Topological classification of $\mathbf{Z}_p^m$ actions on surfaces

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Abstract. Let $\widetilde{S}$ be a closed (compact without boundary) oriented surface with genus g, and G be a group isomorphic to $\mathbf{Z}_p^m$, where p is a prime integer. An action of G on S is a pair $(\widetilde{S}, f)$, where f is a representation of G in the group of orientation preserving autohomeomorphisms of $\widetilde{S}$. Two actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ are called strongly (resp. weakly) equivalent if there is a homeomorphism, $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}'$, sending the orientation of $\widetilde{S}$ to the orientation of $\widetilde{S}'$, such that $f'(h) = \widetilde{\psi} \circ f(h) \circ \widetilde{\psi}^{-1}$, (resp. there is an automorphism $\alpha \in Aut(G)$ such that $f' \circ \alpha(h) = \widetilde{\psi} \circ f(h) \circ \widetilde{\psi}^{-1}$) for all $h \in G$. We give the full description of strong and weak equivalence classes. The main idea of our work is the fact that a fixed point free action of $\mathbf{Z}_p^m$ on a surface provides a bilinear antisymmetric form on $\mathbf{Z}_p^m$. For instance we prove that the weakly equivalence classes of actions of G on surfaces with orbit space of genus g are in one to one correspondence with the set of pairs which consist in a positive integer number k, $k \leq m-n$, $k = (m-n) \bmod 2$, $g \geq \frac{1}{2}(m-n+k)$, and an orbit of the action of $Aut(G)$ on the set of unordered r-tuples $[C_1, ..., C_r]$ of non-trivial elements generating a subgroup isomorphic to $\mathbf{Z}_p^n$ and such that $\sum_1^r C_i = 0$. We use this result for the description of the moduli space of complex algebraic curves admitting a group of automorphisms isomorphic to $\mathbf{Z}_p^m$.

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1 Introduction

The abelian group actions on surfaces is a classical subject and it is studied in the papers [N], [E], [J1], [J2], [Na1], [S], [Z]. In [E], [J1], [J2], [Z], it is found a connection between the topological equivalence classes of actions and the second homology of the group that is acting. But some attemps to use these results for the classification of abelian actions give wrong results in some cases (compare Remark 4.5 of [E] with Corollary 12 in our Section 4). The full classification has been found in the cyclic case by J. Nielsen in [N] and for $\mathbf{Z}_2^m$ in [Na1]. In this paper we present a direct way to deal with the topological classification of $\mathbf{Z}_p^m$ actions, where p is a prime integer and we obtain a complete answer ($\mathbf{Z}_p^m = \mathbf{Z}_p \oplus \overset{m}{\cdots} \oplus \mathbf{Z}_p$ and $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$). The main idea of our work is the fact that a fixed point free action of $\mathbf{Z}_p^m$ provides a bilinear antisymmetric form on $\mathbf{Z}_p^m$.

Let $\widetilde{S}$ be a closed (compact without boundary) oriented surface with genus $g = g(\widetilde{S})$, and G be a group isomorphic to $\mathbf{Z}_p^m$, where p is a prime integer. An action of G on $\widetilde{S}$ is a pair $(\widetilde{S}, f)$, where f is a representation of G in the group of orientation preserving autohomeomorphisms of $\widetilde{S}$. Two actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ are called strongly equivalent if there is a homeomorphism, $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}'$, sending the orientation of $\widetilde{S}$ to the orientation of $\widetilde{S}'$, such that $f'(h) = \widetilde{\psi} \circ f(h) \circ \widetilde{\psi}^{-1}$, for all $h \in G$. We give the full description of strong equivalence classes and in particular in the case of fixed point free actions, the set of such equivalence classes of actions on surfaces of a given genus is in bijection with the set of bilinear antisymmetric forms $(.,.) : G^* \times G^* \to \mathbf{Z}_p$, where G^* is the group of forms of G on $\mathbf{Z}_p$ (Theorems 8 and 9). The case of actions having elements with fixed points is considered in Theorems 13 and 14.

A motivation for our study is to describe the set of connected components in the moduli space $M^{p,m}$ of pairs (C, G), where C is a complex algebraic curve and $G \cong \mathbf{Z}_p^m$ is a group of automorphisms of C. According to [Na2] the description of connected components of $M^{p,m}$ is reduced to the description of topological classes of pairs $(\widetilde{S}, K)$ where K is a group of autohomeomorphisms of $\widetilde{S}$ and K is isomorphic to $\mathbf{Z}_p^m$. We consider that $(\widetilde{S}, K)$ and $(\widetilde{S}', K')$ are equivalent if there exist a homeomorphism $\varphi : \widetilde{S} \to \widetilde{S}'$ such that $K' = \varphi \circ K \circ \varphi^{-1}$. These equivalence classes are in one to one correspondence with classes of weak equivalence (in the terminology of Edmonds [E]). Two actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ of a group $G \cong \mathbf{Z}_p^m$ are weakly equivalent if there is a homeomorphism $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}'$ and an automorphism $\alpha \in Aut(G)$ such that $f' \circ \alpha(h) = \widetilde{\psi} \circ f(h) \circ \widetilde{\psi}^{-1}$ for all $h \in G$. We prove that the weakly equivalence classes of actions of G on surfaces with orbit space of genus g are in one to one correspondence with the set of pairs $(k, Aut(G)[C_1, ..., C_r])$, such that $k \leq m - n$, $k = (m - n) \bmod 2$, $g \geq \frac{1}{2}(m - n + k)$, $r \geq n$ and $[C_1, ..., C_r]$ is an unordered r-tuple of non-trivial elements generating $H \cong \mathbf{Z}_p^n$ such that $\sum_1^r C_i = 0$ (Theorem 16).

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2 Algebraic preliminaries.

Let us consider the standard lattice $\mathbf{Z}^{2g} = \mathbf{Z} \oplus \overset{2g}{\dots} \oplus \mathbf{Z}$ with the standard basis $(e_i) = ((0, ..., 1^{(i)}, ...0))$. We define the bilinear antisymmetric form $(.,.) : \mathbf{Z}^{2g} \times \mathbf{Z}^{2g} \to \mathbf{Z}$, by $(e_i, e_j) = \delta_{i+j,2g+1}$ for $i < j$.

We consider also the group $\mathbf{Z}_p^{2g} = \mathbf{Z}_p \oplus \overset{2g}{\dots} \oplus \mathbf{Z}_p$ where p is a prime and $\mathbf{Z}_p = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{p-1}\}$. Let $\varphi : \mathbf{Z}^{2g} \to \mathbf{Z}_p^{2g}$ be the natural projection defined by $\varphi(e_i) = \overline{e_i}$, where $\overline{e_i} = (0, ..., \overline{1}^{(i)}, ...0)$. Then we have a bilinear antisymmetric form $(.,.)_p : \mathbf{Z}_p^{2g} \times \mathbf{Z}_p^{2g} \to \mathbf{Z}_p$ defined by $(\overline{e_i}, \overline{e_j}) = (\varphi(e_i), \varphi(e_j))_p = (e_i, e_j) \bmod p$.

Let $SL(2g, \mathbf{Z})$ and $SL_p(2g, \mathbf{Z}_p)$ be the subgroups of the automorphisms groups of $\mathbf{Z}^{2g}$ and $\mathbf{Z}_p^{2g}$ that preserve the bilinear forms $(.,.)$ and $(.,.)_p$ respectively. The natural projection $\varphi : \mathbf{Z}^{2g} \to \mathbf{Z}_p^{2g}$ induces a homomorphism $\varphi_* : SL(2g, \mathbf{Z}) \to SL_p(2g, \mathbf{Z}_p)$ such that $\varphi_*(f) \circ \varphi = \varphi \circ f$ for all $f \in SL(2g, \mathbf{Z})$.

The following result is elementary:

Theorem 1 $\varphi_*(SL(2g, \mathbf{Z})) = SL_p(2g, \mathbf{Z}_p)$.

Proof. (Sketch). We say that an element of $\mathbf{Z}^{2g}$ is *primitive* if $e \neq nf$ for all $n \in \mathbf{Z}$ and $f \in \mathbf{Z}^{2g}$. For every $a_p \in \mathbf{Z}_p^{2g}$ there is a primitive $a \in \mathbf{Z}^{2g}$ such that $\varphi(a) = a_p$.

The proof makes use of the two following claims:

1. Assume that $\langle \widetilde{a}_1, \widetilde{a}_2 \rangle$ is a subgroup of $\mathbf{Z}_p^{2g}$, such that $\langle \widetilde{a}_1, \widetilde{a}_2 \rangle \cong \mathbf{Z}_p^2$ and the bilinear form restricted to $\langle \widetilde{a}_1, \widetilde{a}_2 \rangle$ is not trivial. Then there are primitive elements $a_1, a_2 \in \mathbf{Z}^{2g}$ such that $(a_1, a_2) = 1$ and $\varphi(a_1), \varphi(a_2) \in \langle \widetilde{a}_1, \widetilde{a}_2 \rangle$.

2. Let $a, b \in \mathbf{Z}^2$, $\varphi(a) \neq 0$, $\varphi(b) \neq 0$ and $(a, b) = mp$ with $m \in \mathbf{Z}$. Then $\langle\varphi(a)\rangle = \langle\varphi(b)\rangle$.

Using induction and the claims 1 and 2 it is easy to prove:

Let G be a subgroup of $\mathbf{Z}_p^{2g}$. Then there is $\Delta = ((a_i, (i = 1, ..., r), b_j, (j = 1, ..., k \leq s)) \subset \mathbf{Z}^{2g}$, (where k may be 0), such that $\varphi(\Delta)$ generate G, Δ is linear independent and $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}$.

From this fact and induction the Theorem follows. ■

We shall also need the following results of linear algebra over finite fields.

Lemma 2 *Let $H \cong \mathbf{Z}_p^m$ and $(.,.) : H \times H \to \mathbf{Z}_p$ be a bilinear antisymmetric form. Let $\Delta = (a_i, (i = 1, ..., r)) \subset H$ be a maximal set of linear independent elements such that $(a_i, a_j) = 0$. Then there is a basis of H, $((a_i, (i = 1, ..., r), b_j, (j = 1, ..., s))$, $0 \leq s \leq r$, such that $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}$.*

Proof. Let us consider all systems $\Delta' = (a_i, (i = 1, ..., r), b_j, (j = 1, ..., k))$, $0 \leq s \leq r$, such that $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}$. Between them we choose a system Λ with maximal k. Then Λ is a basis with the conditions that we need. ■

Theorem 3 *Let G, G' be subgroups of $\mathbf{Z}_p^{2g}$ and $\psi : G \to G'$ be an isomorphism such that $(\psi(a), \psi(b))_p = (a, b)_p$ for all $a, b \in G$. Then there is an automorphism $\widetilde{\psi} \in SL_p(2g, \mathbf{Z}_p)$ such that $\widetilde{\psi}$ restricted to G is ψ.*

Proof. It is a consequence of Theorem 1 and Lemma 2. ■

3 Strong classification of fixed point free orientation preserving actions of $\mathbf{Z}_p^m$ on surfaces

Let $\widetilde{S}$ be a closed (compact without boundary) oriented surface with genus g, and G be a group isomorphic to $\mathbf{Z}_p^m$, where p is a prime integer. An action of G on $\widetilde{S}$ is a pair $(\widetilde{S}, f)$, where f is a monomorphism of G in the group of orientation preserving autohomeomorphisms of $\widetilde{S}$.

Definition 4 *(Strong equivalence). Two actions $(\widetilde{S}, f)$ and $(\widetilde{S'}, f')$ are called strongly equivalent if there is a homeomorphism, $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}'$, sending the* *orientation of $\widetilde{S}$ to the orientation of $\widetilde{S}'$ and such that $f'(h) = \widetilde{\psi} \circ f(h) \circ \widetilde{\psi}^{-1}$, for all $h \in G$.*

We are interested in to find all the strong equivalence classes of actions of $\mathbf{Z}_p^m$.

We denote by $S = \widetilde{S}/f(G)$ and by $\varphi = \varphi(f) : \widetilde{S} \to S$ the natural projection. We shall consider first the case when $f(h)$ has no fixed points for any $h \in G$, i. e., the action of $(\widetilde{S}, f)$ is fixed point free. The general case will be considered in Section 5. Then the projection $\varphi(f) : \widetilde{S} \to S$ is an unbranched covering with deck group of transformations $f(G)$.

Let us consider $\pi_1(S)$ as the group of deck transformations of the universal covering of S. Then we have:

$$\omega(\widetilde{S}, f) : \pi_1(S) \to \pi_1(S)/\pi_1(\widetilde{S}) = f(G) \stackrel{f^{-1}}{\to} G.$$

The resulting epimorphism $\omega(\widetilde{S}, f) : \pi_1(S) \to G \cong \mathbf{Z}_p^m$ is the monodromy epimorphism of the covering $\varphi(f) : \widetilde{S} \to S$. The epimorphism $\omega(\widetilde{S}, f) : \pi_1(S) \to G$, induces the epimorphism $\theta_p(\widetilde{S}, f) : H_1(S, \mathbf{Z}_p) \to G$, since G is abelian.

Conversely, given an epimorphism $\theta_p : H_1(S, \mathbf{Z}_p) \to G$, there is an action $(\widetilde{S}, f)$ such that $\theta_p = \theta_p(\widetilde{S}, f)$. To obtain $\widetilde{S}$ is enough to consider the monodromy $\omega : \pi_1(S) \to H_1(S, \mathbf{Z}_p) \stackrel{\theta_p}{\to} G$ and then $\widetilde{S} = U/\ker\omega$, where U is the universal covering of S and the action of G is given by $G = \pi_1(S)/\ker\omega$.

Definition 5 *Let S and S' be two surfaces. Two epimorphisms $\theta : H_1(S, \mathbf{Z}_p) \to G$ and $\theta' : H_1(S', \mathbf{Z}_p) \to G$ are called strongly equivalent if there is a homeomorphism $\psi : S \to S'$ such that induces an isomorphism $\psi_p : H_1(S, \mathbf{Z}_p) \to H_1(S', \mathbf{Z}_p)$ such that $\theta = \theta' \circ \psi_p$*

Theorem 6 *(P. A. Smith [S]). Two actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ are strongly equivalent if and only if the epimorphisms $\theta_p(\widetilde{S}, f)$ and $\theta_p(\widetilde{S}', f')$ are strongly equivalent.*

Definition 7 *Let $(\widetilde{S}, f)$ be an action of G, $S = \widetilde{S}/f(G)$, and $\theta = \theta_p(\widetilde{S}, f) : H_1(S, \mathbf{Z}_p) \to G$ be the epimorphism defined by the action $(\widetilde{S}, f)$. Let us consider the spaces of homomorphisms $G^* = \{e : G \to \mathbf{Z}_p\}$ and $H^1(S, \mathbf{Z}_p) = \{e : H_1(S, \mathbf{Z}_p) \to \mathbf{Z}_p\}$. Then θ generates a monomorphism $\theta^* = \theta^*(\widetilde{S}, f)$:*

$G^* \to H^1(S, \mathbf{Z}_p)$. *The intersection form* $(.,.)_p = (.,.)_p^S$ *on* $H_1(S, \mathbf{Z}_p)$ *induces an isomorphism* $i : H^1(S, \mathbf{Z}_p) \to H_1(S, \mathbf{Z}_p)$ *defined by* $(a, .) \to a$ *and a form* $(.,.)_{(\widetilde{S},f)} : G^* \times G^* \to \mathbf{Z}_p$ *such that* $(a, b)_{(\widetilde{S},f)} = (i \circ \theta^*(a), i \circ \theta^*(b))_p$.

Theorem 8 *Two actions* $(\widetilde{S}, f)$ *and* $(\widetilde{S}', f')$ *of the group* G *are strongly equivalent if and only if* $\widetilde{S}$ *and* $\widetilde{S}'$ *have the same genus and* $(.,.)_{(\widetilde{S},f)} = (.,.)_{(\widetilde{S}',f')}$.

Proof. Let S and S' denote $\widetilde{S}/f(G)$ and $\widetilde{S}'/f'(G)$ respectively. Assume that $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ are strongly equivalent, then according to Theorem 6 there exists a homeomorphism $\psi : S \to S'$, which induces an isomorphism $\psi_p : H_1(S, \mathbf{Z}_p) \to H_1(S', \mathbf{Z}_p)$ such that $\theta = \theta' \circ \psi_p$. Since ψ_p is induced by a homeomorphism then ψ_p preserves the intersection form and induces an isomorphism $\psi^* : H^1(S', \mathbf{Z}_p) \to H^1(S, \mathbf{Z}_p)$ such that $(a, b)_p^{S'} = (\psi^*(a), \psi^*(b))_p^S$ and $\theta^*(\widetilde{S}, f) = \psi^* \circ \theta^*(\widetilde{S}', f')$. Hence, for $a, b \in G^*$ we have

$$(a, b)_{(\widetilde{S},f)} = (i \circ \theta^*(\widetilde{S}, f)(a), i \circ \theta^*(\widetilde{S}, f)(b))_p^S =$$
$$= (i \circ \psi^* \circ \theta^*(\widetilde{S}', f')(a), i \circ \psi^* \circ \theta^*(\widetilde{S}', f')(b))_p^S =$$
$$= (i' \circ \theta^*(\widetilde{S}', f')(a), i' \circ \theta^*(\widetilde{S}', f')(b))_p^{S'} = (a, b)_{(\widetilde{S}',f')}.$$

Assume now $(.,.)_{(\widetilde{S},f)} = (.,.)_{(\widetilde{S}',f')}$. We consider some isomorphisms

$Q : H^1(S, \mathbf{Z}_p) \to (\mathbf{Z}_p^{2g}, (.,.))$ and $Q' : H^1(S', \mathbf{Z}_p) \to (\mathbf{Z}_p^{2g}, (.,.))$,

such that $(Q(a), Q(b))_p = (a, b)_p^S$ and $(Q'(a'), Q'(b'))_p = (a', b')_p^S$, for any $a, b \in H^1(S, \mathbf{Z}_p)$ and $a', b' \in H^1(S', \mathbf{Z}_p)$.

We note $\widetilde{G} = Q \circ \theta^*(\widetilde{S}, f)(G^*) \subset \mathbf{Z}_p^{2g}$, and $\widetilde{G}' = Q' \circ \theta^*(\widetilde{S}', f')(G^*) \subset \mathbf{Z}_p^{2g}$. Let $\psi : \widetilde{G} \to \widetilde{G}'$ be the isomorphism given by $\psi = Q' \circ Q^{-1}$. Then, for every $a, b \in \widetilde{G}$, we have $(\psi(a), \psi(b))_p = (a, b)_p$. From Theorem 3 follows that there is $\widetilde{\psi} \in SL_p(2g, \mathbf{Z})$ such that $\widetilde{\psi}$ restricted to $\widetilde{G}$ is ψ. Consider now $\Psi = Q^{-1} \circ \widetilde{\psi} \circ Q' : H^1(S', \mathbf{Z}_p) \to H^1(S, \mathbf{Z}_p)$. Since $\widetilde{\psi} \in SL_p(2g, \mathbf{Z})$ then Ψ comes from an isomorphism $\psi_* : H_1(S, \mathbf{Z}) \to H_1(S', \mathbf{Z})$ sending the intersection form of $H_1(S, \mathbf{Z})$ to the intersection form of $H_1(S', \mathbf{Z})$ (Theorem 1). Then by a classical result of H. Burkardt in 1890 (see [MKS], pg. 178), there is some homeomorphism $\psi : S \to S'$ inducing ψ_* and Ψ, and by construction $\theta^*(\widetilde{S}, f) = \Psi \circ \theta^*(\widetilde{S}', f')$. Then by Theorem 6 the actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ are strongly equivalent. ■

Theorem 9 *Let* $G \cong \mathbf{Z}_p^m$ *and* $(.,.) : G^* \times G^* \to \mathbf{Z}_p$ *be a bilinear antisymmetric form where* $k = \dim\{h \in G^* : (h, G^*) = 0\}$. *Then an action* $(\widetilde{S}, f)$ *such* *that* $(.,.) = (.,.)_{(\widetilde{S},f)}$ *and* $g = g(\widetilde{S}/f(G))$ *exists if and only if* $g \geq \frac{1}{2}(m+k)$, $k = m \bmod 2$, $k \leq m$.

Proof. First we construct the action from the form and the numerical conditions. Applying Lemma 2 we have a basis of G^*, $(a_i^*, (i = 1, ..., r), b_j^*, (j = 1, ..., s))$ $0 \leq s \leq r$, such that $(a_i^*, a_j^*) = (b_i^*, b_j^*) = 0$, $(a_i^*, b_j^*) = \delta_{ij}$ and $s - r = k$. Let $(a_i, (i = 1, ..., r), b_j, (j = 1, ..., s))$ be the dual basis of the above one. Now consider a surface S of genus g and a basis of $H_1(S, \mathbf{Z}_p)$, $(\alpha_i, (i = 1, ..., g), \beta_i, (i = 1, ..., g))$. Then we construct the epimorphism $\theta : H_1(S, \mathbf{Z}_p) \to G^*$, defined by $\theta(\alpha_i) = a_i$, if $i \leq r$, $\theta(\alpha_i) = 0$, if $i > r$ and $\theta(\beta_i) = b_i$, if $i \leq s$, $\theta(\beta_i) = 0$, if $i > s$. Then the epimorphism θ defines a regular covering $\widetilde{S} \to S$ with automorphism group G and the action of G on $\widetilde{S}$ satisfies $(.,.)_{(\widetilde{S},f)} = (.,.)$.

Conversely if there is an action $(\widetilde{S}, f)$ such that $(.,.) = (.,.)_{(\widetilde{S},f)}$ it is obvious that $g = g(\widetilde{S}/f(G)) \geq \frac{1}{2}(m+k)$, $k = m \bmod 2$, $k \leq m$. ■

4 Weak classification of fixed point free orientation preserving actions of $\mathbf{Z}_p^m$ on surfaces

Definition 10 *(Weak equivalence) Let* $(\widetilde{S}, f)$ *and* $(\widetilde{S}', f')$ *be two actions of a group* $G \cong \mathbf{Z}_p^m$. *We shall say that* $(\widetilde{S}, f)$ *and* $(\widetilde{S}', f')$ *are weakly equivalent if there is a homeomorphism* $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}'$ *and an automorphism* $\alpha \in Aut(G)$ *such that* $f' \circ \alpha(h) = \widetilde{\psi} \circ f(h) \circ \widetilde{\psi}^{-1}$, $h \in G$.

The next Theorem solves the problem of weak classification of actions of $\mathbf{Z}_p^m$ on surfaces:

Theorem 11 *Let* $(\widetilde{S}, f)$ *and* $(\widetilde{S}', f')$ *be two actions of a group* $G \cong \mathbf{Z}_p^m$. *Let* $(.,.)_{(\widetilde{S},f)}$ *and* $(.,.)_{(\widetilde{S}',f')}$ *be the antisymmetric forms induced by the two actions,* $k(\widetilde{S}, f) = \dim\{h \in G^* : (h, G^*)_{(\widetilde{S},f)} = 0\}$ *and* $k(\widetilde{S}', f') = \dim\{h \in G^* : (h, G^*)_{(\widetilde{S}',f')} = 0\}$. *Then the actions* $(\widetilde{S}, f)$ *and* $(\widetilde{S}', f')$ *are weakly equivalent if and only if* $g(\widetilde{S}/f(G)) = g(\widetilde{S}'/f'(G))$ *and* $k(\widetilde{S}, f) = k(\widetilde{S}', f')$.

Proof. Let us call $S = \widetilde{S}/f(G)$ and $S' = \widetilde{S}'/f'(G)$, $g = g(S)$ and $g' = g(S')$. Let $\theta^*(\widetilde{S}, f)$ and $\theta^*(\widetilde{S}', f')$ be the epimorphisms defined by the two actions, $\widetilde{G}$ be the image of G^* in $H_1(S, \mathbf{Z}_p)$ by $\theta^*(\widetilde{S}, f)$ and $\widetilde{G}'$ be the image of G^* in $H_1(S', \mathbf{Z}_p)$ by $\theta^*(\widetilde{S}', f')$.

Assume that $g(S) = g(S')$ and $k(\widetilde{S}, f) = k(\widetilde{S}', f')$. Since $k(\widetilde{S}, f) = k(\widetilde{S}', f')$ then there exists an isomorphism $\psi : \widetilde{G}' \to \widetilde{G}$ such that
$(\psi(a), \psi(b))_{(\widetilde{S}', f')} = (a, b)_{(\widetilde{S}, f)}$.

Then, using Theorem 3, and that $g(S) = g(S')$, there is an isomorphism $\widetilde{\psi} : H^1(S', \mathbf{Z}_p) \to H^1(S, \mathbf{Z}_p)$ giving by restriction ψ and sending the intersection form of $H^1(S', \mathbf{Z}_p)$ to the intersection form of $H^1(S, \mathbf{Z}_p)$. By [MKS, pag 178], there exists a homeomorphism $\varphi : S \to S'$ inducing $\widetilde{\psi}$ on cohomology. Then by Theorem 6, the actions $(\widetilde{S}, f)$ and $(\widetilde{S}, \varphi^{-1} \circ f' \circ \varphi)$ are strongly equivalent. The isomorphism ψ, defines an automorphism of G giving the weak equivalence between $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$. ■

Corollary 12 *The weak equivalence classes of $\mathbf{Z}_p^m$ actions are in bijection with the set of pairs of positive integer numbers (k, g) such that $k \le m$, $k = m \bmod 2$ and $g \ge \frac{1}{2}(m + k)$.*

5 Classification of orientation preserving actions of $\mathbf{Z}_p^m$ with elements having fixed points.

Let G be a group isomorphic to $\mathbf{Z}_p^m$ and $(\widetilde{S}, f)$ be an action of G on an oriented closed surface $\widetilde{S}$. We shall call G_{fix} to the subgroup of G generated by the elements of $f(G)$ having fixed points.

The projection $\varphi = \varphi(f) : \widetilde{S} \to S = \widetilde{S}/f(G)$ is a covering branched on a finite set of points $\mathcal{B} = \{b_1, ..., b_r\}$. The covering φ is now determined by an epimorphism $\theta_p(\widetilde{S}, f) : H_1(S - \mathcal{B}, \mathbf{Z}_p) \to G$.

We shall call X_i, $i = 1, ..., r$, the element of $H_1(S - \mathcal{B}, \mathbf{Z}_p)$ represented by the boundary of a small disc in S around the branched point b_i, and with the orientation given by the orientation of S. Then the set $\{\theta_p(\widetilde{S}, f)(X_i)\}$ is a topological invariant for the action $(\widetilde{S}, f)$ and $\left\langle \theta_p(\widetilde{S}, f)(X_i), i = 1, ..., r \right\rangle = G_{fix}$. Then we have an epimorphism $\vartheta : H_1(S, \mathbf{Z}_p) \to G_{free}$, defined by

$$H_1(S, \mathbf{Z}_p) \to H_1(S - \mathcal{B}, \mathbf{Z}_p)/ \langle X_i, i = 1, ..., r \rangle \to G/G_{fix} = G_{free}.$$

In fact the epimorphism ϑ is the epimorphism defined by the fixed point free action defined by the unbranched covering $\widetilde{S}/f(G_{fix}) \to S$. If $G_{free} = G/G_{fix}$ then ϑ defines, as in Section 2, a bilinear form $(.,.)_{(\widetilde{S},f)} : G^*_{free} \times G^*_{free} \to \mathbf{Z}_p$.

Theorem 13 *Two actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ of the group $G \cong \mathbf{Z}_p^m$ are strongly equivalent if and only if:*

1. $\widetilde{S}$ and $\widetilde{S}'$ have the same genus, the number of branched points $r = \#\mathcal{B}$, of the covering $\widetilde{S} \to S = \widetilde{S}/f(G)$, is the same than the number of branched points $r' = \#\mathcal{B}'$, of the covering $\widetilde{S}' \to S' = \widetilde{S}'/f'(G)$.

2. $[\theta_p(\widetilde{S}, f)(X_1), \theta_p(\widetilde{S}, f)(X_2), ..., \theta_p(\widetilde{S}, f)(X_r)] =$
$= [\theta_p(\widetilde{S}, f)(X_1), \theta_p(\widetilde{S}, f)(X_2), ..., \theta_p(\widetilde{S}, f)(X_r)]$, where $[., ..., .]$ means unordered r-tuple. As a consequence $G^f_{free} = G^{f'}_{free} = G_{free}$.

3. The intersection form on G_{free}, induced by f and f' is the same, $(.,.)_{(\widetilde{S},f)} = (.,.)_{(\widetilde{S}',f')}$.

Proof. Using Dehn twists along curves around the branched points (see [C], pg. 151, move (6)) it is possible to obtain a basis $(A_i, (i = 1, ..., g), B_i, (i = 1, ..., g), X_i, (i = 1, ..., r))$, of $H_1(S - \mathcal{B}, \mathbf{Z}_p)$ such that
$\theta_p(\widetilde{S}, f)(A_i) \in G_{free}, \theta_p(\widetilde{S}, f)(B_i) \in G_{free}$, $i = 1, ..., g$,
and $(A_i, A_j) = 0$, $(B_i, B_j) = 0$, $(A_i, B_j) = \delta_{ij}$.

In the same way, we can construct a basis $(A'_i, (i = 1, ..., g), B'_i, (i = 1, ..., g), X'_i, (i = 1, ..., r))$, of $H_1(S' - \mathcal{B}', \mathbf{Z}_p)$ such that
$\theta_p(\widetilde{S}', f')(A'_i) \in G_{free}, \theta_p(\widetilde{S}', f')(B'_i) \in G_{free}$, $i = 1, ..., g$,
and $(A'_i, A'_j) = 0$, $(B'_i, B'_j) = 0$, $(A'_i, B'_j) = \delta_{ij}$,
remark that by condition 1, $g = g'$ and $r = r'$.

By condition 3 and Theorem 8, then the fixed point free action of G_{free} on $\widetilde{S}/f(G_{fix})$ given by f and the fixed point free action of G_{free} on $\widetilde{S}'/f'(G_{fix})$ given by f' are strongly equivalent. Then there exists a homeomorphism, preserving the orientations, $\varphi : S \to S'$ inducing on homology an isomorphism $\psi : H_1(S, \mathbf{Z}_p) \to H_1(S', \mathbf{Z}_p)$ and by the proof of the Theorem 8 we can construct φ such that $\psi(A_i) = A'_i$, and $\psi(B_i) = B'_i$. We now consider a disc D on S containing $\mathcal{B}$ and a disc D' on S' containing $\mathcal{B}'$. Then we can modify φ by composing with an isotopy in S' in order that $\varphi(D) = D'$, and $\varphi(b_i) = b_{\sigma(i)}$ where σ is a permuntation of $\{1, ..., r\}$ such that
$(\theta_p(\widetilde{S}, f)(X_1), \theta_p(\widetilde{S}, f)(X_2), ..., \theta_p(\widetilde{S}, f)(X_r)) =$
$= (\theta_p(\widetilde{S}, f)(X_{\sigma(1)}), \theta_p(\widetilde{S}, f)(X_{\sigma(2)}), ..., \theta_p(\widetilde{S}, f)(X_{\sigma(r)}))$.

Now φ defines an isomorphim $\widetilde{\psi} : H_1(S - \mathcal{B}, \mathbf{Z}_p) \to H_1(S' - \mathcal{B}', \mathbf{Z}_p)$ such that $\widetilde{\psi}(A_i) = A'_i$, $\widetilde{\psi}(B_i) = B'_i$, $\widetilde{\psi}(X_i) = X'_i$, then $\theta_p(\widetilde{S}, f) = \theta_p(\widetilde{S}', f') \circ \varphi$. Hence the actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ are strongly equivalent. ■

As a consequence of Theorem 13 and Theorem 9 we have:

Theorem 14 *Let $G \cong \mathbf{Z}_p^m$, and $H \cong \mathbf{Z}_p^n$ be a subgroup of G. Assume that $[C_1, ..., C_r]$, $r \geq n$, be an unordered element of $(H - \{0\})^r$, where $\{C_1, ..., C_r\}$ generates H and $\sum_1^r C_i = 0$. Let $(.,.)$ be a bilinear antisymmetric form on G/H with $k = \dim\{h \in (G/H)^* : (h, (G/H)^*) = 0\}$. Then for $g \geq \frac{1}{2}(m - n + k)$ and only for all such g there is an action $(\widetilde{S}, f)$ with $g = g(\widetilde{S}/f(G))$, $(.,.) = (.,.)_{(\widetilde{S},f)}$, where $(.,.)_{(\widetilde{S},f)}$ is the bilinear form induced by the fixed point free action on G/H, the elements acting with fixed points generate H and $[\theta_p(\widetilde{S}, f)(X_1), \theta_p(\widetilde{S}, f)(X_2), ..., \theta_p(\widetilde{S}, f)(X_r)] = [C_1, C_2, ..., C_r]$.*

Remark. The unordered elements of H^r, are in one to one correspondence with the functions $F : H \to (\mathbf{Z}_+)^{p-1}$. From $[C_1, C_2, ..., C_r]$ we define $F(h) = (k_1, k_2, ..., k_{p-1})$ if the element h^i appears k_i times in $[C_1, C_2, ..., C_r]$. The function F gives the topological type of the action of H.

Theorem 15 *Two actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ of the group $G \cong \mathbf{Z}_p^m$ are weakly equivalent if and only if:*

1. $\widetilde{S}$ and $\widetilde{S}'$ have the same genus, the number of branched points $r = \#\mathcal{B}$ of the covering $\widetilde{S} \to S = \widetilde{S}/f(G)$, is the same than the number of branched points $r' = \#\mathcal{B}'$ of the covering $\widetilde{S}' \to S' = \widetilde{S}'/f'(G)$.

*2. $(\theta_p(\widetilde{S}, f)(X_1), \theta_p(\widetilde{S}, f)(X_2), ..., \theta_p(\widetilde{S}, f)(X_r)) =$
$= (\gamma \circ \theta_p(\widetilde{S}, f)(X_{\sigma(1)}), \gamma \circ \theta_p(\widetilde{S}, f)(X_{\sigma(2)}), ..., \gamma \circ \theta_p(\widetilde{S}, f)(X_{\sigma(r)}))$, where σ is a permutation of $\{1, ..., r\}$ and γ is an automorphism of G.*

3. $\dim\{h \in G^*_{free} : (h, G^*_{free})_{(\widetilde{S},f)} = 0\} = \dim\{h \in G^*_{free} : (h, G^*_{free})_{(\widetilde{S}',f')} = 0\}$.

Proof. It is similar to the proof of Theorem 13 but using Theorem 8. ■

As a consequence of Corollary 12 we have:

Theorem 16 *Let $G \cong \mathbf{Z}_p^m$. The weak equivalence classes of actions of G are in bijection with the set of triples*

$(k, g, Aut(G)[C_1, ..., C_r])$,

such that $k \leq m - n$, $k = (m - n) \bmod 2$, $g \geq \frac{1}{2}(m - n + k)$, $r \geq n$ and $[C_1, ..., C_r]$ is an unordered r-tuple of non-trivial elements of H^r such that $\{C_1, ..., C_r\}$ generates a group isomorphic to $\mathbf{Z}_p^n$ and $\sum_1^r C_i = 0$.

Let $M^{p,m}$ be the space of pairs $(\widetilde{R}, G)$, where $\widetilde{R}$ is a Riemann surface and G is a group of automorphims of $\widetilde{R}$. The covering $\widetilde{R} \to \widetilde{R}/G$ defines a projection $p : M^{p,m} \to M$, where M is the moduli space of Riemann surfaces. The projection $p : M^{p,m} \to M$, gives a topology on $M^{p,m}$, the weakest topology where p is continuous.

From Theorem 16 and [Na2], Section 6, we have:

Consequence *There exists a one to one correspondence between the connected components of $M^{p,m}$ with such topology and triples*

$$(k, g, Aut(G)[C_1, ..., C_r])$$

described in Theorem 16. Each connected component of $M^{p,m}$ is homeomorphic to the quotient R^n/Mod of a vector space R^n by the discontinuous action of a group Mod.

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