Holomorphic rank-2 vector bundles over non-Kähler elliptic surfaces I

Dedicated to Professor F. Hirzebruch for his 75th birthday.

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Introduction

In this paper, our goal is to study holomorphic rank-2 vector bundles over non-Kähler elliptic surfaces from the point of view of two classical problems: the existence and classification problems. Let X be a smooth compact complex surface. The existence problem consists in determining which topological complex vector bundles admit holomorphic structures, or equivalently, in finding all triples $(r, c_1, c_2) \in \mathbb{N} \times NS(X) \times \mathbb{Z}, r \geq 2$, for which there exists a rank-r holomorphic vector bundle E on X with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$. For projective surfaces, Schwarzenberger [S] proved that any triple $(r, c_1, c_2) \in \mathbb{N} \times NS(X) \times \mathbb{Z}, r \geq 2$, comes from a rank-$r$ holomorphic (algebraic) vector bundle. In contrast, for non-projective surfaces, there is a natural necessary condition for the existence problem:

$$\Delta(r, c_1, c_2) := \frac{1}{r}\left(c_2 - \frac{r-1}{2r}c_1^2\right) \geq 0$$

(cf. [B-L] for the general case; for rank-2 case, see [B-F, E-F]). Using extensions of coherent sheaves, one proves the following sufficient condition (cf. [B-L], see also [B-F, LP]):

$$\Delta(r, c_1, c_2) \geq m(r, c_1),$$

where

$$m(r, c_1) := -\frac{1}{2r}\max\left\{\sum_1^n \left(\frac{c_1}{r} - \mu_i\right)^2, \mu_1, \ldots, \mu_r \in NS(X), \sum_1^n \mu_i = c_1\right\};$$

this is true for any surface other than a K3 surface with algebraic dimension zero, for c_1 divisible by r in $NS(X)$ and $\Delta(r, c_1, c_2) = \frac{1}{r}$. This result also has a converse [B-L, B-F, LP]: any filtrable rank-r holomorphic vector bundle E on a non-algebraic surface X, with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$, satisfies the inequality

$$\Delta(E) := \Delta(r, c_1, c_2) \geq m(r, c_1).$$

Thus, the only unknown situations are in the range $\Delta(r, c_1, c_2) \in [0, m(r, c_1))$. If $m(r, c_1) \neq 0$, this interval is non-empty and to solve the existence problem, one has to construct holomorphic vector bundles with discriminant $\Delta(r, c_1, c_2)$ smaller than $m(r, c_1)$. All these vector bundles will, of course, be non-filtrable and the difficulty of the problem resides in the lack of a general construction method for this case. One is therefore compelled to focus on particular classes of surfaces, to find some specific construction methods.

The existence problem is completely solved in only two cases: the first is the case where the range for non-filtrable bundles is the same as the one for filtrable ones (i.e., $m(r, c_1) = 0$ for all $c_1 \in NS(X)$), and the second is the case of primary Kodaira surfaces (cf. [A-B-T]). In the rank-2 case, the existence problem was solved in [T3] for complex 2-tori, in [A-B-T] for primary Kodaira surfaces and in [T-T] for surfaces of class VII and K3 surfaces. Since the method used in [T-T] (Donaldson polynomials) seems to work also in the case of (non-algebraic) Kähler elliptic surfaces, only the case of non-Kähler elliptic surfaces remains. This article is divided into two parts. In the first, we close the existence problem in the rank-2 case, by giving necessary and sufficient conditions for the existence of holomorphic rank-2 vector bundles over non-Kähler elliptic surfaces. In the second part, we tackle the classification problem by studying the moduli spaces of stable rank-2 vector bundles over non-Kähler elliptic surfaces.

For any non-Kähler elliptic surface X, there exists a holomorphic map $X \xrightarrow{\pi} B$, over a curve B, whose general fibre is an elliptic curve T. By restricting a vector bundle E on X to the smooth fibres of π, one can consider E as a family of bundles over T; such families can be studied using Atiyah's classification [A] of vector bundles over elliptic curves. Furthermore, one can construct two objects, which have proven very useful in the study of vector bundles over elliptic surfaces (cf. [B-H, F1, F-M, F-M-W, M, T]). The first is an effective divisor on the Jacobian surface $J(X) = B \times T^*$ of X, called the *spectral curve* of E, that encodes the holomorphic type of E over each smooth fibre of π. In the case of rank-2 bundles, the second is a section of a ruled surface, defined as the quotient of the Jacobian surface by an involution (which depends on E).

The two parts of this article are organised as follows.

Part I. In the first section, we establish a correspondence between line bundles over a non-Kähler elliptic surface X and sections of its associated Jacobian surface $J(X) = B \times T^* \xrightarrow{p_1} B$ (cf. [F-M] for the algebraic case); this follows from results of [B1, B2, B3, B-U] regarding the Neron-Severi and Picard groups of non-Kähler elliptic surfaces. The construction of the spectral curve of a holomorphic rank-2 vector bundle over X is presented in the second section. Over an elliptic curve, a bundle is said to be *regular* if its group of automorphisms is of the smallest possible dimension. We show that if a vector bundle E is regular on all fibres of π, then it is completely determined by its spectral curve S and a certain line bundle on S. Finally, the third section is devoted to the proof of the existence theorem.

Part II. We now consider the moduli spaces $\mathcal{M}_{c_1,c_2}$ of stable holomorphic rank-2 vector bundles over X, with fixed Chern classes c_1 and c_2. In the first section, we define a map from each non-empty moduli space $\mathcal{M}_{c_1,c_2}$ to a projective space (the latter is the linear system of associated spectral curves on the Jacobian surface $J(X)$); we also describe the general fibre of this map. Section 2 is devoted to the study of rank-2 vector bundles, over a non-Kähler elliptic surface X, that are irregular over at least one fibre of π. In section 3, we define a Poisson structure on the moduli spaces of stable bundles over Kodaira surfaces and we prove that the map defined in section 1 is an algebraically completely integrable Hamiltonian system, with respect to this Poisson structure. In the last section, we introduce Fourier-Mukai transforms in the context of non-Kähler elliptic surfaces and study their properties.

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1 Line bundles over non-Kähler elliptic surfaces

Let $X \stackrel{\pi}{\to} B$ be a minimal non-Kähler elliptic surface (B a smooth compact connected curve). It is well-known that $X \to B$ is a quasi-bundle over B, i.e. all the smooth fibres are pairwise isomorphic and the singular fibres are multiples of elliptic curves (see [K], [B3]).

Let T denote the general fibre of $X \stackrel{\pi}{\to} B$, which is an elliptic curve and let T^* denote the dual of T (i.e. $T^* := \mathrm{Pic}^0(T) \cong T$ non-canonically). It is known that the Jacobian surface associated to $X \stackrel{\pi}{\to} B$, in this case, is simply

$$J(X) = B \times T^* \stackrel{p_1}{\to} B$$

(see, for example [K], [B-P-V], [B1]) and the surface $X \stackrel{\pi}{\to} B$ is obtained from its Jacobian surface $B \times T^*$ by a finite number of logarithmic transformations (see [K], [B-P-V], [B-U] and the considerations below).

In this section we establish a correspondence between line bundles over X and sections of its associated Jacobian surface $J(X)$.

1.1 Principal elliptic bundles

Let us suppose for a moment that $X \stackrel{\pi}{\to} B$ is a principal elliptic bundle, i.e. it has no multiple fibres and so, no singular fibres. By [B1, B2] we have

$$\text{Tors } H^2(X, \mathbb{Z}) = \text{Tors } NS(X) = \pi^* H^2(B, \mathbb{Z}).$$

Also, by [B3, T], the restriction morphism $H^2(X, \mathbb{Z}) \to H^2(T, \mathbb{Z})$ vanishes. Let $b \in B$ and let $T_b = \pi^{-1}(b)$; then, if $\mathcal{L} \in \mathrm{Pic}(X)$ is a line bundle on X, it follows that $c_1(\mathcal{L}|_{T_b}) = 0$, i.e., any line bundle on X can be seen as a family of line bundles of $\mathrm{Pic}^0(T)$.

More generally, let T be a n-torus and let $X \stackrel{\pi}{\to} B$ be a principal torus bundle over the curve B. Every T-principal bundle is described by a cohomology class $\xi \in H^1(B, \mathcal{O}_B(T))$, where $\mathcal{O}_B(T)$ denotes the sheaf of germs of locally holomorphic maps from B to T (see, for example [B-U]). For a Cech 1-cocycle (ξ_{ij}) the function

$$\xi_{ij} : U_i \cap U_j \to T$$

identifies $(z, t) \in U_i \times T$ with $(z, t') = (z, \xi_{ij}(z) + t) \in U_j \times T$ for all $z \in U_i \cap U_j$.

Now, for any line bundle $L \in \mathrm{Pic}(T)$, we have the homomorphism $\varphi_L : T \to \mathrm{Pic}^0(T) = T^*$,

$$\varphi_L(x) = \text{isom.class of}(T_x^* L \otimes L^{-1}),$$

where $T_x : T \to T$ is the translation with $x \in T$. The T-principal bundle $X \to B$ being fixed, we can associate to any line bundle $L \in \mathrm{Pic}(T)$ an element in $H^1(B, \mathcal{O}_B(T^*))$ in the following way: For the Cech 1-cocycle (ξ_{ij}) defining our T-principal bundle, $\xi_{ij} : U_i \cap U_j \to T$, we put

$$\eta_{ij}^L := \varphi_L \circ \xi_{ij} : U_i \cap U_j \to T^*.$$

Therefore, (η_{ij}^L) is a Cech 1-cocycle, which defines a cohomology class in $H^1(B, \mathcal{O}_B(T^*))$ denoted $\tilde{\varphi}_L(\xi)$. The T^*-principal bundle over B described by $\tilde{\varphi}_L(\xi)$ is called the associate T^*-bundle to L. We have (see [B-U]):

Lemma 1.1 *Let $L \in Pic(T)$ be a line bundle. Then, the obstruction to extend L to a line bundle on the total space of the fixed T-principal bundle $X \xrightarrow{\pi} B$ is the associated T^*-bundle to L, $\tilde{\varphi}_L(\xi)$.*

Remark 1.2 If $L \in \mathrm{Pic}^0(T)$ then its associated T^*-bundle is trivial (since L is invariant by translations) and, by the above lemma, L can be extended to a line bundle on the total space of the fixed T-principal bundle.

Let $X \xrightarrow{\pi} B$ be a T-principal bundle with T a n-torus. Choose a point $b \in B$ and fix it. We shall identify the torus T with the fibre $\pi^{-1}(b)$. Restricting a line bundle $\mathcal{L}$ on X to the fibre $\pi^{-1}(b)$ we have a natural group homomorphism:

$$\mathrm{Pic}(X) \xrightarrow{r} \mathrm{Pic}(\pi^{-1}(b)) = \mathrm{Pic}(T).$$

Then $\mathrm{Ker}(r)$ consists of isomorphism classes of line bundles whose restriction to the fibre $\pi^{-1}(b)$ is trivial, hence the restriction to each fibre of π is a line bundle of degree zero.

Let (U_j) be an open covering of B with trivializations

$$\pi^{-1}(U_j) \cong U_j \times T.$$

For each line bundle $\mathcal{L}$ belonging to $\ker(r)$ there exists a holomorphic mapping

$$\varphi_j : U_j \to \mathrm{Pic}^0(T) = T^*$$

with

$$\mathcal{L}|_{\pi^{-1}(U_j)} \cong (\varphi_j \times id_T)^*(\mathcal{P}),$$

where $\mathcal{P}$ is the Poincaré bundle on $\mathrm{Pic}^0(T) \times T$, by the universal property of the Poincaré bundle and the fact that $\mathcal{L}|_{\pi^{-1}(U_j)} \in \mathrm{Pic}(U_j \times T)$ has the property that $\mathcal{L}|_{\pi^{-1}(x)}$ has degree zero for any $x \in U_j$.

Since any line bundle of degree zero on the torus is invariant by the translations, on $U_j \cap U_k \neq \emptyset$ we have

$$\varphi_j = \varphi_k.$$

Hence, the line bundle $\mathcal{L}$ defines a holomorphic mapping

$$\varphi : B \to T^* , \tag{1.3}$$

which satisfies

$$\varphi(b) = [0] , \tag{1.4}$$

since the restriction $\mathcal{L}|_{\pi^{-1}(b)}$ is trivial.

Remark 1.5 If X is a surface, i.e. T is an elliptic curve then, since the restriction $\mathcal{L}|_{\pi^{-1}(b)} \in \mathrm{Pic}^0(T)$, we can obtain a holomorphic mapping $\varphi : B \to T^*$ as above without the normalization condition (1.4).

Note that by the arguments of the proof of Lemma 1.1 (see [B-U]) we can patch together $(\varphi_j \times id_T)^*(\mathcal{P})$ to get a line bundle on X, denoted by $\varphi^*(\mathcal{P})$, since the line bundles of degree zero on a torus are invariant by translations. The line bundle $\mathcal{L}$ and the holomorphic mapping φ are related by

$$\mathcal{L} \cong \pi^*(\mathcal{M}) \otimes \varphi^*(\mathcal{P}) , \tag{1.6}$$

where $\mathcal{M}$ is a line bundle on the curve B. Also note that there is a one to one correspondence between the set of holomorphic maps (1.3) with property (1.4) and $\mathrm{Hom}(J_B, T^*)$, where J_B is the Jacobian of the curve B (by the universal property of the Jacobian of a curve).

Let us consider a group homomorphism

$$R : \mathrm{Pic}(X) \xrightarrow{r} \mathrm{Pic}(T) \xrightarrow{c_1} H^2(T, \mathbb{Z}).$$

A line bundle $\mathcal{L}$ belonging to $\mathrm{Ker}(R)$ is the one whose restriction to each fibre of π is of degree zero (if $X \xrightarrow{\pi} B$ is a surface, then $\mathrm{Ker}(R) = \mathrm{Pic}(X)$). Each line bundle $L \in \mathrm{Pic}^0(T)$ can be extended to a line bundle $\mathcal{L}$ on X in such a way that its restriction to each fibre is isomorphic to L (see Remark 1.2). Hence, there is an isomorphism

$$\mathrm{Ker}(R)/\mathrm{Ker}(r) \cong \mathrm{Pic}^0(T).$$

Define subgroups P_j of $\mathrm{Pic}(X)$ by

$$P_0 = \mathrm{Pic}(X),\ P_1 = \mathrm{Ker}(r),\ P_2 = \pi^*\mathrm{Pic}(B).$$

Then, (P_j) defines a decreasing filtration of $\mathrm{Pic}(X)$ and we have the isomorphisms:

$$P_1/P_2 \cong \mathrm{Hom}(J_B, \mathrm{Pic}^0(T))$$

$$P_0/P_1 \cong \{L \in \mathrm{Pic}(T) |\ \tilde{\varphi}_L(\xi) = 0\}\ ,$$

where $\xi \in H^1(B, \mathcal{O}_B(T))$ is the cohomology class corresponding to the T-principal bundle $X \xrightarrow{\pi} B$ and $\tilde{\varphi}_L(\xi)$ is defined above (see [B-U], Theorem 6).

Remark 1.7 In the case of surfaces we have $P_0/P_1 \cong \mathrm{Pic}^0(T)$ since $P_0 = \mathrm{Pic}(X) = \mathrm{Ker}(R)$ and holomorphic mappings (1.3) are in one to one correspondence with the sections of the Jacobian surface $J(X) = B \times T^* \xrightarrow{p_1} B$.

1.2 Multiple fibres

Let us now consider quasi T-bundles $X \xrightarrow{\pi} B$ over a curve B, where $T = \mathbb{C}^n/\Lambda$ is a n-torus. By [B-U], every quasi T-bundle over B is obtained from the product $B \times T$ by means of logarithmic transformations as follows:

Choose points $b_1,\ b_2, ..., b_k$ on B and put $B' = B \setminus \{b_1,\ b_2, ..., b_k\}$. For each point b_i fix a positive integer m_i. Let a_i be an element of $\frac{1}{m_i}\Lambda$ such that the order of the point $[a_i]$ of the torus T corresponding to a_i is precisely m_i. Let

$$D_i = \{t_i \in \mathbb{C} |\ |t_i| < \varepsilon\}$$

be a coordinate neighbourhood of the point b_i and put

$$\hat{D}_i = \{s_i \in \mathbb{C} |\ |s_i| < \varepsilon^{1/m_i}\}.$$

By the mapping

$$\lambda_i : \hat{D}_i \to D_i,\ s_i \mapsto s_i^{m_i},$$

$\hat{D}_i$ is a m_i-sheeted ramified covering of D_i. A holomorphic mapping $g_i : \hat{D}_i \times T \to \hat{D}_i \times T$ defined by

$$g_i : (s_i, [\zeta]) \mapsto (e_{m_i} s_i, [\zeta + a_i])$$

is an analytic automorphism of order m_i and generates the cyclic group $G_i = \langle g_i \rangle$ of order m_i where

$$e_{m_i} = \exp(2\pi\sqrt{-1}/m_i).$$

Since the automorphism g_i has no fixed points, the quotient $\hat{D}_i \times T/G_i$ is a complex manifold. Let

$$\mu_i : \hat{D}_i \times T \to \hat{D}_i \times T/G_i$$

be the canonical mapping. We have a holomorphic mapping

$$\pi_i : \hat{D}_i \times T/G_i \to D_i\ ,\ [s_i, [\zeta]] \mapsto s_i^{m_i}.$$

Over the punctured disk D_i^* the holomorphic mapping π_i gives a T-principal bundle and over the origin 0 the equation $\pi_i = 0$ defines a divisor of the form $m_i T_i$ where $T_i = T/\langle [a_i] \rangle$ is a torus obtained as the quotient by a finite subgroup generated by the point $[a_i]$.

The mapping

$$\ell_{a_i} : \hat{D}_i^* \times T/G_i \to D_i^* \times T\ ,\ [s_i, [\zeta]] \mapsto (s_i^{m_i}, [\zeta - \frac{m_i a_i}{2\pi\sqrt{-1}} \log s_i])$$

is a well-defined holomorphic mapping and an isomorphism. Therefore, we can patch together $\hat{D}_i \times T/G_i,\ i = 1, 2, ..., k$ and $B' \times T$ by the isomorphisms ℓ_{a_i} to obtain a compact complex manifold X, which is called the manifold obtained from $B \times T$ by means of logarithmic transformations. There is a natural holomorphic mapping $\pi : X \to B$ given by π_i on $\hat{D}_i \times T/G_i$ and the projection to the first factor on $B' \times T$. The fiber space $\pi : X \to B$ is a T-principal bundle over B' and has multiple fibres with multiplicity m_i, if $m_i \geq 2$. Let us assume that our quasi T-bundle $X \xrightarrow{\pi} B$ has

$$m_i \geq 2,\ i = 1, 2, ..., \ell,\ m_{\ell+1} = ... = m_k = 1.$$

Now, we consider line bundles on X. Choose as above a general point $b \in B$ and consider a natural restriction homomorphism

$$r : \mathrm{Pic}(X) \to \mathrm{Pic}(\pi^{-1}(b)) = \mathrm{Pic}(T).$$

Note that for the multiple fibre m_iT_i the line bundle $\mathcal{O}_X(T_i)$ associated with the divisor T_i of X is an element of $\mathrm{Ker}(r)$ and $\mathcal{O}_X(T_i)^{m_i} = \mathcal{O}_X(m_iT_i)$ is the pull-back of the line bundle $\mathcal{O}_B(b_i)$ on the curve B.

Let P_2 be a subgroup of $\mathrm{Pic}(X)$ generated by $\pi^*\mathrm{Pic}(B)$ and $\mathcal{O}_X(T_i)$, $i = 1, 2, ..., \ell$. A line bundle $\mathcal{L}$ belonging to P_2 is characterized by the fact that the restriction of $\mathcal{L}$ to each fibre $\pi^{-1}(c)$, $c \in B'$ is the trivial line bundle.

To a line bundle $\mathcal{L} \in \mathrm{Ker}(r)$, by the same argument as above, we can associate a holomorphic mapping

$$\varphi' : B' \to \mathrm{Pic}^0(T) = T^*.$$

The pull-back $\mu_i^*(\mathcal{L}|_{\pi^{-1}(D_i)})$ defines also a holomorphic mapping

$$\hat{\varphi}_i : \hat{D}_i \to \mathrm{Pic}^0(T),$$

where μ_i is the above natural quotient mapping. Then, on $\hat{D}_i^*$ we have $\hat{\varphi}_i = \varphi' \circ \lambda_i$, where λ_i is defined above. It follows that the holomorphic mapping φ' can be extended to a holomorphic mapping

$$\varphi : B \to \mathrm{Pic}^0(T) = T^*.$$

As $\mathcal{L}|_{\pi^{-1}(b)}$ is a trivial line bundle, we have

$$\varphi(b) = [0].$$

Let H be a subgroup of the torus T generated by $[a_1]$, $[a_2], ..., [a_\ell]$. The group H is isomorphic to Λ_0/Λ where Λ_0 is the lattice generated by Λ and a_i's. To any H-invariant line bundle L on the torus T, we associate a cohomology class (η_{ij}^L) in $H^1(B, \mathcal{O}_B(T^*))$ as follows (cf. [B-U]):

Let (U_j) be an open covering of the curve B such that $U_i = D_i$ for $i = 1, 2, ..., \ell$ and that $b_i \notin U_i \cap U_j$ for $i \neq j$. Since the line bundle L is invariant by the translations $T_{[a_i]}$, though

$$\left[\frac{a_i}{2\pi\sqrt{-1}} \log t_i\right]$$

is multivalued

$$T^*_{[\frac{a_i}{2\pi\sqrt{-1}}\log t_i]} L \otimes L^{-1}$$

is a well-defined line bundle on $\pi^{-1}(U_i \cap U_j)$ for $i = 1, 2, ..., \ell$ and $j \neq i$. Then there exists a holomorphic mapping φ_{ij} from $U_{ij} = U_i \cap U_j$ to T^* such that the line bundle above is the pull-back $(\varphi_{ij} \times id_T)^*(\mathcal{P})$ of the Poincaré bundle. Put

$$\eta^L_{ij} := \begin{cases} \varphi_{ij} & : \quad 1 \leq i \leq \ell, \ \ell < j \\ 0 & : \quad \ell < i, \ j \end{cases}$$

Then, it follows that (η^L_{ij}) is a 1-cocycle and defines a cohomology class in $H^1(B, \mathcal{O}_B(T^*))$. We have (see [B-U]) the extension of Lemma 1.1:

Lemma 1.8 *If a line bundle L of T can be extended to a line bundle $\mathcal{L}$ on X, then L is invariant by H. An H-invariant line bundle on the torus $T = \pi^{-1}(b)$ can be extended to the one of X if and only if the cohomology class $[(\eta^L_{ij})]$ is zero.*

Now, as above, we define a decreasing filtration (P_j) of $\mathrm{Pic}(X)$ by

$$P_0 = \mathrm{Pic}(X), \ P_1 = \mathrm{Ker}(r),$$

$$P_2 = \text{the subgroup generated by } \pi^*\mathrm{Pic}(B) \text{ and } \mathcal{O}_X(T_i)'s,$$

where $m_i T_i$, $i = 1, 2, ..., \ell$ are all the multiple fibres of the quasi T-bundle $\pi : X \to B$. We have (see [B-U], Theorem 10) the isomorphisms:

$$P_1/P_2 \cong \mathrm{Hom}(J_B, \mathrm{Pic}^0(T))$$

$$P_0/P_1 \cong \{L \in \mathrm{Pic}(T)^H | \ [(\eta^L_{ij})] = 0\}.$$

Now we consider again the case of surfaces. We have:

Lemma 1.9 *Let $\pi : X \to B$ be a non-Kähler elliptic surface (i.e. a quasi T-bundle, where T is an elliptic curve). Let $\mathcal{L}$ be a line bundle on X. Then, for any smooth fibre $T_b = \pi^{-1}(b)$, the restriction $\mathcal{L}|_{T_b}$ has degree zero.*

Proof. Let $m_1T_1,\ m_2T_2, ...,\ m_\ell T_\ell$ be all multiple fibres of π and let $b_i = \pi(T_i)$. Let m denote the least common multiple of $m_1,\ m_2, ...,\ m_\ell$ and choose an integer $e \geq 0$ such that m divides $\ell + e$. Take distinct points $b_{\ell+1},\ b_{\ell+e}$, which are different from $b_i\ i = 1, ...,\ \ell$ and a fixed point b with T_b smooth. Then there exists at least one line bundle M on B with

$$M^{\otimes m} \cong \mathcal{O}_B(b_1 + ... + b_{\ell+e}).$$

Such an M defines a cyclic covering $\varepsilon : C \to B$ of degree m, totally ramified at $b_1, ...,\ b_{\ell+e}$ (see B-P-V], Chapter I, Lemma 17.1). By the Lemma 3.18 in [B3] we know that there exist a principal T-bundle $\pi' : Y \to C$ and a cyclic covering with group $\mathbb{Z}_m$, $\psi : Y \to X$ over the covering $\varepsilon : C \to B$. Let $\tilde{T}$ be a connected component of $\psi^{-1}(T_b)$. Then $\tilde{T}$ is a fibre of π' and the restriction $\tilde{T} \to T_b$ of ψ is an isomorphism. Now we have

$$c_1(\mathcal{L}|_{T_b}) = c_1(\psi^*(\mathcal{L})|_{\tilde{T}}) = 0$$

because $\pi' : Y \to C$ is a principal elliptic bundle. □

Remark 1.10 If X is a non-Kähler elliptic surface, by using the Lemma 1.9 and the construction above, we can obtain for *any line bundle* $\mathcal{L} \in \mathrm{Pic}(X)$ a holomorphic mapping $\varphi : B \to T^*$ with

$$\mathcal{L}|_{T_b} = \varphi(b)$$

without the normalization condition (1.4).

Proposition 1.11 *Let $\pi : X \to B$ be a non-Kähler elliptic surface, T the general fibre and let $J(X) = B \times T^*$ be the associated Jacobian surface of X. Then:*

(i) For any section $\Sigma \subset J(X)$ there exists a line bundle $\mathcal{L}$ on X whose restriction to every smooth fibre T_b is the same as the line bundle Σ_b of degree zero on $T = T_b$.

(ii) The set of all line bundles on X which induce the same line bundle of degree zero on every smooth fibre as that corresponding to the section Σ is a principal homogeneous space over $Pic^\tau(X)$, where $Pic^\tau(X)$ is the subgroup of line bundles on X whose first Chern class is torsion.

Proof. Let (P_j) be the filtration of $\mathrm{Pic}(X)$ defined above. Put

$$N(X) := \{L \in \mathrm{Pic}(T)^H |\ [(\eta_{ij}^L)] = 0\}$$

and

$$\tilde{N}(X) := \{c_1(L) |\ L \in N(X)\}.$$

By Theorem 17 [B-U] there exists an exact sequence

$$0 \to \mathrm{Hom}(J_B, T^*) \to NS(X)/\tilde{F_2} \to \tilde{N}(X) \to 0 \ ,$$

where $NS(X)$ is the Neron-Severi group of X and $\tilde{F_2}$ is a subgroup of $H^2(X, \mathbb{Z})$ generated by $c_1(\mathcal{O}_X(T_i))$, $i = 1, 2, ..., \ell$. By [B1] and [B-U] we know that $\tilde{N}(X) = 0$ and

$$NS(X)/\mathrm{Tors}NS(X) \cong \mathrm{Hom}(J_B, T^*).$$

It follows that $N(X) \subset \mathrm{Pic}^0(T)$ and $P_2 = \mathrm{Pic}^\tau(X)$. Since any line bundle in $\mathrm{Pic}^0(T)$ is invariant by translations by Lemmas 1.8, 1.9 we get

$$N(X) = \mathrm{Pic}^0(T).$$

Choose a point b with T_b smooth and fix it. Let $\lambda = \Sigma_b \in T^*$ and let Σ^λ be the constant section $B \times \lambda \subset J(X)$. By the construction above to the line bundle $\mathcal{L}^\lambda$, the extension on X of $\lambda \in T^*$, corresponds the constant section Σ^λ. By the isomorphism

$$P_1/P_2 \cong \mathrm{Hom}(J_B, T^*),$$

it follows that there exists a line bundle $\mathcal{L}_1 \in P_1 = \mathrm{Ker}(r)$ whose corresponding element in $\mathrm{Hom}(J_B, T^*)$ is $\Sigma - \Sigma^\lambda$. Then, the line bundle $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}^\lambda$ has the property that its restriction to every smooth fibre T_b is the same as the line bundle $\Sigma_b \in T^*$, hence (i). If the line bundles $\mathcal{L}'$ and $\mathcal{L}$ on X have the above property then, by the same isomorphism, it follows that $\mathcal{L}' \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^\tau(X)$, hence (ii). $\square$

2 Holomorphic vector bundles over elliptic fibre bundles.

Let $X \xrightarrow{\pi} B$ be a principal elliptic bundle over a curve B of genus g and with fibre T. Therefore, X is isomorphic to a quotient of the form

$$X = \Theta^*/\langle\tau\rangle,$$

where Θ is a line bundle on B with non-negative Chern class d, Θ^* is the complement of the zero section in the total space of Θ, and $\langle\tau\rangle$ is the multiplicative cyclic group generated by a fixed complex number $\tau \in \mathbb{C}$, with $|\tau| > 1$. The standard fibre of this bundle is

$$T \cong \mathbb{C}^*/\langle\tau\rangle \cong \mathbb{C}/(2\pi i\mathbb{Z} + \ln(\tau)\mathbb{Z}).$$

Throughout this section, we assume that $d(X) = d > 0$. Therefore, the surface X is non-Kählerian and the restriction $H^2(X,\mathbb{Z}) \to H^2(T,\mathbb{Z})$ vanishes (see Corollary 1.5 in [T]). Moreover, we have the following invariants

$$\begin{aligned} td(B) &= 1 + (1-g)h, \\ td(X) &= 1 + (1-g)\pi^* h, \end{aligned}$$

where h is the positive generator of $H^2(B,\mathbb{Z})$.

2.1 Line bundles.

The set of all holomorphic line bundles on X with trivial Chern class is given by the zero component of the Picard group $\mathrm{Pic}^0(X)$. Referring to Proposition 1.6 in [T], one has

$$\mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(B) \times \mathbb{C}^*.$$

Any line bundle in $\mathrm{Pic}^0(X)$ is therefore of the form $H \otimes L_\alpha$, where H is the pullback to X of an element of $\mathrm{Pic}^0(B)$ and L_α the line bundle corresponding to the constant automorphy factors $\alpha \in \mathbb{C}^*$. We illustrate this by constructing the restriction of the universal (Poincaré) line bundle V over $X \times \mathrm{Pic}^0(X)$ to $X \times \mathbb{C}^* := X \times \{0\} \times \mathbb{C}^*$. One starts with a trivial line bundle $\bar{\mathbb{C}}$ on $\Theta^* \times \mathbb{C}^*$, and applies to it the following $\mathbb{Z}$-action

$$\begin{array}{ccc} \Theta^* \times \mathbb{C}^* \times \mathbb{Z} & \longrightarrow & \Theta^* \times \mathbb{C}^* \\ (z,\alpha,n) & \longmapsto & (\tau^n z, \alpha). \end{array}$$

Since this action is trivial on $\mathbb{C}^*$, the Poincaré line bundle V is obtained identifying $s \in \bar{\mathbb{C}}_{(z,\alpha)}$ with $\alpha s \in \bar{\mathbb{C}}_{(\tau z,\alpha)}$.

Notation: In the remainder, we shall denote by L_α the line bundle corresponding to the automorphy factor $\alpha \in \mathbb{C}^*$.

Remarks. 1) For any vector bundle on X, restriction to a fibre of π is a natural operation. Since line bundles are given by constant automorphy factors, their restriction to any fibre is an element of $\mathrm{Pic}^0(T)$. Referring to the above, we have the exact sequence

$$0 \to \mathrm{Pic}^0(B) \times \langle \tau \rangle \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(T) \to 0,$$

$$0 \to \mathrm{Pic}^0(B) \times \mathbb{Z} \to \mathrm{Pic}^0(B) \times \mathbb{C}^* \to T \to 0.$$

2) Although L_{τ^m} is trivial over the fibres of π, one cannot define an action of $\mathbb{Z}$ on $\mathbb{C}^*$ that leaves restriction of the Poincaré line bundle V to $X \times \mathbb{C}^*$ invariant. If we let $\mathbb{Z}$ act on $\mathbb{C}^*$, then the multiplication by τ is defined on the fibres of $\bar{\mathbb{C}}$ as

$$\begin{array}{ccc} \Theta^* \times \mathbb{C}^* \times \mathbb{C} & \xrightarrow{\tau} & \Theta^* \times \mathbb{C}^* \times \mathbb{C} \\ (z, \alpha, t) & \longmapsto & (\tau z, \tau\alpha, \alpha t). \end{array} \tag{2.1}$$

On the surface X, z and τz define the same point x. However, (2.1) indicates that τ sends $V_{(x,\alpha)}$ to $V_{(x,\tau\alpha)} \otimes L_{\tau^{-1}}$. Hence, the Poincaré line bundle is not invariant under such an action.

Since the principal elliptic fibre bundle $X \xrightarrow{\pi} B$ that is topologically non-trivial, we have the following identification (cf. [B1, B2]):

$$NS(X)/Tors(NS(X)) \cong Hom(J_B, Pic^0(T)), \tag{2.2}$$

where J_B denotes the Jacobian variety of B. The torsion of $H^2(X, \mathbb{Z})$ is generated by the class of the fibre. In the remainder, the class modulo $Tors(H^2(X, \mathbb{Z}))$ of an element $c \in H^2(X, \mathbb{Z})$ will be denoted $\widehat{c}$. Given these considerations, we have (see also [A-B-T]):

Lemma 2.3 *For any element $c \in NS(X)$, $\pi_*(c) = 0$ and $c^2 = -2\deg(\widehat{c})$.*

Proof: The lemma is certainly true for torsion classes. Let us then assume that $c \notin Tors(NS(X)$ and choose a line bundle $L \in Pic(X)$ such that $c_1(L) = c$. Then $\widehat{c} \neq 0$ and, by fixing a base-point in B, the cohomology class $\widehat{c}$ can be considered as a covering map $\widehat{c} : B \to Pic^0(T)$ such that

$$\widehat{c}^{\,-1}(\lambda_0) = \{b \in B \mid L|_{F_b} \simeq \lambda_0\}.$$

Since $\widehat{c} \neq 0$, we have $\widehat{c}^{\,-1}(\mathcal{O}_T) \neq B$. Therefore, the stalk of π_*L is zero at the generic point in B and $\pi_*L = 0$. Furthermore, $R^1\pi_*L$ is a torsion sheaf supported on $\widehat{c}^{\,-1}(\mathcal{O}_T)$ and, by Grothendieck-Riemann-Roch, we have:

$$c_1(R^1\pi_*L) = -\pi_*(c) - \frac{1}{2}c^2 \cdot h,$$

where $\pi_*(c) \in H^0(B, \mathbb{Z})$ and h is the positive generator of $H^2(B, \mathbb{Z})$. Hence, $\pi_*(c) = 0$ and $\deg(\widehat{c}) = \#(\widehat{c}^{\,-1}(\mathcal{O}_T)) = -\frac{1}{2}c^2$. $\square$

2.2 The spectral curve of a rank 2 bundle.

Consider a pair $(c_1, c_2) \in NS(X) \times \mathbb{Z}$ and its corresponding *discriminant* $\Delta(2, c_1, c_2) := \frac{1}{2}\left(c_2 - \frac{c_1^2}{4}\right) \geq 0$. In the following, we also associate to such a pair the integer

$$(2.4) \qquad n(2, c_1, c_2) \;:=\; 2\Delta(2, c_1, c_2) - \frac{c_1^2}{4}.$$

Referring to lemma 2.3, we immediately obtain
Remark 2.5 If $\Delta(2, c_1, c_2) \geq 0$, then $n(2, c_1, c_2) \geq 0$.

Let E be a rank 2 vector bundle over X, with $c_1(E) = c_1$ and $c_2(E) = c_2$, where $(c_1, c_2) \in NS(X) \times \mathbb{Z}$. We fix the following notation:

$$\Delta(E) := \Delta(2, c_1, c_2) \quad \text{and} \quad n_E := n(2, c_1, c_2).$$

Lemma 2.6 *Given the above notations, the vector bundle E satisfies*

$$\pi_*(ch(E) \cdot td(X)) = -n_E h,$$

where h is the positive generator of $H^2(B, \mathbb{Z})$.

Proof: Given (2.4), we note that $n_E = -\frac{1}{2}(c_1^2 - 2c_2)$ and $ch(E) = 2 + c_1 - n_E$. Therefore,

$$ch(E) \cdot td(X) = 2 + (c_1 + 2(1-g)\pi^* h) + (-n_E + (1-g)\pi^* h \cdot c_1)$$

and, by the Projection formula,

$$\pi_*(ch(X) \cdot td(X)) = \pi_*(c_1) + (-n_E + (1-g)\pi_*(c_1)) \cdot h.$$

The result then follows from the fact that $\pi_*(c_1) = 0$ (see lemma 2.3). $\square$
Note: The above is also true for rank r vector bundles.

To study bundles on X, one of our main tools will be restriction of the bundle to the fibres $\pi^{-1}(x) \cong T$ of the fibration $\pi : X \to B$. Since the restriction of any bundle on X to a fibre T has first Chern class zero, we consider E as family of degree zero bundles over the elliptic curve T, parametrised by B. Holomorphic vector bundles over elliptic curves have been completely classified by Atiyah [A]. For rank two bundles $\mathcal{E}$ with $c_1(\mathcal{E}) = 0$ we have:

Proposition 2.7 (Atiyah) *Bundles over an elliptic curve T with trivial first Chern class are of the following types :*

(i) $\lambda_0 \oplus \lambda_0'$*, with* $\lambda_0, \lambda_0' \in Pic^0(T)$*.*

(ii) Non trivial extensions $0 \to \lambda_0 \to \mathcal{E} \to \lambda_0 \to 0$*, where* $\lambda_0 \in Pic^0(T)$*.*

(iii) $\lambda \oplus (\lambda')^*$*, with* $\lambda, \lambda' \in Pic^{-k}(T)$*,* $k > 0$*.*

Given a rank two bundle over X, its restriction to a generic fibre of π is of type (i) or (ii). More precisely, we have

Proposition 2.8 *Let E be a rank 2 holomorphic vector bundle over X; then $E|_{\pi^{-1}(x)}$ is of type (iii) on at most an isolated set of points $x \in B$.*

Proof: Similar to that of Remark 2.4 in [T]; see also [B-H] and Proposition 3.1 of the next section.

Note: We shall call these isolated points the *jumps* of the bundle E.

Let $L = L_\alpha$ be a line bundle in $Pic^0(X)$ such that $h^0(\pi^{-1}(x), L^*E) = 0$, for generic x. Referring to Proposition 2.7, the direct image sheaf $R^1\pi_*(L^*E)$ is a torsion sheaf supported on isolated points x such that $E|_{\pi^{-1}(x)}$ has $L|_{\pi^{-1}(x)}$ as a subline bundle and is of type (i) or (ii), or $E|_{\pi^{-1}(x)}$ is of type (iii). If h is the positive generator of $H^2(B, \mathbb{Z})$, then

$$c_1(R^1\pi_*(L^*E)) = -\pi_*(ch(E) \cdot td(X)) \cdot td(B)^{-1} = n_E h.$$

However, if $\Delta(E) \geq 0$, then $n_E \geq 0$ (see remark 2.2): the sheaf $R^1\pi_*(L^*E)$ is supported on n_E points, counting multiplicity.

To obtain a complete description of the restriction of E to the fibres of π, this construction must be repeated for every line bundle on X; this is done by taking the direct image $R^1\pi_*$ for all line bundles simultaneously. Let π also denote the projection

$$\begin{array}{rcl} \pi : X \times \mathrm{Pic}^0(B) \times \mathbb{C}^* & \longrightarrow & B \times \mathrm{Pic}^0(B) \times \mathbb{C}^* \\ (x, a, \alpha) & \longmapsto & (\pi(x), a, \alpha), \end{array}$$

and let $s : X \times \mathrm{Pic}^0(B) \times \mathbb{C}^* \to X$ be the projection onto the first factor. If V is the universal (Poincaré) line bundle over $X \times \mathrm{Pic}^0(B) \times \mathbb{C}^*$, one defines

$$\mathcal{L} := R^1\pi_*(s^*E \otimes V).$$

This sheaf is supported on a divisor $\tilde{C}$ that is defined with multiplicity. We have the following remarks:

– Let H be the pullback to X of a line bundle of degree zero on B. The restriction of H to any fibre T is then trivial, implying that the support of $R^1\pi_*(s^*E \otimes V \otimes H)$ is also $\tilde{C}$. We can therefore restrict ourselves to $X \times \mathbb{C}^* := X \times \{0\} \times \mathbb{C}^*$ and the divisor $\tilde{C}$ descends to $B \times \mathbb{C}^*$. In the remainder, we will use the same notation for the restriction to $X \times \mathbb{C}^*$ of the above construction.

– Consider the $\mathbb{Z}$-action on $B \times \mathbb{C}^*$ induced from the one on $X \times \mathbb{C}^*$, described in section 2.1. For any $(x, \alpha) \in B \times \mathbb{C}^*$, multiplication by τ sends the fibre $\mathcal{L}_{(z,\alpha)}$ to $\mathcal{L}_{(z,\tau\alpha)} \otimes \mathcal{L}_{\tau^{-1}}$, implying that it does not change the support of $\mathcal{L}$: the divisor $\tilde{C}$ is invariant under multiplication by τ. (However, $\mathcal{L}$ is not invariant under this action.)

The quotient of $\mathbb{C}^*$ by the $\mathbb{Z}$-action is $\mathrm{Pic}^0(T) := \mathbb{C}^*/\langle\tau\rangle$. By the above remarks, the support $\tilde{C}$ of $\mathcal{L}$ descends to a divisor C on $B \times \mathrm{Pic}^0(T)$, as summarised in proposition:

Proposition 2.9 *Let $(c_1, c_2) \in NS(X) \times \mathbb{Z}$ be such that $\Delta(2, c_1, c_2) \geq 0$. To each holomorphic vector bundle E on X with $c_1(E) = c_1$ and $c_2(E) = c_2$, there is associated a divisor C in $B \times Pic^0(T)$ called the* spectral curve *of E. This divisor is an n-to-one cover of $Pic^0(T)$, where $n = n(2, c_1, c_2)$, and a double cover of B, denoted $f : C \to B$, of genus $g(C) = b/2 + n(g-1) + 1$, where b is the branching number of f. In other words, the divisor C is an element of the linear system $|2B + nPic^0(T)|$ over $B \times Pic^0(T)$ and we have the following picture:*

$$\begin{tikzcd} Pic^0(T) & C \arrow[l, "n:1"'] \arrow[d, "2:1"] & \subset \; B \times Pic^0(T) \\ & B. & \end{tikzcd}$$

2.3 Regular rank two bundles.

Over a smooth elliptic curve, a rank two vector bundle of degree zero is said to be *regular* if its group of automorphisms is of the smallest dimension. If E is a holomorphic vector bundle over X, then E is regular if and only if its restriction to a fibre T is *never* isomorphic to $\lambda_0 \oplus \lambda_0$, with $\lambda_0 \in \mathrm{Pic}^0(T)$, or of type (iii) (see section 2.2). Furthermore, if a regular bundle E satisfies the conditions of Proposition 2.9, then the restriction of the torsion sheaf $\mathcal{L} = R^1\pi_{1*}(s^*E \otimes V)$ to its support $\tilde{C}$ is a line bundle, and the associated spectral curve C is smooth and irreducible. In this case, one also has that the bundle E can be recovered from the sheaf $\mathcal{L}$, as we shall prove in this section.

Let X_1 and X_2 be two copies of the elliptic bundle X, with projections onto B denoted by π_1 and π_2, respectively. Hence, $X_1 \times_B X_2$ is the fibred product induced by $\pi_i : X_i \longrightarrow B$, for $i = 1, 2$. Let $p_i : X_1 \times_B X_2 \longrightarrow X_i$, $i = 1, 2$, be the natural projections associated to this product. For $i = 1, 2$, we will also denote by p_i the following projections

$$\begin{array}{rcccc} p_i & : & X_1 \times_B X_2 \times \mathbb{C}^* & \longrightarrow & X_i \times \mathbb{C}^* \\ & & (x_1, x_2, \alpha) & \longmapsto & (p_i(x_1, x_2), \alpha), \end{array}$$

and by π_i the projections

$$\begin{array}{rcccc} \pi_i & : & X_i \times \mathbb{C}^* & \longrightarrow & B \times \mathbb{C}^* \\ & & (x_i, \alpha) & \longmapsto & (\pi_i(x_i), \alpha). \end{array}$$

We have the following commutative diagram

$$\begin{tikzcd} & X_1 \times_B X_2 \times \mathbb{C}^* \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_2 \times \mathbb{C}^* \arrow[d, "\pi_2"] \\ s^*E \arrow[r] & X_1 \times \mathbb{C}^* \arrow[r, "\pi_1"] \arrow[d, "s"'] & B \times \mathbb{C}^* \arrow[d] \\ E \arrow[r] & X_1 \arrow[r, "\pi_1"] & B. \end{tikzcd}$$

Moreover, if $\tilde{s}$ is the canonical identification of $X_2 \times \mathbb{C}^*$ with $X_1 \times \mathbb{C}^*$

$$\begin{array}{rcccc} \tilde{s} & : & X_2 \times \mathbb{C}^* & \longrightarrow & X_1 \times \mathbb{C}^* \\ & & (x_2, \alpha) & \longmapsto & (x_2, \alpha) \end{array},$$

we also have the following commutative diagram

$$\begin{tikzcd}
X_1 \times \mathbb{C}^* \arrow[d, "s"'] & X_2 \times \mathbb{C}^* \arrow[l, "\tilde{s}"'] \arrow[dl, "\sigma"] \\
X_1. &
\end{tikzcd}$$

Let E be a regular holomorphic rank 2 vector bundle on X such that $\Delta(E) \geq 0$. We consider the torsion sheaf $\mathcal{L} = R^1\pi_{1*}(s^*E \otimes V)$. Referring to section 2.2, its support $\tilde{C} = \text{supp}(\mathcal{L}) \subset B \times \mathbb{C}^*$ induces the spectral curve C associated to E and the double cover $C \xrightarrow{f} B$. Let $Y = X_1 \times_B C \to C$ be the induced principal elliptic bundle. Then $Y \to C$ is also topologically non-trivial and its total space is a double cover of X_1, by the natural map $Y \xrightarrow{\varphi} X_1$. We then get E back from $\mathcal{L}$ in four steps:

Step 1. Pullback $\mathcal{L}$ to $X_2 \times \mathbb{C}^*$ and then push down to $X_1 \times \mathbb{C}^*$: we obtain the torsion sheaf $\tilde{s}_*(\pi_2^*\mathcal{L})$ whose support is now $\pi_1^{-1}(\tilde{C}) = X_1 \times_B \tilde{C}$.

Step 2. Tensor $\tilde{s}_*(\pi_2^*\mathcal{L})$ by V^* to counter the influence of V in $\mathcal{L}$.

Step 3. Finally, push down to X_1. The fibre of the map $s : X_1 \times \mathbb{C}^* \to X_1$ is not proper. However, since $\tilde{s}_*(\pi_2^*\mathcal{L}) \otimes V^* = \tilde{s}_*(\pi_2^* R^1\pi_{1*}(s^*E \otimes V)) \otimes V^*$, the presence of both V and V^* implies that the $\mathbb{Z}$-action, induced from $X_1 \times \mathbb{C}^*$, is fibre preserving. Taking the quotient with respect to this action, we obtain a torsion sheaf on the compact manifold $(X_1 \times \mathbb{C}^*)/\mathbb{Z} = X_1 \times \text{Pic}^0(T)$ that we denote $L := (\tilde{s}_*(\pi_2^*\mathcal{L}) \otimes V^*)/_\sim$. The support of L is $X_1 \times_B C := Y$ and the restriction of L to Y is a line bundle.

Step 4. Push down L to X_1 to obtain $E \cong s_*(L) = \varphi_*(L|_Y)$.

The above construction is summarised in

Proposition 2.10 *Let E be a regular holomorphic vector bundle on X_1 such that $\Delta(E) \geq 0$ and let C be the spectral curve associated to E. If $\mathcal{L} = R^1\pi_{1*}(s^*E \otimes V)$, we then consider the sheaf $L := (\tilde{s}_*(\pi_2^*\mathcal{L}) \otimes V^*)/_\sim$, where the quotient is taken with respect to the $\mathbb{Z}$-action induced from $X_1 \times \mathbb{C}^*$. Therefore, L is a torsion sheaf with support $Y = X_1 \times_B C$ and the restriction of L to Y is a line bundle. Conversely, $E \cong \varphi_*(L|_Y)$, where $\varphi : Y \to X_1$ is the natural map.*

Proof: Almost identical to that of proposition 3.1 in [M].

3 Existence of rank-2 holomorphic vector bundles over non-Kähler elliptic surfaces

Let $X \xrightarrow{\pi} B$ be a non-Kähler elliptic surface, i.e., the fibration $\pi : X \to B$ is a quasi-bundle over the curve B with general fibre an elliptic curve T and multiple fibres over points $b_1, \ldots, b_\ell \in B$; the multiple fibres are $m_1T_1, \ldots, m_\ell T_\ell$, where m_i is an integer ≥ 2 and T_i is an elliptic curve (corresponding to the quotient of T by a cyclic group of order m_i).

Let E be a holomorphic 2-vector bundle over X and let $\delta = \det E$ be its determinant line bundle. Denote by $c_1 = c_1(E) = c_1(\delta)$ and $c_2 = c_2(E)$ the Chern classes of E. If we let

$$\Delta(E) := \frac{1}{2}(c_2 - c_1^2/4)$$

be the discriminant of E, then a well-known result states that $\Delta(E) \geq 0$ (cf. [B-L, E-F, B-F, B3, LP]).

3.1 The bisection of a rank 2 bundle

Referring to Lemma 1.9, the restriction of δ to any smooth fibre of π has degree zero; hence, E restricts to a rank-2 vector bundle of degree zero over an elliptic curve, on the smooth fibres π. Such bundles have been classified by Atiyah [A]. For a smooth fibre $T_b = \pi^{-1}(b)$, we denote

$$\delta_b = \delta|_{T_b} \in \mathrm{Pic}^0(T) = T^*.$$

Recall that the restriction $E|_{T_b}$ is then one of the following three types:

(i) $\lambda_b \oplus (\delta_b \otimes \lambda_b^{-1})$, $\lambda_b \in \mathrm{Pic}^0(T)$.

(ii) Non-split extensions $0 \to \lambda_b \to E|_{T_b} \to \lambda_b \to 0$, $\lambda_b^2 = \delta_b$, $\lambda_b \in \mathrm{Pic}^0(T)$.

(iii) $\lambda_b \oplus (\delta_b \otimes \lambda_b^{-1})$, $\lambda_b \in \mathrm{Pic}^{-k}(T)$, $k > 0$.

Proposition 3.1 *Let E be a rank-2 vector bundle over $X \xrightarrow{\pi} B$ with discriminant $\Delta(E) \geq 0$. Then $E|_{\pi^{-1}(b)}$ is of type (iii) on at most an isolated (finite) set of points $b \in B$.*

Proof: Suppose that $b \in B$ is a point such that $E|_{\pi^{-1}(b)}$ is of type (iii). Let us consider the elementary modification

$$0 \to E' \to E \to j_* \lambda_b \to 0,$$

where $j : T_b \to X$ is the natural inclusion. Referring to [F2] (Chapter II, Lemma 16), the discriminant of E' is given by

$$\Delta(E') = \Delta(E) + \frac{1}{2} j_* c_1(\lambda_b) < \Delta(E),$$

because $\lambda_b \in \mathrm{Pic}^{-k}(T), k > 0$. Since $\Delta(E') \geq 0$, we get the result. $\square$

Remark 3.2 After performing all (finitely many) necessary elementary modifications, the restriction of the holomorphic rank-2 vector bundle E is of type (i) or (ii) over any smooth fibre of $\pi : X \to B$ (cf. also [A-T]). Hence, to prove an existence theorem, we just have to consider bundles whose restrictions are of type (i) or (ii) over the smooth fibres of π.

Let $J(X) = B \times T^* \overset{p_1}{\to} B$ be the Jacobian surface associated to the elliptic surface X and denote by $B_0 = B \times 0 \subset J(X)$ its zero-section. Consider the bisection (spectral cover) $\overline{C} \subset J(X)$ corresponding to the rank-2 vector bundle E: it has the following geometrical description. For each smooth fibre T_b of the fibration π, we associate to the point b the pair of line bundles $\{\lambda_b, \delta_b \otimes \lambda_b^{-1}\}$, if $E|_{T_b}$ is of type (i), or the line bundle λ_b with multiplicity 2, if $E|_{T_b}$ is of type (ii). More intrinsically, one can obtain the bisection $\overline{C}$ by constructing a local relative Poincaré line bundle on the fibred product $J(X) \times_B X$ (cf. [F-M, M]).

Remark 3.3 The bisection $\overline{C}$ is defined on the complement of the fibres of $J(X)$ that correspond to multiple fibres of π, i.e., on $p_1^{-1}(B \setminus \{b_1, \cdots, b_\ell\})$. In fact, it also extends across such fibres. To see this, it suffices to show that the pull-back of $\overline{C}$ to some finite base change of $J(X) \overset{p_1}{\to} B$ extends to $J(X)$. According to the proof of Lemma 1.9, there exists a principal T-bundle $\pi' : Y \to C$, over a cyclic covering $\varepsilon : C \to B$ with group $\mathbb{Z}_m$, which in turn induces a cyclic covering $\psi : Y \to X$ with group $\mathbb{Z}_m$. In light of this, the Jacobian surface associated to Y is $J(Y) = C \times T^*$ and the covering $\varepsilon : C \to B$ induces a covering $J(Y) \to J(X)$. Therefore, by replacing $X \to B$ with $Y \to C$, which has no multiple fibres, the above construction (see section 2) applies over every fibre.

Let us now define on $J(X) = B \times T^*$ the following involution:

$$\begin{array}{rcl} i : J(X) & \to & J(X) \\ (b, \lambda) & \to & (b, \delta_b \otimes \lambda^{-1}). \end{array}$$

For fixed $b \in B$, the involution $\lambda \mapsto \delta_b \otimes \lambda^{-1}$ induced on the corresponding fibre of $J(X) \xrightarrow{p_1} B$ has four fixed points (the solutions of $\lambda^2 = \delta_b$). If we take the quotient of $J(X)$ with respect to this involution, then each fibre of p_1 becomes $T^*/i \cong \mathbb{P}^1$: the quotient $J(X)/i$ is isomorphic to a ruled surface $\mathbb{F}$ over B. Let $\eta : J(X) \to \mathbb{F}$ be the canonical map. Since the bisection $\overline{C}$ is invariant under the involution i, it is the pull-back of a curve A on the quotient $\mathbb{F}$, which is a section of the ruling. (Indeed, the bisection $\overline{C}$ satisfies $\overline{C}.T^* = 2$ and we have $A \cdot f = \frac{1}{2}\eta^* A \cdot \eta^* f = \frac{1}{2}\overline{C} \cdot T^* = 1$, for a general fibre $f \cong \mathbb{P}^1$ of the ruled surface $\mathbb{F}$.) As a section, A must be smooth; however $\overline{C}$ need not be smooth. It may be reducible or multiple with multiplicity 2.

We can actually say more about the Jacobian surface $J(X)$ and the ruled surface $\mathbb{F}$, as stated in the following two lemmas.

Lemma 3.4 *Let $\mathcal{L}$ be a line bundle over a non-Kähler elliptic surface $\pi : X \to B$. Let Σ be the section of the Jacobian surface $J(X)$ of X defined by the line bundle $\mathcal{L}$ as in section 1. Then*

$$c_1^2(\mathcal{L}) = (\Sigma - B_0)^2,$$

where B_0 is the zero section of $J(X)$.

Proof: The invariants of the surface $J(X) = B \times T^*$ are

$$p_g(J(X)) = g,\ q(J(X)) = g + 1,\ K_{J(X)} = p_1^* K_B,$$

where g is the genus of the base curve B. For any section Σ of $J(X)$, the adjunction formula gives $\Sigma^2 = 0$; hence,

$$(\Sigma - B_0)^2 = -2\Sigma \cdot B_0.$$

By [B1, B-U], there is an isomorphism

$$NS(X)/\mathrm{Tors}NS(X) \cong \mathrm{Hom}(J_B, T^*),$$

as in the case of principal non-Kähler elliptic bundles. If we denote by $\hat{c}_1$ the class modulo torsion of $c_1(\mathcal{L})$, then we can think of $\hat{c}_1 : J_B \to T^*$ as a covering map $\hat{c}_1 : B \to T^*$; furthermore, the proof of Lemma 1.3 in [A-B-T] implies that $\hat{c}_1^2 = -2\deg(\hat{c}_1)$. But $\deg(\hat{c}_1) = \Sigma \cdot B_0$, and the result follows. □

Lemma 3.5 *The ruled surface $\mathbb{F}$ has the invariant $e = 0$.*

Proof: Let A be a section of the ruled surface $\mathbb{F}$. Then η^*A is a bisection of $J(X)$. If this bisection is reducible, then $\eta^*A = C_1 + C_2$, where C_1 and C_2 are sections of $J(X)$, and

$$2A^2 = (\eta^*A)^2 = (C_1 + C_2)^2 = 2C_1 \cdot C_2 \geq 0.$$

If the bisection is irreducible, then by the construction given in the proof of Theorem 3.6, there is an induced map $\gamma' : J(W) \to J(X)$ that is two-to-one and is such that $\gamma'^*(\eta^*(A)) = C_1 + C_2$, where C_1 and C_2 are sections of the Jacobian surface $J(W) = C \times T^*$ (we use the notation of that proof). Then, we obtain again

$$4A^2 = 2C_1 \cdot C_2 \geq 0,$$

and since

$$-e = \min\{A^2 | A \text{ section of } \mathbb{F}\}$$

(cf. Proposition 12 of Chapter 5 in [F2]), it follows that $e \leq 0$.

Let R be the fixed point set of the involution $i : J(X) \to J(X)$ defined above. Given that $J(X)$ is smooth surface with no singular fibres, the fixed point set R is a smooth curve of $J(X)$, which is a 4-section. But R is the ramification divisor of the double covering $\eta : J(X) \to \mathbb{F}$; therefore, the branch divisor $\eta_*(R) = D'$ of η is a smooth 4-section. Referring to [B-P-V] (Proposition 9.1, Chapter V), there is an isomorphism $J(X) \to J(X)$ such that the zero section B_0 is mapped onto the fixed section Σ_δ corresponding to the fixed line bundle $\delta \in \mathrm{Pic}(X)$ (cf. Remark 1.10). In fact, the isomorphism is simply given by $(b, t) \mapsto (b, \delta_b^{-1} t)$, for any $(b, t) \in J(X)$, and the section Σ_δ is now the zero section of the Jacobian $J(X)$: it is fixed by the involution i. Hence, $A_0 = \eta_*\Sigma_\delta$ is a section of the ruled surface $\mathbb{F}$ and $\eta^*A_0 = 2\Sigma_\delta$; the branch divisor of η is the sum of A_0 with a smooth 3-section D''. In other words,

$$2A_0^2 = (\eta^*A_0)^2 = 4(\Sigma_\delta)^2 = 0,$$

and $e = 0$. $\square$

3.2 Existence theorems

Let us now consider the case where the bisection $\overline{C}$ is reducible or a section of the Jacobian surface counted with multiplicity 2. Choosing a component of $\overline{C}$ determines a section Σ of $J(X)$, i.e., there exists a holomorphic map $\varphi : B \to T^*$ such that $\Sigma = (id_B \times \varphi)(B)$. By Proposition 1.11, there exists a line bundle $\mathcal{D}'$ on X such that $\mathcal{D}'|_{T_b} = \Sigma_b = \varphi(b)$, on any smooth fibre T_b of π. By construction, the vector bundle $E \otimes (\mathcal{D}')^{-1}|_{T_b}$ has a section. Therefore, the vector bundle $\pi_*(E \otimes (\mathcal{D}')^{-1}) \neq 0$ and by tensoring with a very ample line bundle $\mathcal{A}$ on B, the vector bundle $\pi_*(E \otimes (\mathcal{D}')^{-1}) \otimes \mathcal{A}$ will have a non-zero section. Consequently, the vector bundle $E \otimes (\mathcal{D}')^{-1} \otimes \pi^*(\mathcal{A})$ also has a non-zero section. By standard arguments, the initial holomorphic rank-2 vector bundle E is given as an extension

$$0 \to \mathcal{D} \to E \to \delta \otimes \mathcal{D}^{-1} \otimes I_Z \to 0,$$

where $Z \subset X$ is a locally complete intersection of codimension 2. Indeed, if we denote $\mathcal{D}'' = \mathcal{D}' \otimes \pi^*(\mathcal{A}^{-1})$, we have an injective map $\mathcal{D}'' \to E$. If the quotient $E/\mathcal{D}''$ is not torsion-free, then by taking the inverse image of the torsion of $E/\mathcal{D}''$, we obtain another line bundle $\mathcal{D}$ and two injective maps

$$\mathcal{D}'' \to \mathcal{D}, \ \mathcal{D} \to E.$$

Furthermore, $\mathcal{D} \cong \mathcal{D}'' \otimes \mathcal{O}_X(D)$, where D is an effective divisor on X. However, any divisor on X is a combination of fibres; therefore, the line bundles $\mathcal{D}'$, $\mathcal{D}''$ and $\mathcal{D}$ have the same restriction Σ_b to any smooth fibre T_b of $\pi : X \to B$ (see Proposition 1.11). Changing $\mathcal{D}''$ by $\mathcal{D}$ we get a torsion free quotient and the above exact sequence.

Let us now consider the case where the bisection $\overline{C}$ is irreducible and let C be the normalisation of $\overline{C}$. The map $p_1 : J(X) \to B$ induces a map $\gamma : C \to B$ of degree 2. Let W be the normalisation of $X \times_B C$. Since the surface $\pi : X \to B$ has no singular fibres besides the multiple fibres of type $m_i T_i$ (where T_i is a smooth elliptic curve), W is smooth and we have a commutative diagram

$$\begin{tikzcd} W \arrow[r, "\tilde{\gamma}"] \arrow[d, "\rho"'] & X \arrow[d, "\pi"] \\ C \arrow[r, "\gamma"'] & B, \end{tikzcd}$$

where $\tilde{\gamma}$ is generically two-to-one. The map $C \to \overline{C} \subset J(X)$ defines a section $\tilde{\Sigma}_1$ of $J(X) \times_B C$, which is the Jacobian surface $J(W)$ of W (W is again a non-Kähler elliptic surface). By using the section $\tilde{\Sigma}_1$, we obtain an exact sequence

$$0 \to \tilde{\mathcal{D}} \to \tilde{\gamma}^* E \to \tilde{\gamma}^* \delta \otimes \tilde{\mathcal{D}}^{-1} \otimes I_{\tilde{Z}} \to 0,$$

where $\tilde{Z} \subset W$ is a locally complete intersection of codimension 2.

Since $\deg(\tilde{\gamma}) = 2$, we have

$$2c_1^2(E) = c_1^2(\tilde{\gamma}^* E), 2c_2(E) = c_2(\tilde{\gamma}^* E),$$

which implies

$$4\Delta(E) = 2\Delta(\tilde{\gamma}^* E) = c_1(\tilde{\mathcal{D}}).c_1(\tilde{\gamma}^* \delta \otimes \tilde{\mathcal{D}}^{-1}) + \ell(\tilde{Z}) - \frac{1}{4} c_1^2(\tilde{\gamma}^* \delta).$$

Therefore,

$$4\Delta(E) = -\frac{1}{4}(c_1(\tilde{\mathcal{D}}) - c_1(\tilde{\gamma}^* \delta \otimes \tilde{\mathcal{D}}^{-1}))^2 + \ell(\tilde{Z}).$$

By Proposition 1.11, both $\tilde{\mathcal{D}}$ and $\tilde{\gamma}^* \delta \otimes \tilde{\mathcal{D}}^{-1}$ naturally correspond to sections of $J(W)$. In fact, if we denote by $\gamma' : J(W) \to J(X)$ the obvious map, then the bisection $\overline{C}$ satisfies $\gamma'^*(\overline{C}) = \tilde{\Sigma}_1 + \tilde{\Sigma}_2$, where $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are sections of $J(W)$ corresponding to the line bundles $\tilde{\mathcal{D}}$ and $\tilde{\gamma}^* \delta \otimes \tilde{\mathcal{D}}^{-1}$, respectively. By an argument similar to the one used to prove Claims 1.17 and 1.18 in Chapter 7 of [F-M] (and that extends the Lemma 3.4) we show that

$$(c_1(\tilde{\mathcal{D}}) - c_1(\tilde{\gamma}^* \delta \otimes \tilde{\mathcal{D}}^{-1}))^2 = (\tilde{\Sigma}_1 - \tilde{\Sigma}_2)^2.$$

Referring to the proof of Lemma 3.4, for any section Σ of $J(W)$ we have $\Sigma^2 = 0$. Then

$$(\tilde{\Sigma}_1 - \tilde{\Sigma}_2)^2 = 2\tilde{\Sigma}_1^2 + 2\tilde{\Sigma}_2^2 - (\tilde{\Sigma}_1 + \tilde{\Sigma}_2)^2 = -\gamma'^* \overline{C}^2 = -2\overline{C}^2,$$

and it follows that

$$\Delta(E) = \frac{1}{8}\overline{C}^2 + \frac{1}{4}\ell(\tilde{Z}).$$

We have proven the following result, which is similar to Theorem 1.3, Chapter VII, [F-M]:

Theorem 3.6 *Let $\pi : X \to B$ be a non-Kähler elliptic surface and E be a holomorphic rank-2 vector bundle over X whose restriction to any smooth fibre of π is of type (i) or (ii). Let δ be the determinant line bundle of E. Then E satisfies one of the following two cases:*

(A) There exists a line bundle $\mathcal{D}$ on X and a locally complete intersection Z of codimension 2 in X such that E is given by an extension

$$0 \to \mathcal{D} \to E \to \delta \otimes \mathcal{D}^{-1} \otimes I_Z \to 0.$$

(B) There exists: (i) a smooth irreducible curve C and a birational map $C \to \overline{C} \subset J(X)$, where $\overline{C}$ is a bisection that is invariant under the involution i on $J(X)$ defined by $\delta = det(E)$;

(ii) a line bundle $\tilde{\mathcal{D}}$ on the normalisation W of $X \times_B C$, whose restriction to a smooth fibre of $W \to C$ is the same as that induced by the section of $J(W)$ that corresponds to the map $C \to J(X)$;

(iii) a codimension 2 locally complete intersection $\tilde{Z}$ in W and an exact sequence

$$0 \to \tilde{\mathcal{D}} \to \tilde{\gamma}^* E \to \tilde{\gamma}^* \delta \otimes \tilde{\mathcal{D}}^{-1} \otimes I_{\tilde{Z}} \to 0,$$

where $\tilde{\gamma} : W \to X$ is the natural, generically two-to-one map.

Furthermore, we have

$$\Delta(E) = \frac{1}{8}\overline{C}^2 + \frac{1}{4}\ell(\tilde{Z}).$$

□

The bound of $\Delta(E)$, given in Theorem 3.6, can be improved as follows:

Theorem 3.7 *Let $\pi : X \to B$ be a non-Kähler elliptic surface and δ be a line bundle in $Pic(X)$. Furthermore, let $i : J(X) \to J(X)$ be the involution defined by δ and suppose that $\overline{C}$ is an irreducible bisection of $J(X) \to B$, which is invariant by the involution i. Then, there exists a rank-2 holomorphic vector bundle V on X such that*

$$c_1(V) = c_1(\delta), \ \Delta(V) = \frac{1}{8}\overline{C}^2 = \frac{1}{4}A^2,$$

where A is a section of the ruled surface $\mathbb{F}$ with $\eta^* A = \overline{C}$.

Proof. Let $C \to \overline{C}$ be the normalisation of $\overline{C}$ and let $\gamma : C \to B$ be the two-to-one map induced from the map $C \to \overline{C} \subset J(X) = B \times T^*$. If W is the normalisation of the fibred product $C \times_B J(X)$, then $W \to C$ is a non-Kähler elliptic surface with associated Jacobian surface $J(W) = C \times T^*$. Also, there is a two-to-one covering map $\tilde{\gamma} : W \to X$, which induces the covering map $\gamma' : J(W) \to J(X)$. The map $C \to C \times_B J(X)$ gives a section Σ_1 of $J(W) \to C$ and $\gamma'^*\overline{C} = \Sigma_1 + \Sigma_2$, where Σ_2 is another section of $J(W)$, i.e., $\gamma'^*\overline{C}$ is reducible. By Proposition 1.11, there exists a line bundle $\mathcal{L}$ such that the restriction of $\mathcal{L}$ to any smooth fibre T_c of W is Σ_{2c}. Let $\mathcal{D}$ be the line bundle defined by the equality

$$\mathcal{L} \cong \tilde{\gamma}^*\delta \otimes \mathcal{D}^{-1}.$$

In addition, let $\bar{i}$ be the involution on W that interchanges the sheets of $\tilde{\gamma}$. If $G \subset X$ is the (smooth) branch divisor of the double covering $\tilde{\gamma} : W \to X$, then there exists a line bundle $L_0 \in \mathrm{Pic}(X)$ such that $L_0^2 = \mathcal{O}_X(G)$.

Define the holomorphic 2-vector bundle V on X by

$$V := \tilde{\gamma}_*(\mathcal{L}).$$

By Lemma 29, Chapter 2 of [F2] or by [B4], there is an exact sequence:

$$0 \to \bar{i}^*\mathcal{L} \otimes \tilde{\gamma}^* L_0^{-1} \to \tilde{\gamma}^*\tilde{\gamma}_*(\mathcal{L}) \to \mathcal{L} \to 0.$$

Since the involution $\bar{i}$ on W is induced by interchanging the sheets of the double cover $C \to B$, and $\bar{i}$ is induced by the involution i on $J(X)$ defined by δ, it follows that the restriction of $\bar{i}^*\mathcal{L}$ to any smooth fibre T_c of W (which is not in the ramification locus of $\tilde{\gamma}$), is the same as the restriction of $\mathcal{D}$, namely as Σ_{1c}. From the preceding exact sequence, we get

$$0 \to \mathcal{D} \otimes \mathcal{O}_W(F) \to \tilde{\gamma}^*V \to \tilde{\gamma}^*\delta \otimes \mathcal{D}^{-1} \to 0,$$

where F is a divisor on W (hence a combination of fibres of the non-Kähler elliptic surface $W \to C$).

Furthermore, as in the last part of the proof of Theorem 3.6, we have

$$\Delta(V) = \frac{1}{8}\overline{C}^2 = \frac{1}{4}A^2,$$

where A is the section of the ruled surface $\mathbb{F}$ defined by the bisection $\overline{C}$. By [A-B-T], we also get

$$c_1(V) \equiv c_1(\delta) \mod \mathrm{Tors}\,(NS(X)).$$

To get rid of the torsion, we need to add multiples of classes of fibres. Then, as in [A-B-T], we can modify the line bundle $\mathcal{L}$, by tensoring it with line bundles of the form $\mathcal{O}_W(T_c)$ or $\mathcal{O}_W(T_i)$, and obtain the desired result

$$c_1(V) = c_1(\delta).$$

Note that the discriminant remains unchanged (see the formula in [A-B-T] for the direct image of a line bundle). $\square$

Remark 3.8 Let $\pi : X \to B$ be a non-Kähler elliptic surface and E a holomorphic rank-2 vector bundle on X with determinant δ, which satisfies case (B) of Theorem 3.6. If the following conditions are satisfied: (1) the bisection $\overline{C}$ is smooth (i.e., $\overline{C} = C$), (2) the branch points of the map $C \to B$ do not lie under the multiple fibres of π (so that $W = X \times_B C$), and (3) $\tilde{Z} = \emptyset$, then one can prove that $E \cong V$ (where V is the holomorphic 2-vector bundle of Theorem 3.7), as in the proof of Theorem 1.12, Chapter VII of [F-M].

Remark 3.9 The necessary and sufficient condition, for the existence of filtrable holomorphic vector bundles on the compact complex surfaces, was given in [B-L] (see also [B-F],[LP]). Therefore, only the case of non-filtrable holomorphic vector bundles remains to be studied. For non-Kähler elliptic surfaces, the above results imply that the existence problem for vector bundles is reduced to the existence problem of irreducible bisections, in the Jacobian surface associated to the given elliptic surface; moreover, these bisections must be invariant under the involution constructed above. Equivalently, the problem is reduced to the existence of sections of a ruled surface, whose inverse images are irreducible.

We obtained the following existence result:

Theorem 3.10 *Let $\pi : X \to B$ be a non-Kähler elliptic surface and let $c_1 \in NS(X)$ and $c_2 \in \mathbb{Z}$ be fixed such that $m(2, c_1) > 0$ and $\Delta(2, c_1, c_2) \in [0, m(2, c_1))$. Then, there exists a holomorphic rank-2 vector bundle over X with Chern classes c_1, c_2 if and only if there exists a line bundle $\delta \in Pic(X)$ with $c_1(\delta) = c_1$ and such that, if $i : J(X) \to J(X)$ is the involution defined by δ and $\eta : J(X) \to \mathbb{F}$ is the quotient map, there exists a section A of the ruled surface $\mathbb{F}$ with $\eta^* A$ irreducible in $J(X)$ and $A^2 = 4\Delta(2, c_1, c_2)$.*

Remark 3.11 The case $\Delta(2, c_1, c_2) = 0$ was studied in [A-B]. In contrast to the case of primary Kodaira surfaces, for non-Kähler elliptic surfaces with base curve of genus $g(B) > 1$ there are "gaps" for the discriminant of holomorphic rank-2 vector bundles. For example, if $g(B) = 3$, there are no sections A of the ruled surface $B \times \mathbb{P}^1$ with $A^2 = 2$. Also, just one of the two value 4, 6 is possible for A^2, depending on whether B is hyperelliptic or not (see [Ha], Chapter V). Thus, the existence of holomorphic vector bundles on X depends on the geometry of the base curve B.

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