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Preprint Series 2022 (65)

Date of submission: October 03, 2022

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MPIM 22-65

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September 30, 2022

Abstract. Let Γ be an affine group acting properly discontinuously on $\mathbb{R}^n, n \leq 5$. Then Γ contains a free non-commutative group if and only if the semisimple part of the Zariski closure of Γ contains $SO(2,1)$ as a normal subgroup.

1 Introduction

Let X be a topological space and let Γ be a subgroup of the group of homeomorphisms of X. A subgroup Γ is said to act *properly discontinuously* on X if for every compact subset K of X the set $\{g \in \Gamma : gK \cap K \neq \emptyset\}$ is finite. A subgroup Γ is called *crystallographic* if Γ acts properly discontinuously on X and the orbit space X/Γ is compact. Recall that a torsion free affine group Γ acts properly discontinuously on $\mathbb{R}^n$ if and only if Γ is the fundamental group of a complete locally flat affine manifold M. Obviously, $M = \mathbb{R}^n/\Gamma$. In 1964 L. Auslander conjectured that if Γ is an affine group acting properly discontinuously on $\mathbb{R}^n$ and $\mathbb{R}^n/\Gamma$ is compact then Γ is virtually solvable.

In 1977 J.Milnor asked if the fundamental group $\pi(M)$ of a complete locally flat affine manifold M contains a free non-commutative subgroup?

The Tits' alternative implies, that if the answer to Milnor's question is negative then the fundamental group $\pi(M)$ is virtually solvable. Thus the answer to Milnor's question negatively means that the Auslander conjecture is true without the assumption that M is compact. This is obviously true for n = 1 and is not difficult to prove for n = 2.

In 1983 Margulis showed that for $n = 3$ the answer to Milnor's question is positive by constructing a free (non-commutative) discrete subgroup of the affine group that acts properly discontinuously on $\mathbb{R}^3$ and leaves a quadratic form of signature $(2,1)$ invariant. Moreover, the linear part of Γ is Zariski dense in $SO(2,1)$. Actually, this example came as a surprise and is sometimes called "the Margulis phenomenon".

The aim of this paper is to prove the following theorem.

Main Theorem *Let Γ be an affine group acting properly discontinuously on $\mathbb{R}^n$, $n \leq 5$. Then Γ contains a free subgroup if and only if the semisimple part of the Zariski closure of Γ contains $SO(2,1)$ as a normal subgroup.*

Let us give a short description of the proof. It easily follows from [M1], that if the semisimple part of the Zariski closure of Γ contains $SO(2,1)$ as a normal subgroup then Γ contains a free subgroup. The difficult part is to show that if the Zariski closure of the linear part of Γ does not contain $SO(2,1)$ as a normal subgroup then Γ does not contains a free subgroup. For the proof, we look at the semisimple parts S of the Zariski closure of Γ, where Γ is an affine group acting properly discontinuously in dimension at most 5 and S does not contain $SO(2,1)$ as a normal subgroup. We give a complete list and can exclude all cases except one, based on our earlier work [AMS3]. The remaining case, $SL_2(\mathbb{R}) \times SO(3)$ is dealt with in section 3.

We remark that there exists a affine group Γ which acts properly discontinuously on $\mathbb{R}^6$ and contains a free subgroup such that the semisimple part of the Zariski closure of Γ does not contain $SO(2,1)$ as a normal subgroup [DGK].

The authors would like to thank several institutions for their support during the preparation of this paper: Bielefeld University, Bar-Ilan University, Yale University, Max Planck Institute for Mathematics (Bonn). Without all these supports, the paper whose authors live on three different continents could not have seen the light of day.

2 Linear parts of affine groups groups acting properly discontinuously

2.1. Notation and terminology. In this section we introduce the terminology we will use throughout the paper. Let $V = \mathbb{R}^n, n > 1$, and let $GL(V)$ be the group of all linear transformations of the vector space V. Let $\mathrm{Aff}(\mathbb{R}^n)$ be the group of affine transformations of the affine space $\mathbb{R}^n$. Since the group $\mathrm{Aff}(\mathbb{R}^n)$ is the semidirect product $GL(V) \ltimes V$ every element $g \in \mathrm{Aff}(\mathbb{R}^n)$ is a pair $g = (l(g), v_g)$ where $l(g) \in GL(V), v_g \in V$. The linear transformation $l(g)$ is called the linear part of g and v_g is called a translational vector. Let $[l(g)]$ be the matrix of $l(g)$ and let $[v_g]$ be the coordinates of v_g in the same basis. Then we obtain a group isomorphism

$$\phi(g) = \begin{pmatrix} [l(g)] & [v_g] \\ 0 & 1 \end{pmatrix} \qquad (*)$$

between $\mathrm{Aff}(\mathbb{R}^n)$ and a subgroup of $GL_{n+1}(\mathbb{R})$.
Denote by l the natural homomorphism $l : \mathrm{Aff}(\mathbb{R}^n) \to GL(V)$. The set $l(X)$ where $X \subseteq \mathrm{Aff}(\mathbb{R}^n)$ is called the linear part of X.

Let Γ be an affine group and let G be the Zariski closure of Γ. Let S be a semisimple part of G. Clearly, S is a semisimple part of the connected component of the linear part $l(G)$ of G. The goal of this section is to give a complete list of all possible non- trivial semisimple subgroups $S, S < GL(V)$, which might be a semisimple part of an affine group which acts properly discontinuously. The semisimple subgroups of $l(G)$, which occur in our list have to fulfill the following assumptions$(P1), (P2)$ and $(P3)$ below.

(P1) $S < GL(V)$, $dim\, V \leq 5$.

(P2) *S does not contain $SO(2,1)$ as a normal subgroup*

(P3) *Every element $g \in l(G)^0$ has one as an eigenvalue.*

The motivations for $(P1)$ and $(P2)$ are obvious. The justification for $(P3)$ follows from Proposition 2.2 [AMS1] that says: if Γ acts properly discontinuously then every element of the connected component $l(G)^0$ of $l(G)$ has one as an eigenvalue.

2.2. Linear parts and decompositions. Let $l(G)$ be a subgroup of $GL(V)$

$\dim V \leq 5$. Let V_0 be the maximal subspace in V such that S acts trivially on V_0. Let V_1 be the unique S–invariant subspace such that $V = V_0 \oplus V_1$.

Case 1 Assume that for every regular element $s \in S$ the restriction $s|_{V_1}$ does not have 1 as an eigenvalue. Thus $V_0 \neq 0$. Consider the inclusion $i_s : S \longrightarrow GL(V_1)$ as a representation of the semisimple Lie group S.

Assume first that S is a simple group. It follows from [AMS4] that all possible semisimple parts of G which have property (P2) are:

(1) $S = SL_l(\mathbb{R}), V_1 = \mathbb{R}^l, 2 \leq l < 5, 2 < n \leq 5, l < n$,

(2) $S = Sp_4(\mathbb{R}), V_1 = \mathbb{R}^4$

(3) $S = SL_2(\sigma(\mathbb{C})), V_1 = \mathbb{R}^4$

where $\sigma : \mathbb{C} \to M_2(\mathbb{R})$ is the standard embedding.

Suppose that the group S is semisimple, but not simple. It follows from [AMS4] that all possible semisimple parts in this case that have property (P3) are:

(4) $S = SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), V_1 = \mathbb{R}^4, n = 5$.

Case 2. Assume that for every regular element $s \in S$ the restriction $s|_{V_1}$ has 1 as an eigenvalue. It follows from [AMS4, 2.4, 2.5] that in this case

(1) $S = SO(3,2), \dim V_1 = 5$

(2) $S = SO(4,1), \dim V_1 = 5$.

(3) $S = SO(3) \times SL_2(\mathbb{R})$, $n = 5$.

Case 1 and 2 give us a complete list of all possible semisimple parts of G that have properties (P1),(P2) and (P3).

Our strategy is to show case by case that none of the semisimple groups listed in Case 1 and Case 2 is a semisimple part of G. Thus, $S = 1$ and Γ is virtually solvable.

3 The dynamics of the action of an affine group

We recall here some basic definitions and notions of the dynamics of the action of an affine group (see. [AMS1]. [AMS2]). Let Γ be an affine group acting properly discontinuously on $\mathbb{R}^n$. Let G be the Zariski closure of Γ. Obviously, Γ acts properly discontinuously if a subgroup of Γ of finite index acts properly discontinuously. Therefore from now on we will assume that the linear part $l(G)$ of G is a connected algebraic group.

Let $g \in G$. Let $l(g)$ be a semisimple element of $l(G)$. Then $\mathbb{R}^n$ is the direct sum of $A^+(g)$, $A^-(g)$ and $A^0(g)$, where $A^+(g)$ (resp. $A^-(g)$, $A^0(g)$) is the subspace of $\mathbb{R}^n$ such that all eigenvalues of the restriction $l(g)\mid_{A^+(g)}$ have modulus > 1 (resp. $l(g)\mid_{A^-(g)}$ have modulus < 1, and $l(g)\mid_{A^0(g)}$ have modulus 1). Set $D^+(g) = A^+(g) \oplus A^0(g)$ and $D^-(g) = A^-(g) \oplus A^0(g)$. Clearly, $D^-(g) = D^+(g^{-1})$ and $D^-(g) \cap D^+(g) = A^0(g)$. Let $\| \centerdot \|$ and d

denote the norm and metric on $\mathbb{R}^n$ corresponding to an inner product on $\mathbb{R}^n$. Let $\|g\|_-$ be the norm of the restriction $g|_{A^-(g)}$. Set $\|g\|_+ = \|g^{-1}\|_-$ and put $s(g) = \max\{\|g\|_+, \|g\|_-\}$. Obviously, $s(g) = s(g^{-1})$. Let $g \in GL(V)$. Set $V_g^0 = \{v \in V; gv = v\}$. Let G be a subgroup of $GL(V)$. A semisimple element $g \in G$ is called ***regular*** in G if

$$\dim V_g^0 = \min\{\dim V_t^0 |\, t \in G, t \ \ \textit{semisimple} \ \}$$

Let us remark that the set of regular elements of an algebraic group is Zariski open.
Let $g \in G$ be a semisimple element. such that

$$\dim(A^0(g)) = \min\{\dim A^0(t) | t \in G, t\, \textit{semisimple,}\}$$

then g is called $\mathbb{R}$-***regular*** in G. Let G be an affine group, $G <$Aff$\mathbb{R}^n$. An affine transformation $g \in G$ is called regular (respectively $\mathbb{R}$-regular) if $l(g)$ is a regular (respectively $\mathbb{R}$-regular) element of $l(G)$.

Our definition of $\mathbb{R}$-regular element slightly differs from that of [P] were it was first introduced. Note that the set of $\mathbb{R}$-regular elements in an algebraic group G need not be Zariski open in G. Nevertheless under some conditions a Zariski dense subgroup of an algebraic group G contains an $\mathbb{R}$-regular element [P],[AMS1],[AMS4]. For example this is true if $G = SO(B)$ where B is a non degenerate form of signature (p, q) and Γ is a Zariski dense subgroup of G. Note that in case $p = 2, q = 1$ every hyperbolic element is regular and $\mathbb{R}$-regular.

The metric $\|\,\centerdot\,\|$ on $\mathbb{R}^n$ induces the standard metric $\widehat{d}$ on the projective space $P = \mathbb{P}(\mathbb{R}^n)$ by the formula (see [T]).

$$\widehat{d}([v], [w]) = \frac{\|v \wedge w\|}{\|v\|\|w\|} = \sin \angle(v, w)$$

for any two points $[v], [w] \in P$ where v, w are non-zero vectors in $\mathbb{R}^n$. Let X, Y be two closed subset of P. Set and $\underline{d}(X, Y) = \min_{x \in X, y \in Y} \widehat{d}(x, y)$ and $\overline{d}(X, Y) = \max \min_{x \in X, y \in Y} \widehat{d}(x, y)$. We can and will consider a linear subspace $W \neq \{0\}$ of $\mathbb{R}^n$as a closed subset of P.

A *hyperbolic* element g is called ε *–hyperbolic* if $\underline{d}(A^+(g), A^-(g)) > \varepsilon$. There exists a positive constant $\overline{s}(\varepsilon)$ such that for every ε *-hyperbolic* element g and $n \in \mathbb{Z}$, we have

$$s(g^n) \leq \overline{s}(\varepsilon) s(g)^{|n|}.$$

Two hyperbolic elements g and h are called *transversal* if $D^+(g) \cap A^-(h) = D^-(g) \cap A^+(h) = D^+(h) \cap A^-(g) = D^-(h) \cap A^+(g) = \{0\}$. Two transversal elements g and h are called ε*-transversal* if $\underline{d}(D^+(g), A^-(h)) > \varepsilon$, $\underline{d}(D^-(g), A^+(h)) > \varepsilon$, $\underline{d}(D^+(h), A^-(g)) > \varepsilon$, $\underline{d}(D^-(h), A^+(g)) > \varepsilon$. Obviously, g and h are transversal (resp. ε-transversal) if and only if g^{-1} and h^{-1} are transversal (resp. ε-transversal). Two transversal elements g and h are called ***very transversal*** if g and h^{-1} are transversal. Therefore if g and h are ***very transversal*** then h and g^{-1} are transversal.

For any $g \in G$ there exists an eigenvector vector $v, v \in A^0(g)$ such that $l(g)v = v$ by Proposition 2.2 [AMS3]. Hence for any semisimple element g of G there exists a g-invariant line L_g. The restriction of g to L_g is the translation by a non- zero vector t_g. Let us note that for a given $g \in G$ all such lines are parallel and the vector t_g does not depend on the choice of L_g. We take for g the g-invariant line L_g closest to the origin. Let us define the following affine subspaces: $E_g^+ = D^+(g) + L_g$, $E_g^- = D^-(g) + L_g$, $E_g^+ \cap E_g^- = C_g$. Let $p \in L_g$ be a point. Then $t_g = \overrightarrow{p\, gp}$. Clearly $t_g = -t_{g^{-1}}$, $L_g = L_{g^{-1}}$. Let s be an affine transformation such that $t_s = 0$. Then $L_h = l(s)L_g$, $t_h = l(s)t_g$ for $h = sgs^{-1}$. We denote by $o(g)$ the restriction of g to C_g.

Let g be an ε-hyperbolic element of G. Assume that $x \in E_g^-$ and $y \in L_g$ such that $\overrightarrow{xy} \in D^-(g)$. Then there exists a constant $c(\varepsilon)$ such that for $n \in \mathbb{Z}, n > 0$, we have $d(g^n(x), g^n(y)) \leq c(\varepsilon) s(g)^n d(x, y)$.

Let $\{g_0, h_1, \ldots, h_m\} \subset G$ be ε–hyperbolic elements, pairwise very ε– transversal. Set $s = \max\{s(g_0), s(h_1), \ldots, s(h_m)\}$ and $s_0 = s^{1/2}$. Let $g_\ell = h_{i_\ell}^{n_\ell} \cdot \cdots \cdot h_{i_1}^{n_1} \cdot g_0$, $1 \leq i_k \leq m$, $i_k \neq i_{k+1}, n_k \in \mathbb{Z}, 1 \leq k \leq (l-1)$, and $M_\ell = |n_1| + \cdots + |n_\ell|$. From Lemma 1.3 [AMS2] follows then that there exists a constant $s(\varepsilon) < 1$ such that if $s_0 < s(\varepsilon)$,

$$s(g_\ell) \leq s_0^{M_\ell + 1} \tag{1}$$

$$\overline{d}(A^+(g_{\ell-1}), A^+(g_\ell)) \leq \frac{\varepsilon}{2} s_0^{M_{\ell-1}} \tag{2}$$

$$\overline{d}(A^-(g_0) A^-(g_\ell)) \leq \frac{\varepsilon}{2} s_0 \tag{3}$$

$$\overline{d}(A^+ g_\ell), A^+(h_{i_\ell})) \leq \frac{\varepsilon}{2} s_0^{i_\ell} \tag{4}$$

$$\underline{d}(A^+(g_\ell), A^+(h_i) \cup A^-(h_i)) \geq \frac{\varepsilon}{2}, i \neq i_\ell \tag{5}$$

$$\underline{d}(A^+(g_\ell), A^-(g_\ell)) \geq \varepsilon/2 \tag{6}$$

It is well known that there exists a positive constant $s_1(\varepsilon)$ such that for $s_0 \leq s_1(\varepsilon)$ the group G_1 generated by $g_0, h_1, \dots, h_m$ is free with free generators $g_0, h_1, \dots, h_m$. There is a choice of $g_0, h_1, \dots, h_m$ such that the group generated by $g_0, h_1, \dots, h_m$ is Zariski dense in G. The proof is based on the so-called Ping-Pong Lemma. For details see [AMS1], [AMS2].

Let $q_0 \in \mathbb{R}^n$ be the origin. Let q_ℓ be the point of C_{g_ℓ} such that $d(q_0, q_\ell) = d(q_0, C_{g_\ell})$. Set $d_{g_\ell} = d(q_\ell, g_\ell q_\ell)$. From Lemma 1.6 [AMS2] follows that there exist constants $s_2(\varepsilon)$, $d_1(\varepsilon)$ and $d_2(\varepsilon)$ such that for $s_0 < \min\{s(\varepsilon), s_2(\varepsilon)\}$ we have

$$d(q_0, C_{g_\ell}) < d_1(\varepsilon) \tag{7}$$

and

$$d_{g_\ell} \leq d_2(\varepsilon)|M_l| \tag{8}$$

The identification procedure. Let g and h be two hyperbolic, transversal elements of G. Following [AMS2, chapter 3] we consider the following subspaces and projections. Let $C_{h,g} = E_h^+ \cap E_g^-$ and $C_{g,h} = E_h^- \cap E_g^+$. Set $\pi_h^- : C_{g,h} \to C_h$ along $A^-(h)$ $\pi_h^+ : C_h \to C_{h,g}$ along $A^+(h)$, $\pi_g^- : C_{h,g} \to C_g$ along $A^-(g)$ and $\pi_g^+ : C_g \to C_{g,h}$ Define the following transformation $\overline{o}(gh)$ of $C_{g,h}$ as $\overline{o}(gh) = \pi_g^+\overline{o}(g)\pi_g^-\pi_h^+\overline{o}(h)\pi_h^-$. Obviously, $\overline{o}(g^n h^m) = \pi_g^+\overline{o}(g)^n\pi_g^-\pi_h^+\overline{o}(h)^m\pi_h^-$ for positive $n, m \in \mathbb{Z}$.

The reasons for this definition are the following. The map $\overline{o}(g^n h^m)$ of $C_{g,h}$ approximates $g^n h^m$ in the following sense. For positive integers n, m such that $n \to \infty, m \to \infty$ we have $E^+_{g^n h^m} \to E^+_g$ and $E^-_{g^n h^m} \to E^-_h$. Therefore $C_{g^n h^m} \to C_{g,h}$. For a given $q \in C_{g,h}$ and $\overline{q} = \overline{o}(g^n h^m)q$ for every positive numbers ε_k such that $\varepsilon_k \to 0$, there exists $\delta_k, \delta_k > 0$, $\delta_k \to 0$, positive numbers N_k, $N_k \to \infty$ and $q_k \in U(q, \delta_k)$ such that for $n_k, m_k > N_k$ we have $d(\overline{o}(g^{n_k} h^{m_k})q, g^{n_k} h^{m_k} q_k) < \varepsilon_k$. We can thus approximate $g^n h^m$ for certain points near $C_{g,h}$ by the orthogonal map $\overline{o}(g^n h^m)$ for sufficiently big n, m.

Let $\{g_0, h_1, \ldots, h_m\} \subset G$ be ε–hyperbolic elements, pairwise very ε–transversal. and let $g_\ell = h_{i_\ell}^{n_\ell} \cdot \cdots \cdot h_{i_1}^{n_1} \cdot g_0$, $1 \leq i_k \leq m$, $i_k \neq i_{k+1}, n_k \in \mathbb{Z}, 1 \leq k \leq (l-1)$, and $M_\ell = |n_1| + \cdots + |n_\ell|$. Set $\overline{o}(g_\ell) = \pi^+_{h_{i_l}}\overline{o}(h_{i_\ell}^{n_\ell})\pi^-_{h_{i_l}} \ldots \pi^+_{h_{i_1}}\overline{o}(h_{i_1}^{n_1})\pi^-_{h_{i_1}}\pi^+_{g_0}\overline{o}(g_0)\pi^-_{g_0} = \pi^+_{h_{i_l}}\overline{o}(h_{i_\ell})^{n_\ell} \ldots \overline{o}(h_{i_1})^{n_1}\pi^-_{h_{i_1}}\pi^+_{g_0}\overline{o}(g_0)\pi^-_{g_0}$ and let $\pi_\ell = \pi^+_{h_{i_l}}\pi^-_{h_{i_l}} \ldots \pi^+_{h_{i_1}}\pi^+_{g_0}\pi^-_{g_0}$.

From now on we will assume that Γ is an affine group such that the linear part $l(\Gamma)$ is Zariski dense in $SL_2(\mathbb{R}) \times SO(3)$. Hence $l(G) = SL_2(\mathbb{R}) \times SO(3)$. In this case for a $\mathbb{R}$–regular element $g \in G$ we have $\dim A^+(g) = \dim A^-(g) = 1$, $\dim A^0(g) = 3$ and the restriction $l(g) \mid_{A^0(g)} \in SO(3)$. Let V_1 and V_2 be two $l(G)$–invariant subspaces of $\mathbb{R}^5$ such that $\mathbb{R}^5 = V_1 \oplus V_2$ and $l(G)|_{V_1} = SL_2(\mathbb{R})$ and $l(G)|_{V_2} = SO(3)$. Denote by π_i the map $\pi_i : l(\Gamma)) \to l(G)|_{V_i}, i = 1, 2$. Let $g \in SO(3)$ be an element of infinite order. Then there exists an eigenvector $v_0(g) \in \mathbb{R}^3$ with eigenvalue 1. Let $V_0(g)$ be the one- dimensional subspace of $\mathbb{R}^3$ spanned by $v_0(g)$. Let p_g be the set $V_0(g) \cap S^2$. Let $g, h \in SO(3)$ be two elements of infinite order which do not commute. Let P be the subspace of $\mathbb{R}^3$ spanned by $v_0(g)$ and $v_0(h)$. Obviously, $\dim P = 2$.

Lemma 3.1 Let $g, h \in SO(3)$ be two non-commuting elements of infinite order. Let

$g(t)$ and $h(s)$ be the one parameter subgroups, such that $g(1) = g$ and $h(1) = h$. Let P be the subspace of $\mathbb{R}^3$ spanned by $v_0(g)$ and $v_0(h)$. Then for every vector $v \in \mathbb{R}^3 \backslash P$ there exist $t, s \in \mathbb{R}, t, s > 0$ such that $g(t)h(s)v = v$.

Proof Let σ be the reflection in P. Then there exist two rotations $g(t)$ and $h(s)$ such that $h(s)v = \sigma v$ and $g(t)\sigma v = v$. Thus, $g(t)h(s)v = v$.

Let $\gamma_a, \gamma_b \in \Gamma$ be two $\mathbb{R}$-regular elements. Denote by $V_0(\pi_2(l(\gamma_a^m \gamma_b^n)))$ the space spanned by $v_0(\pi_2(l(\gamma_a^m \gamma_b^n)))$ and put $p_{(n,m)} = V_0(\pi_2(l(\gamma_a^m \gamma_b^n))) \cap S^2$

Proposition 3.2. There exist two very transversal hyperbolic elements $\gamma_a, \gamma_b \in \Gamma$ such that the set $\{p_{(n,m)}, n, m \in \mathbb{Z}, n > 0, m > 0\}$, is dense in S^2.

Proof. Let γ_a and γ_b be two very transversal elements. Then the group Γ_1 generated by $l(\gamma_a)$ and $l(\gamma_b)$ contains the free group generated by $l(\gamma_a^n)$ and $l\gamma_b^n)$ for some enough big n. Let us show that the group generated by $\pi_2(l(\gamma_a))$ and $\pi_2(l(\gamma_b))$ is dense in $SO(3)$. Indeed, if the subgroup generated by $\pi_2(l(\gamma_a))$ and $\pi_2(l(\gamma_b))$ is not dense in $SO(3)$ then it is virtually abelian. Therefore there exists G_1 a subgroup of finite index in G and nonzero vector $v, v \in V_2$ such that $\pi_2(l(g))v = v$ for every $g \in G_1$. Assume that V_1 is $l(G)$–invariant. Then L_{g_a} and L_{g_b} are parallel. Hence by the same arguments we use in the proof of Proposition 2.9, [AMS3] we conclude that Γ does not act properly discontinuously. Assume that V_2 is $l(G)$–invariant. Since the restriction $l(G)|_{V_2}$ is virtually abelian, the infinite group $[G_1, G_1]$ acts trivially on V_2. Hence $[G_1, G_1]$ has a fixed point fixed point in $\mathbb{R}^5$ that is impossible because an infinite subgroup $\Gamma \cap G_1$ acts properly discontinuously. Thus we will assume that elements $\pi_2(l(\gamma_a))$ and $\pi_2(l(\gamma_b))$ fulfill the requirements of Lemma 3.1. Let $\overline{\gamma_a} = \pi_2(\gamma_a)$ and $\overline{\gamma_b} = \pi_2(\gamma_b)$ and $\overline{\gamma_a}(t)$ and $\overline{\gamma_b}(t)$ be one parameter subgroups such that $\overline{\gamma_a}(1) = \overline{\gamma_a}$ and $\overline{\gamma_b}(1) = \overline{\gamma_b}$. The semigroup generated by $\overline{\gamma_a}$ (resp. $\overline{\gamma_b}$) is dense in $\overline{\gamma_a}(t)$ (resp. $\overline{\gamma_b}(t)$). Therefore by lemma 3.1 the set $p_{(n,m)}$ is dense in S^2.

Remark 2 It is obvious that for very transversal elements γ_a and γ_b we have $A^+(\gamma_a^n \gamma_b^m) \to A^+(\gamma_a)$, $A^+(\gamma_a^{-n} \gamma_b^{-m}) \to A^-(\gamma_b)$ $A^-(\gamma_a^n \gamma_b^m) \to A^-(\gamma_b)$, $A^-(\gamma_a^{-n} \gamma_b^{-m}) \to A^+(\gamma_a)$,

$E^+_{\gamma_a^n\gamma_b^m} \to E^+_{\gamma_a}$ and $E^-_{\gamma_a^n\gamma_b^m} \to E^-_{\gamma_b}$, $E^+_{\gamma_a^{-n}\gamma_b^{-m}} \to E^-_{\gamma_{ab}}$ and $E^-_{\gamma_a^{-n}\gamma_b^{-m}} \to E^+_{\gamma_a}$ when $m, n \to \infty$.
There exist ε and a set of ε–hyperbolic, pairwise very ε–transversal elements $\{\gamma_0, \gamma_1, \dots, \gamma_m\} \subset \Gamma$, such that the group generated by the set $\{l(\gamma_0), l(\gamma_1), l(\gamma_2) \dots, l(\gamma_m)\}$ is a free Zariski dense subgroup of $l(G)$ freely generated by$\{l(\gamma_0), l(\gamma_1), \dots, l(\gamma_m)\}$ (see [AMS1, Proposition 3.7]). Denote by Γ_0 the group generated by the set $\{\gamma_1, \dots, \gamma_m\}$ and put $\Gamma_n = \Gamma_0\gamma_0^n, n \in \mathbb{Z}, n > 0$. Recall that any element $\gamma \in \Gamma_n, n \geq 1$, is $\varepsilon/2$–hyperbolic.
Let q_0 be the point of origin. By (8) that there exists a constant $d^* = d(\varepsilon)$ such that

$$d_\Gamma = \max_{n\in\mathbb{Z},n>0}\{d(q_0, C_\gamma), \gamma \in \Gamma_n, n \geq 1\} \leq d^*. \tag{11}$$

By our definition above, $d_\gamma = d(q_\gamma, \gamma q_\gamma)$, where $q_\gamma \in C_\gamma$ such that $d(q_0, C_\gamma) = d(q_0, q_\gamma)$. Obviously $\{\gamma_0^n, n \in \mathbb{Z}\} \cap \Gamma_1 = \emptyset$. Thus we have $\Gamma_n \cap \Gamma_m = \emptyset$ for $n \neq m$. Since Γ acts properly discontinuously, from (11) follows that for every Γ_n there exists an element $\gamma_n \in \Gamma_n$ such that $d_{\gamma_n} = \min\{d_\gamma, , \gamma \in \Gamma_n\}$. Set $d_n = d_{\gamma_n}$.

Set $I_M = \{m, m > 0, m \in \mathbb{Z} \,|\, d_m < M\}$

Lemma 3.3 . For every $M \in \mathbb{Z}, M > 0$ the set $I_M = \{m, m > 0, m \in \mathbb{Z}|d_m < M\}$ is finite.

Proof. Suppose that there exists a positive number M such that the set $I_M = \{m, m > 0, m \in \mathbb{Z}|d_m < M\}$ is infinite. It is obvious that $d(q_0, \gamma_m q_{\gamma_m}) \leq d_\Gamma + M$ Hence for all $\gamma_m, m \in I_M$ we have $B(q_0, d_\Gamma + M) \cap \gamma_m B(q_0, d_\Gamma + M) \neq \emptyset$. This is a contradiction since Γ acts properly discontinuously.

From Lemma 3.3 follows that there exists an infinite sequence $\{\gamma_m, \gamma_m \in \Gamma_m\}$ such that $d_m = d_{\gamma_m} \to \infty$ when $m \to \infty$.

Remark 3 Recall that $A^-(\gamma_m) \to A^-(\gamma_0)$ and $E^-_{\gamma_m} \to E^-_{\gamma_0}$ when $m \to \infty$. Since the projective space is compact we can and will assume that there are two subspaces A^+ and

E^+ such that $A^+(\gamma_m) \to A^+$ and $E^+_{\gamma_m} \to E^+$ when $m \to \infty$.

Proposition 3.4. If $l(\Gamma)$ is Zariski dense in $SL_2(\mathbb{R}) \times SO(3)$ then Γ does not act properly discontinuously.

Proof. Our proof follows the same strategy that we used in the proof of [Lemma 5.1 AMS2.] Namely, we will show that there exists a constant $C = C(\varepsilon)$ such that if $d_m > C$ there exist an element γ of the group generated by $\gamma_a, \gamma_b \in \Gamma_0$ and positive number t such that $d_{\gamma^t \gamma_m} < d_{\gamma_m} = d_m$. Since, $\gamma^t \in \Gamma_0$ we will have $\gamma^t \gamma_m \in \Gamma_m$. This will contradict the definition $d_{\gamma_m} = \min\{d_\gamma, \gamma \in \Gamma_m\}$.

Using the notations from Remark 3 we set $E^+_s = C_{\gamma_s} \oplus A^+$, $C_s(n,m) = E^+_s \cap E^-_{g_{(n,m)}}$, where $\gamma_{(n,m)} = \gamma_a^n \gamma_b^m$ and $C_{s,n,m} = (A^-(\gamma_0) \oplus C^-_{\gamma_s}) \cap E^+_{\gamma_{(n,m)}}$, $C_{\gamma_{(n,m)}} = E^-_{\gamma_{(n,m)}} \cap (C_{\gamma_m} \oplus A^+.)$ Let us set the following projections $\pi^-_s : C_{s,n,m} \to C_{\gamma_s}$ along $A^-(\gamma_s)$, $\pi^+_s : C_{\gamma_s} \to C_s(n,m)$ along A^+, $\pi^-_{\gamma(n,m)} : C_s(n,m) \to C_{\gamma_{(n,m)}}$ along $A^-(\gamma_{(n,m)})$ and $\pi^+_{(n,m)} : C_{\gamma_{(n,m)}} \to C_{s,n,m}$. Since elements $\gamma_{(n,m)}, \gamma_s$ are ε–transversal and ε–hyperbolic all these projections are $L(\varepsilon)$– Lipschitz. From Proposition 3.2 follows that for every positive θ there exist a finite subset $S^* \subseteq \{\gamma_a^n \gamma_b^m, n, m \in \mathbb{Z}\}$ such that $\Pi = \{p_{(n,m)}, \gamma_a^n \gamma_b^m \in S^*\}$ is a θ–net of the sphere $S^2 \subset \mathbb{R}^3$. Namely, for every vector of norm one in V_2 there exists an element $\gamma \in S^*$ such that $|\sin \angle(v, \tau_\gamma)| < \theta$. We choose θ such that

$$\theta L(\varepsilon) < 1/4 \tag{12}$$

Let $q_{s,n,m}$ be a point in $C_{s,n,m}$ such that $\pi^-_{\gamma_s}(q_{s,n,m}) = q_s$. Then

$$q_{s,n,m}(k) = \pi^+_{\gamma(n,m)} o(\gamma_{(n.m)})^k \pi^-_{\gamma(n,m)} \pi^+_s o(\gamma_s) \pi^-_s(q_{s,n,m}) \in C_{s,n,m}$$

and

$$\pi^-_{\gamma(n,m)} \pi^+_s o(\gamma_s)(q_s) - \pi^-_{\gamma(n,m)} \pi^+_s(q_s) = \pi^-_{\gamma(n,m)} \pi^+_s \gamma_s q_s - \pi^-_{\gamma(n,m)} \pi^+_s(q_s) = \pi^-_{\gamma(n,m)} \pi^+_s(\gamma_s q_s - q_s).$$

Set $\pi_k : C_{s,n,m} \to C_{\gamma^k_{(n,m)}\gamma_s}$ along $A^+(\gamma^k_{(n,m)}\gamma_s) \oplus A^-(\gamma^k_{(n,m)}\gamma_s)$. Let $q_1 = \pi_k(q_{s,n,m})$, $q_2 = \pi_k(\gamma^k_{(n,m)}\gamma_s q_1)$. Then $q_2 = \gamma^k_{(n,m)}\gamma_s q_1$ It is easy to see that if the scalar product

$(\tau_{\gamma_{(m,n)}}, \pi_{s,n,m}(\tau_{\gamma_s})) > 0$ then the scalar product $(\tau_{\gamma_{(-m,-n)}}, \pi_{s,-n,-m}(\tau_{\gamma_s})) < 0$. Thus we can and will assume that we take an element $\gamma_{(m,n)} \in S^*$ such that the scalar product is negative. Using the same argument we used in the proof of Lemma 5.7 [AMS2] we conclude from (12) that there exists an element $\gamma_{(n,m)} \in S^*$, a positive number $k = k(\gamma_s)$, and constants $c(\varepsilon)$ and $c(S^*)$ such that we have

$$d_{\gamma_{(n,m)}^k \gamma_s} \leq \frac{1}{4} d_{\gamma_s} + c(\varepsilon) + c(S^*)$$

Therefore if $d_{\gamma_s} > 2[c(\varepsilon) + c(S^*)]$ then $d_{\gamma_{(n,m)}^k \gamma_s} < d_{\gamma_s}$. Since $\gamma_{(n,m)} \in \Gamma_0$ this contradicts the definition of d_{γ_s} and proves the proposition.

4 The main theorem.

Main theorem. Let Γ be an affine group acting properly discontinuously on the affine space $\mathbb{R}^n, n \leq 5$. Then Γ does not contain a free non- abelian subgroup if and only if the Zariski closure G of Γ does not contain $SO(2,1)$ as a normal subgroup.

Proof. . Let G be the Zariski closure of Γ. Assume that Γ acts properly discontinuously and the semisimple part of G is not trivial. Then the possible cases for the linear realization of $l(G)$ are listed in Case 1, (1) -(4) and Case 2, (1)-(3). By the same arguments we used in [AMS 3, Proposition 3.6] we conclude that Case 1, (1) -(4) are impossible. Let $l(G)$ be as in Case 2. If $l(G) = SO(3,2)$ then by [AMS1] Γ does not act properly discontinuously. Assume that $l(G) = SO(4,1)$. Then G leaves invariant a form of signature $p = 4, q = 1$. Then Γ does not act properly discontinuously by [AMS2] since $p - q > 2$. In case 2 (3) Γ does not act properly discontinuously by Proposition 3.4. This proves the statement. □

Corollary. Let Γ be a crystallographic group, $\Gamma < \mathrm{Aff}\mathbb{R}^n, n \leq 5$. Then Γ is virtually solvable.

Proof. Let G be the Zariski closure of Γ. Assume that $l(G)$ does not contain $SO(2,1)$ as a normal subgroup,. Then Γ does not contain a free subgroup by our Main Theorem. Thus by the Tits alternative, Γ is virtually solvable. Assume that $l(G)$ contains $SO(2,1)$ as a normal subgroup. Then the space $\mathbb{R}^5$ is the direct sum of two $l(G)$–invariant subspace $\mathbb{R}^5 = V_1 \oplus V_2, \dim V_1 = 3$, $\dim V_2 = 2$. Then the real rank of every simple subgroup of $l(G)$ is ≤ 1. Hence Γ is virtually solvable [S],[To].

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