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WREATH MACDONALD OPERATORS

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Abstract. We construct a novel family of difference-permutation operators and prove that they are diagonalized by the wreath Macdonald P-polynomials; the eigenvalues are written in terms of elementary symmetric polynomials of arbitrary degree. Our operators arise from integral formulas for the action of the horizontal Heisenberg subalgebra in the vertex representation of the corresponding quantum toroidal algebra.

1. Introduction

Let $X_N = \{x_1, \ldots, x_N\}$ be a set of variables. The *Macdonald polynomials* $\{P_\lambda[X_N; q, t]\}$ are a basis of the ring of (q,t)-deformed symmetric polynomials $\mathbb{Q}(q,t)[X_N]^{\mathfrak{S}_N}$ that have appeared in a remarkably broad collection of mathematical fields. They can be characterized as eigenfunctions of a commuting family of difference operators, the *Macdonald operators*: for $1 \leq n \leq N$,

$$D_n(X_N; q, t) := t^{\frac{n(n-1)}{2}} \sum_{\substack{I \subset \{1,\ldots,N\} \\ |I|=n}} \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \right) \prod_{i \in I} T_{q,x_i} \tag{1.1}$$

$$D_n(X_N; q, t) P_\lambda[X_N; q, t] = e_n(q^{\lambda_1} t^{N-1}, q^{\lambda_2} t^{N-2}, \ldots, q^{\lambda_N}) P_\lambda[X_N; q, t] \tag{1.2}$$

Here, T_{q,x_i} is the q-shift operator

$$T_{q,x_i} x_j = q^{\delta_{i,j}} x_j$$

and e_n is the nth elementary symmetric polynomial. The Macdonald operators are themselves distinguished as Hamiltonians of the quantum trigonometric *Ruijsenaars-Schneider* integrable system.

This paper is concerned with the *wreath* Macdonald polynomials, a generalization of the Macdonald polynomials proposed by Haiman [H1]. Fix an integer $r > 0$ and partition the variables $x_1, \ldots, x_N$ into r subsets:

$$X_{N_\bullet} := \bigsqcup_{i=0}^{r-1} \left\{ x_l^{(i)} \right\}_{l=1,\ldots N_i} = \{x_1, \ldots, x_N\}$$

where $\sum_{i=0}^{r-1} N_i = N$. We call the index i the *color* of $x_l^{(i)}$, and it will be helpful to view it as an element of $I := \mathbb{Z}/r\mathbb{Z}$. The number of variables is recorded by the vector $N_\bullet := (N_0, \ldots, N_{r-1})$ and we set $|N_\bullet| := N$. Consider the action of the product of symmetric groups

$$\mathfrak{S}_{N_\bullet} := \prod_{i \in I} \mathfrak{S}_{N_i}$$

on the polynomial ring $\mathbb{Q}(q,t)\,[X_{N_\bullet}]$ wherein $\mathfrak{S}_{N_i}$ only permutes the variables of color i. The wreath Macdonald polynomials can be viewed as a set of *color-symmetric* polynomials that are again indexed by a single partition:

$$P_\lambda[X_{N_\bullet}; q, t] \in \mathbb{Q}(q,t)\,[X_{N_\bullet}]^{\mathfrak{S}_{N_\bullet}}.$$

The combinatorics of r-cores and r-quotients play a key role in this subject, which we review in Section 2 below. When we restrict λ to range over partitions with a fixed r-core and $\ell(\lambda) \leq |N_\bullet|$, we obtain a basis of color-symmetric polynomials. For reasons that seem technical at first, the r-core and $N_\bullet$ must satisfy a compatibility condition (see 2.9). The original Macdonald polynomials are the case $r = 1$.

Haiman's proposed definition characterizes $P_\lambda[X_{N_\bullet}; q, t]$ using a pair of triangularity conditions. In contrast with the usual Macdonald theory, we *a priori* do not have an analogous characterization as the joint

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eigenfunction of an explicit family of difference operators. The present work remedies this situation: we produce a novel family of difference-permutation operators that are diagonalized by the wreath Macdonald polynomials and whose eigenvalues are written in terms of the elementary symmetric polynomials. In addition to the degree n, they also carry a color parameter $p \in I$:

$$
\begin{aligned}
D_{p,n}(X_{N_\bullet};q,t) &:= \frac{(-1)^{\frac{n(n-1)}{2}}}{\prod_{k=1}^{n}(1-q^kt^{-k})} \\
&\times \sum_{\underline{\mathbf{J}}\in Sh_p^{[n]}(X_{N_\bullet})} \overset{\curvearrowright}{\prod_{a=1}^{n}} \left\{ (1-qt^{-1})^{|\underline{J}_a|} \left(\frac{x^{(r-1)}_{\underline{J}_a^{\triangledown}}}{x^{(p)}_{\underline{J}_a}}\right) \frac{\displaystyle\prod_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|_{\geq a}}}^{N_p} \left(tx^{(p-1)}_{\underline{J}_a^{\vartriangle}} - x_l^{(p)}\right)}{\displaystyle\prod_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|_{\leq a}}}^{N_p} \left(x^{(p)}_{\underline{J}_a} - x_l^{(p)}\right)} \right. \\
&\times \left.\left(\prod_{i\in I\setminus\{p\}} \prod_{\substack{l=1\\ x_l^{(i)}\neq x^{(i)}_{\underline{J}_a^{\vartriangle}}}}^{N_i} \frac{\left(tx^{(i-1)}_{\underline{J}_a^{\vartriangle}} - x_l^{(i)}\right)}{\left(x^{(i)}_{\underline{J}_a^{\vartriangle}} - x_l^{(i)}\right)}\right)\left(\prod_{i\in J_a\setminus\{p\}} \frac{q^{-1}tT_{\underline{J}_a}x^{(i)}_{\underline{J}_a}}{\left(x^{(i)}_{\underline{J}_a} - T_{\underline{J}_a}x^{(i)}_{\underline{J}_a}\right)}\right) T_{\underline{J}_a} \right\}.
\end{aligned}
\tag{1.3}
$$

The notation used in this formula is outlined in 5.1.4. Our main result is the following:

Theorem (see Theorem 5.11)**.** *For λ having r-core compatible with $N_\bullet$ and $\ell(\lambda) \leq |N_\bullet|$, $P_\lambda[X_{N_\bullet};q,t]$ satisfies the eigenfunction equation*

$$
D_{p,n}(X_{N_\bullet};q,t)P_\lambda[X_{N_\bullet};q,t] = e_n\left[\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1 \bmod r}}^{|N_\bullet|} q^{\lambda_b}t^{|N_\bullet|-b}\right]P_\lambda[X_{N_\bullet};q,t]. \tag{1.4}
$$

For the eigenvalues, we have used plethystic notation—we merely mean the elementary symmetric function e_n evaluated at the characters appearing in the summation. In earlier work [OS1], the first two authors constructed the first order dual operators $D^*_{p,1}$ and their eigenfunction equation in Theorem 5.11.

Our operators (1.3) are much more complicated than the original Macdonald operators (1.1). In the case $r = 1$, we do indeed obtain (1.1) after some simplification (see Remark 5.7). When $r > 1$, the vanilla q-shift operator T_{q,x_i} is replaced with what we call a *cyclic-shift operator* $T_{\underline{J}_a}$, which cyclically permutes variables of different colors in a addition to multiplying by a power of q. Because of this extra permutation, the cyclic-shift operators might not commute. Note now the *ordered* product in (1.3)—we expect the formula to simplify meaningfully after taking to account the (non)commutativity of the constituent cyclic-shift operators. Moving beyond the intricacies of our formula, let us now highlight some nice conceptual aspects of our operators.

1.1. **Integral formulas.** Our strategy for deriving (1.3) and establishing the eigenfunction equation uses work of the third author [W]. Namely, we study the wreath Macdonald polynomials using the *quantum toroidal algebra* $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ and its *vertex representation* W. The aforementioned work proves that infinite-variable wreath Macdonald polynomials can be naturally embedded inside W such that they diagonalize a large commutative subalgebra of $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$, the *horizontal Heisenberg subalgebra*. This alone is insufficient for obtaining explicit formulas—we also need work of Neguț [N] realizing $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ in terms of a *shuffle algebra*. The shuffle algebra is a space of rational functions endowed with an exotic product structure, and it is isomorphic to a part of $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ via a map that is morally (but not precisely) an integration map. Writing its action on W and then specializing from infinite to finite variables, we obtain actual integral formulas. Finally, to pin down the eigenvalues, we use the (twisted) isomorphism established by Tsymbaliuk [T] between the vertex representation and the *Fock representation*.

We apply this process to the shuffle realizations of well-chosen elements of the horizontal Heisenberg subalgebra which were found in [W]. Our operators are the highest degree parts (see Lemma 5.8), and we can write their action as follows: for a factored element

$$f=\prod_{i\in I} f_i\left(x_\bullet^{(i)}\right)\in\mathbb{C}(q,t)\left[X_{N_\bullet}\right]^{\mathfrak{S}_{N_\bullet}}$$

$$\begin{aligned}D_{p,n}(X_{N_\bullet};q,t)f=&\oint_C \frac{(-1)^{\frac{n(n-1)}{2}}t^{-\frac{n(n+1)}{2}}(1-qt^{-1})^{nr}}{\prod_{a=1}^n(1-q^at^{-a})}\prod_{i\in I}\prod_{a=1}^n\prod_{l=1}^{N_i}\left(\frac{tw_{i,a}-x_l^{(i)}}{w_{i+1,a}-x_l^{(i)}}\right)\\&\times\prod_{1\le a<b\le n}\left\{\frac{(w_{p,a}-w_{p,b})(w_{p,a}-qt^{-1}w_{p,b})}{(w_{p,b}-t^{-1}w_{p+1,a})(w_{p-1,a}-t^{-1}w_{p,b})}\right.\\&\times\left.\prod_{i\in I\backslash\{p\}}\frac{(w_{i,a}-w_{i,b})(w_{i,a}-qt^{-1}w_{i,b})}{(w_{i+1,a}-qw_{i,b})(w_{i-1,a}-t^{-1}w_{i,b})}\right\}\\&\times\prod_{a=1}^n\left\{\left(\frac{w_{0,a}}{w_{p+1,a}}\right)\left(\frac{w_{p+1,a}}{w_{p,a}-t^{-1}w_{p+1,a}}\right)\right.\\&\times\left.\prod_{i\in I\backslash\{p\}}\left(\frac{w_{i,a}}{w_{i,a}-t^{-1}w_{i+1,a}}\right)\left(\frac{w_{i+1,a}}{w_{i+1,a}-qw_{i,a}}\right)\right\}\\&\times\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}+\sum_{a=1}^n qw_{i,a}-\sum_{a=1}^n w_{i+1,a}\right]\prod_{a=1}^n\frac{dw_{i,a}}{2\pi\sqrt{-1}w_{i,a}}\end{aligned}$$

where for each variable $w_{i,a}$, the cycle C only encloses poles of the form $(w_{i,a}-qw_{i-1,b})$ and $(w_{i,a}-x_{i-1,l})$. Explicit evaluation of this integral leads to (1.4). We also carry this out for its dual counterpart in Theorem 5.11.

Using other shuffle elements from [W], we obtain similar integral formulas for wreath analogues of the *Noumi-Sano operators* [NSa], although we are only able to evaluate the integral and obtain formulas for the operators in degree $n=1$. We note that our approach is similar to [FHHSY] in the $r=1$ case, although our *a priori* knowledge and endgoals are different. In [FHHSY], the authors use the well-known Macdonald operators to study the action of certain shuffle elements, whereas we use $r>1$ analogues of their shuffle elements to discover new operators. In [T2], Tsymbaliuk has also produced difference operators out of $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ through very different means. The relation between Tsymbaliuk's operators to wreath Macdonald theory does not seem straightforward but could be interesting.

1.2. **Towards bispectral duality.** In the case $r=1$, the eigenfunction equation (1.2) is particularly interesting when juxtaposed with the *Pieri rules* [Mac]. To make this apparent, introduce a continuous extension of the discrete parameters $\lambda=(\lambda_1,\ldots,\lambda_N)$:

$$s_i:=q^{\lambda_i}t^{N-i},\ S_N:=\{s_1,\ldots,s_N\}.$$

We call the variables X_N the *position variables* and S_N the *spectral variables.* Note that applying the q-shift $T_{q,s_i}P_\lambda[X_N;q,t]$ amounts to adding a box to row i of the partition λ. For a certain renormalization $\tilde{P}_\lambda[X_N;q,t]$ of $P_\lambda[X_N;,q,t]$, we can write the Pieri rules as

$$e_n(x_1,\ldots,x_N)\tilde{P}_\lambda[X_N;q,t]=t^{\frac{n(n-1)}{2}}\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=n}}\left(\prod_{\substack{i\in I\\j\notin I}}\frac{ts_i-s_j}{s_i-s_j}\right)\prod_{i\in I}T_{q,s_i}\tilde{P}_\lambda[X_N;q,t]. \tag{1.5}$$

The fact that no shift operator T_{q,s_i} appears more than once enforces the well known support condition of the Pieri rules: the $\tilde{P}_\mu[X_N;q,t]$ that appear on the right hand side of (1.5) are such that $\mu\backslash\lambda$ contains no horizontally adjacent boxes. On the other hand, we can view the eigenfunction equation (1.2) as describing multiplication by $e_n(s_1,\ldots,s_N)$. The similarity between (1.2) and (1.5) is reflective of a symmetry $X_N\leftrightarrow S_N$.

This symmetry is the subject of many beautiful works in Macdonald theory. A totalizing perspective on this was given by Noumi and Shiraishi [NS], who produced an explicit function $f_N(s_1,\ldots,s_N|x_1,\ldots,x_N)$

satisfying

$$f_N(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2}, \dots, q^{\lambda_N}|x_1, \dots, x_N) = \tilde{P}_\lambda[X_N; q, t]$$
$$f_N(s_1, \dots, s_N|x_1, \dots, x_N) = f_N(x_1, \dots, x_N|s_1, \dots, s_N).$$

Discretizing the x-variables as well, we obtain the well-known *evaluation duality* [Mac]:

$$\tilde{P}_\lambda(q^{\mu_1}t^{N-1}, q^{\mu_2}t^{N-2}, \dots, q^{\mu_N}) = \tilde{P}_\mu(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2}, \dots, q^{\lambda_N}).$$

The evaluation duality is also a consequence of the *Cherednik-Macdonald-Mehta formula* [C], which can be regarded as a remarkable statement about the quantum toroidal algebra $U_{q,t}(\ddot{\mathfrak{gl}}_1)$ and its *Miki automorphism*. The $X_N \leftrightarrow S_N$ symmetry has also been extended by Etingof and Varchenko [EV] to the much broader context of traces of intertwiners for quantum groups, although we note that in their setting, finding explicit formulas is difficult. Finally, the symmetry is also a case of *3d mirror symmetry* as proposed by Okounkov [AO].

For the wreath case $r > 1$, the spectral variables should also have color. We assign $s_l^{(i)}$ to some b such that $b - \lambda_b \equiv i + 1 \bmod r$:

$$s_l^{(i)} := q^{\lambda_b}t^{|N_\bullet|-b}.$$

Here, we point out a natural motivation for imposing our compatibility condition between $\mathrm{core}_r(\lambda)$ and $N_\bullet$—it forces there to also be N_i spectral variables of color i. The eigenfunction equation (1.4) then describes multiplication by $e_n(s_1^{(p)}, \dots, s_{N_p}^{(p)})$. Note that adding a box to a row will not only contribute a q-shift but also change the color, and that is precisely what the cyclic-shift operators $T_{\underline{J}_a}$ do. Work of the third author [W] provides one constraint on the support of the wreath Pieri rules. Namely, for a box (a, b), if we call the class of $b - a \bmod r$ its color, then $P_\mu[X_{N_\bullet}; q, t]$ appears as a summand of

$$e_n(x_{p,1}, \dots, x_{p,N_p})P_\lambda[X_{N_\bullet}; q, t]$$

only if $\mu\backslash\lambda$ consists of n boxes of each color such that no boxes of color p and $p + 1$ are horizontally adjacent. One can check that the combinations of $T_{\underline{J}_a}$ appearing in (1.3) enforce this condition after swapping $x_l^{(i)} \leftrightarrow s_l^{(i)}$. Computer calculations done by the second author also confirm a wreath analogue of evaluation duality. While we are still a long way from establishing a wreath analogue of the $X_N \leftrightarrow S_N$ symmetry, our strange operators seem to go out of their way to say it must be true. Generalizing any of the aforementioned perspectives for understanding this symmetry must surely lead to interesting mathematics.

1.3. **Outline.** Section 2 introduces the wreath Macdonald polynomials. It includes a review of the combinatorics of r-cores and r-quotients. Section 3 focuses on the quantum toroidal algebra and its representations. We derive eigenvalues for the infinite-variable analogues of our operators. Section 4 moves onto the shuffle algebra. We write the action of a shuffle element on the vertex representation as the constant term of a series. Section 5 is the technical heart of the paper. We derive integral formulas for our operators and compute the integral. Some additional efforts are needed to go from the infinite-variable eigenvalues to their finite-variable versions. Finally, in the Appendix, we derive integral formulas for wreath analogues of Noumi-Sano operators. Unfortunately, for these operators, we were only able to evaluate the integrals for degree $n = 1$. Throughout, we present examples following the derivation of each of our operators.

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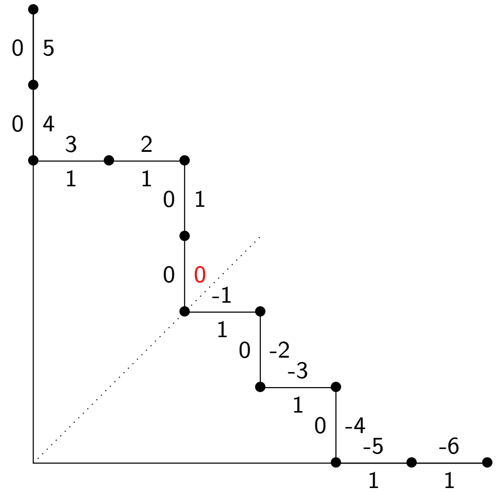
2. Wreath Macdonald functions

Fix a positive integer r and let $I = \mathbb{Z}/r\mathbb{Z}$.

2.1. **Partitions.** Let $\mathbb{Y}$ be the set of all integer partitions. We define the diagram of a partition $\mu = (\mu_1, \mu_2, \dots) \in \mathbb{Y}$ to be $D(\mu) = \{(a, b) \in (\mathbb{Z}_{\geq 0})^2 : 0 \leq a < \mu_{b+1}\}$. The *residue* of $(a, b) \in \mathbb{Z}^2$ is the element $b - a \in \mathbb{Z}/r\mathbb{Z}$.



i	$\cdots$	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	$\cdots$
b_i	0	0	0	1	1	0	0	1	0	1	0	1	1	1

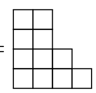
$\mathrm{charge}(b) = 0$ $\qquad$ $\mathrm{shape}(b) =$ 

FIGURE 1. The shape of an edge sequence

2.2. **Edge sequences and partitions.** A function $b : \mathbb{Z} \to \{0,1\}$ can be viewed as an infinite indexed binary word $\cdots b(1)b(0)b(-1)\cdots$; notice that in writing such a word we index the positions in reverse order. An *inversion* of b is a pair of integers $i > j$ such that $b(i) > b(j)$, a 1 to the left of a 0. An *edge sequence* is a function $b : \mathbb{Z} \to \{0,1\}$ such that $b(i) = 0$ for $i \gg 0$ and $b(i) = 1$ for $i \ll 0$, that is, b has finitely many inversions. Let ES denote the set of edge sequences. The *shape* of b is the partition whose French partition diagram has boundary traced out by the values of b from northwest to southeast where 0 (resp. 1) indicates a vertical (resp. horizontal) unit segment; see Figure 2.1. Its parts are given by the number of 1's to the left of each 0 in the edge sequence. The *charge* of b is the index of the segment that touches the main diagonal from the northwest, or equivalently the index of the last 0 in the edge sequence of the form $\cdots 0011 \cdots$ obtained from b by repeatedly swapping adjacent pairs 10 to 01 until none remain. There is a bijection

$$\begin{aligned} \mathrm{ES} &\to \mathbb{Z} \times \mathbb{Y} \\ b &\mapsto (\mathrm{charge}(b), \mathrm{shape}(b)). \end{aligned} \tag{2.1}$$

Example 2.1. An edge sequence b and its charge and shape are pictured in Figure 2.1.

2.3. **Cores and quotients.** Our goal is to define the bijection

$$\begin{aligned} \mathbb{Y} &\cong \mathcal{C}_r \times \mathbb{Y}^r \\ \lambda &\mapsto (\mathrm{core}_r(\lambda), \mathrm{quot}_r(\lambda)) \end{aligned} \tag{2.2}$$

where core_r is the r-*core* and quot_r is the r-*quotient* map.

In the following diagram all horizontal maps are bijections and vertical maps are inclusions.

$$\begin{tikzcd}
\mathbb{Z}\times\mathbb{Y} & \mathrm{ES} \arrow[l, "\mathrm{charge}\times\mathrm{shape}"'] \arrow[r] & \mathrm{ES}^r \arrow[r, "c^\bullet\times\mathrm{quot}_r"] & \mathbb{Z}^r\times\mathbb{Y}^r & \\
\{0\}\times\mathbb{Y} \arrow[u] & \mathrm{ES}_0 \arrow[l] \arrow[r] \arrow[u] & (\mathrm{ES}^r)_0 \arrow[r] \arrow[u] & Q\times\mathbb{Y}^r \arrow[u] \arrow[r, "\kappa^{-1}\times\mathrm{id}"] & \mathcal{C}_r\times\mathbb{Y}^r \\
\{0\}\times\mathcal{C}_r \arrow[u] \arrow[rrr, "\kappa"] & & & Q\times\varnothing^r \arrow[u] &
\end{tikzcd}$$

Elements $b^\bullet = (b^0, b^1, \dots, b^{r-1}) \in \mathrm{ES}^r$ are called *abaci*. We may write them as $\{0, 1, \dots, r-1\} \times \mathbb{Z}$ matrices with entries in $\{0, 1\}$ where a 0 is a bead and a 1 is a hole (position with no bead) and the i-th row represents the edge sequence b^i and is the i-th runner in the abacus.

There is a bijection $\mathrm{ES} \to \mathrm{ES}^r$ sending b to $(b^0, b^1, \dots, b^{r-1})$ by letting b^i select the bits in b indexed by integers congruent to i mod r: $b^i(j) = b(rj + i)$ for $0 \le i < r$ and $j \in \mathbb{Z}$. The inverse map is given by interleaving the sequences $b^0, b^1, \dots, b^{r-1}$. This bijection is charge-additive: $\mathrm{charge}(b) = \sum_{j=0}^{r-1} \mathrm{charge}(b^j)$. The r-fold product of the bijection (2.1) yields the bijection $\mathrm{ES}^r \cong \mathbb{Z}^r \times \mathbb{Y}^r$. Denote this by $b^\bullet = (b^0, \dots, b^{r-1}) \mapsto ((c_0, \dots, c_{r-1}), \lambda^\bullet)$. We write $\lambda^\bullet = \mathrm{quot}_r(b^\bullet)$; this is the r-quotient. Call $(c_0, \dots, c_{r-1}) = c^\bullet(b^\bullet)$ the charge vector. This indicates the position on each runner where the beads end after pushing all beads to the left. This defines the bijections going across the top row of the diagram.

We now restrict all these bijections. Let $\mathrm{ES}_0 = \{b \in \mathrm{ES} \mid \mathrm{charge}(b) = 0\}$ and $(\mathrm{ES}^r)_0 = \{b^\bullet \in \mathrm{ES}^r \mid \sum_{i=0}^{r-1} c_i(b^\bullet) = 0\}$. Then $c^\bullet(b^\bullet)$ can be viewed as an element of the $\mathfrak{sl}_r$ root lattice Q (and belongs to the zero lattice $Q = 0$ when $r = 1$). The second row of the diagram (save the last map) is given by suitable restrictions of the top row of bijections.

An r-core is a partition γ which does not have r as a hook length. That is, $h_\gamma(i, j) \neq r$ for all $(i, j) \in \gamma$. We denote by $\mathcal{C}_r \subset \mathbb{Y}$ the set of r-cores. Let γ be a partition and let $b \in \mathrm{ES}$ be such that $\mathrm{shape}(b) = \gamma$. Then γ has a box $(i, j) \in \gamma$ with hook-length r, that is, $h_\gamma(i, j) = r$, if and only if there is an index k such that $b(k) = 1$ and $b(k + r) = 0$. This is equivalent to $\mu^{(k)} \neq \varnothing$ where $\mu^\bullet = \mathrm{quot}_r(\gamma)$ and we take superscripts mod r. This proves that γ is an r-core if and only if the r-quotient of γ is empty: $\mathrm{quot}_r(\gamma) = (\varnothing^r)$.

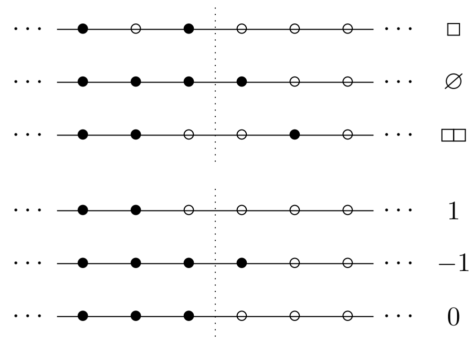
Therefore the bijection $\{0\} \times \mathbb{Y} \cong Q \times \mathbb{Y}^r$ restricts to the bijection $\{0\} \times \mathcal{C}_r \cong Q \times (\varnothing^r)$, that is, $\mathcal{C}_r \cong Q$. We call this bijection κ.

Example 2.2. Let $b \in \mathrm{ES}_0$ be as in the previous example. We have $\lambda = \mathrm{shape}(b) = (4, 3, 2, 2)$. Set $r = 3$. We map $b \mapsto (b^0, b^1, b^2)$ which are pictured in the matrix below. Reading up the columns of the $\{0, 1, 2\} \times \mathbb{Z}$ matrix we recover b. Each runner of the abacus is an edge sequence; the corresponding shapes give the 3-quotient of $(4, 3, 2, 2)$, which is $(1, \varnothing, 2)$.

To get the 3-core of λ we move all beads to the left in each runner. This produces the second abacus. Reading up columns we obtain the edge sequence $\cdots 0001|1011 \cdots$. Therefore $\mathrm{core}_3(4, 3, 2, 2) = (2)$. The charge sequence is $(1, -1, 0) \in Q$.

i	$\cdots$	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	$\cdots$
b_i	$\cdots$	0	0	1	1	0	0	1	0	1	0	1	1	$\cdots$

	2	1	0	-1	-2	-3
b^0	0	1	0	1	1	1
b^1	0	0	0	0	1	1
b^2	0	0	1	1	0	1



core

i	$\cdots$	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	$\cdots$
a_i	0	0	0	0	0	0	1	1	0	1	1	1	1	1

Remark 2.3. Our map quot_r and our definition of charge are the same as in [W], except that we interchange the roles of black and white dots in our Maya diagrams.

When considering a fixed r, we simply write $\mathrm{core} = \mathrm{core}_r$ and $\mathrm{quot} = \mathrm{quot}_r$.

2.4. **Cores and ribbons.** Consider $\mu, \lambda \in \mathbb{Y}$ such that $\mu \subset \lambda$. The skew shape $\lambda/\mu := D(\lambda) - D(\mu)$ is a μ-addable and λ-removable r-ribbon if $|\lambda| - |\mu| = r$ and the set of boxes λ/μ is rookwise connected with at most one element on each southwest-northeast diagonal. Then an r-core is precisely a partition that has no removable r-ribbon. One way to obtain $\mathrm{core}(\mu)$ is to repeatedly remove (removable) r-ribbons starting with μ until an r-core is reached; by definition this is $\mathrm{core}(\mu)$. This is well-defined: one obtains the same r-core independently of the order of removal of r-ribbons. It is the same as moving the beads in the abacus to the left.

2.5. **Cores to root lattice.** Recall that Q denotes the $\mathfrak{sl}_r$ root lattice (or $Q = 0$ in the case $r = 1$), realized as the zero sum elements in the lattice $\mathbb{Z}^I$:

$$Q := \left\{ (c_0, \dots, c_{r-1}) \in \mathbb{Z}^I \;\middle|\; \sum_{i \in I} c_i = 0 \right\}.$$

Let $\epsilon_i \in \mathbb{Z}^I$ be the i-th coordinate vector. Then Q is the spanned by the elements

$$\alpha_i := \epsilon_{i-1} - \epsilon_i, \qquad i \in I.$$

We realize the simple roots of $\mathfrak{sl}_r$ as the α_i for $i \neq 0$.

Another way to compute the bijection $\kappa : \mathcal{C} \to Q$ is as follows. Define the map $\kappa : \mathbb{Y} \to Q$ by

$$\kappa(\mu) = - \sum_{(p,q) \in \mu} \alpha_{q-p}.$$

It is not difficult to show that the restriction of κ to $\mathcal{C}$ is the same as the bijection $\mathcal{C} \cong Q$ constructed above.

Example 2.4. Let $r = 3$ and consider the 3-core (2). We put α_{q-p} into the box (p, q):

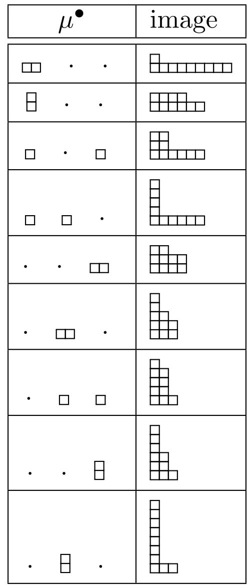
$$\begin{array}{|c|c|} \hline \alpha_0 & \alpha_2 \\ \hline \end{array}$$

Thus $\kappa((2)) = -(\alpha_0 + \alpha_2) = \alpha_1$, which agrees with the charge sequence $(1, -1, 0) \in Q$ computed above.

Define the bijection $\mathrm{big} : Q \times \mathbb{Y}^I \to \mathbb{Y}$ via the following commutative diagram:

$$\begin{tikzcd} \mathbb{Y} \arrow[r, "(\mathrm{core},\mathrm{quot})"] & \mathcal{C} \times \mathbb{Y}^I \arrow[d, "\kappa \times \mathrm{id}"] \\ & Q \times \mathbb{Y}^I \arrow[ul, "\mathrm{big}"] \end{tikzcd} \tag{2.3}$$

Example 2.5. We list the elements $\mu^\bullet \in \mathbb{Y}^I$ of total size 2 and their images under $\mu^\bullet \mapsto$ big$(-\alpha_1, \mu^\bullet)$.



2.6. **Symmetric functions.** Let Λ be the algebra of symmetric functions over $\mathbb{K} = \mathbb{Q}(q,t)$ [Mac, §I.2]. Denote by $\Lambda^I = \Lambda^{\otimes I}$ the I-fold tensor power of Λ over $\mathbb{K}$, which is a graded $\mathbb{K}$-algebra with grading given by the sum of degrees in each tensor factor. For $f \in \Lambda$, we write $f[X^{(i)}]$ to indicate the element of Λ^I with 1 in tensor factors $j \neq i$ and f in factor i. The power sums $p_k[X^{(i)}]$ for $i \in I$ and $k > 0$ generate Λ^I as a $\mathbb{K}$-algebra. We write $X^\bullet$ for the I-tuple of alphabets $(X^{(0)}, \dots, X^{(r-1)})$ and often denote by $f[X^\bullet]$ a generic element of Λ^I.

For an I-tuple of partitions $\lambda^\bullet = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(r-1)}) \in \mathbb{Y}^I$, define the tensor Schur function $s_{\lambda^\bullet} = \bigotimes_{i\in I} s_{\lambda^{(i)}} = \prod_{i\in I} s_{\lambda^{(i)}}[X^{(i)}]$. Let $\langle -, - \rangle$ be the Hall pairing on Λ^I, which is given by $\langle s_{\lambda^\bullet}, s_{\mu^\bullet} \rangle = \delta_{\lambda^\bullet,\mu^\bullet}$. For $f \in \Lambda^I$, we denote by $f^\perp$ be the adjoint under the Hall pairing to the operator of multiplication by f. Explicitly,

$$p_n^\perp[X^{(i)}] p_m[X^{(j)}] = n\delta_{n,m}\delta_{i,j}$$

For any $a \in \mathbb{K}$, define the $\mathbb{K}$-algebra automorphism $\mathcal{P}_{\mathrm{id}-a\chi^{-1}}$ of Λ^I by

$$\mathcal{P}_{\mathrm{id}-a\chi^{-1}}(p_k[X^{(i)}]) = p_k[X^{(i)}] - a^k p_k[X^{(i-1)}] \tag{2.4}$$

for all $i \in I$ and $k > 0$. (The notation $\mathcal{P}_{\mathrm{id}-a\chi^{-1}}$ arises from more general matrix plethysms $\mathcal{P}_A$ for $A \in \mathrm{Mat}_{I\times I}(\mathbb{K})$ defined in [OS2].)

2.7. **Wreath Macdonald functions.** Let $H_\lambda[X^\bullet; q, t]$ be the wreath Macdonald functions [H1], as defined in [W].[1] These are characterized by the conditions

$$\mathcal{P}_{\mathrm{id}-q\chi^{-1}} H_\lambda[X^\bullet; q, t] \in \mathbb{K}^\times s_{\mathrm{quot}(\lambda)} + \bigoplus_{\substack{\nu > \lambda \\ \kappa(\nu)=\kappa(\lambda)}} \mathbb{K} s_{\mathrm{quot}(\nu)} \tag{2.5}$$

$$\mathcal{P}_{\mathrm{id}-t^{-1}\chi^{-1}} H_\lambda[X^\bullet; q, t] \in \mathbb{K}^\times s_{\mathrm{quot}(\lambda)} + \bigoplus_{\substack{\nu < \lambda \\ \kappa(\nu)=\kappa(\lambda)}} \mathbb{K} s_{\mathrm{quot}(\nu)} \tag{2.6}$$

$$\langle s_{(n)}[X^{(0)}], H_\lambda[X^\bullet; q, t] \rangle = 1. \tag{2.7}$$

where $n = |\mathrm{quot}(\lambda)|$ and $<$ is the (strict) dominance order on partitions [Mac, §I.1].

For any $\lambda \in \mathbb{Y}$, the wreath Macdonald P-function $P_\lambda[X^\bullet; q, t^{-1}]$ is defined to be the scalar multiple of $\mathcal{P}_{\mathrm{id}-t^{-1}\chi^{-1}}(H_\lambda[X^\bullet; q, t])$ in which the coefficient of $s_{\mathrm{quot}(\lambda)}$ is 1. In particular, $P_\lambda[X^\bullet; q, t^{-1}]$ satisfies the

[1] In the more general framework of [OS2] (due to Haiman), these are the wreath Macdonald functions attached to translation elements in the affine Weyl group of type A_{r-1}.

unitriangularity

$$P_\lambda[X^\bullet; q, t^{-1}] \in s_{\mathrm{quot}(\lambda)} + \bigoplus_{\substack{\nu<\lambda \\ \kappa(\nu)=\kappa(\lambda)}} \mathbb{K}s_{\mathrm{quot}(\nu)}$$

For any fixed $\alpha \in Q$, the $P_\lambda[X^\bullet; q, t^{-1}]$ such that $\kappa(\lambda) = \alpha$ form a homogeneous basis of Λ^I, with $P_\lambda[X^\bullet; q, t^{-1}]$ having degree $|\mathrm{quot}(\lambda)|$.

Our notation $P_\lambda[X^\bullet; q, t^{-1}]$ agrees with the usual conventions in the classical $r = 1$ case. For technical reasons, it is often convenient to work with $P_\lambda[X^\bullet; q, t^{-1}]$ rather than $P_\lambda[X^\bullet; q, t]$, though we will eventually switch to the latter.

2.8. **Symmetric polynomials.** For any $N_\bullet = (N_0, \dots, N_{r-1}) \in (\mathbb{Z}_{\geq 0})^I$, we can consider a finite set of variables

$$X_{N_\bullet} := \{x_l^{(i)}\}_{1\leq l\leq N_i}^{i\in I}$$

and the corresponding restriction map

$$\pi_{N_\bullet} : \Lambda^I \to \Lambda^I_{N_\bullet} := \bigotimes_{i\in I} \mathbb{K}\left[x_1^{(i)}, \dots, x_{N_i}^{(i)}\right]^{\mathfrak{S}_{N_i}} \tag{2.8}$$

given by the tensor product $\pi_{N_\bullet} = \otimes_{i\in I}\pi_{N_i}$, where $\pi_N : \Lambda \to \mathbb{K}[x_1, \dots, x_N]^{\mathfrak{S}_N}$ is the standard projection to symmetric polynomials. We also write $\pi_{N_\bullet}(f) = f[X_{N_\bullet}]$.

2.9. **Finitization.** Our main result will characterize the images $P_\lambda[X_{N_\bullet}; q, t] := \pi_{N_\bullet}(P_\lambda[X^\bullet; q, t])$ as eigenfunctions of explicit q-difference operators. For reasons which are clarified below, we will only consider variable number vectors $N_\bullet$ for P_λ which are compatible with $\mathrm{core}(\lambda)$ in the following way. If $\kappa(\lambda) = \alpha = (c_0, c_1, \dots, c_{r-1})$, then we stipulate that P_λ will only be assigned variables $X_{N_\bullet}$ where $N_\bullet$ is equivalent to $-\kappa(\lambda)$ modulo $\mathbb{Z}(1, \dots, 1)$, i.e.,

$$N_i - N_{i-1} = (h_i, \kappa(\lambda)) = (h_i, \alpha) = c_{i-1} - c_i, \quad \text{for all } i \in I, \tag{2.9}$$

where $h_i = \alpha_i^\vee$ for all $i \in I$ and $(-,-) : Q^\vee \times Q \to \mathbb{Z}$ is the standard pairing between $\mathfrak{sl}_r$ root and coroot lattices. Identifying the lattices $Q^\vee \cong Q$ and realizing Q inside $\mathbb{Z}^I$ as above, $(-,-)$ becomes the dot product on $\mathbb{Z}^I$ and $h_i = \epsilon_{i-1} - \epsilon_i$ for all $i \in I$.

Example 2.6. In the setting of Example 2.2, the root lattice element is $\kappa(\lambda) = (1, -1, 0)$. The smallest variable number vector which we allow for $\lambda = (4, 3, 2, 2)$ is therefore $N_\bullet = (0, 2, 1)$. To this we can add the vector $(1, 1, 1)$ any number of times.

Lemma 2.7. *Under the compatibility condition* (2.9) *between $N_\bullet \in (\mathbb{Z}_{\geq 0})^I$ and $\alpha \in Q$, we have the following:*

(1) *The quantity*

$$|N_\bullet| := \sum_{i\in I} N_i$$

is divisible by r.

(2) *For $\lambda \in \mathbb{Y}$ with $\kappa(\lambda) = \alpha$ and $\ell(\lambda) \leq |N_\bullet|$,*

$$N_i = \#\left\{1 \leq b \leq |N_\bullet| : b - \lambda_b \equiv i + 1 \ \mathit{mod}\ r\right\}$$

where we count $\lambda_b = 0$ if $\ell(\lambda) < b \leq |N_\bullet|$; in particular, $\mathrm{quot}(\lambda) = \lambda^\bullet$ *satisfies $\ell(\lambda^{(i)}) \leq N_i$ for all $i \in I$.*

(3) *For any $\lambda^\bullet \in \mathbb{Y}^I$ satisfying $\ell(\lambda^{(i)}) \leq N_i$ for all i, the partition $\lambda = \mathrm{big}(\lambda^\bullet, \alpha)$ satisfies $\ell(\lambda) \leq |N_\bullet|$.*

Proof. (1) This follows from the fact that $N_\bullet$ and $\kappa(\lambda)$ are congruent modulo $\mathbb{Z}(1, \dots, 1)$, and the coordinates of the latter sum to zero.

(2) This follows from [Mac, I.1, Ex. 8] after taking our labeling conventions into account.

(3) For any edge sequence b, the length of shape(b) is precisely the number of 0's positioned to the right of at least one 1. Given $\alpha \in Q$, our choice of $N_\bullet$ ensures that the number of 0's positioned to the right of 1's in the interleaved edge sequence defining λ will not exceed $|N_\bullet|$. ∎

An immediate consequence of parts (2) and (3) of Lemma 2.7 is the following:

Proposition 2.8. *Under the compatibility condition (2.9) between $N_\bullet \in (\mathbb{Z}_{\geq 0})^I$ and $\alpha \in Q$, the wreath Macdonald polynomials $P_\lambda[X_{N_\bullet}; q, t]$ indexed by $\lambda \in \mathbb{Y}$ satisfying $\ell(\lambda) \leq |N_\bullet|$ and $\kappa(\lambda) = \alpha$ form a basis of $\Lambda^I_{N_\bullet}$.*

3. Quantum toroidal algebra

To ensure compatibility with [W] and [T], we assume that $r \geq 3$ from this point on.[2]

3.1. The algebra $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$.

Let $\mathfrak{q}$ and $\mathfrak{d}$ be two indeterminates, and set $\mathbb{F} := \mathbb{C}(\mathfrak{q}^{\frac{1}{2}}, \mathfrak{d}^{\frac{1}{2}})$.

3.1.1. *Generators and relations.* For $i, j \in I = \mathbb{Z}/r\mathbb{Z}$, we set

$$a_{i,j} = \begin{cases} 2 & j = i \\ -1 & j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$m_{i,j} = \begin{cases} \mp 1 & j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

and we define

$$g_{i,j}(z) := \frac{\mathfrak{q}^{a_{i,j}} z - 1}{z - \mathfrak{q}^{a_{i,j}}}.$$

The *quantum toroidal algebra* $U_{s,u}(\ddot{\mathfrak{sl}}_r)$ is a unital associative $\mathbb{F}$-algebra with generators

$$\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm\frac{1}{2}}, \mathfrak{q}^{\pm d_1}, \mathfrak{q}^{\pm d_2}\}_{i \in I}^{k \in \mathbb{Z}}.$$

Its relations are described in terms of *currents*:

$$e_i(z) := \sum_{k \in \mathbb{Z}} e_{i,k} z^{-k}$$

$$f_i(z) := \sum_{k \in \mathbb{Z}} f_{i,k} z^{-k}$$

$$\psi_i^\pm(z) := \psi_{i,0}^{\pm 1} + \sum_{k > 0} \psi_{i,\pm k} z^{\mp k}.$$

The relations are then

$$[\psi_i^\pm(z), \psi_j^\pm(w)] = 0, \ \gamma^{\pm\frac{1}{2}} \text{ are central},$$

$$\psi_{i,0}^{\pm 1}\psi_{i,0}^{\mp 1} = \gamma^{\pm\frac{1}{2}}\gamma^{\mp\frac{1}{2}} = \mathfrak{q}^{\pm d_1}\mathfrak{q}^{\mp d_1} = \mathfrak{q}^{\pm d_2}\mathfrak{q}^{\mp d_2} = 1,$$

$$\mathfrak{q}^{d_1} e_i(z) \mathfrak{q}^{-d_1} = e_i(\mathfrak{q}^{-1}z), \ \mathfrak{q}^{d_1} f_i(z)\mathfrak{q}^{-d_1} = f_i(\mathfrak{q}^{-1}z), \ \mathfrak{q}^{d_1}\psi_i^\pm(z)\mathfrak{q}^{-d_1} = \psi_i^\pm(\mathfrak{q}^{-1}z),$$

$$\mathfrak{q}^{d_2} e_i(z)\mathfrak{q}^{-d_2} = \mathfrak{q} e_i(z), \ \mathfrak{q}^{d_2} f_i(z)\mathfrak{q}^{-d_2} = \mathfrak{q}^{-1} f_i(z), \ \mathfrak{q}^{d_2}\psi_i^\pm(z)\mathfrak{q}^{-d_2} = \psi_i^\pm(z),$$

$$g_{i,j}(\gamma^{-1}\mathfrak{d}^{m_{i,j}} z/w)\psi_i^+(z)\psi_j^-(w) = g_{i,j}(\gamma\mathfrak{d}^{m_{i,j}} z/w)\psi_j^-(w)\psi_i^+(z),$$

$$e_i(z)e_j(w) = g_{i,j}(\mathfrak{d}^{m_{i,j}} z/w)e_j(w)e_i(z),$$

$$f_i(z)f_j(w) = g_{i,j}(\mathfrak{d}^{m_{i,j}} z/w)^{-1} f_j(w)f_i(z),$$

$$(\mathfrak{q} - \mathfrak{q}^{-1})[e_i(z), f_j(w)] = \delta_{i,j}\left(\delta(\gamma w/z)\psi_i^+(\gamma^{\frac{1}{2}}w) - \delta(\gamma z/w)\psi_i^-(\gamma^{\frac{1}{2}}z)\right),$$

$$\psi_i^\pm(z)e_j(w) = g_{i,j}(\gamma^{\pm\frac{1}{2}}\mathfrak{d}^{m_{i,j}} z/w)e_j(w)\psi_i^\pm(z),$$

$$\psi_i^\pm(z)f_j(w) = g_{i,j}(\gamma^{\mp\frac{1}{2}}\mathfrak{d}^{m_{i,j}} z/w)^{-1} f_j(w)\psi_i^\pm(z),$$

$$\mathrm{Sym}_{z_1,z_2}[e_i(z_1), [e_i(z_2), e_{i\pm 1}(w)]_{\mathfrak{q}}]_{\mathfrak{q}^{-1}} = 0, \ [e_i(z), e_j(w)] = 0 \text{ for } j \neq i, i \pm 1,$$

$$\mathrm{Sym}_{z_1,z_2}[f_i(z_1), [f_i(z_2), f_{i\pm 1}(w)]_{\mathfrak{q}}]_{\mathfrak{q}^{-1}} = 0, \ [f_i(z), f_j(w)] = 0 \text{ for } j \neq i, i \pm 1,$$

Here, $\delta(z)$ denotes the delta function

$$\delta(z) = \sum_{k \in \mathbb{Z}} z^k$$

and for $v \in \mathbb{F}$, $[a, b]_v = ab - vba$ is the v-commutator. Finally, we denote by:

[2] See Remark 5.7 and Remark 5.12 for discussion of the cases $r = 1, 2$.

- $'\ddot{U}$ the subalgebra obtained by dropping the generator $\mathfrak{q}^{d_1}$;
- $\ddot{U}'$ the subalgebra obtained by dropping the generator $\mathfrak{q}^{d_2}$;
- $'\ddot{U}'$ the subalgebra obtained by dropping both generators $\mathfrak{q}^{d_1}$ and $\mathfrak{q}^{d_2}$.

3.1.2. *Miki automorphism.* We recall that $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ contains two copies of the quantum affine algebra $U_{\mathfrak{q}}(\dot{\mathfrak{sl}}_r)$. The first, called the *vertical* copy, is generated by currents where $i \neq 0$. This copy is given in the new Drinfeld presentation. On the other hand, the second copy, called the *horizontal* copy, is generated by the constant terms of all the currents. This copy is given in the Drinfeld-Jimbo presentation. We do not go into detail on these two subalgebras as we will not need them in the sequel. However, we mention them because they give the 'two loops' of the quantum toroidal algebra. Let η denote the $\mathbb{C}(\mathfrak{q})$-linear antiautomorphism of $'\ddot{U}'$ defined by

$$\begin{gathered}\eta(\mathfrak{d}) = \mathfrak{d}^{-1}\\ \eta(e_{i,k}) = e_{i,-k},\ \eta(f_{i,k}) = f_{i,-k},\ \eta(h_{i,k}) = -\gamma^k h_{i,-k},\\ \eta(\psi_{i,0}) = \psi_{i,0}^{-1},\ \eta(\gamma^{\frac{1}{2}}) = \gamma^{\frac{1}{2}}.\end{gathered} \tag{3.1}$$

The following beautiful result of Miki gives the 'S-transformation' of the torus:

Theorem 3.1 ([M])**.** *There is an algebra automorphism ς of $'\ddot{U}'$ that sends the horizontal copy of $U_{\mathfrak{q}}(\dot{\mathfrak{sl}}_r)$ to the vertical copy. Moreover, ς satisfies $\varsigma^{-1} = \eta\varsigma\eta$.*

3.1.3. *Heisenberg subalgebras.* We will also work with elements $\{h_{i,k}\}_{i\in I}^{k\neq 0}$ defined by

$$\psi_i^{\pm}(z) = \psi_{i,0}^{\pm 1}\exp\left(\pm(\mathfrak{q}-\mathfrak{q}^{-1})\sum_{k>0} h_{i,\pm k}z^{\mp k}\right).$$

Together with $\gamma^{\pm\frac{1}{2}}$, these elements generate a rank r Heisenberg algebra. The relations are

$$\begin{gathered}[h_{i,k}, h_{j,k'}] = \delta_{k,-k'}\frac{(\gamma^k-\gamma^{-k})\mathfrak{d}^{-km_{i,j}}[ka_{i,j}]_{\mathfrak{q}}}{(\mathfrak{q}-\mathfrak{q}^{-1})k}\\ \gamma^{\frac{1}{2}} \text{ is central}\end{gathered} \tag{3.2}$$

where $[n]_v$ is the usual quantum number:

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}$$

Because the $r\times r$ matrix with (i,j) entry given by $u^{-2km_{i,j}}[ka_{i,j}]_{s^{-2}}$ is invertible, we can define *dual elements* $\{h_{i,k}^{\perp}\}_{i\in I}^{k\neq 0}$ characterized by

$$[h_{i,k}^{\perp}, h_{j,-k'}] = [h_{j,k'}, h_{i,-k}^{\perp}] = \delta_{i,j}\delta_{k,k'}(\gamma^k - \gamma^{-k}) \tag{3.3}$$

for $k > 0$. We denote by $\ddot{U}^0$ the subalgebra generated by $\{\gamma^{\pm\frac{1}{2}}\}\cup\{h_{i,k}\}_{i\in I}^{k\neq 0}$ an call it the *vertical Heisenberg subalgebra.* In analogy with 3.1.2, we call $\varsigma(\ddot{U}^0)$ the *horizontal Heisenberg subalgebra.*

Remark 3.2. In [W], the author defines elements $\{b_{i,k}^{\perp}\}$ in terms of a pairing that is not used in this paper. By comparing the commutator (3.2) to the pairing in *loc. cit.*, we have that

$$h_{i,k}^{\perp} = -b_{i,k}^{\perp}$$

3.2. **Vertex representation.** $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ directly interacts with the wreath Macdonald polynomials via its *vertex representation*, originally constructed by Yoshihisa Saito [S].

3.2.1. *Twisted group algebra.* Recall that Q and $Q^\vee$ denote the $\mathfrak{sl}_r$ root and coroot lattices, respectively, with simple roots $\{\alpha_j\}_{j=1}^{r-1}$, simple coroots $\{h_j\}_{j=1}^{r-1}$, and canonical pairing $(-,-) : Q^\vee \times Q \to \mathbb{Z}$:

$$(h_i, \alpha_j) = a_{i,j}.$$

Let P denote the $\mathfrak{sl}_r$ weight lattice and $\{\Lambda_p\}_{j=1}^{r-1}$ the fundamental weights. We will also need

$$\alpha_0 = -\sum_{j=1}^{r-1}\alpha_j, \quad h_0 = -\sum_{j=1}^{r-1} h_j, \quad \Lambda_0 := 0.$$

We have that $\{\alpha_2,\dots,\alpha_{r-1},\Lambda_{r-1}\}$ is a basis of P.

The *twisted group algebra* $\mathbb{F}\{P\}$ is the $\mathbb{F}$-algebra generated by $\{e^{\alpha_j}\}_{j=2}^{r-1}\cup\{e^{\Lambda_{r-1}}\}$ satisfying the relations

$$\begin{aligned} e^{\alpha_i}e^{\alpha_j} &= (-1)^{(h_i,\alpha_j)}e^{\alpha_j}e^{\alpha_i} \\ e^{\alpha_i}e^{\Lambda_{r-1}} &= (-1)^{\delta_{i,r-1}}e^{\Lambda_{r-1}}e^{\alpha_i}. \end{aligned}$$

Given a general $\alpha\in P$, we write $\alpha=\sum_{j=2}^{r-1}m_j\alpha_j+m_r\Lambda_{r-1}$ and then set

$$e^{\alpha}=e^{m_2\alpha_2}\cdots e^{m_{r-1}\alpha_{r-1}}e^{m_r\Lambda_{r-1}}.$$

For example,

$$\begin{aligned} e^{\alpha_1} &= e^{-2\alpha_2}e^{-3\alpha_3}\cdots e^{-(r-1)\alpha_{r-1}}e^{r\Lambda_{r-1}} \\ e^{\alpha_0} &= e^{\alpha_2}e^{2\alpha_3}\cdots e^{(r-2)\alpha_{r-1}}e^{-r\Lambda_{r-1}}. \end{aligned} \tag{3.4}$$

Define $\mathbb{F}\{Q\}$ to be the subalgebra of $\mathbb{F}\{P\}$ generated by $\{e^{\alpha_i}\}_{i=1}^{r-1}$.

3.2.2. *Vertex operators.* The vertical Heisenberg subalgebra $\ddot{U}^0$ has a Fock representation F_r defined as follows. Let $\ddot{U}^0_+$ denote the subalgebra generated by $\gamma^{\frac{1}{2}}$ and $\{h_{i,k}\}_{i\in I}^{k>0}$. $\ddot{U}^0_+$ has a one-dimensional representation $\mathbb{F}_{\mathfrak{q}}$ where $\gamma^{\frac{1}{2}}$ acts by $\mathfrak{q}^{\frac{1}{2}}$ while $h_{i,k}$ acts by 0. F_r is then the induced representation

$$F_r:=\mathrm{Ind}_{\ddot{U}^0_+}^{\ddot{U}^0}\mathbb{F}_{\mathfrak{q}}\cong\mathbb{K}[h_{i,-k}]_{i\in I}^{k>0}.$$

The vertex representation is defined on the space $W:=F_r\otimes\mathbb{F}\{Q\}$. For $v\otimes e^{\alpha}\in W$ where

$$\begin{aligned} v &= h_{i_1,-k_1}\cdots h_{i_N,-k_N}v_0 \\ \alpha &= \sum_{j=1}^{r-1}m_j\alpha_j \end{aligned}$$

we define the operators $h_{i,k}$, e^{β}, ∂_{α_i}, $z^{H_{i,0}}$, and d by

$$\begin{gathered} h_{i,k}(v\otimes e^{\alpha}):=(h_{i,k}v)\otimes e^{\alpha},\ e^{\beta}(v\otimes e^{\alpha}):=v\otimes(e^{\beta}e^{\alpha}), \\ \partial_{\alpha_i}(v\otimes e^{\alpha}):=(h_i,\alpha+\Lambda_p)\,v\otimes e^{\alpha}, \\ z^{H_{i,0}}(v\otimes e^{\alpha}):=z^{(h_i,\alpha+\Lambda_p)}\mathfrak{d}^{\frac{1}{2}\sum_{j=1}^{r-1}(h_i,m_j\alpha_j)m_{i,j}}v\otimes e^{\alpha}, \\ d(v\otimes e^{\alpha}):=-\left(\frac{(\alpha,\alpha)}{2}+(\alpha,\Lambda_p)+\sum_{i=1}^{N}k_i\right)v\otimes e^{\alpha}. \end{gathered}$$

Theorem 3.3 ([S]). *Let $\vec{c}=(c_0,\dots,c_{r-1})\in(\mathbb{F}^{\times})^r$. The following formulas endow W with an action of $\ddot{U}'$:*

$$\begin{aligned} \rho_{\vec{c}}(e_i(z)) &= c_i\exp\left(\sum_{k>0}\frac{\mathfrak{q}^{-\frac{k}{2}}}{[k]_{\mathfrak{q}}}h_{i,-k}z^k\right) \\ &\quad\times\exp\left(-\sum_{k>0}\frac{\mathfrak{q}^{-\frac{k}{2}}}{[k]_{\mathfrak{q}}}h_{i,k}z^{-k}\right)e^{\alpha_i}z^{1+H_{i,0}}, \\ \rho_{\vec{c}}(f_i(z)) &= \frac{(-1)^{r\delta_{i,0}}}{c_i}\exp\left(-\sum_{k>0}\frac{\mathfrak{q}^{\frac{k}{2}}}{[k]_{\mathfrak{q}}}h_{i,-k}z^k\right) \\ &\quad\times\exp\left(\sum_{k>0}\frac{\mathfrak{q}^{\frac{k}{2}}}{[k]_{\mathfrak{q}}}h_{i,k}z^{-k}\right)e^{-\alpha_i}z^{1-H_{i,0}}, \\ \rho_{\vec{c}}(\psi_i^{\pm}(z)) &= \exp\left(\pm(\mathfrak{q}-\mathfrak{q}^{-1})\sum_{k>0}h_{i,\pm k}z^{\mp k}\right)\mathfrak{q}^{\pm\partial_{\alpha_i}}, \\ \rho_{\vec{c}}(\gamma^{\frac{1}{2}}) &= \mathfrak{q}^{\frac{1}{2}},\ \rho_{p,\vec{c}}(\mathfrak{q}^{d_1})=\mathfrak{q}^{d}. \end{aligned}$$

3.2.3. *Embedding symmetric functions.* We can let Λ^I act on F_r via multiplication operators given by

$$p_k[X^{(i)}] \mapsto \frac{k}{[k]_{\mathfrak{q}}} h_{i,-k}. \tag{3.5}$$

To obtain an identification $W \cong \Lambda^I \otimes \mathbb{F}\{Q\}$, we need to embed $\mathbb{K}$ into $\mathbb{F}$:

$$q = \mathfrak{q}\mathfrak{d},\ t = \mathfrak{q}\mathfrak{d}^{-1}. \tag{3.6}$$

As operators on Λ^I, we have from (3.3) the identification

$$p_k[X^{(i)}]^\perp \mapsto k h_{i,k}^\perp.$$

Now consider transforming the formulas for $\rho_{\vec{c}}$ using matrix plethysms on $\{p_k[X^{(i)}]\}$. We can obtain an isomorphic representation as long as we perform a corresponding transformation on $\{h_{i,k}\}$ to maintain the commutation relations, using (3.2) as a guide. First, we define $\rho_{\vec{c}}^+$ by performing the plethysm

$$p_k[X^{(i)}] \mapsto \mathfrak{q}^{\frac{k}{2}} \left(p_k[X^{(i)}] - t^{-k} p_k[X^{(i-1)}] \right).$$

For $\rho_{\vec{c}}^+$, we will only be interested in the currents $\{e_i(z)\}$, although we have a representation for the entire algebra:

$$\begin{aligned} E_i(z) := \rho_{\vec{c}}^+(e_i(z)) = c_i \exp&\left[\sum_{k>0} \left(p_k[X^{(i)}] - t^{-k} p_k[X^{(i-1)}] \right) \frac{z^k}{k} \right] \\ &\times \exp\left[\sum_{k>0} \left(-p_k[X^{(i)}]^\perp + q^{-k} p_k[X^{(i-1)}]^\perp \right) \frac{z^{-k}}{k} \right] e^{\alpha_i} z^{1+H_{i,0}}. \end{aligned} \tag{3.7}$$

Similarly, we define $\rho_{\vec{c}}^-$ by performing the plethysm

$$p_k[X^{(i)}] \mapsto \mathfrak{q}^{-\frac{k}{2}} \left(t^k p_k[X^{(i)}] - p_k[X^{(i-1)}] \right).$$

Here, we will only be interested in the action of the currents $\{f_i(z)\}$:

$$\begin{aligned} F_i(z) := \rho_{\vec{c}}^-(f_i(z)) = \frac{(-1)^{r\delta_{i,0}}}{c_i} \exp&\left[\sum_{k>0} \left(-t^k p_k[X^{(i)}] + p_k[X^{(i-1)}] \right) \frac{z^k}{k} \right] \\ &\times \exp\left[\sum_{k>0} \left(q^k p_k[X^{(i)}]^\perp - p_k[X^{(i-1)}]^\perp \right) \frac{z^{-k}}{k} \right] e^{-\alpha_i} z^{1-H_{i,0}}. \end{aligned} \tag{3.8}$$

The following is a consequence of the main result of [W]:

Theorem 3.4. *Under both representations $\rho_{\vec{c}}^\pm$, $\varsigma(\ddot{U}^0)$ acts diagonally on $\{P_\lambda[X^\bullet; q, t^{-1}] \otimes e^{\kappa(\lambda)}\}$.*

Remark 3.5. The paper [W] is concerned with the *transformed* wreath Macdonald functions $\{H_\lambda[X^\bullet; q, t]\}$. The plethysms used to define $\rho_{\vec{c}}^\pm$ are both scalar multiples of the plethysm $\mathcal{P}_{\mathrm{id}-t^{-1}\chi^{-1}}$ which sends $H_\lambda[X^\bullet; q, t]$ to a scalar multiple of $P_\lambda[X^\bullet; q, t^{-1}]$.

3.2.4. *Normal ordering.* Later, we will make use of a particular expression for products of the currents $\{E_i(z)\}$ and $\{F_i(z)\}$. We will need notation for an ordered product or composition of noncommuting operators $a_1, \dots, a_m$:

$$\begin{aligned} \prod_{j=1}^{\overset{\curvearrowright}{m}} a_j &:= a_1 a_2 \cdots a_m \\ \prod_{j=1}^{\overset{\curvearrowleft}{m}} a_j &:= a_m a_{m-1} \cdots a_1 \end{aligned} \tag{3.9}$$

Proposition 3.6. *For $p \in I$, we have*

$$
\begin{aligned}
\overset{\curvearrowright}{\prod_{a=1}^{n}}\overset{\curvearrowright}{\prod_{i=1}^{r}} E_{p+i}(z_{p+i,a}) &= \left((-1)^{\frac{(r-2)(r-3)}{2}}\mathfrak{d}^{\frac{r}{2}-1}\prod_{i\in I}c_i\right)^n \\
&\times \prod_{1\le a<b\le n}\prod_{i\in I}\frac{(1-z_{i,b}/z_{i,a})\left(1-q^{-1}t^{-1}z_{i,b}/z_{i,a}\right)}{\left(1-t^{-1}z_{i+1,b}/z_{i,a}\right)\left(1-q^{-1}z_{i-1,b}/z_{i,a}\right)} \\
&\times \prod_{a=1}^{n}\frac{z_{p,a}/z_{p+1,a}}{(1-q^{-1}z_{p,a}/z_{p+1,a})\prod_{i\in I\setminus\{p+1\}}(1-t^{-1}z_{i,a}/z_{i-1,a})} \\
&\times \prod_{i\in I}\exp\left(\sum_{a=1}^{n}\sum_{k>0}\left(p_k[X^{(i)}]-t^{-k}p_k[X^{(i-1)}]\right)\frac{z_{i,a}^{k}}{k}\right) \\
&\times \prod_{i\in I}\exp\left(\sum_{a=1}^{n}\sum_{k>0}\left(-p_k[X^{(i)}]^{\perp}+q^{-k}p_k[X^{(i-1)}]^{\perp}\right)\frac{z_{i,a}^{-k}}{k}\right)\prod_{i\in I}\prod_{a=1}^{n}z_{i,a}^{H_{i,0}}
\end{aligned}
\tag{3.10}
$$

where all rational functions are Laurent series expanded assuming

$$|z_{i,a}|=1,\ |q|>1,\ |t|>1. \tag{3.11}$$

For the F-currents, we have

$$
\begin{aligned}
\overset{\curvearrowleft}{\prod_{a=1}^{n}}\overset{\curvearrowleft}{\prod_{i=1}^{r}} F_{p+i}(z_{p+i,a}) &= \left(\frac{(-1)^{\frac{(r-2)(r-3)}{2}}}{\mathfrak{d}^{\frac{r}{2}-1}\prod_{i\in I}c_i}\right)^n \\
&\times \prod_{1\le a<b\le n}\prod_{i\in I}\frac{(1-z_{i,a}/z_{i,b})(1-qtz_{i,a}/z_{i,b})}{(1-tz_{i-1,a}/z_{i,b})(1-qz_{i+1,a}/z_{i,b})} \\
&\times \prod_{a=1}^{n}\frac{z_{p+1,a}/z_{p,a}}{(1-qz_{p+1,a}/z_{p,a})\prod_{i\in I\setminus\{p\}}(1-tz_{i,a}/z_{i+1,a})} \\
&\times \prod_{i\in I}\exp\left(\sum_{a=1}^{n}\sum_{k>0}\left(-t^{k}p_k[X^{(i)}]+p_k[X^{(i-1)}]\right)\frac{z_{i,a}^{k}}{k}\right) \\
&\times \prod_{i\in I}\exp\left(\sum_{a=1}^{n}\sum_{k>0}\left(q^{k}p_k[X^{(i)}]^{\perp}-p_k[X^{(i-1)}]^{\perp}\right)\frac{z_{i,a}^{-k}}{k}\right)\prod_{i\in I}\prod_{a=1}^{n}z_{i,a}^{-H_{i,0}}
\end{aligned}
\tag{3.12}
$$

where all rational functions are Laurent series expanded assuming

$$|z_{i,a}|=1,\ |q|<1,\ |t|<1. \tag{3.13}$$

Proof. The computation is standard. We will only go over the signs and powers of $\mathfrak{d}$. The sign comes from the commutation of $\{e^{\alpha_i}\}$; in both cases, these factors simplify to $\pm e^0$. For the E-currents, if $p=0$, then by (3.4),

$$e^{\alpha_1}=e^{r\Lambda_{r-1}}e^{-(r-1)\alpha_{r-1}}\cdots e^{-3\alpha_3}e^{-2\alpha_2}.$$

Thus,

$$e^{\alpha_1}e^{\alpha_2}\cdots e^{\alpha_{r-1}}=(-1)^{\frac{(r-2)(r-3)}{2}}e^{r\Lambda_{r-1}}e^{-(r-2)\alpha_{r-1}}\cdots e^{-2\alpha_3}e^{-\alpha_2}.$$

On the other hand, if $p\neq 0$, we have

$$
\begin{aligned}
e^{\alpha_0}e^{\alpha_1} &= (-1)^{\frac{(r-1)(r-2)}{2}-1}e^{-\alpha_2}\cdots e^{-\alpha_{r-1}} \\
&= (-1)^{\frac{(r-1)(r-2)}{2}+r-3}e^{-\alpha_{r-1}}\cdots e^{-\alpha_2} \\
&= (-1)^{\frac{(r-2)(r-3)}{2}}e^{-\alpha_{r-1}}\cdots e^{-\alpha_2}
\end{aligned}
$$

which also leads to a sign of $(-1)^{\frac{(r-2)(r-3)}{2}}$. For the F-currents, first consider the case $p=0$.

$$\begin{aligned} e^{-\alpha_0}e^{-\alpha_{r-1}}\cdots e^{-\alpha_3}e^{-\alpha_2} &= (-1)^{r+\frac{(r-2)(r-3)}{2}}e^{-2\alpha_2}e^{-3\alpha_3}\cdots e^{-(r-1)\alpha_{-r-1}}e^{r\Lambda_{r-1}} \\ &= (-1)^{r+\frac{(r-2)(r-3)}{2}}e^{r\Lambda_{r-1}}e^{-(r-1)\alpha_{r-1}}\cdots e^{-3\alpha_2}e^{-2\alpha_2}. \end{aligned}$$

If $p \neq 0$, then we use that

$$\begin{aligned} e^{-\alpha_1}e^{-\alpha_0} &= (-1)^r e^{-\alpha_1}e^{r\Lambda_{r-1}}e^{-(r-2)\alpha_{r-1}}\cdots e^{-2\alpha_3}e^{-\alpha_2} \\ &= (-1)^{r+\frac{(r-2)(r-3)}{2}}e^{\alpha_2}e^{\alpha_3}\cdots e^{\alpha_{r-1}}. \end{aligned}$$

Finally, note that $F_0(z)$ also has a sign of $(-1)^r$. The power of $\mathfrak{d}$ comes from the interaction between $\{z^{\pm H_{i,0}}\}$ and $\{e^{\pm\alpha_j}\}$. First observe that when considering $E_i(z_{i,a})$ and $E_j(z_{j,b})$ for $a \neq b$, the powers of $\mathfrak{d}$ from $j=i-1$ and $j=i+1$ cancel out. When $a=b$, there is a total power of $\mathfrak{d}^{\frac{r}{2}-1}$. The case for $\{F_i(z)\}$ is similar but inverted. □

3.3. **Fock representation.** While our main focus will be on the vertex representation, we will consider another representation of $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$, called the *Fock representation*. Our goal will be gain some knowledge on the eigenvalues implicit in the statement of Theorem 3.4.

3.3.1. *Definition.* In order to define the Fock representation, we will need some notation for partitions. For a partition λ, let $\square = (a,b) \in D(\lambda)$. We set:

(1) $\chi_\square = q^{a-1}t^{b-1}$, the character of the box;
(2) $c_\square = b-a$ modulo r (its color);
(3) $d_i(\lambda)$ the number of elements of $D(\lambda)$ with content equivalent to i modulo r;
(4) $A_i(\lambda)$ and $R_i(\lambda)$ the addable and removable i-nodes of λ, respectively.

Finally, we will abbreviate $a \equiv b$ mod r by simply $a \equiv b$ and use the Kronecker delta function $\delta_{a\equiv b} := \delta_{a-b,0}$.

Let $v \in \mathbb{F}^\times$. The Fock representation $\mathcal{F}(v)$ has a basis $\{|\lambda\rangle\}$ indexed by partitions.

Theorem 3.7 ([FJMM], cf. [W])**.** *We can define a $'\ddot{U}$-action τ_v on $\mathcal{F}(v)$ where the only nonzero matrix elements of the generators are*

$$\langle\lambda|e_i(z)|\lambda+\square\rangle = \delta_{c_\square = i}(-\mathfrak{d})^{d_{i+1}(\lambda)}\delta\left(\frac{z}{\chi_\square v}\right)\frac{\prod\limits_{\blacksquare\in R_i(\lambda)}(\chi_\square - \mathfrak{q}^2\chi_\blacksquare)}{\prod\limits_{\substack{\blacksquare\in A_i(\lambda)\\ \blacksquare\neq\square}}(\chi_\square-\chi_\blacksquare)}$$

$$\langle\lambda+\square|f_i(z)|\lambda\rangle = \delta_{c_\square = i}(-\mathfrak{d})^{-d_{i+1}(\lambda)}\delta\left(\frac{z}{\chi_\square v}\right)\frac{\prod\limits_{\substack{\blacksquare\in A_i(\lambda)\\ \blacksquare\neq\square}}(\mathfrak{q}\chi_\square - \mathfrak{q}^{-1}\chi_\blacksquare)}{\prod\limits_{\blacksquare\in R_i(\lambda)}\mathfrak{q}\left(\chi_\square-\chi_\blacksquare\right)}$$

$$\langle\lambda|\psi_i^\pm(z)|\lambda\rangle = \prod_{\blacksquare\in A_i(\lambda)}\frac{(\mathfrak{q}z-\mathfrak{q}^{-1}\chi_\blacksquare v)}{(z-\chi_\blacksquare v)}\prod_{\blacksquare\in R_i(\lambda)}\frac{(\mathfrak{q}^{-1}z-\mathfrak{q}\chi_\blacksquare v)}{(z-\chi_\blacksquare v)},$$

$$\langle\lambda|\gamma^{\frac{1}{2}}|\lambda\rangle = 1,\ \langle\lambda|\mathfrak{q}^{d_2}|\lambda\rangle = \mathfrak{q}^{-|\lambda|}.$$

3.3.2. *Tsymbaliuk isomorphism.* The representation τ_v on $\mathcal{F}(v)$ has a cyclic vector $|\varnothing\rangle$. On the other hand, $\rho_{\vec{c}}$ and $\rho_{\vec{c}}^\pm$ also have $1\otimes 1 \in F_r \otimes \mathbb{F}\{Q\}$ as a cyclic vector. Both can be considered vacuum vectors for their respective representations. The following theorem was proved by Tsymbaliuk:

Theorem 3.8 ([T])**.** *Let*

$$v = (-1)^{\frac{(\ell-2)(\ell-3)}{2}}\frac{\mathfrak{q}\mathfrak{d}^{-\frac{\ell}{2}}}{c_0\cdots c_{\ell-1}} \tag{3.14}$$

The vacuum-to-vacuum map

$$\mathcal{F}(v) \ni |\varnothing\rangle \mapsto 1\otimes 1 \in W$$

induces an isomorphism between the $'\ddot{U}'$-module τ_v and the ς-twisted modules $\rho_{\vec{c}}\circ\varsigma$, $\rho_{\vec{c}}^\pm\circ\varsigma$.

The Tsymbaliuk isomorphism is only a vacuum-to-vacuum map. In light of Remark 3.5, the following result from [W] provides more detail on the Tsymbaliuk isomorphisms:

Theorem 3.9. *The Tsymbaliuk isomorphisms (Theorem 3.8) between τ_v and $\rho^{\pm}_{\vec{c}}$ send*

$$\mathbb{F}|\lambda\rangle \to \mathbb{F}\left(P_\lambda[X^\bullet; q, t^{-1}] \otimes e^{\kappa(\lambda)}\right).$$

Thus, we can study the eigenvalues of $\varsigma(\ddot{U}^0)$ on P_λ by instead studying the eigenvalues of $\ddot{U}^0$ on the basis $\{|\lambda\rangle\}$.

3.3.3. *Infinite-variable eigenvalues.* From the formulas in Theorem 3.7, we can see that

$$\langle\lambda|\psi_0^{\pm 1}|\lambda\rangle = \mathfrak{q}^{\pm(|A_i(\lambda)|-|R_i(\lambda)|)}.$$

Therefore,

$$\begin{aligned}
&\left\langle \lambda \middle| \exp\left(\pm(\mathfrak{q}-\mathfrak{q}^{-1})\sum_{k>0} h_{i,\pm k} z^{\mp k}\right) \middle| \lambda \right\rangle \\
&= \prod_{\blacksquare\in A_i(\lambda)} \frac{\mathfrak{q}^{\mp 1}\left(\mathfrak{q}z - \mathfrak{q}^{-1}\chi_\blacksquare v\right)}{(z-\chi_\blacksquare v)} \prod_{\blacksquare\in R_i(\lambda)} \frac{\mathfrak{q}^{\pm 1}\left(\mathfrak{q}^{-1}z - \mathfrak{q}\chi_\blacksquare v\right)}{(z-\chi_\blacksquare v)} \\
&= \exp\left[\sum_{k>0}\left(\sum_{\blacksquare\in A_i(\lambda)} (1-\mathfrak{q}^{\mp 2k})\chi_\blacksquare^{\pm k} + \sum_{\blacksquare\in R_i(\lambda)} (1-\mathfrak{q}^{\pm 2k})\chi_\blacksquare^{\pm k}\right)\frac{v^{\pm k} z^{\mp k}}{k}\right].
\end{aligned}$$

Taking logarithms, we see that for $k > 0$,

$$\begin{aligned}
\langle\lambda|h_{i,\pm k}|\lambda\rangle &= \frac{v^{\pm k}[k]_\mathfrak{q}}{k}\left(\sum_{\blacksquare\in A_i(\lambda)} \mathfrak{q}^{\mp k}\chi_\blacksquare^{\pm k} - \sum_{\blacksquare\in R_i(\lambda)} \mathfrak{q}^{\pm k}\chi_\blacksquare^{\mp k}\right) \\
&= \frac{v^{\pm k}\mathfrak{q}^{\mp k}[k]_\mathfrak{q}}{k}\left(\sum_{\blacksquare\in A_i(\lambda)} \chi_\blacksquare^{\pm k} - \sum_{\blacksquare\in R_i(\lambda)} (qt\chi_\blacksquare)^{\pm k}\right).
\end{aligned} \tag{3.15}$$

Using (3.15), we can try to piece together elements of $\ddot{U}^0$ whose eigenvalues are analogues of those of Macdonald operators in infinitely many variables.

Lemma 3.10. *Assume $|t^{\pm 1}| < 1$ (where '+' and '−' are separate cases). For $p \in I$, we have*

$$\begin{aligned}
&\left\langle \lambda \middle| \exp\left[-\sum_{k>0}\left(\frac{\sum_{i=0}^{r-1} t^{\pm k(i+1)} h_{p-i,\pm k}}{(1-t^{\pm kr})}\right)\frac{\mathfrak{q}^{\pm k}(-z)^{\mp k}}{v^{\pm k}[k]_\mathfrak{q}}\right] \middle| \lambda \right\rangle \\
&= \exp\left[-\sum_{k>0}\left(\sum_{\substack{b>0\\ b-\lambda_b\equiv p+1}} q^{\pm k\lambda_b} t^{\pm kb}\right)\frac{(-z)^{\mp k}}{k}\right] \\
&= \prod_{\substack{b>0\\ b-\lambda_b\equiv p+1}} \left(1+q^{\pm\lambda_b} t^{\pm b} z^{\mp 1}\right)
\end{aligned} \tag{3.16}$$

where we set $\lambda_b = 0$ for all $b > \ell(\lambda)$.

Proof. Comparing (3.16) to (3.15), we need to establish the equality

$$\frac{1}{1-t^{\pm kr}}\sum_{i=0}^{r-1} t^{\pm k(i+1)}\left(\sum_{\blacksquare\in A_{p-i}(\lambda)} \chi_\blacksquare^{\pm k} - \sum_{\blacksquare\in R_{p-i}(\lambda)} (qt\chi_\blacksquare)^{\pm k}\right) = \left(\sum_{\substack{b>0\\ b-\lambda_b\equiv p+1}} q^{\pm k\lambda_b} t^{\pm kb}\right). \tag{3.17}$$

We note that here, we consider $(1-t^{\pm kr})^{-1}$ as a geometric series. The summands on the right hand side of (3.17) are t-shifts of the characters of the boxes immediately to the right of the rows of $D(\lambda)$ whose contents are congruent to p modulo r. We can account for these coordinates by starting at each addable box of $D(\lambda)$, going straight up, and ending the search once we are diagonally adjacent to a removable box. This is exactly what the left hand side of (3.17) does, although we note that the addable and removable boxes that cancel each other out need not have the same color. □

4. Shuffle algebra

We will obtain difference operators by computing the action of $\varsigma(\ddot{U}^0)$ on the vertex representation. However, computing the images of elements under ς is difficult. The *shuffle algebra* provides another avatar of the quantum toroidal algebra with which we can access the horizontal Heisenberg subalgebra.

4.1. **Definition and structures.** Let $k_\bullet = (k_0, \ldots, k_{r-1}) \in (\mathbb{Z}_{\geq 0})^I$ and consider the function space

$$\mathbb{S}_{k_\bullet} := \mathbb{F}(z_{i,a})_{i\in I}^{1\leq a\leq k_i}.$$

The product of symmetric groups

$$\mathfrak{S}_{k_\bullet} := \mathfrak{S}_{k_0} \times \cdots \times \mathfrak{S}_{k_{r-1}}$$

acts on $\mathbb{S}_{k_\bullet}$ where the factor $\mathfrak{S}_{k_i}$ only permutes the variables $\{z_{i,a}\}_{a=1}^{k_i}$. We call i the *color* of $z_{i,r}$, so $\mathfrak{S}_{k_\bullet}$ acts by *color-preserving permutations* Finally, let

$$\mathbf{S}_{k_\bullet} := (\mathbb{S}_{k_\bullet})^{\mathfrak{S}_{k_\bullet}}$$

$$\mathbf{S} := \bigoplus_{k_\bullet \in (\mathbb{Z}_{\geq 0})^I} \mathbf{S}_{k_\bullet}.$$

Unless we say otherwise, an element of $\mathbb{S}$ with k_i variables of color i for all i is assumed to be in $\mathbb{S}_{k_\bullet}$.

4.1.1. *Shuffle product.* We endow $\mathbf{S}$ with the *shuffle product* $\star$, defined as follows. For $i, j \in I$, we define the *mixing terms*:

$$\omega_{i,j}(z,w) := \begin{cases} \left(z - \mathfrak{q}^2 w\right)^{-1}(z-w)^{-1} & \text{if } i = j \\ \left(\mathfrak{q}w - \mathfrak{d}^{-1}z\right) & \text{if } i+1 = j \\ \left(z - \mathfrak{q}\mathfrak{d}^{-1}w\right) & \text{if } i-1=j \\ 1 & \text{otherwise.} \end{cases}$$

For $F \in \mathbf{S}_{k_\bullet}$ and $G \in \mathbf{S}_{l_\bullet}$, let $F \star G \in \mathbf{S}_{k_\bullet + l_\bullet}$ be defined by

$$F \star G := \frac{1}{k_\bullet! l_\bullet!} \mathrm{Sym}_{k_\bullet + l_\bullet}\left[F\left(\{z_{i,a}\}_{i\in I}^{1\leq a\leq k_i}\right) G\left(\{z_{j,b}\}_{j\in I}^{k_j < b \leq k_j + l_j}\right) \prod_{i,j\in I} \prod_{\substack{1\leq a\leq k_i \\ k_j < b \leq k_j + l_j}} \omega_{i,j}(z_{i,a}, z_{j,b}) \right]$$

where for $n_\bullet \in (\mathbb{Z}_{\geq 0})^I$,

$$n_\bullet! = \prod_{i\in I} n_i! = |\mathfrak{S}_{n_\bullet}|$$

and $\mathrm{Sym}_{n_\bullet}$ denotes the *color symmetrization*, i.e. the symmetrization over $\mathfrak{S}_{n_\bullet}$.

4.1.2. *The shuffle algebra.* Consider now for each $k_\bullet$ the subspace $\mathcal{S}_{k_\bullet} \subset \mathbf{S}_{k_\bullet}$ consisting of functions F satisfying the following two conditions:

(1) *Pole conditions:* F is of the form

$$F = \frac{f(\{z_{i,r}\})}{\prod_{i\in I} \prod_{\substack{1\leq r,r'\leq k_i \\ r\neq r'}} (z_{i,r} - \mathfrak{q}^2 z_{i,r'})} \tag{4.1}$$

for a color-symmetric *Laurent polynomial* f.

(2) *Wheel conditions:* F has a well-defined finite limit when

$$\frac{z_{i,r_1}}{z_{i+\epsilon,s}} \to \mathfrak{q}\mathfrak{d}^{\epsilon} \text{ and } \frac{z_{i+\epsilon,s}}{z_{i,r_2}} \to \mathfrak{q}\mathfrak{d}^{-\epsilon}$$

for any choice of i, r_1, r_2, s, and ϵ, where $\epsilon \in \{\pm 1\}$. This is equivalent to specifying that the Laurent polynomial f in the pole conditions evaluates to zero.

We set

$$\mathcal{S} := \bigoplus_{k_\bullet \in (\mathbb{Z}_{\geq 0})^I} \mathcal{S}_{k_\bullet}.$$

The following is standard:

Proposition 4.1. *The shuffle product $\star$ defines an associative product on $\mathbf{S}$ and $\mathcal{S}$ is closed under $\star$.*

We call $(\mathcal{S}, \star)$ the *shuffle algebra of type $\hat{A}_{r-1}$*.

4.1.3. *Relation to $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$.* Let

- $\ddot{U}^+ \subset U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ be the subalgebra generated by $\{e_i(z)\}_{i\in I}$ and
- $\ddot{U}^- \subset U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ be the subalgebra generated by $\{f_i(z)\}_{i\in I}$.

Correspondingly, we set $\mathcal{S}^+ := \mathcal{S}$ and $\mathcal{S}^- := \mathcal{S}^{op}$. The following key structural result was proved by Neguț:

Theorem 4.2 ([N]). *There are algebra isomorphisms $\Psi_\pm : \mathcal{S}^\pm \to \ddot{U}^\pm$ such that*

$$\Psi_+(z_{i,1}^n) = e_{i,n}$$
$$\Psi_-(z_{i,1}^n) = f_{i,n}.$$

Finally, note that the subalgebras $\ddot{U}^\pm$ are each closed under η. We will need to understand how the antiautomorphism η is manifested on the shuffle side:

Proposition 4.3. *For $F \in \mathcal{S}^\pm_{k_\bullet}$, we have*

$$\Psi_+^{-1}\eta\Psi_+(F) = \Psi_-^{-1}\eta\Psi_-(F) = F(z_{i,r}^{-1}) \prod_{i\in I}\prod_{r=1}^{k_i} (-\mathfrak{d})^{k_{i+1}k_i} z_{i,r}^{k_{i+1}+k_{i-1}-2(k_i-1)}\Big|_{\mathfrak{d}\mapsto\mathfrak{d}^{-1}}.$$

Proof. This clearly agrees on the generators $e_{i,n}$ and $f_{i,n}$. One needs to check that the formula above defines a $\mathbb{C}(\mathfrak{q})$-linear algebra antiautomorphism that inverts $\mathfrak{d}$. □

4.1.4. *Shuffle presentation of horizontal Heisenberg elements.* Recall the vertical Heisenberg elements (3.16) whose action on $\mathcal{F}(v)$ are related to infinite-variable Macdonald operators. Previous work [W] gives us a better understanding of ς^{-1} of such elements. However, we need ς instead, and thus we will apply $\varsigma = \eta\varsigma^{-1}\eta$ (cf. Theorem 3.1) and Proposition 4.3. To that end, observe that

$$\begin{aligned}
&\varsigma \exp\left[-\sum_{k>0}\left(\frac{\sum_{i=0}^{r-1} t^{\pm k(i+1)} h_{p-i,\pm k}}{(1-t^{\pm kr})}\right)\frac{\mathfrak{q}^{\pm k}(-z)^{\mp k}}{[k]_{\mathfrak{q}}}\right]\\
&= \eta \exp\left[\sum_{k>0}\left(\frac{\sum_{i=0}^{r-1} q^{\pm k(i+1)} \varsigma^{-1}(h_{p-i,\mp k})}{(1-q^{\pm kr})}\right)\frac{\mathfrak{q}^{\pm k}(-z)^{\mp k}}{[k]_{\mathfrak{q}}}\right] \qquad (4.2)\\
&= \eta \exp\left[(\mathfrak{q}-\mathfrak{q}^{-1})^{-1}\sum_{k>0}\left(\varsigma^{-1}(h^\perp_{p,\mp k}) - t^{\pm k}\varsigma^{-1}(h^\perp_{p+1,\mp k})\right)\frac{q^{\pm k}(-z)^{\mp k}}{k}\right]
\end{aligned}$$

where in the last line, we use (3.2). Let $\delta = (1,\dots,1) \in (\mathbb{Z}_{\geq 0})^I$ be the diagonal vector and consider the elements $\mathcal{E}^\pm_{p,n} \in \mathcal{S}^\pm$ given by

$$\begin{aligned}
\mathcal{E}^+_{p,n} &:= \mathrm{Sym}_{n\delta}\left(\prod_{1\leq a<b\leq n}\left\{\frac{z_{p+1,a}-q^{-1}z_{p,b}}{z_{p+1,a}-tz_{p,b}}\prod_{i,j\in I}\omega_{i,j}\left(z_{i,a},z_{j,b}\right)\right\}\right.\\
&\quad\left.\times\prod_{a=1}^n\left\{\left(\frac{z_{0,a}}{z_{p,a}}-q^{-1}\frac{z_{0,a}}{z_{p+1,a}}\right)\prod_{i\in I} z_{i,a}\right\}\right)\\
\mathcal{E}^-_{p,n} &:= \mathrm{Sym}_{n\delta}\left(\prod_{1\leq a<b\leq n}\left\{\frac{z_{p+1,a}-q^{-1}z_{p,b}}{z_{p+1,a}-tz_{p,b}}\prod_{i,j\in I}\omega_{i,j}\left(z_{i,a},z_{j,b}\right)\right\}\right.\\
&\quad\left.\times\prod_{a=1}^n\left\{\left(q\frac{z_{p+1,a}}{z_{0,a}}-\frac{z_{p,a}}{z_{0,a}}\right)\prod_{i\in I} z_{i,a}\right\}\right).
\end{aligned}$$

Lemma 4.4. *We have*

$$\sum_{n=0}^{\infty}\frac{(-1)^n\mathfrak{q}^{n(r-1)}t^{-n}(1-q^{-1}t^{-1})^{nr}}{v^{-n}\prod_{a=1}^{n}(1-q^{-a}t^{-a})}\Psi_+\left(\mathcal{E}^+_{p,n}\right)z^{-n}=\varsigma\exp\left[-\sum_{k>0}\left(\frac{\sum_{i=0}^{r-1}t^{-k(i+1)}h_{p-i,-k}}{(1-t^{-kr})}\right)\frac{\mathfrak{q}^{-k}(-z)^k}{v^{-k}[k]_{\mathfrak{q}}}\right]$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{nr-n}\mathfrak{d}^{-n(r-1)}t^n(1-qt)^{nr}}{v^nq^n\prod_{a=1}^{n}(1-q^{-a}t^{-a})}\Psi_-\left(\mathcal{E}^-_{p,n}\right)z^{n}=\varsigma\exp\left[-\sum_{k>0}\left(\frac{\sum_{i=0}^{r-1}t^{k(i+1)}h_{p-i,k}}{(1-t^{kr})}\right)\frac{\mathfrak{q}^{k}(-z)^k}{v^{k}[k]_{\mathfrak{q}}}\right].$$

Proof. In [W], it was shown that

$$\begin{aligned}&\exp\left[(\mathfrak{q}-\mathfrak{q}^{-1})^{-1}\sum_{k>0}\left(\varsigma^{-1}(h^{\perp}_{p,k})-t^{-k}\varsigma^{-1}(h^{\perp}_{p+1,k})\right)\frac{q^{-k}(-z)^k}{k}\right]\\&=\sum_{n=0}^{\infty}\frac{(-1)^{nr}(-q)^{-n}t^{nr}(1-q^{-1}t^{-1})^{nr}}{\mathfrak{q}^n\prod_{a=1}^{n}(1-q^{-a}t^{-a})}\Psi_+(\mathcal{H}^+_{p,n})z^n\end{aligned}$$

and

$$\begin{aligned}&\exp\left[(\mathfrak{q}-\mathfrak{q}^{-1})^{-1}\sum_{k>0}\left(\varsigma^{-1}(h^{\perp}_{p,-k})-t^{k}\varsigma^{-1}(h^{\perp}_{p+1,-k})\right)\frac{q^{k}(-z)^{-k}}{k}\right]\\&=\sum_{n=0}^{\infty}\frac{(-q)^{n}(1-qt)^{nr}}{\mathfrak{q}^n\prod_{a=1}^{n}(1-q^{-a}t^{-a})}\Psi_-(\mathcal{H}^-_{p,n})z^{-n}\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}^+_{p,n}&=\mathrm{Sym}_{n\delta}\left(\prod_{1\le a<b\le n}\left\{\frac{t^{-1}z_{p+1,b}-z_{p,a}}{qz_{p+1,b}-z_{p,a}}\prod_{i,j\in I}\omega_{i,j}\left(z_{i,a},z_{j,b}\right)\right\}\right.\\&\qquad\left.\times\prod_{a=1}^{n}\left\{\left(\frac{z_{p,a}}{z_{0,a}}-t^{-1}\frac{z_{p+1,a}}{z_{0,a}}\right)\prod_{i\in I}z_{i,a}\right\}\right)\\\mathcal{H}^-_{p,n}&=\mathrm{Sym}_{n\delta}\left(\prod_{1\le a<b\le n}\left\{\frac{t^{-1}z_{p+1,b}-z_{p,a}}{qz_{p+1,b}-z_{p,a}}\prod_{i,j\in I}\omega_{i,j}\left(z_{i,a},z_{j,b}\right)\right\}\right.\\&\qquad\left.\times\prod_{a=1}^{n}\left\{\left(t\frac{z_{0,a}}{z_{p+1,a}}-\frac{z_{0,a}}{z_{p,a}}\right)\prod_{i\in I}z_{i,a}\right\}\right).\end{aligned}\tag{4.3}$$

It is helpful to recall Remark 3.2 when making comparisons with [W]. The result follows from applying Proposition 4.3 to (4.2). We note that the mixing terms in 4.3 contribute a power of $\mathfrak{d}^{-rn(n-1)}$ before inverting $\mathfrak{d}$. ☐

4.2. **Action on the vertex representation.** For $F\in\mathcal{S}^+$ and $G\in\mathcal{S}^-$, we will present a way to compute the actions of $\rho^+_{\vec{c}}(\Psi_+(F))$ and $\rho^-_{\vec{c}}(\Psi_-(G))$. Our approach was inspired by Lemma 3.2 of [FJM] in the case $r=1$.

4.2.1. *Matrix elements.* The following is a consequence of computations similar to those done for Proposition 3.6:

Proposition 4.5. *For $v_1,v_2\in W$, we have*

$$\left\langle v_1\left|\prod_{i=0}^{\overset{\frown}{r-1}}\prod_{a=1}^{\overset{\frown}{k_i}}E_i(z_{i,a})\right|v_2\right\rangle=\frac{f\left(\{z_{i,a}\}_{i\in I}^{1\le a\le k_i}\right)\prod_{i\in I}\prod_{1\le a<b\le k_i}\left(z_{i,a}-z_{i,b}\right)\left(z_{i,a}-q^{-1}t^{-1}z_{i,b}\right)}{\prod_{\substack{1\le a\le k_0\\1\le b\le k_{r-1}}}\left(z_{0,a}-t^{-1}z_{r-1,b}\right)\prod_{i\in I\setminus\{r-1\}}\prod_{\substack{1\le a\le k_i\\1\le b\le k_{i+1}}}\left(z_{i,a}-q^{-1}z_{i+1,b}\right)}\tag{4.4}$$

for some Laurent polynomial f, where the rational functions are expanded into Laurent series assuming

$$|z_{i,a}|=1,\ |q|>1,\ |t|>1.\tag{4.5}$$

On the other hand,

$$\left\langle v_1 \left| \prod_{i=0}^{\overset{\curvearrowleft}{r-1}} \prod_{a=1}^{\overset{\curvearrowleft}{k_i}} F_i(z_{i,a}) \right| v_2 \right\rangle = \frac{g\left(\{z_{i,a}\}_{i\in I}^{1\le a\le k_i}\right) \prod_{i\in I} \prod_{1\le a<b\le k_i} (z_{i,b}-z_{i,a})(z_{i,b}-qtz_{i,a})}{\prod_{\substack{1\le a\le k_{r-1}\\ 1\le b\le k_0}} (z_{r-1,b}-tz_{0,a}) \prod_{i\in I\setminus\{0\}} \prod_{\substack{1\le b\le k_i\\ 1\le a\le k_{i-1}}} (z_{i,b}-qz_{i-1,a})} \tag{4.6}$$

for some Laurent polynomial g, where the rational functions are now expanded into Laurent series assuming

$$|z_{i,a}|=1,\ |q|<1,\ |t|<1. \tag{4.7}$$

Notice that $\omega_{i,i+1}(z,w)^{-1}$ and $\omega_{i,i-1}(z,w)^{-1}$ are rational functions that we can also expand according to (4.5) and (4.7). Thus, we can make sense of matrix elements of products of currents multiplied by these inverted mixing terms. We do not claim that such products yield well-defined series of operators—just that their matrix elements make sense. The following is a consequence of the toroidal relations:

Proposition 4.6. *When computing matrix elements, we have the relations*

$$\frac{E_i(z)E_{i+1}(w)}{\omega_{i,i+1}(z,w)} = \frac{E_{i+1}(w)E_i(z)}{\omega_{i+1,i}(w,z)}$$
$$\frac{F_i(z)F_{i+1}(w)}{\omega_{i,i+1}(w,z)} = \frac{F_{i+1}(w)F_i(z)}{\omega_{i+1,i}(z,w)}.$$

4.2.2. *Constant term formula.* For $F\in\mathcal{S}^+_{k_\bullet}$ and $G\in\mathcal{S}^-_{k_\bullet}$, consider the rational functions:

$$\frac{F\times\left\langle v_1 \left| \prod_{i=0}^{\overset{\curvearrowright}{r-1}} \prod_{a=1}^{\overset{\curvearrowright}{k_i}} E_i(z_{i,a}) \right| v_2 \right\rangle}{\left(\prod_{i\in I}\prod_{1\le a<a'\le k_i} \omega_{i,i}(z_{i,a},z_{i,a'})\right)\left(\prod_{0\le i<j\le r-1} \prod_{\substack{1\le a\le k_i\\ 1\le b\le k_j}} \omega_{i,j}(z_{i,a},z_{j,b})\right)}$$

$$\frac{G\times\left\langle v_1 \left| \prod_{i=0}^{\overset{\curvearrowright}{r-1}} \prod_{a=1}^{\overset{\curvearrowright}{k_i}} E_i(z_{i,a}) \right| v_2 \right\rangle}{\left(\prod_{i\in I}\prod_{1\le a<a'\le k_i} \omega_{i,i}(z_{i,a},z_{i,a'})\right)\left(\prod_{0\le i<j\le r-1} \prod_{\substack{1\le a\le k_i\\ 1\le b\le k_j}} \omega_{i,j}(z_{i,a},z_{j,b})\right)}.$$

We can expand these rational functions into Laurent series according to the assumptions (4.5) and (4.7), respectively. For any Laurent series, we denote by $\{-\}_0$ this operation of taking constant terms.

Lemma 4.7. *For $F\in\mathcal{S}^+_{k_\bullet}$ and $G\in\mathcal{S}^-_{k_\bullet}$, we have*

$$\rho^+_{\vec{c}}(\Psi_+(F)) = \frac{1}{k_\bullet!}\left\{\frac{F\times \prod_{i=0}^{\overset{\curvearrowright}{r-1}} \prod_{a=1}^{\overset{\curvearrowright}{k_i}} E_i(z_{i,a})}{\left(\prod_{i\in I}\prod_{1\le a<a'\le k_i} \omega_{i,i}(z_{i,a},z_{i,a'})\right)\left(\prod_{0\le i<j\le r-1} \prod_{\substack{1\le a\le k_i\\ 1\le b\le k_j}} \omega_{i,j}(z_{i,a},z_{j,b})\right)}\right\}_0 \tag{4.8}$$

where the right-hand side is expanded according to (4.5) and

$$\rho_{\vec{c}}^{-}(\Psi_-(G)) = \frac{1}{k_\bullet!}\left\{\frac{G\times \prod\limits_{i=0}^{\overset{\curvearrowleft}{r-1}}\prod\limits_{a=1}^{\overset{\curvearrowleft}{k_i}} F_i(z_{i,a})}{\left(\prod\limits_{i\in I}\prod\limits_{1\le a<a'\le k_i}\omega_{i,i}(z_{i,a},z_{i,a'})\right)\left(\prod\limits_{0\le i<j\le r-1}\prod\limits_{\substack{1\le a\le k_i\\ 1\le b\le k_j}}\omega_{i,j}(z_{i,a},z_{j,b})\right)}\right\}_0 \tag{4.9}$$

where the right-hand side is expanded according to (4.7). In particular, the expressions on the right-hand side are well-defined operators on W.

Proof. A consequence of Theorem 4.2 and the toroidal relations is that $\mathcal{S}^{\pm}$ are both spanned by shuffle monomials

$$z_{0,1}^{n(0,1)}\star z_{0,1}^{n(0,2)}\star\cdots\star z_{0,1}^{n(0,k_0)}\star z_{1,1}^{n(1,1)}\star\cdots\star z_{r-1,1}^{n(r-1,k_{r-1})}$$

since

$$\Psi_+\left(z_{0,1}^{n(0,1)}\star\cdots\star z_{r-1,1}^{n(r-1,k_{r-1})}\right) = e_{0,n(0,1)}\cdots e_{r-1,n(r-1,k_{r-1})}$$
$$\Psi_-\left(z_{0,1}^{n(0,1)}\star\cdots\star z_{r-1,1}^{n(r-1,k_{r-1})}\right) = f_{r-1,n(r-1,k_{r-1})}\cdots f_{0,n(0,1)}.$$

We will check that the matrix elements coincide for these monomials, from which the lemma follows. For the '+' case, the proposed formula gives us

$$\frac{1}{k_\bullet!}\left\{\mathrm{Sym}_{k_\bullet}\left(\left\{\prod_{i\in I}\prod_{a=1}^{k_i} z_{i,a}^{n(i,a)}\prod_{1\le a<a'\le k_i}\omega_{i,i}(z_{i,a},z_{i,a'})\right\}\left\{\prod_{0\le i<j\le r-1}\prod_{\substack{1\le a\le k_i\\ 1\le b\le k_j}}\omega_{i,j}(z_{i,a},z_{j,b})\right\}\right)\right.$$
$$\left.\times\frac{\left\langle v_1\left|\prod\limits_{i=0}^{\overset{\curvearrowright}{r-1}}\prod\limits_{a=1}^{\overset{\curvearrowright}{k_i}}E_i(z_{i,a})\right|v_2\right\rangle}{\left(\prod\limits_{i\in I}\prod\limits_{1\le a<a'\le k_i}\omega_{i,i}(z_{i,a},z_{i,a'})\right)\left(\prod\limits_{0\le i<j\le r-1}\prod\limits_{\substack{1\le a\le k_i\\ 1\le b\le k_j}}\omega_{i,j}(z_{i,a},z_{j,b})\right)}\right\}_0.$$

From the toroidal relations, we have

$$\frac{E_i(z_{i,a})E_i(z_{i,b})}{\omega_{i,i}(z_{i,a},z_{i,b})} = \frac{E_i(z_{i,b})E_i(z_{i,a})}{\omega_{i,i}(z_{i,b},z_{i,a})}.$$

Using this to swap variables, we can move both the matrix element and the mixing terms inside the symmetrization, where the mixing terms will all cancel out. Notice that taking the constant term is insensitive to the labeling of the variables, and thus the constant terms of all the summands of the symmetrization are equal. The end result is

$$\left\{\prod_{i\in I}\prod_{a=1}^{k_i} z_{i,a}^{n(i,a)}\left\langle v_1\left|\prod_{i=0}^{\overset{\curvearrowright}{r-1}}\prod_{a=1}^{\overset{\curvearrowright}{k_i}}E_i(z_{i,a})\right|v_2\right\rangle\right\}_0 = \left\langle v_1\left|\rho_{\vec{c}}^{+}\left(e_{0,n(0,1)}\cdots e_{r-1,n(r-1,k_{r-1})}\right)\right|v_2\right\rangle.$$

The '−' case is similar. ∎

5. Difference operators

5.1. **Setup.** Now, we will fix $\alpha \in Q$, which also fixes a core. The previous two sections were concerned with symmetric functions in infinitely many variables. Here, we will shift to working with finitely many variables

$$\left\{x_l^{(i)}\right\}_{i\in I}^{1\leq l\leq N_i} = X_{N_\bullet}.$$

We will impose the compatibility (2.9) between α and the vector $N_\bullet$ recording the number of variables of each color. Our approach for finding difference operators is straightforward: we use Lemma 4.7 to compute the action of $\rho_{\vec{c}}^{\pm}(\Psi_\pm(\mathcal{E}_{p,n}^{\pm}))$ on a function $f\left[X_{N_\bullet}\right]$. We assume that $n \leq N_i$ for all $i \in I$.

5.1.1. *Finitized vertex operators.* Recall that $\Lambda_{N_\bullet}^I$ denotes the tensor product over $i \in I$ of rings of symmetric polynomials in N_i variables and $\pi_{N_\bullet} : \Lambda^I \to \Lambda_{N_\bullet}^I$ is the natural projection. We will abuse notation and also denote the map $(\pi_{N_\bullet} \otimes 1) : \Lambda^I \otimes \mathbb{K}\{Q\} \to \Lambda_{N_\bullet}^I \otimes \mathbb{K}\{Q\}$ by $\pi_{N_\bullet}$. Recall Proposition 3.6. The projection $\pi_{N_\bullet}$ interacts with the operator components of (3.10) as follows. From the 'left' halves, we have

$$\pi_{N_\bullet}\left(\exp\left[\sum_{k>0}\left(p_k[X^{(i)}] - t^{-k}p_k[X^{(i-1)}]\right)\frac{z^k}{k}\right] z^{H_{i,0}}(f\otimes e^\alpha)\right)$$

$$= \frac{\prod_{l=1}^{N_{i-1}}\left(1-t^{-1}zx_l^{(i-1)}\right)}{\prod_{l=1}^{N_i}\left(1-zx_l^{(i)}\right)} z^{N_i - N_{i-1}}\left(\pi_{N_\bullet}(f)\otimes e^\alpha\right) = \frac{\prod_{l=1}^{N_{i-1}}\left(z^{-1}-t^{-1}x_l^{(i-1)}\right)}{\prod_{l=1}^{N_i}\left(z^{-1}-x_l^{(i)}\right)}\left(\pi_{N_\bullet}(f)\otimes e^\alpha\right) \tag{5.1}$$

For this to hold, we will need to impose conditions on $|x_l^{(i)}|$. Recall that we have the conditions (3.11) when working with $\{E_i(z)\}$. We extend this to

$$|z| = 1,\ |q| > 1,\ |t| > 1,\ |x_l^{(j)}| < 1$$

for (5.1) to hold. We also point out that the compatibility condition (2.9) is used to obtain the first equality.

Next, let $f = \prod_{i\in I} f_i[X^{(i)}]$, where $f_i \in \Lambda$ for all $i \in I$. From the 'right' halves, we have

$$\pi_{N_\bullet}\left(\exp\left[\sum_{k>0}\left(-p_k[X^{(i)}]^\perp + q^{-k}p_k[X^{(i-1)}]^\perp\right)\frac{z^{-k}}{k}\right]\cdot f\right)$$

$$= \pi_{N_\bullet}\left(f_i[X^{(i)} - z^{-1}]f_{i-1}[X^{(i-1)} + q^{-1}z^{-1}]\prod_{\substack{j\in I\\ j\neq i,i-1}} f_j[X^{(j)}]\right) \tag{5.2}$$

$$= f_i\left[x_\bullet^{(i)} - z^{-1}\right] f_{i-1}\left[x_\bullet^{(i-1)} + q^{-1}z^{-1}\right]\prod_{\substack{j\in I\\ j\neq i,i-1}} f_j\left[x_\bullet^{(j)}\right].$$

Here, (5.2) follows from checking on power sums $p_k[X^{(i)}]$ and $p_k[X^{(i-1)}]$.

For (3.12), we have

$$\pi_{N_\bullet}\left(\exp\left[\sum_{k>0}\left(-t^k p_k[X^{(i)}] + p_k[X^{(i-1)}]\right)\frac{z^k}{k}\right] z^{-H_{i,0}}(f\otimes e^\alpha)\right)$$

$$= \frac{\prod_{l=1}^{N_i}\left(1-tzx_l^{(i)}\right)}{\prod_{l=1}^{N_{i-1}}\left(1-zx_l^{(i-1)}\right)} z^{N_{i-1}-N_i}\left(\pi_{N_\bullet}(f)\otimes e^\alpha\right) = \frac{\prod_{l=1}^{N_i}\left(z^{-1}-tx_l^{(i)}\right)}{\prod_{l=1}^{N_{i-1}}\left(z^{-1}-x_l^{(i-1)}\right)}\left(\pi_{N_\bullet}(f)\otimes e^\alpha\right).$$

Here, we extend (3.13) to

$$|z| = 1,\ |q| < 1,\ |t| < 1,\ |x_l^{(j)}| < 1.$$

On the other hand, from the right halves appearing in (3.12), we have

$$\begin{aligned}
&\pi_{N_\bullet}\left(\exp\left[\sum_{k>0}\left(q^k p_k[X^{(i)}]^\perp - p_k[X^{(i-1)}]^\perp\right)\frac{z^{-k}}{k}\right]\cdot f\right)\\
&=\pi_{N_\bullet}\left(f_i[X^{(i)}+qz^{-1}]f_{i-1}[X^{(i-1)}-z^{-1}]\prod_{\substack{j\in I\\ j\neq i,i-1}} f_j[X^{(j)}]\right)\\
&=f_i\left[x_\bullet^{(i)}+qz^{-1}\right]f_{i-1}\left[x_\bullet^{(i-1)}-z^{-1}\right]\prod_{\substack{j\in I\\ j\neq i,i-1}} f_j\left[x_\bullet^{(j)}\right].
\end{aligned}$$

5.1.2. *Applying the constant term formula.* Plugging in $\mathcal{E}_{p,n}^{\pm}$ into the formula from Lemma 4.7, we can use the toroidal relations and Proposition 4.6 to reorder the currents in alignment with Proposition 3.6. We remind the reader that the formulas for $\mathcal{E}_{p,n}^{\pm}$ are given by (4.3). As in the proof of Lemma 4.7, we can use the toroidal relations to remove the symmetrizations in $\mathcal{E}_{p,n}^{\pm}$. Suppose $f=\prod_{i\in I} f_i[X^{(i)}]$. Taking the result for $\mathcal{E}_{p,n}^{+}$, acting on $f\otimes e^\alpha$, and then applying $\pi_{N_\bullet}$ gives us:

$$\begin{aligned}
\pi_{N_\bullet}\left((\rho_{\vec{c}}^{+}\circ\Psi_+)(\mathcal{E}_{p,n}^{+})(f\otimes e^\alpha)\right) &= \left((-1)^{\frac{(r-2)(r-3)}{2}}\mathfrak{d}^{\frac{r}{2}-1}\prod_{i\in I}c_i\right)^n \Bigg\{\prod_{i\in I}\prod_{a=1}^{n}\prod_{l=1}^{N_i}\left(\frac{z_{i+1,a}^{-1}-t^{-1}x_l^{(i)}}{z_{i,a}^{-1}-x_l^{(i)}}\right)\\
&\times\prod_{1\le a<b\le n}\left[\frac{1-q^{-1}z_{p,b}/z_{p+1,a}}{1-tz_{p,b}/z_{p+1,a}}\prod_{i\in I}\frac{(1-z_{i,b}/z_{i,a})\left(1-q^{-1}t^{-1}z_{i,b}/z_{i,a}\right)}{(1-t^{-1}z_{i+1,b}/z_{i,a})\,(1-q^{-1}z_{i-1,b}/z_{i,a})}\right]\\
&\times\prod_{a=1}^{n}\Bigg[\left(\frac{z_{0,a}}{z_{p+1,a}}\right)\left(\frac{z_{p+1,a}}{\omega_{p+1,p}(z_{p+1,a},z_{p,a})}\right)\\
&\times\prod_{i\in I\backslash\{p+1\}}\frac{z_{i,a}}{(1-t^{-1}z_{i,a}/z_{i-1,a})\,\omega_{i-1,i}(z_{i-1,a},z_{i,a})}\Bigg]\\
&\times\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}-\sum_{a=1}^{n}z_{i,a}^{-1}+q^{-1}\sum_{a=1}^{n}z_{i+1,a}^{-1}\right]\Bigg\}_0\otimes e^\alpha
\end{aligned}$$

where all rational functions are expanded as Laurent series assuming

$$|z_{i,a}|=1,\ |x_l^{(j)}|<1,\ |t|>1,\ |q|>1. \tag{5.3}$$

For $\mathcal{E}_{p,n}^{-}$, we instead have

$$\begin{aligned}
\pi_{N_\bullet}\left((\rho_{\vec{c}}^{-}\circ\Psi_-)(\mathcal{E}_{p,n}^{-})(f\otimes e^\alpha)\right) &= \left(\frac{(-1)^{\frac{(r-2)(r-3)}{2}}}{\mathfrak{d}^{\frac{r}{2}-1}\prod_{i\in I}c_i}\right)^n \Bigg\{\prod_{i\in I}\prod_{a=1}^{n}\prod_{l=1}^{N_i}\left(\frac{z_{i,a}^{-1}-tx_l^{(i)}}{z_{i+1,a}^{-1}-x_l^{(i)}}\right)\\
&\times\prod_{1\le a<b\le n}\left[\frac{q^{-1}-z_{p+1,a}/z_{p,b}}{t-z_{p+1,a}/z_{p,b}}\prod_{i\in I}\frac{(1-z_{i,a}/z_{i,b})\,(1-qtz_{i,a}/z_{i,b})}{(1-tz_{i-1,a}/z_{i,b})\,(1-qz_{i+1,a}/z_{i,b})}\right]\\
&\times\prod_{a=1}^{n}\Bigg[\left(\frac{z_{p+1,a}}{z_{0,a}}\right)\left(\frac{-z_{p,a}}{\omega_{p+1,p}(z_{p+1,a},z_{p,a})}\right)\\
&\times\prod_{i\in I\backslash\{p\}}\frac{z_{i,a}}{(1-tz_{i,a}/z_{i+1,a})\,\omega_{i,i+1}(z_{i,a},z_{i+1,a})}\Bigg]\\
&\times\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}+q\sum_{a=1}^{n}z_{i,a}^{-1}-\sum_{a=1}^{n}z_{i+1,a}^{-1}\right]\Bigg\}_0\otimes e^\alpha
\end{aligned}$$

where all rational functions are expanded into Laurent series assuming.

$$|z_{i,a}| = 1,\ |x_l^{(j)}| < 1,\ |q| < 1,\ |t| < 1. \tag{5.4}$$

In both formulas, we are taking constant terms in the z-variables.

We will slightly modify these equations as follows:

- Using our assignment $q = \mathfrak{q}\mathfrak{d}$, $t = \mathfrak{q}\mathfrak{d}^{-1}$, we have
$$\omega_{i,i+1}(z,w) = \mathfrak{q}w - \mathfrak{d}^{-1}z = \mathfrak{d}^{-1}(qw - z) = \mathfrak{q}\left(w - q^{-1}z\right)$$
$$\omega_{i,i-1}(z,w) = z - \mathfrak{q}\mathfrak{d}^{-1}w = z - tw$$
We will substitute this for the mixing terms.
- Recall the extra constants in Lemma 4.4. Let
$$c_n^+ := \frac{(-1)^n\mathfrak{q}^{n(r-1)}t^{-n}(1-q^{-1}t^{-1})^{nr}}{v^{-n}\prod_{a=1}^n(1-q^{-a}t^{-a})}, \quad c_n^- := \frac{(-1)^{nr-n}\mathfrak{d}^{-n(r-1)}t^n(1-qt)^{nr}}{v^nq^n\prod_{a=1}^n(1-q^{-a}t^{-a})}$$
where $v = (-1)^{\frac{(r-2)(r-3)}{2}}\mathfrak{q}\mathfrak{d}^{-\frac{r}{2}}(c_0\cdots c_{r-1})^{-1}$. We will instead work with $c_n^+\mathcal{E}_{p,n}^+$ and $c_n^-\mathcal{E}_{p,n}^-$.

Doing all this, we obtain

$$\begin{aligned}
c_n^+\pi_{N_\bullet}\Big((\rho_{\vec{c}}^+\circ\Psi_+)(\mathcal{E}_{p,n}^+)(f\otimes e^\alpha)\Big) &= \frac{(1-q^{-1}t^{-1})^{nr}}{t^n\prod_{a=1}^n(1-q^{-a}t^{-a})}\left\{\prod_{i\in I}\prod_{a=1}^n\prod_{l=1}^{N_i}\left(\frac{z_{i+1,a}^{-1}-t^{-1}x_l^{(i)}}{z_{i,a}^{-1}-x_l^{(i)}}\right)\right.\\
&\times\prod_{1\le a<b\le n}\left[\frac{(1-z_{p+1,b}/z_{p+1,a})\left(1-q^{-1}t^{-1}z_{p+1,b}/z_{p+1,a}\right)}{(1-tz_{p,b}/z_{p+1,a})\,(1-t^{-1}z_{p+2,b}/z_{p+1,a})}\right.\\
&\times\left.\prod_{i\in I\backslash\{p+1\}}\frac{(1-z_{i,b}/z_{i,a})\left(1-q^{-1}t^{-1}z_{i,b}/z_{i,a}\right)}{(1-q^{-1}z_{i-1,b}/z_{i,a})\,(1-t^{-1}z_{i+1,b}/z_{i,a})}\right]\\
&\times\prod_{a=1}^n\left[\left(\frac{z_{0,a}}{z_{p,a}}\right)\left(\frac{1}{1-t^{-1}z_{p+1,a}/z_{p,a}}\right)\right.\\
&\times\left.\prod_{i\in I\backslash\{p+1\}}\left(\frac{1}{1-t^{-1}z_{i,a}/z_{i-1,a}}\right)\left(\frac{1}{1-q^{-1}z_{i-1,a}/z_{i,a}}\right)\right]\\
&\times\left.\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}-\sum_{a=1}^nz_{i,a}^{-1}+q^{-1}\sum_{a=1}^nz_{i+1,a}^{-1}\right]\right\}_0\otimes e^\alpha
\end{aligned}\tag{5.5}$$

and

$$\begin{aligned}
c_n^-\pi_{N_\bullet}\Big((\rho_{\vec{c}}^-\circ\Psi_-)(\mathcal{E}_{p,n}^-)(f\otimes e^\alpha)\Big) &= \frac{t^n(1-qt)^{nr}}{\prod_{k=1}^n(1-q^kt^k)}\left\{\prod_{i\in I}\prod_{a=1}^n\prod_{l=1}^{N_i}\left(\frac{z_{i,a}^{-1}-tx_l^{(i)}}{z_{i+1,a}^{-1}-x_l^{(i)}}\right)\right.\\
&\times\prod_{1\le a<b\le n}\left[\frac{(1-z_{p,a}/z_{p,b})\,(1-qtz_{p,a}/z_{p,b})}{(1-t^{-1}z_{p+1,a}/z_{p,b})\,(1-tz_{p-1,a}/z_{p,b})}\right.\\
&\times\left.\prod_{i\in I\backslash\{p\}}\frac{(1-z_{i,a}/z_{i,b})\,(1-qtz_{i,a}/z_{i,b})}{(1-qz_{i+1,a}/z_{i,b})\,(1-tz_{i-1,a}/z_{i,b})}\right]\\
&\times\prod_{a=1}^n\left[\left(\frac{z_{p,a}}{z_{0,a}}\right)\left(\frac{1}{1-tz_{p,a}/z_{p+1,a}}\right)\right.\\
&\times\left.\prod_{i\in I\backslash\{p\}}\left(\frac{1}{1-tz_{i,a}/z_{i+1,a}}\right)\left(\frac{1}{1-qz_{i+1,a}/z_{i,a}}\right)\right]\\
&\times\left.\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}+q\sum_{a=1}^nz_{i,a}^{-1}-\sum_{a=1}^nz_{i+1,a}^{-1}\right]\right\}_0\otimes e^\alpha.
\end{aligned}\tag{5.6}$$

5.1.3. *Integral formula.* Regardless of f, the formulas obtained in 5.1.2 are constant terms of Laurent series expansions of some rational function. Note that all poles are simple except for the poles at zero possibly coming from the plethystic modifications done to f. Thus, it will be advantageous to invert all the z-variables: let

$$w_{i,a} := z_{i,a}^{-1}.$$

The right hand side of (5.5) is equal to

$$\left\{ g^+_{p,n}(w_{\bullet,\bullet}, X_{N_\bullet}) \prod_{i\in I} f_i \left[\sum_{l=1}^{N_i} x_l^{(i)} - \sum_{a=1}^{n} w_{i,a} + \sum_{a=1}^{n} q^{-1} w_{i+1,a} \right] \right\}_0 \otimes e^\alpha \tag{5.7}$$

where

$$g^+_{p,n}(w_{\bullet,\bullet}, X_{N_\bullet}) := \frac{(-1)^{\frac{n(n-1)}{2}} (1-q^{-1}t^{-1})^{nr}}{t^{\frac{n(n+1)}{2}} \prod_{a=1}^n (1-q^{-a}t^{-a})} \prod_{i\in I} \prod_{a=1}^{n} \prod_{l=1}^{N_i} \left(\frac{w_{i+1,a} - t^{-1} x_l^{(i)}}{w_{i,a} - x_l^{(i)}} \right) \tag{5.8}$$

$$\times \prod_{1\le a<b\le n} \left[\frac{(w_{p+1,b} - w_{p+1,a})(w_{p+1,b} - q^{-1}t^{-1} w_{p+1,a})}{(w_{p+1,a} - t^{-1} w_{p,b})(w_{p+2,b} - t^{-1} w_{p+1,a})} \right. \tag{5.9}$$

$$\left. \times \prod_{i\in I\setminus\{p+1\}} \frac{(w_{i,b} - w_{i,a})(w_{i,b} - q^{-1}t^{-1} w_{i,a})}{(w_{i-1,b} - q^{-1} w_{i,a})(w_{i+1,b} - t^{-1} w_{i,a})} \right] \tag{5.10}$$

$$\times \prod_{a=1}^{n} \left[\left(\frac{w_{p,a}}{w_{0,a}} \right) \left(\frac{w_{p+1,a}}{w_{p+1,a} - t^{-1} w_{p,a}} \right) \right. \tag{5.11}$$

$$\left. \times \prod_{i\in I\setminus\{p+1\}} \left(\frac{w_{i,a}}{w_{i,a} - t^{-1} w_{i-1,a}} \right) \left(\frac{w_{i-1,a}}{w_{i-1,a} - q^{-1} w_{i,a}} \right) \right]. \tag{5.12}$$

Now, all the poles appearing in (5.7) are simple. Similarly, the right hand side of (5.6) becomes

$$\left\{ g^-_{p,n}(w_{\bullet,\bullet}, X_{N_\bullet}) \prod_{i\in I} f_i \left[\sum_{l=1}^{N_i} x_l^{(i)} + \sum_{a=1}^{n} q w_{i,a} - \sum_{a=1}^{n} w_{i+1,a} \right] \right\}_0 \otimes e^\alpha \tag{5.13}$$

where

$$g^-_{p,n}(w_{\bullet,\bullet}, X_{N_\bullet}) := \frac{(-1)^{\frac{n(n-1)}{2}} t^{\frac{n(n+1)}{2}} (1-qt)^{nr}}{\prod_{a=1}^n (1-q^a t^a)} \prod_{i\in I} \prod_{a=1}^{n} \prod_{l=1}^{N_i} \left(\frac{w_{i,a} - t x_l^{(i)}}{w_{i+1,a} - x_l^{(i)}} \right) \tag{5.14}$$

$$\times \prod_{1\le a<b\le n} \left[\frac{(w_{p,a} - w_{p,b})(w_{p,a} - qt w_{p,b})}{(w_{p,b} - t w_{p+1,a})(w_{p-1,a} - t w_{p,b})} \right. \tag{5.15}$$

$$\left. \times \prod_{i\in I\setminus\{p\}} \frac{(w_{i,a} - w_{i,b})(w_{i,a} - qt w_{i,b})}{(w_{i+1,a} - q w_{i,b})(w_{i-1,a} - t w_{i,b})} \right] \tag{5.16}$$

$$\times \prod_{a=1}^{n} \left[\left(\frac{w_{0,a}}{w_{p,a} - t w_{p+1,a}} \right) \prod_{i\in I\setminus\{p\}} \left(\frac{w_{i,a}}{w_{i,a} - t w_{i+1,a}} \right) \left(\frac{w_{i+1,a}}{w_{i+1,a} - q w_{i,a}} \right) \right]. \tag{5.17}$$

Remark 5.1. Without the compatibility condition (2.9) between $N_\bullet$ and α, we would have to contend with an additional Laurent monomial factor in the variables $w_{i,a}$ in (5.7) and (5.13). This would prevent us from obtaining a manageable formula due to the presence of non-simple poles at zero.

Lemma 5.2. *We have*

$$c_n^+\pi_{N_\bullet}\left((\rho^+_{\vec{c}}\circ\Psi_+)(\mathcal{E}^+_{p,n})(f\otimes e^\alpha)\right)$$
$$=\left(\oint_{|w_{i,a}|=1}\cdots\oint g^+_{p,n}(w_{\bullet,\bullet},X_{N_\bullet})\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}-\sum_{a=1}^n w_{i,a}+\sum_{a=1}^n q^{-1}w_{i+1,a}\right]\prod_{a=1}^n\frac{dw_{i,a}}{2\pi\sqrt{-1}w_{i,a}}\right)\otimes e^\alpha.$$

We assume

$$|x_l^{(i)}|<1,\ |q|>1,\ |t|>1$$

and orient the unit circle $|w_{i,a}|=1$ counter-clockwise. In the '$-$' case, we have

$$c_n^-\pi_{N_\bullet}\left((\rho^-_{\vec{c}}\circ\Psi_-)(\mathcal{E}^-_{p,n})(f\otimes e^\alpha)\right)$$
$$=\left(\oint_{|w_{i,a}|=1}\cdots\oint g^-_{p,n}(w_{\bullet,\bullet},X_{N_\bullet})\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}+\sum_{a=1}^n qw_{\bullet,\bullet}-\sum_{a=1}^n w_{\bullet+1,\bullet}\right]\prod_{a=1}^n\frac{dw_{i,a}}{2\pi\sqrt{-1}w_{i,a}}\right)\otimes e^\alpha$$

where we now assume

$$|x_l^{(i)}|<1,\ |q|<1,\ |t|<1$$

and also orient the unit circle clockwise.

Proof. In case '$\pm$', the integrands are given by series in the $x_l^{(i)}$ and $q^{\mp1}$, $t^{\mp1}$, with coefficients which are Laurent polynomials in the $w_{i,a}$. Under the given assumptions, these series converge uniformly absolutely on the integration cycle and thus we can exchange the order of summation and integration. This turns the above integrals into the constant term formulas (5.7) and (5.13). □

5.1.4. *Cyclic-shift operators.* To describe the results of our computation, we need to introduce some difference operators that also permute variables. As before, let $X_{N_\bullet}=\{x_l^{(i)}\}_{i\in I}^{1\le l\le N_i}$ denote our set of variables compatible with our r-core. Define a *shift pattern of* $X_{N_\bullet}$ to be a subset of $X_{N_\bullet}$ that contains no more than one variable of each color. A shift pattern *contains color* $p\in I$ if it contains a variable of color p. Let $Sh_p(X_{N_\bullet})$ denote the set of all shift patterns containing color p.

For a shift pattern $\underline{J}$, let $J\subset I$ denote the colors of the variables in $\underline{J}$. We denote the variables in $\underline{J}$ by $x_{\underline{J}}^{(i)}$, so $\underline{J}=\{x_{\underline{J}}^{(i)}\}_{i\in J}$. To $\underline{J}$ we associate the following:

(1) *Gap labels:* For $i\in I$, let $i^\vartriangle\in J$ be first element greater than *or* equal to i in the cyclic order. Similarly, let $i^\triangledown\in J$ be the first element less than or equal to i in the cyclic order. We stipulate that $0\le i^\vartriangle-i, i-i^\triangledown\le r-1$. With this set, we define:
$$x^{(i)}_{\underline{J}^\vartriangle}=q^{(i-i^\vartriangle)}x_{\underline{J}}^{(i^\vartriangle)}$$
$$x^{(i)}_{\underline{J}^\triangledown}=q^{(i-i^\triangledown)}x_{\underline{J}}^{(i^\triangledown)}.$$
To clarify, $x^{(i)}_{\underline{J}^\vartriangle}=x^{(i)}_{\underline{J}^\triangledown}=x^{(i)}_{\underline{J}}$ if $i\in J$. Thus, while $\underline{J}$ gives a list of variables colored by $J\subset I$, we 'fill in the gaps' for values $i\in I\backslash J$ with certain q-shifts of the elements of $\underline{J}$. Note that the q-shifts are negative for $x^{(i)}_{\underline{J}^\vartriangle}$ and positive for $x^{(i)}_{\underline{J}^\triangledown}$.

(2) *A cyclic-shift operator:* For $i\in J$, let $i^\blacktriangledown\in J$ be the first element *strictly* less than i in the cyclic order. We set $1\le i-i^\blacktriangledown\le r$, where r occurs if and only if $|J|=\{i\}$. We then define the operator $T_{\underline{J}}$ on $\mathbb{K}[X_{N_\bullet}]$ as the algebra map induced by
$$T_{\underline{J}}(x_l^{(i)})=\begin{cases}q^{(i-i^\blacktriangledown)}x_{\underline{J}}^{(i^\blacktriangledown)} & \text{if } i\in J \text{ and } x_l=x_{\underline{J}}^{(i)}\\ x_l^{(i)} & \text{otherwise.}\end{cases}$$
Note that this q-shift is positive. If we let $i^\blacktriangle\in J$ be the first element strictly greater than i in the cyclic order, then observe that
$$T_{\underline{J}}^{-1}(x_l^{(i)})=\begin{cases}q^{(i-i^\blacktriangle)}x_{\underline{J}}^{(i^\blacktriangle)} & \text{if } i\in J \text{ and } x_l^{(i)}=x_{\underline{J}}^{(i)}\\ x_l^{(i)} & \text{otherwise}\end{cases}$$

where as before, we view $1 \leq i^{\blacktriangle} - i \leq r$. Finally, we note the following: for $i \in J$

$$
\begin{aligned}
T_{\underline{J}}(x_{\underline{J}}^{(i)}) &= q x_{\underline{J}^{\triangledown}}^{(i-1)} \\
T_{\underline{J}}^{-1}(x_{\underline{J}}^{(i)}) &= q^{-1} x_{\underline{J}^{\vartriangle}}^{(i+1)}.
\end{aligned}
\tag{5.18}
$$

We will also make use of n-tuples of shift patterns. For such an n-tuple $\underline{\mathbf{J}} = (\underline{J}_1, \ldots, \underline{J}_n)$ and $0 \leq k \leq n$, we denote

$$
\begin{aligned}
|\underline{\mathbf{J}}| &= \underline{J}_1 \cup \cdots \cup \underline{J}_n \subset X_{N_\bullet} \\
|\underline{\mathbf{J}}|_{\leq k} &= \underline{J}_1 \cup \cdots \cup \underline{J}_k \subset X_{N_\bullet} \\
|\underline{\mathbf{J}}|_{\geq k} &= \underline{J}_k \cup \cdots \cup \underline{J}_n \subset X_{N_\bullet}.
\end{aligned}
$$

If $\underline{\mathbf{J}}$ is an n-tuple of shift patterns all containing color p, we say $\underline{\mathbf{J}}$ is *p-distinct* if the p-colored variables $x_{\underline{J}_k}^{(p)}$ are all distinct. Let $Sh_p^{[n]}(X_{N_\bullet})$ denote the set of all p-distinct n-tuples of shift patterns containing color p.

5.2. **Degree one case.** We will first compute the integrals from Lemma 5.2 for the case $n = 1$. In the '+' case, we have:

$$
t^{-1}(1-q^{-1}t^{-1})^{r-1} \oint_{|w_{i,1}|=1} \cdots \oint \prod_{i\in I}\prod_{l=1}^{N_i} \frac{\left(w_{i+1,1} - t^{-1}x_l^{(i)}\right)}{\left(w_{i,1} - x_l^{(i)}\right)} \tag{5.19}
$$

$$
\times \left(\frac{w_{p,1}}{w_{0,1}}\right)\left(\frac{w_{p+1,1}}{w_{p+1,1} - t^{-1}w_{p,1}}\right) \prod_{i\in I\setminus\{p+1\}} \left(\frac{w_{i,1}}{w_{i,1} - t^{-1}w_{i-1,1}}\right)\left(\frac{w_{i-1,1}}{w_{i-1,1} - q^{-1}w_{i,1}}\right) \tag{5.20}
$$

$$
\times \prod_{i\in I} f_i \left[\sum_{l=1}^{N_i} x_l^{(i)} - w_{i,1} + q^{-1}w_{i+1,1}\right] \frac{dw_{i,1}}{2\pi\sqrt{-1}w_{i,1}}. \tag{5.21}
$$

We will first integrate $w_{p,1}$. Based on (5.3), the residues within the unit circle $|w_{p,1}| = 1$ come from the factors:

$$
\frac{1}{\underbrace{\left(w_{p,1} - t^{-1}w_{p-1,1}\right)}_{(5.20)} \underbrace{\prod_{l=1}^{N_p}\left(w_{p,1} - x_l^{(p)}\right)}_{(5.19)}}.
$$

We will call the first type of pole a t-pole and the second type an x-pole. For the computations we will perform later on, we will further restrict our parameters so that

$$
|x_l^{(i)}| < 1,\ |q| \gg 1,\ |t| \gg 1.
$$

We will address how to relax this later on in the proof of Theorem 5.3 below.

The '−' case is

$$
t(1-qt)^{r-1} \oint_{|w_{i,1}|=1} \cdots \oint \prod_{i\in I}\prod_{l=1}^{N_i} \frac{\left(w_{i,1} - tx_l^{(i)}\right)}{\left(w_{i+1,1} - x_l^{(i)}\right)} \tag{5.22}
$$

$$
\times \left(\frac{w_{0,1}}{w_{p,1} - tw_{p+1,1}}\right) \prod_{i\in I\setminus\{p\}} \left(\frac{w_{i,1}}{w_{i,1} - tw_{i+1,1}}\right)\left(\frac{w_{i+1,1}}{w_{i+1,1} - qw_{i,1}}\right) \tag{5.23}
$$

$$
\times \prod_{i\in I} f_i \left[\sum_{l=1}^{N_i} x_l^{(i)} + qw_{i,1} - w_{i+1,1}\right] \frac{dw_{i,1}}{2\pi\sqrt{-1}w_{i,1}}. \tag{5.24}
$$

Here, we will instead start by integrating $w_{p+1,1}$. As before, there are x-poles and a t-pole coming from:

$$\frac{1}{\underbrace{(w_{p+1,1}-tw_{p+2,1})}_{(5.23)}\underbrace{\prod_{l=1}^{N_{p+1}}\left(w_{p+1,1}-x_l^{(p)}\right)}_{(5.22)}}.$$

In this case, we will restrict to

$$|x_l^{(i)}|<1,\ |q|\ll 1,\ |t|\ll 1.$$

5.2.1. *The t-poles.* First consider the '+' case. Here, we begin with the residue $w_{p,1}=t^{-1}w_{p-1,1}$. Let us group together the factors

$$\frac{w_{p,1}w_{p-1,1}}{w_{p,1}\left(w_{p-1,1}-q^{-1}w_{p,1}\right)\left(w_{p,1}-t^{-1}w_{p-1,1}\right)\prod_{l=1}^{N_p}\left(w_{p,1}-x_l^{(p)}\right)}\prod_{l=1}^{N_{p-1}}\frac{\left(w_{p,1}-t^{-1}x_l^{(p-1)}\right)}{\left(w_{p-1,1}-x_l^{(p-1)}\right)}.$$

Upon taking taking the residue, this becomes

$$\frac{t^{-N_{p-1}}}{(1-q^{-1}t^{-1})\prod_{l=1}^{N_p}\left(t^{-1}w_{p-1,1}-x_l^{(p)}\right)}.$$

Because of the additional restriction $|t|\gg 1$, the poles above will be outside the unit circle $|w_{p-1,1}|=1$.

This pattern persists as we continue *downwards* in cyclic order until we reach $w_{p+1,1}$. Here, we have

$$\left.\frac{w_{p+1,1}w_{p,1}}{w_{p+1,1}w_{0,1}\left(w_{p+1,1}-t^{-1}w_{p,1}\right)}\right|_{\substack{w_{0,1}\mapsto t^{p+1-r}w_{p+1,1}\\ w_{p,1}\mapsto t^{-(r-1)}w_{p+1,1}}}\prod_{l=1}^{N_p}\frac{\left(w_{p+1,1}-t^{-1}x_l^{(p)}\right)}{\left(t^{-r+1}w_{p+1,1}-x_l^{(p)}\right)}$$
$$=\frac{t^{-p}}{1-t^{-r}}\cdot\frac{1}{w_{p+1,1}}\prod_{l=1}^{N_p}\frac{\left(w_{p+1,1}-t^{-1}x_l^{(p)}\right)}{\left(t^{-r+1}w_{p+1,1}-x_l^{(p)}\right)}.$$

The only pole here is the simple pole at $w_{p+1,1}=0$. After taking this residue, (5.21) becomes just $f\left[X_{N_\bullet}\right]$. Bringing in the front matter in (5.19), we are left with

$$\frac{t^{-p-1-|N_\bullet|}}{1-t^{-r}}f\left[X_{N_\bullet}\right]. \tag{5.25}$$

Here, we recall that $N_\bullet=(N_0,\dots,N_{r-1})$ records the number of x-variables and $|N_\bullet|=\sum_{i\in I}N_i$.

For the '$-$' case, recall that we begin at $w_{p+1,1}$ and take the residue $w_{p+1,1}=tw_{p+2}$. We group together the factors

$$\frac{w_{p+1,1}w_{p+2,1}}{w_{p+1,1}\left(w_{p+2,1}-qw_{p+1,1}\right)\left(w_{p+1,1}-tw_{p+2,1}\right)\prod_{l=1}^{N_p}\left(w_{p+1,1}-x_l^{(p)}\right)}\prod_{l=1}^{N_{p+1}}\frac{\left(w_{p+1,1}-tx_l^{(p+1)}\right)}{\left(w_{p+1,2}-x_l^{(p+1)}\right)}$$

which upon taking the residue becomes

$$\frac{t^{N_{p+1}}}{(1-qt)\prod_{l=1}^{N_p}\left(tw_{p+2,1}-x_l^{(p)}\right)}.$$

The remaining poles above lie outside the unit circle $|w_{p+2,1}|=1$ because we have assumed $|t|\ll 1$. We continue *upwards* in cyclic order, yielding similar calculations until we arrive at $w_{p,1}$. Here, we have the factors

$$\left.\frac{w_{0,1}}{w_{p,1}\left(w_{p,1}-tw_{p+1,1}\right)}\right|_{\substack{w_{0,1}\mapsto t^{p}w_{p,1}\\ w_{p+1,1}\mapsto t^{r-1}w_{p,1}}}\prod_{l=1}^{N_p}\frac{\left(w_{p,1}-tx_l^{(p)}\right)}{\left(t^{r-1}w_{p,1}-x_l^{(p)}\right)}$$
$$=\frac{t^p}{1-t^r}\cdot\frac{1}{w_{p,1}}\prod_{l=1}^{N_p}\frac{\left(w_{p,1}-tx_l^{(p)}\right)}{\left(t^{r-1}w_{p,1}-x_l^{(p)}\right)}.$$

The only pole within the unit circle $|w_{p,1}|=1$ is $w_{p,1}=0$. After taking this residue, the final result (after including the front matter) is

$$\frac{t^{p+1+|N_\bullet|}}{1-t^r}f\left[X_{N_\bullet}\right]. \tag{5.26}$$

5.2.2. *The x-poles.* We will first work out the '+' case. Thus, we have taken the residue of $w_{p,1}$ at the pole $w_{p,1}=x_l^{(p)}$ for some $1\le l\le N_p$. This variable $x_l^{(p)}$ will be an element of a shift pattern $\underline{J}$. Therefore, we call it $x_{\underline{J}}^{(p)}$. It will be advantageous to now group together the factors

$$\frac{w_{p,1}w_{p+1,1}}{w_{0,1}w_{p,1}\left(w_{p+1,1}-t^{-1}w_{p,1}\right)}\prod_{l=1}^{N_p}\frac{\left(w_{p+1,1}-t^{-1}x_l^{(p)}\right)}{\left(w_{p,1}-x_l^{(p)}\right)}.$$

After taking the residue, we leave behind

$$\frac{w_{p+1,1}}{w_{0,1}}\prod_{\substack{l=1\\ x_l^{(p)}\neq x_{\underline{J}}^{(p)}}}^{N_p}\frac{\left(w_{p+1,1}-t^{-1}x_1^{(p)}\right)}{\left(x_{\underline{J}}^{(p)}-x_l^{(p)}\right)}.$$

Next, we consider $w_{p+1,1}$. We group together the factors

$$\frac{w_{p+1,1}w_{p+2,1}}{w_{p+1,1}\left(w_{p+2,1}-t^{-1}w_{p+1,1}\right)\underbrace{\left(w_{p+1,1}-q^{-1}w_{p+2,1}\right)}_{(1)}}\prod_{l=1}^{N_{p+1}}\frac{\left(w_{p+2,1}-t^{-1}x_l^{(p+1)}\right)}{\underbrace{\left(w_{p+1,1}-x_l^{(p+1)}\right)}_{(2)}}.$$

The only (nonremovable) poles within the unit circle $|w_{p+1,1}|=1$ are marked (1) and (2). We thus have two cases:

(1) *Residue at* $w_{p+1,1}=q^{-1}w_{p+2,1}$: In this case, $\left(w_{p+2,1}-t^{-1}w_{p+1,1}\right)$ cancels with a $w_{p+2,1}$ in the numerator, leaving behind

$$\left.\frac{1}{\left(1-q^{-1}t^{-1}\right)}\prod_{l=1}^{N_{p+1}}\frac{\left(w_{p+2,1}-t^{-1}x_l^{(p+1)}\right)}{\left(w_{p+1,1}-x_l^{(p+1)}\right)}\right|_{w_{p+1,1}\mapsto q^{-1}w_{p+2,1}}.$$

Because $|q|\gg 1$, the poles above lie outside the unit circle $|w_{p+2,1}|=1$.

(2) *Residue at* $w_{p+1,1}=x_l^{(p+1)}=:x_{\underline{J}}^{(p+1)}$: Here, $\left(w_{p+2,1}-t^{-1}w_{p+1,1}\right)$ cancels with a factor in the numerator, leaving behind

$$\frac{w_{p+2,1}}{\left(x_{\underline{J}}^{(p+1)}-q^{-1}w_{p+2,1}\right)}\prod_{\substack{l=1\\ x_l^{(p+1)}\neq x_{\underline{J}}^{(p+1)}}}^{N_{p+1}}\frac{\left(w_{p+2,1}-t^{-1}x_1^{(p+1)}\right)}{\left(x_{\underline{J}}^{(p+1)}-x_l^{(p+1)}\right)}. \tag{5.27}$$

Again because $|q|\gg 1$, the first pole above lies outside the unit circle $|w_{p+2,1}|=1$.

This pattern and dichotomy for residues continues *upwards* in cyclic order. The x-variables in the type (2) residues constitute a shift pattern $\underline{J}$ and our gap labels $x_{\underline{J}^\vartriangle}^{(i)}$ incorporate the q-shifts from the type (1) residues. Therefore, $w_{i,1}$ is always evaluated at $x_{\underline{J}^\vartriangle}^{(i)}$. Finally, observe that (5.21) becomes $T_{\underline{J}}^{-1}f\left[X_{N_\bullet}\right]$.

The '$-$' case is similar. Our first variable is $w_{p+1,1}$, for which we take the residue at $w_{p+1,1} = x_l^{(p)} =: x_{\underline{J}}^{(p)}$. We consider the factors

$$\left(\frac{w_{0,1}}{w_{p+1,1}}\right)\frac{1}{(w_{p,1}-tw_{p+1,1})}\prod_{l=1}^{N_p}\frac{\left(w_{p,1}-tx_l^{(p)}\right)}{\left(w_{p+1,1}-x_l^{(p)}\right)}.$$

After taking the residue, the pole from $(w_{p,1}-tw_{p+1,1})$ cancels with a factor in the numerator, leaving behind

$$\frac{w_{0,1}}{x_{\underline{J}}^{(p)}}\prod_{\substack{l=1\\ x_l^{(p)}\neq x_{\underline{J}}^{(p)}}}^{N_p}\frac{\left(w_{p,1}-tx_l^{(p)}\right)}{\left(x_{\underline{J}}^{(p)}-x_l^{(p)}\right)}.$$

We now proceed *downward* in cyclic order. For each $w_{i,1}$, we consider the factors

$$\frac{w_{i,1}w_{i-1,1}}{w_{i,1}\,(w_{i-1,1}-tw_{i,1})\underbrace{(w_{i,1}-qw_{i-1,1})}_{(1)}}\prod_{l=1}^{N_{i-1}}\frac{\left(w_{i-1,1}-tx_l^{(i-1)}\right)}{\underbrace{\left(w_{i,1}-x_l^{(i-1)}\right)}_{(2)}}.$$

Because $(w_{i,1}-tw_{i+1,1})$ has been canceled at this point, the only poles within the unit circle $|w_{i,1}|=1$ are those marked (1) and (2). The analysis is as before:

(1) *Residue at* $w_{i,1}=qw_{i-1,1}$: This leaves behind

$$\frac{1}{(1-qt)}\prod_{l=1}^{N_{i-1}}\frac{\left(w_{i-1,1}-tx_l^{(i-1)}\right)}{\left(w_{i,1}-x_l^{(i-1)}\right)}\Bigg|_{w_{i,1}\mapsto qw_{i-1,1}}.$$

(2) *Residue at* $w_{i,1}=x_l^{(i-1)} =: x_{\underline{J}}^{(i-1)}$: The leftovers are now

$$\frac{w_{i-1,1}}{\left(x_{i-1,\underline{J}}-qw_{i-1,1}\right)}\prod_{\substack{l=1\\ x_l^{(i-1)}\neq x_{\underline{J}}^{(i-1)}}}^{N_{i-1}}\frac{\left(w_{i-1,1}-tx_l^{(i-1)}\right)}{\left(x_{\underline{J}}^{(i-1)}-x_l^{(i-1)}\right)}. \tag{5.28}$$

The x-variables where we have taken residues constitute a shift pattern $\underline{J}$ and $w_{i,1}$ is always evaluated at $x_{\underline{J}^\triangledown}^{(i-1)}$. Finally, (5.24) becomes $T_{\underline{J}}f\,[X_{N_\bullet}]$.

5.2.3. *Degree one wreath Macdonald operators.* As seen from 5.2.1, taking residues at t-poles yields a constant times $f\,[X_{N_\bullet}]$. On the other hand, taking residues at x-poles yields a sort of difference operator applied to $f\,[X_{N_\bullet}]$. Notice that in both the '+' and '$-$' cases, the x-poles always gives a shift pattern containing p. We record the resulting shift operators below. Let

$$\begin{aligned}D^*_{p,1}(X_{N_\bullet};q,t^{-1}) &:= \frac{1}{1-q^{-1}t^{-1}}\sum_{\underline{J}\in Sh_p(X_{N_\bullet})}\left(1-q^{-1}t^{-1}\right)^{|J|}\frac{x_{\underline{J}^\vartriangle}^{(p+1)}}{x_{\underline{J}^\vartriangle}^{(0)}}\\ &\times\left(\prod_{i\in I}\prod_{\substack{l=1\\ x_l^{(i)}\neq x_{\underline{J}^\vartriangle}^{(i)}}}^{N_i}\frac{\left(tx_{\underline{J}^\vartriangle}^{(i+1)}-x_l^{(i)}\right)}{\left(x_{\underline{J}^\vartriangle}^{(i)}-x_l^{(i)}\right)}\right)\left(\prod_{i\in J\setminus\{p\}}\frac{qtT_{\underline{J}}^{-1}(x_{\underline{J}}^{(i)})}{\left(x_{\underline{J}}^{(i)}-T_{\underline{J}}^{-1}(x_{\underline{J}}^{(i)})\right)}\right)T_{\underline{J}}^{-1}\\ D_{p,1}(X_{N_\bullet};q,t^{-1}) &:= \frac{1}{1-qt}\sum_{\underline{J}\in Sh_p(X_{N_\bullet})}(1-qt)^{|J|}\frac{x_{\underline{J}^\triangledown}^{(r-1)}}{x_{\underline{J}}^{(p)}}\\ &\times\left(\prod_{i\in I}\prod_{\substack{l=1\\ x_l^{(i)}\neq x_{\underline{J}^\triangledown}^{(i)}}}^{N_i}\frac{\left(t^{-1}x_{\underline{J}^\triangledown}^{(i-1)}-x_l^{(i)}\right)}{\left(x_{\underline{J}^\triangledown}^{(i)}-x_l^{(i)}\right)}\right)\left(\prod_{i\in J\setminus\{p\}}\frac{q^{-1}t^{-1}T_{\underline{J}}(x_{\underline{J}}^{(i)})}{\left(x_{\underline{J}}^{(i)}-T_{\underline{J}}(x_{\underline{J}}^{(i)})\right)}\right)T_{\underline{J}}.\end{aligned}$$

Observe that when $r = 1$, $D_{0,1}(x_{0,\bullet}; q, t)$ and $D^*_{0,1}(x_{0,\bullet}; q, t)$ are the first Macdonald and dual Macdonald operators, respectively.

Theorem 5.3. *For generic values of q, t,*

$$D^*_{p,1}\left(X_{N_\bullet}; q, t\right) P_\lambda[X_{N_\bullet}; q, t] = \left(\sum_{\substack{b=1 \\ b-\lambda_b \equiv p+1}}^{|N_\bullet|} q^{-\lambda_b} t^{-|N_\bullet|+b} \right) P_\lambda[X_{N_\bullet}; q, t] \tag{5.29}$$

$$D_{p,1}\left(X_{N_\bullet}; q, t\right) P_\lambda[X_{N_\bullet}; q, t] = \left(\sum_{\substack{b=1 \\ b-\lambda_b \equiv p+1}}^{|N_\bullet|} q^{\lambda_b} t^{|N_\bullet|-b} \right) P_\lambda[X_{N_\bullet}; q, t]. \tag{5.30}$$

Proof. The difference operators we computed in 5.2.2 are

$$t^{-|N_\bullet|} D^*_{p,1}(X_{N_\bullet}; q, t^{-1})$$

in the '+' case and

$$t^{|N_\bullet|} D_{p,1}(X_{N_\bullet}; q, t^{-1})$$

in the '−' case. We note that after taking residues, the w-variables in the first halves of (5.27) and (5.28) can be rewritten using (5.18).

Combining Lemma 3.10, Theorem 4.4, (5.25), (5.26), and the computations of 5.2.2, we have for $\lambda \in \mathbb{Y}$ with $\kappa(\lambda) = \alpha$ and $|\mathrm{quot}(\lambda)| \le |N_\bullet|$,

$$\left(t^{-|N_\bullet|} D^*_{p,1}(X_{N_\bullet}; q, t^{-1}) + \frac{t^{-p-1-|N_\bullet|}}{1 - t^{-r}} \right) P_\lambda\left[X_{N_\bullet}; q, t^{-1}\right] = \left(\sum_{\substack{b>0 \\ b-\lambda_b \equiv p+1}} q^{-\lambda_b} t^{-b} \right) P_\lambda[X_{N_\bullet}; q, t^{-1}]$$

where we assume $|q| \gg 1$, $|t| \gg 1$, and $|x_{i,l}| < 1$. Even here, it is essential that $|t| \gg 1$ as we are working with series in t^{-1}. We can do away with this once we notice that since $|N_\bullet|$ is divisible by r (Proposition 2.7) and $\ell(\lambda) \le |N_\bullet|$,

$$\left(\sum_{\substack{b>0 \\ b-\lambda_b \equiv p+1}} q^{-\lambda_b} t^{-b} \right) = \frac{t^{-p-1-|N_\bullet|}}{1 - t^{-r}} + \left(\sum_{\substack{b=1 \\ b-\lambda_b \equiv p+1}}^{|N_\bullet|} q^{-\lambda_b} t^{-b} \right).$$

Thus (5.30) holds under our conditions on $|q|$, $|t|$, and $|x_l^{(i)}|$.

Finally, we address the genericity of parameters. The equations (5.29) and (5.30) are equalities of rational functions in the space $(X_{N_\bullet}, q, t)$. We have established them over an analytic open subset of $(X_{N_\bullet}, q, t)$. After subtracting one side to the other, this is equivalent saying a rational function is zero on a codimension zero subspace, and thus it must be zero. □

The eigenvalues of $\{D_{p,1}(X_{N_\bullet}; q, t)\}_{p\in I}$ on $\{P_\lambda[X_{N_\bullet}; q, t]\}$ are nondegenerate. Therefore, we have

Corollary 5.4. *For λ with core $\kappa(\lambda)$ compatible with $N_\bullet$ (cf. 2.9), the line spanned by $P_\lambda[X_{N_\bullet}; q, t]$ is characterized by the eigenfunction equations (5.29) ranging over all $p \in I$.*

Example 5.5. Let $r = 1$, $p = 1$, $N_\bullet = (2, 1, 0)$, and $\lambda = (3, 1, 1)$. In this case, λ is a 3-core and so

$$P_\lambda[X_{N_\bullet}; q, t] = 1.$$

There are three shift patterns containing 1:

$$\underline{J}_1 = \{x_1^{(1)}\}$$
$$\underline{J}_2 = \{x_1^{(0)}, x_1^{(1)}\}$$
$$\underline{J}_3 = \{x_2^{(0)}, x_1^{(1)}\}$$

The operator $D_{1,1}(X_{N_\bullet};q,t)$ is then

$$D_{1,1}(X_{N_\bullet};q,t) = q\left(\frac{qtx_1^{(1)}-x_1^{(0)}}{q^2x_1^{(1)}-x_1^{(0)}}\right)\left(\frac{qtx_1^{(1)}-x_2^{(0)}}{q^2x_1^{(1)}-x_2^{(0)}}\right)T_{\underline{J}_1} \tag{5.31}$$

$$+(1-qt^{-1})q\left(\frac{qtx_1^{(1)}-x_2^{(0)}}{x_1^{(0)}-x_2^{(0)}}\right)\left(\frac{qtx_1^{(1)}}{x_1^{(0)}-q^2x_1^{(1)}}\right)T_{\underline{J}_2} \tag{5.32}$$

$$+(1-qt^{-1})q\left(\frac{qtx_1^{(1)}-x_1^{(0)}}{x_2^{(0)}-x_1^{(0)}}\right)\left(\frac{qtx_1^{(1)}}{x_2^{(0)}-q^2x_1^{(1)}}\right)T_{\underline{J}_3}. \tag{5.33}$$

The cyclic-shift operators act trivially on $P_\lambda(X_{N_\bullet};q,t)$. Consolidating (5.32) and (5.33) gets us

$$\begin{aligned}
&(1-qt^{-1})q\left\{\left(\frac{qtx_1^{(1)}-x_2^{(0)}}{x_1^{(0)}-x_2^{(0)}}\right)\left(\frac{qtx_1^{(1)}}{x_1^{(0)}-q^2x_1^{(1)}}\right)+\left(\frac{qtx_1^{(1)}-x_1^{(0)}}{x_2^{(0)}-x_1^{(0)}}\right)\left(\frac{qtx_1^{(1)}}{x_2^{(0)}-q^2x_1^{(1)}}\right)\right\}\\
&=(1-qt^{-1})q\left\{\frac{qtx_1^{(1)}\left(qtx^{(1)}x_2^{(0)}-qtx_1^{(1)}x_1^{(0)}-x_2^{(0)}x_2^{(0)}+x_1^{(0)}x_1^{(0)}+q^2x_2^{(0)}x_1^{(1)}-q^2x_1^{(0)}x_1^{(1)}\right)}{(x_1^{(0)}-x_2^{(0)})(x_1^{(0)}-q^2x_1^{(1)})(x_2^{(0)}-q^2x_1^{(1)})}\right\}\\
&=(1-qt^{-1})q\left\{\frac{qtx_1^{(1)}\left(-qtx_1^{(1)}+x_1^{(0)}+x_2^{(0)}-q^2x_1^{(1)}\right)}{(x_1^{(0)}-q^2x_1^{(1)})(x_2^{(0)}-q^2x_1^{(1)})}\right\}\\
&=(1-qt^{-1})q\left\{\frac{-(q^2t^2+q^3t)x_1^{(1)}x_1^{(1)}+qtx_1^{(0)}x_1^{(1)}+qtx_2^{(0)}x_1^{(1)}}{(x_1^{(0)}-q^2x_1^{(1)})(x_2^{(0)}-q^2x_1^{(1)})}\right\}\\
&=q\left\{\frac{(-q^2t^2+q^4)x_1^{(1)}x_1^{(1)}+(qt-q^2)x_1^{(0)}x_1^{(1)}+(qt-q^2)x_2^{(0)}x_1^{(1)}}{(x_1^{(0)}-q^2x_1^{(1)})(x_2^{(0)}-q^2x_1^{(1)})}\right\}. &(5.34)
\end{aligned}$$

On the other hand, (5.31) becomes

$$\begin{aligned}
&q\left(\frac{qtx_1^{(1)}-x_1^{(0)}}{q^2x_1^{(1)}-x_1^{(0)}}\right)\left(\frac{qtx_1^{(1)}-x_2^{(0)}}{q^2x_1^{(1)}-x_2^{(0)}}\right)\\
&=q\left\{\frac{q^2t^2x_1^{(1)}x_1^{(1)}-qtx_1^{(1)}x_2^{(0)}-qtx_1^{(0)}x_1^{(1)}+x_1^{(0)}x_2^{(0)}}{(x_1^{(0)}-q^2x_1^{(1)})(x_2^{(0)}-q^2x_1^{(1)})}\right\}. &(5.35)
\end{aligned}$$

Combining (5.34) and (5.35) gets us

$$\begin{aligned}
D_{1,1}(X_{N_\bullet};q,t)P_\lambda[X_{N_\bullet};q,t] &= q\left\{\frac{q^4x_1^{(1)}x_1^{(1)}-q^2x_1^{(1)}x_2^{(0)}-q^2x_1^{(0)}x_1^{(1)}+x_1^{(0)}x_2^{(0)}}{(x_1^{(0)}-q^2x_1^{(1)})(x_2^{(0)}-q^2x_1^{(1)})}\right\}\\
&= q\frac{(x_1^{(0)}-q^2x_1^{(1)})(x_2^{(0)}-q^2x_1^{(1)})}{(x_1^{(0)}-q^2x_1^{(1)})(x_2^{(0)}-q^2x_1^{(1)})}\\
&= qP_\lambda[X_{N_\bullet};q,t].
\end{aligned}$$

Example 5.6. Let $r=2$, $p=0$, $N_\bullet=(1,1)$, and $\lambda=(1,1)$. Here,

$$P_\lambda[X_{N_\bullet};q,t]=x_1^{(1)}.$$

There are two shift patterns containing 0:

$$\begin{aligned}
\underline{J}_1 &= \{x_1^{(0)}\}\\
\underline{J}_2 &= \{x_1^{(0)},x_1^{(1)}\}
\end{aligned}$$

We then have

$$D_{0,1}(X_{N_\bullet};q,t)=q\left(\frac{tx_1^{(0)}-x_1^{(1)}}{qx_1^{(0)}-x_1^{(1)}}\right)T_{\underline{J}_1}+(1-qt^{-1})\frac{x_1^{(1)}}{x_1^{(0)}}\left(\frac{tx_1^{(0)}}{x_1^{(1)}-qx_1^{(0)}}\right)T_{\underline{J}_2}.$$

Observe that

$$T_{\underline{J}_1} x_1^{(1)} = x_1^{(1)}$$
$$T_{\underline{J}_2} x_1^{(1)} = q x_1^{(0)}.$$

Altogether then,

$$\begin{aligned} D_{0,1}(X_{N_\bullet}; q,t) P_\lambda[X_{N_\bullet}; q,t] &= q\left(\frac{t x_1^{(0)} - x_1^{(1)}}{q x_1^{(0)} - x_1^{(1)}}\right) x_1^{(1)} + (1 - qt^{-1}) \frac{x_1^{(1)}}{x_1^{(0)}} \left(\frac{t x_1^{(0)}}{x_1^{(1)} - q x_1^{(0)}}\right) q x_1^{(0)} \\ &= q x_1^{(1)} \left(\frac{t x_1^{(0)} - x_1^{(1)} - (t-q) x_1^{(0)}}{q x_1^{(0)} - x_1^{(1)}}\right) \\ &= q x_1^{(1)} \left(\frac{q x_1^{(0)} - x_1^{(1)}}{q x_1^{(0)} - x_1^{(1)}}\right) \\ &= q P_\lambda[X_{N_\bullet}; q,t]. \end{aligned}$$

5.3. **Higher degree operators.** Now we consider general values of n. In the '+' case, we will start with taking the constant terms of the p-colored variables $\{w_{p,\bullet}\}$. There are two kinds of poles inside the unit circle $|w_{p,b}| = 1$:

(x) the poles $\left(w_{p,b} - x_l^{(p)}\right)$ in (5.8) and

(t) the poles $\left(w_{p,b} - t^{-1} w_{p-1,a}\right)$ for $a \le b$ in (5.10) and (5.12).

As in 5.2, we call them x- and t-poles, respectively. We note that evaluating two variables $w_{p,b}$ and $w_{p,b'}$ at the same pole will result in zero due to the factor $(w_{p,b} - w_{p,b'})$ in (5.10). Besides that, for $r > 1$, these residues can be evaluated independently and we elect to do so. For the '−' case, we instead start with $\{w_{p+1,\bullet}\}$, for which the relevant poles are now

(x) $\left(w_{p+1,a} - x_l^{(p)}\right)$ in (5.14) and

(t) $(w_{p+1,a} - t w_{p+2,b})$ for $a \le b$ in (5.16) and (5.17).

As in 5.2, we will further restrict our parameters to $|q|, |t| \gg 1$ in the '+' case and $|q|, |t| \ll 1$ in the '−' case.

5.3.1. *Only x-poles.* As in 5.2, the case where we only start with x-poles yields a new difference operator. In both the '+' and '−' cases, each of the n variables $\{x_{l_a}^{(p)}\}_{a=1}^n$ will become part of a shift pattern containing p, so we set $x_{\underline{J}_a}^{(p)} := x_{l_a}^{(p)}$. Furthermore, as these variables must be distinct, we have that the tuple $\underline{\mathbf{J}} := (\underline{J}_1, \dots, \underline{J}_n)$ will be p-distinct. After taking these residues, we will proceed as in 5.2.2 for a specific value of a.

First consider the '+' case. To see the effect of taking the residues $w_{p,b} = x_{\underline{J}_b}^{(p)}$, we group together the factors

$$\left(\frac{w_{p,b}}{w_{0,b}}\right) \frac{w_{p+1,b}}{w_{p,b}\left(w_{p+1,b} - t^{-1} w_{p,b}\right)} \prod_{l=1}^{N_p} \frac{\left(w_{p+1,b} - t^{-1} x_l^{(p)}\right)}{\left(w_{p,b} - x_l^{(p)}\right)}$$
$$\times \frac{1}{\prod_{b<c} \left(w_{p+1,b} - t^{-1} w_{p,c}\right)} \prod_{a<b} \frac{\left(w_{p,b} - w_{p,a}\right)\left(w_{p+1,b} - q^{-1} t^{-1} w_{p+1,a}\right)}{\left(w_{p+1,a} - t^{-1} w_{p,b}\right)}.$$

Upon taking residues, this becomes

$$\left(\frac{w_{p+1,b}}{w_{0,b}}\right) \frac{\displaystyle\prod_{\substack{l=1 \\ x_l^{(p)} \notin |\underline{\mathbf{J}}|}}^{N_p} \left(w_{p+1,b} - t^{-1} x_l^{(p)}\right)}{\displaystyle\prod_{\substack{l=1 \\ x_l^{(p)} \notin |\underline{\mathbf{J}}|_{\le b}}}^{N_p} \left(x_{\underline{J}_b}^{(p)} - x_l^{(p)}\right)} \underbrace{\prod_{a<b} \left(w_{p+1,b} - q^{-1} t^{-1} w_{p+1,a}\right)}_{(\dagger)}. \tag{5.36}$$

The next variable we consider is $w_{p+1,1}$. Notice that we have canceled the poles $\left(w_{p+1,1} - t^{-1}w_{p,b}\right)$ for all $b \geq 1$, and consequently, the only two kinds of poles within the unit circle $|w_{p+1,1}| = 1$ are as before in 5.2.2. We group together the factors

$$\frac{w_{p+1,1}w_{p+2,1}}{w_{p+1,1}\left(w_{p+2,1} - t^{-1}w_{p+1,1}\right)\left(w_{p+1,1} - q^{-1}w_{p+2,1}\right)} \times \underbrace{\prod_{1<b} \frac{\left(w_{p+1,b} - w_{p+1,1}\right)\left(w_{p+2,b} - q^{-1}t^{-1}w_{p+2,1}\right)}{\left(w_{p+2,b} - t^{-1}w_{p+1,1}\right)\left(w_{p+1,b} - q^{-1}w_{p+2,1}\right)}}_{(*)} \underbrace{\prod_{b=1}^{n}\prod_{l=1}^{N_{p+1}} \frac{\left(w_{p+2,b} - t^{-1}x_l^{(p+1)}\right)}{\left(w_{p+1,b} - x_l^{(p+1)}\right)}}_{(**)}.$$

The residues are

(1) *Residue at* $w_{p+1,1} = q^{-1}w_{p+2,1}$: In this case, the factors in $(*)$ cancel out, leaving behind

$$\frac{1}{(1 - q^{-1}t^{-1})} \prod_{b=1}^{n}\prod_{l=1}^{N_{p+1}} \frac{\left(w_{p+2,b} - t^{-1}x_l^{(p+1)}\right)}{\left(w_{p+1,b} - x_l^{(p+1)}\right)}\Bigg|_{w_{p+1,1}\mapsto q^{-1}w_{p+2,1}}.$$

As in 5.2.2, $w_{p+1,1}$ will ultimately be evaluated at $x^{(p+1)}_{\underline{J}_1^{\Delta}}$ and the poles above lie outside the unit circle $|w_{p+2,1}| = 1$ because $|q| \gg 1$.

(2) *Residue at* $w_{p+1,1} = x_l^{(p+1)} =: x^{(p+1)}_{\underline{J}_1}$: Here, the factors in $(*)$ cancel with those in $(**)$ containing $x^{(p+1)}_{\underline{J}_1}$. We are left with

$$\frac{w_{p+2,1}}{\left(x^{(p+1)}_{\underline{J}_1} - q^{-1}w_{p+2,1}\right)} \prod_{\substack{l=1\\ x_l^{(p+1)} \neq x^{(p+1)}_{\underline{J}_1}}}^{N_{p+1}} \frac{\left(w_{p+2,1} - t^{-1}x_l^{(p+1)}\right)}{\left(x_{p+1,\underline{J}_1} - x_l^{(p+1)}\right)}$$
$$\times \prod_{1<b}\left\{\frac{\left(w_{p+2,b} - q^{-1}t^{-1}w_{p+2,1}\right)}{\left(w_{p+1,b} - q^{-1}w_{p+2,1}\right)} \prod_{\substack{l=1\\ x_l^{(p+1)} \neq x^{(p+1)}_{\underline{J}_1}}}^{N_{p+1}} \frac{\left(w_{p+2,b} - t^{-1}x_l^{(p+1)}\right)}{\left(w_{p+1,b} - x_l^{(p+1)}\right)}\right\}. \tag{5.37}$$

Because $|q| \gg 1$, the pole $\left(x^{(p+1)}_{\underline{J}_1} - q^{-1}w_{p+2,1}\right)$ lies outside the unit circle $|w_{p+2,1}| = 1$. Our key organizational trick here is that when $w_{p+2,1}$ is ultimately evaluated at $x^{(p+2)}_{\underline{J}_1^{\Delta}}$, then we can use (5.18) to write (5.37) as

$$T^{-1}_{\underline{J}_1}\left(\prod_{1<b}^{n}\prod_{l=1}^{N_{p+1}} \frac{\left(w_{p+2,b} - t^{-1}x_l^{(p+1)}\right)}{\left(w_{p+1,b} - x_l^{(p+1)}\right)}\right)$$

since $T_{\underline{J}_1}$ will only affect $x^{(p+1)}_{\underline{J}_1}$.

This pattern continues *upwards* in cyclic order for the variables $w_{i,1}$. The x-variables where we take residues gives a shift pattern $\underline{J}_1$ containing p and $w_{i,1}$ is evaluated at $x^{(i)}_{\underline{J}_1^{\Delta}}$. In (5.36), the term in (†) for $a = 1$ can be rewritten as $\left(w_{p+1,b} - t^{-1}T^{-1}_{\underline{J}_1}x^{(p)}_{\underline{J}_1}\right)$ Finally, we note that these residues result in $T^{-1}_{\underline{J}_1}$ applied to $f\left[X_{N_\bullet}\right]$. Thus, we can rewrite the result after taking the residues for $a=1$ as:

$$\frac{(-1)^{\frac{n(n-1)}{2}}t^{-\frac{n(n+1)}{2}}(1-q^{-1}t^{-1})^{r(n-1)}}{\prod_{a=1}^{n}(1-q^{-a}t^{-a})}\sum_{\underline{J}_1\in Sh_p}(1-q^{-1}t^{-1})^{|\underline{J}_1|}\left(\frac{x^{(p+1)}_{\underline{J}_1^\Delta}}{x^{(0)}_{\underline{J}_1^\Delta}}\right)$$

$$\times\frac{\prod_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|_{\geq 1}}}^{N_p}\left(x^{(p+1)}_{\underline{J}_1^\Delta}-t^{-1}x_l^{(p)}\right)}{\prod_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|_{\leq 1}}}^{N_p}\left(x^{(p)}_{\underline{J}_1}-x_l^{(p)}\right)}\left(\prod_{\substack{i\in I\\ i\neq p}}\prod_{\substack{l=1\\ x_l^{(i)}\neq x^{(i)}_{\underline{J}_1^\Delta}}}^{N_i}\frac{\left(x^{(i+1)}_{\underline{J}_1^\Delta}-t^{-1}x_l^{(i)}\right)}{\left(x^{(i)}_{\underline{J}_1^\Delta}-x_l^{(i)}\right)}\right)\left(\prod_{i\in J_1\setminus\{p\}}\frac{qT^{-1}_{\underline{J}_1}x^{(i)}_{\underline{J}_1}}{\left(x^{(i)}_{\underline{J}_1}-T^{-1}_{\underline{J}_1}x^{(i)}_{\underline{J}_1}\right)}\right)$$

$$\times\oint_{|w_{i,a}|=1}\cdots\oint T^{-1}_{\underline{J}_1}\left(\prod_{a=1}^{n}\left\{\left(w_{p+1,a}-t^{-1}x^{(p)}_{\underline{J}_1}\right)\prod_{1<a<b\leq n}\left(w_{p+1,b}-q^{-1}t^{-1}w_a^{(p+1)}\right)\right.\right.$$

$$\left.\times\frac{\prod_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|}}^{N_p}\left(w_{p+1,a}-t^{-1}x_l^{(p)}\right)}{\prod_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|_{\leq a}}}^{N_p}\left(x^{(p)}_{\underline{J}_a}-x_l^{(p)}\right)}\prod_{i\in I\setminus\{p\}}\prod_{l=1}^{N_i}\frac{\left(w_{i+1,a}-t^{-1}x_l^{(i)}\right)}{\left(w_{i,a}-x_l^{(i)}\right)}\right\}$$

$$\times\prod_{1<a<b\leq n}\prod_{i\in I\setminus\{p\}}\frac{(w_{i,b}-w_{i,a})(w_{i+1,b}-q^{-1}t^{-1}w_{i+1,a})}{(w_{i+1,b}-t^{-1}w_{i,a})(w_{i,b}-q^{-1}w_{i+1,a})}$$

$$\times\prod_{a=2}^{n}\left\{\left(\frac{w_{p+1,a}}{w_{0,a}}\right)\prod_{i\in I\setminus\{p\}}\frac{w_{i,a}w_{i+1,a}}{(w_{i+1,a}-t^{-1}w_{i,a})\,(w_{i,a}-q^{-1}w_{i+1,a})}\right\}$$

$$\left.\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}-\sum_{a=2}^{n}w_{i,a}+\sum_{a=2}^{n}q^{-1}w_{i+1,a}\right]\right)\prod_{i\in I\setminus\{p\}}\prod_{a=2}^{n}\frac{dw_{i,a}}{2\pi\sqrt{-1}w_{i,a}}.$$

We have written this so that we can repeat the calculation for $a=1$ for general a in increasing order. Note that as we do this, we can rewrite factors in (†) of (5.36) in terms of $T^{-1}_{\underline{J}_a}x_{p,\underline{J}_a}$ using (5.18). It is cleaner to present the end result as a sum of compositions of operators. Namely, let

$$D^*_{p,n}(x_{\bullet,\bullet};q,t^{-1}):=\frac{(-1)^{\frac{n(n-1)}{2}}}{\prod_{k=1}^{n}(1-q^{-k}t^{-k})}$$

$$\times\sum_{\underline{\mathbf{J}}\in Sh_p^{[n]}(X_{N_\bullet})}\overset{\curvearrowleft}{\prod_{a=1}^{n}}\left\{(1-q^{-1}t^{-1})^{|\underline{J}_a|}\left(\frac{x^{(p+1)}_{\underline{J}_a^\Delta}}{x^{(0)}_{\underline{J}_a^\Delta}}\right)\frac{\prod_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|_{\geq a}}}^{N_p}\left(tx^{(p+1)}_{\underline{J}_a^\Delta}-x_l^{(p)}\right)}{\prod_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|_{\leq a}}}^{N_p}\left(x^{(p)}_{\underline{J}_a}-x_l^{(p)}\right)}\right.\tag{5.38}$$

$$\left.\times\left(\prod_{\substack{i\in I\\ i\neq p}}\prod_{\substack{l=1\\ x_l^{(i)}\neq x^{(i)}_{\underline{J}_a^\Delta}}}^{N_i}\frac{\left(tx^{(i+1)}_{\underline{J}_a^\Delta}-x_l^{(i)}\right)}{\left(x^{(i)}_{\underline{J}_a^\Delta}-x_l^{(i)}\right)}\right)\left(\prod_{i\in J_a\setminus\{p\}}\frac{qtT^{-1}_{\underline{J}_a}(x^{(i)}_{\underline{J}_a})}{\left(x^{(i)}_{\underline{J}_a}-T^{-1}_{\underline{J}_a}(x^{(i)}_{\underline{J}_a})\right)}\right)T^{-1}_{\underline{J}_a}\right\}.$$

Here, recall our notation for ordered products/compositions (3.9).

The end result of the residue calculation is

$$t^{-n|N_\bullet|}D^*_{p,n}(X_{N_\bullet};q,t^{-1})f\left[X_{N_\bullet}\right].$$

The '$-$' case is similar. We begin by taking residues of $\{w_{p+1,\bullet}\}$ and then start instead at $x_n^{(p)}$. Afterwards, we continue *downwards* in cyclic order until we have taken constant terms of all variables with $a=n$. We then continue *downwards* in a. Let

$$D_{p,n}(X_{N_\bullet};q,t^{-1}) := \frac{(-1)^{\frac{n(n-1)}{2}}}{\prod_{k=1}^n(1-q^kt^k)} \times \sum_{\underline{\mathbf{J}}\in Sh_p^{[n]}(X_{N_\bullet})} \prod_{a=1}^{\overset{\curvearrowright}{n}} \left\{ (1-qt)^{|\underline{J}_a|}\left(\frac{x^{(r-1)}_{\underline{J}_a^\triangledown}}{x^{(p)}_{\underline{J}_a}}\right) \frac{\prod\limits_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|_{\geq a}}}^{N_p}\left(t^{-1}x^{(p-1)}_{\underline{J}_a^\triangledown}-x_l^{(p)}\right)}{\prod\limits_{\substack{l=1\\ x_l^{(p)}\notin|\underline{\mathbf{J}}|_{\leq a}}}^{N_p}\left(x^{(p)}_{\underline{J}_a}-x_l^{(p)}\right)} \right. \tag{5.39}$$
$$\left. \times\left(\prod_{\substack{i\in I\\ i\neq p}}\prod_{\substack{l=1\\ x_l^{(i)}\neq x^{(i)}_{\underline{J}_a^\triangledown}}}^{N_i}\frac{\left(t^{-1}x^{(i-1)}_{\underline{J}_a^\triangledown}-x_l^{(i)}\right)}{\left(x^{(i)}_{\underline{J}_a^\triangledown}-x_l^{(i)}\right)}\right)\left(\prod_{i\in J_a\setminus\{p\}}\frac{q^{-1}t^{-1}T_{\underline{J}_a}(x^{(i)}_{\underline{J}_a})}{\left(x_{i,\underline{J}_a}-T_{\underline{J}_a}(x^{(i)}_{\underline{J}_a})\right)}\right)T_{\underline{J}_a}\right\}.$$

The end result is then

$$t^{n|N_\bullet|}D_{p,n}(X_{N_\bullet};q,t^{-1})f\left[X_{N_\bullet}\right].$$

Remark 5.7. In contrast with the $n=1$ case, it is less obvious that these yield the higher order Macdonald operators with t inverted when $r=1$. When $r=1$, note that our sum is over *ordered* n-tuples of distinct shift operators, whereas the usual formula for the nth Macdonald operator is over *unordered* n-tuples. Summing over the orderings for a given n-tuple, the numerator will contain a factor that is antisymmetric, while the denominator will contain a Vandermonde determinant. The quotient of these two will yield $(-t)^{\pm\frac{n(n-1)}{2}}$ times the $(qt)^{\mp 1}$-generating function of lengths of elements in $\mathfrak{S}_n$. After consolidating all constants, one is indeed left with the nth Macdonald operator.

5.3.2. *Mixed poles.* In the case where there are t-poles, our goal is to show that the result is a linear combination of the lower order operators applied to $f\left[X_{N_\bullet}\right]$: $D^*_{p,k}(X_{N_\bullet};q,t^{-1})$ in the '$+$' case and $D_{p,k}(X_{N_\bullet};q,t^{-1})$ in the '$-$' case, where $k<n$. Unlike in the case of $n=1$, we will not try to compute the coefficients of this linear combination—we will compute them indirectly in 5.4. As in all the previous cases, the initial residues force a string of other residues, and we will first compute these strings that start from the initial t-poles. Once these variables are evaluated, the remaining terms will evaluate like 5.3.1.

In the '$+$' case, let $1\leq b_1^p\leq n$ be any index where the residue for w_{p,b_1^p} is taken at a t-pole. Denote this pole by $w_{p,b_1^p}=t^{-1}w_{p-1,b_1^{p-1}}$. In contrast to our previous calculations, we will not always cancel out factors but rather remark on why taking residues at certain poles will result in zero. The poles contributing within the unit circle $|w_{p-1,b_1^{p-1}}|=1$ are as follows.

(1) $(w_{p-1,b_1^{p-1}}-q^{-1}w_{p,a})$ *for* $a\geq b_1^{p-1}$: If $a<b_1^p$, then the factor $(w_{p,b_1^p}-q^{-1}t^{-1}w_{p,a})$ in the numerator of (5.10) becomes zero when taking this residue. If $a=b_1^{p-1}=b_1^p$, then this is a pole at 0, which cancels with the extra factor of $w_{p-1,b_1^{p-1}}$ as in 5.2.1.
(2) $(w_{p-1,b_1^{p-1}}-x_l^{(p-1)})$: The factor $(w_{p,b_1^p}-t^{-1}x_l^{(p-1)})$ in the numerator of (5.8) will evaluate to zero.
(3) $(w_{p-1,b_1^{p-1}}-t^{-1}w_{p-2,a})$ *for* $a\leq b_1^{p-1}$: These poles possibly yield nonzero residues.

Taking a residue of the third kind, we evaluate $w_{p-1,b_1^{p-1}}=t^{-1}w_{p-2,b_1^{p-2}}$ for some $b_1^{p-2}\leq b_1^{p-1}$.

This pattern continues *downwards* in cyclic order, picking out variables w_{i,b_1^i} where $b_1^p\geq b_1^{p-1}\geq\cdots\geq b_1^{p+1}$. At $w_{p+1,b_1^{p+1}}$, the pole of type (3) becomes

(3') $(w_{p+1,b_1^{p+1}} - t^{-1}w_{p,a})$ *for all* a: If $w_{p,a}$ is evaluated at an x-variable $x_l^{(p)}$, then as in 5.3.1, the factor $(w_{p+1,b_1^{p+1}} - t^{-1}x_l^{(p)})$ will evaluate to zero upon taking this residue. Thus, only the case where $w_{p,a}$ is evaluated at a t-pole yields a nonzero residue. For $a = b_1^p$, this is a pole at $w_{p+1,b_1^{p+1}} = 0$. If $b_1^{p+2} = b_1^{p+1}$, then because of the analogue of case (1), there are no extra powers of $w_{p+1,b_1^{p+1}}$ to cancel this pole.

If we take the residue in (3') at $w_{p,a}$ evaluated at a t-pole but $a \neq b_1^p$, then we set $b_2^p := a$. Letting the t-pole be $(w_{p,b_2^p} - t^{-1}w_{p-1,b_2^{p-1}})$ for $b_2^{p-1} \leq b_2^p$, the process is similar to as before. There is just one alteration to the poles of type (3):

(3") $(w_{i,b_2^i} - t^{-1}w_{i-1,b_1^{i-1}})$: This is a pole at 0, which cancels with the factor $\pm(w_{i,b_2^i} - w_{i,b_1^i})$ in the numerator of (5.9).

Thus, we avoid variables that we have already evaluated. Note that at first glance, the product of factors in (5.9) and (5.10) involving w_{i,b_2^i} and w_{j,b_1^j} may contribute a pole at 0, but in fact, their products have total degree zero and thus become a constant. There is an outlier case of $(w_{p+1,b_1^{p+1}} - t^{-1}w_{p,b_2^p})$, which has been removed when we take residues, but this can be replaced with $(w_{p+1,b_1^{p+1}} - t^{-1}w_{p,b_1^p})$ to restore the degree zero balance. We continue like this to new indices $\{b_3^i\}_{i\in I}, \{b_4^i\}_{i\in I}$, etc. until either there are no more nonzero residues or we finally take the residue at 0 of $z_{p+1,b_k^{p+1}}$ for some final value k.

For $1 \leq m < m' \leq k$, we note that as in the $(m,m') = (1,2)$ case, the product of the binomials in (5.9) and (5.10) involving one variable from $\{w_{i,b_m^i}\}_{i\in I}$ and another variable from $\{w_{j,b_{m'}^j}\}_{i\in I}$ has degree zero provided we make the same adjustment for $i = p+1$ and $m' = m+1$. Thus, these factors turn into a constant. To consider binomials involving only $\{w_{i,b_m^i}\}_{i\in I}$ for one value of m, we note that when we take the residues, we remove

$$\frac{1}{w_{i,b_m^i} - t^{-1}w_{i+1,b_m^{i+1}}} \qquad \text{for } i \neq p+1,$$

$$\frac{1}{w_{p+1,b_m^{p+1}} - t^{-1}w_{p,b_{m+1}^p}} \qquad \text{for } 1 \leq m < k.$$

There is a leftover power of w_{i,b_m^i} for $i \neq p$ from (5.11) and (5.12), and as discussed in the pole of type (1) above, these are only absorbed when $b_m^{i+1} = b_m^i$. These unabsorbed powers turn the entire integral zero when we take the final residue $w_{p+1,b_k^{p+1}} = 0$. Thus, we only need to consider the case where for each m,

$$b_m^p = b_m^{p-1} = \cdots = b_m^{p+1} =: b_m.$$

In this case, all factors only involving $\{w_{i,b_m}\}_{i\in I}^{1\leq m\leq k}$ leave behind a constant. Evidently, the corresponding terms in (5.11) and (5.12) disappear. The terms involving w_{i,b_m} and an x-variable in (5.8) leave behind a power of t when we cancel

$$\prod_{m=1}^{k}\prod_{i\in I}\prod_{l=1}^{N_i} \frac{\left(w_{i+1,b_m} - t^{-1}x_l^{(i)}\right)}{\left(w_{i,b_m} - x_l^{(i)}\right)}.$$

Finally, the product of terms in (5.9) involving any index $1 \leq a \leq n$ and b_m leave behind a constant when we evaluate $w_{i,b_m} = 0$ for all $i \in I$. The remaining factors are a scalar multiple of the calculation for $\mathcal{E}_{p,n-k}$.

The '$-$' case is analyzed similarly. We summarize our findings from this subsection with the following:

Lemma 5.8. *Assuming* $|x_l^{(i)}| < 1$ *and* $|q|, |t| \gg 1$, *we have*

$$\pi_{N_\bullet}\left((\rho_{\vec{c}}^+ \circ \Psi_+)(c_n^+\mathcal{E}_{p,n}^+)f\right) = \left(t^{-n|N_\bullet|}D_{p,n}^*(X_{N_\bullet}; q, t^{-1}) + \sum_{k=0}^{n-1} c_{p,k,n}^+ D_{p,k}^*(X_{N_\bullet}; q, t^{-1})\right) f\left[X_{N_\bullet}\right]$$

for some $c_{p,k,n}^+ \in \mathbb{C}(q,t)$. *If we assume instead* $|q|, |t| \ll 1$, *we have*

$$\pi_{N_\bullet}\left((\rho_{\vec{c}}^- \circ \Psi_-)(c_n^-\mathcal{E}_{p,n}^-)f\right) = \left(t^{n|N_\bullet|}D_{p,n}(X_{N_\bullet}; q, t^{-1}) + \sum_{k=0}^{n-1} c_{p,k,n}^- D_{p,k}(X_{N_\bullet}; q, t^{-1})\right) f\left[X_{N_\bullet}\right]$$

for some $c_{p,k,n}^- \in \mathbb{C}(q,t)$.

5.4. **Eigenvalues.** To describe the eigenvalues of the operators (5.38) and (5.39), we will use the elementary symmetric functions e_k. As in the proof of Theorem 5.3, Lemma 5.8 gives us

$$\begin{aligned}(5.40)\qquad &\left(t^{-n|N_\bullet|}D^*_{p,n}(X_{N_\bullet};q,t^{-1})+\sum_{k=0}^{n-1}c^+_{p,k,n}D^*_{p,k}(X_{N_\bullet};q,t^{-1})\right)P_\lambda[X_{N_\bullet};q,t]\\ &=e_n\left[\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1}}^{\infty}q^{-\lambda_b}t^{-b}\right]P_\lambda[X_{N_\bullet};q,t^{-1}]\end{aligned}$$

for $|x_l^{(i)}|<1$, $|q|\gg 1$, and $|t|\gg 1$ and

$$\begin{aligned}(5.41)\qquad &\left(t^{n|N_\bullet|}D_{p,n}(X_{N_\bullet};q,t^{-1})+\sum_{k=0}^{n-1}c^-_{p,k,n}D_{p,k}(X_{N_\bullet};q,t^{-1})\right)P_\lambda[X_{N_\bullet};q,t^{-1}]\\ &=e_n\left[\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1}}^{\infty}q^{\lambda_b}t^{b}\right]P_\lambda[X_{N_\bullet};q,t^{-1}]\end{aligned}$$

for $|x_l^{(i)}|<1$, $|q|\ll 1$, and $|t|\ll 1$. Using induction starting with the case $n=1$ from Theorem 5.3, we have that $D_{p,n}(X_{N_\bullet};q,t)$ and $D^*_{p,n}(X_{N_\bullet};q,t)$ act diagonally on $P_\lambda[X_{N_\bullet};q,t]$ under the appropriate conditions on variables and parameters. Our goal here is to extract their eigenvalues from (5.40) and (5.41) and extend their validity to generic values.

5.4.1. *Spectral variables.* Letting λ vary over partitions with $\mathrm{core}(\lambda)$ compatible with $N_\bullet$ and $\ell(\lambda)\le|N_\bullet|$, we note that by Proposition 2.7, the stabilized eigenvalues

$$e_n\left[\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1}}^{\infty}q^{-\lambda_b}t^{-b}\right]\text{ and }e_n\left[\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1}}^{\infty}q^{\lambda_b}t^{b}\right]$$

depend only on the N_p values of b where $1\le b\le|N_\bullet|$ and $b-\lambda_b=p+1$. We define the *color p spectral variables* $\{s_a^{(p)}\}_{a=1}^{N_p}$ by setting

$$s_a^{(p)}=q^{\lambda_{b_a}}t^{b_a}$$

where $1\le b_a\le|N_\bullet|$ is the ath number where $b_a-\lambda_{b_a}\equiv p+1$. Using these variables, we can rewrite

$$e_n\left[\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1}}^{\infty}q^{-\lambda_b}t^{-b}\right]=e_n\left[\frac{t^{-|N_\bullet|-p-1}}{1-t^{-nr}}+\sum_{a=1}^{N_p}\left(s_a^{(p)}\right)^{-1}\right]$$

where $t\gg 1$ and

$$e_n\left[\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1}}^{\infty}q^{\lambda_b}t^{b}\right]=e_n\left[\frac{t^{|N_\bullet|+p+1}}{1-t^{nr}}+\sum_{a=1}^{N_p}s_a^{(p)}\right]$$

where $t\ll 1$. The following is but a slight alteration of Lemma 3.2 from [FHHSY]:

Lemma 5.9. *For $|t|\gg 1$, we have*

$$(5.42)\qquad e_n\left[\frac{t^{-|N_\bullet|-p-1}}{1-t^{-nr}}+\sum_{a=1}^{N_p}\left(s_a^{(p)}\right)^{-1}\right]=\sum_{k=0}^{n}\frac{t^{-n|N_\bullet|-(n-k)(p+1)-r\binom{n-k}{2}}}{\prod_{l=1}^{n-k}(1-t^{-rl})}e_k\left[\sum_{a=1}^{N_p}t^{|N_\bullet|}\left(s_a^{(p)}\right)^{-1}\right].$$

while for $|t| \ll 1$, we have

$$e_n\left[\frac{t^{|N_\bullet|+p+1}}{1-t^{nr}}+\sum_{a=1}^{N_p} s_a^{(p)}\right]=\sum_{k=0}^{n}\frac{t^{n|N_\bullet|+(n-k)(p+1)+r\binom{n-k}{2}}}{\prod_{l=1}^{n-k}(1-t^{rl})}e_k\left[\sum_{a=1}^{N_p} t^{-|N_\bullet|}s_a^{(p)}\right] \tag{5.43}$$

Proof. As in *loc. cit.*, it is an easy application of the quantum binomial theorem. □

5.4.2. *Spectral shift.* By Lemma 5.9, the stabilized eigenvalues are polynomial in the spectral variables. Moreover, its degree k part is given by e_k evaluated at $\{t^{-|N_\bullet|}s_\bullet^{(p)}\}$. We would like to show that the summations in (5.40) and (5.41) correspond in some sense to this decomposition by the degree. The degree of a homogeneous polynomial can be measured using q-shifts. On the other hand, by the definition of the spectral variables, multiplying $s_a^{(p)}$ by q corresponds to adding a node to the end of a row. However, we must do this in a way that is color-insensitive. This motivates the following:

Proposition 5.10. *Let λ be a partition with core $\kappa(\lambda)$ compatible with $N_\bullet$ and $\ell(\lambda) \le |N_\bullet|$. Then*

$$\left(\prod_{i\in I}\prod_{l=1}^{N_i} x_l^{(i)}\right)P_\lambda[X_{N_\bullet};q,t]=P_{\lambda+r^{|N_\bullet|}}[X_{N_\bullet};q,t].$$

Here, $\lambda + r^{|N_\bullet|}$ denotes the partition obtained by adding r boxes to the first $|N_\bullet|$ rows of λ.

Proof. By Corollary 5.4, $P_{\lambda+r^{|N_\bullet|}}[X_{N_\bullet};q,t]$ is characterized by the eigenvalue equations

$$D_{p,1}(X_{N_\bullet};q,t)P_{\lambda+r^{|N_\bullet|}}[X_{N_\bullet};q,t]=\left(\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1}}^{|N_\bullet|} q^{\lambda_b+r}t^{|N_\bullet|-b}\right)P_{\lambda+r^{|N_\bullet|}}[X_{N_\bullet};q,t]$$

ranging over all $p \in I$. Note that we have used $b-\lambda_b \equiv b-\lambda_b+r$. Now, for a shift pattern $\underline{J}$, it is easy to see that

$$T_{\underline{J}}\left(\prod_{i\in I}\prod_{l=1}^{N_i} x_l^{(i)}\right)=q^r\left(\prod_{i\in I}\prod_{l=1}^{N_i} x_l^{(i)}\right) \tag{5.44}$$

from which the proposition follows. □

5.4.3. *Eigenfunction equation.* We are now ready to derive the eigenvalues of the higher order wreath Macdonald operators.

Theorem 5.11. *For λ with core $\kappa(\lambda)$ compatible with $N_\bullet$ according to (2.9) and $\ell(\lambda) \le |N_\bullet|$, the wreath Macdonald polynomial $P_\lambda[X_{N_\bullet};q,t]$ satisfies the equations:*

$$D^*_{p,n}(X_{N_\bullet};q,t)P_\lambda[X_{N_\bullet};q,t]=e_n\left[\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1}}^{|N_\bullet|} q^{-\lambda_b}t^{-|N_\bullet|+b}\right]P_\lambda[X_{N_\bullet};q,t]$$

$$D_{p,n}(X_{N_\bullet};q,t)P_\lambda[X_{N_\bullet};q,t]=e_n\left[\sum_{\substack{b=1\\ b-\lambda_b\equiv p+1}}^{|N_\bullet|} q^{\lambda_b}t^{|N_\bullet|-b}\right]P_\lambda[X_{N_\bullet};q,t].$$

Here, $x_{i,l}$, q, and t take generic values.

Proof. Let $\mathfrak{e}_{p,n}(\lambda;q,t^{-1})$ and $\mathfrak{e}^*_{p,n}(\lambda;q,t^{-1})$ be the eigenvalues of $D_{p,n}(X_{N_\bullet};q,t^{-1})$ and $D^*_{p,n}(X_{N_\bullet};q,t^{-1})$, respectively, at $P_\lambda[X_{N_\bullet};q,t^{-1}]$. By (5.40), (5.41), and Lemma 5.9, we can induct on n to show that, as functions of λ, $\mathfrak{e}_{p,n}(\lambda;q,t^{-1})$ is polynomial in $\{s_\bullet^{(p)}\}$ and $\mathfrak{e}^*_{p,n}(\lambda;q,t^{-1})$ is polynomial in $\{(s_\bullet^{(p)})^{-1}\}$. Applying

(5.44) n times, we have

$$D^*_{p,n}(X_{N_\bullet};q,t^{-1})\prod_{i\in I}\prod_{l=1}^{N_i}x_l^{(i)} = q^{-nr}\prod_{i\in I}\prod_{l=1}^{N_i}x_l^{(i)}$$
$$D_{p,n}(X_{N_\bullet};q,t^{-1})\prod_{i\in I}\prod_{l=1}^{N_i}x_l^{(i)} = q^{nr}\prod_{i\in I}\prod_{l=1}^{N_i}x_l^{(i)}.$$

It then follows from Proposition 5.10 that $\mathfrak{e}_{p,n}(\lambda;q,t^{-1})$ is homogeneous of degree n and $\mathfrak{e}^*_{p,n}(\lambda;q,t^{-1})$ is homogeneous of degree $-n$. By induction, the eigenvalues of the other terms in (5.40) have strictly higher degrees and those of the other terms in (5.41) have strictly lower degrees. It follows that $\mathfrak{e}^*_{p,n}(\lambda;q,t^{-1})$ is the degree $-n$ piece of (5.42) and $\mathfrak{e}_{p,n}(\lambda;q,t^{-1})$ is the degree n piece of (5.43). This establishes the eigenvalue equations under the appropriate conditions (5.3) and (5.4) on $x_l^{(i)}$, q, and t. We extend to generic values as in the proof of 5.3. □

Remark 5.12. Even though $r \geq 3$ was assumed throughout, we have verified experimentally that Theorem 5.11 continues to hold as stated for $r = 2$. The $r = 1$ case is discussed in Remark 5.7 above.

Example 5.13. Let $r = 2$, $p = 1$, $N_\bullet = (0,2)$, and $\lambda = (1)$. Because λ is a 2-core,

$$P_\lambda[X_{N_\bullet};q,t] = 1$$

There are only two shift patterns containing 1:

$$\underline{J}_1 = \{x_1^{(1)}\}$$
$$\underline{J}_2 = \{x_2^{(1)}\}.$$

Note that

$$T_{\underline{J}_1}x_1^{(1)} = q^2x_1^{(1)} \qquad T_{\underline{J}_2}x_1^{(1)} = x_1^{(1)}$$
$$T_{\underline{J}_1}x_2^{(1)} = x_1^{(1)} \qquad T_{\underline{J}_2}x_2^{(1)} = q^2x_2^{(1)}.$$

Therefore,

$$\begin{aligned} D_{1,1}(X_{N_\bullet};q,t)P_\lambda[X_\bullet;q,t] &= \frac{(-1)(1-qt^{-1})}{1-q^2t^{-2}}\left\{\frac{qtx_2^{(1)}-q^2x_1^{(1)}}{x_1^{(1)}-x_2^{(1)}} + \frac{qtx_1^{(1)}-q^2x_2^{(1)}}{x_2^{(1)}-x_1^{(1)}}\right\} \\ &= \frac{(-1)(1-qt^{-1})(-qt-q^2)}{1-q^2t^{-2}} \\ &= qtP_\lambda[X_{N_\bullet};q,t]. \end{aligned}$$

Appendix A. Wreath Noumi-Sano operators

In this appendix, we apply our methods to study wreath analogues of the trigonometric *Noumi-Sano operators* [NSa]. We obtain explicit formulas for degree $n = 1$ and an integral formula for general n.

A.1. **Infinite-variable eigenvalues.** Let $(x;y)_\infty$ denote the infinite y-Pochammer symbol:

$$(x;y)_\infty = \prod_{i=0}^{\infty}(1-xy^i).$$

Lemma A.1. *Assume $|q^{\pm 1}|<1$ and $|t^{\pm 1}|<1$ (where '+' and '−' are separate cases). For $p\in I$, we have*

$$
\begin{aligned}
&\left\langle \lambda \middle| \exp\left[-\sum_{k>0}\left(\frac{\sum_{i=1}^r q^{\pm k(i-1)}h_{p+i,\pm k}}{(1-q^{\pm kr})}\right)\frac{\mathfrak{q}^{\pm k}z^{\mp k}}{v^{\pm k}[k]_{\mathfrak{q}}}\right] \middle| \lambda\right\rangle \\
&= \exp\left[\sum_{k>0}\left(\sum_{i=1}^r \frac{q^{\pm k(i-1)}}{1-q^{\pm kr}}\left\{\sum_{\substack{b>0\\ b-\lambda_b\equiv p+i}} q^{\pm k\lambda_b}t^{\pm kb} - t^{\mp k}\sum_{\substack{b>0\\ b-\lambda_b\equiv p+i+1}} q^{\pm k\lambda_b}t^{\pm kb}\right\}\right)\frac{z^{\mp k}}{k}\right] \\
&= \prod_{i=1}^r \frac{\prod_{\substack{b>0\\ b-\lambda_b\equiv p+i+1}}\left(q^{\pm(\lambda_b+i)}t^{\pm(b-1)}z^{\mp 1};q^{\pm r}\right)_\infty}{\prod_{\substack{b>0\\ b-\lambda_b\equiv p+i}}\left(q^{\pm(\lambda_b+i)}t^{\pm b}z^{\mp 1};q^{\pm r}\right)_\infty}
\end{aligned}
\tag{A.1}
$$

where we set $\lambda_b=0$ for all $b>\ell(\lambda)$.

A.2. **Shuffle elements.** We rewrite

$$
\begin{aligned}
&\varsigma\exp\left[-\sum_{k>0}\left(\frac{\sum_{i=1}^r q^{\pm k(i-1)}h_{p+i,\pm k}}{(1-q^{\pm kr})}\right)\frac{\mathfrak{q}^{\pm k}z^{\mp k}}{[k]_{\mathfrak{q}}}\right] \\
&= \varsigma\exp\left[\sum_{k>0}\left(\frac{\sum_{i=0}^{r-1} q^{\mp k(i+1)}h_{p-i,\pm k}}{(1-q^{\mp kr})}\right)\frac{\mathfrak{q}^{\pm k}z^{\mp k}}{[k]_{\mathfrak{q}}}\right] \\
&= \eta\exp\left[-\sum_{k>0}\left(\frac{\sum_{i=0}^{r-1} t^{\mp k(i+1)}\varsigma^{-1}(h_{p-i,\mp k})}{(1-t^{\mp kr})}\right)\frac{\mathfrak{q}^{\pm k}z^{\mp k}}{[k]_{\mathfrak{q}}}\right] \\
&= \eta\exp\left[(\mathfrak{q}-\mathfrak{q}^{-1})^{-1}\sum_{k>0}\left(-q^{\pm k}\varsigma^{-1}(h^\perp_{p,\mp k})+\varsigma^{-1}(h^\perp_{p+1,\mp k})\right)\frac{z^{\mp k}}{k}\right].
\end{aligned}
$$

Recall the formulas (4.3) and (4.3) for $\mathcal{E}^\pm_{p,n}$ and $\mathcal{H}^\pm_{p,n}$. In [W], it was shown that

$$
\begin{aligned}
&\exp\left[(\mathfrak{q}-\mathfrak{q}^{-1})^{-1}\sum_{k>0}\left(-q^{-k}\varsigma^{-1}(h^\perp_{p,k})+\varsigma^{-1}(h^\perp_{p+1,k})\right)\frac{z^{k}}{k}\right] \\
&= \sum_{n=0}^\infty \frac{(-1)^{nr}t^{nr}\mathfrak{d}^{-n}\left(1-q^{-1}t^{-1}\right)^{nr}}{\mathfrak{q}^{2n}\prod_{r=1}^n(1-q^{-r}t^{-r})}\Psi_+\left(\mathcal{E}^-_{p,n}\right) \\
&\exp\left[(\mathfrak{q}-\mathfrak{q}^{-1})^{-1}\sum_{k>0}\left(-q^{k}\varsigma^{-1}(h^\perp_{p,-k})+\varsigma^{-1}(h^\perp_{p+1,-k})\right)\frac{z^{-k}}{k}\right] \\
&= \sum_{n=0}^\infty \frac{\mathfrak{d}^{n}\left(1-qt\right)^{nr}}{\prod_{r=1}^n(1-q^{-r}t^{-r})}\Psi_-\left(\mathcal{E}^+_{p,n}\right).
\end{aligned}
$$

Applying η, we get

$$
\begin{aligned}
\sum_{n=0}^\infty \frac{\mathfrak{q}^{n(r-1)}t^{-n}\left(1-q^{-1}t^{-1}\right)^{nr}}{v^{-n}\prod_{r=1}^n(1-q^{-r}t^{-r})}\Psi_+(\mathcal{H}^-_{p,n}) &= \varsigma\exp\left[-\sum_{k>0}\left(\frac{\sum_{i=0}^{r-1}q^{-ki}h_{p+i,-k}}{(1-q^{-kr})}\right)\frac{\mathfrak{q}^{-k}z^k}{v^{-k}[k]_{\mathfrak{q}}}\right] \\
\sum_{n=0}^\infty \frac{(-1)^{nr}\mathfrak{d}^{-n(r-1)}t^n\left(1-qt\right)^{nr}}{v^nq^n\prod_{r=1}^n(1-q^{-r}t^{-r})}\Psi_-\left(\mathcal{H}^+_{p,n}\right) &= \varsigma\exp\left[-\sum_{k>0}\left(\frac{\sum_{i=0}^{r-1}q^{ki}h_{p+i,k}}{(1-q^{kr})}\right)\frac{\mathfrak{q}^{k}z^{-k}}{v^{k}[k]_{\mathfrak{q}}}\right].
\end{aligned}
$$

A.3. **Normal ordering.** It will be slightly nicer to reorder our currents differently from Proposition 3.6:

Proposition A.2. *For $p \in I$, we have*

$$\begin{aligned}
\prod_{a=1}^{\overset{\curvearrowright}{n}} \prod_{i=1}^{\overset{\curvearrowleft}{r}} E_{p+i}(z_{p+i,a}) = &\left((-1)^{\frac{(r-2)(r-3)}{2}+r}\mathfrak{d}^{-\frac{r}{2}+1}\prod_{i\in I} c_i\right)^n \\
&\times \prod_{1\le a<b\le n}\prod_{i\in I}\frac{(1-z_{i,b}/z_{i,a})\left(1-q^{-1}t^{-1}z_{i,b}/z_{i,a}\right)}{(1-t^{-1}z_{i+1,b}/z_{i,a})\left(1-q^{-1}z_{i-1,b}/z_{i,a}\right)} \\
&\times \prod_{a=1}^{n}\frac{z_{p+1,a}/z_{p,a}}{(1-t^{-1}z_{p+1,a}/z_{p,a})\prod_{i\in I\setminus\{p\}}(1-q^{-1}z_{i,a}/z_{i+1,a})} \\
&\times \prod_{i\in I}\exp\left(\sum_{a=1}^{n}\sum_{k>0}\left(p_k[X^{(i)}]-t^{-k}p_k[X^{(i-1)}]\right)\frac{z_{i,a}^k}{k}\right) \\
&\times \prod_{i\in I}\exp\left(\sum_{a=1}^{n}\sum_{k>0}\left(-p_k[X^{(i)}]^\perp+q^{-k}p_k[X^{(i-1)}]^\perp\right)\frac{z_{i,a}^{-k}}{k}\right)\prod_{i\in I}\prod_{a=1}^{n} z_{i,a}^{H_{i,0}}
\end{aligned}$$

where all rational functions are Laurent series expanded assuming

$$|z_{i,a}|=1,\ |q|>1,\ |t|>1. \tag{A.2}$$

For the F-currents, we have

$$\begin{aligned}
\prod_{a=1}^{\overset{\curvearrowleft}{n}} \prod_{i=1}^{\overset{\curvearrowright}{r}} F_{p+i}(z_{p+i,a}) = &\left(\frac{(-1)^{\frac{(r-2)(r-3)}{2}+r}\mathfrak{d}^{\frac{r}{2}-1}}{\prod_{i\in I} c_i}\right)^n \\
&\times \prod_{1\le a<b\le n}\prod_{i\in I}\frac{(1-z_{i,a}/z_{i,b})(1-qtz_{i,a}/z_{i,b})}{(1-tz_{i-1,a}/z_{i,b})(1-qz_{i+1,a}/z_{i,b})} \\
&\times \prod_{a=1}^{n}\frac{z_{p,a}/z_{p+1,a}}{(1-tz_{p,a}/z_{p+1,a})\prod_{i\in I\setminus\{p+1\}}(1-qz_{i,a}/z_{i-1,a})} \\
&\times \prod_{i\in I}\exp\left(\sum_{a=1}^{n}\sum_{k>0}\left(-t^kp_k[X^{(i)}]+p_k[X^{(i-1)}]\right)\frac{z_{i,a}^k}{k}\right) \\
&\times \prod_{i\in I}\exp\left(\sum_{a=1}^{n}\sum_{k>0}\left(q^kp_k[X^{(i)}]^\perp-p_k[X^{(i-1)}]^\perp\right)\frac{z_{i,a}^{-k}}{k}\right)\prod_{i\in I}\prod_{a=1}^{n} z_{i,a}^{-H_{i,0}}
\end{aligned}$$

where all rational functions are Laurent series expanded assuming

$$|z_{i,a}|=1,\ |q|<1,\ |t|<1. \tag{A.3}$$

A.4. **Integral formula.** Let

$$d_n^+ = \frac{\mathfrak{q}^{n(r-1)}t^{-n}\left(1-q^{-1}t^{-1}\right)^{nr}}{v^{-n}\prod_{r=1}^{n}(1-q^{-r}t^{-r})}, \qquad d_n^- = \frac{(-1)^{nr}\mathfrak{d}^{-n(r-1)}t^n(1-qt)^{nr}}{v^nq^n\prod_{r=1}^{n}(1-q^{-r}t^{-r})}.$$

We have

$$
\begin{aligned}
d_n^+\pi_{N_\bullet}\Big((\rho_{\vec{c}}\circ\Psi_+)(\mathcal{H}^-_{p,n})(f\otimes e^\alpha)\Big) &= \frac{(-1)^n\left(1-q^{-1}t^{-1}\right)^{nr}}{\prod_{r=1}^n(1-q^{-r}t^{-r})}\left\{\prod_{i\in I}\prod_{a=1}^n\prod_{l=1}^{N_i}\left(\frac{z_{i+1,a}^{-1}-t^{-1}x_l^{(i)}}{z_{i,a}^{-1}-x_l^{(i)}}\right)\right.\\
&\times\prod_{1\le a<b\le n}\left[\frac{(1-z_{p,b}/z_{p,a})\left(1-q^{-1}t^{-1}z_{p,b}/z_{p,a}\right)}{(1-qz_{p+1,b}/z_{p,a})\left(1-q^{-1}z_{p-1,b}/z_{p,a}\right)}\right.\\
&\times\left.\prod_{i\in I\setminus\{p\}}\frac{(1-z_{i,b}/z_{i,a})\left(1-q^{-1}t^{-1}z_{i,b}/z_{i,a}\right)}{(1-q^{-1}z_{i-1,b}/z_{i,a})\left(1-t^{-1}z_{i+1,b}/z_{i,a}\right)}\right]\\
&\times\prod_{a=1}^n\left[\left(\frac{z_{0,a}}{z_{p+1,a}}\right)\left(\frac{1}{1-q^{-1}z_{p,a}/z_{p+1,a}}\right)\right.\\
&\times\left.\prod_{i\in I\setminus\{p\}}\left(\frac{1}{1-q^{-1}z_{i,a}/z_{i+1,a}}\right)\left(\frac{1}{1-t^{-1}z_{i+1,a}/z_{i,a}}\right)\right]\\
&\times\left.\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}-\sum_{a=1}^n z_{i,a}^{-1}+q^{-1}\sum_{a=1}^n z_{i+1,a}^{-1}\right]\right\}_0\otimes e^\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
d_n^-\pi_{N_\bullet}\Big((\rho_{\vec{c}}\circ\Psi_-)(\mathcal{H}^+_{p,n})(f\otimes e^\alpha)\Big) &= \frac{(-1)^n\left(1-qt\right)^{nr}}{\prod_{r=1}^n(1-q^{r}t^{r})}\left\{\prod_{i\in I}\prod_{a=1}^n\prod_{l=1}^{N_i}\left(\frac{z_{i,a}^{-1}-tx_l^{(i)}}{z_{i+1,a}^{-1}-x_l^{(i)}}\right)\right.\\
&\times\prod_{1\le a<b\le n}\left[\frac{(1-z_{p+1,a}/z_{p+1,b})\left(1-qtz_{p+1,a}/z_{p+1,b}\right)}{(1-q^{-1}z_{p,a}/z_{p+1,b})\left(1-qz_{p-2,a}/z_{p+1,b}\right)}\right.\\
&\times\left.\prod_{i\in I\setminus\{p+1\}}\frac{(1-z_{i,a}/z_{i,b})\left(1-qtz_{i,a}/z_{i,b}\right)}{(1-qz_{i+1,a}/z_{i,b})\left(1-tz_{i-1,a}/z_{i,b}\right)}\right]\\
&\times\prod_{a=1}^n\left[\left(\frac{z_{p+1,a}}{z_{0,a}}\right)\left(\frac{1}{1-qz_{p+1,a}/z_{p,a}}\right)\right.\\
&\times\left.\prod_{i\in I\setminus\{p+1\}}\left(\frac{1}{1-qz_{i,a}/z_{i-1,a}}\right)\left(\frac{1}{1-tz_{i-1,a}/z_{i,a}}\right)\right]\\
&\times\left.\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}+\sum_{a=1}^n qz_{i,a}^{-1}-\sum_{a=1}^n z_{i+1,a}^{-1}\right]\right\}_0\otimes e^\alpha.
\end{aligned}
$$

Finally, we make the substitution

$$w_{i,a}=z_{i,a}^{-1}$$

and rewrite these formulas in terms of integrals. This gets us

$$
\begin{aligned}
&d_n^+\pi_{N_\bullet}\Big((\rho_{\vec{c}}\circ\Psi_+)(\mathcal{H}^-_{p,n})(f\otimes e^\alpha)\Big)\\
&=\left(\oint_{|w_{i,a}|=1}\cdots\oint\varrho^+_{p,n}(w_{\bullet,\bullet},X_{N_\bullet})\prod_{i\in I}f_i\left[\sum_{l=1}^{N_i}x_l^{(i)}-\sum_{a=1}^n w_{i,a}+\sum_{a=1}^n q^{-1}w_{i+1,a}\right]\prod_{a=1}^n\frac{dw_{i,a}}{2\pi\sqrt{-1}w_{i,a}}\right)\otimes e^\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
& d_n^- \pi_{N_\bullet}\left((\rho_{\vec{c}} \circ \Psi_-)\,(\mathcal{H}^+_{p,n})(f\otimes e^\alpha)\right) \\
&= \left(\oint_{|w_{i,a}|=1} \cdots \oint \varrho^-_{p,n}(w_{\bullet,\bullet}, X_{N_\bullet}) \prod_{i\in I} f_i \left[\sum_{l=1}^{N_i} x_l^{(i)} + \sum_{a=1}^{n} q w_{i,a} - \sum_{a=1}^{n} w_{i+1,a} \right] \prod_{a=1}^{n} \frac{dw_{i,a}}{2\pi\sqrt{-1}w_{i,a}} \right) \otimes e^\alpha
\end{aligned}
$$

where

$$
\begin{aligned}
\varrho^+_{p,n}(w_{\bullet,\bullet}, X_{N_\bullet}) &= \frac{(-1)^{\frac{n(n+1)}{2}}\left(1-q^{-1}t^{-1}\right)^{nr}}{q^{\frac{n(n-1)}{2}}\prod_{r=1}^{n}(1-q^{-r}t^{-r})} \left\{ \prod_{i\in I}\prod_{a=1}^{n}\prod_{l=1}^{N_i} \left(\frac{w_{i+1,a}-t^{-1}x_l^{(i)}}{w_{i,a}-x_l^{(i)}} \right) \right. \\
&\times \prod_{1\le a<b\le n} \left[\frac{(w_{p,b}-w_{p,a})\left(w_{p,b}-q^{-1}t^{-1}w_{p,a}\right)}{(w_{p,a}-q^{-1}w_{p+1,b})\,(w_{p-1,b}-q^{-1}w_{p,a})} \right. \\
&\times \left. \prod_{i\in I\setminus\{p\}} \frac{(w_{i,b}-w_{i,a})\left(w_{i,b}-q^{-1}t^{-1}w_{i,a}\right)}{(w_{i-1,b}-q^{-1}w_{i,a})\,(w_{i+1,b}-t^{-1}w_{i,a})} \right] \\
&\times \prod_{a=1}^{n} \left[\left(\frac{w_{p+1,a}}{w_{0,a}} \right) \left(\frac{w_{p,a}}{w_{p,a}-q^{-1}w_{p+1,a}} \right) \right. \\
&\times \left. \prod_{i\in I\setminus\{p\}} \left(\frac{w_{i,a}}{w_{i,a}-q^{-1}w_{i+1,a}} \right) \left(\frac{w_{i+1,a}}{w_{i+1,a}-t^{-1}w_{i,a}} \right) \right]
\end{aligned}
$$

and

$$
\begin{aligned}
\varrho^-_{p,n}(w_{\bullet,\bullet}, x_{\bullet,\bullet}) &= \frac{(-1)^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} (1-qt)^{nr}}{\prod_{r=1}^{n}(1-q^r t^r)} \left\{ \prod_{i\in I}\prod_{a=1}^{n}\prod_{l=1}^{N_{i-1}} \left(\frac{w_{i-1,a}-t x_l^{(i-1)}}{w_{i,a}-x_l^{(i-1)}} \right) \right. \\
&\times \prod_{1\le a<b\le n} \left[\frac{(w_{p+1,a}-w_{p+1,b})\,(w_{p+1,a}-qtw_{p+1,b})}{(w_{p+1,b}-qw_{p,a})\,(w_{p+2,a}-qw_{p+1,b})} \right. \\
&\times \left. \prod_{i\in I\setminus\{p+1\}} \frac{(w_{i,a}-w_{i,b})\,(w_{i,a}-qtw_{i,b})}{(w_{i+1,a}-qw_{i,b})\,(w_{i-1,a}-tw_{i,b})} \right] \\
&\times \prod_{a=1}^{n} \left[\left(\frac{w_{0,a}}{w_{p+1,a}} \right) \left(\frac{w_{p+1,a}}{w_{p+1,a}-qw_{p,a}} \right) \right. \\
&\times \left. \prod_{i\in I\setminus\{p+1\}} \left(\frac{w_{i,a}}{w_{i,a}-qw_{i-1,a}} \right) \left(\frac{w_{i-1,a}}{w_{i-1,a}-tw_{i,a}} \right) \right].
\end{aligned}
$$

A.5. **Degree one.** We compute the integral and record the resulting action on f when $n=1$.

A.5.1. *Difference operators.* Let $Sh(X_{N_\bullet}) = Sh$ denote the set of *all* shift patterns. Define

$$H^*_{p,1}(X_{N_\bullet};q,t^{-1}) := -\sum_{\substack{\underline{J}\in Sh\\ \underline{J}\neq\varnothing}} (1-q^{-1}t^{-1})^{|J|-\delta_{p\in J}} \frac{x^{(p+1)}_{\underline{J}^\vartriangle}}{x^{(0)}_{\underline{J}^\vartriangle}} \left(\prod_{i\in I}\prod_{\substack{l=1\\ x_l^{(i)}\neq x^{(i)}_{\underline{J}^\vartriangle}}}^{N_i} \frac{\left(x_l^{(i)} - t x^{(i+1)}_{\underline{J}^\vartriangle}\right)}{\left(x_l^{(i)} - x^{(i)}_{\underline{J}^\vartriangle}\right)}\right)$$

$$\times \left(\frac{qtT^{-1}_{\underline{J}}(x^{(p)}_{\underline{J}}) - x^{(p)}_{\underline{J}}}{x^{(p)}_{\underline{J}} - T^{-1}_{\underline{J}}(x^{(p)}_{\underline{J}})}\right)^{\delta_{p\in J}} \left(\prod_{i\in J\setminus\{p\}} \frac{qtT^{-1}_{\underline{J}}(x^{(i)}_{\underline{J}})}{\left(x^{(i)}_{\underline{J}} - T^{-1}_{\underline{J}}(x^{(i)}_{\underline{J}})\right)}\right) T^{-1}_{\underline{J}}$$

$$H_{p,1}(X_{N_\bullet};q,t^{-1}) := -\sum_{\substack{\underline{J}\in Sh\\ \underline{J}\neq\varnothing}} (1-qt)^{|J|-\delta_{p\in J}} \frac{x^{(r-1)}_{\underline{J}^\triangledown}}{x^{(p)}_{\underline{J}^\triangledown}} \left(\prod_{i\in I}\prod_{\substack{l=1\\ x_l^{(i)}\neq x^{(i)}_{\underline{J}^\triangledown}}}^{N_i} \frac{\left(x_l^{(i)} - t^{-1} x^{(i-1)}_{\underline{J}^\triangledown}\right)}{\left(x_l^{(i)} - x^{(i)}_{\underline{J}^\triangledown}\right)}\right)$$

$$\times \left(\frac{q^{-1}t^{-1}T_{\underline{J}}(x^{(p)}_{\underline{J}}) - x^{(p)}_{\underline{J}}}{x^{(p)}_{\underline{J}} - T_{\underline{J}}(x^{(p)}_{\underline{J}})}\right)^{\delta_{p\in J}} \left(\prod_{i\in J\setminus\{p\}} \frac{q^{-1}t^{-1}T_{\underline{J}}(x^{(i)}_{\underline{J}})}{\left(x^{(i)}_{\underline{J}} - T_{\underline{J}}(x^{(i)}_{\underline{J}})\right)}\right) T_{\underline{J}}.$$

Setting $r=1$ and inverting t, we indeed obtain the first Noumi-Sano operator.

A.5.2. *Eigenvalues.* For a series $f(z)$ in z, let $[z^n]f(z)$ denote the coefficient of z^n. Methods similar to those in 5.4 allow us to establish the following.

Theorem A.3. *For $|q|>1$, we have*

$$H^*_{p,1}(X_{N_\bullet};q,t)P_\lambda[X_{N_\bullet};q,t] = [z]\left(\prod_{i=1}^{r} \frac{\prod\limits_{\substack{b=1\\ b-\lambda_b\equiv p+i+1}}^{|N_\bullet|}\left(q^{-(\lambda_b+i)}t^{-|N_\bullet|+(b-1)}z;q^{-r}\right)_\infty}{\prod\limits_{\substack{b=1\\ b-\lambda_b\equiv p+i}}^{|N_\bullet|}\left(q^{-(\lambda_b+i)}t^{-|N_\bullet|+b}z;q^{-r}\right)_\infty}\right)P_\lambda[X_{N_\bullet};q,t].$$

On the other hand, for $|q|<1$, we have

$$H_{p,1}(X_{N_\bullet};q,t)P_\lambda[X_{N_\bullet};q,t] = [z^{-1}]\left(\prod_{i=1}^{r} \frac{\prod\limits_{\substack{b=1\\ b-\lambda_b\equiv p+i+1}}^{|N_\bullet|}\left(q^{\lambda_b+i}t^{|N_\bullet|-(b-1)}z^{-1};q^{r}\right)_\infty}{\prod\limits_{\substack{b=1\\ b-\lambda_b\equiv p+i}}^{|N_\bullet|}\left(q^{\lambda_b+i}t^{|N_\bullet|-b}z^{-1};q^{r}\right)_\infty}\right)P_\lambda[X_{N_\bullet};q,t].$$

Remark A.4. We have presented the eigenvalues in terms of our original spectral variables $q^{\lambda_b}t^{|N_\bullet|-b}$. However, we can give a more natural combinatorial expression for the eigenvalues if we forgo this and use instead the *transpose* partition λ' [Mac, (I.1.3)]. Let

$$f_\lambda(q,t) = \frac{1}{1-q} - \sum_{j\geq 1} q^{j-1}t^{|N_\bullet|-\lambda'_j}. \tag{A.4}$$

It can be viewed as a series or as a rational function since $(1-q^r)f_\lambda(q,t)$ is a polynomial. Let $\Gamma=\mathbb{Z}/r\mathbb{Z}$ be the cyclic group and let χ be the generator of $R(\Gamma)$. Define $f^{(p)}_\lambda(q,t)$ by the following expression in $\mathbb{Q}(q,t)\otimes R(\Gamma)$.

$$f_\lambda(q\chi^{-1},t\chi^{-1}) = \chi^{-1}\sum_{p\in I} f^{(p)}_\lambda(q,t)\chi^p. \tag{A.5}$$

Then the eigenvalues are given by

$$H^*_{p,1}(X_{N_\bullet};q,t)P_\lambda[X_{N_\bullet};q,t] = f^{(p)}_\lambda(q^{-1},t^{-1})P_\lambda[X_{N_\bullet};q,t] \tag{A.6}$$

$$H_{p,1}(X_{N_\bullet};q,t)P_\lambda[X_{N_\bullet};q,t] = f^{(p)}_\lambda(q,t)P_\lambda[X_{N_\bullet};q,t]. \tag{A.7}$$

Example A.5. Let $r=2$ and $\alpha=0$ (empty core). We use $N_0=N_1=1$. There are three nonempty shift patterns: $\underline{J}_1=\{x_1^{(0)}\}$, $\underline{J}_2=\{x_1^{(1)}\}$, and $\underline{J}_3=\{x_1^{(0)},x_1^{(1)}\}$. We apply $H_{0,1}[X_{N_\bullet};q,t^{-1}]$ to $P_\emptyset[X_{N_\bullet};q,t]=1$ using summands $\underline{J}_1,\underline{J}_2,\underline{J}_3$:

$$\begin{aligned}
-H_{0,1}(X_{N_\bullet};q,t^{-1})\cdot P_\emptyset[X_{N_\bullet};q,t^{-1}] &= (1-qt)^0\frac{qx_1^{(0)}}{x_1^{(0)}}\frac{x_1^{(1)}-t^{-1}x_1^{(0)}}{x_1^{(1)}-qx_1^{(0)}}\frac{q^{-1}t^{-1}q^2x_1^{(0)}-x_1^{(0)}}{x_1^{(0)}-q^2x_1^{(0)}}\\
&+(1-qt)^1\frac{x_1^{(1)}}{qx_1^{(1)}}\frac{x_1^{(0)}-t^{-1}x_1^{(1)}}{x_1^{(0)}-qx_1^{(1)}}\frac{q^{-1}t^{-1}q^2x_1^{(1)}}{x_1^{(1)}-q^2x_1^{(1)}}\\
&+(1-qt)^1\frac{q^{-1}t^{-1}q^2x_1^{(0)}-x_1^{(0)}}{x_1^{(0)}-q^2x_1^{(0)}}\frac{q^{-1}t^{-1}q^2x_1^{(1)}}{x_1^{(1)}-q^2x_1^{(1)}}\\
&=\frac{q(qt^{-1}-1)}{1-q^2}\frac{x_1^{(1)}-t^{-1}x_1^{(0)}}{x_1^{(1)}-qx_1^{(0)}}\\
&+\frac{(1-qt)t^{-1}}{1-q^2}\frac{x_1^{(0)}-t^{-1}x_1^{(1)}}{x_1^{(0)}-qx_1^{(1)}}\\
&+\frac{(1-qt)(qt^{-1}-1)qt^{-1}}{(1-q^2)^2}\\
&=\frac{q}{t^2}\frac{1-t^2}{1-q^2}P_\emptyset[X_{N_\bullet};q,t^{-1}].
\end{aligned}$$

We have

$$\begin{aligned}
f_\emptyset(q,t) &= \frac{1}{1-q}-\frac{t^2}{1-q}=\frac{1-t^2}{1-q}\\
f_\emptyset(q\chi^{-1},t\chi^{-1}) &= \frac{1-t^2}{1-q^2}(1+q\chi)\\
f_\emptyset^{(0)}(q,t) &= q\frac{1-t^2}{1-q^2}\\
f_\emptyset^{(0)}(q,t^{-1}) &= q\frac{1-t^{-2}}{1-q^2}=-\frac{q}{t^2}\frac{1-t^2}{1-q^2}.
\end{aligned}$$

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