)$. It follows that b and the coefficients of a' have the required form. □

5. Proof of Theorem 1.1

This section is the proof of Theorem 1.1. More precisely, since we already proved Proposition 3.4, we prove here that $\operatorname{Im}\varphi \subset (\mathrm{Ch}(BT)^W)^2$ for the valus of n listed in the statement of Theorem 1.1. We do this by employing the upper bound on $\operatorname{Im}\varphi$ given in Proposition 4.3. Besides, we continue to employ the fact that $\operatorname{Im}\varphi$ is stable under the Steenrod operations St^i on $\mathrm{Ch}(BT)$. (To get Proposition 4.3, we only used St^1.) Note that the subring $\mathrm{Ch}(BT)^W \subset \mathrm{Ch}(BT)$ is stable under the Steenrod operations because W acts on $\mathrm{Ch}(BT)$ through automorphisms of approximations of BT.

Continuing the analogy between the operation St^1 on $\mathrm{Ch}(BT)$ and the Bockstein operation Sq^1 in the motivic cohomology, let us note that every odd operation St^{2i+1} on $\mathrm{Ch}(BT)$ is the composition $\mathrm{St}^1 \circ \mathrm{St}^{2i}$. (See [24, Lemma 9.6] for the corresponding property of Sq^1.) Since we already exhausted (in Proposition 4.3) stability of $\operatorname{Im}\varphi$ under St^1, the additional restrictions on $\operatorname{Im}\varphi$ will come from the action of the even Steenrod operations. More exactly, we will be using St^2 and St^4 only.

Recall that the image of φ is a subring of the ring

$$A := \mathrm{Ch}(BT)^W = \mathbb{F}[t, c_2,\dots,c_l],$$

where $t := \tilde{z}_1$ for odd n and $t := \check{z}_1$ for n divisible by 4. (We do not consider the values of n congruent to 2 modulo 4 because they do not appear in Theorem 1.1.) The generators $t, c_2,\dots,c_l$ of A are algebraically independent. By Proposition 2.1, $\operatorname{Im}\varphi$ is actually inside the smaller ring $A' := \mathbb{F}[t^2, c_2,\dots,c_l]$. Note that A' is stable under the Steenrod operations on A. By Proposition 3.4, $\operatorname{Im}\varphi$ contains the subring A^2 of squares in A, which is also stable under the Steenrod operations. As an A^2-module, A' is free with the basis given by the 2^{l-1} products $\prod_{i\in I} c_i$, where I runs over the subsets in $\{2,\dots,l\}$.

$\mathbf{n=7}$.

Since $l=3$ in this case, the statement follows directly from Proposition 4.3.

$\mathbf{n=8,9}$.

We have $\operatorname{Ch}(BT)^W=\mathbb{F}[t,c_2,c_3,c_4]=:A$. By Proposition 4.3, $\operatorname{Im}\varphi$ is contained in the A^2-subalgebra generated by c_4. Therefore any element α of $\operatorname{Im}\varphi$ has the form $\alpha=a^2+b^2c_4$ with $a,b\in A$. We have

$$A^2[c_4]\ni \operatorname{St}^2(\alpha)=(\operatorname{St}^1(a))^2+(\operatorname{St}^1(b))^2c_4+b^2c_4c_2$$

because $\operatorname{St}^2(c_4)=c_2c_4$. It follows that $b^2c_4c_2\in A^2[c_4]$ and therefore $b=0$ meaning that $\alpha\in A^2$.

$\mathbf{n=11}$.

We have $\operatorname{Ch}(BT)^W=\mathbb{F}[t,c_2,c_3,c_4,c_5]=:A$. By Proposition 4.3, $\operatorname{Im}\varphi$ is contained in $A^2[c_3c_5]$. We are going to use the formula

$$\operatorname{St}^4(c_3c_5)=\operatorname{St}^2(c_3)\cdot\operatorname{St}^2(c_5)+c_3\cdot\operatorname{St}^4(c_5)=(c_2c_3+c_5)\cdot(c_2c_5)+c_3\cdot(c_4c_5).$$

Any element α of $\operatorname{Im}\varphi$ has the form $\alpha=a^2+b^2c_3c_5$ with $a,b\in A^2$. Applying St^4 to it, we get

$$A^2[c_3c_5]\ni\operatorname{St}^4(\alpha)=(\operatorname{St}^2(a))^2+(\operatorname{St}^2(b))^2c_3c_5+b^2\operatorname{St}^4(c_3c_5).$$

It follows that $b^2\operatorname{St}^4(c_3c_5)$ is in $A^2[c_3c_5]$ and therefore $a_2=0$ meaning that $\alpha\in A^2$.

$\mathbf{n=12,13}$.

Here we have $\operatorname{Ch}(BT)^W=\mathbb{F}[t,c_2,c_3,c_4,c_5,c_6]=:A$. By Proposition 4.3, $\operatorname{Im}\varphi$ is contained in $A^2[c_3c_5,c_6]$. So, any element α of $\operatorname{Im}\varphi$ has the form

$$\alpha=a^2+b^2c_3c_5+c^2c_6+d^2c_3c_5c_6$$

with $a,b,c,d\in A$. With the formulas

$$\operatorname{St}^2(c_6)=c_2c_6\quad\text{and}\quad\operatorname{St}^2(c_3c_5)=(c_2c_3+c_5)\cdot c_5+c_3\cdot(c_2c_5)=c_5^2\in A^2,$$

the inclusion $A^2[c_3c_5,c_6]\supset\operatorname{Im}\varphi\ni\operatorname{St}^2(\alpha)$ tells us that that

$$c^2\operatorname{St}^2(c_6)+d^2c_3c_5\operatorname{St}^2(c_6)=c^2c_2c_6+d^2c_3c_5c_2c_6\in A^2[c_3c_5,c_6]$$

and implies $c=d=0$.

As a next and final step, we use the operation St^4, satisfying the formula

$$\operatorname{St}^4(c_3c_5)=\operatorname{St}^2(c_3)\cdot\operatorname{St}^2(c_5)+c_3\cdot\operatorname{St}^4(c_5)=(c_2c_3+c_5)\cdot(c_2c_5)+c_3\cdot(c_4c_5+c_6).$$

Applying St^4 to $\alpha=a^2+b^2c_3c_5$, we see that $b^2\operatorname{St}^4(c_3c_5)$ is in $A^2[c_3c_5,c_6]$. Therefore $b=0$ as well so that $\alpha\in A^2$.

$\mathbf{n=15}$.

Here we have $\operatorname{Ch}(BT)^W=\mathbb{F}[t,c_2,c_3,c_4,c_5,c_6,c_7]=:A$. By Proposition 4.3, $\operatorname{Im}\varphi$ is inside the ring $B:=A^2[c_3c_5,c_3c_7,c_5c_7]$, where the three A^2-algebra generators together with 1 are (free) A^2-module generators of B. For any odd i,j, $\mathrm{St}^2(c_ic_j)$ is a linear combination with coefficients in $\mathbb{F}$ of $c_{i+2}c_j$ and c_ic_{j+2}. Therefore the ring B is stable under the operation St^2 and we proceed to employment of St^4 for which the formulas are:

$$\begin{aligned}
&\mathrm{St}^4(c_3c_5) = \mathrm{St}^2(c_3)\,\mathrm{St}^2(c_5) + c_3\,\mathrm{St}^4(c_5) = (c_2c_3+c_5)(c_2c_5) + c_3(c_4c_5+c_3c_6+c_2c_7),\\
&\mathrm{St}^4(c_3c_7) = \mathrm{St}^2(c_3)\,\mathrm{St}^2(c_7) + c_3\,\mathrm{St}^4(c_7) = (c_2c_3+c_5)(c_2c_7) + c_3(c_4c_7),\\
&\mathrm{St}^4(c_5c_7) = \mathrm{St}^4(c_5)c_7 + \mathrm{St}^2(c_5)\,\mathrm{St}^2(c_7) + c_5\,\mathrm{St}^4(c_7) =\\
&\qquad\qquad = (c_4c_5+c_3c_6+c_2c_7)c_7 + (c_2c_5)(c_2c_7) + c_5(c_4c_7).
\end{aligned} \tag{5.1}$$

Any element α of $\operatorname{Im}\varphi \subset B$ has the form

$$\alpha = a^2 + b_{35}^2c_3c_5 + b_{37}^2c_3c_7 + b_{57}^2c_5c_7$$

with $a, b_{35}, b_{37}, b_{57} \in A$, and $\mathrm{St}^4(\alpha)$ is in $\operatorname{Im}\varphi \subset B$ again. It follow that

$$b_{35}^2\,\mathrm{St}^4(c_3c_5) + b_{37}^2\,\mathrm{St}^4(c_3c_7) + b_{57}^2\,\mathrm{St}^4(c_5c_7) \in B.$$

The product $c_3c_4c_5$ (which is one of the basis elements of the free A^2-module A') occurs in the first formula of (5.1) only and shows that $b_{35} = 0$. The product $c_3c_4c_7$ occurs in the second formula only and shows that $b_{37} = 0$. The product $c_3c_6c_7$ occurs in the third formula only and shows that $b_{57} = 0$. It follows that $\alpha \in A^2$.

$\mathbf{n = 16, 17}$.

Here we have $\mathrm{Ch}(BT)^W = \mathbb{F}[t, c_2, c_3, c_4, c_5, c_6, c_7, c_8] =: A$. By Proposition 4.3, $\operatorname{Im}\varphi$ is inside the ring $A^2[c_3c_5, c_3c_7, c_5c_7, c_8]$. The observation on $\mathrm{St}^2(c_ic_j)$ for odd i,j is still valid so that the subring $B := A^2[c_3c_5, c_3c_7, c_5c_7]$ (without c_8) is stable under St^2. Any element of $\alpha \in \operatorname{Im}\varphi$ has the form $a + bc_8$ with $a, b \in B$. Since $\mathrm{St}^2(\alpha)$ is in $\operatorname{Im}\varphi$ as well and

$$\mathrm{St}^2(\alpha) = \mathrm{St}^2(a) + \mathrm{St}^2(b)c_8 + bc_2c_8,$$

we get that $b = 0$, i.e., $\alpha = a \in B$.

Now we show that $\alpha \in A^2$ exactly as in the case $n = 15$, using the formulas for St^4 which are almost the same as in (5.1):

$$\begin{aligned}
&\mathrm{St}^4(c_3c_5) = (c_2c_3+c_5)(c_2c_5) + c_3(c_4c_5+c_3c_6+c_2c_7),\\
&\mathrm{St}^4(c_3c_7) = (c_2c_3+c_5)(c_2c_7) + c_3(c_4c_7+c_3c_8),\\
&\mathrm{St}^4(c_5c_7) = (c_4c_5+c_3c_6+c_2c_7)c_7 + (c_2c_5)(c_2c_7) + c_5(c_4c_7+c_3c_8).
\end{aligned}$$

Appendix A. Erratum to [11]

It is claimed in [11, Proof of Theorem 3] that for any even $n = 2l \geq 8$, the modulo 2 Euler class c_l is outside the image of $\mathrm{Ch}(BG) \to \mathrm{Ch}(BT)$, where T is the standard split maximal torus in $G := \mathrm{Spin}(n)$. But the proof of this claim, given there, is only valid for n divisible by 4. Lemma A.1 shows that the claim actually fails for $n = 10$. For all $n \neq 10$ however, the claim holds. To see it, assume that $n = 2l > 10$ with odd l. In particular, $l \geq 7$. By Proposition 2.4, any odd degree homogeneous element in $\mathrm{CH}(BT)^W$ is divisible by e in $\mathrm{CH}(BT)^W$. Assume that $c_l \in \operatorname{Im}\varphi$. Then $\mathrm{St}^6(c_l) = c_6c_l \in \operatorname{Im}\varphi$ and it follows that c_6 is in the image of $\mathrm{CH}(BT)^W \to \mathrm{Ch}(BT)$. However, in degree up to 6, this image is generated by c_2, c_4, and c_3^2 (see Remark 3.3). Therefore $c_l \notin \operatorname{Im}\varphi$.

Lemma A.1. *For* $G = \mathrm{Spin}(10)$*, the image of* $\Phi\colon \mathrm{CH}(BG) \to \mathrm{CH}(BT)$ *contains the Euler class* $e = x_1x_2x_3x_4x_5 \in \mathrm{CH}(BT)^W$*.*

Proof. Let P be the standard parabolic subgroup in $G' := \mathrm{Spin}(12)$ such that the quotient variety G'/P is the projective quadric X given by the standard split quadratic form q of dimension 12. The group P contains the standard split maximal torus T' of G'. The group G is the semisimple part of (the reductive part of) P and has the same as P Weyl group W acting on the polynomial ring $\mathrm{CH}(BT') = \mathbb{Z}[z, x_1, \dots, x_5]$ in the 6 (independent) variables the way described in §2. As already mentioned in §2, generators of the ring of W-invariants $\mathrm{CH}(BT')^W$ are constructed in [9, Proposition 2.4] and [13, Proposition 5.1] (see also [5]). One of them is the Euler class $e' := x_1 \dots x_5 \in \mathrm{CH}(BT')^W$. One other – the orbit product $\check{z}$ of z.

In view of the commutative square

$$\begin{array}{ccc} \mathrm{CH}(BP) & \xrightarrow{\Phi'} & \mathrm{CH}(BT')^W \\ \downarrow & & \downarrow \\ \mathrm{CH}(BG) & \longrightarrow & \mathrm{CH}(BT)^W \end{array}$$

we prove Lemma A.1 by finding in the image of Φ' an element mapped e. Note that e' is mapped to e.

A homomorphism of graded rings $\Psi\colon\ \mathrm{CH}(BT')^W \to \mathrm{CH}(X)$ is constructed in [5, Lemma 2.2]. It is uniquely determined by the property that the composition $\Psi \circ \Phi'$ is (a particular case of) the homomorphism $\mathrm{CH}(BP) \to \mathrm{CH}(G/P)$ considered in [12, §6]. It is shown in [13, Propositions 5.3 and 5.4] that all the generators of $\mathrm{CH}(BT')^W$ other than e and $\check{z}$ are mapped to the subring in $\mathrm{CH}(X)$ generated by the class $h \in \mathrm{CH}^1(X)$ of a hyperplane section of the quadric. The image of $f_0 = 2z - (x_1 + \ldots + x_6) \in \mathrm{Im}\,\Phi'$ under Ψ equals h so that the subring generated by h is inside the image of the composition $\Psi \circ \Phi'$. By [12, Theorem 6.4], the cokernel of $\Psi \circ \Phi'$ is killed by the torsion index of G' equaling 2. Therefore the image of $\Psi \circ \Phi'$ contains $2\,\mathrm{CH}(X)$. The cokernel of Φ' is killed by the torsion index of P, which is also equal to 2, and so, $\mathrm{Im}\,\Phi' \supset 2\,\mathrm{CH}(BT')^W$.

Since the degree 2^4 of $\check{z}$ is higher that the degree 5 of the Euler class, we do not care about $\check{z}$. The image in $\mathrm{CH}^5(X)$ of the Euler class $e \in \mathrm{CH}^5(BT')$ is the difference $\lambda - \lambda'$ of two distinct classes of maximal totally isotropic subspaces of q. Since $h^5 = \lambda + \lambda'$ (see, e.g., [10, §2.1]), we have $\lambda - \lambda' = 2\lambda - h^5 \in \mathrm{Im}(\Psi \circ \Phi')$. It follows that $\mathrm{Im}\,\Phi'$ contains an element of the form $e + a$, where $a \in \mathrm{CH}^5(BT')^W$ is a polynomial in the generators of $\mathrm{CH}(BT')^W$ of degree < 5. Since f_0 is the only generator of odd degree < 5, a is divisible by f_0 and therefore vanishes in $\mathrm{CH}(BT)^W$. □

References

[1] Benson, D. J., and Wood, J. A. Integral invariants and cohomology of $B\mathrm{Spin}(n)$. *Topology 34*, 1 (1995), 13–28.

[2] Borel, A. La cohomologie mod 2 de certains espaces homogènes. *Comment. Math. Helv. 27* (1953), 165–197.

[3] Brosnan, P. Steenrod operations in Chow theory. *Trans. Amer. Math. Soc. 355*, 5 (2003), 1869–1903 (electronic).

[4] Demazure, M. Invariants symétriques entiers des groupes de Weyl et torsion. *Invent. Math. 21* (1973), 287–301.
[5] Devyatov, R. A., Karpenko, N. A., and Merkurjev, A. S. Maximal indexes of flag varieties for spin groups. *Forum Math. Sigma 9* (2021), Paper No. e34, 12.
[6] Garibaldi, S. Cohomological invariants: exceptional groups and spin groups. *Mem. Amer. Math. Soc. 200*, 937 (2009), xii+81. With an appendix by Detlev W. Hoffmann.
[7] Guillot, P. The Chow rings of G_2 and Spin(7). *J. Reine Angew. Math. 604* (2007), 137–158.
[8] Karpenko, N. A. Envelopes and classifying spaces. Preprint (26 May 2022, 10 pages). Available on author's web page.
[9] Karpenko, N. A. On generic flag varieties for odd spin groups. Preprint (final version of 23 Nov 2021, 11 pages). To appear in Publ. Mat.
[10] Karpenko, N. A. Algebro-geometric invariants of quadratic forms. *Algebra i Analiz 2*, 1 (1990), 141–162.
[11] Karpenko, N. A. On classifying spaces of spin groups. *Results Math. 77*, 4 (2022), Paper No. 144, 8 pages.
[12] Karpenko, N. A., and Merkurjev, A. S. Canonical p-dimension of algebraic groups. *Adv. Math. 205*, 2 (2006), 410–433.
[13] Karpenko, N. A., and Merkurjev, A. S. Indexes of generic Grassmannians for spin groups. *Proc. Lond. Math. Soc. (3) 125*, 4 (2022), 825–840.
[14] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.
[15] Merkurjev, A. S. Invariants of algebraic groups and retract rationality of classifying spaces. In *Algebraic groups: structure and actions*, vol. 94 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 2017, pp. 277–294.
[16] Molina Rojas, L. A., and Vistoli, A. On the Chow rings of classifying spaces for classical groups. *Rend. Sem. Mat. Univ. Padova 116* (2006), 271–298.
[17] Primozic, E. Motivic Steenrod operations in characteristic p. *Forum Math. Sigma 8* (2020), Paper No. e52, 25.
[18] Quillen, D. The mod 2 cohomology rings of extra-special 2-groups and the spinor groups. *Math. Ann. 194* (1971), 197–212.
[19] Rojas, L. A. M. The Chow ring of the classifying space of Spin_8. Ph.D. Thesis (2006), 71 pages. Available at `matfis.uniroma3.it/Allegati/Dottorato/TESI/molina/TesiMolina.pdf`.
[20] Smirnov, A., and Vishik, A. Subtle characteristic classes. arXiv:1401.6661v2 [math.AG] (4 Oct 2021), 32 pages.
[21] Tanania, F. Subtle characteristic classes for *Spin*-torsors. *Math. Z. 301*, 1 (2022), 41–74.
[22] Totaro, B. The Chow ring of a classifying space. In *Algebraic K-theory (Seattle, WA, 1997)*, vol. 67 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.
[23] Totaro, B. The torsion index of the spin groups. *Duke Math. J. 129*, 2 (2005), 249–290.
[24] Voevodsky, V. Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.*, 98 (2003), 1–57.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
Email address: `karpenko@ualberta.ca`, *web page*: `www.ualberta.ca/~ karpenko`