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First Betti number and collapse

by

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Abstract

We show that when a sequence of Riemannian manifolds collapses under a lower Ricci curvature bound, the first Betti number cannot drop more than the dimension.

1 Introduction

For $n \in \mathbb{N}$, $c \in \mathbb{R}$, $D > 0$, let $\mathfrak{M}_{Ric}(n, c, D)$ (resp. $\mathfrak{M}_{sec}(n, c, D)$) denote the class of closed n-dimensional Riemannian manifolds of Ricci curvature $\geq c$ (resp. sectional curvature $\geq c$) and diameter $\leq D$. A significant proportion of the subject consists of understanding the relationship between sequences $X_i \in \mathfrak{M}_{Ric}(n, c, D)$ and their Gromov–Hausdorff limits. Our main result concerns the first Betti number of such limit space.

Theorem 1. Let $X_i \in \mathfrak{M}_{Ric}(n, c, D)$ be a sequence with $\beta_1(X_i) \geq r$ for each i. If X_i converges in the Gromov–Hausdorff sense to a space X containing a k-regular point, then

$$\beta_1(X) \geq r + k - n.$$

It has been known that for a Riemannian manifold M of almost non-negative Ricci curvature, if its first Betti number equals its dimension then M is homeomorphic to a torus. This result has been recently extended to singular spaces by Mondello, Mondino, and Perales [9]. A consequence of their work and Theorem 1 is the following.

Corollary 2. For each $n \in \mathbb{N}$, there is $\varepsilon > 0$ such that if $X_i \in \mathfrak{M}_{sec}(n, -\varepsilon, 1)$ is a sequence of spaces with $\beta_1(X_i) \geq n$ that converges in the Gromov–Hausdorff sense to a space X of Hausdorff dimension k, then X is bi-Hölder homeomorphic to a flat k-dimensional torus.

Remark 3. Theorem 1 shows that the first Betti number cannot drop more than the dimension. Contrastingly, the fundamental group can decrease in the limit even if there is no collapse: Otsu has constructed a sequence of metrics in $\mathbb{S}^3 \times \mathbb{R}P^2$ of positive Ricci curvature that converges in the Gromov–Hausdorff sense to a simply connected 5-dimensional space [10].

Theorem 1 is an improvement of the main result of [14]. On the other hand, the goal of this program is to solve following problem.

Question 4. Assume a sequence $X_i \in \mathfrak{M}_{Ric}(n, c, D)$ of spaces homeomorphic to the n-dimensional torus converges in the Gromov–Hausdorff sense to a space X. Is X necessarily homeomorphic to a torus?

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2 Preliminaries

In this section we recall the required material for Theorem 1 and Corollary 2, which we prove in the following section.

2.1 Gromov–Hausdorff topology

The basics on the subject can be found in ([2], Chapter 7).

Definition 5. We say that a function $f : X \to Y$ between metric spaces is an ε-isometry if for all $x_1, x_2 \in X$ one has $|d^X(x_1, x_2) - d^Y(fx_1, fx_2)| \leq \varepsilon$, and $f(X)$ intersects each closed ball of radius ε in Y. We say that a sequence of functions $f_i : X_i \to Y_i$ between metric spaces are *Gromov–Hausdorff approximations* if f_i is an ε_i-isometry for some sequence $\varepsilon_i \to 0$.

Proposition 6. (Gromov) Let X_i be a sequence of compact metric spaces, and let X be a complete metric space. Then the following are equivalent:

- There is a sequence $f_i : X_i \to X$ of Gromov–Hausdorff approximations.
- There is a sequence $h_i : X \to X_i$ of Gromov–Hausdorff approximations.

In either case, X is compact and one says that the sequence X_i *converges to* X *in the Gromov–Hausdorff sense.* Furthermore, there is a metric on the class of compact metric spaces modulo isometry that yields this topology.

Definition 7. We say that a function $f : (X, x) \to (Y, y)$ between pointed metric spaces is an ε-isometry if $fx = y$, for all $x_1, x_2 \in B^X(x, 2/\varepsilon)$ one has $|d^X(x_1, x_2) - d^Y(fx_1, fx_2)| \leq \varepsilon$, and $f(B^X(x, 2/\varepsilon))$ intersects each closed ball of radius ε in $B^Y(y, 1/\varepsilon)$. We say that a sequence of functions $f_i : (X_i, x_i) \to (Y_i, y_i)$ between pointed metric spaces are *pointed Gromov–Hausdorff approximations* if f_i is a pointed ε_i-isometry for some sequence $\varepsilon_i \to 0$.

Proposition 8. (Gromov) Let (X_i, x_i) be a sequence of proper pointed metric spaces, and let (X, x) be a complete pointed metric space. Then the following are equivalent:

- There is a sequence $f_i : (X_i, x_i) \to (X, x)$ of pointed Gromov–Hausdorff approximations.
- There is a sequence $h_i : (X, x) \to (X_i, x_i)$ of pointed Gromov–Hausdorff approximations.

In either case, X is proper and one says that the sequence (X_i, x_i) *converges to* (X, x) *in the pointed Gromov–Hausdorff sense.* Furthermore, there is a metric on the class of proper pointed metric spaces modulo isometry that yields this topology.

For $n \in \mathbb{N}$, $c \in \mathbb{R}$, we denote by $\mathfrak{M}_{Ric}(n, c)$ the class of complete n-dimensional Riemannian manifolds of Ricci curvature $\geq c$. One reason we know so much about these families of spaces is because they are pre-compact with respect to the Gromov–Hausdorff topology.

Theorem 9. (Gromov) Let (Y_i, y_i) be a sequence with $Y_i \in \mathfrak{M}_{Ric}(n, c)$ for each i. Then one can find a subsequence that converges in the pointed Gromov–Hausdorff sense to some proper metric space (Y, y).

2.2 Equivariant Gromov–Hausdorff convergence

There is a well studied notion of convergence of group actions in this setting. For a proper metric space X, the topology that we use on its group of isometries $Iso(X)$ is the compact-open topology, which in this setting coincides with both the topology of pointwise convergence and the topology of uniform convergence on compact sets. This topology makes $Iso(X)$ a locally compact second countable metrizable group.

Definition 10. Let (Y_i, q_i) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space (Y, q). Consider pointed Gromov–Hausdorff approximations $f_i : (Y_i, q_i) \to (Y, q)$ and $h_i : (Y, q) \to (Y_i, q_i)$ such that $d^Y(f_i \circ h_i(y), y) \to 0$ for all $y \in Y$. Also let $\Gamma_i \leq Iso(Y_i)$ be a sequence of groups of isometries. We say that Γ_i *converges in the equivariant Gromov–Hausdorff sense to* a closed group $\Gamma \leq Iso(Y)$ if for all $R, \varepsilon > 0$, one has the following:

- For each $g \in \Gamma$, there is $i_0 \in \mathbb{N}$ such that for each $i \geq i_0$ there is $g_i \in \Gamma_i$ with $d^Y(f_i \circ g_i \circ h_i(y), g(y)) \leq \varepsilon$ for all $y \in B^Y(q, R)$.
- There is $i_0 \in \mathbb{N}$ such that if $i \geq i_0$, $g \in \Gamma_i$ with $d^Y(gq_i, q_i) \leq R$, then there is $\gamma \in \Gamma$ such that $d^Y(f_i \circ g \circ h_i(y), \gamma(y)) \leq \varepsilon$ for all $y \in B^Y(q, 10R)$.

Although this definition clearly depends on f_i and h_i, we usually omit this when we state that Γ_i converges to Γ.

This definition of equivariant convergence allows one to take limits before or after taking quotients.

Lemma 11. Let (Y_i, q_i) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space (Y, q), and $\Gamma_i \leq Iso(Y_i)$ a sequence of isometry groups that converges in the equivariant Gromov–Hausdorff sense to a closed group $\Gamma \leq Iso(Y)$. Then the sequence $(Y_i/\Gamma_i, [q_i])$ converges in the pointed Gromov–Hausdorff sense to $(Y/\Gamma, [q])$.

Since the isometry groups of proper metric spaces are locally compact, one has an Arzelá-Ascoli type result ([5], Proposition 3.6).

Theorem 12. (Fukaya–Yamaguchi) Let (Y_i, q_i) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space (Y, q), and take a sequence $\Gamma_i \leq Iso(Y_i)$ of groups of isometries. Then there is a subsequence $(Y_{i_k}, q_{i_k}, \Gamma_{i_k})_{k\in\mathbb{N}}$ such that Γ_{i_k} converges in the equivariant Gromov–Hausdorff sense to a closed group $\Gamma \leq Iso(Y)$.

In [6], Gromov studied which is the structure of discrete groups that act transitively on spaces that look like $\mathbb{R}^n$. Using the Malcev embedding theorem, he showed that they look essentially like lattices in nilpotent Lie groups. In [1], Breuillard–Green–Tao studied in general what is the structure of discrete groups that have a large portion acting on a space of controlled doubling. It turns out that the answer is still essentially just lattices in nilpotent Lie groups. In ([13], Sections 7-9) the ideas from [6] and [1] are used to obtain the following structure result.

Theorem 13. Let (Z, p) be a proper pointed geodesic space of topological dimension $\ell \in \mathbb{N}$ and let (D_i, p_i) be a sequence of discrete metric spaces converging in the pointed Gromov–Hausdorff sense to (Z, p). Assume there is a sequence of isometry groups $\Gamma_i \leq Iso(D_i)$ that act transitively and for each i, Γ_i is generated by its elements that move p_i at most 10. Then for large enough i, there are finite index subgroups $G_i \leq \Gamma_i$ and finite normal subgroups $F_i \triangleleft G_i$ such that G_i/F_i is isomorphic to a quotient of a lattice in a nilpotent Lie group of dimension ℓ. In particular, if the groups Γ_i are abelian, for large enough i their rank is at most ℓ.

For $k \in \mathbb{N}$, a proper metric space X, we say that $x \in X$ is a *k-regular point* if for any sequence $\lambda_i \to \infty$, the sequence $(\lambda_i X, x)$ converges in the pointed Gromov–Hausdorff sense to $\mathbb{R}^k$. For limits of sequences in $\mathfrak{M}_{Ric}(n, c)$, almost all points are regular [3].

Theorem 14. (Cheeger–Colding) Let $X_i \in \mathfrak{M}_{Ric}(n, c)$ converge in the pointed Gromov–Hausdorff sense to a space X. If $\mathcal{R}_k$ denotes the set of k-regular points of X, then $\mathcal{R}_k \neq \emptyset$ implies $k \leq n$, and $\cup_{j=0}^{n} \mathcal{R}_k$ is dense in X.

Arguably the most used tool in the theory of Riemannian manifolds of non-negative Ricci curvature is the Cheeger–Gromoll splitting theorem. It was later generalized by Cheeger and Colding to limits of Riemannian manifolds [4]. Using this, one could understand how $\mathbb{R}^k$ arises as a quotient of such spaces.

Theorem 15. (Cheeger–Colding) Let $\varepsilon_i \to 0$ and $(Y_i, q_i) \in \mathfrak{M}_{Ric}(n, -\varepsilon_i)$ a sequence that converges in the pointed Gromov–Hausdorff sense to (Y, q). If Y contains an isometric copy of $\mathbb{R}^k$, then Y split as a metric space as $\mathbb{R}^k \times Z$ for some proper geodesic space Z of Hausdorff dimension $\leq n - k$.

Corollary 16. Let $\varepsilon_i \to 0$ and $(Y_i, q_i) \in \mathfrak{M}_{Ric}(n, -\varepsilon_i)$ be a sequence that converges in the pointed Gromov–Hausdorff sense to (Y, q). Assume there is a sequence of groups of isometries $\Gamma_i \leq Iso(Y_i)$ such that $(Y_i/\Gamma_i, [q_i])$ converges in the pointed Gromov–Hausdorff sense to $\mathbb{R}^k$ and Γ_i converges in the equivariant Gromov–Hausdorff sense to a group $\Gamma \leq Iso(Y)$. Then Y splits as a metric space as $\mathbb{R}^k \times Z$ for some proper geodesic space Z of Hausdorff dimension $\leq n - k$, and the Z-fibers given by this product coincide with the orbits of Γ.

Proof. One can use the submetry $\phi : Y \to Y/\Gamma = \mathbb{R}^k$ to lift the lines of $\mathbb{R}^k$ to lines in Y passing through q. By Theorem 15, we get the desired splitting $Y = \mathbb{R}^k \times Z$ with $\phi(z_0, x) = x$ for all $x \in \mathbb{R}^k$ and some $z_0 \in Z$.

Let $g \in \Gamma$ and assume $g(z_0, x) = (z, y)$ for some $z_0, z \in Z$, $x, y \in \mathbb{R}^k$. Then for all $t \geq 1$, one has

$$\begin{aligned} t|y-x| &= |\phi(z_0, x+t(y-x) - \phi((z_0,x))| \\ &= |\phi(z_0, x+t(y-x)) - \phi(z,y)| \\ &\leq d^Y((z_0, x+t(y-x)), (z,y)) \\ &= \sqrt{d^Z(z_0,z)^2 + |(t-1)(y-x)|^2}. \end{aligned}$$

As $t \to \infty$, this is only possible if $x = y$, and we conclude that the action of Γ respects the splitting $Y = \mathbb{R}^k \times Z$. □

2.3 Homology and Ricci curvature bounds

We define the *content* of a map $A \to X$ between topological spaces to be the image of the natural map $H_1(A) \to H_1(X)$. If $\mathcal{U}$ is a family of subsets of X, we denote by $H_1(\mathcal{U} \prec X) \leq H_1(X)$ the subgroup generated by the contents of the inclusions $U \to X$ with $U \in \mathcal{U}$. This group satisfies a natural monotonicity property.

Lemma 17. Let X be a topological space, and $\mathcal{U}$, $\mathcal{V}$ two families of subsets of X. If for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ with $U \subset V$, then $H_1(\mathcal{U} \prec X) \leq H_1(\mathcal{V} \prec X)$.

If $\varepsilon > 0$, X is a metric space, and $\mathcal{U}$ is the family of balls of radius ε in X, then we denote $H_1(\mathcal{U} \prec X)$ simply by $H_1^\varepsilon(X)$. It has been recently shown that limits of sequences in $\mathfrak{M}_{Ric}(n,c,D)$ are semi-locally-simply-connected [12].

Theorem 18. (Pan–Wang) Let $X_i \in \mathfrak{M}_{Ric}(n,c,D)$ converge in the Gromov–Hausdorff sense to a space X. Then X is semi-locally-simply-connected. In particular, $H_1^\varepsilon(X)$ is trivial for small enough ε.

Theorem 19. (Sormani–Wei) Let X be a compact geodesic space. Assume there is $\varepsilon > 0$ such that $H_1^{2\varepsilon}(X)$ is trivial, and let Y be a compact geodesic space with $f : Y \to X$ an $\varepsilon/100$-approximation. Then there is a surjective morphism $H_1(Y) \to H_1(X)$ (independent of ε) whose kernel is precisely $H_1^\varepsilon(Y)$.

Proof sketch: We follow the lines of ([11], Theorem 2.1), where they prove this result for π_1 instead of H_1. Each 1-cycle in Y can be thought as a family of loops $\mathbb{S}^1 \to Y$ with integer multiplicity. For each map $\gamma : \mathbb{S}^1 \to Y$, by uniform continuity one could pick finitely many cyclically ordered points $\{z_1, \ldots, z_m\} \subset \mathbb{S}^1$ such that $\gamma([z_{j-1}, z_j])$ is contained in a ball of radius $\varepsilon/10$ for each j. Then set $\phi(\gamma) : \mathbb{S}^1 \to X$ to be the loop with $\phi(\gamma)(z_j) = f(\gamma(z_j))$ for each j, and $\phi(\gamma)|_{[z_{j-1},z_j]}$ a minimizing geodesic from $\phi(\gamma)(z_{j-1})$ to $\phi(\gamma)(z_j)$.

Clearly, $\phi(\gamma)$ depends on the choice of the points z_j and the minimizing paths $\phi(\gamma)|_{[z_{j-1},z_j]}$. However, the homology class of $\phi(\gamma)$ in $H_1(X)$ does not depend on these choices, since different choices yield curves that are ε-uniformly close, which by hypothesis are homologous.

Assume that a 1-cycle c in Y is the boundary $\partial\sigma$ of a 2-chain σ. After taking iterated barycentric subdivision, one could assume that each simplex of σ is contained in a ball of radius $\varepsilon/10$. By recreating σ in X via f simplex by simplex, one could find a 2-chain whose boundary is $\phi(c)$. This means that ϕ induces a map $\tilde{\phi} : H_1(Y) \to H_1(X)$.

In a similar fashion, if a 1-cycle c in Y is such that $\phi(c)$ is the boundary of a 2-chain σ, one could again apply iterated barycentric subdivision to obtain a 2-chain σ' in X whose boundary is $\phi(c)$ and such that each simplex is contained in a ball of radius $\varepsilon/10$. Using f one could recreate the 1-skeleton of σ' in Y in such a way that expresses c as a linear combination with integer coefficients of 1-cycles contained in balls of radius ε in Y. This implies that the kernel of $\tilde{\phi}$ is contained in $H_1^{\varepsilon}(Y)$.

If a 1-cycle c in Y is contained in a ball of radius ε, then $\phi(c)$ is contained in a ball of radius 2ε and then by hypothesis, $\phi(c)$ is a boundary. This shows that the kernel of $\tilde{\phi}$ is precisely $H_1^{\varepsilon}(Y)$.

Lastly, for any loop $\gamma : \mathbb{S}^1 \to X$, one can create via f a loop $\gamma_1 : \mathbb{S}^1 \to Y$ such that $\phi(\gamma_1)$ is uniformly close (and hence homologous) to γ, so $\tilde{\phi}$ is surjective. □

Corollary 20. Let X be a compact geodesic space. Assume there is $\rho > 0$ such that $H_1^{2\rho}(X)$ is trivial, and consider a sequence X_i of compact geodesic spaces that converges to X in the Gromov–Hausdorff sense. Then there is a sequence $\rho_i \to 0$ such that $H_1^{\rho_i}(X_i) = H_1^{\rho}(X_i)$ for each i.

Proof. For large enough i, let $\rho_i \in (0, \rho]$ be such that $\rho_i \to 0$ and there is a $\rho_i/100$-approximation $X_i \to X$. One could then apply Theorem 19 for $\varepsilon \in [\rho_i, \rho]$ to get a map $H_1(X_i) \to H_1(X)$ whose kernel equals both $H_1^{\rho_i}(X_i)$ and $H_1^{\rho}(X_i)$. For small i, simply set $\rho_i = \rho$. □

The following results were obtained in [8], and are stated in terms of π_1. The first one states that for $M \in \mathfrak{M}_{Ric}(n, c, D)$, there is a subgroup $N \leq H_1(M)$ that can be detected anywhere. The second one states that at regular points, there is a gap phenomenon.

Theorem 21. (Kapovitch–Wilking) For each $n \in \mathbb{N}$, $c \in \mathbb{R}$, $D > 0$, $\varepsilon_1 > 0$, there are $\varepsilon_0 > 0$, $C \in \mathbb{N}$, such that the following holds. For each $M \in \mathfrak{M}_{Ric}(n, c, D)$, there is $\varepsilon \in [\varepsilon_0, \varepsilon_1]$ and a subgroup $N \leq H_1(M)$ such that for all $x \in M$,

- N lies in the content of the inclusion $B^M(x, \varepsilon/1000) \to M$.
- The index of N in the content of the inclusion $B^M(x, \varepsilon) \to M$ is $\leq C$.

Lemma 22. (Kapovitch–Wilking) Let $X_i \in \mathfrak{M}_{Ric}(n, c, D)$ converge in the Gromov–Hausdorff sense to a space X. Consider a k-regular point $x \in X$, and $h_i : X \to X_i$ a sequence of Gromov–Hausdorff approximations. Then there is $\eta > 0$ and a sequence $\eta_i \to 0$ such that the contents of the inclusions $B^{X_i}(h_i(x), \eta_i) \to X_i$, $B^{X_i}(h_i(x), \eta) \to X_i$ coincide.

For the proof of Corollary 2 we require the following result from [9].

Theorem 23. (Mondello–Mondino–Perales) For each $n \in \mathbb{N}$ there is $\varepsilon > 0$ such that if $X_i \in \mathfrak{M}_{sec}(n, -1, \varepsilon)$ converges in the Gromov–Hausdorff sense to a space X of Hausdorff dimension k and $\beta_1(X) \geq k$, then X is bi-Hölder homeomorphic to a flat k-dimensional torus.

3 Proof of the main results

Proof of Theorem 1: Let $p \in X$ be a k-regular point, $h_i : X \to X_i$ a sequence of Gromov–Hausdorff approximations, and set $p_i := h_i(p)$. Then by Theorem 22, there is $\varepsilon_2 > 0$ and a sequence $\eta_i \to 0$ such that the contents of the maps $B^{X_i}(p_i, \eta_i) \to X_i$, $B^{X_i}(p_i, \varepsilon_2) \to X_i$ coincide.

By Theorem 18, there is $\varepsilon_1 \in (0, \varepsilon_2]$ such that for each $x \in X$, the content of the inclusion $B^X(x, 2\varepsilon_1) \to X$ is trivial. By Theorem 19, all we need to show is that for large enough i, $H_1^{\varepsilon_1}(X_i)$ has rank $\leq n - k$. By Corollary 20, there is a sequence $\rho_i \to 0$ with the property that $H_1^{\rho_i}(X_i) = H_1^{\varepsilon_1}(X_i)$ for each i.

By Theorem 21, there are $\varepsilon_0 > 0$, $C \in \mathbb{N}$, subgroups $N_i \leq H_1(X_i)$, and a sequence $\delta_i \in [\varepsilon_0, \varepsilon_1]$ with the property that for each $x \in X_i$, the content of the map $B^{X_i}(x, \delta_i) \to X_i$ contains N_i as a subgroup of index $\leq C$

Let $x_1, \ldots, x_m \in X$ be such that $X = \cup_{j=1}^m B^X(x_j, \varepsilon_0/3)$, and set $x_j^i := h_i(x_j)$. Then for large enough i, the balls $B^{X_i}(x_j^i, \varepsilon_0/2)$ cover X_i. This implies that for large enough i, each ball of radius ρ_i in X_i is contained in a ball of the form $B^{X_i}(x_j^i, \varepsilon_0)$. Hence if we let $\mathcal{U}_i$ denote the family $\{B^{X_i}(x_j^i, \delta_i)\}_{j=1}^m$, then by Lemma 17 we get

$$H_1^{\rho_i}(X_i) \leq H_1(\mathcal{U}_i \prec X_i) \leq H_1^{\varepsilon_1}(X_i) = H_1^{\rho_i}(X_i).$$

Since $H_1^{\mathcal{U}_i}(X_i)$ is generated by the contents of the inclusions $B^{X_i}(x_j^i, \delta_i) \to X_i$ with $j \in \{1, \ldots, m\}$, the index of N_i in $H_1^{\mathcal{U}_i}(X_i)$ is at most C^m. Therefore, the rank of $H_1^{\varepsilon_1}(X_i)$ equals the rank of N_i for all large enough i.

Let $\Gamma_i \leq H_1(X_i)$ denote the content of the inclusion $B^{X_i}(p_i, \varepsilon_2) \to X_i$. Since $\varepsilon_2 \geq \varepsilon_1$, Γ_i contains N_i, and since Γ_i equals the content of the inclusion $B^{X_i}(p_i, \eta_i) \to X_i$, and $\eta_i \leq \varepsilon_0$ for large enough i, the index of N_i in Γ_i is finite. Hence Theorem 1 will follow from the following claim.

Claim: For large enough i, Γ_i has rank $\leq n - k$.

Let $\lambda_i \to \infty$ be a sequence that diverges so slowly that $\lambda_i \eta_i \to 0$ and the sequence $(\lambda_i X_i, p_i)$ converges in the pointed Gromov–Hausdorff sense to $\mathbb{R}^k$. We can achieve this since p is k-regular and $\eta_i \to 0$.

Let (Y_i, q_i) denote the regular cover of $(\lambda X_i, p_i)$ with Galois group $H_1(X_i)$. By Theorem 9 and Theorem 12, we can assume that the sequence (Y_i, q_i) converges in the pointed Gromov–Hausdorff sense to a proper geodesic space (Y, q), and the groups $H_1(X_i)$ converge in the equivariant Gromov–Hausdorff sense to some closed group $\Gamma \leq Iso(Y)$. Since all elements of $H_1(X_i) \backslash \Gamma_i$ move q_i at least $\varepsilon_2 \lambda_i$ away, the equivariant Gromov–Hausdorff limit of Γ_i equals Γ as well. Note that from the definition of equivariant Gromov–Hausdorff convergence, it follows that the Γ_i-orbits of q_i converge in the pointed Gromov–Hausdorff sense to the Γ-orbit of q.

By Corollary 16, Y splits isometrically as a product $\mathbb{R}^k \times Z$ with Z a proper geodesic space of Hausdorff dimension $\leq n - k$, such that the Z-fibers coincide with the Γ-orbits. Since the topological dimension is always dominated by the Hausdorff dimension ([7], Chapter 7), the topological dimension of Z is at most $n - k$. Then by Theorem 13, the rank of Γ_i is at most $n - k$ for large enough i. $\square$

Proof of Corollary 2: Let $\varepsilon > 0$ be given by Theorem 23. By Theorem 1, $\beta_1(X) \geq k$, and the result follows. $\square$

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