First Betti number and collapse

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We show that when a sequence of Riemannian manifolds collapses under a lower Ricci curvature bound, the first Betti number cannot drop more than the dimension.

1 Introduction

For \( n \in \mathbb{N}, c \in \mathbb{R}, D > 0 \), let \( \mathcal{M}_{\text{Ric}}(n, c, D) \) (resp. \( \mathcal{M}_{\text{sec}}(n, c, D) \)) denote the class of closed \( n \)-dimensional Riemannian manifolds of Ricci curvature \( \geq c \) (resp. sectional curvature \( \geq c \)) and diameter \( \leq D \). A significant proportion of the subject consists of understanding the relationship between sequences \( X_i \in \mathcal{M}_{\text{Ric}}(n, c, D) \) and their Gromov–Hausdorff limits. Our main result concerns the first Betti number of such limit space.

**Theorem 1.** Let \( X_i \in \mathcal{M}_{\text{Ric}}(n, c, D) \) be a sequence with \( \beta_1(X_i) \geq r \) for each \( i \). If \( X_i \) converges in the Gromov–Hausdorff sense to a space \( X \) containing a \( k \)-regular point, then

\[
\beta_1(X) \geq r + k - n.
\]

It has been known that for a Riemannian manifold \( M \) of almost non-negative Ricci curvature, if its first Betti number equals its dimension then \( M \) is homeomorphic to a torus. This result has been recently extended to singular spaces by Mondello, Mondino, and Perales [9]. A consequence of their work and Theorem 1 is the following.

**Corollary 2.** For each \( n \in \mathbb{N} \), there is \( \varepsilon > 0 \) such that if \( X_i \in \mathcal{M}_{\text{sec}}(n, -\varepsilon, 1) \) is a sequence of spaces with \( \beta_1(X_i) \geq n \) that converges in the Gromov–Hausdorff sense to a space \( X \) of Hausdorff dimension \( k \), then \( X \) is bi-Hölder homeomorphic to a flat \( k \)-dimensional torus.

**Remark 3.** Theorem 1 shows that the first Betti number cannot drop more than the dimension. Contrastingly, the fundamental group can decrease in the limit even if there is no collapse: Otsu has constructed a sequence of metrics in \( S^3 \times \mathbb{R}P^2 \) of positive Ricci curvature that converges in the Gromov–Hausdorff sense to a simply connected 5-dimensional space [10].

Theorem 1 is an improvement of the main result of [14]. On the other hand, the goal of this program is to solve following problem.

**Question 4.** Assume a sequence \( X_i \in \mathcal{M}_{\text{Ric}}(n, c, D) \) of spaces homeomorphic to the \( n \)-dimensional torus converges in the Gromov–Hausdorff sense to a space \( X \). Is \( X \) necessarily homeomorphic to a torus?

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2 Preliminaries

In this section we recall the required material for Theorem 1 and Corollary 2, which we prove in the following section.

2.1 Gromov–Hausdorff topology

The basics on the subject can be found in ([2], Chapter 7).

Definition 5. We say that a function $f : X \to Y$ between metric spaces is an $\varepsilon$-isometry if for all $x_1, x_2 \in X$ one has $|d^X(x_1, x_2) - d^Y(fx_1, fx_2)| \leq \varepsilon$, and $f(X)$ intersects each closed ball of radius $\varepsilon$ in $Y$. We say that a sequence of functions $f_i : X_i \to Y_i$ between metric spaces are Gromov–Hausdorff approximations if $f_i$ is an $\varepsilon_i$-isometry for some sequence $\varepsilon_i \to 0$.

Proposition 6. (Gromov) Let $X_i$ be a sequence of compact metric spaces, and let $X$ be a complete metric space. Then the following are equivalent:

- There is a sequence $f_i : X_i \to X$ of Gromov–Hausdorff approximations.
- There is a sequence $h_i : X \to X_i$ of Gromov–Hausdorff approximations.

In either case, $X$ is compact and one says that the sequence $X_i$ converges to $X$ in the Gromov–Hausdorff sense. Furthermore, there is a metric on the class of compact metric spaces modulo isometry that yields this topology.

Definition 7. We say that a function $f : (X, x) \to (Y, y)$ between pointed metric spaces is an $\varepsilon$-isometry if $fx = y$, for all $x_1, x_2 \in B^X(x, 2/\varepsilon)$ one has $|d^X(x_1, x_2) - d^Y(fx_1, fx_2)| \leq \varepsilon$, and $f(B^X(x, 2/\varepsilon))$ intersects each closed ball of radius $\varepsilon$ in $B^Y(y, 1/\varepsilon)$. We say that a sequence of functions $f_i : (X_i, x_i) \to (Y_i, y_i)$ between pointed metric spaces are pointed Gromov–Hausdorff approximations if $f_i$ is a pointed $\varepsilon_i$-isometry for some sequence $\varepsilon_i \to 0$.

Proposition 8. (Gromov) Let $(X_i, x_i)$ be a sequence of proper pointed metric spaces, and let $(X, x)$ be a complete pointed metric space. Then the following are equivalent:

- There is a sequence $f_i : (X_i, x_i) \to (X, x)$ of pointed Gromov–Hausdorff approximations.
- There is a sequence $h_i : (X, x) \to (X_i, x_i)$ of pointed Gromov–Hausdorff approximations.

In either case, $X$ is proper and one says that the sequence $(X_i, x_i)$ converges to $(X, x)$ in the pointed Gromov–Hausdorff sense. Furthermore, there is a metric on the class of proper pointed metric spaces modulo isometry that yields this topology.

For $n \in \mathbb{N}$, $c \in \mathbb{R}$, we denote by $\mathcal{M}_{\text{Ric}}(n, c)$ the class of complete $n$-dimensional Riemannian manifolds of Ricci curvature $\geq c$. One reason we know so much about these families of spaces is because they are pre-compact with respect to the Gromov–Hausdorff topology.
Theorem 9. (Gromov) Let \((Y_i, y_i)\) be a sequence with \(Y_i \in \mathcal{M}_{Ric}(n, c)\) for each \(i\). Then one can find a subsequence that converges in the pointed Gromov–Hausdorff sense to some proper metric space \((Y, y)\).

2.2 Equivariant Gromov–Hausdorff convergence

There is a well studied notion of convergence of group actions in this setting. For a proper metric space \(X\), the topology that we use on its group of isometries \(Iso(X)\) is the compact-open topology, which in this setting coincides with both the topology of pointwise convergence and the topology of uniform convergence on compact sets. This topology makes \(Iso(X)\) a locally compact second countable metrizable group.

Definition 10. Let \((Y_i, q_i)\) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space \((Y, q)\). Consider pointed Gromov–Hausdorff approximations \(f_i : (Y_i, q_i) \to (Y, q)\) and \(h_i : (Y, q) \to (Y_i, q_i)\) such that \(d^Y(f_i \circ h_i(y), y) \to 0\) for all \(y \in Y\). Also let \(\Gamma_i \leq Iso(Y_i)\) be a sequence of groups of isometries. We say that \(\Gamma_i\) converges in the equivariant Gromov–Hausdorff sense to a closed group \(\Gamma \leq Iso(Y)\) if for all \(R, \varepsilon > 0\), one has the following:

- For each \(g \in \Gamma\), there is \(i_0 \in \mathbb{N}\) such that for each \(i \geq i_0\) there is \(g_i \in \Gamma_i\) with \(d^Y(f_i \circ g_i \circ h_i(y), g(y)) \leq \varepsilon\) for all \(y \in B^Y(q, R)\).

- There is \(i_0 \in \mathbb{N}\) such that if \(i \geq i_0\), \(g \in \Gamma_i\) with \(d^Y(gq_i, q_i) \leq R\), then there is \(\gamma \in \Gamma\) such that \(d^Y(f_i \circ g \circ h_i(y), \gamma(y)) \leq \varepsilon\) for all \(y \in B^Y(q, 10R)\).

Although this definition clearly depends on \(f_i\) and \(h_i\), we usually omit this when we state that \(\Gamma_i\) converges to \(\Gamma\).

This definition of equivariant convergence allows one to take limits before or after taking quotients.

Lemma 11. Let \((Y_i, q_i)\) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space \((Y, q)\), and \(\Gamma_i \leq Iso(Y_i)\) a sequence of isometry groups that converges in the equivariant Gromov–Hausdorff sense to a closed group \(\Gamma \leq Iso(Y)\). Then the sequence \((Y_i/\Gamma_i, [q_i])\) converges in the pointed Gromov–Hausdorff sense to \((Y/\Gamma, [q])\).

Since the isometry groups of proper metric spaces are locally compact, one has an Arzelà-Ascoli type result ([5], Proposition 3.6).

Theorem 12. (Fukaya–Yamaguchi) Let \((Y_i, q_i)\) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space \((Y, q)\), and take a sequence \(\Gamma_i \leq Iso(Y_i)\) of groups of isometries. Then there is a subsequence \((Y_{i_k}, q_{i_k}, \Gamma_{i_k})_{k \in \mathbb{N}}\) such that \(\Gamma_{i_k}\) converges in the equivariant Gromov–Hausdorff sense to a closed group \(\Gamma \leq Iso(Y)\).
In [6], Gromov studied which is the structure of discrete groups that act transitively on spaces that look like \( \mathbb{R}^n \). Using the Malcev embedding theorem, he showed that they look essentially like lattices in nilpotent Lie groups. In [1], Breuillard–Green–Tao studied in general what is the structure of discrete groups that have a large portion acting on a space of controlled doubling. It turns out that the answer is still essentially just lattices in nilpotent Lie groups. In ([13], Sections 7-9) the ideas from [6] and [1] are used to obtain the following structure result.

**Theorem 13.** Let \((Z,p)\) be a proper pointed geodesic space of topological dimension \( \ell \in \mathbb{N} \) and let \((D_i,p_i)\) be a sequence of discrete metric spaces converging in the pointed Gromov–Hausdorff sense to \((Z,p)\). Assume there is a sequence of isometry groups \( \Gamma_i \leq \text{Iso}(D_i) \) that act transitively and for each \( i \), \( \Gamma_i \) is generated by its elements that move \( p_i \) at most 10. Then for large enough \( i \), there are finite index subgroups \( G_i \leq \Gamma_i \) and finite normal subgroups \( F_i \triangleleft G_i \) such that \( G_i/F_i \) is isomorphic to a quotient of a lattice in a nilpotent Lie group of dimension \( \ell \). In particular, if the groups \( \Gamma_i \) are abelian, for large enough \( i \) their rank is at most \( \ell \).

For \( k \in \mathbb{N} \), a proper metric space \( X \), we say that \( x \in X \) is a \( k \)-regular point if for any sequence \( \lambda_i \to \infty \), the sequence \( (\lambda_i X, x) \) converges in the pointed Gromov–Hausdorff sense to \( \mathbb{R}^k \). For limits of sequences in \( \mathcal{M}_{\text{Ric}}(n,c) \), almost all points are regular [3].

**Theorem 14.** (Cheeger–Colding) Let \( X_i \in \mathcal{M}_{\text{Ric}}(n,c) \) converge in the pointed Gromov–Hausdorff sense to a space \( X \). If \( \mathcal{R}_k \) denotes the set of \( k \)-regular points of \( X \), then \( \mathcal{R}_k \neq \emptyset \) implies \( k \leq n \), and \( \bigcup_{j=0}^{n} \mathcal{R}_k \) is dense in \( X \).

Arugably the most used tool in the theory of Riemannian manifolds of non-negative Ricci curvature is the Cheeger–Gromoll splitting theorem. It was later generalized by Cheeger and Colding to limits of Riemannian manifolds [4]. Using this, one could understand how \( \mathbb{R}^k \) arises as a quotient of such spaces.

**Theorem 15.** (Cheeger–Colding) Let \( \varepsilon_i \to 0 \) and \((Y_i,q_i) \in \mathcal{M}_{\text{Ric}}(n,-\varepsilon_i)\) a sequence that converges in the pointed Gromov–Hausdorff sense to \((Y,q)\). If \( Y \) contains an isometric copy of \( \mathbb{R}^k \), then \( Y \) split as a metric space as \( \mathbb{R}^k \times Z \) for some proper geodesic space \( Z \) of Hausdorff dimension \( \leq n-k \).

**Corollary 16.** Let \( \varepsilon_i \to 0 \) and \((Y_i,q_i) \in \mathcal{M}_{\text{Ric}}(n,-\varepsilon_i)\) be a sequence that converges in the pointed Gromov–Hausdorff sense to \((Y,q)\). Assume there is a sequence of groups of isometries \( \Gamma_i \leq \text{Iso}(Y_i) \) such that \( (Y_i/\Gamma_i, [q_i]) \) converges in the pointed Gromov–Hausdorff sense to \( \mathbb{R}^k \) and \( \Gamma_i \) converges in the equivariant Gromov–Hausdorff sense to a group \( \Gamma \leq \text{Iso}(Y) \). Then \( Y \) splits as a metric space as \( \mathbb{R}^k \times Z \) for some proper geodesic space \( Z \) of Hausdorff dimension \( \leq n-k \), and the \( Z \)-fibers given by this product coincide with the orbits of \( \Gamma \).

**Proof.** One can use the submetry \( \phi : Y \to Y/\Gamma = \mathbb{R}^k \) to lift the lines of \( \mathbb{R}^k \) to lines in \( Y \) passing through \( q \). By Theorem 15, we get the desired splitting \( Y = \mathbb{R}^k \times Z \) with \( \phi(z_0,x) = x \) for all \( x \in \mathbb{R}^k \) and some \( z_0 \in Z \).
Let $g \in \Gamma$ and assume $g(z_0, x) = (z, y)$ for some $z_0, z \in Z$, $x, y \in \mathbb{R}^k$. Then for all $t \geq 1$, one has
\[
t |y - x| = |\phi(z_0, x + t(y - x)) - \phi(z_0, x)| \\
= |\phi(z_0, x + t(y - x)) - \phi(z, y)| \\
\leq d^1((z_0, x + t(y - x)), (z, y)) \\
= \sqrt{d^2(z_0, z)^2 + |(t - 1)(y - x)|^2}.
\]
As $t \to \infty$, this is only possible if $x = y$, and we conclude that the action of $\Gamma$ respects the splitting $Y = \mathbb{R}^k \times Z$. 

2.3 Homology and Ricci curvature bounds

We define the content of a map $A \to X$ between topological spaces to be the image of the natural map $H_1(A) \to H_1(X)$. If $U$ is a family of subsets of $X$, we denote by $H_1(U \prec X) \leq H_1(X)$ the subgroup generated by the contents of the inclusions $U \to X$ with $U \in U$. This group satisfies a natural monotonicity property.

**Lemma 17.** Let $X$ be a topological space, and $U, V$ two families of subsets of $X$. If for each $U \in U$ there is $V \in V$ with $U \subset V$, then $H_1(U \prec X) \leq H_1(V \prec X)$.

If $\varepsilon > 0$, $X$ is a metric space, and $U$ is the family of balls of radius $\varepsilon$ in $X$, then we denote $H_1(U \prec X)$ simply by $H^1_1(X)$. It has been recently shown that limits of sequences in $\mathfrak{M}_{\text{Ric}}(n, c, D)$ are semi-locally-simply-connected [12].

**Theorem 18.** (Pan–Wang) Let $X_i \in \mathfrak{M}_{\text{Ric}}(n, c, D)$ converge in the Gromov–Hausdorff sense to a space $X$. Then $X$ is semi-locally-simply-connected. In particular, $H^1_1(X)$ is trivial for small enough $\varepsilon$.

**Theorem 19.** (Sormani–Wei) Let $X$ be a compact geodesic space. Assume there is $\varepsilon > 0$ such that $H^1_\varepsilon(X)$ is trivial, and let $Y$ be a compact geodesic space with $f : Y \to X$ an $\varepsilon/100$-approximation. Then there is a surjective morphism $H_1(Y) \to H_1(X)$ (independent of $\varepsilon$) whose kernel is precisely $H^1_\varepsilon(Y)$.

**Proof sketch:** We follow the lines of ([11], Theorem 2.1), where they prove this result for $\pi_1$ instead of $H_1$. Each 1-cycle in $Y$ can be thought as a family of loops $S^1 \to Y$ with integer multiplicity. For each map $\gamma : S^1 \to Y$, by uniform continuity one could pick finitely many cyclically ordered points $\{z_1, \ldots, z_m\} \subset S^1$ such that $\gamma([z_{j-1}, z_j])$ is contained in a ball of radius $\varepsilon/10$ for each $j$. Then set $\phi(\gamma) : S^1 \to X$ to be the loop with $\phi(\gamma)(z_j) = f(\gamma(z_j))$ for each $j$, and $\phi(\gamma)|_{[z_{j-1}, z_j]}$ a minimizing geodesic from $\phi(\gamma)(z_{j-1})$ to $\phi(\gamma)(z_j)$.

Clearly, $\phi(\gamma)$ depends on the choice of the points $z_j$ and the minimizing paths $\phi(\gamma)|_{[z_{j-1}, z_j]}$. However, the homology class of $\phi(\gamma)$ in $H_1(X)$ does not depend on these choices, since different choices yield curves that are $\varepsilon$-uniformly close, which by hypothesis are homologous.

Assume that a 1-cycle $c$ in $Y$ is the boundary $\partial \sigma$ of a 2-chain $\sigma$. After taking iterated barycentric subdivision, one could assume that each simplex of $\sigma$ is contained in a ball
of radius $\varepsilon/10$. By recreating $\sigma$ in $X$ via $f$ simplex by simplex, one could find a 2-chain whose boundary is $\phi(c)$. This means that $\phi$ induces a map $\tilde{\phi} : H_1(Y) \to H_1(X)$.

In a similar fashion, if a 1-cycle $c$ in $Y$ is such that $\phi(c)$ is the boundary of a 2-chain $\sigma$, one could again apply iterated barycentric subdivision to obtain a 2-chain $\sigma'$ in $X$ whose boundary is $\phi(c)$ and such that each simplex is contained in a ball of radius $\varepsilon/10$. Using $f$ one could recreate the 1-skeleton of $\sigma'$ in $Y$ in such a way that expresses $c$ as a linear combination with integer coefficients of 1-cycles contained in balls of radius $\varepsilon$ in $Y$. This implies that the kernel of $\phi$ is contained in $H^f_1(Y)$.

If a 1-cycle $c$ in $Y$ is contained in a ball of radius $\varepsilon$, then $\phi(c)$ is contained in a ball of radius $2\varepsilon$ and then by hypothesis, $\phi(c)$ is a boundary. This shows that the kernel of $\tilde{\phi}$ is precisely $H^f_1(Y)$.

Lastly, for any loop $\gamma : S^1 \to X$, one can create via $f$ a loop $\gamma_1 : S^1 \to Y$ such that $\tilde{\phi}(\gamma_1)$ is uniformly close (and hence homologous) to $\gamma$, so $\tilde{\phi}$ is surjective.

**Corollary 20.** Let $X$ be a compact geodesic space. Assume there is $\rho > 0$ such that $H^{\rho}_1(X)$ is trivial, and consider a sequence $X_i$ of compact geodesic spaces that converges to $X$ in the Gromov–Hausdorff sense. Then there is a sequence $\rho_i \to 0$ such that $H^{\rho_i}_1(X_i) = H^{\rho}_1(X)$ for each $i$.

**Proof.** For large enough $i$, let $\rho_i \in (0, \rho]$ be such that $\rho_i \to 0$ and there is a $\rho_i/100$-approximation $X_i \to X$. One could then apply Theorem 19 for $\varepsilon \in [\rho_i, \rho]$ to get a map $H_1(X_i) \to H_1(X)$ whose kernel equals both $H^{\rho_i}_1(X_i)$ and $H^{\rho}_1(X)$. For small $i$, simply set $\rho_i = \rho$.

The following results were obtained in [8], and are stated in terms of $\pi_1$. The first one states that for $M \in \mathcal{M}_{Ric}(n, c, D)$, there is a subgroup $N \leq H_1(M)$ that can be detected anywhere. The second one states that at regular points, there is a gap phenomenon.

**Theorem 21.** (Kapovitch–Wilking) For each $n \in \mathbb{N}$, $c \in \mathbb{R}$, $D > 0$, $\varepsilon_1 > 0$, there are $\varepsilon_0 > 0$, $C \in \mathbb{N}$, such that the following holds. For each $M \in \mathcal{M}_{Ric}(n, c, D)$, there is $\varepsilon \in [\varepsilon_0, \varepsilon_1]$ and a subgroup $N \leq H_1(M)$ such that for all $x \in M$,

- $N$ lies in the content of the inclusion $B^M(x, \varepsilon/1000) \to M$.
- The index of $N$ in the content of the inclusion $B^M(x, \varepsilon) \to M$ is $\leq C$.

**Lemma 22.** (Kapovitch–Wilking) Let $X_i \in \mathcal{M}_{Ric}(n, c, D)$ converge in the Gromov–Hausdorff sense to a space $X$. Consider a $k$-regular point $x \in X$, and $h_i : X \to X_i$ a sequence of Gromov–Hausdorff approximations. Then there is $\eta > 0$ and a sequence $\eta_i \to 0$ such that the contents of the inclusions $B^X(h_i(x), \eta_i) \to X_i$, $B^X(h_i(x), \eta) \to X_i$ coincide.

For the proof of Corollary 2 we require the following result from [9].

**Theorem 23.** (Mondello–Mondino–Perales) For each $n \in \mathbb{N}$ there is $\varepsilon > 0$ such that if $X_i \in \mathcal{M}_{Ric}(n, -1, \varepsilon)$ converges in the Gromov–Hausdorff sense to a space $X$ of Hausdorff dimension $k$ and $\beta_1(X) \geq k$, then $X$ is bi-Hölder homeomorphic to a flat $k$-dimensional torus.
3 Proof of the main results

Proof of Theorem 1: Let \( p \in X \) be a \( k \)-regular point, \( h_i : X \to X_i \) a sequence of Gromov–Hausdorff approximations, and set \( p_i := h_i(p) \). Then by Theorem 22, there is \( \varepsilon_2 > 0 \) and a sequence \( \eta_i \to 0 \) such that the contents of the maps \( B_X(p_i, \eta_i) \to X_i, B_X(p_i, \varepsilon_2) \to X_i \) coincide.

By Theorem 18, there is \( \varepsilon_1 \in (0, \varepsilon_2] \) such that for each \( x \in X \), the content of the inclusion \( B_X(x, 2\varepsilon_1) \to X \) is trivial. By Theorem 19, all we need to show is that for large enough \( i \), \( H^{1i}_1(X_i) \) has rank \( \leq n - k \). By Corollary 20, there is a sequence \( \rho_i \to 0 \) with the property that \( H^{1i}_1(X_i) = H^{1i}_1(X_i) \) for each \( i \).

By Theorem 21, there are \( \varepsilon_0 > 0, C \in \mathbb{N} \), subgroups \( N_i \leq H_1(X_i) \), and a sequence \( \delta_i \in [\varepsilon_0, \varepsilon_1] \) with the property that for each \( x \in X_i \), the content of the map \( B_X(x, \delta_i) \to X_i \) contains \( N_i \) as a subgroup of index \( \leq C \). Let \( x_1, \ldots, x_m \in X \) be such that \( X = \bigcup_{j=1}^m B_X(x_j, \varepsilon_0/3) \), and set \( x^i_j := h_i(x_j) \). Then for large enough \( i \), the balls \( B_X(x^i_j, \varepsilon_0/2) \) cover \( X_i \). This implies that for large enough \( i \), each ball of radius \( \rho_i \) in \( X_i \) is contained in a ball of the form \( B_X(x^i_j, \varepsilon_0) \). Hence if we let \( U_i \) denote the family \( \{ B_X(x^i_j, \delta_i) \}_{j=1}^m \), then by Lemma 17 we get

\[
H^{1i}_1(X_i) \leq H_1(U_i \setminus X_i) \leq H^{1i}_1(X_i) = H^{1i}_1(X_i).
\]

Since \( H^{1i}_1(X_i) \) is generated by the contents of the inclusions \( B_X(x^i_j, \delta_i) \to X_i \) with \( j \in \{1, \ldots, m\} \), the index of \( N_i \) in \( H^{1i}_1(X_i) \) is at most \( C^m \). Therefore, the rank of \( H^{1i}_1(X_i) \) equals the rank of \( N_i \) for all large enough \( i \).

Let \( \Gamma_i \leq H_1(X_i) \) denote the content of the inclusion \( B_X(p_i, \varepsilon_2) \to X_i \). Since \( \varepsilon_2 \geq \varepsilon_1 \), \( \Gamma_i \) contains \( N_i \), and since \( \Gamma_i \) equals the content of the inclusion \( B_X(p_i, \eta_i) \to X_i \), and \( \eta_i \leq \varepsilon_0 \) for large enough \( i \), the index of \( N_i \) in \( \Gamma_i \) is finite. Hence Theorem 1 will follow from the following claim.

Claim: For large enough \( i \), \( \Gamma_i \) has rank \( \leq n - k \).

Let \( \lambda_i \to \infty \) be a sequence that diverges so slowly that \( \lambda_i \eta_i \to 0 \) and the sequence \( (\lambda_i X_i, p_i) \) converges in the pointed Gromov–Hausdorff sense to \( \mathbb{R}^k \). We can achieve this since \( p \) is \( k \)-regular and \( \eta_i \to 0 \). Let \( (Y_i, q_i) \) denote the regular cover of \( (\lambda_i X_i, p_i) \) with Galois group \( H_1(X_i) \). By Theorem 9 and Theorem 12, we can assume that the sequence \( (Y_i, q_i) \) converges in the pointed Gromov–Hausdorff sense to a proper geodesic space \( (Y, q) \), and the groups \( H_1(X_i) \) converge in the equivariant Gromov–Hausdorff sense to some closed group \( \Gamma \leq Iso(Y) \). Since all elements of \( H_1(X_i)\backslash \Gamma_i \) move \( q_i \) at least \( \varepsilon_2 \lambda_i \), away, the equivariant Gromov–Hausdorff limit of \( \Gamma_i \) equals \( \Gamma \) as well. Note that from the definition of equivariant Gromov–Hausdorff convergence, it follows that the \( \Gamma_i \)-orbits of \( q_i \) converge in the pointed Gromov–Hausdorff sense to the \( \Gamma \)-orbit of \( q \).

By Corollary 16, \( Y \) splits isometrically as a product \( \mathbb{R}^k \times Z \) with \( Z \) a proper geodesic space of Hausdorff dimension \( \leq n - k \), such that the \( Z \)-fibers coincide with the \( \Gamma \)-orbits. Since the topological dimension is always dominated by the Hausdorff dimension ([7], Chapter 7), the topological dimension of \( Z \) is at most \( n - k \). Then by Theorem 13, the rank of \( \Gamma_i \) is at most \( n - k \) for large enough \( i \).
Proof of Corollary 2: Let $\varepsilon > 0$ be given by Theorem 23. By Theorem 1, $\beta_1(X) \geq k$, and the result follows.

References


