ON THE COHOMOLOGY OF POLYCYCLIC-BY-FINITE GROUPS

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Abstract.

Let $k[G]$ be the group algebra of a polycyclic-by-finite group G over a field k. We show that, for any finitely generated $k[G]$-module V, $H^*(G,V) = \bigoplus_{n\geq 0} H^n(G,V)$ is a Noetherian module over the cohomology algebra $H^*(G,k)$. In particular, the power series $P_G(V;t) = \sum_{n\geq 0} \dim_k H^n(G,V)t^n \in \mathbb{Z}[[t]]$ is a rational function in t of the form $f(t) / \prod_{i=1}^{s} (1-t^{k_i})$, where $f(t) \in \mathbb{Z}[t]$. For $G = D_\infty$, the infinite dihedral group, and char $k = 2$ we show that $P_G(V;-1)$ is defined and equals $-\frac{1}{2}\rho(V)$, where $\rho(V)$ denotes the Goldie rank of V.

Introduction.

A celebrated theorem due to Farkas, Snider [F-S], and Cliff [C] asserts that the group algebra $k[G]$ of a torsion-free polycyclic-by-finite group G over a field k has no zero divisors. The proof heavily uses the fact that, for G torsion-free, $k[G]$ has finite global dimension. In the presence of torsion in G, however, $k[G]$ has infinite global dimension if char k divides the order of an element of $G \setminus \{1\}$. The goal of this note is to establish a cohomological finiteness result which holds for general polycyclic-by-finite G and arbitrary coefficient fields. We also briefly discuss the relations between the cohomology of a finitely generated $k[G]$-module and its so-called Goldie rank in the case when $k[G]$ is prime.

Specifically, let R be a commutative Noetherian ring and let V be a finitely generated (left) module over the group ring $R[G]$, where G is polycyclic-by-finite. Then we will show in Section 1 that $H^*(G,V) = \bigoplus_{n\geq 0} H^n(G,V)$ is a Noetherian module over the cohomology ring $H^*(G,R) = \bigoplus_{n\geq 0} H^n(G,R)$ under the action given by the cup product $\smile: H^*(G,R) \times H^*(G,V) \to H^*(G,V)$. As a standard consequence, it follows that for any additive $\mathbb{Z}$-valued function λ on the class of all finitely generated R-modules, the power series

$$P(V;t) = \sum_{n\geq 0} \lambda(H^n(G,V))t^n \in \mathbb{Z}[[t]]$$

is a rational function in t of the form $f(t)/\prod_{i=1}^{s}(1-t^{k_i})$ with $f(t)\in \mathbb{Z}[t]$. For finite groups, these results are due to Evens [E]. Now each polycyclic-by-finite group G contains a normal subgroup N of finite index which is poly-(infinite cyclic), and hence of finite cohomological dimension. Thus our strategy is to use Evens's theorem and extend it by means of the Lyndon-Hochschild-Serre (LHS) spectral sequence for $1 \to N \to G \to G/N \to 1$. The arguments here are similar to the ones used by Evens.

In Section 2, we consider finitely generated modules V over the group algebra $k[G]$ of G over a field k, under the assumption that $k[G]$ is prime. (This holds if and only if G has no finite normal subgroups $\neq \langle 1\rangle$ [P, Theorem 4.2.10].) We discuss the relations between the Poincaré series $P(V;t)=\sum_{n\geq 0} \dim_k H^n(G,V)t^n$ and the Goldie rank $\rho(V)$ of V. Here, by definition, $\rho(V)$ is the composition lenght of $Q(k[G])\otimes_{k[G]} V$ over $Q(k[G])$, the simple Artinian quotient ring of $k[G]$. In the special case where $G=D_\infty$ is the infinite dihedral group and char $k=2$ it is shown that, for any finitely generated $k[G]$-mudule V, $P(V;-1)=-\frac{1}{2}\rho(V)=-\rho(V)/\rho(k[G])$.

The crucial facts about polycyclic-by-finite groups that will be used in this article are the following:

(1) group rings of polycyclic-by-finite groups over Noetherian rings are Noetherian [P, 10.2.7] , and

(2) polycyclic-by-finite groups are virtual Poincaré duality groups [Br, Chap VIII Sec. 10 and 11] ,[B].

A good deal of the material in Section 1 holds for more general classes of groups, provided one assumes the coefficient modules V in $H^*(G,V)$ to be finitely generated over the ground ring R (see Remark 3). Since we are interested in applications to infinite-dimensional modules, we will concentrate on polycyclic-by-finite groups.

Notations and Conventions. Throughout, R will be a commutative Noetherian ring, k will denote a commutative field, and G will be a polycyclic-by-finite group. All modules over the group rings R[G] and k[G] will be left modules. In general, our notation follows [P] and [C-E].

1. The Finiteness Theorem

The proof of the first lemma has been shown to us by R.Bieri.

Lemma 1. Let V be a finitely generated R[G]-module. Then for all $n \geq 0$, $H^n(G,V)$ and $H_n(G,V)$ are Noetherian as R-modules.

Proof. Since R[G] is Noetherian, we can choose a projective resolution $\underline{P} = (P_n)_{n\geq 0}$ of V over R[G] with each P_n finitely generated over R[G]. Now $H_n(G,V) \cong \mathrm{Tor}_n^{R[G]}(R,V) \cong$ $\cong H_n(R\otimes_{R[G]}\underline{P})$, and $R\otimes_{R[G]}\underline{P}$ is a complex of finitely generated R-modules. Since R is Noetherian, each $H_n(R\otimes_{R[G]}\underline{P})$ is Noetherian over R, which proves the assertion for $H_n(G,V)$.

Now let N be a torsion-free polycyclic normal subgroup of G having finite index in G. Then, by [B, Satz 3.1.2 and Bemerkung 1], there exist R-isomorphisms

$$H^n(N,V) \cong H_{cdN-n}(N,\tilde{R}\otimes_R V),$$

where cdN denotes the cohomological dimension of N an $\tilde{R}$ is an R[N]-module such that $\tilde{R} \cong R$ as R-modules but each $x \in N$ acts as Id or -Id. Clearly, $\tilde{R} \otimes_R V$ is finitely generated over R[N], as V is, and so the foregoing implies that $H^n(N,V)$ is a Noetherian R-module. To prove the assertion for $H^n(G,V)$ consider the LHS-spectral sequence for $1 \to N \to G \to G/N \to 1$. Its E_2-term $E_2^{p,q}(V) = H^p(G/N, H^q(N,V))$ is finitely generated over R. Hence $E_\infty^{p,q}(V)$ is Noetherian over R, being a subquotient of $E_2^{p,q}(V)$, and we conclude that $H^{p+q}(G,V)$ is Noetherian over R. □

Now consider $H^*(G,R) = \bigoplus_{n \geq 0} H^n(G,R)$, with the trivial action of G on R, and $H^*(G,V) = \bigoplus_{n \geq 0} H^n(G,V)$. The multiplication of R gives rise to a cup product $\smile: H^p(G,R) \times H^q(G,R) \longrightarrow H^{p+q}(G,R)$ which makes $H^*(G,R)$ an R-algebra and, similarly, the action of R on V yields an action of $H^*(G,R)$ on $H^*(G,V)$ via cup products. The following result is the promised extension of Evens's theorem [E, Theorem 6.1].

<u>Theorem 2.</u> Let V be a finitely generated R[G]-module. Then $H^*(G,V)$ is a Noetherian module over $H^*(G,R)$.

<u>Proof.</u> Let N be a torsion-free polycyclic normal subgroup of G having finite index in G. Then N has finite cohomdogical dimension [G, §8.8, Lemma 8] and so Lemma 1 implies that $H^*(N,V) = \bigoplus_{n=0}^{cdN} H^n(N,V)$ is Noetherian over R. By Evens's theorem, we conclude that $H^*(G/N, H^*(N,V))$ is Noetherian over $H^*(G/N,R)$.

Now consider the LHS-spectral sequence for $1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$, with E_2-term $E_2^{p,q}(.) \cong H^p(G/N, H^q(N,.))$. For each $r \geq 2$, there is a canonical pairing

$$. : \quad E_r^{p,q}(R) \times E_r^{s,t}(V) \longrightarrow E_r^{p+s,q+t}(V)$$

induced by the action of R on V. If d_r denotes the differential of $E_r(.)$, then one has the following product rule

$$d_r(a \cdot b) = (d_r a) \cdot b + (-1)^{p+q} a \cdot (d_r b) \quad (a \in E_r^{p,q}(R),\ b \in E_r^{s,t}(V)).$$

Moreover, since $E_r^{p,q}(.) = E_\infty^{p,q}(.)$ for $r > \max\{p,q+1\}$, we get an analogous pairing for the E_∞-terms. For $r=2$, the above pairing coincides with the cup product pairing

$$\smile : H^p(G/N, H^q(N,R)) \times H^s(G/N, H^t(N,V)) \longrightarrow H^{p+s}(G/N, H^{q+t}(N,V)),$$

except for a $\pm$ sign [H-S, Section II.5]. Specifically, letting Φ_V denote the isomorphism $E_2(V) \xrightarrow{\sim} H^*(G/N, H^*(N,V))$ and similarly for R, we have

$$\Phi_V(a \cdot b) = (-1)^{pt}\, \Phi_R(a) \smile \Phi_V(b) \quad (a \in E_2^{p,q}(R),\ b \in E_2^{s,t}(V)).$$

In particular, $E_2^{*,0}(R)$ becomes a graded ring, isomorphic to the cohomology ring $H^*(G/N,R)$, and $E_2(V) = \bigoplus_{t=0}^{\mathrm{cd}N} E^{*,t}(V)$ is a direct sum of graded modules over $E_2^{*,0}(R)$.

Altering the action by a ±sign as above doesn't affect the lattice of graded submodules and so we conclude from the first paragraph of the proof that $E_2(V)$ has the ascending chain condition for graded $E_2^{*,0}(R)$-submodules. Hence, by [N-VO, Chap. A, Theorem II. 3.5], $E_2(V)$ is Noetherian over $E_2^{*,0}(R)$.

Let $h_r : \ker d_r \longrightarrow E_{r+1}(.)$ be the canonical epimorphism and set

$$E_{2,\infty}(V) = \{b \in E_2(V) \mid h_{r-1} \cdot \ldots \cdot h_2(b) \in \ker d_r \text{ for all } r \geq 2\},$$

and similarly for $E_{2,\infty}(R)$. Note that $E_2^{*,0}(R) \subseteq E_{2,\infty}(R)$, since the spectral sequence lies in the first quadrant. The product rule for the differentials d_r implies that $E_{2,\infty}(R) \cdot E_{2,\infty}(V) \subseteq E_{2,\infty}(V)$. In particular, $E_{2,\infty}(V)$ is an $E_2^{*,0}(R)$-submodule of $E_2(V)$ and as such is Noetherian. Under the projection maps

$$E_{2,\infty}(V) \twoheadrightarrow E_\infty(V) \quad \text{and} \quad E_2^{*,0}(R) \twoheadrightarrow E_\infty^{*,0}(R),$$

the action of $E_2^{*,0}(R)$ on $E_{2,\infty}(V)$ induces the pairing $\cdot : E_\infty^{*,0}(R) \times E_\infty(V) \longrightarrow E_\infty(V)$ mentioned above. Thus $E_\infty(V)$ is Noetherian as a module over $E_\infty^{*,0}(R)$ and, a fortiori, over $E_\infty(R)$.

Finally, $E_\infty(.)$ is the associated graded module of $H^*(G,.)$ with respect to a suitable filtration of $H^*(G,.)$ which makes $H^*(G,R)$ a filtered ring and $H^*(G,V)$ a filtered module over $H^*(G,R)$ [H-S, Section II.1]. Therefore, $H^*(G,V)$ is Noetherian over $H^*(G,R)$ [N-VO, Chap. D, Corollary IV.4] , and the theorem is proved. □

Remark 3. If V in the above theorem is assumed to be finitely generated over R, then the same conclusion holds for much wider classes of groups G. Namely, it certainly suffices to assume that G has a subgroup N of finite index such that the trivial $R[N]$-module R has a finite resolution by finitely generated projectives over $R[N]$ (so, in the terminology of [Br, Chap. VIII], G is of type VFP over R). Indeed, the above proof can be copied literally, with Lemma 1 becoming superfluous, as $H^*(N,V)$ is clearly Noetherian over R. Therefore, part(i) of the following corollary holds more generally and so does part(ii), provided the coefficient module V is finitely generated over R.

Corollary 4. i. $H^*(G,R)$ is a Noetherian ring and a finitely generated R-algebra (Quillen [Q, Proposition 14.5]).

ii. Let V be a finitely generated $R[G]$-module. Then for any additive $\mathbb{Z}_{\geq 0}$-valued function on the class of all finitely generated R-modules, the power series (cf. Lemma 1)

$$P_G(V,t) = \sum_{n\geq 0} \lambda(H^n(G,V))t^n \in \mathbb{Z}[[t]]$$

is a rational function in t of the form $f(t)/\prod_{i=1}^{s}(1-t^{k_i})$ with $f(t) \in \mathbb{Z}[t]$.

Moreover, if $cr_G(V)$ denotes the order of the pole of $P_G(V;t)$ at $t=1$, then

$$cr_G(V) = \inf\{r \in \mathbb{R} \mid \lambda(H^n(G,V)) \leq c \cdot n^{r-1} \text{ for some } c > 0 \text{ and all } n \gg 0\}.$$

Furthermore, $cr_G(V) \leq cr_G(R)$ where R is the trivial $R[G]$-module.

Proof. (i). Theorem 2 implies that $H^*(G,R)$ is a Noetherian ring. In particular, the ideal $H^+(G,R) = \bigoplus_{n>0} H^n(G,R)$ is finitely generated as a left ideal, say by the elements $x_i \in H^{k_i}(G,R)$ $(i=1,2,\ldots,s)$. It is then easy to see that $x_1, x_2, \ldots, x_s$ together with 1 generate $H^*(G,R)$ as an R-algebra (cf. [A-M, Proposition 10.7]).

(ii). In view of part (i) and Theorem 2, the assertion concerning the rationality of $P_G(V;t)$ follows from the Hilbert-Serre theorem [A-M, Theorem 11.1]. Indeed, s and the exponents k_i in $f(t)/\prod_{i=1}^{s}(1-t^{k_i})$ can be chosen as in part(i). (The proof given in [A-M] works for $H^*(G,R)$, since $H^*(G,R)$ satisfies $xy = (-1)^{nm}yx$ for $x \in H^n(G,R)$, $y \in H^m(G,R)$.)

As a consequence of the specific form of the rational function $P_G(V;t)$ it follows that, for large enough n, the n-th coefficient $d_n = \lambda(H^n(G,V))$ can be written as

$$d_n = \sum_{j=1}^{r} P_j(n)\alpha_j^n .$$

Here, the $\alpha_j^{'}$s are roots of $\prod_{i=1}^{s}(1-t^{k_i})$ and each $P_j(n)$ is a polynomial in n with rational coefficients [H, Section 3.1]. Set $k=$ l.c.m.$\{k_i \mid i=1,2,\dots,s\}$ and $\rho = \inf\{r\in\mathbb{R} \mid \lambda(H^n(G,V)) \leq c\cdot n^{r-1}$ for some $c>o$ and all $n>>o\}$. Then $\alpha_j^k = 1$ for all j and there are functions $c_\ell : \mathbb{Z}/k\mathbb{Z} \longrightarrow \mathbb{Q}$ $(\ell=o,1,\dots,h)$ with $c_h \neq 0$ and

$$d_n = \sum_{\ell=o}^{h} c_\ell(n+k\mathbb{Z})\, n^\ell$$

for all $n>>o$. Clearly, $\rho-1=h$. Now consider

$$Q(t) = P_G(V;t)/(1-t) = \sum_{n\geq o} D_n t^n \quad \text{with}$$
$$D_n = \sum_{m\leq n} d_n = \sum_{\ell=o}^{h+1} C_\ell(n+k\mathbb{Z})\ n^\ell \quad (n>>o)$$

for suitable functions $C_\ell : \mathbb{Z}/k\mathbb{Z} \longrightarrow \mathbb{Q}$. Since $D_{n+1} \geq D_n \geq 0$ for all n, it follows that C_{h+1} is constant >0.
Write $Q(t) = Q_1(t) + Q_2(t)$, where $Q_1(t)$ has coefficients of absolute value $< D\cdot n^h$ for a suitable constant $D > o$ and $Q_2(t) = C_{h+1}\cdot\sum_{n\geq o} n^{h+1}t^n$. Then, for $o<x<1$, we have

$$|Q_1(x)|\cdot(1-x)^{h+1} \leq D\cdot(\sum_{n\geq o} n^h x^n)\cdot(1-x)^{h+1}$$
$$\leq D\cdot\left(\frac{d^h}{dx^h}\frac{1}{1-x}\right)\cdot(1-x)^{h+1} = D\cdot h!$$

Therefore, $Q_1(t)$ has a pole of order at most $h+1$ at $t=1$. Similarly, $Q_2(t)$ has a pole of order exactly $h+2$ at $t=1$ and so the same holds for $Q(t)$. This shows that $P_G(V;t)$ has a pole of order $h+1$ at $t=1$, whence $cr_G(V) = h+1 = \rho$, as we have claimed.

Finally, if $\rho(R)$ is defined in analogy with $\rho=\rho(V)$ above, then Theorem 2 and the foregoing together imply that $cr_G(V) = \rho(V) \leq \rho(R) = cr_G(R)$. This completes the proof. □

<u>Example 5</u> Suppose G does not contain any subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and let the coefficient ring R be of characteristic p. Then, by [Br, Chap X. Sec. 6], there exists a positive integer d such that $H^n(G,V) \cong H^{n+d}(G,V)$ holds for any $R[G]$-module V and all $n>h(G)$, the Hirsch number of G. In particular, if V is finitely generated over $R[G]$ and $P_G(V;t)$ is as in the above corollary, then

$$P_G(V;t) = \frac{f(t)}{1-t^d}$$

for some polynomial $f(t)\in\mathbb{Z}[t]$ of degree $\leq h(G)+d$. For concrete computations in the case when $G=D_\infty$ is the infinite dihedral group and R is a field of characteristic 2 see Example 10.

We conclude this section with a brief discussion of the case where $R=k$ is a commutative field. There are several notions of dimension for $H^*(G,k)$ at our disposal. Namely, (i) the Gelfand-Kirillov dimension over k (see [B-K], (ii) the prime length or classical Krull dimension of $H^*(G,k)$, and (iii) the Gabriel-Rentschler Krull dimension [R-G]. However, using the fact that $H^*(G,k)$ is a finitely generated module over the central subalgebra $\bigoplus_{n\geq 0} H^{2n}(G,k)$ which in turn is a finitely generated k-algebra, by Corollary 4(i) and the Artin-Tate lemma, it is easy to show that (i), (ii), and (iii) all coincide with $cr_G(k)$. The following result of Quillen's (again valid for G of type VFP over k) determines this number [Q, Theorem 14.1].

<u>Proposition 6 (Quillen).</u> If char $k=p$ then $cr_G(k)$ equals the p-rank of G, i.e. the largest integer r such that G contains an elementary abelian subgroup of order p^r.

<u>2. Relations with Goldie Ranks</u>

Throughout this section, <u>G will be a polycyclic-by-finite group without finite normal subgroups ≠ <1>.</u> Thus, for any field k, the group algebra $k[G]$ is prime [P, Theorem 4.2.10]. If V is a finitely generated $k[G]$-module, then we let $\rho(V)$ denote its <u>Goldie rank,</u> i.e.

$$\rho(V) = \rho_G(V) = \text{composition length of } Q(k[G]) \otimes_{k[G]} V,$$

where $Q(k[G])$ is the classical simple Artinian ring of quotients of $k[G]$. Rurthermore, we define the <u>normalized Goldie rank</u> $\chi(V)$ of V to be

$$\chi(V) = \chi_G(V) = \rho(V)/\rho(k[G]).$$

Since $Q(k[G])$ is flat over $k[G]$, ρ is additive on short exact sequences and hence χ is also additive. The following lemma describes some further properties of χ. Part (ii) is due to Rosset [R, Proposition 4] and will not be needed in the sequel.

<u>Lemma 7.</u>

i. Let V be a finitely generated $k[G]$-module and let $H \leq G$ be a subgroup of finite index. Then

$$\chi_G(V) = [G:H]^{-1}\chi_H(V).$$

ii. Let $H \leq G$ be a subgroup of G and let W be a finitely generated $k[H]$-module. Then

$$\chi_G(k[G] \otimes_{k[H]} W) = \chi_H(W).$$

<u>Proof of (i).</u> Note that H has no finite normal subgroups. Choose a torsion-free normal subgroup N of G with $N \leq H$ and $[G:N]$ finite. We can clearly assume that $H = N$ so that H is normal in G and torsion-free.

Let C denote the set of nonzero elements of $k[H]$. Since $k[H]$ is a Noetherian domain [F-S,C], C is an Ore set of regular elements in $k[H]$, and in $k[G]$ [P, proof of Lemma 13.3.5 (ii)]. Moreover, $Q(k[G]) = C^{-1}k[G] \supseteq C^{-1}k[H] = Q(k[H])$, and $Q = Q(k[G])$ is simple Artinian and has (right and left) demension $[G:H]$ over the division subring $D = Q(k[H])$. If V is a finitely generated $k[G]$-module, then $Q \otimes_{k[G]} V$ is finitely generated over Q and hence of the form $Q \otimes_{k[G]} V \cong B^{(r)}$, where B is a simple left ideal of Q and $r = \rho(V)$. In particular, $Q \cong B^{(s)}$ with $s = \rho(k[G])$ and so we have

$$\dim_D B = \frac{1}{s} \dim_D Q = [G:H]/s.$$

Therefore, $\chi_G(V) = r/s = [G:H]^{-1} \dim_D(Q \otimes_{k[G]} V)$. To complete the proof, note that $Q \otimes_{k[G]} V \cong D \otimes_{k[H]} V$ as D-modules, since $Q = C^{-1}k[G] \cong D \otimes_{k[H]} k[G]$, and so $\dim_D(Q \otimes_{k[G]} V) = \rho_H(V) = \chi_H(V)$. □

We remark that if V is finitely generated projective over $k[G]$ and char $k=0$, then $\chi_G(V)$ is the Euler characteristic of V as discussed in [P, Chap. 13, Sec. 4]. The following lemma gives a cohomological discription of $\chi_G(V)$.

<u>Lemma 8.</u> Let V be a finitely generated $k[G]$-module. Let $H \leq G$ be a poly-(infinite cyclic) subgroup of finite index in G and let $h = \mathrm{cd} H$ denote the Hirsch number of H. Then

$$\begin{aligned}\chi_G(V) &= [G:H]^{-1} \cdot \sum_{i=0}^{h} (-1)^i \dim_k H_i(H,V) \\ &= (-1)^h \cdot [G:H]^{-1} \cdot \sum_{i=0}^{h} (-1)^i \dim_k H^i(H,V).\end{aligned}$$

(Recall that, by Lemma 1, each $H_i(H,V)$ and $H^i(H,V)$ is finite-dimensional over k.)

<u>Proof.</u> We first consider homology. (This part is also contained in [R, Proposition 2].) Let $\underline{\underline{P}} = (P_i)_{i=0}^{h}$ be a projective resolution of V over $k[H]$ such that each P_i is finitely generated over $k[H]$. Then $H_i(H,V) = H_i(k \otimes_{k[H]} \underline{\underline{P}})$, and hence we get

$$\sum_{i=0}^{h} (-1)^i \dim_k H_i(H,V) = \sum_{i=0}^{h} (-1)^i \dim_k k \otimes_{k[H]} P_i .$$

By [P, Theorem 13.4.9], each P_i is stably free. Thus, as in [P, Lemma 13.4.10], we have $\dim_k k \otimes_{k[H]} P_i = \rho_H(P_i)$ and the above equality becomes

$$\sum_{i=0}^{h} (-1)^i \dim_k H_i(H,V) = \rho_H(V),$$

using the additivity of ρ_H. Since $\rho_H(V) = \chi_H(V)$, the assertion for homology now follows from Lemma 7(i).

To deal with cohomology, we use Poincaré duality as in the proof of Lemma 1. By [B], there exists a one-dimensional $k[H]$-module $\tilde{k}$ such that each $x \in H$ acts as Id or -Id and such that there are k-isomorphisms

$$H^i(H,V) \cong H_{h-i}(H,\tilde{k} \otimes_k V) \qquad (0 \leq i \leq h).$$

By the foregoing, we conclude that

$$\sum_{i=0}^{h} (-1)^i \dim_k H^i(H,V) = (-1)^h \sum_{i=0}^{h} (-1)^i \dim_k H_i(H,\tilde{k} \otimes_k V)$$

$$= (-1)^h \rho_H(\tilde{k} \otimes_k V).$$

Finally, $\rho_H(\tilde{k} \otimes_k V) = \rho_H(V)$. For, if $N \leq H$ denotes the kernel of the action of H on $\tilde{k}$, then N has index at most 2 in H and the restrictions of V and $\tilde{k} \otimes_k V$ to $k[N]$ are isomorphic. Lemma 7(i) now implies that $\rho_H(\tilde{k} \otimes_k V) = \rho_H(V)$ which completes the proof. □

Presumably Lemma 8 is true for any torsion-free subgroup $H \leq G$ of finite index in G. This is certainly so for $k = \mathbb{Q}$, because finitely generated projective $\mathbb{Q}[H]$-modules are stably free for arbitrary torsion-free polycyclic-by-finite groups H [F-H]. For general coefficient fields, however, this seems to be unknown at present.

Using the notation $P_G(V;t) = \sum_{n\geq 0} \dim_k H^n(G,V)t^n$ introduced in Corollary 4, the second equality in Lemma 7 can be expressed as

$$\chi_G(V) = (-1)^h \cdot [G:H]^{-1} \cdot P_H(V;-1) ,$$

where $P_H(V;-1)$ denotes the evaluation of the polynomial $P_H(V;t)$ at $t=-1$.

Lemma 9. Let V be a finitely generated $k[G]$-module which has a finite resolution by finitely generated stably free projectives over $k[G]$. Then $P_G(V;t)$ is a polynomial in t of degree at most $h=h(G)$, the Hirsch number of G, and

$$\chi_G(V) = (-1)^h \cdot P_G(V;-1)$$

Proof. Choose a poly-(infinite cyclic) normal subgroup $H \leq G$ of finite index in G. By [B, Satz 3.1.2] and [Br, Chap. VIII Theorem 10.1], $H^n(H,k[H]) = 0$ for $n \neq h$ and so $H^n(G,k[G]) = 0$ for $n \neq h$, by the Shapiro lemma. Therefore, for any finitely generated $k[G]$-module W which is stably free, $n \neq h$ implies $H^n(G,W) = 0$. An application of the long cohomology sequence now implies that if V has a resolution $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to V \to 0$, where all P_i are finitely generated stably free projectives over $k[G]$, then, $P_G(V;t)$ is a polynomial of degree at most h in t. It also follows that $P_G(V;-1) = \sum_{i=o}^{h} (-1)^i P_G(P_i;-1)$. Since χ_G is additive on short exact sequences, we further have $\chi_G(V) = \sum_{i=o}^{n} (-1)^i \chi_G(P_i)$. Therefore, we can assume that V is stably free, and even free over $k[G]$. The Shapiro lemma now implies that, for all $n \geq o$, $\dim_k H^n(H,V) = [G:H] \dim_k H^n(G,V)$. Thus $P_G(V;t) = [G:H]^{-1} \cdot P_H(V;t)$ and Lemma 8 yields $\chi_G(V) = (-1)^h \cdot P_G(V;-1)$. □

We now consider the infinite dihedral group

$$G = D_\infty = < x,y \mid y^2 = (yx)^2 = 1 >$$

and assume the ground field k to be of characteristic 2 Then $k[G]$ has infinite global dimension. Nevertheless, it turns out that, in this case, the equality $\chi_G(V) = (-1)^{h(G)} \cdot P_G(V;-1)$ holds for arbitrary finitely generated $k[G]$-modules V.

<u>Example 10.</u> Let $G = D_\infty$ be the infinite dihedral group and let $\operatorname{char} k=2$. Then, for any finitely generated $k[G]$-module V, $P_G(V;t)$ has the form $P_G(V;t) = f(t)/1-t$ with $f(t) \in \mathbb{Z}[t]$ of degree ≤ 2, and $P_G(V;-1) = -\chi_G(V)$.

<u>Proof.</u> Since $G \cong C_1 * C_2$ with $C_1 = \langle y \rangle$ and $C_2 = \langle yx \rangle$ cyclic of order 2, we get $H^n(G,V) = H^n(C_1,V) \oplus \oplus H^n(C_2,V)$ for all $n \geq 2$ [G', § 8.6 Theorem 3]. Also, for all $n \geq 1$, $H^n(C_1,V) \cong H^1(C_1,V) \cong \mathrm{Fix}_V(y)/(y-1)V$ and similarly for C_2. Therefore,

$$H^n(G,V) \cong H^2(G,V) \cong \mathrm{Fix}_V(y)/(y-1)V \oplus \mathrm{Fix}_V(yx)/(yx-1)V$$

for all $n \geq 2$, and setting $h_n = \dim_k H^n(G,V)$ it follows that

$$P_G(V;t) = h_o + h_1 t + h_2 t^2 (1+t+t^2+\ldots) = h_o + h_1 t + h_2 t^2/(1-t).$$

Thus $P_G(V;t)$ has the desired form, and $P_G(V;-1) = h_o - h_1 + \frac{1}{2} h_2$.

Next, we show that $P_G(\cdot\ ,-1)$ is additive on short exact sequences

$$0 \to U \to V \xrightarrow{\pi} W \to 0 ,$$

where U,V,W are finitely generated $k[G]$-modules. The connecting homomorphism $\omega_n: H^n(G,W) \longrightarrow H^{n+1}(G,U)$ for $n \geq 2$ sends $[(w_1,w_2)] \in \mathrm{Fix}_W(y) \oplus \mathrm{Fix}_W(yx)/(y-1)W \oplus (yx-1)W \cong H^n(G,W)$ to $[((y-1)v_1, (yx-1)v_2)] \in \mathrm{Fix}_U(y) \oplus \mathrm{Fix}_U(yx)/(y-1)U \oplus (yx-1)U \cong \cong H^{n+1}(G,U)$, where v_1 and $v_2 \in V$ are chosen so that $\pi(v_i) = w_i$. Thus we see that $\dim_k \mathrm{Im}\omega_n$ is constant for all $n \geq 2$.

In particular, $Q(t) = \sum_{n\geq o} \dim_k(\mathrm{Im}\,\omega_n)t^n \in \mathbb{Z}[\![t]\!]$ is rational and defined for $t=-1$. By the long cohomology sequence, we have

$$O = \dim_k(\mathrm{Im}\,\omega_{n-1}) - \dim_k H^n(G,U) + \dim_k H^n(G,V) - \dim_k H^n(G,W) + \dim_k(\mathrm{Im}\,\omega_n)$$

for all $n\geq o$. Thus $P_G(V;t) = P_G(U;t) + P_G(W;t) - Q(t)\cdot(t+1)$ and evaluating at $t=-1$, we obtain $P_G(V;-1) = P_G(U;-1) + P_G(W;-1)$.

To compute $\chi_G(V)$, let $H = \langle x \rangle \leq G$ so that H is infinite cyclic and normal of index 2 in G. By Lemma 6(i),

$$\chi_G(V) = \frac{1}{2}\,\rho_H(V) = \frac{1}{2}\dim_F(F\otimes_{k[H]}V),$$

where $F = Q(k[H]) = k(x)$ is the field of rational functions in x. Set $V_o = \{v\in V \mid \alpha v = 0 \text{ for some } \alpha\neq o\}$. Then V_o is a $k[G]$-submodule of V and is finitely generated over $k[G]$, and over $k[H]$. Hence V_o is finite-dimensional over k and, moreover, $\chi_G(V_o) = 0$. Thus $\chi_G(V) = \chi_G(W)$ where $W=V/V_o$ is finitely generated free over $k[H]$. Therefore, $\dim_F(F\otimes_{k[H]}W) = \dim_k W/(x-1)W$. In view of the additivity of $P_G(\cdot\,,-1)$ it remains to show that

$$P_G(V;-1) = -\frac{1}{2}\dim_k V/(x-1)V \quad \text{if V is free over } k[H]$$

and

$$P_G(V;-1) = 0 \quad \text{for V finite-dimensional over k.}$$

First assume that V is finite-dimensional over k. Then, using the specific form of $H^2(G,V)$, we have

$$h_2 = \dim_k \mathrm{Fix}_V(y) - \dim_k (y-1)V + \dim_k \mathrm{Fix}_V(yx) - \dim_k (yx-1)V.$$

Similarly, using the isomorphism $H^1(G,V) \cong \mathrm{Fix}_V(y) \oplus \mathrm{Fix}_V(yx)/U$ where $U = \{((y-1)v, (yx-1)v) \mid v \in V\} \cong V/H^0(G,V)$, we get

$$h_1 = h_0 + \dim_k \mathrm{Fix}_V(y) + \dim_k \mathrm{Fix}_V(yx) - \dim_k V.$$

Thus

$$P_G(V;-1) = \dim_k V - \frac{1}{2}\, d(y) - \frac{1}{2}\, d(yx),$$

where we have set $d(y) = \dim_k \mathrm{Fix}_V(y) + \dim_k (y-1)V$ and similarly for $d(yx)$. But $V|_{k<y>} \cong k<y>^r \oplus k^s$ for suitable r and s and so $d(y) = (r+s) + r = \dim_k V$. Similarly, $d(yx) = \dim_k V$, whence $P_G(V;-1) = 0$.

Finally, assume that V is free over $k[H]$. Then $H^n(H,V) = 0$ for $n \neq 1$ and $H^1(H,V) \cong V/(x-1)V = \overline{V}$. Therefore, $H^0(G,V) = 0$ and $H^n(G,V) \cong H^{n-1}(G/H,\overline{V})$ $(n \geq 1)$. Since $\overline{V}$ is finite-dimensional over k, we can write $\overline{V} \cong k<y>^r \oplus k^s$, as above, and we see that $H^1(G,V) \cong$ $\cong H^0(<y>,\overline{V}) \cong \mathrm{Fix}_{\overline{V}}(y)$ has dimension $r+s$, and $H^2(G,V) \cong H^1(<y>,\overline{V}) \cong \mathrm{Fix}_{\overline{V}}(y)/(y-1)\overline{V}$ has dimension s. Therefore, $P_G(V;-1) = -(r+s) + \frac{1}{2}s = -\frac{1}{2}\dim_k \overline{V}$, as it was to be shown.. □

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